GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION

PROJECT ADMINISTRATION DATA SHEET

Project No./(Center No.)  E-20-642 (R5903-1A0)

Project Director:  Dr. S. N. Atluri

Sponsor:  AFOSR Bolling AFB, D. C.

Agreement No.:  Grant No. AFOSR-84-0020 Amendment C

Award Period:  From 12/1/86 To 11/30/88 (Performance) 12/30/87

Sponsor Amount:

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Cost Sharing No./(Center No.)  E-20-330 (F5903-1A0) Cost Sharing: $9,917

Title:  Control, Nonlinear Dynamics, and Modeling Problems in Large Space Structures

ADMINISTRATIVE DATA

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Military Security Classification:  N/A
(or) Company/Industrial Proprietary:  N/A

ONR Resident Rep. is ACO:  Yes  X  No

Defense Priority Rating:  N/A

RESTRICTIONS

See Attached  N/A
Supplemental Information Sheet for Additional Requirements.

Travel:  Foreign travel must have prior approval — Contact OCA in each case. Domestic travel requires sponsor approval where total will exceed greater of $500 or 125% of approved proposal budget category.

Equipment:  Title vests with GIT, prior written authorization is required for items over $5,000 if not specifically included in approved budget.

COMMENTS:
Continuation of project E-20-621

*Total contract value (including all amendments) now: $367,281

SPONSOR'S I.D. NO.  02.104.001.87.001

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NOTICE OF PROJECT CLOSEOUT

Closeout Notice Date 01/23/91

Project No. E-20-642
Project Director ATLURI S N
Sponsor AIR FORCE/BOLLING AFB, DC
Contract/Grant No. AFOSR-84-0020
Prime Contract No. 
Center No. R5903-1A0
School/Lab CIVIL ENGR
Contract Entity GTRC

Title STUDIES CONTROL NONLINEAR DYNAMICS & MODELING PROBLEMS IN LARGE SPACE STR

Effective Completion Date 900315 (Performance) 900514 (Reports)

Closeout Actions Required: 

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Comments

Subproject Under Main Project No.

Continues Project No.

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NOTE: Final Patent Questionnaire sent to PDPI.

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Abstract

Constraint equations arise in the dynamics of mechanical systems whenever there is the need to restrict kinematically possible motions of the system. In practical applications, constraint equations can be used to simulate complex, connected systems. If the simulation must be carried out numerically, it is useful to look for a formulation that leads straightforwardly to a numerical approximation.

In this paper, we extend the methodology of our previous work to incorporate the dynamics of holonomically and nonholonomically constrained systems. The constraint equations are cast in a variational form, which may be included easily, in the time finite element framework. The development of the weak constraint equations and their associated "tangent" operators is presented. We also show that this approach to constraint equations may be employed to develop time finite elements using a quaternion parametrization of finite rotation. Familiarity with the notation and methodology of our previously presented work is assumed.

1 Introduction

Very often the vectorial and variational theories of mechanics are considered to be completely equivalent. Many times, variational principles are used only as an alternative approach for obtaining differential equations of motion.

Here, we assert that variational methods provide more robust formulations, which not only afford a general, unified approach, but are also more easily implemented, numerically.

During the last decade the variational formulations for dynamical systems and their numerical approximations, have known a renewed interest (e.g. Bailey (1981), Simkins (1981), Bauch and Riff (1982) and others). Borri et al (1985) apply a weak formulation of Hamilton's principle to the dynamics of holonomic systems, using time finite elements. Although many authors, such as Neimark and Fufaev (1972), have treated nonholonomic systems, a general variational formulation suitable for a direct numerical approximation is not yet available.

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As a possible solution to this problem, we suggest the adoption of a new variational principle for holonomically and nonholonomically constrained dynamical systems and show how two different time finite element approximations can be derived.

For the sake of simplicity the formulations presented here are for finite degree of freedom systems, but they can be easily extended to continuous systems as in Iura, Borri and Atluri (1988).

2 Different Forms of Hamilton's Principle

It is shown in Borri, Mello and Atluri (1990), that Hamilton's principle for unconstrained dynamics can be written as:

\[ \int_{t_1}^{t_2} (\delta \mathcal{L}(\dot{q}, q, t) + \delta q \cdot f)dt = \delta q \cdot p_b |_{t_1} \]

where \( t_1, t_2 \) are the ends of the time interval of interest; \( q \) and \( p \) are respectively the generalized coordinates and the momenta of the system. \( \mathcal{L}(\dot{q}, q, t) \) denotes the Lagrangian function and \( f \) the external forces not included in \( \mathcal{L} \).

In the same work, it is shown that through a Legendre transformation, Eq. (1) can be rewritten in the following mixed form:

\[ \int_{t_1}^{t_2} (\delta \dot{q} \cdot p - \delta \dot{p} \cdot q - \delta H + \delta q \cdot f)dt = (\delta q \cdot p_b - \delta p \cdot q_b) |_{t_1} \]

where \( H(p, q, t) = p \cdot \dot{q} - \mathcal{L} \) denotes the Hamiltonian of the system.

Eq. (1) and Eq. (2) are, respectively, primal and mixed forms of Hamilton's principle and are very suitable for numerical approximations in the context of finite elements in the time domain (e.g. Borri and Montegazza (1986) and Borri and Atluri (1988)). We note that Eq. (1) implies that displacement continuity is satisfied "a priori" while in Eq. (2) displacement continuity is enforced in a weak sense.

For purposes of illustration, we consider the class of constraints which are functions of the generalized coordinates and time, but only linear functions of the generalized velocities. As pointed out in Lanczos (1964), this class of constraints encompasses a large number of practical situations.

\[ \psi(\dot{q}, q, t) = A(q, t) \cdot \dot{q} + a(q, t) = 0 \]

If calculating the work of constraint forces is to be avoided, the virtual displacements must satisfy the constraints:

\[ A \cdot \delta q = 0 \]

Constraints expressed in the form of Eq. (3) can be either nonholonomic, or the time derivatives of holonomic constraints. In order to enforce Eq. (3) and Eq. (4), they are cast in weak form, with an appropriate choice of test functions. Let \( \mu \) be the Lagrange multipliers. Then, weighting Eq. (3) with the variation \( \delta \mu \) and Eq. (4) with the time derivative \( \dot{\mu} \), the following relation is obtained:
The benefit of this form is that it allows an integration by parts that reduces the continuity requirements for the Lagrangian multipliers. Carrying out this integration yields:

\[
\int_{t_1}^{t_2} \left( \delta \mu \cdot \psi - \mu \cdot \frac{\partial \psi}{\partial q} \cdot \delta q \right) dt = 0
\]  

Combining Eq. (6) with the primal form, Eq.(1), results in a modified primal form:

\[
\int_{t_1}^{t_2} \left\{ \delta \mathcal{L} + \delta q \cdot f + \delta(\mu \cdot \psi) + \mu \cdot \left( \frac{d}{dt} \frac{\partial \psi}{\partial q} - \frac{\partial \psi}{\partial q} \right) \cdot \delta q \right\} dt
\]

\[
= \delta q \cdot \left( p_b + \mu \cdot \frac{\partial \psi}{\partial q} \right) \bigg|_{t_1}^{t_2}
\]

This may be written more concisely as:

\[
\int_{t_1}^{t_2} (\delta \mathcal{C} + \delta q \cdot \mathbf{f}) dt = \delta q \cdot \mathbf{p}_b \bigg|_{t_1}^{t_2}
\]

where:

\[
\mathcal{C} = \mathcal{L} + \mu \cdot \psi \quad \mathbf{p} = p + \mu \cdot \frac{\partial \psi}{\partial q} \quad \mathbf{f} = f + \mathbf{f}_c
\]

and

\[
\mathbf{f}_c = \mu \cdot \left( \frac{d}{dt} \frac{\partial \psi}{\partial q} - \frac{\partial \psi}{\partial q} \right)
\]

Eq.(8) is a modified primal form of Hamilton’s Principle for constrained systems and \(\mathcal{C}, \mathbf{p}, \mathbf{f}\) are respectively, the modified Lagrangian function, the modified generalized momenta, and the external forces as modified by the reactions due to the nonholonomic constraints. The constraint reaction force \(\mathbf{f}_c\) is typical of nonholonomic constraints, since it is identically zero for the holonomic case. In fact, the holonomic constraint implies \(\psi = \phi\) from which it is clear that:

\[
\frac{\partial \phi}{\partial q} = \frac{\partial \phi}{\partial q}
\]

so:

\[
\mathbf{f}_c = \mu \cdot \left( \frac{d}{dt} \frac{\partial \phi}{\partial q} - \frac{\partial}{\partial q} \frac{d \phi}{dt} \right) \equiv 0
\]
In the nonholonomic case however, \( f_c \) is different from zero. Substituting Eq.(3) into Eq.(10), the modification of the force due to nonholonomic constraints may be written as:

\[
f_c = C \cdot \dot{q} + c \\
C = -C^t
\]  

(13a)

where:

\[
C_{ik} = \mu_a \cdot \left( \frac{\partial A_{ai}}{\partial q_k} - \frac{\partial A_{ak}}{\partial q_i} \right) \\
c_i = \mu_a \cdot \left( \frac{\partial}{\partial t} A_{ai} - \frac{\partial a_a}{\partial q_i} \right)
\]  

(13b)

It is interesting to note that \( \bar{p} \) are actually the generalized momenta of the augmented Lagrangian. In fact, it can be seen that:

\[
\bar{p} = \frac{\partial \bar{L}}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} + \mu \cdot \frac{\partial \psi}{\partial \dot{q}}
\]  

(14)

Taking this property into account, the modified Hamiltonian function can be defined as:

\[
\bar{H} = \bar{p} \cdot \dot{q} - \bar{L}
\]  

(15)

and Eq.(2) may be rewritten in the following modified mixed form:

\[
\int_{t_1}^{t_2} \left( \delta \dot{q} \cdot \bar{p} - \delta \ddot{p} \cdot q - \delta \bar{H} + \delta q \cdot \bar{f} \right) dt = (\delta q \cdot \bar{p}_b - \delta \bar{p} \cdot q_b)|_{t_1}^{t_2}
\]  

(16)

where the velocity \( \dot{q} \) is eliminated from the expression for \( f_c \) in favor of the momentum. It is worth emphasizing that the modified momenta \( \bar{p} \) are no longer constrained and can be viewed as independent variables. The compatible momenta \( p \) and the Lagrangian multipliers \( \mu \) can be recovered from the modified momenta by a simple projection.

In order to better understand the role of the Lagrangian multipliers, consider the Euler equations corresponding to the principle Eq.(8). These are:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - f + A^t \cdot \frac{d}{dt} \mu = 0
\]  

(17a)

\[
A \cdot \dot{q} + \alpha = 0
\]  

(17b)

It is immediately seen that the Lagrange multipliers enter into these expressions only through their time derivatives. It is therefore only the derivative which has a meaningful physical interpretation. Therefore the initial value of modified momentum \( \bar{p} \) may be selected in a convenient way. Specifically, a scaling approach may be used, which at the end of each integration step resets \( \mu = 0 \) and maintains \( \bar{p} = p \). Further, the constraint
equations can be written in terms of the modified momentum \( \tilde{p} \) and solved for the Lagrange multipliers. In order to accomplish this, the following equations must be solved in terms of the velocity \( \dot{q} \):

\[
\tilde{p} = \frac{\partial \mathcal{L}}{\partial \dot{q}}
\]  

(18)

which leads to:

\[
\dot{q} = B \cdot (\tilde{p} - A^t \cdot \mu) + b
\]  

(19)

Substituting Eq.(19) back into Eq.(3) and solving for \( \mu \) results in:

\[
\mu = \mu_o + D^{-1} \cdot A \cdot B \cdot \tilde{p}
\]

(20a)

where:

\[
D = A \cdot B \cdot A^t, \quad \mu_o = D^{-1} \cdot (A \cdot b + a)
\]

(20b)

The true momentum can then be recovered from the modified momentum through the expression:

\[
p = \tilde{p} - A^t \cdot \mu,
\]

(21)

thus obtaining:

\[
p = p_o + P \cdot \tilde{p}
\]

(22a)

where:

\[
P = I - A^t \cdot D^{-1} \cdot A \cdot B, \quad p_o = -A^t \cdot \mu_o
\]

(22b)

Clearly, \( P \) is a projection, since:

\[
P^2 = P
\]

(23)

From the knowledge of \( \tilde{p}_b \) the corresponding \( \mu_b \) can be recovered and using Eq.(22), the boundary momentum \( p_b \) may be found.

3 Examples

In order to verify this methodology for both holonomic and nonholonomic constraints, the classical cases of a spinning top and rolling coin are considered. The mass center of the top is chosen as the reference point, so that the motion is described by six degrees of
freedom, with the fixity of the suspension point being enforced as a constraint. For the coin, the mass center is again chosen as the reference, with the nonholonomic constraint of zero velocity of the contact point, enforced in weak form. In each case the constraint is put in the form of Eq.(3), and included as described above. The numerical results are compared for two, three and four noded time finite elements with various step sizes. The time finite element approach applied here, is developed in Borri, Mello and Atluri (1990), using the finite rotation vector as rotation coordinates. It is assumed that the reader is familiar with those results. In the following discussion, only the residual vectors and tangent matrices associated with the constraints are discussed.

3.1 Spinning Top

The case of a spinning top is treated in virtually all texts on dynamics. The motion is typically described in terms of three Euler angles and the translational degrees of freedom are eliminated by using the suspension point as a reference. Since the intent of this example is to demonstrate the application of constraint equations, the mass center is chosen as the reference point. The constraint that the suspension point is fixed may then be expressed as:

\[ \dot{x} = \rho \times \omega \]  

(24)

where \( \dot{x} \) is linear velocity of the mass center, \( \omega \) is the angular velocity of the top and \( \rho \) is the radius vector from the mass center to the suspension point.

Following the above procedure the constraint is cast in weak form along with the constraint on virtual motion:

\[ \delta x = \rho \times \delta \theta \]  

(25)

Letting the multipliers be \( \mu \), Eq.(24) is weighted with the variation, \( \delta \mu \), while the virtual constraint is weighted with the time derivative, \( \dot{\mu} \). Combining the weak forms of the constraints and integrating over the time interval leads to:

\[ \int_{t_1}^{t_2} \delta \mu \cdot (\dot{x} - \rho \times \omega) - \dot{\mu} \cdot (\delta x - \rho \times \delta \theta) \, dt = 0 \]  

(26)

Integrating by parts the term involving \( \dot{\mu} \), results in an expression of the form:

\[ \int_{t_1}^{t_2} \delta \mu \cdot (\dot{x} - \rho \times \omega) + \mu \cdot (\delta \dot{x} - \rho \times \delta \dot{\theta} - \rho \times \delta \theta) \, dt = \mu \cdot (\delta x - \rho \times \delta \theta) \big|_{t_1}^{t_2} \]  

(27)

The residual vector contribution, due to the constraint, is then:

\[ \left\{ (\dot{x} - \rho \times \omega), \mu, (\rho \times \mu), 0, (\dot{\rho} \times \mu) \right\} \]  

(28)
where the organization of the test functions is $(\delta \mu, \dot{\delta} \mathbf{x}, \dot{\theta}_\delta, \delta \mathbf{x}, \theta_\delta)$.

