On Approximately Optimal $H^\infty$ Controllers for Distributed Systems

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2 Preliminary Remarks

The sensitivity minimization problem is to find an internally stabilizing controller $C$ such that the following optimum performance is achieved:

$$
\inf_{C \in H^\infty} \| W(1 + PC)^{-1}\|_{\infty} = \mu
$$

See Figure 1 for the closed loop set-up, where $P$ is the plant to be controlled and $W$ is the weight modeling the disturbances. Assuming that the weight is an outer function and the plant $P$ is stable we can transform this problem to a Nehari problem (in the usual way of first inverting the Youla parametrization for the controller $C = Q_0(1 + PQ_0)^{-1}$; $Q_0 \in H^\infty$, $(1 - PQ_0) \neq 0$; and then finding an inner/outer factorization for $P = mP_0$, where $m$ is inner and $P_0$ is outer):

$$
\mu = \inf_{Q_0 \in H^\infty} \| W - mQ_0 \|_{\infty}
$$

Conversely, from (1) by finding $Q$ realizing $\mu$, and by inverting $W$ and $P_0$, we get the optimal controller $C_0$ which internally stabilizes the system and satisfies

$$
\| W(1 + PC_0)^{-1}\|_{\infty} = \mu.
$$

Given a tolerance $\varepsilon > 0$, we say that $C_0$ is approximately optimal (or sub-optimal), with tolerance $\varepsilon$, if it internally stabilizes the system and satisfies the bound

$$
\| W(1 + PC)^{-1}\|_{\infty} \leq \mu + \varepsilon =: \rho.
$$

One important point which should be emphasized is that when the plant is strictly proper its outer part $P_0$ is only approximately invertible in $H^\infty$ as a stable causal transfer function, so a proper optimal controller does not exist in general. Nevertheless, even if $P_0$ is infinite dimensional, there are good rational approximations for the inverse of such outer functions, that can be used in the implementation of an approximately optimal controller. However, this is not the only problem in constructing the $H^\infty$ controllers. The issue to be discussed first in this paper is what happens when the inner part $m$ of the plant is infinite dimensional. So, in Section 4 the question of approximating the inverse of $P_0$ will be left aside, and it will be assumed that $P_0^{-1}$ is in $H^\infty$. Then, in Sections 5 and 6 we consider the case where $P_0$ is only approximately invertible.

The difficulty in the case of infinite dimensional $m$ comes from the fact that $C_0$ in (2) is infinite dimensional. Hence, implementation of the optimal controller is not easy. Another (possibly more serious) problem is that $C_0$ is very sensitive to the parameters of $m$. In other words, if we use approximate values for those parameters (instead of the exact ones) in the controller, the resulting closed loop system may not be stable.

In the light of the above discussion we now assume that $m$ is finite dimensional, and consider the following problem: given $\rho \geq \mu$, find the set of all $Q \in H^\infty$ such that

$$
\| W - mQ \|_{\infty} \leq \rho.
$$

Let us summarize the results of [5] in connection with the above problem. Suppose that the weight is rational: $W(z) = p(z)/q(z)$ where $p(z) = p_0 + zp_1 + \ldots + z^n p_n$ and $q(z) = q_0 + zq_1 + \ldots + z^n q_n$ (i.e., $n$ is the maximum of the degrees of $p$ and $q$, so some of the above coefficients may well be zero).

Let $S$ denote the unilateral shift on $H^2$ and define the space $H(m) = H^2 \ominus mH^2$. Then the compressed shift associated with $H(m)$ is defined as $T := P_{H(m)}S|_{H(m)}$, where $P_{H(m)}$ denotes orthogonal projection onto $H(m)$.

