R	T_{max}
0.2	0.17
0.5	0.13
1.0	0.10
1.5	0.08
2.0	0.06
4.0	0.04

Fig. 1. Maximum sample time for global Ω_D-holdability in X.

and the holding set be given by

$$X=\{(x^1, x^2) : |x^1+x^2|\le R,\ |x^1-x^2|\le R,\ R\ge 0\}.$$

For a given value of R, it was determined from Theorem 1 that the discrete-time system is globally Ω_D-holdable in X if the sample time T satisfies $T \le T_{max}$. If $T > T_{max}$, the discrete-time system is not globally Ω_D-holdable in X. Fig. 1 gives some values of R and the corresponding values of T_{max}. Note that as T_{max} decreases, R increases. Hence, the size of the set in which the system is globally Ω_D-holdable is dependent on the sample time.

Next, let $x_0 = [7/10, 1/5]'$ and $T = 0.1$. Using Theorem 2 and the modified algorithm from [4], an admissible control that holds the state of the discrete-time system in X with $R = 1.0$ from x_0 for all $k \in N$ was computed. This holding control is given by

$$u_k=\begin{cases} 1.00 & k=0, 1, \cdots, 5 \\ 0.13 & k=6 \\ 1.00 & k=7, 8 \\ 1.00 & k=9, 10 \\ 0.00 & k=11, 12, \cdots . \end{cases}$$

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Weighted Sensitivity Minimization for Delay Systems

CIPRIAN FOIAS, ALLEN TANNENBAUM, AND GEORGE ZAMES

Abstract—**In this note we discuss the H^∞-sensitivity minimization problem for linear time-invariant delay systems. While the unweighted case reduces to simple Nevanlinna–Pick interpolation, the weighted case turns out to be much more complicated and demands certain functional-analytic techniques for its solution.**

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NOMENCLATURE

D	open unit disk
$\bar{D}$	closed unit disk
∂D	unit circle
H	open right half plane
$\bar{H}$	closed right half plane
$\tilde{H}$	$\bar{H} \cup \{\infty\}$
$H^p(X)$	the standard Hardy p-space ($1 \le p \le \infty$) on X where $X = D$ or H. (See [2] or [15] for details.) We will also use some elementary facts about L^p-spaces. Again see [2] or [15] for details. Finally, if $u \in H^\infty(X)$ is an inner function, then $H^2(X) \ominus uH^2(X)$ will denote the orthogonal complement of $uH^2(X)$ in $H^2(X)$.

I. INTRODUCTION

Since the paper of Zames [16], there has been much literature on weighted H^∞-sensitivity minimization in control (see [6] for an extensive list of references). In the papers [1], [7], [9] explicit algorithms are derived for computing the optimal sensitivity and controllers for LTI finite-dimensional systems.

In this note, we will consider applying H^∞-minimization techniques to systems with delays. Because of the distributed nature of the problem, in order to solve the weighted case, we will need infinite-dimensional techniques. We will therefore use some of the results from Sarason [14], and Sz. Nagy-Foias [15]. For strictly proper weights and a plant consisting of a delay, we will derive explicit formulas for the minimal H^∞-sensitivity and the optimal controller. In contrast to the above infinite-dimensional nature of weighted H^∞-minimization of delay systems, we will also show that unweighted sensitivity minimization amounts to a simple interpolation problem for a large class of distributed plants.

II. WEIGHTED SENSITIVITY MINIMIZATION

In this section we will show that the weighted sensitivity H^∞-minimization problem for even the simplest delay systems is nontrivial and demands infinite-dimensional techniques. We should note that D. Flamm in his M.I.T. thesis proposal [4], has independently derived a result similar to (2.1).

We begin by recalling the general weighted sensitivity H^∞-minimization problem for SISO, LTI plants. We are given a SISO, LTI plant $P_0(s)$, and a stable "weighting" function $W(s)$. Let $C(s)$ be an internally stabilizing LTI controller for $P_0(s)$ in the feedback system of Fig. 1. Then following [16], we define the weighted sensitivity

$$S_W(s)=W(s)(1+P_0(s)C(s))^{-1}. \tag{1}$$

The problem we are interested in then is in determining the existence of and computing

$$\inf \{\|S_W(s)\|_\infty : C \text{ stabilizing}\} \tag{2}$$

where $\| \ \|_\infty$ denotes the H^∞-norm in the right half plane. In the finite-dimensional case, sufficient conditions are given for this infimum to exist and an optimal controller (in general, nonproper) is computed in [1], [7]–[9].