The tangent matrix for this constraint is obtained by linearizing the residual vector. In carrying out the linearization process, the following identities are used.

\begin{align*}
    d\rho &= \theta_d \times \rho = -\rho \times \theta_d \\
    \dot{\rho} &= \omega \times \rho \\
    d\dot{\rho} &= \theta_d \times \rho + \theta_d \times \rho = -\rho \times \theta_d - \dot{\rho} \times \theta_d \\
    d\omega &= \theta_d - \omega \times \theta_d
\end{align*}

With these relations in mind, it is straightforward to write down the linearization. The first group of terms corresponding to the test functions $\delta \mu$, leads to:

\begin{equation}
    \delta \mu \cdot \left( d\dot{\mathbf{x}} - d\rho \times \omega - \rho \times d\omega \right)
\end{equation}

which, in view of the above identities, simplifies to:

\begin{equation}
    \delta \mu \cdot \left[ I, (\rho \times I), (\rho \times \omega) \times I \right] \cdot \left\{ \begin{array}{c} d\dot{\mathbf{x}} \\ \theta_d \\ \theta_d \end{array} \right\}
\end{equation}

Similarly, the second group of terms in the residual vector, corresponding to the test functions $\delta \dot{\mathbf{x}}$, lead to:

\begin{equation}
    \delta \dot{\mathbf{x}} \cdot I \cdot d\mu
\end{equation}

The terms involving the test functions $\dot{\theta}_\delta$ linearize as:

\begin{equation}
    \dot{\theta}_\delta \cdot (d\rho \times \mu + \rho \times d\mu)
\end{equation}

which reduces to:

\begin{equation}
    \dot{\theta}_\delta \cdot [\rho \times I, \mu \times \rho \times I] \cdot \left\{ \begin{array}{c} d\mu \\ \theta_d \end{array} \right\}
\end{equation}
Finally, the last group of terms in the residual vector leads to:

$$\theta_6 \cdot (d \dot{\rho} \times \mu + \dot{\rho} \times d\mu)$$

which may be written as:

$$\theta_6 \cdot \left[ \dot{\rho} \times I, \mu \times \rho \times \dot{\theta}_d, \mu \times \dot{\rho} \times I \right] \cdot \begin{bmatrix} d\mu \\ \dot{\theta}_d \\ \theta_d \end{bmatrix}$$

Combining these relations provides the tangent matrix for the constraint. The resulting tangent matrix is then:

$$T_{top} = \begin{bmatrix} 0 & I & -\rho \times I & 0 & \dot{\rho} \times I \\ I & 0 & 0 & 0 & 0 \\ \rho \times I & 0 & 0 & 0 & \mu \times \rho \times I \\ 0 & 0 & 0 & 0 & 0 \\ \dot{\rho} \times I & 0 & \mu \times \rho \times I & 0 & \mu \times \dot{\rho} \times I \end{bmatrix}$$

The exact solution for a spinning top involves the evaluation of an elliptic integral. This is not done here. Rather, the regions of different behavior are explored. Initial conditions for the top are altered, in order to produce the following three situations (see Figure 1). Case 1 exhibits precession which is always in the same direction throughout the motion. Case 2 exhibits precession which changes sign during the motion. Case 3 does not reverse its direction of precession but does stop precessing at points in its motion (cuspidal motion).

For each of these cases, the problem is solved with different time steps and with two, three and four noded time elements. The input data for each case are given as:

Case 1:
- The mass is 1.0
- The axial moment of inertia is .40
- The transverse moment of inertia is .75
- The initial orientation is in the yz plane, inclined 10 degrees from vertical.
- The initial angular velocity is (0, .9888, 7.5167) rad/sec.
- The distance from the mass center to the support is .2
- Gravity is 3.0

Case 2:
- The initial angular velocity is (0, .20905, 6.2964) rad/sec.
- All other data are the same.

Case 3:
The initial angular velocity is \((0, 0, 6.3794) \text{ rad/sec}\).

All other data are the same.

The solutions obtained by time finite elements are compared to the solution found by integrating the equations of motion in terms of Euler angles, using the fourth order Runge-Kutta integrator RKM45. The allowable error per step, used in the Runge-Kutta integration was \(10^{-10}\), and this solution will be referred to as the exact solution in the following comparisons.

Figure 2 shows the results for Case 1, using two noded elements and with the maximum time step restricted to .06 seconds. This represents about 28 degrees of proper rotation per step. For a direct comparison, Figure 3 shows the result for three noded elements with the step size restricted to .06 seconds. Even though a three noded element spans twice the time of a two noded element, the improvement in the solution is clear. Figure 4 presents the results for the four noded element, which are also quite good. The three and four noded elements are very accurate and demonstrate superior performance compared to the two noded elements. The errors in the \(x, y\) and \(z\) coordinates of the mass center, at the end of the simulation, were \((.976\%, 1.345\%, .0539\%)\) for the two noded elements, \((.308\%, .176\%, .0068\%)\) for the three noded elements and \((.068\%, .095\%, .0022\%)\) for the four noded elements.

The same type of behavior is exhibited in the other two motion cases. Figures 5 and 6 show the results for case 2, by three-noded and four-noded elements, respectively. The cuspidal motion is shown in Figures 7 and 8.

3.2 Rolling Coin

As an example of a nonholonomic or velocity constraint, the classic problem of a rolling coin is considered (see Figure 9). Here again the mass center is taken as the reference point. The constraint equations, enforcing zero velocity of the contact point, appear identical to those for the top. However, the radius vector from the mass center to the contact point is not embedded in the body fixed frame, as it was for the top.

\[ \dot{\mathbf{x}} = \rho \times \omega \]  

(41)

where \(\dot{\mathbf{x}}\) is linear velocity of the mass center, \(\omega\) is the angular velocity of the coin and \(\rho\) is the instantaneous radius vector from the mass center to the contact point.

The constraint is again cast in a weak form, along with the constraint on virtual motion:

\[ \delta \mathbf{x} = \rho \times \theta_{\delta} \]  

(42)

The result is identical to Eq.(26).

\[ \int_{t_1}^{t_2} \delta \mu \cdot (\dot{\mathbf{x}} - \rho \times \omega) - \dot{\mu} \cdot (\delta \mathbf{x} - \rho \times \theta_{\delta}) \, dt = 0 \]  

(43)

Integrating by parts the term involving \(\dot{\mu}\) results in an expression of the form
\[
\int_{t_1}^{t_2} \delta \mu \cdot (\ddot{x} - \rho \times \omega) + \mu \cdot (\delta \dot{x} - \rho \times \dot{\theta}_\delta - \dot{\rho} \times \theta_\delta) \, dt = \mu \cdot (\delta x - \rho \times \theta_\delta) \bigg|_{t_1}^{t_2}
\]  

(44)

Before Eq.(44) can be simplified further, the radius must be expressed in terms of known directions. This can be done by first denoting the unit vector in the direction of gravity by \( e_g \), the unit vector normal to the plane of the coin by \( e_n \) and the unit vector along the line of the radius as \( e_\rho \). The unit vector \( e_\rho \) is then written as:

\[
e_\rho = \frac{(I - e_n \cdot e_n^t) \cdot e_g}{\sqrt{e_g \cdot (I - e_n \cdot e_n^t) \cdot e_g}} = \frac{c}{\sqrt{c \cdot c}}
\]  

(45)

where \( c = (I - e_n \cdot e_n^t) \cdot e_g \). The variation and time derivative of \( e_\rho \) can then be expressed as:

\[
\delta e_\rho = \frac{1}{\sqrt{c \cdot c}} \left[ I - \frac{c \cdot c^t}{c \cdot c} \right] \cdot \delta c = A \cdot B \cdot \theta_\delta
\]  

(46)

\[
\dot{e}_\rho = \frac{1}{\sqrt{c \cdot c}} \left[ I - \frac{c \cdot c^t}{c \cdot c} \right] \cdot \dot{c} = A \cdot B \cdot \omega
\]  

(47)

where:

\[
A = \frac{1}{\sqrt{c \cdot c}} \left[ I - e_\rho \cdot e_\rho^t \right]
\]  

(48)

\[
B = \left[ e_n^t \cdot e_n \cdot I - e_n \cdot e_n^t \cdot e_g \times I \right]
\]  

(49)

Using these relations in Eq.(44), the residual vector for the rolling constraint is expressed as:

\[
\left\{ (\dot{x} - \rho \times \omega), \mu, (\rho \times \mu), 0, (\dot{\rho} \times \mu) \right\}
\]  

(50)

where, again, the organization of the test functions is \((\delta \mu, \delta x, \dot{\theta}_\delta, \delta x, \theta_\delta)\). In Eq.(50) \( \dot{\rho} \) is understood to be \( \rho \dot{e}_\rho \) with \( \dot{e}_\rho \) given by Eq.(47) and \( \rho \) being the modulus of \( \rho \).

The tangent matrix for the rolling constraint is obtained by linearizing the residual vector. In carrying out the linearization, the following identities and notations are used:

\[
d\omega = \dot{\theta}_d - \omega \times \theta_d
\]  

(51)

\[
d\rho = \rho d e_\rho = \rho A \cdot B \cdot \theta_d
\]  

(52)
\[ \dot{p} = \rho \dot{e}_p = \rho A \cdot B \cdot \omega \] (53)

\[ d \dot{p} = \rho A \cdot B \dot{\theta}_d + \left( \dot{A} \cdot B + A \cdot \dot{B} \right) \cdot \theta_d \] (54)

where:

\[ \dot{A} = -\frac{c \cdot B \cdot \omega}{c \cdot c} A - \frac{1}{\sqrt{c \cdot c}} \left( \dot{e}_p \cdot e'_p + e_p \cdot e'_p \right) \] (55)

\[ \dot{B} = e_g \cdot \omega \times e_n \times I + e_g \cdot e_n \cdot (\omega \times e_n) \times I \]

\[ + e_n \cdot e'_n \cdot \omega \times e_g \times I - \omega \times e_n \cdot e'_n \cdot e_g \times I \] (56)

In view of the above relations, the development of the tangent matrix for this constraint proceeds in exactly the same way as for the top. The details of the algebra are omitted, but the steps are straightforward to verify. The resulting tangent matrix is given by:

\[
\begin{bmatrix}
0 & I & -\rho \times I & 0 & \rho \times \omega \times I + \omega \times \rho A \cdot B \\
I & 0 & 0 & 0 & 0 \\
\rho \times I & 0 & 0 & 0 & -\mu \times \rho A \cdot B \\
0 & 0 & 0 & 0 & 0 \\
\rho A \cdot B \cdot \omega \times I & 0 & -\mu \times \rho A \cdot B & -\mu \times \rho \left( \dot{A} \cdot B + A \cdot \dot{B} \right)
\end{bmatrix}
\] (57)

The problem definition for the rolling coin example is as follows: The coin was started with an initial angular velocity of \(-2\) radians/sec. about the axis of symmetry of the coin, and an initial velocity of the mass center of .4 units/sec. in the positive X direction. The coin has unit mass and axial and transverse moments of inertia given by .4 and .75, respectively. The coin has a radius of .2 units, and is initially inclined at 45 degrees.

Just as with the top, the solution for the rolling coin was evaluated with the integrator RKM45, imposing an accuracy requirement of \(10^{-10}\) to establish an "exact" solution. The same type of behavior observed in the top problem is found here. The two-noded elements produced good results for reasonably small time steps, while the three, and four-noded elements were much more accurate. Figures 10 through 12 show the plots of the mass center, indicated by circles and the contact point, indicated by plus signs. The two noded element used a time step of .3 seconds which corresponds to approximately 35 degrees of proper rotation per element. The three and four noded elements were similarly restricted to a maximum time step of .3 seconds and clearly produce much more accurate solutions.
Rigid Body Dynamics Using Quaternions

The methodology for including constraints presented above, may be applied when using quaternions as rotation parameters. The use of quaternions as a four dimensional parameterization of rotation is well known. However, their use in a time finite element approximation, has not been reported. Quaternions, used in this context result in a very simple expression for the linearization of the governing equations, as will be shown in the following discussion. This simplicity may justify the increased number of parameters. For this choice of rotation coordinates, a free tumbling body constitutes a constrained problem, if the restriction that the quaternion have unit magnitude, is enforced in a weak sense.

For the case of a free tumbling body, attention is focused on the angular motion. The rotation tensor $R$, which describes the orientation of a body fixed frame, may be expressed in terms of a unit quaternion. This is demonstrated in Appendix A, along with some other fundamental relations in quaternion algebra.

A generic quaternion is indicated by $q$ and has a scalar part $g$, and a vector part $q_v$. In this case, the quaternion constitutes the generalized coordinates. A unit quaternion has a scalar part $g_v = \cos \frac{\phi}{2}$ and a vector part $q_v = e \sin \frac{\phi}{2}$. Where $\phi$ is the magnitude of rotation and $e$ is the unit vector along the axis of rotation. As shown in Appendix A, a four dimensional angular velocity is defined as $\omega = 2B'(q) \cdot \dot{q}$. Expressed in reference coordinates, the pull back of the angular velocity is $\omega^* = 2A^t(q) \cdot \dot{q}$. When $q$ is a unit quaternion, $\omega_s = \omega^*_s = 0$ and the vectorial part $\omega_v = R \cdot \omega^*_v$ coincides with the three dimensional angular velocity. An a priori satisfaction of the unit condition on the quaternion is not convenient in a finite element context. This is due to the fact that polynomial interpolation of unit quaternions does not result in a unit quaternion. The unity constraint that $q \cdot q = 1$ will be put in the differential form, $2 \cdot \dot{q} \cdot q = 0$ and enforced in a weak sense.

As with the angular velocity, we define a four dimensional momentum $h$ as $h = J \cdot \omega$, where $J$ is a four dimensional inertia tensor which can be defined in a number of ways. The simplest is:

$$J = \begin{bmatrix} J_s & 0 \\ 0 & J_3 \end{bmatrix}$$

where $J_3$ is the three dimensional inertia tensor and the scalar $J_s$ is an arbitrary constant, different from zero. With this definition, the scalar part of $h$ is zero for a unit quaternion. The vectorial part then coincides with the usual three dimensional momentum regardless of the value of $J_s$. Choosing $J_s \neq 0$ allows the calculation of the inverse relation $\omega = J^{-1} \cdot h$.

The high degree of nonlinearity in rigid body dynamics arises from the fact that $J_3$ is not constant but satisfies the following:

$$J_3 = R \cdot J_3 \cdot R^t$$

(59)
where $J_3$ is constant. The four dimensional inertia tensor is then defined as:

$$ J = G \cdot \bar{J} \cdot G^t $$

(61)

where:

$$ G = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \quad \text{and} \quad \bar{J} = \begin{bmatrix} J_z & 0 \\ 0 & J_3 \end{bmatrix} $$

(61)

Two matrices, $A(q)$ and $B(q)$, which will be used extensively in this section, are defined to be:

$$ A(q) = \begin{bmatrix} q_x & -q_y \\ q_y & q_x + q_y \times I \end{bmatrix} = \begin{bmatrix} q & A_\perp(q) \end{bmatrix} $$

(62)

$$ B(q) = \begin{bmatrix} q_x & -q_y \\ q_y & q_x - q_y \times I \end{bmatrix} = \begin{bmatrix} q & B_\perp(q) \end{bmatrix} $$

(63)

The modified primal form, Eq. (7), and the modified mixed form, Eq. (16), may now be written in terms of the quaternion parameters.

**PRIMAL FORM**

In the case of the free tumbling body, the kinetic energy is the Lagrangian and may be written as:

$$ T = \frac{1}{2} \omega^* \cdot \bar{J} \cdot \omega^* = 2 \dot{q} \cdot A(q) \cdot \bar{J} \cdot A'(q) \cdot \dot{q} = 2q \cdot C(q) \cdot \bar{J} \cdot C'(\dot{q}) \cdot q $$

(64)

The momentum, $p$, which is conjugate to $\dot{q}$, is defined as:

$$ p = \frac{\partial T}{\partial \omega} = 2A(q) \cdot h^* = 2B(h^*) \cdot q $$

(65)

The modified kinetic energy is then:

$$ \bar{T} = T + 2\mu \cdot q \cdot \dot{q} $$

(66)

and the modified momentum is:

$$ \bar{p} = \frac{\partial \bar{T}}{\partial \omega} + 2\mu \cdot q = 2A(q) \cdot h^* + 2\mu \cdot q = 2A(q) \cdot \bar{h}^* = 2B(\bar{h}^*) \cdot q $$

(67)

where:

$$ \bar{h}^* = (h^*_x + \mu, h^*_y) $$

(65)
is the modified four dimensional angular momentum. The primal form of Hamilton's principle, with a quaternion parametrization of rotation, may be linearized quite simply as:

\[
\int_{t_1}^{t_2} \delta(\dot{q}, q, \mu) \cdot T_q \cdot d(\dot{q}, q, \mu) \, dt = \delta q \cdot \mathbf{p}_b|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta(q, q, \mu) \cdot R_q \, dt
\]

where:

\[
R_q = \left( 2B(\vec{h}^*) \cdot q, 2B^t(\vec{h}^*) \cdot \dot{q} + f, 2\dot{q}^t \cdot q \right)
\]

\[
T_q = \begin{bmatrix}
4A(q) \cdot \vec{J} \cdot A^t(q) & 2B(\vec{h}^*) + 4A(q) \cdot \vec{J} \cdot C^t(q) & 2q \\
2B^t(\vec{h}^*) + 4C(q) \cdot \vec{J} \cdot A^t(q) + f \cdot \dot{q} & 4C(\dot{q}) \cdot \vec{J} \cdot C^t(\dot{q}) + f \cdot \dot{q} & 2\dot{q}^t \\
2q^t & 2\dot{q}^t & 0
\end{bmatrix}
\]

**MIXED FORM**

The discussion of the mixed formulation begins with the definition of the modified Hamiltonian:

\[
\vec{H} = \frac{1}{2} h^* \cdot J^{-1} \cdot h^*
\]

where \( h^* \) is given by the following expression:

\[
h^* = (\vec{h}^*_s - \mu, \vec{h}^*_v) = \frac{1}{2} A^t(q) \cdot (\vec{p} - 2\mu \vec{q}) = \frac{1}{2} C^t(\vec{p} - 2\mu \vec{q}) \cdot \vec{q}
\]

It is interesting in this case to see the Euler equations corresponding to the mixed form. They are:

\[
\dot{q} = \frac{1}{2} B(\omega^*) \cdot q
\]

\[
\dot{p} = -\frac{1}{2} B^t(\omega^*) \cdot \vec{p} + 2\mu \omega_s q + f
\]

\[
0 = \frac{1}{2} q \cdot (B(\omega^*) + B^t(\omega^*)) \cdot \vec{q} = q^2 \omega_s
\]
where \( \omega^* = \frac{1}{2} J^{*-1} \cdot A^t(q) \cdot (\bar{p} - 2\mu q) \). From the last equation it is immediately clear that \( \omega_s = 0 \). The equations can then be reduced to the first two sets in terms of \( \bar{p} \), and \( q \) leading to:

\[
\begin{align*}
\dot{q} &= \frac{1}{2} B(\omega^*) \cdot q \\
\dot{\bar{p}} &= -\frac{1}{2} B^t(\omega^*) \cdot \bar{p} + f
\end{align*}
\]  

(73)

Solving for \( \mu \) from the equation \( \omega_s = 0 \) leads to:

\[
\mu = \frac{1}{2} q^{-2} q \cdot \bar{p}
\]  

(76)

Clearly, the multiplier, \( \mu \), is independent of \( J^*_s \). Moreover, by substitution into the definition \( p = \bar{p} - 2\mu q \), the true momentum may be written as:

\[
p = P \cdot \bar{p}
\]  

(77)

where:

\[
P = I_4 - q^{-2} q \cdot q^t
\]  

(78)

is the projection for the present case. This projection is not required during the solution process, but only when the true momentum must be recovered.