First, consider the optimal case: $\rho = \mu$. The optimal interpolant $Q_{opt}$, which makes $\| B_{opt}\|_{\infty} = \rho$, where

$$
B_{opt} = W - mQ_{opt}.
$$
can be computed using Sarason's theorem ([8]) which states that
\[ \mu = \|W(T)\| = \|W(T)q(T)\|^{-1}. \]
The essential norm can be defined as follows:
\[ \|W(T)\|_0 = \sup \{ \|W(z)\| : \mu \text{ is a singular point of } m \}. \]
We need to assume \( \mu > \|W(T)\|_0 \), see [6], to conclude that \( W(T) \) attains its norm at a singular value \( \mu = \mu \). In this case there exists a singular vector \( h_0 \), for the so-called ([5]) skew Toeplitz operator
\[ A_\mu = \rho^*q(T)p(T)^* - p(T)p(T)^*. \]
(*) denotes adjoint) which makes
\[ A_\mu h_0 = 0. \]
The vector \( h_0 \) can be computed explicitly from the problem data \( W = \rho/q \) and \( m \) in terms of a determinantal formula, see [6], and [8]. Then, \( B_{opt} \) can be found via Sarason's result as
\[ B_{opt} = \rho^*q(T)p(T)^*h_0 / \rho p(T)^*h_0. \]

Let us now consider the case where: \( \rho > \mu \). It is obvious that in this case \( A_\mu \) is invertible and its inverse can be computed explicitly; again, the formula is given in [8]. This is going to be used in the characterization of all the suboptimal solutions \( Q_\mu \in H^m \) which make
\[ \|W - mQ_\mu\|_\infty \leq \rho. \]
(4)

This characterization is obtained using the one step extension procedure of [1]. Here we want to summarize the method briefly. Set \( m(z) := \rho m(z) \) and let \( T_c \) denote the compression of \( S \) to \( H(m) \cap \mathbb{C}^C \). For \( \alpha \in \mathbb{C} \) fixed, the problem of finding \( B_{opt}(\cdot, \alpha) = (W - \alpha m - Q_{opt}(\cdot))(z) \) such that
\[ \|B_{opt}(\cdot, \alpha)w\|_\infty = \|\|W - \alpha m\|\|T_c\|\| = \rho \]
can be solved using the technique described above for the optimal case. From the one step extension theory ([1]) we know that the set of all such \( \alpha \in \mathbb{C} \) form a circle, say \( \Gamma \). Furthermore, the equation of \( \Gamma \) can be explicitly calculated. Then, the set of all suboptimal solutions \( Q_\mu \in H^m \) satisfying (4) is obtained in terms of \( B_{opt}(\cdot, \alpha)(u) \),
\[ W - mQ_\mu = B_{opt}(\cdot, \alpha)(u), \]
where \( \phi(z) \) is a linear fractional map taking the unit circle to \( \Gamma \), and \( u \in H^m \), \( \|\phi\|_\infty \leq 1 \) is the free parameter. The explicit characterization is as follows. Set
\[ g_1 := (\rho^*q(T)p_{m_1}(\cdot)S^* - p_{m_1}(\cdot)p_{m_2}(S))m, \]
\[ g_2 := p_{m_2}(T^*)(1 - m m(0)), \]
and
\[ h_1 := A_{g_1}^+ g_2, \]
\[ h_2 := A_{g_2}^+ g_1. \]
For a given \( \alpha \in \Gamma \) define
\[ h_0(z) := m(z) - h_1(z) - \alpha h_2(z), \]
and
\[ B(z, \alpha) := (\rho^*(\cdot)^* h_0) / (\rho S^* h_0 - S h_0). \]
Then we have the following result.

Theorem 1 ([5]) The set of all functions of the form
\[ B(z) = W(z) - m(z)Q_\mu(z) \]
with \( Q_\mu \in H^m \), such that \( \|B(z)\|_\infty \leq \rho \), is given by
\[ \{ B(z, \alpha) = (\rho^*q(T)p_{m_1}(\cdot)S^* - p_{m_1}(\cdot)p_{m_2}(S))m / (\rho S^* h_0 - S h_0), \alpha \in \Gamma, \|\phi\|_\infty \leq 1 \} \]
where \( \tau \) and \( \eta \) are certain explicitly computable constants. See [8] for the formulae.

