THE FOUR BLOCK PROBLEM FOR DISTRIBUTED SYSTEMS

1. INTRODUCTION

In this paper we will study the singular values of a certain operator, the four block operator, (to be defined below), which appears naturally in many engineering $H^\infty$ control problems. Indeed following the framework of the monograph of Francis [7], almost all such robust design problems can be formulated in terms of the spectral properties of such an operator. This includes the problems of sensitivity and mixed sensitivity minimization, model-matching and certain tracking problems. Our techniques give the optimal solution to this problem of mixed sensitivity minimization, model-matching and certain tracking problems. Indeed following the line of argument given in [1], [4], and [6]. Thus we must first identify the essential norm of $A$ (denoted by $\|A\|_E$), and then give an algorithmic procedure for determining a singular value of $A$, $\rho > \|A\|_E$. We are using the standard notation from operator theory as, for example, given in [9], [10]. In particular $\sigma$ will denote the essential spectrum. We begin with the following result whose proof may be found in [5]:

PROPOSITION 1. Notation as in the Introduction. Set

$$\alpha := \max \left\{ \begin{array}{l} \mathcal{W}(f) \mathcal{H}(g) : \xi \in \sigma_T(T) \\ \mathcal{G}(g) \mathcal{H}(h) : \xi \in \tilde{\sigma}_T(T) \end{array} \right\}$$

$$\beta := \max \left\{ \begin{array}{l} \mathcal{W}(f) \mathcal{H}(g) : \xi \in \tilde{\sigma}_T(T) \\ \mathcal{G}(g) \mathcal{H}(h) : \xi \in \tilde{\sigma}_T(T) \end{array} \right\}$$

where the norms in (1) and (2) are those in $L^2(C^2)$. Then

$$\|A\|_E = \max(\alpha, \beta).$$

We will see in the next section how Proposition 1 leads to an algorithm for finding $\|A\|_E$.

2. PRELIMINARY RESULTS AND NOTATION

In this section we make some remarks, and prove a result which will allow us to give the determinantal formula for the singular values and vectors of $A$ in Section 3. We are basically following the line of argument given in [1], [4], and [6]. Thus we must first identify the essential norm of $A$ (denoted by $\|A\|_E$), and then give an algorithmic procedure for determining a singular value of $A$, $\rho > \|A\|_E$. We are using the standard notation from operator theory as, for example, given in [9], [10]. In particular $\sigma$ will denote the essential spectrum. We begin with the following result whose proof may be found in [5]:

$$\|A\|_E = \max(\alpha, \beta).$$

3. ALGORITHM FOR NORM OF FOUR BLOCK OPERATOR

In this section we will discuss an algorithm for finding the singular values of $A$, which we will implement via a determinantal formula in Section 4. Again the line of argument we use here follows very closely that of [1], [4], and [6]. Using the notation of Section 2, we let $\rho > \max(\alpha, \beta)$. Notice that if $\|A\|_E > \|A\|_E$, then $\|A\|$ is an eigenvalue of $A^*A$.

So we begin by writing $w = aS, f = bS, g = cS, h = dS$, where $a, b, c, d, k$ are polynomials of degree $\leq n$. Then $\rho^2$ is an eigenvalue of $A^*A$ if and only if

$$\begin{bmatrix}
 0 & f(S) \\
 g(S) & h(S)
\end{bmatrix}
\begin{bmatrix}
 a(S) \\
bk(S) \\
c(S) \\
k(S)
\end{bmatrix} = 0
$$

for some non-zero

$$\begin{bmatrix}
 x \\
y
\end{bmatrix} \in H^2 \otimes H^2.$$

Next for any polynomial $\rho$ of degree $\leq n$, we set

$$\tilde{\rho}(z) = \rho^z \frac{1}{\bar{\rho}(z)}$$

for $z \in C$, $z \neq 0$. With this notation if we multiply (4) by $\tilde{\rho}(z), z \neq 0$, we see that

$$\begin{bmatrix}
 x \\
y
\end{bmatrix} \in H^2 \otimes H^2.$$
But (5) and (8) can be re-written as

\[ \begin{bmatrix} a(s) x(T) & a(s) y(T) \\ b(s) x(T) & b(s) y(T) \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} B(s) p_x \\ B(s) p_y \end{bmatrix} = \begin{bmatrix} a_1 v_1 + \cdots + a_n v_n \\ b_1 v_1 + \cdots + b_n v_n \end{bmatrix}. \]

where

\[ B(s) = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^{-1} \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}. \]

Multiplying (7) by \( \zeta^* \), and applying the orthogonal projection onto \( H^2 \oslash H^2 \), it is easy to see that

\[ \begin{bmatrix} a(s) x(T) \\ b(s) x(T) \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} a(s) x(T) \\ b(s) x(T) \end{bmatrix} \begin{bmatrix} \phi' \\ \psi' \end{bmatrix}, \]

and hence,

\[ \begin{bmatrix} a(s) x(T) \\ b(s) x(T) \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} a(s) x(T) \\ b(s) x(T) \end{bmatrix} \begin{bmatrix} \phi' \\ \psi' \end{bmatrix}, \]

where \( \phi' = k(s) \phi \) and \( \phi' = k(s) \phi \). Since

\[ \begin{bmatrix} a(s) x(T) \\ b(s) x(T) \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} a(s) x(T) \\ b(s) x(T) \end{bmatrix} \begin{bmatrix} \phi' \\ \psi' \end{bmatrix}, \]

we deduce that \( \phi' = \psi' = 0 \), and thus \( x = y = 0 \).