The linearization of the modified mixed principle, around a given state \((\bar{p}_g, q_g)\); can be written as:

\[
\int_{t_1}^{t_2} \left[ \frac{d}{dt} (\delta \bar{p}, \delta q) \cdot I_S \cdot (d\bar{p}, dq) - \delta(\bar{p}, q, \mu) \cdot T_h \cdot d(\bar{p}, q, \mu) \right] dt
\]  

(79)

\[
= (\delta \bar{p}, \delta q) \cdot I_S \cdot (\bar{p}_g, q_g) - \int_{t_1}^{t_2} \delta(\bar{p}, q, \mu) \cdot R_h \, dt
\]

where:

\[
I_S = \begin{bmatrix} 0 & -I_4 \\ I_4 & 0 \end{bmatrix}
\]  

(80)

\[
R_h = \left( \frac{1}{2} B(\omega^*) \cdot q, \frac{1}{2} B^t(\omega^*) \cdot \bar{p} - 2\mu \omega^*_s q, -q^2 \omega^*_s \right)
\]  

(81)
\[ T_h = \begin{bmatrix}
\frac{1}{4} A(q) \cdot J^{-1} \cdot A(q) & \frac{1}{2} A(q) \cdot J^{-1} \cdot \frac{\partial h}{\partial q} & -\frac{1}{2} q^2 J_s^{-1} q \\
-\frac{1}{2} \frac{\partial h}{\partial q} \cdot J^{-1} \cdot A'(q) + \frac{1}{2} B'(\omega^*) + f \cdot \bar{p} & \frac{\partial h}{\partial q} \cdot J^{-1} - \frac{\partial h}{\partial q} \cdot \mu \omega_s I_4 + f \cdot q & -2\omega_s^* \cdot q - \frac{\partial h}{\partial q} \cdot J^{-1} \cdot \frac{\partial h}{\partial \mu} \\
-\frac{1}{2} q^2 J_s^{-1} \cdot q^t & \frac{\partial h}{\partial \mu} \cdot J^{-1} \cdot \frac{\partial h}{\partial q} & q^t J_s^{-1}
\end{bmatrix} \]

in which:

\[
\frac{\partial h}{\partial q} = \frac{1}{2} \left( C'(p) - 2\mu A'(q) \right) \tag{S3}
\]

\[
\frac{\partial h}{\partial \mu} = -A'(q) \cdot q \tag{S4}
\]

These tangent matrices and residual vectors have been verified by numerical studies. The formulations for a free tumbling body, using quaternions, produce the same results as reported in Borri, Mello and Atluri (1990), were the finite rotation vector is employed.

It is straight forward to incorporate the translational degrees of freedom and impose physical constraints, as was done for the top and coin in the previous discussion. The results obtained are identical to those presented above. One difference worth noting is that the quaternion approach was much faster for the simple single body problems considered. While extensive timing studies were not made, the quaternion approach for simulating the spinning top required approximately 45 percent of the time required by the finite rotation vector approach. One factor in this "speed-up" is the simplicity of forming the tangent matrices. Another is the fact that an incremental approach is not required since the four parameter representation of rotation is not singular. This cuts down on the amount of internal book keeping required by the program.

5 Conclusions

A consistent variational approach to both holonomic and nonholonomic constraints is presented. The methodology is demonstrated for both types of constraint, through numerical simulations. It is also shown that this approach may be used to develop primal and mixed time finite elements, using a quaternion representation for finite rotation. For the single rigid-body problems investigated, the quaternion formulations are approximately twice as fast as those using the finite rotation vector.
A - Relevant Formulae on Quaternions

Let \( q \) be a quaternion, and \( q_s \) and \( q_v \) be respectively the scalar and vectorial parts. In the following \( q = (q_s, q_v) \) is understood. Provided \( q \neq 0 \), it may be normalized to a unit quaternion. Let \( q^\# \), be the unit quaternions associated with \( q \) i.e. \( q^\# = \frac{q}{|q|} \) where \( |q| = (q \cdot q)^{1/2} \) is the modulus of \( q \). Moreover, let \( \bar{q} \) be the quaternion conjugate to \( q \), i.e., \( \bar{q}_s = q_s \) and \( \bar{q}_v = -q_v \).

It is well known that quaternion multiplication can be cast in the following way. Let \( q, r, s \) be quaternions and \( s = qr \) (\( \bar{s} = \bar{r}\bar{q} \)). This multiplication may be expressed as:

\[
s = A(q) \cdot r = B(r) \cdot q
\]

where the operators \( A(\cdot) \) and \( B(\cdot) \), for a given quaternion \( q \) are defined as:

\[
A(q) = \begin{bmatrix} q_s & -q_v^t \\ q_v^t & q_s I + q_v \times I \end{bmatrix} = [q|A_\perp(q)]
\]

\[
B(q) = \begin{bmatrix} q_s & -q_v^t \\ q_v^t & q_s I - q_v \times I \end{bmatrix} = [q|B_\perp(q)]
\]

It is interesting to note that \( A_\perp(q) \) and \( B_\perp(q) \) are orthogonal to \( q \), i.e., \( q \cdot A_\perp(q) = q \cdot B_\perp(q) = 0 \ \forall q \). The \( \cdot \) denotes the four dimensional dot product. The operators \( A(\cdot) \) and \( B(\cdot) \) follow some rules that are reported here for convenience:

\[
A(q) \cdot B^t(r) = B^t(r) \cdot A(q)
\]

\[
A^t(q) \cdot r = C^t(r) \cdot q
\]

\[
B^t(q) \cdot r = C(r) \cdot q
\]

where \( C(\cdot) \) is defined as:

\[
C(q) = [q| - A^t_\perp(q)] = \begin{bmatrix} q^t \\ -B^t_\perp(q) \end{bmatrix}
\]

\[
C^t(q) = [q| - B^t(q)] = \begin{bmatrix} q^t \\ -A^t_\perp(q) \end{bmatrix}
\]

Moreover, the following relations hold:
\[ A'(q) \cdot A(q) = A(q) \cdot A'(q) = q^2 I_4 \]
\[ B'(q) \cdot B(q) = A(q) \cdot B'(q) = q^2 I_4 \]
\[ C'(q) \cdot C(q) = C(q) \cdot C'(q) = q^2 I_4 \]

while:
\[ A_\perp(q) \cdot A_\perp(q) = q^2 I \]
\[ B_\perp(q) \cdot B_\perp(q) = q^2 I \]

where \( q^2 = q \cdot q \) and \( I \) and \( I_4 \) denote respectively the three-dimensional and four-dimensional identity tensors.

It is well known that a unit quaternions may represent a rotation. In fact consider the following quaternion product:
\[ n = q^# m q^# = A(q^#) \cdot A(m) \cdot q^# = A(q^#) \cdot B(q^#) \cdot m = A(q^#) \cdot B'(q^#) \cdot m \]

It is easily shown that \( n \cdot n = m \cdot m \) which means that \( n \) may be obtained from \( m \) by a rotation. Clearly this rotation may be expressed as \( A(q^#) \cdot B'(q^#) \). Then in general, we can write:
\[ G(q^#) = A(q^#) \cdot B'(q^#) = B'(q^#) \cdot A(q^#) = C(q^#) \cdot C(q^#) = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \]

where \( R = B_\perp(q^#) \cdot A_\perp(q^#) \) is a proper three dimensional rotation tensor, i.e. \( RR' = R' R = I \) and \( \det R = 1 \).

In terms of the rotation vector \( r = \phi e \) the unit quaternion or Euler parameters are defined as:
\[ q_s^# = \cos \phi / 2 \quad q_v^# = \sin \phi / 2 e \]

In the previous section it is shown that the rotation tensor may be expressed as:
\[ R = I + \sin \phi e \times I + (1 - \cos \phi) e \times e \times I \]

Recalling the following trigonometric identities:
\[
\sin \phi = 2 \sin \frac{\phi}{2} \cos \frac{\phi}{2}, \quad 1 - \cos \phi = 2 \sin^2 \frac{\phi}{2}
\]  

(96)

the rotation tensor is expressed as:

\[
\mathbf{R} = \mathbf{I} + 2 \mathbf{q}_3^\# \mathbf{q}_v^\# \times \mathbf{I} + 2 \mathbf{q}_v^\# \times \mathbf{q}_v^\# \times \mathbf{I}
\]  

(97)

Successive rotations can be easily handled with quaternions. Suppose that: \( \mathbf{R}(r_3) = \mathbf{R}(r_2) \cdot \mathbf{R}(r_1) \) where \( r_3 \) is the rotation vector associated with the total rotation resulting from the sequence of rotations, \( r_1 \) followed by \( r_2 \). Then let \( \mathbf{q}_1^\#, \mathbf{q}_2^\#, \mathbf{q}_3^\# \) be the unit quaternions corresponding to \( r_1, r_2 \) and \( r_3 \) respectively. For each of them, a relation of the form Eq.(93) holds. This leads to the relation \( \mathbf{G}(q_3^\#) = \mathbf{G}(q_2^\#) \cdot \mathbf{G}(q_1^\#) \). The expansion of \( \mathbf{r}_3 \) in terms of \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) is very involved but the relation between \( \mathbf{q}_3^\# \) and \( \mathbf{q}_1^\# \) and \( \mathbf{q}_2^\# \) is simply a multiplication. In fact \( q_3^\# = q_2^\# q_1^\# \), which in matrix form is:

\[
\mathbf{q}_3^\# = \mathbf{B}(\mathbf{q}_1^\#) \cdot \mathbf{q}_2^\# = \mathbf{A}(\mathbf{q}_2^\#) \cdot \mathbf{q}_1^\#
\]  

(98)

Moreover, in this case the following relations hold:

\[
\mathbf{B}(\mathbf{q}_3^\#) = \mathbf{B}(\mathbf{q}_1^\#) \cdot \mathbf{B}(\mathbf{q}_2^\#)
\]  

(99)

\[
\mathbf{A}(\mathbf{q}_3^\#) = \mathbf{A}(\mathbf{q}_2^\#) \cdot \mathbf{A}(\mathbf{q}_1^\#)
\]

The composite rotation is then easily obtained.

\[
\mathbf{G}(q_3^\#) = \mathbf{A}(q_3^\#) \cdot \mathbf{B}(q_3^\#)
\]  

(100)

then Eq.(93) yields:

\[
\mathbf{G}(q_3^\#) = \mathbf{A}(q_2^\#) \cdot \mathbf{A}(q_1^\#) \cdot \mathbf{B}(q_2^\#) \cdot \mathbf{B}(q_1^\#)
\]

\[
= \mathbf{A}(q_2^\#) \cdot \mathbf{B}(q_2^\#) \cdot \mathbf{A}(q_1^\#) \cdot \mathbf{B}(q_1^\#)
\]

\[
= \mathbf{G}(q_2^\#) \cdot \mathbf{G}(q_1^\#)
\]

(101)

Quaternions and Angular Velocity

Let \( \mathbf{\omega} \) be the spinning velocity of \( \mathbf{R} \) defined through \( \mathbf{\omega} \times \mathbf{I} = \dot{\mathbf{R}}(r) \mathbf{R}'(r) \) then using the relations developed above:
\[ \omega_v = R(r) \cdot \dot{r} \quad \omega_v^* = R^t(r) \cdot \dot{r} \]

where: \( \omega_v^* = R^t \cdot \omega_v \) and:

\[
R(r) = I + \frac{1 - \cos \phi}{\phi^2} r \times I + \frac{1}{\phi^2} \left( 1 - \frac{\sin \phi}{\phi} \right) r \times r \times I \\
= I + \frac{1 - \cos \phi}{\phi} e \times I + \left( 1 - \frac{\sin \phi}{\phi} \right) e \times e \times I \\
= \frac{\sin \phi}{\phi} I + \frac{1 - \cos \phi}{\phi} e \times I + \left( 1 - \frac{\sin \phi}{\phi} \right) e \cdot e'
\]

and \( r = \phi e \). Taking into account the following identities:

\[
\dot{r} = \phi e + \phi \dot{e} \quad e \cdot e = 1 \quad \dot{e} \cdot e = 0
\]

\[
\dot{\phi} = \frac{\phi^*}{2} \sin \frac{\phi}{2} \quad \dot{e} = \frac{\phi^*}{\sin \frac{\phi}{2}} - \frac{1}{\sin \frac{\phi}{2}} \frac{1}{2} \cos \phi \quad \phi \quad \phi
\]

into Eq.(102) and Eq.(103) leads to:

\[
\omega_v = 2B^t_1(q^#) \cdot \dot{q}^# \quad \omega_v^* = 2A^t_1(q^#) \cdot \dot{q}^#
\]

Eq.(105) can be written in four dimensional form as follows:

\[
\omega = 2B^t(q) \cdot \dot{q} \quad \omega^* = 2A^t(q) \cdot \dot{q}
\]

where \( \omega = (\omega_s, \omega_v) \), \( \omega^*(\omega_s, \omega_v^*) \) and \( \omega_v = 2q \cdot \dot{q} \). It is recognized that \( \omega_s = \frac{d}{dt}[q \cdot q] \) which is zero when \( q \) is a unit quaternion. If \( q \) is not a unit quaternion then \( \omega_s = 0 \) may be interpreted as the differential form of the constraint of unit magnitude. The four dimensional vector \( \omega \) is the spinning velocity corresponding to \( q \), and \( \omega^* \) is its pullback that can be performed by the tensor \( G(q) \) through the relation:

\[
\omega^* = G^t(q) \omega
\]

Composition of angular velocity can be handled easily. In fact, supposing \( q_3^# = q_2^# q_1^# \) then:

\[
q_3^# = B(q_1^#)q_2^# = A(q_2^#)q_1^#
\]
Further, let \( \omega_1, \omega_2, \omega_3 \) be the corresponding velocities defined through the Eq.(106), that are related by the formula:

\[
\omega_3 = \omega_2 + G(q_2^\#) \cdot \omega_1
\]  

Which, in terms of derivatives may be written as:

\[
\omega_3 = 2B'(q_2^\#)(q_2^\# + A(q_2^\#) \cdot B'(q_1^\#) \cdot q_2^\#)
\]

\[
= 2G(q_2^\#)(A'(q_2^\#) \cdot q_2^\# + B'(q_1^\#) \cdot q_1^\#)
\]

and also:

\[
G'(q_3^\#)\omega_3 = 2G'(q_1^\#)(A'(q_2^\#) \cdot q_2^\# + B'(q_1^\#) \cdot q_2^\#)
\]

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Figure 1: Spinning Top - Motion Cases
Figure 2: Top Case 1 - Two Noded Elements
Figure 3: Top Case 1 - Three Noded Elements
Figure 4: Top Case 1 - Four Noded Elements
Figure 5: Top Case 2 - Three Noded Elements
Figure 6: Top Case 2 - Four Noded Elements
Figure 7: Top Case 3 - Three Noded Elements
Figure 8: Top Case 3 - Four Noded Elements
Figure 9: Rolling Coin Problem
Figure 10: Rolling Coin - Two Noded Elements
Figure 11: Rolling Coin - Three Noded Elements
Figure 12: Rolling Coin - Four Noded Elements
On a consistent theory, and variational formulation of finitely stretched and rotated 3-D space-curved beams

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Abstract. This paper deals with finite rotations, and finite strains of three-dimensional space-curved elastic beams, under the action of conservative as well as nonconservative type external distributed forces and moments. The plausible deformation hypothesis of "plane sections remaining plane" is invoked. Exact expressions for the curvature, twist, and transverse shear strains are given, as is a consistent set of boundary conditions. General mixed variational principles, corresponding to the stationarity of a functional with respect to the displacement vector, rotation tensor, stress-resultants, stress-couples, and their conjugate strain-measures, are stated for the case when conservative-type external moments act on the beam. The momentum-balance conditions arising out of these functionals, either coincide exactly with, or are equivalent to, those from the "static method". The incremental variational functionals, governing both the Total and Updated Lagrangian incremental finite element formulations, are given. An example of the case of the buckling of a beam subject to axial compression and non-conservative type axial twisting couple, is presented and discussed.