3 Structure of the suboptimal controllers
From the above parametrization we are going to obtain the structure of all suboptimal \( H^m \) controllers. Using the notation of Theorem 1, we set \( B_{opt}(z) := B(z, \alpha) \). We can find the controller from \( C := Q_\mu(1 - PQ_\mu)^{-1} \), the Youla parametrization, where \( Q_\mu \) is such that
\[ B_{opt} = W - mQ_\mu. \]

Therefore,
\[ C = (S - P)Q_\mu^{-1}. \]
We now study \( B_{opt} \),
\[ B_{opt}(z) = \rho^*q(z)h_0(z) / \rho h_0(z) - h_0(z) + \alpha h_0(z) \]
where \( h_0(z) \) and \( h_0(z) \) are polynomials of degree \( 1 \) and \( \alpha = 0 \). Let \( C \) be the set of all \( C \) such that \( C \) is \( \rho^*q(z)+\alpha h_0(z) \). Then,
\[ P\alpha Q_{\mu} = \left( \frac{p_{\mu}(\cdot)h_0(z) - h_0(z) - \alpha h_0(z)}{p_{\mu}(\cdot)h_0(z) - h_0(z)} \right)^{-1} \]
\[ = \frac{1}{\alpha} \left( \frac{\rho^*q(z)h_0(z) - h_0(z) - \alpha h_0(z)}{p_{\mu}(\cdot)h_0(z) - h_0(z)} \right), \]
where \( (\cdot) = \rho^*q(z)h_0(z) - h_0(z) \). Recall that \( h_0(z) = m(z) - h_1(z) - \alpha h_2(z) \). It is easy to see from the invariance of the skew Toeplitz operator \( A_{\mu} \) that \( h_1 \) and \( h_2 \) have the following form (see e.g. Lemma 2.1 and Corollary 2.5 of [3])
\[ h_1(z) = f_1(z) + m(z)F_1(z) \]
and
\[ h_2(z) = f_2(z) + m(z)F_2(z) \]
for some \( f_1, f_2, F_1, F_2 \) polynomials of degree \( \leq 2n \). This leads us to the following expression:
\[ P\alpha Q_{\mu} = \left( \frac{\rho^*q(z)h_0(z) - h_0(z) - \alpha h_0(z)}{p_{\mu}(\cdot)h_0(z) - h_0(z)} \right) \]
\[ = \frac{1}{\alpha} \left( \frac{\rho^*q(z)h_0(z) - h_0(z) - \alpha h_0(z)}{p_{\mu}(\cdot)h_0(z) - h_0(z)} \right), \]
Note that
\[ -\frac{1}{\rho^*q(z)h_0(z) - h_0(z) - \alpha h_0(z)} = \frac{1}{\rho^*q(z)h_0(z) - h_0(z)} - \frac{1}{\rho^*q(z)h_0(z) - h_0(z) - \alpha h_0(z)}. \]
We summarize the above formulae with the following.

Corollary 1 The set of all controllers which internally stabilise the plant \( P \), and satisfy the bound
\[ \|W(1 + PC)^{-1}\|_\infty \leq \rho \]
for \( \rho \geq \mu \), have the form
\[ C = \left( \frac{W(\cdot)W^{-1}(\cdot)}{\rho^2} - 1 \right) \frac{G_0(\cdot)}{1 + m(z)G_0(\cdot)} P^{-1}(\cdot) \]
\[ u \in H^m, \|u\| \leq 1, \text{ where } G_0(\cdot) \text{ is a linear fractional transformation in the free parameter } \]
\[ \text{free parameter } u \text{ depending on } m, \text{ moreover } u \text{ itself must be infinite dimensional so that the infinite dimensional parts of the controller gets cancelled.} \]
4 Finite dimensional approximately optimal controllers

Recall from the Theorem 1 that the structure described in the Corollary 1 is valid for the optimal controller as well. In the case of the optimal controller, however, the free parameter is absent in the term $G_m$, that is, instead of $G_m$, we have a fixed rational function, say $G$. Hence, if we replace $m$ by a rational function $m_f$ in the feedback path around $G$ we obtain a finite dimensional controller. We now study the effects of this approximation of $m$ by $m_f$. More specifically we want to answer the following questions: under which conditions the stability is preserved, and what is the deviation from the optimal performance?