Now let us assume that

\[ \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in H(m) \oslash H(m), \]

and that

\[ \begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix} \in H(m) \oslash H(m) \]

for some \( \zeta, \eta \in H^2 \).

But (5) and (8) can be re-written as

\[ \begin{bmatrix} a(s) x(T) & a(s) y(T) \\ b(s) x(T) & b(s) y(T) \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} B(s) p_x \\ B(s) p_y \end{bmatrix} = \begin{bmatrix} a_1 v_1 + \cdots + a_n v_n \\ b_1 v_1 + \cdots + b_n v_n \end{bmatrix}. \]

for some \( \zeta, \eta \in H^2 \).
Now it is well-known (and easy to compute) that
\[ T_j p = \overline{p} - m \beta_j, \]
and similarly, of course, for \( q \) and \( P_1 = 1 - m \beta_0 \). (Red that \( P \) denotes orthogonal projection.) By applying \( P \) to (15), we get that
\[ C(T) \left[ \begin{array}{c} p \\ q \\ v_j \end{array} \right] = 0. \]  
(16)

Notice that this means
\[ C \left[ \begin{array}{c} p \\ q \\ v_j \end{array} \right] = 0. \]  
(16a)

Consequently, we see that (11) is equivalent to
\[ C \left[ \begin{array}{c} p \\ q \\ v_j \end{array} \right] = 0. \]  
(16b)

Thus (16a) and (16b) are equivalent to
\[ C \left[ \begin{array}{c} p \\ q \\ v_j \end{array} \right] = 0. \]  
(16c)

and
\[ B \left[ \begin{array}{c} p \\ q \\ v_j \end{array} \right] = 0. \]  
(16d)

Taking a respite (16) from all of these computations, let us summarize the above discussion with:

**Proposition 3.** Equality (11) is equivalent to the two equalities (16c) and (16d). The eigenvalue equation (4) with
\[ \begin{cases} a = 0 \\ b = 0 \end{cases} \]  
(18)
is thus equivalent (see Proposition 2) to (16c), (16d), and
\[ \begin{cases} a = 0 \\ b = 0 \end{cases} \]  
(19)

Next set
\[ d_4 \left( \frac{p}{q} \right) = \det C \left( \frac{p}{q} \right) \]  
(20)

where
\[ C^{(c)} \left( \frac{p}{q} \right) = \sum_{j=0}^{2n} C_j^{(c)} \]  
(21)

and
\[ d_j \left( \frac{p}{q} \right) = \sum_{j=1}^{2n} C_j \]  
(22)

for \( 0 \leq j \leq 2n \). Note that \( C^{(c)} \left( \frac{p}{q} \right) = d_4 \left( \frac{p}{q} \right) \).

We will now make a technical assumption in order to simplify our exposition. Below we will discuss how to remove this assumption of genericity. Explicitly, we assume that
\[ d_j \]  
(23)
has distinct roots all of which are non-zero.

We now have the following result (see [5] for the proof):

**Lemma 1.** Under assumption (20), \( d_4 \left( \frac{p}{q} \right) \) has \( r \) zeros \( \alpha_1, \ldots, \alpha_r \in D, r \) zeros \( \beta_1, \ldots, \beta_r \in D \). For \( 0 < r \leq 2n \).

Continuing our computation, we note that by multiplying by \( C^{(c)} \left( \frac{p}{q} \right) \), we can express (16c) equivalently as
\[ \left( \begin{array}{c} E_j \\ F_j \\ G_j \end{array} \right) = 0 \]  
(22a)

where the matrix-valued polynomials \( E_j, F_j, G_j \) are given by
\[ E_j = \sum_{k=0}^{2n} c_k \]  
(23a)

and
\[ F_j = \sum_{k=0}^{2n} c_k \]  
(23b)

for \( 1 \leq j \leq n \).