1 Introduction

There exist three approaches that are commonly used for describing the large displacements and large rotations of space-curved beams. The first approach is based on a direct use of the 3-dimensional finite-elasticity theory. The second one is based on certain plausible hypotheses such as the Euler-Bernoulli hypotheses, while the last one, on the work of Reissner (1973). From a mathematical viewpoint, the first approach may lead to a consistent beam theory; however, it is not so easy to derive the kinematic relations. Therefore, asymptotic expansions have been used in the first approach (Parker 1979, Pleus and Sayir 1983). Each of the variables employed in such theories does not always carry a physical meaning, and their interpretation becomes difficult especially in highly nonlinear problems. The approach based on plausible deformation hypotheses, on the other hand, may not yield a beam theory consistent with 3-D finite-elasticity theory; however, as indicated by the theory of the "elastica", this approach is often found to be practically useful. Also in buckling problems, the second and the third approach enables one to easily take into account the prebuckling deformations, since each variable has a clear physical meaning.

There has been a limited number of earlier works concerning theories for beams undergoing large deformations, large rotations, and large strains. Notable among these is due to Reissner (1973, 1981), who developed a finite strain beam theory based on the differential equations of force and moment equilibrium for elements of the deformed curve. The exact definitions for kinematic relationships have been derived (Reissner 1973, 1981), while the expressions for boundary conditions consistent with the equilibrium equations have been obtained implicitly.

In this paper, using plausible and consistent kinematic hypotheses, a large deformation (and large rotation) beam theory is developed. The effects of stretching, bending, torsion and transverse shear, are taken into account, while the cross-sectional warping deformations are neglected. In the present formulation, we do not restrict the magnitude of strains, but assume that the material is linearly elastic. Using the principle of virtual work, we present a set of boundary conditions which are consistent with the presently developed finite strain beam theory.

As indicated by Argyris et al. (1979), the use of an arbitrary set of mathematical variables to describe rotations may lead to unsymmetric geometric stiffnesses of finite beam elements, even when
the beam is subjected to a conservative system of external moments. One of the objectives of this paper is to present well-defined variational functionals, and associated ‘principles’ corresponding to the vanishing of the first variation of such functionals, when a conservative system of external forces and moments act on the finitely deformed beam. Using these functionals, one may construct a symmetric geometric stiffness of a beam element in its current equilibrium state. It is noted however, that when a nonconservative system of external forces and moments act on the beam, the geometric stiffness of a beam element, in its current equilibrium state will be unsymmetric. A systematic approach to solve such problems of nonconservative loading, has been discussed by Kondoh and Atluri (1987), based on a direct statement of the weak form of the associated balance laws.

The variational functionals, in the presence of a conservative system of external forces and moments, which are presented in this paper, form the bases of general mixed-hybrid finite element methods for finitely strained and rotated space-curved beams. The modus operandi for such finite element methods, involving finite rotation kinematics, has been discussed earlier by Atluri and Murakawa (1977), Murakawa and Atluri (1978).

The remainder of the paper is as follows. Section 2 deals with preliminaries; Sect. 3 with the geometry of the undeformed beam; and Sect. 4 with the geometry of the deformed beam. In Sect. 5 we deal with the principle of virtual work for the finitely strained beam; and discuss how this virtual work principle may be cast in the form of a condition of stationarity of well-defined functional, even when only a system of conservative external forces and moments act on the beam undergoing finite rotations. Depending on the form of virtual variations of the rotation parameters considered, (if R is the rotation tensor, one form of rotational variation corresponds to the vector \( \delta \phi \) such that \( \delta \phi \times I = \delta R \cdot R' \) and the other form corresponds directly to \( \delta \omega \) where \( \omega \) are the three parameters that describe the Lagrangian components of \( R \)), the linear and angular momentum balance conditions take on different but equivalent forms; with only one of these forms coinciding with those derived a priori from the so-called “static-method”. In Sect. 6 we deal with the constitutive equations; and Sect. 7 deals with the most general mixed variational principles under conservative loading, and their “incremental” counterparts. In the general variational principles, the variables are: the displacement vector, the rotation tensor, the stress-resultant vector acting on the beam cross-section, the stress-couple vector acting on the beam cross-section, and the appropriate strain and curvature measures that are conjugate to these mechanical variables.

To demonstrate the novel features of the presently developed theory, we consider, in Sect. 8, a problem of buckling of a beam subjected to an axial compression and a nonconservative twisting couple with the emphasis on the boundary conditions. The effects of prebuckling and shear deformations are manifested in the presently derived buckling load.

2 Preliminaries

The fundamental hypotheses for deriving the present finite-strain beam theory are itemized as follows:

1. The plane cross-sections of the beam remain plane and do not undergo any shape-change during the deformation.
2. The cross-sections are constant along the beam axis which remains a smooth space curve throughout the deformation.

Throughout this paper, the summation convention is adopted; and the Latin indices will have the range 1, 2, and 3, and the Greek indices the range 1 and 2.

3 The geometry of the undeformed beam

Consider a naturally curved and twisted beam in a fixed Cartesian coordinate system \( X^m \), with base vectors \( I_m \), as shown in Fig. 1. An orthogonal curvilinear coordinate system \( Y^m \), with base vectors \( E_m \), is introduced to describe the motion of the beam. The coordinates \( Y^m \) are taken in the cross sections, while the coordinate \( Y^3 \) is taken along the beam axis. The way to select the origin of the coordinates \( Y^m \) will be discussed in Sect. 6. The orientation of the present coordinate systems follows the familiar “right hand rule”. 
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The position vector of a point at the beam-axis is represented as
\[ X = X^m(Y^3) E_m. \]  
(1)

The tangent base vector \( \bar{E}_3 \) is defined by:
\[ \bar{E}_3 = \frac{dX}{dY^3}. \]  
(2)

In general, the base vector \( \bar{E}_3 \) is not a unit vector, while the base vectors \( \bar{E}_m \) are chosen to be unit vectors without loss of generality. For latter convenience, we introduce the unit vectors \( E_m \) defined by
\[ E_4 = \bar{E}_4, \quad E_3 = \frac{\bar{E}_3}{|\bar{E}_3|}. \]  
(3a, b)

The well-known Frenet-Serret formulae lead to the relations:
\[ \bar{E}_{m,3} = K \times \bar{E}_m; \quad K = K_m E_m, \]  
(4a, b)

where \( (\ )_3 = \frac{d(\ )}{dL} \) where \( dL = |\bar{E}_3| dY^3; K_a \) are the components of initial curvature, and \( K_3 \) is the initial twist.

The position vector of an arbitrary point in a cross-section of the beam is given by:
\[ R = X + Y^2 E_4. \]  
(5)

Then, the base vectors at an arbitrary point in a cross-section of the beam are given by:
\[ A_4 = E_4, \quad A_3 = -Y^2 K_3 E_1 + Y^1 K_3 E_2 + g_0 E_3, \]  
(6a, b)

where
\[ g_0 = 1 - Y^1 K_2 + Y^2 K_1. \]  
(7)

The contravariant base vectors \( A^m \) are defined through the relation:
\[ A^m \cdot A_n = \delta^m_n \]  
where \( \delta^m_n \) is the Kronecker delta.
4 The geometry of the deformed beam

Let \( \mathbf{\delta}_3 \) be the unit vector tangential to the deformed beam axis. After deformation, the unit base vectors \( \mathbf{E}_a \) are transformed to the unit base vectors \( \mathbf{e}_a \), as shown in Fig. 1. Without loss of generality, the base vectors \( \mathbf{e}_a \) and \( \mathbf{e}_3 \) are assumed to be the maps of the base vectors \( \mathbf{E}_a \) and \( \mathbf{E}_3 \) after a purely rigid rotation, denoted by the tensor \( \mathbf{R} \), alone. Accordingly, we have

\[
\mathbf{e}_a \cdot \mathbf{e}_3 = \delta_{a3}; \quad \mathbf{e}_a \cdot \mathbf{\delta}_3 = 0.
\] (8a, b)

Equation (8a) is consistent with assumption (1). The nonorthogonality condition given by Eq. (8b) is due to the transverse shear deformation which renders \( \mathbf{e}_3 \neq \mathbf{\delta}_3 \).

The relationship between the unit orthogonal vectors \( \mathbf{e}_m \) and \( \mathbf{E}_m \) is written, in terms of a finite rotation tensor \( \mathbf{R} \), (Atluri 1984; Pietraszkiewicz and Badur 1983), as:

\[
\mathbf{e}_m = \mathbf{R} \cdot \mathbf{E}_m; \quad \mathbf{R} = \mathbf{R}_{ij} \mathbf{E}_i \mathbf{E}_j.
\] (9a, b)

Because of the condition that \( \mathbf{R} \cdot \mathbf{R}^T = \mathbf{I} \), where \( \mathbf{I} \) is the identity tensor and ( ) \(^T\) a transpose, the number of independent components of \( \mathbf{R} \) are three.

Finite rotation vectors are also used by Atluri (1984), Kane, Likins and Levinson (1983), Pietraszkiewicz and Badur (1983), Reissner (1973) and Simmonds and Danielson (1972) to represent the relationship between \( \mathbf{e}_m \) and \( \mathbf{E}_m \). Let \( \mathbf{e} \) be a unit vector satisfying \( \mathbf{R} \cdot \mathbf{e} = \mathbf{e} \) and \( \omega \) a magnitude of rotation about the axis of rotation defined by \( \mathbf{e} \). The alternate representations commonly used for finite rotation vectors are:

\[
\Omega = \sin \omega \mathbf{e}; \quad \theta = 2\tan \frac{\omega}{2} \mathbf{e}; \quad \text{and} \quad \omega = \omega \mathbf{e}.
\] (10a–c)

In terms of the finite rotation vectors \( \Omega \), \( \theta \), and \( \omega \), the relationship between \( \mathbf{e}_m \) and \( \mathbf{E}_m \) may be written as (Pietraszkiewicz and Badur 1983).

\[
\mathbf{e}_m = \mathbf{E}_m + \Omega \times \mathbf{E}_m + \frac{1}{2 \cos^2 \omega} \Omega \times (\Omega \times \mathbf{E}_m);
\]
\[
= \mathbf{E}_m + \frac{1}{1 + \frac{1}{4} \theta^2} \theta \times (\mathbf{E}_m + \frac{1}{2} \theta \times \mathbf{E}_m);
\] (11a–c)
\[
= \mathbf{E}_m + \frac{\sin \omega}{\omega} \omega \times \mathbf{E}_m + \frac{1 - \cos \omega}{(\omega)^2} \omega \times (\omega \times \mathbf{E}_m).
\]

The finite rotation tensor \( \mathbf{R} \) is often expressed very conveniently, in terms of \( \omega \), as:

\[
\mathbf{R} = \exp(\omega \times \mathbf{I}).
\] (12)

It is well known that no 3-parameter representation of \( \mathbf{R} \) can be both global and nonsingular (Stuelpnagel 1964); for this reason the four quaternion or Euler parameters have been introduced to describe the finite rotations (Kane, Likins and Levinson 1983; Stuelpnagel 1964). In spite of this drawback, the concept of a finite rotation vector is useful for the study of finite rotations in structural members. For example, the three Rodrigues parameters, defined by \( \mathbf{E}_m \cdot \theta/2 \), are not global, i.e., \( \omega = \pi \) rad yields the Rodrigues vector of infinite magnitude. However, the finite rotation tensor \( \mathbf{R} \) is determined uniquely even if \( \omega = \pi \) rad (Kane, Likins and Levinson 1983). In this way, the finite rotation vectors have been frequently employed to describe the finite rotations in structural mechanics.

In the present case of a space-curved beam, we define the angles of shear deformations, denoted by \( \beta_a \), in the following fashion:

\[
\sin \beta_a = \mathbf{e}_a \cdot \mathbf{\delta}_3.
\] (13)
Since $|\hat{e}_3| = 1$, we have

\[ \hat{e}_3 = \sin \beta_3 \varepsilon_3 + \beta_3 \varepsilon_3, \]  \tag{14} 

where \( \beta_3 = \hat{e}_3 \cdot e_3. \)  \tag{15} 

From the definition of covariant base vectors, the deformed unit tangent vector \( \hat{e}_3 \) takes the natural form, in terms of the displacement components, \( u^m \), as:

\[ \hat{e}_3 = (X + u)_3/(X + u),_3 = u^m|_3 E_m; \quad u^m|_3 = (\delta^m_3 + u^m|_3)/g; \]

\[ g = \sqrt{(u^1|_3)^2 + (u^2|_3)^2 + (1 + u^3|_3)^2} \]  \tag{16a–c} 

where \( u = u^m E_m \) is the displacement vector at the beam axis and \( (\ )|_3 \) the covariant differentiation with respect to the metric tensor \( E_{ij} = E_i \cdot E_j. \) With the use of Eqs. (9) and (16), the angles of shear deformations \( \beta_a \) and \( \beta_3 \) are represented, in terms of the displacement components and the finite rotation tensor, as

\[ \sin \beta_a = (R \cdot E_a) \cdot (u^m|_3 E_m); \quad \beta_3 = (R \cdot E_3) \cdot (u^m|_3 E_m). \]  \tag{17a, b} 

According to the assumption (1), the deformed base vectors at an arbitrary material point are given by

\[ a_m = (X + u + Y^s e_a)_m, \]  \tag{18} 

where \( (\ )_a \) denotes a differentiation with respect to \( Y^s. \) It follows from Eqs. (14), (16) and (18) that

\[ a_3 = e_3; \quad a_1 = (g \sin \beta_1 - Y^2 k_3) e_1 + (g \sin \beta_2 + Y^1 k_3) e_2 + (g \beta_3 - Y^1 k_2 + Y^2 k_1) e_3, \]  \tag{19a, b} 

where

\[ e_{m, 3} = k \times e_m; \quad k = k_m e_m. \]  \tag{20a, b} 

The components \( k_m \) of \( k \) are expressed in terms of the finite rotation tensor as

\[ k_i = \frac{1}{2} e_{ijk} [(R \cdot E_{j,3})_i] [(R \cdot E_k)], \]  \tag{21} 

where \( e_{ijk} \) is the permutation symbol. It should be noted that the components \( k_m \) are not exactly the curvatures and twist after the deformation since the deformed beam axis has undergone extension.

5 Equilibrium equations and boundary conditions

To derive the equilibrium equations, the so-called "static method" or alternatively, the energy method are often employed. Once the appropriate stress resultants and moments are defined, the static method yield the equilibrium equations from a consideration of the free-body diagram of a differential element of the beam. On the basis of the static method, however, it is difficult to derive the boundary conditions consistent with the resulting equilibrium equations. The energy method, on the other hand, may lead to the equilibrium equations and the associated boundary conditions without difficulty, but with tedious calculations. Since no explicit boundary conditions consistent with a finite beam theory are currently available in literature, the energy method is employed herein to derive them.