For simplicity of the notation and the computations, we will restrict ourselves to the following case:

$$ W(z) = \frac{p(z)}{q(z)} \quad q(z) = 1, \quad p(z) = p_0 + p_1 z,$$

and $m(z)$ is any arbitrary inner function. Then, the optimal controller can be computed as (see Appendix A),

$$ C(z) = \left( \frac{W(z) W(z^{-1})}{p^2} - 1 \right) \frac{G(z)}{1 + m(z) G(z)} P_m^{-1}(z) \quad (5b) $$

with $G(z) = \frac{R(z)}{R(z)+\Delta}$. Consequently, the optimal sensitivity is

$$ B_m(z) = W(z) \left( 1 + m(z) p(z) / \rho \right) / \left( 1 + m(z) p(z) / \rho \right) P_m^{-1}(z). $$

Now we replace $m$ by $m_f$ in the expression for the controller, so that the controller becomes a finite dimensional transfer function:

$$ C_f(z) := \left( \frac{W(z) W(z^{-1})}{p^2} - 1 \right) \frac{p(z)}{1 + m_f(z) p(z)} P_m^{-1}(z). $$

It is easy to see that if we use $C_f$, in the closed loop as a controller, then the sensitivity function becomes

$$ B_f(z) = W(z) \left( 1 + m(z) p(z) / \rho \right) / \left( 1 + m(z) p(z) / \rho \right) P_m^{-1}(z), $$

where $\Delta = (m(z) - m_f(z))/p$.

Set

$$ R(z) := \frac{p(z)}{\rho \rho(z) R(z)}, \quad \text{and} \quad \Delta_m(z) := m(z) - m_f(z). $$

Then we can rewrite $B_f$ as

$$ B_f(z) = B_m(z) \frac{R(z)}{R(z) + \Delta_m(z)} + W(z) \frac{\Delta_m(z)}{R(z) + \Delta_m(z)}. $$

This expression shows that a rational function $m_f$, which makes $C_f$ approximately optimal, can be found by studying the relation between the terms $R$ and $\Delta_m$.

From this point on, in the examples that we are going to consider, we will conduct our analysis and design in the right half plane, which is more natural for continuous time systems. When we do this we transform the problem data by using the conformal map $z = e^{j\theta}$ between the right half plane and the unit circle. In particular $R(z)$ denotes $R(1+e^{j\theta})$, and $\Delta_m(z) = \Delta_m(1+e^{j\theta})$, and similarly for all the other transfer functions.

Let us now compare $R$ and $\Delta_m$ to analyze the approximate optimality of $C_f$.

First of all in order to guarantee stability we should have

$$ R(z) + \Delta_m(z) \neq 0 $$

inside the closed right half plane. Also, since we are looking for a performance close to the optimum, the $H^\infty$ norm of $B_f$ should be close to $\rho$. Note that if we could make $|B_m(j\omega)| \geq |B_f(j\omega)|$ for all $\omega \geq 0$, then we would have $|B_f(j\omega)| \approx |B_m(j\omega)|$ for all $\omega \geq 0$, which implies that $\|B_f\|_\infty \approx \|B_m\|_\infty = \rho$.

However this is not possible in general, because there is no good uniform (on the imaginary axis) rational approximation for an irrational inner function which has essential singularities on the boundary. This is the main difficulty in finding the finite dimensional approximately optimal $H^\infty$ controllers for distributed systems with invertible outer part. In the next section, we generalize the above idea, of designing $m_f$ by comparing $R$ with $\Delta_m$, to plants which can be approximated uniformly on the imaginary axis.