We are almost done now! Indeed arguing precisely as in [5], we note that since \( p, q \in H(m) \), they must be analytic in a neighborhood of \( \mathbb{C} \setminus \{ \alpha \} \) as well as in \( \mathbb{D} \). Hence using Lemma 1, from (22) we see that
\[ \sum_{j=1}^{2n} \left( E_j + m F_j \right) \frac{p_j}{q_j} + \sum_{j=1}^{2n} G_j \frac{p_j}{q_j} = 0 \]  
(23a)

for \( 1 \leq j \leq r \), and for \( 2r+1 \leq j \leq 4n \). For \( \alpha \in \mathbb{D} \), we multiply (22) by \( \alpha \) to get
\[ \sum_{j=1}^{2n} \left( E_j + m F_j \right) \frac{p_j}{q_j} + \sum_{j=1}^{2n} G_j \frac{p_j}{q_j} = 0 \]  
(24)

But this last equation admits an analytic extension to the complement of the disc, i.e. all the functions are analytic in \( \mathbb{C} \). Set
\[ E_j(\alpha) := \left( \frac{p_j(q_j)}{q_j(p_j)} \right) \]  
(25)

and similarly for \( F_j, G_j \). Hence we can express (16c) equivalently as

\[ \sum_{j=1}^{2n} \left( E_j + m F_j \right) \frac{p_j}{q_j} + \sum_{j=1}^{2n} G_j \frac{p_j}{q_j} = 0 \]  
(26)

for \( 1 \leq j \leq n \).
Then from (24a) and Lemma 1, we get that

\[ \sum_{j=1}^r \sum_{i=1}^r \left( F_i^1(Q) + F_i^2(Q) \right) \chi_j \chi_i = 0 \]  

(44b)

for \( 1 \leq i \leq r \).

We now play the analogous game with the \( C \) operator that we just did with the \( C \) operator. Indeed, setting

\[ d_b(Q) := \text{det} B(Q) \]

we have the analogy of (21), and since \( p > \beta \), we have that \( d_b(Q) \neq 0 \) for \( \xi \in D \). Again we make the assumption of genericity that \( d_b \) has distinct roots all of which are non-zero. (25)

Just like for lemma 1 above, we see that \( d_b \) has \( 2n \) zeros \( \beta_1, \ldots, \beta_{2n} \in D \), and \( 2n \) zeros \( 1/\beta_1, \ldots, 1/\beta_{2n} \) in the complement of \( D \).

We now set (as above)

\[ B^+(Q) := \sum_{j=0}^{2n} B_j \]

where

\[ B_j = \begin{bmatrix} b_{j,11} & b_{j,12} \\ b_{j,21} & b_{j,22} \end{bmatrix} \]

\[ B_j^* = \begin{bmatrix} b_{j,11} & -b_{j,12} \\ -b_{j,21} & b_{j,22} \end{bmatrix} \]

for \( 0 \leq j \leq 2n \). Note that \( B^+(Q) = d_b(Q) \). Finally, since

\[ \begin{bmatrix} \xi \\ \eta \end{bmatrix} \in H^1(D) \cap H^2(D) \]

implies for \( 1 \leq i \leq 2n \), the equations

\[ \sum_{j=1}^{2n} \sum_{i=1}^{2n} \left( F_i^1(Q) + F_i^2(Q) \right) \chi_j \chi_i = 0 \]  

(46)

where

\[ H_j(Q) := \sum_{k=1}^{2n} \sum_{l=1}^{2n} C_{j,k} \chi_k \chi_l \]

\[ K_j(Q) := \sum_{k=1}^{2n} \sum_{l=1}^{2n} \chi_k \chi_l \]

We are at long last ready to state our main result (see [5] for the proof):

**THEOREM 1.** Assume the genericity conditions (20) and (25) hold. Then \( p \) is a singular value of the four block operator \( A \) if and only if

\[ \det \begin{bmatrix} M_1 & M_2 \\ M_4 & M_5 \end{bmatrix} = 0 \]

where

\[ M_1 := \begin{bmatrix} E_1(\alpha) + m(\alpha)F_1(\alpha) & E_2(\alpha) + m(\alpha)F_2(\alpha) \\ E_3(\alpha) + m(\alpha)F_3(\alpha) & E_4(\alpha) + m(\alpha)F_4(\alpha) \end{bmatrix} \]

\[ M_2 := \begin{bmatrix} m(\alpha)E_1(\alpha) & m(\alpha)E_2(\alpha) \\ m(\alpha)E_3(\alpha) & m(\alpha)E_4(\alpha) \end{bmatrix} \]

\[ M_5 := \begin{bmatrix} K_1(\beta_1) & K_2(\beta_1) \\ K_1(\beta_2) & K_2(\beta_2) \end{bmatrix} \]

\[ M_4 := \begin{bmatrix} M_1 & M_2 \\ M_3 & M_5 \end{bmatrix} \]

\[ M_3 := \begin{bmatrix} m(\alpha)G_1(\alpha) & m(\alpha)G_2(\alpha) \\ m(\alpha)G_3(\alpha) & m(\alpha)G_4(\alpha) \end{bmatrix} \]

\[ M_3 := \begin{bmatrix} m(\alpha)G_1(\alpha) & m(\alpha)G_2(\alpha) \\ m(\alpha)G_3(\alpha) & m(\alpha)G_4(\alpha) \end{bmatrix} \]

\[ G_1(\alpha) & G_2(\alpha) \\ G_3(\alpha) & G_4(\alpha) \end{bmatrix} \]

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REFERENCES