In this section, we will be interested in taking $P_0(s) = e^{-hs}$, and $W(s) = 1/(as + 1)$ for $a > 0$. The general technique which we give below in the proof of (2.1) goes through immediately for *any* $W(s)$, a stable strictly proper real rational weighting function with stable inverse. In point of fact, the proof of (2.1) gives an explicit procedure for solving the Nehari problem of computing the distance of an L^∞-function of the form $e^{hs}Q(s)$ (regarded as defined on the $j\omega$-axis) to H^∞, where $Q(s)$ is a stable strictly

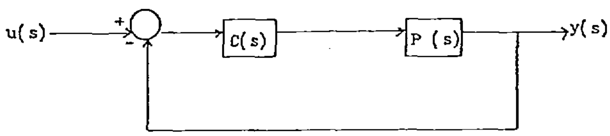

Fig. 1.

proper real rational function. However, in order to clearly illustrate our ideas and techniques we will take $W(s)$ to be the above linear weight. In Example (2.4) at the end of this section, we will write down a differential equation derived from the proof of (2.1) which allows one to solve the weighted sensitivity H^∞-minimization problem for a quadratic weight.

We will now compute infimum of (2), and then in (2.2) the corresponding optimal compensator.

Theorem (2.1): For $W(s) = 1/(as + 1)$, we have

$$\inf_{C \text{ stabilizing}} \| W(1+e^{-hs}C)^{-1}\|_\infty = \frac{1}{\sqrt{1+(a^2y_{ah}^2)/h^2}}$$

where y_{ah} is the unique root of the equation

$$\tan y + ay/h = 0 \tag{3}$$

lying between $\pi/2$ and π. [y_{ah} may also be characterized as the smallest strictly positive root of (3).]

Proof: Following [16], we are reduced to the following Nehari-type problem. Compute (H denotes the right half plane)

$$\inf_{q\in H^\infty(H)} \left\| \frac{1}{as+1} - e^{-hs}q \right\|_\infty . \tag{4}$$

From general theory (see, e.g., [2]), the infimum exists, and there exists a unique optimal $\tilde{q}_{\rm opt} \in H^\infty(H)$ which attains it. (In (2.2) we will compute $\tilde{q}_{\rm opt}$.)

In order to compute (4), we transfer the problem to the unit disk D. Accordingly, set

$$s = \frac{1+z}{1-z}.$$

Then

$$W\left(\frac{1+z}{1-z}\right) = \frac{1-z}{(a-1)z+(a+1)} =: \tilde{W}(z).$$

Set, moreover,

$$m(z) := e^{h(z+1)/(z-1)}.$$

Finally, let T denote the compression (i.e., the projection) of the unilateral shift operator on H^2 (defined by multiplication by z) onto $H^2 \ominus mH^2$, where $H^2 := H^2(D)$. Then by [14, Theorem 1]

$$\inf_{q\in H^\infty(D)} \| \tilde{W}(z) - m(z)q(z)\|_\infty = \|\Gamma\| \tag{5}$$

where

$$\Gamma := (1-T)((a-1)T+(a+1))^{-1}.$$

(Note that symbolically $\tilde{W}(T) = \Gamma$.)

In order to compute $\|\Gamma\|$ we will use some nice results from Sarason [13]. First, we note that $H^2 \ominus mH^2 \overset{\sim}{\rightarrow} L^2[0, h]$ (i.e., the two spaces are isometrically isomorphic). Moreover, via this isometry [13],

$$\frac{1+T}{2} \cong (1+V)^{-1}$$

where $V: L^2[0, h] \rightarrow L^2[0, h]$ is the Volterra operator

$$Vf(x) := \int_0^x f(t)\, dt$$

and "$\cong$" denotes "is unitarily equivalent to."