5 On the outer part of the plant

In this section we consider the case where the plant $P(z)$ is continuous on the unit circle. In other words the outer part of the plant is such that the essential singularities of the inner part gets killed. For example if a plant with transportation delay has strictly proper outer part $P_0$, then it becomes continuous on the unit circle. This kind of plants can be approximated perfectly (up to a certain tolerance) by finite dimensional transfer functions. However, in this situation $P_0$ is not invertible in $H^\infty$, so we must find an approximate inverse.

Recall the structure of the optimal controller:

$$ C(z) = \left( \frac{W(z) W(z^{-1})}{p^2} - 1 \right) \frac{G(z)}{1 + m(z) G(z)} P_m^{-1}(z). $$

We can rewrite this as

$$ C(z) = \left( \frac{W(z) W(z^{-1})}{p^2} - 1 \right) \frac{G(z) P_m^{-1}(z)}{1 + P(z) G(z) P_m^{-1}(z)}. $$

A finite dimensional controller can be obtained by approximating $P$ and $P^{-1}$ separately:

$$ C_f(z) = \left( \frac{W(z) W(z^{-1})}{p^2} - 1 \right) \frac{G(z) P_f^{-1}(z)}{1 + P(z) G(z) P_f^{-1}(z)}, $$

where $P_0$ and $P_0^{-1}$ are finite dimensional proper approximations for $P$ and $P^{-1}$ respectively. See Figure 3 for the implementation of $C_f$.

Let us now analyze the performance of the closed loop system under this finite dimensional controller. After a simple algebra similar to the one in Section 4 we see that the sensitivity function $B_f := W(1+PG^{-1})^{-1}$ is in the form

$$ B_f(z) = B_m(z) \frac{R(z)}{R(z) + \Delta(z)} + W(z) \frac{\Delta(z)}{R(z) + \Delta(z)}. $$

where

$$ R(z) := \left( 1 + P(z) R(z) \right) m(z) G(z) (1 - \rho^2), $$

$$ \delta(z) := m(z) \delta_m + \delta_f P_f^{-1}(z), $$

$$ \Delta_m(z) := \frac{R(z) G(z)}{P(z) \rho^2 \rho(z) R(z)} m(z) G(z), $$

$$ \delta_f(z) := P_m P_f^{-1}(z) - 1, \quad \delta_f(z) := P_f(z) - P(z). $$

Following the ideas of Sections 4 and 5, to make $C_f$ approximately optimal we can design $P_f$ and $P^{-1}$ by comparing $R$ with $\Delta$ and $\delta$. One important point to note is that since $P(\rho^2)$ is continuous on $[0,\infty)$, we can approximate it up to a given tolerance by a finite dimensional transfer function $P_f$, uniformly on the unit circle. Also we can choose a proper rational $P_f$ such that $P_f$ is close to 1 on the unit circle excluding some arbitrarily small neighborhood of the point $\rho^2$. Moreover, when $P_f$ is strictly proper and $P^{-1}$ is proper we have $\delta_f(\rho^2) = \delta_f(\rho^2) = 0$ as $\delta = 0$. Then, it is not difficult to see that if $B_f(\rho^2) = W(1+PG^{-1})^{-1}$ as $\delta \rightarrow 0$. On the other hand, when $P_f$ is strictly proper, by definition of $\rho$ in (1), we necessarily have $|W(\rho^2)| \leq \rho$. Therefore, in this case having a proper $P_f$ guarantees a good performance in the high frequency range.

6 Example: low pass weights and delay systems

In this section we will consider a first order low pass weight and a plant with delay. We take the outer part of the plant to be strictly proper, so that the transfer function $P(z)$ becomes continuous on the imaginary axis.