The Green strain tensor is defined as

\[ e = e_{ij} A^i A^j; \quad e_{ij} = \frac{1}{2} (a_i \cdot a_j - A_i \cdot A_j). \]  \tag{22a, b} 

Substituting Eqs. (6) and (19) into Eqs. (22) leads to

\[ \varepsilon_{ij} = \frac{1}{2} (g \sin \beta_1 - Y^2 k_3); \quad \varepsilon_{23} = \frac{1}{2} (g \sin \beta_2 + Y^1 k_3); \quad \varepsilon_{33} = \frac{1}{2} [ (g \sin \beta_1 - Y^2 k_3)^2 \\
+ (g \sin \beta_2 + Y^1 k_3)^2 + (g \beta_3 - Y^1 k_2 + Y^2 k_1)^2 - g (K_3)^2 - (g_0)^2]; \quad e_{ab} = 0. \]  \tag{23a–d}
where
\[ E_m = k_m - K_m; \quad q = (Y^1)^2 + (Y^2)^2. \] (24a, b)

The stress resultants and moments are defined, following Atluri (1984), as:
\[ T = \int g_0 A^3 \cdot (S_1 \cdot F^T) dA; \quad M = \int Y^2 e_z \times [g_0 A^2 \cdot (S_1 \cdot F^T)] dA, \] (25a, b)

where \( S_1 = (S''\alpha_{1} A_{\alpha}) \) is the second Piola-Kirchhoff stress tensor, \( F \) the deformation gradient tensor and \( dA = dY^1 dY^2 \). By using the component representation, we obtain the stress resultants and moments in the form (Appendix I)
\[ T = \int g_0 \pi \cdot e_z, \quad M = \int \pi \cdot e_z \times [g_0 \pi \cdot e_z \times \pi \cdot e_z] dA, \] (26a–c)

\[ M^1 = \int \pi \cdot e_z \times [g_0 \pi \cdot e_z \times \pi \cdot e_z] dA, \quad M^2 = - \int \pi \cdot e_z \times [g_0 \pi \cdot e_z \times \pi \cdot e_z] dA, \quad M^3 = \int \pi \cdot e_z \times [g_0 \pi \cdot e_z \times \pi \cdot e_z] dA, \] (26d–f)

where \( \pi \cdot e_z \) are the stress components defined by
\[ \pi = \pi \cdot e_n. \] (27)

The \( (\cdot) \) in superscript is used to emphasize that these are not components in convected coordinates \( y^m \).

The internal virtual work is written as (Washizu 1982)
\[ IVW = \int S^i_j \cdot \delta e_{ij} dV, \] (28)

where \( dV = g_0 dY^1 dY^2 dL \). In this paper, following Atluri (1984), we introduce a tensor \( (\delta R \cdot R^T) \) as a rotational variation. Since \( R \cdot R^T = I, \) \( \delta R \cdot R^T \) is a skew-symmetric tensor. There exists, therefore, a vector \( \delta \phi \) satisfying \( \delta R \cdot R^T = \delta \phi \times I \). Through some straightforward algebra it may be shown that the variation of the finite rotation vector \( \omega \), denoted as \( \delta \omega \), is related to the vector \( \delta \phi \) as
\[ \delta \phi = \frac{\sin \omega}{\omega} \omega - \frac{1 - \cos \omega}{(\omega)^2} \delta \omega \times \omega + \left\{ \frac{1}{\omega} - \frac{\sin \omega}{(\omega)^2} \right\} \delta \omega \omega. \] (29)

Substituting Eqs. (23) into Eq. (28) and using Eqs. (25), one is lead, after some straightforward algebra, to:
\[ IVW = - \int [T_3 \cdot \delta u + (M_3 + (X + u)_3 \times T) \cdot \delta \phi] dL + [\int_{S_u+S_e} \delta u \cdot \delta \phi] \bigg|_{S_u}^{S_e}, \] (30)

where \( l \) is the length of the beam axis before the deformation, \( S_u \) and \( S_e \) are parts of boundary on which geometrical and mechanical boundary conditions are prescribed respectively.

Let \( P_b \) be the vector of body force defined per unit volume of the undeformed beam, \( P_c \) the vector of distributed surface traction defined per unit area of the undeformed cylindrical surface of the beam, denoted as \( S_t \); and \( P_e \) the vector of distributed surface tractions at the end cross sections denoted as \( S_e \). Then the external virtual work is written as (Washizu 1982):
\[ EVW = \int P_b \cdot \delta V dV + \int_{S_e} P_c \cdot \delta V dS_c + \int_{S_e} \left[ \int_{S_e} P_e \delta V dS_e \right] \bigg|_{S_u}^{S_e}, \] (31)

where \( V \) is the displacement vector at an arbitrary material point defined by \( V = u + Y^e (e_x - E) \) and \( dS_c = |R_x \times R_3| ds dL \) in which \( s \) is the coordinate taken along the bounding curve of the cross sections.

At first, we assume that the directions of external forces do not change during the deformation. Therefore we may write the external forces in the form
\[ P_b = P'_b E_j, \quad P_c = P'_c E_j, \quad P_e = P'_e E_j, \] (32a–c)

where \( P'_b, P'_c \) and \( P'_e \) are constant. Introducing Eqs. (32) into Eq. (31) yields
\[ EVW = \int q \cdot \delta u + m \cdot \delta \phi dL + \int_{S_e} [\bar{q} \cdot \delta u + \bar{m} \cdot \delta \phi] \bigg|_{S_u}^{S_e}, \] (33)

where
\[ q = q^l E_j, \quad \bar{q} = \bar{q}^l E_j, \quad m = m_{aj} e_z \times E_j, \quad \bar{m} = \bar{m}_{aj} e_z \times E_j, \] (34a–d)
\[ q' = \int P_0 g_0 dA + \int P_1 |R_x \times R_3| ds, \quad \tilde{q}' = \int P_4 dS_e, \quad (34\,e,\,f) \]
\[ m_{aj} = \int Y^\kappa P_0 g_0 dA + \int Y^\kappa P_1 |R_x \times R_3| ds, \quad \tilde{m}_{aj} = \int Y^\kappa P_4 dS_e, \quad (34\,g,\,h) \]

From the principle of virtual work, i.e., \( IVW = EVW \) (Washizu 1982), we obtain the linear momentum balance \( (LMB) \) and angular momentum balance \( (AMB) \) conditions, expressed as:
\[ T_3 + \varrho = 0, \quad (\text{for arbitrary } \delta u) \]
\[ M_3 + (X + u)_3 \times T + m = 0 \quad (\text{for arbitrary } \delta \phi). \quad (35\,a,\,b) \]

The associated boundary conditions are written as
\[ T = \tilde{q}; \quad M = \tilde{m} \ on \ S_o, \quad u = \bar{u}; \quad \phi = \bar{\phi} \ on \ S_u, \quad (36\,a-d) \]

where \( \bar{u} \) and \( \bar{\phi} \) denote the prescribed values on \( S_u \).

It is well known that the displacement field and the variations of variables determine whether the energy method yields the same equilibrium equations as those derived by the static method. In this paper, because of using \( \delta R \cdot R' \) as the rotational variation, we can derive the same \( LMB \) and \( AMB \) conditions as those derived by the static method (Ericksen and Truesdell 1958; Reissner 1973). The consistent boundary conditions are also obtained in the process. It follows from the above derivation, Eqs. (33-36), that the “external moment” vectors \( m \) (Eqs. 35 b, 34 c) and \( \tilde{m} \) (Eqs. 36 b, 34 d) are dependent on deformations even though the components of external moments \( m_{aj} \) and \( \tilde{m}_{aj} \) are independent of deformations. In the existing literature (Ericksen and Truesdell 1958; Reissner 1973, 1981), where static method has been employed, the existence of an “external moment” vector is assumed, a priori, and then the \( AMB \) condition takes the same form as that given by Eq. (35 b).

Next, for later convenience, we consider the “non-conservative” type follower forces defined by
\[ \tilde{P}_b = \tilde{P}'_b e_j; \quad \tilde{P}_c = \tilde{P}'_c e_j; \quad \tilde{P}_s = \tilde{P}'_s e_j, \quad (37\,a-c) \]
where \( \tilde{P}_b \) is the vector of body force defined per unit volume of the undeformed beam, \( \tilde{P}_c \) the vector of distributed surface force defined per unit area of the undeformed cylindrical surface of the beam, \( S_c \); and \( \tilde{P}_s \) the vector of distributed surface tractions at the end cross section, \( S_e \). In this case, the \( EVW \) is written as
\[ EVW = \int [q \cdot \delta u + \tilde{m} \cdot \delta \phi] dL + \int \left[ \tilde{q}' \cdot \delta u + \tilde{m}' \cdot \delta \phi \right] dS, \quad (38) \]
where
\[ q = \hat{q}^t e_j; \quad \tilde{q} = \hat{q}'^t e_j; \quad \tilde{m} = \tilde{m}_{aj} e_x \times e_j; \quad \tilde{m}' = \tilde{m}_{aj}' e_x \times e_j; \]
\[ \tilde{m}' = \hat{m}' e_x \times e_j; \quad \tilde{m}' = \hat{m}' e_x \times e_j; \quad \tilde{m}' = \hat{m}' e_x \times e_j; \quad \tilde{m}' = \hat{m}' e_x \times e_j; \quad \tilde{m}' = \hat{m}' e_x \times e_j; \quad \tilde{m}' = \hat{m}' e_x \times e_j; \quad (39\,e,\,f) \]
\[ \tilde{m}_{aj} = \int Y^\kappa \tilde{P}'_0 g_0 dA + \int Y^\kappa \tilde{P}'_1 |R_x \times R_3| ds; \quad \tilde{m}_{aj}' = \int Y^\kappa \tilde{P}'_4 dS_e, \quad (39\,g-h) \]

Since \( IVW = EVW \), the equilibrium equations are obtained as
\[ T_3 + \varrho = 0; \quad M_3 + (X + u)_3 \times T + \tilde{m} = 0. \quad (40,\,b) \]

The associated boundary conditions are written as
\[ T = \tilde{q}; \quad M = \tilde{m} \ on \ S_o, \quad u = \bar{u}; \quad \phi = \bar{\phi} \ on \ S_u, \quad (41\,a-d) \]

As is well known, in the case of follower forces, not only the external moment vector \( \tilde{m} \) but also the external force vector \( \tilde{q} \) are dependent on deformations.

In summary, the \( IVW \) is given by Eq. (30); and \( EVW \) is given in the case of fixed-directional ('dead') loading by Eq. (33) while in the case of follower loading it is given by Eq. (38). In both the cases of loading, the principle of virtual work is
\[ IVW = EVW = 0. \]

The question then arises if the above equation can be written, equivalently, as the stationary condition or the vanishing of the first variation of a well-defined functional. It will be shown in
Sect. 6 and 7 of this paper that when $T$ and $M$ are expressed in terms of appropriate kinematic variables, $IVW$ of Eq. (30) can be expressed as the first variation of an internal energy functional, written in terms of $U$ and $R$. On the other hand, even in the case of conservative loading, the $EVW$, especially of the moments, i.e., the term $m \cdot \delta \phi$ (with the $m$ as defined in Eq. (34c) and $\delta \phi$ as defined through $\delta \phi \times I = \delta R \cdot R^T$) does not, on first sight, appear to correspond to the first variation of an external energy functional. This has certain implications in constructing weak solutions, say based on the finite element method. It is well-known that if the governing equations of the problem can be written as the Euler-Lagrange equations corresponding to the vanishing of the first variation of a well-defined functional, and if similar basis functions are used for the trial and test functions, the tangent-stiffness matrix at an equilibrium state is always symmetric. If arbitrary trial functions are used for $R$, and arbitrary and different test functions are used for $\delta \phi$, it is seen from Eqs. (30), (33), and (34c), that the geometric stiffness matrix derived from the principle of virtual work will be unsymmetric even for conservative loading. Further, the contributions to the unsymmetric stiffness arise not only from the $IVW$ of Eq. (30), but also from the $m \cdot \delta \phi$ term in $EVW$ of Eq. (33). In the case of non-conservative loading, the $EVW$ of Eq. (38) will in any case not correspond to the first variation of an energy functional, and will lead to a contribution to the unsymmetric stiffness matrix at an equilibrium state.

In order to express $(m \cdot \delta \phi)$ of Eq. (33) as the first variation of an energy functional, we adopt a strategy wherein $(m \cdot \delta \phi)$ can be expressed in terms of components of $m$ and $\delta \phi$ in the undeformed basis, $E_j$. Thus, from (34c) and the definition of $\delta \phi$ [i.e., $\delta \phi \times I = \delta R \cdot R^T$], we have:

$$m \cdot \delta \phi = m_{aj} (e_a \times E_j) \cdot \delta \phi = m_{aj} E_j \cdot (\delta \phi \times e_a) = m_{aj} E_j \cdot (\delta R \cdot R^T \cdot e_a) = m_{aj} [E_j \cdot \delta R \cdot E_j].$$

If further, one writes $R$ in terms of Lagrangean components, i.e., $R = R_{ik} E_i E_k$, then

$$\delta R = \delta R_{jk} E_j E_k,$$ and hence,

$$m \cdot \delta \phi = m_{aj} \delta R_{ja} = \delta [m_{aj} R_{ja}],$$

where $m_{aj}$ are given constants as defined in (34g). We assume that the Lagrangean components $R_{ik}$ of $R$ are expressed in terms of three arbitrary parameters $\alpha^i$, such that $(\delta R = R_{ikj} E_j E_k \delta \alpha^i)$, where $(.)^i_j$ denotes the differentiation with respect to $\alpha^i$. When a dead-load system of forces as in Eq. (32) is considered, the principle of virtual work $IVW = EVW$ [with $IVW$ as in (30) and $EVW$ as in (33)] leads to the linear momentum balance conditions as in (35a); however, the angular momentum balance conditions corresponding to arbitrary variations $\delta \alpha^i$ become:

$$N_j + r_j = 0, \quad [AMB \text{ for } (\delta \alpha^i)],$$

where

$$N_j = Q^1 (e_1 \cdot E_j) R_{11,j} + Q^2 (e_1 \cdot E_j) R_{13,j} + Q^3 (e_2 \cdot E_j) R_{11,j},$$

$$r_j = m_{si} R_{sij}, \quad Q^1 e_1 = M_3 + (X + u)_3 \times T,$$

in which $m_{si}$ are defined in Eq. (34g). The associated boundary conditions are written as

$$L_j = \bar{r}_j \text{ on } S_u; \quad \bar{\alpha} = \bar{\alpha}^i \text{ on } S_u,$$

where $\bar{\alpha}$ denote the prescribed values on $S_u$ and

$$L_j = M^1 (e_1 \cdot E_j) R_{11,j} + M^2 (e_1 \cdot E_j) R_{13,j} + M^3 (e_2 \cdot E_j) R_{11,j}.$$ (46a, b)

The external moments denoted by $\bar{r}_j$ are obtained from Eq. (45b) by replacing $m_{si}$ by $\bar{m}_{si}$, where $\bar{m}_{si}$ are defined in Eq. (34h).

To show the equivalence of $AMB$ condition given by Eq. (35b), associated with $\delta \phi$, and by Eq. (44), associated with $\delta \alpha^i$, we consider the tensor equations of $AMB$ conditions.

Since $\delta R \cdot R^T = \delta \phi \times I$, we have

$$\{M_3 + (X + u)_3 \times T + m\} \cdot \delta \phi = C : (\delta R \cdot R^T),$$

where the use of Eq. (45c) is made and

$$C = Q^1 e_3 e_2 + Q^2 e_1 e_3 + Q^1 e_2 e_1 + m_{am} E_m e_a.$$
On the other hand, with simple manipulation, we have

\[(N_j + r_j)\delta \alpha^j = \{ C : (R_j \cdot R^T) \} \delta \alpha^j. \tag{50}\]

As shown in Eq. (48), the \(AMB\) condition associated with \(\delta \phi\) is that \(C - C^T = 0\). While, as shown in Eq. (50), the \(AMB\) conditions associated with \(\delta \alpha^j\) is that \(C : (R_j \cdot R^T) = 0\). It is shown consequently that, since \(R_j \cdot R^T\) is a skewsymmetric tensor, the \(AMB\) conditions associated with \(\delta \phi\) is equivalent to that associated with \(\delta \alpha^j\).

6 Constitutive equations

In a finite displacement theory, a variety of stress tensor has been used. As a result, a number of constitutive equations have been proposed. We postulate herein that the present materials are homogeneous, isotropic and linearly elastic.