Simple computations then show

$$T \cong (1-V)(1+V)^{-1} \text{ and } \Gamma \cong V(V+a)^{-1}.$$

Now clearly, $V^{-1} = \hat{D}$ where $\hat{D}$ is the derivative operator, $\hat{D}f := f'$, with domain consisting of

$$\{f \in L^2[0, h] : f' \in L^2[0, h],\ f(0)=0\}.$$

Hence $\Gamma \cong (a\hat{D} + 1)^{-1} =: \hat{\Gamma}$.

Since Γ is compact (because $W(s)$ is strictly proper; see [14]) we have $\|\Gamma\| = \|\hat{\Gamma}\| = \sqrt{\mu}$ where μ is the maximal eigenvalue of $\hat{\Gamma}^*\hat{\Gamma}$. But then, $\lambda_{\min} := 1/\mu$ is the minimal positive eigenvalue of $(\hat{\Gamma}^*\hat{\Gamma})^{-1}$. (We will see below that the inverse of $\hat{\Gamma}^*\hat{\Gamma}$ can be identified with a differential closed operator with dense domain.) In other words, we are reduced to computing the minimal positive eigenvalue of the operator

$$(\hat{\Gamma}^*\hat{\Gamma})^{-1} = (a\hat{D}+1)(a\hat{D}^*+1)$$

where (using integration by parts; see [15]) $\hat{D}^* = -\hat{D}$ with domain

$$\{f \in L^2[0, h] : f' \in L^2[0, h],\ f(h)=0\}.$$

Consequently, we may make the identification

$$(\hat{\Gamma}^*\hat{\Gamma})^{-1} = -a^2\hat{D}^2 + 1$$

on the space of functions

$$\{f \in L^2[0, h] : f', f'' \in L^2[0, h],\ f(h)=0,\ (-af'+f)(0)=0\}.$$

We have therefore shown that in order to compute the minimal positive eigenvalue of $(\hat{\Gamma}^*\hat{\Gamma})^{-1}$, we are required to solve the eigenvalue problem

$$(-a^2\hat{D}^2+1)f = \lambda f$$

with boundary conditions

$$f(h) = 0$$

$$-af'(0) + f(0) = 0.$$

Note that since Γ (and, hence, $\hat{\Gamma}$) is a contraction $\lambda \geq 1$. Since $\lambda = 1$ is clearly not an eigenvalue, we have $\lambda > 1$.

Therefore, from the ordinary differential equation

$$a^2\hat{D}^2 f + (\lambda-1)f = 0 \qquad (\lambda > 1)$$

we get

$$f(t) = A\cos\left(\frac{\sqrt{\lambda-1}}{a}\right)t + B\sin\left(\frac{\sqrt{\lambda-1}}{a}\right)t.$$

From the boundary conditions, we see

$$f(h) = A\cos\left(\frac{\sqrt{\lambda-1}}{a}\right)h + B\sin\left(\frac{\sqrt{\lambda-1}}{a}\right)h = 0$$

and

$$-af'(0) + f(0) = -B\sqrt{\lambda-1} + A = 0.$$

But these imply, that

$$\sqrt{\lambda-1}\cos\left(\frac{\sqrt{\lambda-1}}{a}\right)h + \sin\left(\frac{\sqrt{\lambda-1}}{a}\right)h = 0. \tag{6}$$

Set

$$y := \frac{\sqrt{\lambda-1}}{a}h. \tag{7}$$

Then from (6), we immediately derive the equation

$$\tan y + ay/h = 0 \tag{8}$$

and from (7) the minimal strictly positive root y_{ah} of (8) (i.e., the unique root of (8) between $\pi/2$ and π) corresponds to the required minimal eigenvalue $\lambda_{\min}$. Working our way back through the definitions then shows that $\|\Gamma\|$ has the required value. □

Using (2.1) we now find the optimal compensator.