Let us choose the weight $W$ and plant $P$ to be

$$ W(z) = \frac{r_\pi z + 1}{r_\pi z + 1}, \quad \text{and} \quad P(z) = \frac{1}{r_\pi z + 1 + \Delta_0}. $$

Here $\Delta$ is the amount of the time delay, $1/r_\pi$ is the bandwidth of the plant, and $r_\pi$ is the bandwidth of the weight (determining the band on which...
This puts the weight in the form $W(z)$ dimensional approximation of the inner part of the plant. For the inverse of the outer part we choose the proper function

$$P_{sp}(s) = \frac{\hat{p}_{s}}{\hat{p}_{s} + 1}$$

where $\hat{p}_{s} > 0$ is very small (we discuss later how small this should be). Now we check under which conditions the finite dimensional controller $C_1(s)$ is approximately optimal. Recall the equation

$$B_1 = B_{opt} \frac{1}{\hat{p}_{s}} + \frac{X_3}{1 + X_3}$$

where $X_3 = X_3$. Therefore, if the conditions

(a) $X_1 \in H^\infty$;  
(b) $X_3 \in H^\infty$;  
(c) $\|X_1\|_{\infty} < 1$

are satisfied then $B_1 \in H^\infty$. Assuming $m_1 \in H^\infty$, then for (a) and (b) to hold it is necessary and sufficient to have

$$m_1(s) = \hat{m}(s)/(1 + \hat{m}(s))$$

(see Appendix B), where $\omega_1$ is determined by the zero $\hat{p}(s)\hat{p}(s) - \hat{p}_{s}^2$ for $s = (\hat{m}_s - 1)/(\hat{m}_s + 1)$. Simple computations give that

$$\omega_1 = \frac{\nu}{\omega_0}$$

with $\nu = \sqrt{1 - \hat{p}_{s}^2}$.

We will choose a Padé approximation for the delay term and add a filter to this to take into account the effect of $(1 + \hat{m}(s))$:

$$m_1(s) = \frac{\hat{m}_s - \hat{m}(s)}{\hat{m}_s + \hat{m}(s)}$$

and $\hat{m}(s)$ is a Padé approximation which is going to be defined below. The choice of $\hat{m}_s = \omega_1 \nu (\hat{m}_s Y_{\hat{m}_s})$ makes $\hat{m}(s) = \hat{m}(s)/(1 + \hat{m}(s))$. So, we need only to check if any the first order Padé approximation $(1 + \hat{m}(s))/(1 + \hat{m}(s))$ is actually equal to $e^{\hat{s}m(s)}$. This does not in general hold, however, when $\hat{s}$ is order of magnitude .01 (and $\omega_1$ is less then .1) then the difference is so small (less then $10^{-10}$) that we can fix the problem by changing the term $(1 + \hat{m}(s))/(1 + \hat{m}(s))$ to

$$K \frac{1 - (\hat{d}_s + 1)}{1 + \nu} = m_1(s).$$

Here for such small values of $\nu$ and $\omega_1$ we have $1 - K$ and $d_s$ are less then $10^{-10}$. So, in the frequency range of interest (we will see that for such small $\nu$ this is $0 \leq \nu \leq 10^3$) $m_1(s)$ can practically be seen to be equal to the first order Padé approximation of $e^{\hat{s}m(s)}$.

In summary, we are going to use

$$m_1(s) = \frac{\hat{m}_s - \hat{m}(s)}{\hat{m}_s + \hat{m}(s)}$$

in the controller, where $\hat{m}_s$ is a first order approximation for the delay term. This takes care of conditions (a) and (b).

One other condition we need to satisfy is $\|X_1\|_{\infty} < 1$. After substitution of the terms we see that

$$X_1(s) = \frac{\hat{s}_s}{\hat{s}_s + \hat{m}_s + 1} \hat{m}_s(s) + \frac{\hat{s}_s - \hat{m}_s}{\hat{s}_s + \hat{m}_s} \hat{m}(s)/(1 + \hat{m}(s))$$

Define

$$\hat{R}_s(s) := \frac{1}{\hat{s}_s + \hat{m}_s + 1}$$

and

$$\hat{D}_s(s) := X_1(s)\hat{R}_s(s).$$

Plot $[R_1(s)]$ versus $\omega$, and choose $\epsilon_1\tau_s$ small enough such that

$$\|D_1(s)\| < \|[R_1(s)]\|$$

for $\omega \leq \omega_s$.