Equations (26) indicate that the use of the stress tensor \(t_{mn}\) yields the compact definition for the stress resultants and moments. Therefore, we utilize the stress tensor \(t_{mn}\) and the conjugate strain tensors \(\gamma_{mn}\) to construct the constitutive equations. The conjugate strain tensor \(\gamma_{mn}\) are given by (Appendix 2)

\[\gamma_{mn} = a_m \cdot e_n - A_m \cdot E_n. \tag{51}\]

For one-dimensional beams, we assume the following relationships:

\[t^{34} = G\gamma_{33}; \quad t^{13} = E\gamma_{33}, \tag{52a, b}\]

where \(G\) is the shearing modulus and \(E\) the Young modulus. Substituting Eq. (51) and (52) into Eqs. (26) and modifying the shear rigidity yields

\[
\begin{bmatrix}
T^1 \\
T^2 \\
T^3 \\
M^1 \\
M^2 \\
M^3
\end{bmatrix}
= 
\begin{bmatrix}
GA_0 & 0 & 0 & 0 & 0 & -GI_1 \\
. & GA_0 & 0 & 0 & 0 & GI_2 \\
EA & EI_1 & -EI_2 & 0 & 0 & h_3 \\
. & EI_{11} & -EI_{12} & 0 & 0 & k_1 \\
Sym. & EI_{22} & 0 & 0 & 0 & k_2 \\
. & GJ & 0 & 0 & 0 & k_3
\end{bmatrix}
\]

where

\[h_1 = g \sin \beta_1; \quad h_2 = g \sin \beta_2; \quad h_3 = g \beta_3 - 1, \quad A = \int g_0 \, dA; \quad A_0 = k A; \quad I_2 = \int Y^2 g_0 \, dA, \quad J = \int g g_0 \, dA. \tag{54 a-f}\]

\[I_{12} = \int Y^1 Y^2 g_0 \, dA; \quad I_{11} = \int (Y^2)^2 g_0 \, dA; \quad I_{22} = \int (Y^1)^2 g_0 \, dA,
\]

The factor \(k\) is a shear-correction factor (Cowper 1966). It is worth noting that if we introduce the constitutive equations expressed by the second Piola-Kirchhoff stress tensor and the Green strain tensor such that \(S_{11} = Ge_{34}\) and \(S_{11} = Ee_{33}\), the constitutive Eqs. (53) are not obtained.

Next we consider the appropriate choices for the origin of coordinates \(Y^m\). It is possible, even for a naturally curved and twisted beam, to choose the origin so that \(I_2 = I_{12} = 0\). Another way is to choose the origin so that the coordinate \(Y^3\) coincides with the fiber axis of beams. In the latter case, the coefficients \(I_2\) and \(I_{12}\) do not always become zero.

For later convenience, we introduce the strain energy function \(W_s\) expressed as

\[W_s = \frac{1}{2} GA_0 (h_1)^2 + \frac{1}{2} GA_0 (h_2)^2 + \frac{1}{2} EA (h_3)^2 + \frac{1}{2} EI_{11} (k_1)^2 + \frac{1}{2} EI_{12} (k_2)^2 + \frac{1}{2} GJ (k_3)^2 + EI_1 h_1 k_1 - EI_2 h_3 k_2 - EI_{12} h_1 k_3 - GL_1 h_1 k_3 + GL_2 h_2 k_3. \tag{55}\]

It should be emphasized that the present strain energy function is derived from the stress-strain relationships given Eqs. (52).
7 General mixed variational principle

As a basis of a numerical method, a variational principle often plays an important role. In finite elasticity, so far, the principle of stationary potential energy has been more widely used. The pure displacement formulation, however, has now been abandoned by most researchers of finite element formulations (Hibbitt 1986). Therefore, recently, the general mixed variational formulations are receiving a wide attention. In this section, we will derive the functional for general mixed variational principle for finitely deformed beams. Based on the resulting functional, we will present the incremental functionals in the context of a total Lagrangian (TL) formulation and an updated Lagrangian (UL) formulation.

As first shown by Fraeijs de Veubeke (1972), and later generalized by Atluri and Murakawa (1977), a general mixed principle, for a 3-dimensional elastic material, and involving the first Piola-Kirchhoff stress tensor $t_i$, the right stretch tensor $U$ the finite rotation tensor $R$ and the displacement vector $v$ as variables, can be stated as the stationary condition of the functional $F_1$:

$$F_1(t_1, U, R, v) = \int_{V_0} \left[ W_0(U) + t_i^T \left\{ (I + F_0) v - R \cdot U \right\} 
- q_0 \delta \cdot v \right] dV_0 - \int_{s_a} \delta \cdot v dS - \int_{s_u} t \cdot (v - \delta) dS,$$

where $W_0$ is a strain energy function, $q_0$ the mass density in the undeformed state, $\delta$ the body force vector per unit mass, $t$ the traction on the boundary per unit undeformed area and $F_0$ the gradient operator in the undeformed state.

The functional $F_1$ for a finitely deformed shell has been derived by Atluri (1984). Based on the resulting modified functional (Atluri, 1984), some numerical results have been obtained by Punch and Atluri (1986). However, to the best of the author's knowledge, no studies exist on the functional $F_1$ for a finitely deformed beam.

We see that the following relationships hold for the present problem of a beam:

$$V_0(\ ) = A^m(\ )_m; \quad I = A^m A^m; \quad t_1 = t_i^T A_i a_i,$$

$$U = A_e A_e + \{ (h + E_1 - Y^2 k_3 E_1 + Y^1 k_3 E_2 + ( - Y^1 k_2 + Y^2 k_1) E_3 ) A^3; \quad h = h_m E_m. \quad (57a-c)$$

Since $t_1 = S_l A_l A_i, F = a_i A_i$ (Atluri 1984), the components $S_l$ are numerically equal to the components $t_i^T$. Since $R \cdot R^T = I$, it follows that $R_3 \cdot R_T$ is a skew-symmetric tensor. There exists, therefore, the vector $l_3$ such that $R_3 \cdot R_T = l_3 \times I$. In terms of $\omega$ of Eq. (10c), the vector $l_3$ is represented as (Pietraszkiewicz and Badur 1983)

$$l_3 = \frac{\sin \omega}{\omega} \omega_3 - \frac{1 - \cos \omega}{\omega^2} \omega_3 \times \omega + \left\{ \frac{1}{\omega} - \frac{\sin \omega}{\omega^2} \right\} \omega_3 \omega. \quad (58)$$

Since $e_{\omega,3} = l_3 \times e_s + R \cdot E_n,3$, we have

$$t_i^T \{ (1 + F_0) v - R \cdot U \} = t_i^T a_i \cdot [(x + u)_3 - R \cdot (h + E_3) + Y^e l_3 \times e_s - (R \cdot F) \times Y^e e_s], \quad (59)$$

where $k = k_m E_m. \quad (60)$

Using Eqs. (59) and (56), and employing the notations given earlier in this paper, the functional $F_1$ for a finitely deformed beam is expressed, after some algebra, as

$$F_1(T, M, h, {\phi}, u, R) = \int \left[ W_0(h, \{ R \cdot h \}) + T \cdot \{ (x + u)_3 - R \cdot (h + E_3) \} + M \cdot \{ l_3 - R \cdot l_3 \} 
- q \cdot u \right] dL 
+ [ M \cdot (\phi - \phi) ]_{Z_3} - [ l_3 \cdot u ]_{Z_3} - [ \phi \cdot T \cdot (x + u)_3 \cdot T ] \cdot \delta \phi \right] dL,$$

where $l_3$ is a vector function of $R$. We now consider the first variation of $F_1$, which is:

$$\delta F_1 = \int \left[ \left( \frac{\partial W_0}{\partial h_m} - T^m \right) \delta h_m + \left( \frac{\partial W_0}{\partial k_m} - M^m \right) \delta k_m + \delta T \cdot \{ (x + u)_3 - R \cdot (h + E_3) \} 
+ \delta M \cdot \{ l_3 - R \cdot l_3 \} + (T_3 + q) \cdot \delta u - \{ M_{3} + (x + u)_3 \cdot T \} \cdot \delta \phi \right] dL,$$

$$\left[ \delta T \cdot (x + u)_3 \cdot T \right] \cdot \delta \phi \right] dL - \left[ \frac{\partial W_0}{\partial h_m} - T^m \right]_{Z_3} - \left[ \frac{\partial W_0}{\partial k_m} - M^m \right]_{Z_3} - [ \phi \cdot T \cdot (x + u)_3 \cdot T ] \cdot \delta \phi \right]_{Z_3}.$$

(62)
It can be seen that the stationary condition, \( \delta F_1 = 0 \), leads to the constitutive Eqs. (53), the compatibility Eqs. (17) and (21), the \( LMB \) and \( AMB \) conditions (35), and the mechanical and geometrical boundary conditions (36). It is reemphasized that no external moments are included in \( F_1 \).

When external moments \( m \), due to dead-load type of forces, as in Eq. (34c) are present, as shown in Sect. 5, the functional \( F_1 \), may be modified as:

\[
G_1(T, M, h, \vec{k}, u, \alpha^m, L^+) = \left[ W_s(h, \vec{k}) + T \cdot \{(x + u)_3 - R \cdot (h + E_3)\} + M \cdot \left\{ I_3 - R \cdot \vec{k} \right\} - q \cdot u - m_{am}R_{am} \right] \frac{dL}{s_a} - \left[ \frac{\partial T}{\partial (x - \vec{u})} \frac{\partial L^+}{\partial (x - \vec{a})} \right] \frac{dL}{s_a} - \left[ \frac{\partial \vec{u} \cdot \vec{u}}{\partial \vec{u}} \right] \frac{dL}{s_a},
\]

where \( L^+ \) is a Lagrangian multiplier and \( R \) and \( I_3 \) are functions of \( \alpha^m \). The first variation of \( G_1 \) takes the form

\[
\delta G_1 = \int \left[ \left( \frac{\partial W_s}{\partial m_{am}} - T \right) \delta h_m + \left( \frac{\partial W_s}{\partial \vec{k}_m} - M \right) \delta \vec{k}_m + \delta T \cdot \{(x + u)_3 - R \cdot (h + E_3)\} \\
+ \delta M \cdot \left\{ I_3 - R \cdot \vec{k} \right\} - (T_3 + q) \cdot \delta u - (N_j + r) \delta \alpha^l \right] \frac{dL}{s_a} \\
- \left[ \frac{\delta T}{\partial (x - \vec{u})} \frac{\delta L^+}{\partial (x - \vec{a})} \right] \frac{dL}{s_a} \\
- \left[ \frac{\partial \vec{u} \cdot \vec{u}}{\partial \vec{u}} \right] \frac{dL}{s_a} - \left[ \frac{\partial T}{\partial (x - \vec{u})} \frac{\partial L^+}{\partial (x - \vec{a})} \right] \frac{dL}{s_a} \\
- \left[ \frac{\partial \vec{u} \cdot \vec{u}}{\partial \vec{u}} \right] \frac{dL}{s_a}.
\]

The physical meaning of the Lagrangian multiplier \( L^+ \) is clear from Eqs. (64). The stationary condition, \( \delta G_1 = 0 \), yields the constitutive Eqs. (53), the compatibility Eqs. (17) and (21), the \( LMB \) condition (35a) and \( AMB \) condition (44), and the mechanical and geometrical boundary conditions (36a), (46a), (36c) and (46b). It should be stressed that the effects of the external moments due to a system of conservative forces, are taken into account in \( G_1 \).

For an incremental approach, we construct the incremental functionals in the context of TL and UL formulations. Let \( C_0 \) be the initial known configuration of beams, and let \( C_N \) and \( C_{N+1} \) respectively, be the configuration prior to, and after, the addition of the \((N+1)\)-th increment of prescribed loads and/or deformations. In the TL formulation, the fixed metric of \( C_0 \) is used to refer to all the state variables in each successive configuration. In the UL formulation, the variables in the state \( C_{N+1} \) are referred to the configuration in \( C_N \). Let \( \alpha^m \) denote the variable in \( C_N \) and \( \Delta( ) \) the incremental variable in passing from \( C_N \) to \( C_{N+1} \). Note that all variables are referred to the convected coordinate system \( Y^m \).

(i) TL formulation. At first we consider the incremental functional of \( F_1 \). In the TL formulation, the finite rotation tensor is required to satisfy the orthogonality condition that \((R^N + \Delta R) \cdot (R^N + \Delta R)^T = I_3 \); or in a variational sense, \( \delta \Delta R \cdot (R^N + \Delta R)^T \) is skewsymmetric. As a result, we obtain the incremental functional of \( F_1 \) in the form

\[
\Delta F_1(\Delta T, \Delta M, \Delta h, \Delta \vec{k}, \Delta R, \Delta u) = \left[ \Delta W_s + \Delta T \cdot \left\{ \Delta (x + u)_3 - R \cdot (h + E_3)\right\} \\
- (T^N \cdot \Delta R \cdot (h^N + E_3 + \Delta h) \right) + (M^N + \Delta M) \cdot \left\{ \Delta I_3 - R \cdot \Delta \vec{k} - \Delta R \cdot (\vec{k}^N + \Delta \vec{k}) \right\} \\
- \Delta q \cdot \Delta u \right] \frac{dL}{s_a} - \left[ \Delta T \cdot (\Delta u - \Delta \vec{u}) + (M^N + \Delta M) \cdot (\Delta \phi - \Delta \varphi) \right] \frac{dL}{s_a} - \left[ \Delta q \cdot \Delta u \right] \frac{dL}{s_a},
\]

where

\[
\Delta W_s = \frac{1}{2} G_{a0}(\Delta h_3)^2 + \frac{1}{2} G_{a0}(\Delta h_2)^2 + \frac{1}{2} E(A \Delta h)^2 + \frac{1}{2} E_{11}(\Delta \vec{k})^2 + \frac{1}{2} E_{22}(\Delta \vec{k})^2 + \frac{1}{2} \left( \Delta \vec{k} \cdot \Delta \vec{k} \right)
+ E_{12}(\Delta \vec{k})^2 + E_{12}(\Delta \vec{k})^2 + G_{a2}(\Delta \vec{k})^2 + (h^N + \Delta h_3) \Delta F_1 - E_{12}(h^N + \Delta h_3) \Delta F_2
- G_{f1}(h^N + \Delta h_1) \Delta F_1 + G_{f2}(h^N + \Delta h_2) \Delta F_2 - E_{12} \{ \Delta \vec{k} \cdot (\Delta \vec{k} + \vec{k}) \} + G_{a2}(\Delta \vec{k})^2.
\]

The linear and third order terms with respect to incremental values are included in \( \Delta F_1 \) to satisfy the orthogonality condition of finite rotation tensor exactly. The variational equations corresponding to \( \delta \Delta F_1 = 0 \) are, as can be shown easily
\[
\frac{\partial \Delta W_i}{\partial \Delta h_m} = \Delta T_m; \quad \frac{\partial \Delta W_i}{\partial \Delta \kappa_m} = M^m + \Delta M^m; \quad \Delta u_3 = R^N \cdot \Delta h^N + \Delta R \cdot (h^N + E^N + \Delta h),
\]

\[
\Delta l_i = R^N \cdot \Delta \kappa + \Delta R \cdot (\kappa^N + \Delta \kappa); \quad \Delta T_3 + \Delta q = 0;
\]

\[
(M^N + \Delta M)_3 + (X + u^N + \Delta u)_3 \times (T^N + \Delta T) = 0,
\]

\[
\Delta T = \Delta \Phi; \quad M^o + \Delta M = 0 \quad \text{on } S_o; \quad \Delta u = \Delta \bar{u}; \quad \Delta \Phi = \Delta \Phi \quad \text{on } S_o,
\]

where the following relation is used:

\[
\delta \Delta R \cdot (R^N + \Delta R)^T = \delta \Delta \phi \times 1.
\]

The present incremental governing equations, except for Eqs. (67 b), (67 f) and (67 h), are exact ones in the state \( \mathcal{C}_{n+1} \). The constitutive Eq. (67 b), the \( AMB \) condition (67 f) and the mechanical boundary condition (67 h) contain the constant terms associated with \( \mathcal{C}_N \). This is, as shown by Atluri and Murakawa (1977), Atluri (1979, 1980) and Murakawa and Atluri (1978), due to the nonlinear orthogonality condition of finite rotation tensor. These constant terms in Eqs. (67 b, f, h) show the governing equations in the state \( \mathcal{C}_N \). Therefore, these constant terms vanish if the state variables satisfy the governing equations in \( \mathcal{C}_N \). Consequently, the present incremental functional \( \Delta F_i \) leads to the exact incremental governing equation in \( \mathcal{C}_{n+1} \).