Computation of Optimal Compensator (2.2): We use the notation of (2.1). Let $\lambda = \lambda_{\min}$ denote the minimal positive eigenvalue of $(\hat{\Gamma}^*\hat{\Gamma})^{-1}$, and f_λ the corresponding eigenfunction,

$$f_\lambda(t) = \sqrt{\lambda - 1}\cos\left(\frac{\sqrt{\lambda-1}}{a}\right)t + \sin\left(\frac{\sqrt{\lambda-1}}{a}\right)t.$$

Now consider $g(t) := \hat{\Gamma} f_\lambda(t)$. Then by definition $g(t) \in L^2[0, h]$ is the (unique) solution of the initial value problem

$$a\frac{dg}{dt} + g(t) = f_\lambda(t), \qquad g(0) = 0.$$

The solution of this is obviously,

$$g(t) = \sin\left(\frac{\sqrt{\lambda-1}}{a}\right)t.$$

Now in the standard way we can regard $L^2[0, h]$ as the closed subspace of functions in $L^2[0, \infty]$ which are identically 0 outside the interval $[0, h]$. With this identification, we can take the Laplace transform of elements of $L^2[0, h]$.

Accordingly, let $F_\lambda(s)$, $G(s)$ denote the Laplace transforms of the functions $f_\lambda(t)$, $g(t) \in L^2[0, h]$, respectively. Then it follows from [14, p. 188] that

$$G(s)/F_\lambda(s) = W(s) - e^{-hs}q_{\text{opt}}(s)$$

where $q_{\text{opt}} \in H^\infty(H)$ is the unique function for which the infimum (4) is attained. From $q_{\text{opt}}(s)$ it is now standard (see [16]) to write down the optimal compensator. We should also note that as in the finite-dimensional case [8], the optimal sensitivity $G(s)/F_\lambda(s)$ is inner (i.e., "allpass"). See [14, p. 188].

Remarks (2.3):

i) The computation of (2.1) easily generalizes to any real rational strictly proper $W(s) \in H^\infty(H)$ with stable inverse. The strict properness ensures that the corresponding operator Γ [see equation (5)] will be compact (see [14]). When $W(s)$ is only proper, then the corresponding Γ will decompose as the sum of a compact operator and a scalar multiple of the identity and once again the above techniques may be used to compute the norm. See [5] for details. In general, the operator "s" corresponds to "$\hat{D}$" (derivation) on $L^2[0, h]$ and so as in (2.1) one is reduced to a simple boundary-value problem from which the optimal sensitivity can be computed from the associated Wronskian determinant [see (2.4)].

ii) In [5] based on the techniques of (2.1), we solve the optimal sensitivity problem for arbitrary real rational proper stable weights and plants of the form $e^{-hs}P_0(s)$, where $P_0(s)$ is real rational and proper.

Example (2.4): We will now apply the procedure of (2.1) to show how to compute the infimum of (2) where $W(s) = 1/(a^2s^2 + 2as + 1)$, $a > 0$. Indeed, following the argument of (2.1), one is reduced to the following eigenvalue problem:

$$(a^4\hat{D}^4 - 2a^2\hat{D}^2 + 1)f = \lambda f$$

with boundary conditions

$$f(h) = f'(h) = 0$$

$$a^2f''(0) - 2af'(0) + f(0) = a^2f'''(0) - 2af''(0) + f'(0) = 0.$$

Computing the minimal positive eigenvalue $\lambda_{\min}$ (which one can do from the associated Wronskian determinant), one gets that the required infimum is $1/\sqrt{\lambda_{\min}}$.

III. Unweighted Sensitivity Minimization

In this section we would like to briefly consider the problem of unweighted sensitivity minimization for certain kinds of delay systems. The close connection between unweighted sensitivity minimization and robust system design in the finite-dimensional case has already been considered in [11], [12]. We wish to show here that for certain kinds of distributed systems, the *unweighted* sensitivity minimization problem amounts to a known finite interpolation problem. As we have seen, this is in striking contrast to the weighted case in which even for the simplest distributed systems, the solution of the weighted problem becomes highly nontrivial.

Since the results of this section will be given mainly to *contrast* them with the much harder weighted problem, they will not be given in full generality. Indeed, the same kinds of ideas used in [10], [3] for the solution of certain kinds of robust stabilization problems for broad classes of distributed systems go through immediately in the context of unweighted sensitivity minimization. However, for our modest heuristic purposes we will consider here a SISO, LTI plant $P_0(s) = e^{-hs}\tilde{P}_0(s)$ where $h > 0$, and $\tilde{P}_0(s)$ is a strictly proper real rational function.