$\omega_s$ is to be defined below. Consider the second term in the numerator of $X_1$. Note that

$$\left|\hat{m}(s)\hat{s}_s - \hat{m}(s)\right| = \left|\hat{m}(s)\right|$$

as $\omega \to \infty$.

So, $\hat{m}_s$ should approximate $\hat{m}$ "reasonably good" at least up to frequency $\omega_s$ where

$$\epsilon_1\tau_s \omega < \omega_s.$$
used as a controller for the plant $P(z)$. In Sections 5 and 6 of this paper we have solved the problem corresponding to the situation where the plant is continuous, e.g. $P(z)$, and the controller is in the form of $G(z)$. So, in summary, for plants with invertible outer parts an approximately optimal finite dimensional controller can be found by introducing a rational outer function $P(z)$, which takes the role of the outer part and makes $P(z)$ continuous on the imaginary axis, and then finding a finite dimensional controller (for $P(z)$) using the theory of Sections 5 and 6. □

7 Conclusions

We have obtained the structure of all suboptimal $H^\infty$ controllers for systems with rational weights and stable arbitrary distributed plants. From this structure we have identified the infinite dimensional parts of these $H^\infty$ controllers. Our main objective in this paper was to illustrate that an approximately optimal finite dimensional controller can be designed by appropriately approximating the infinite dimensional parts of the optimal controller.

An important open problem arising from our results is: Given $p > \mu$ characterize the set of all $u \in H^\infty$, $\|u\|_\infty \leq 1$, such that the transfer function

$$G_u \triangleq \frac{1}{1 + mu_G}$$

in (5a) is finite dimensional, and find the lowest possible dimension. Solution to this problem would give the characterization of all finite dimensional suboptimal $H^\infty$ controllers for stable distributed plants.

8 Appendix A

Optimal Sensitivity for $W(z) := p(z) = p_2 + p_1 z$

We have seen that computation of the optimal sensitivity reduces to finding a nonzero vector $h_0$ such that $A_h h_0 = 0$. That is, in our case,

$$((p_2 + p_1 z) (p_2 + p_1 z^* - p^*) h_0 = 0. \quad (a1)$$

But we also have the following

$$T h_0 = z (h_0 - h_0 (0)) \quad (a2a)$$

and

$$TT^* h_0 = h_0 (z) - h_0 (0) (1 - m(z) m(0)) \quad (a2c)$$

for some constants $u_1$ and $v_1 := h_0 (0)$. Putting these expressions in (a1) we see that (a1) is equivalent to

$$(z^2 + (p_2 + p_1^2 - p^2) z + 1) h_0 (z) = \frac{p(0) u_1}{p_0} + 3 m(z) (u_1 + m(0) v_1) \quad (a2)$$

Recall that by Sarason’s theorem $B_{opt} = p^2 (T^* h_0 / p(T) h_0)$. In our case we then have

$$B_{opt} (z) = \frac{z (h_0 (z) - m(z) h_0 (z))}{(p_2 + p_1) h_0 (z) - p_0 h_0 (z)} \quad (a2)$$

Set $R(z) = r_0 + p_1 z$, $h(z) = -z^2 (p_2 + p_1^2 - p^2) / p_0 p_1$, and $w_1 := (u_1 + m(0) v_1)$. Since

$$h_0 (z) = \frac{p(0) u_1}{p_0} + 3 m(z) (u_1 - m(0) v_1) \quad (a3)$$

in $B_{opt} (z)$, and arranging terms we get

$$B_{opt} (z) = p^2 (z) \frac{1 + m(z) p_0 u_1 / p_0}{1 + \frac{1}{m(z)} p_0 u_1 / p_0} \quad (a4)$$