As can be seen from the functionals \( F_1 \) and \( G_1 \), given in Eqs. (61) and (63), respectively, we can derive the incremental functional of \( G_1 \) by a slight modification of \( \Delta F_i \). To obtain \( \Delta G_1 \) from \( \Delta F_i \) given in Eq. (65), one may introduce \( (\vec{m}^N + \Delta \vec{m}_i) \Delta R_{ia} \) and \( (\tilde{m}^N + \Delta \tilde{m}_i) \Delta R_{ia} \) into the integration and boundary term \( S_o \), respectively, and replace \( \Delta R \) and \( (M^N + \Delta M) \cdot (\Delta \phi - \Delta \Phi) \) on \( S_o \) by \( \Delta \alpha^m \) and \( (L^{N+} + \Delta L^+_i) (\Delta \alpha^m - \Delta \alpha^m) \), respectively. The stationary condition, \( \delta \Delta G_1 = 0 \), leads to the exact incremental governing equation in \( \mathcal{C}_{n+1} \).

(ii) UL formulation. In the UL formulation, the notation \( * ( \cdot ) \) is used to emphasize that these values are referred to the configuration in \( \mathcal{C}_N \). Since \( * R^N = * I \), the orthogonality condition of finite rotation tensor is written as

\[
\delta \Delta^* R \cdot (* I + \Delta^* R)^T = \delta \Delta^* \phi \times * I.
\]

The incremental functional of \( F_1 \) is obtained as

\[
\Delta^* F_i = \{ \Delta^* W_i + \Delta^* T \cdot \{ \Delta^* u_3 - \Delta^* h - \Delta^* R \cdot (X^N + \Delta^* h) \} \}
\]

\[
- \{ \Delta^* T \cdot \{ \Delta^* u - \Delta^* \bar{u} \} + (M^N + \Delta M) \cdot (\Delta^* \phi - \Delta^* \Phi) \} \}
\]

\[
\frac{\partial \Delta^* W_i}{\partial \Delta \kappa_m} = \Delta^* T_m; \quad \frac{\partial \Delta^* W_i}{\partial \Delta \kappa_m} = \Delta^* h^N + \Delta^* R \cdot (X^N + \Delta^* h);
\]

\[
\Delta^* u_3 = \Delta^* h + \Delta^* R \cdot (X^N + \Delta^* h);
\]

The linear and third order terms with respect to incremental values are included in the incremental functional \( \Delta^* F_i \) so as to satisfy the nonlinear orthogonality condition (69). The Euler-Lagrange equations and natural boundary conditions of the statement \( \delta \Delta^* F_i = 0 \) are

\[
\Delta T = \Delta \Phi; \quad \Delta h = \Delta \bar{u}; \quad \Delta \Phi = \Delta \Phi \quad \text{on } S_o,
\]

where \( * \) is the position vector of the beam axis in the reference state \( \mathcal{C}_N \), and

\[
\Delta^* W_i = \frac{1}{2} G A_0 (\Delta^* h_1)^2 + \frac{1}{2} G A_0 (\Delta^* h_2)^2 + \frac{1}{2} E A (\Delta^* h_3)^2 + \frac{1}{2} E l_1 (\Delta^* \kappa_1)^2 + \frac{1}{2} E l_2 (\Delta^* \kappa_2)^2
\]

\[
+ \frac{1}{4} G J (\Delta^* \kappa_3)^2 + E l_1 \Delta^* h_3 \Delta^* \kappa_1 - E l_2 \Delta^* h_3 \Delta^* \kappa_2 - G l_1 \Delta^* h_1 \Delta^* \kappa_3 + G l_2 \Delta^* h_2 \Delta^* \kappa_3
\]

\[
- E l_1 \Delta^* \kappa_1 \Delta^* \kappa_2 + * M^N \Delta^* \kappa_3 + * M^N \Delta^* \kappa_2 + * M^N \Delta^* \kappa_1 + * M^N \Delta^* \kappa_3.
\]
\[ \Delta^* t = (\mathbf{1} + \Delta^* \mathbf{R}) \cdot \Delta^* \mathbf{F} \quad \Delta^* T_3 + \Delta^* q = 0; \]
\[ (\Delta^* M + \Delta^* \mathbf{M})_3 + (\Delta^* X + \Delta^* u)_3 \times (\Delta^* T + \Delta^* T) = 0; \]
\[ \Delta^* T = \Delta^* q; \quad \Delta^* M + \Delta^* \mathbf{M} = 0 \quad \text{on } S_e; \quad \Delta^* u = \Delta^* \mathbf{u}; \quad \Delta^* \phi = \Delta^* \mathbf{\phi} \quad \text{on } S_u. \]

The constant terms associated with the reference state \( C_N \) vanish if the state variables satisfy the governing equations in the reference state. Then Eqs. (72) present the incremental governing equations. Note again that no external moments exist in \( \Delta^* F_1 \).

The incremental functional of \( G_i \) is obtained by a slight modification of \( \Delta^* F_1 \). To obtain \( \Delta^* G_i \) from \( \Delta^* F_1 \) given in Eq. (70), one may introduce
\[ (-\mathbf{m}_3^N + \Delta^* \mathbf{m}_3) \Delta^* \mathbf{R}_s \quad \text{and} \quad (-\mathbf{m}_3^N + \Delta^* \mathbf{m}_3) \Delta^* \mathbf{R}_s \]
into the integration and boundary term \( S_e \), respectively, and replace \( \Delta^* \mathbf{R} \) and \( (\Delta^* \mathbf{M} + \Delta^* \mathbf{M}) \)
\( (\Delta^* \phi - \Delta^* \mathbf{\phi}) \) on \( S_e \) by \( \Delta^* \mathbf{f}^m \) and \( (\Delta^* \mathbf{f}^m + \Delta^* \mathbf{L}^r) (\Delta^* \mathbf{f} - \Delta^* \mathbf{\phi}) \), respectively. The stationary condition,
\[ \delta \Delta^* G_i = 0, \]
yields the exact incremental governing equations in \( C_{N+1} \).

8 Applications

To investigate a validity of the present governing equations, we consider the problem of buckling of an initially straight beam subjected to the action of an axial compressive force \( P_0 \) and a twisting couple \( M_0 \), as shown in Fig. 2. As described before, the twisting couple is generated by external end forces. When the end forces, as shown in Fig. 3, are conservative, the resulting mechanical boundary condition for the moment is dependent on the deformations (see Appendix 3).

In this example, we assume that the end forces are nonconservative so that the torsional moment along the beam is constant before the buckling. In the case of initially straight beams, it is possible to choose the origin of coordinate \( Y^m \) so that \( I_3 = I_{12} = 0 \). As a result, it is easy to show that the nonvanishing stress resultants and moments before buckling are the axial force, \( T_3 = EAh_3 = -P_0 \), and the torsional moment, \( M_3 = GJk_3 = M_0 \).

Let \( \Delta( ) \) be the incremental value after the buckling. In the buckling problem, there exists at least one equilibrium position in the vicinity of the original equilibrium position under the same boundary conditions. The equilibrium equations for forces and moments in the direction of \( e_s \) are written, from Eqs. (35), as
\[ T_3 - k_3 T_2^2 + k_2 T_2^3 = 0; \quad T_3^2 + k_3 T_1^2 - k_1 T_3 = 0; \quad M_3 - k_3 M_2^2 + k_2 M_2^3 - (1 + h_3) T_2^2 + h_2 T_3^3 = 0; \]
\[ M_3^2 + k_3 M_1^2 - k_1 M_2^2 + (1 + h_3) T_1^2 - h_1 T_3^3 = 0. \]

Figs. 2 and 3.2 A straight beam subjected to axial forces and twisting couples. 3 Forces at the end cross section.
These component expressions are the same as those of Reissner (1973, 1981). Keeping in mind that \( T^3, M^3, h_3 \) and \( h_3 \) do not vanish before buckling and retaining the linear terms with respect to the incremental values, we obtain the present buckling equations represented by

\[
\begin{align*}
\Delta T_3 + T^2 \Delta \delta_2 - \delta_3 \Delta T^2 &= 0; \quad \Delta T_3 + \delta_3 \Delta T^1 - T^3 \Delta \delta_1 = 0; \\
\Delta M_3 - \delta_3 \Delta M^2 + M^3 \Delta \delta_2 - (1 + h_3) \Delta T^2 + T^3 \Delta h_2 &= 0; \\
\Delta M_3 + \delta_3 \Delta M^1 - M^3 \Delta \delta_1 + (1 + h_3) \Delta T^1 - T^3 \Delta h_1 &= 0. \\
\end{align*}
\tag{74 a–d}
\]

Eliminating the incremental shear forces and using the incremental relations such as \( \delta M = EI \delta \), leads to

\[
\begin{align*}
\frac{EI_{11}}{1 - \frac{P_0}{EA} + \frac{P_0}{GA_0}} \Delta \delta_{1,33} + & \left\{ P_0 + \frac{\left(1 - \frac{EI_{11}}{GJ}\right) (M_0)^2}{GJ} \right\} \Delta \delta_1 + \frac{\left(1 - \frac{EI_{12}}{GJ}\right) (M_0)^2}{GJ} \Delta \delta_{2,3} = 0, \\
\frac{EI_{22}}{1 - \frac{P_0}{EA} + \frac{P_0}{GA_0}} \Delta \delta_{2,33} + & \left\{ P_0 + \frac{\left(1 - \frac{EI_{12}}{GJ}\right) (M_0)^2}{GJ} \right\} \Delta \delta_2 + \frac{\left(1 - \frac{EI_{11}}{GJ}\right) (M_0)^2}{GJ} \Delta \delta_{1,3} = 0, \quad (75 a, b)
\end{align*}
\]

The expressions (75) denote the buckling equations for the present problem. Following the way of Timoshenko and Gere (1961), it is easy to obtain the buckling load under the prescribed boundary conditions.

For comparison with the existing results, we consider a simply supported beam with equal bending rigidities \( EI \). Since \( \Delta \delta = 0 \) at \( Y^3 = 0 \) and \( l \), we obtain the following expression:

\[
\frac{\left\{ \left(1 - \frac{EI}{GJ}\right) (M_0)^2 \right\}^2}{4 EI} + P_0 \left(1 - \frac{P_0}{EA} + \frac{P_0}{GA_0}\right) + \left(1 - \frac{EI}{GJ}\right) (M_0)^2 \Delta \delta_1 = \frac{\pi^2 EI}{l^2}. \quad (76)
\]

The terms denoted \( ( ) \) and \( ( ) \) indicate the effects of the twist and the stretch before buckling respectively and the term \( ( ) \) is the effects of shear deformations. Neglecting the twisting couple \( M_0 \) in Eq. (76) yields the Euler buckling load in which the effects of shear deformations are taken into account. Ziegler (1982) and Reissner (1982) have discussed the Euler buckling load. It should be noted that the difference among those results depends only on the constitutive equations. The well-known Greenhill equation (Timoshenko and Gere 1961) is derived from neglecting the terms \( ( ) \), \( ( ) \) and \( ( ) \) in Eq. (76).

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Appendix 1

Following Atluri (1984), the differential force vector acting on the deformed cross-section is given as

\[
dT = g_0 dAA^3 \cdot (S_1 \cdot F^T). \quad (A.1)
\]
Since $S_1 = S_1^{mn} A_m A_n$ and $F^T = A' a_i$, integrating Eq. (A.1) leads to
$$T = \int S_1^{mn} a_m g_0 dA.$$  \hspace{1cm} (A.2)
Introducing Eq. (27) into Eq. (A.2) yields
$$T = \int r_m e_m g_0 dA.$$  \hspace{1cm} (A.3)
Equations (26a) and (26c) are derived from Eq. (A.3).
In a similar way, the differential moment vector acting on the deformed cross sections is represented as
$$dM = Y^x e_x \times dT.$$  \hspace{1cm} (A.4)
After some manipulation, we have
$$M = \int \{ (i^{33} Y^2 e_1 - i^{33} Y^1 e_2 + (i^{32} Y^1 - i^{31} Y^2) e_3 ) \} g_0 dA.$$  \hspace{1cm} (A.5)
Equations (26b) and (26d–f) are derived from Eq. (A.5).

**Appendix 2**

When we express the deformed base vectors as
$$a_m = K_{mn} e_n,$$  \hspace{1cm} (A.6)
the internal virtual work per unit volume is presented as
$$S_1^{mn} \delta e_{mn} = \frac{1}{2} S_1^{mn} \delta (K_{ml} K_{ln}).$$  \hspace{1cm} (A.7)
Since $S_1^{mn} = S_1^{nm}$ and $r_{mn} = S_1^{mn} K_{ln}$, we have
$$S_1^{mn} \delta e_{mn} = r_{ml} \delta K_{ml}.$$  \hspace{1cm} (A.8)
Consequently, the conjugate strain tensors $\gamma_{mn}$ are defined by Eq. (51).

**Appendix 3**

We consider, herein, the end forces which give rise to the twisting couple of constant magnitude $M_0$.
For simplicity, we consider a beam with circular cross sections subjected end forces, as shown in Fig. 3. If we treat the end forces as conservative ones defined by Eq. (32b), the mechanical boundary conditions at both end cross section are obtained as
$$M^3 = Pd [R_{11} (R_{11} R_{22} - R_{12} R_{21}) + R_{31} (R_{31} R_{22} - R_{21} R_{32})] / \det |R_{ij}|.$$  \hspace{1cm} (A.9)
On the other hand, if we treat the end forces as nonconservative ones defined by Eq. (37b), the mechanical boundary conditions at both end cross sections become
$$M^3 = Pd.$$  \hspace{1cm} (A.10)
As shown in Eqs. (A.9) and (A.10), the end forces which give rise to the constant twisting couple must be nonconservative ones.

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Dynamic Analysis of Finitely Stretched and Rotated Three-Dimensional Space-Curved Beams

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Abstract—The problem of transient dynamics of highly flexible three-dimensional space-curved beams, undergoing large rotations and stretches, is treated. The case of conservative force loading, which may also lead to configuration-dependent moments on the beam, is considered. Using the three parameters associated with a conformal rotation vector representation of finite rotations, a well-defined Hamilton functional is established for the flexible beam undergoing finite rotations and stretches. This is shown to lead to a symmetric tangent stiffness matrix at all times. In the present total Lagrangian description of motion, the mass-matrix of a finite element depends linearly on the linear accelerations, but nonlinearly on the rotation parameters and attendant accelerations; the stiffness matrix depends nonlinearly on the deformation; and an 'apparent' damping matrix depends nonlinearly on the rotations and attendant velocities. A Newmark time-integration scheme is used to integrate the semi-discrete finite element equations in time. Several examples of transient dynamic response of highly flexible beam-like structures, including those in free flight, are presented to illustrate the validity of the theoretical methodology developed in this paper.

1. Introduction

This paper is concerned with the large deformation dynamic analysis of finitely stretched and rotated three-dimensional elastic space-curved beams, extending model developed by Iura and Atluri[1] for such problems, in the static case.

The model used is based on Timoshenko's hypotheses; the effects of stretching, bending, torsion and transverse shear are taken into account. For simplicity, however, the cross-sectional warping effects are neglected. These kinematic assumptions have been employed also by Antman and Jordan[2], Reissner[3] and Simo and Vu-Quoc[4] to develop a three-dimensional beam theory. In these references, the existence of prescribed external moments has been postulated a priori. Iura and Atluri[1] have utilized the variational method to derive the consistent boundary conditions in which the external moments arise as a consequence of the applied external forces. Iura and Atluri[1] have observed that the conservative moments (using the definition of Schweizerhof and Ramm[5] as to whether the load is conservative or not) are generally configuration dependent. Argyris et al.[6] have employed the same definition for external moments. Using the rotational degrees of freedom referred to fixed axes of a global Cartesian system, Argyris et al.[6] have derived a nonsymmetric tangent stiffness matrix at the element level. Simo and Vu-Quoc[4] have concluded that, in the context of a classical formulation of rotations, the tangent stiffness matrix becomes symmetric only at an equilibrium configuration, provided that no distributed external moments are assumed to exist. Iura and Atluri[1], on the other hand, have shown that the use of any three independent components of the finite rotation tensor, as rotational variables, leads to a symmetric tangent stiffness matrix, not only at the equilibrium but also the nonequilibrium configuration, even if the distributed external moments exist in the problem.