Given $C(s)$ an internally stabilizing proper LTI compensator for $P_0(s)$ in the feedback system of Fig. 1, define the *sensitivity*

$$S(s) := (1 + P_0(s)C(s))^{-1}.$$

Then the problem we are interested in is computing $\inf\{\|S(s)\|_\infty : C \text{ stabilizing}\}$ where $\|\ \|_\infty$ denotes the H^∞-norm in the right half plane.

We now show why this is a simple interpolation problem (see also [11], [10]). In order to do this, following [11] let us define $\alpha_{\max}$. Suppose we are given $a_1, \cdots, a_r \in D$, $a_{r+1}, \cdots, a_{r+l} \in \partial D$, and points $b_i \in C$, $i = 1, \cdots, r + l$. (For simplicity, we assume that all the a_i are distinct. For the general case, see [11].) Given $\alpha \in R$, $\alpha > 0$, we are interested in determining when there exists an analytic $f_\alpha : \bar{D} \to D$ such that $f_\alpha(a_i) = \alpha b_i$, $i = 1, \cdots, r + l$. It is easy to compute the maximal α, $\alpha_{\max}$, such that f_α exists if and only if $\alpha < \alpha_{\max}$. Briefly, define matrices for $1 \le i, j \le r$

$$A = \left[\frac{1}{1 - a_i\bar{a}_j}\right], \quad B = \left[\frac{b_i\bar{b}_j}{1 - a_i\bar{a}_j}\right].$$

Let $\lambda_{\max}$ be the maximal eigenvalue of $A^{-1}B$. Then

$$\alpha_{\max} = \min\left\{\frac{1}{\sqrt{\lambda_{\max}}}, \frac{1}{|b_{r+1}|}, \cdots, \frac{1}{|b_{r+l}|}\right\}.$$

Denote the zeros of $\tilde{P}_0(s)$ in $\bar{H}(= \bar{H} \cup \{\infty\})$ by $z_1, \cdots, z_m$ and the poles in $\bar{H}$ by $p_1, \cdots, p_n$. Let $\phi : \bar{H} \to \bar{D}$ be a fixed conformal equivalence. Set

$$a_i = \phi(z_i) \qquad i = 1, \cdots, m$$
$$a_{j+m} = \phi(p_j) \qquad j = 1, \cdots, n$$
$$b_i = 0 \qquad i = 1, \cdots, m$$
$$b_{j+m} = e^{hp_j} \qquad j = 1, \cdots, n.$$

Let $\alpha_{\max}$ be computed with respect to this interpolation data.

Theorem (3.1):

$$\inf\{\|S(s)\|_\infty : C \text{ stabilizing}\} = \frac{1}{\alpha_{\max}}.$$

Proof: Basically, we follow the same arguments and ideas in [11], [10], [3]. Indeed suppose $r > 0$ is such that

$$S(s) : \bar{H} \to D_r := \{|z| < r\}. \tag{9}$$

Note that since $P_0(s)$ is strictly proper $r > 1$. Clearly, we are interested in computing the infimum over all r such that there exists an internally stabilizing proper compensator $C(s)$ with (9) holding. But (9) holds if and only if

$$e^{-hs}\tilde{P}_0(s)C(s) : \bar{H} \to G$$

where $G := \{z \in C : |z + 1| > 1/r\}$.

Define now the conformal equivalence $\psi : G \to D$ by

$$\psi(z) := \frac{r(z+1) - r}{r^2(z+1) - 1}.$$

Notice that $\psi(0) = 0$, and $\psi(\infty) = 1/r$.