On the other hand from equation (a3) we obtain that

$$\frac{1}{p_0} p_0 u_1 + m(0) v_1 = 0$$

for $i = 1, 2$ where $a_2 = a_2$ are the roots of $h(z) = 0$ on the unit circle. So, we must have

$$p(a_1) m(a_2) = p(a_2) m(a_1) \quad (a5)$$

in order to have a nonzero vector $h_0$ satisfying $A_h h_0 = 0$. Solving the equations, after tedious computations, we get that $p_0 u_1 / p_0 = 0$. Putting this result in (a4), and solving for $Q_{opt}$ from $B_{opt} = W - mQ_{opt}$, and then solving for the optimum controller $C_{opt}$, via Youla parameterization, we end up with

$$C_{opt} = \frac{W(z) W(z^{-1})}{p^2} - 1 - \frac{G(z)}{1 + \frac{1}{m(z)} p_0 u_1 / p_0} \quad (a6)$$

where $G(z) = p^2 / p(z)$. The computations are straightforward but too lengthy to present here.

When $m(z) = \frac{1}{2} e^{-2 \mu}$ we find $p$ from the equation

$$h_{w_1} + \tan^{-1} \frac{w_1}{y_1} + \tan^{-1} y_1 + 2 \tan^{-1} \frac{w_1}{y_1} = \pi$$

where

$$w_1 = \frac{y}{r_0} \quad \text{and} \quad y = \sqrt{1 - p/r_0^2} \quad (a7)$$

This is obtained by writing the equation (a5) explicitly and transforming the data from the unit circle to right half plane using the transformation $z = \frac{e^{\mu}}{1 + 2 \mu}$. Hence in the example considered when $r_0, \omega_0, \mu$ and $k$ are fixed $\delta$ is found by (8).

9 Appendix B

Recall the expression

$$X_0 (s) = \frac{1}{2} \frac{\omega_0 \sin^2 \omega_0}{\omega_0} \frac{1 + \frac{1}{m(z)} p_0 u_1 / p_0}{1 + \frac{1}{m(z)} p_0 u_1 / p_0} \quad (a8)$$

and definitions

$$\delta (s) = 1 + \frac{1}{\omega_0 \sin^2 \omega_0} \quad (a9)$$

From the equations (a3-a), (a4) and (a5) of Appendix A it is easy to see that

$$\delta (s) = 1 + \frac{1}{\omega_0 \sin^2 \omega_0} \quad (a10)$$

Moreover, $\omega_0$ and $-\omega_0$ are the only points where $\delta (s)$ vanishes in the closed right half plane. Therefore for the stability of $\delta (s)$ we need

$$\delta (s) = 1 + \frac{1}{\omega_0 \sin^2 \omega_0} \quad (a11)$$

Using (a10) and re-arranging terms we get that (a12) is equivalent to having

$$(-1) (-c_2 \tau_p \omega_0 / \tau_0) + \frac{1}{\omega_0 \sin^2 \omega_0} - \frac{m(\omega_0)}{m(\omega_0) - m(\omega_0)} = 0 \quad (a12)$$

or

$$m(\omega_0) - m(\omega_0) = (1 + c_2 \tau_p \omega_0 / \tau_0) \quad (a12)$$

It is also routine to check that

$$\delta (s) = 1 + \frac{1}{\omega_0 \sin^2 \omega_0} \quad (a13)$$

So, since $+\omega_0$ and $-\omega_0$ are the only points in the closed right half plane that makes $\delta (s) = 0$, condition (a13) is also sufficient for the stability of $\delta (s)$.

Now let's look at $X_1$:

$$X_1 = \frac{r_0 \tau_0 - 1}{r_0 \tau_0 + 1} \frac{m(z)(-c_2 \tau_p \omega_0 / \tau_0) + (m(z) - m(\omega_0))}{1 + \frac{1}{m(z)} p_0 u_1 / p_0} \quad (a14)$$

From this expression, we see that the stability of $X_1$ also is equivalent to (a13).

References


FIGURES

Figure 1: Closed Loop Control System

Figure 2: Structure of the Suboptimal Controller

Figure 3: Finite dimensional controller

Figure 4: $|\hat{H}(j\omega)|$

Figure 5: $|\beta_s(j\omega)|$

Figure 6: $|\beta_f(j\omega)|$

Figure 7: $|\theta_f(j\omega)|$