Set $u(s) = \tilde{P}_0(s)C(s)$. Then $\psi(e^{-hs}u(s)) = e^{-hs}q(s)$ and since e^{-hs} is inner, we have that $q(s)$ is analytic and $q(s):\tilde{H} \to D$. Moreover, the interpolation conditions of internal stability [11] translate into the following interpolation conditions on $q(s)$:

$$q(z_i)=0 \qquad i=1, \cdots, m$$

$$q(p_j)=\frac{e^{hp_j}}{r} \qquad j=1, \cdots, n.$$

Let now $\alpha_{\max}$ be as above. Then trivially from the definition of $\alpha_{\max}$, $1/r < \alpha_{\max}$, from which we get the theorem. □

Remarks 3.2:

i) The formula derived in (3.1) for the minimal unweighted sensitivity is identical to that derived in [11] in the finite-dimensional case.

ii) Note that from the proof of (3.1), in the unweighted case one can essentially factor out the singular inner part of the plant (i.e., e^{-hs}) in computing the optimal sensitivity. It is this observation, which allows one to reduce the problem of unweighted sensitivity minimization to one of finite interpolation. This breaks down for nonconstant weights.

IV. Conclusion

In this note, we have tried to illustrate some of the difficulties involved in the application of weighted H^∞-minimization techniques to even the simplest distributed systems. While the unweighted case is rather trivial and reduces to Nevanlinna–Pick interpolation, the weighted problem seems to genuinely reflect the distributed nature of the system.

Finally the functional-analytic techniques we have used here in treating a plant consisting of a pure delay generalize to much more general distributed systems. For this, we refer the interested reader to [5].

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A Stability Property

MATEI KELEMEN

Abstract—**In this note we obtain a stability property, not for a point of equilibrium, but for a certain family of asymptotically stable equilibria of a dynamical system with an input.**

We show that this property depends not only on the initial condition, but also on the (time) derivative of the input function.

We suggest possible applications of this result to some control problems.

I. Introduction

Let us consider the system

$$\dot{x}=f(\alpha, x) \tag{1}$$

where $x \in U$ is the state variable and $\alpha \in V$ is the input function of time, U and V are open sets in R^n and R^m, respectively.

Recall that a point $q_0 \in U$ is a sink of (1) corresponding to the constant input $\alpha_0 \in V$ if

$$f(\alpha_0, q_0)=0, \ \mathrm{Re}\ \sigma\left(\frac{\partial f}{\partial x}(\alpha_0, q_0)\right)<0$$

where $\sigma(\cdot)$ denotes the spectrum of a matrix. This means that if the input is "frozen" at α_0, then q_0 is an asymptotically stable equilibrium point.

Roughly speaking, what we are going to prove is the following fact: if $\alpha(\cdot)$ varies slowly and $q(\cdot)$ is C^1 such that $q(t)$ is always a sink corresponding to $\alpha(t)$, then any solution starting near $q(t_0)$ will stay close to $q(\cdot)$. Moreover, $x(t)$ will always be in the domain of attraction of $q(t)$, corresponding to the input $\alpha(t)$. (That is, if from any moment t' on, we would fix the input at $\alpha(t')$, then lim $x(t) = q(t')$ when t goes to infinity.)

In earlier studies (e.g., [1, Theorem 2, sect. 32] and [2, ch. 4, sect. 8]) similar stability results were obtained for slowly varying systems with no input.

We also point out that the set Q, which plays a role in the statement of our theorem, shares some resemblances with a "slow manifold," see [3].

II. The Result

Our stability result is as follows.

Theorem: Suppose that U, V are open sets in R^n, R^m, and U is convex; $f: V \times U \to R^n$ is C^2 such that $M = \{(\beta, y) \in V \times U\}$, where y is a sink of (1) corresponding to β, is not empty; Q is an open, connected subset of M, relatively compact in M (i.e., the closure $\bar{Q}$ of Q in M is bounded); $\alpha:[t_0, \infty) \to V$, $q:[t_0, \infty) \to U$ are C^1 such that $(\alpha(t), q(t)) \in Q$ for every $t \geq t_0$. Let $x(\cdot)$ be a solution of (1), with α as above.

Then, for any $\rho > 0$ there exist $\delta_1 > 0$, $\delta_2 > 0$, independent of t_0, α, q, such that for $|x(t_0) - q(t_0)| \leq \delta_1$, and $\max_{t \geq t_0} |\dot{\alpha}(t)| \leq \delta_2$, we have:

i) $|x(t) - q(t)| < \rho$ for every $t \geq t_0$. If in addition ρ is sufficiently small

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