The large deformation dynamics of a continuum body have, in the past, been formulated with the use of the total Lagrangian formulation, the updated Lagrangian formulation, the Eulerian formulation, the Euler—Lagrangian formulation and the moving coordinate formulation[7—10]. Among these formulations, the inertia effects are readily taken into account in the total Lagrangian formulation. Here, therefore, we employ the total Lagrangian formulation and show the capability of the present formulation to simulate the dynamic behavior of finitely stretched and rotated beams.

In this paper the elasto-static model for finitely stretched and rotated space-curved beams, developed by Iura and Atluri[1], is extended to the case of dynamic analysis. In Sec. 2 we summarize the kinematic relations of the present model briefly. The principle of virtual work for elastodynamics is introduced, in Sec. 3, to derive the linear momentum balance (LMB) and angular momentum balance (AMB) conditions and the associated boundary conditions. Depending on the form of virtual variations of the rotational parameters considered, the LMB...
and AMB conditions take on different but equivalent forms, as in the static problem. A well-defined functional for Hamilton’s principle is obtained by using one form of rotation parameters, or the components of the finite rotation tensor.

In Sec. 4, the finite element formulation is utilized for deriving the semi-discrete equations of motion. The rotation variables used herein are taken as the Lagrangian components of conformal rotation vector [11]. Without using the four quaternion or Euler parameters, the singularity, associated with the finite rotation vector, can be avoided with a simple manipulation. As shown in the existing literature [10, 12], the resulting mass matrix is no longer constant due to the effects of finite rotations. Even though no external damping effects are accounted for in the formulation, the nonlinear terms of the velocity of rotational components appear in the semi-discrete equations.

A variety of time integration schemes has been proposed by many investigators [13]. In this paper, we use the Newmark algorithm to integrate the resulting semi-discrete equations. Although the stability and convergence conditions for the nonlinear dynamic problems have not been established yet, the Newmark family of algorithms has received wide attention.

To demonstrate the validity and the applicability of the present beam theory, numerical examples are presented in Sec. 5. After we confirm the accuracy of the beam model developed herein, we investigate the configuration dependency of the external moments in both planar and nonplanar problems.

2. PRELIMINARIES

2.1. Fundamental hypotheses

The fundamental hypotheses used are itemized as follows:

(1) The plane cross-sections of the beam remain plane and do not undergo any change of shape during the deformation.

(2) The cross-sections are constant along the beam axis, which remains a smooth space-curve throughout the deformation.

It should be noted that no simplification is made in the present formulation; not only the rotatory inertia but also the Coriolis and the centrifugal effects are accounted for.

Throughout this paper, the summation convention is adopted; and the Latin indices will have the range 1, 2 and 3, and the Greek indices the range 1 and 2.

2.2. The geometry of the undeformed and deformed beam

We summarize, for completeness, the kinematic relations of the present beam model developed by Iura and Atluri [1].

Let \( Y^m \) be a convected orthogonal curvilinear coordinate system. The coordinates \( Y^m \) are taken in the cross-section, while the coordinate \( Y^i \) is taken along the beam axis, as shown in Fig. 1. The undeformed base vectors at an arbitrary point in a cross-section of the beam are given, in terms of the undeformed unit base vectors \( E_\alpha \) at the beam axis, by

\[
A_i = E_i, \quad (1a)
\]

\[
A_j = -Y^j K_j E_i + Y^j K_j E_2 + g_j E_1, \quad (1b)
\]

where

\[
g_j = 1 - Y^j K_j + Y^j K_j.
\]

The quantities \( K_j \) are the components of initial curvature, and the \( K_j \) is the initial twist, satisfying the following relations:

\[
E_{m,3} = K \times E_m, \quad K = K_m E_m. \quad (2a, b)
\]

where \( ( )_3 = d( )/dL \) where \( L \) is the parameter of the length of an arc along the line of origin of the coordinate system \( Y^m \), in the reference configuration.

Let \( e_\alpha \) be the maps of the base vectors \( E_\alpha \) after a purely rigid rotation, denoted by the finite rotation tensor \( R_\alpha \), alone; that is, \( e_\alpha = R_\alpha E_\alpha \). In general, due to the shear deformation, the unit vector tangential to the deformed beam-axis does not coincide with the vector \( e_3 \). From the definition of covariant base vectors, the deformed unit tangent vector, denoted by \( e_3 \), may be written as:

\[
\dot{e}_3 = u^m || E_m, \quad (3a)
\]

\[
u^m || = (\delta_3^m + u^m ||) / g, \quad (3b)
\]

\[
g = \sqrt{[(u^m ||)^2 + (u^m ||)^2 + (1 + u^m ||)^2]}, \quad (3c)
\]

where \( u(=u^m E_m) \) is the displacement vector at the beam axis, \( \delta_3^m \) the Kronecker delta, and \( ( )_3 \) denotes a covariant differentiation by using the metric tensor \( E_\alpha = E_\beta E_\gamma \).

According to the hypothesis (1), the displacement vector at an arbitrary material point is represented as

\[
U = u + Y^i (e_i - E_i). \quad (4)
\]

The deformed base vectors at an arbitrary point in a cross-section of the beam are given as

\[
a_i = e_i, \quad (5a)
\]

\[
a_j = (g \sin \beta_j - Y^j k_j) e_i + (g \sin \beta_j + Y^j k_j) e_i + (g \beta_j - Y^j k_j + Y^j k_i) e_i, \quad (5b)
\]
Fig. 1. Kinematic scheme for highly flexible space-beam analysis.

where

\[
\sin \beta_i = (R \cdot E_i) \cdot (u^{m} \mid_{\mathbf{f}}, E_m), \quad \sin \beta_j = (R \cdot E_j) \cdot (u^{m} \mid_{\mathbf{f}}, E_m),
\]

\[
\beta_i = (R \cdot E_i) \cdot (u^{m} \mid_{\mathbf{f}}, E_m), \quad \beta_j = (R \cdot E_j) \cdot (u^{m} \mid_{\mathbf{f}}, E_m),
\]

\[
k_i = \frac{1}{2} \epsilon_{iab} [(R \cdot E_i)_a] [(R \cdot E_i)_b],
\]

in which \( \epsilon_{iab} \) is the permutation symbol. The parameters \( \beta_i \) denote the angles of shear deformations. The vector \( \beta \), defined by \( \beta = k_m \epsilon_m \), satisfies the following differential relation:

\[
\epsilon_m \beta = \kappa \times \epsilon_m.
\]

2.3. Strain-stress relationships

According to Atluri [14], the stress resultants and moments are defined as

\[
T = \int g_0 A^\prime \cdot (S_1 \cdot F^\prime) \, dA, \quad M = \int Y^a e_\gamma \times [g_0 A^\prime \cdot (S_1 \cdot F^\prime)] \, dA,
\]

where \( A^\prime \) is the reciprocal basis of \( A_m \), \( S_1 = S^m m A_m A_n \) is the second Piola–Kirchhoff stress tensor, \( F \) the deformation gradient tensor, \((\cdot)^T\) a transpose, and \( dA = dY^1 \, dY^2 \). The components of \( T \) and \( M \) are given by

\[
T = T^m \epsilon_m, \quad M = M^m \epsilon_m,
\]

\[
T^m = \int t^{1m} g_0 \, dA, \quad M^1 = \int t^{13} Y^1 g_0 \, dA,
\]

\[
M^2 = - \int t^{13} Y^2 g_0 \, dA,
\]

\[
M^3 = \int (t^{11} Y^1 - t^{13} Y^2) g_0 \, dA,
\]

where \( t^{mn} \) are the stress tensors defined by

\[
t^{mn} = s^{mn} a^\gamma \cdot e_\gamma.
\]

The \((\cdot)\) in the contravariant tensor components is used to emphasize that these are not components in the convected coordinates \( Y^m \). The conjugate strain tensors \( \gamma_{mn} \) are expressed as

\[
\gamma_{mn} = a_m \cdot e_n - A_m \cdot E_n.
\]

For one-dimensional beams, we assume that the
following constitutive relationships hold:
\[ \begin{align*}
\tau^{1} & = G\gamma^{1}, \\
\tau^{2} & = E\gamma^{2},
\end{align*} \quad (12a, b) \]
where \( G \) is the shearing modulus and \( E \) the Young modulus. Substituting eqns (11) and (12) into eqns (9) and utilizing eqns (5) leads to
\[ \begin{bmatrix}
T^1 \\
T^2 \\
T^3 \\
M^1 \\
M^2 \\
M^3
\end{bmatrix} = 
\begin{bmatrix}
G A_0 & 0 & 0 & 0 \\
GA_0 & 0 & 0 & 0 \\
E A & E I_1 & -E I_2 & 0 \\
E I_1 & E I_{11} & -E I_{12} & 0 \\
E I_2 & E I_{12} & E J & 0 \\
Sym.
\end{bmatrix}
\begin{bmatrix}
h_1 \\
h_2 \\
h_3 \\
k_1 \\
k_2 \\
k_3
\end{bmatrix} \tag{13}
\]
where
\[ \begin{align*}
h_1 & = g \sin \beta, \\
h_3 & = g \beta - 1, \\
A & = \int g_0 \, dA, \\
A_0 & = kA, \\
I_1 & = \int Y^1 g_0 \, dA, \\
I_{12} & = \int Y^{12} g_0 \, dA, \\
I_{11} & = \int (Y^1)^2 g_0 \, dA, \\
I_{22} & = \int (Y^2)^2 g_0 \, dA, \\
J & = \int \rho g_0 \, dA, \\
\rho & = (Y^1)^2 + (Y^2)^2, \\
\kappa_i & = k_i - K_i. \tag{14k}
\end{align*} \]
The factor \( k \) is the shear-correction factor [15]. The strain-energy function per unit length along the beam axis is obtained as
\[ \begin{align*}
W_r & = \frac{1}{2} G A_0 (h_1)^2 + \frac{1}{2} G A_0 (h_3)^2 + \frac{1}{2} E A (h_2)^2 \\
& + \frac{1}{2} E I_1 (\kappa_1)^2 + \frac{1}{2} E I_{12} (\kappa_2)^2 + \frac{1}{2} G J (\kappa_3)^2 \\
& + E I_1 h_1 \kappa_1 - E I_2 h_2 \kappa_2 - E I_{12} \kappa_3 \\
& - G I_1 h_1 \kappa_1 + G I_2 h_2 \kappa_2, \tag{15}
\end{align*} \]
It should be emphasized that the widely accepted strain-energy function, expressed by eqn (15), is based on the constitutive eqns (12a, b). If we introduce the well-known constitutive equations such that
\[ \begin{align*}
S^{11} & = G I_1, \\
S^{22} & = E I_{12},
\end{align*} \]
where \( \kappa_\text{w} \) denotes the Green strain tensor, the resulting strain-energy function is substantially different from the present one.

2.4. Definition of external moments
Argyris et al. [6] have defined a conservative moment as a moment generated by a couple of two equal and opposite conservative forces acting on a rigid lever. In this paper, the definition of the external moments are obtained from the external virtual work in which \( S \) is the position vector at the undeformed cylindrical surface, and \( \frac{\partial}{\partial s} \), denotes a differentiation with respect to the coordinate \( S \) taken along the bounding curve of the cross-sections. The vector
\[ \begin{align*}
P_b & = P_b E, \\
P_r & = P_r E, \\
P_s & = P_s E, \tag{17a-c}
\end{align*} \]
Introducing eqns (17) into eqn (16) yields
\[ \begin{align*}
EVW & = \int [q \cdot \delta u + M \cdot \delta \phi] \, dL \\
& + \int [q \cdot \delta u + \bar{m} \cdot \delta \phi] \, dS, \tag{18}
\end{align*} \]
where
\[ \begin{align*}
q & = q_1 E, \\
q & = q_1 E, \quad \bar{q} = \bar{q}_1 E, \tag{19a, b}
m & = m_w e_s \times E, \\
m & = \bar{m}_w e_s \times E, \tag{19c, d}
q' & = \int P_b g_0 \, dA + \int P_r |S_s \times S_s| \, ds, \tag{19e}
q' & = \int P_r / ds, \tag{19f}
m_w & = \int Y^w P_b g_0 \, dA + \int Y^w P_r |S_s \times S_s| \, ds, \tag{19g}
m_w & = \int Y^w P_r \, ds, \tag{19h}
\end{align*} \]
Analysis of three-dimensional space-curved beams

3. THE EQUATIONS OF MOTION

3.1. The principle of virtual work

The principle of virtual work for the elastodynamic problem is written as

\[ \int_0^1 [\delta T - IVW + EVW] dt = 0, \]

where \( T \) is the kinetic energy of the beam and IVW the internal virtual work, defined as

\[ T = \frac{1}{2} \int \rho \dot{U} \cdot \dot{U} \, dV, \]

\[ IVW = \int S' \delta e_0 \, dV, \]

in which \( \rho \) is the density in the reference state and (') a differentiation with respect to time. The subsidiary conditions for eqn (20) are the strain-displacement relationships, geometrical boundary conditions and the conventional conditions that the variations of displacements at \( t = t_1 \) and \( t = t_2 \) are equal to zero.

At first, following Atluri [14], we introduce a tensor \( \delta \mathbf{R} \cdot \mathbf{R}' = (\delta \phi \times \mathbf{I}) \) as a rotational variation to derive the AMB condition. Then, using eqn (20), we obtain the LMB and AMB conditions, expressed as

\[ T, + q = \dot{L}, \quad (\text{for arbitrary } \delta \mathbf{u}), \]  
\[ M, + (x + \mathbf{u}) \times T + \dot{m} = \dot{H}, \quad (\text{for arbitrary } \delta \phi). \]  

\( \delta \phi \) is the rotational variation and defined by \( \delta \phi \times \mathbf{I} = \delta \mathbf{R} \cdot \mathbf{R}'[14] \) where \( \mathbf{I} \) is an identity tensor.

It follows from eqns (19c,d) that the external moments \( m \) and \( \dot{m} \) defined herein are dependent on the deformation. Figures 2(a) and (b) show the configuration dependency of the external moments. Before the deformation, as shown in Fig. 2(a), the equivalent load at point 0 (or the origin of the coordinates \( Y'' \)) is the force \( f \) only. After the deformation, as shown in Fig. 2(b), the external loads at point 0 are not only the force \( f \) but also the moment \( m = \mathbf{P} \times \mathbf{R} \) (for arbitrary \( \delta \mathbf{u} \)). This example indicates that the external moments are dependent on the deformation.

Argyris et al. [6] have also pointed out the configuration dependency of external moments. They have derived a nonsymmetric tangent stiffness matrix by using the rotational degrees of freedom referred to fixed axis of a global Cartesian system. In this paper, however, the resulting tangent stiffness matrix is always symmetric, as shown later, as long as any three independent components of \( \mathbf{R} \) are employed as the rotation parameters. The present result agrees with that discussed by Schweizerhof and Ramm [5].

\[ \mathbf{J}_d = \int \rho Y^* Y^* \, dA, \]  
\[ \mathbf{J}_s = \int \rho \mathbf{Y} \cdot \mathbf{r} \, dA, \]

The associated boundary conditions at both the end cross-sections are obtained as

\[ \mathbf{T} = \dot{\mathbf{u}}, \quad \mathbf{M} = \ddot{\mathbf{m}} \quad \text{on } \mathbf{S}_e, \]  
\[ \delta \mathbf{u} = 0, \quad \delta \phi = 0 \quad \text{on } \mathbf{S}_e, \]

where \( \mathbf{S}_e \) is a part of boundary on which geometrical boundary conditions are prescribed.

Iura and Atluri [1] have introduced three components of \( \mathbf{R} \) denoted by \( \alpha' \) as a set of rotation variables. The advantage of the use of \( \alpha' \) in dynamic
problems is that a well-defined functional is obtained for Hamilton’s principle, as shown later. To obtain the AMB and the boundary conditions for \( a' \), we consider, at first, the tensor equation of the AMB condition corresponding to \( \delta \phi \). The inner product between the AMB condition and the variation \( \delta \phi \) is written as

\[
\{ M + (x + u), x + T + m - \vec{H} \} \cdot \delta \phi
\]

\[
= C(\delta R \cdot R^T),
\]

(25)

where

\[
C = Q^1 e_1 e_1 + Q^2 e_1 e_2 + Q^3 e_1 e_3
+ m_1 e_1 - J_s \vec{e}_2 e_3 - J_s \vec{e}_3 e_2,
\]

(26a)

\[
Q^r e_m = M + (x + u), x + T.
\]

(26b)

Since \( a' \) are taken as the Lagrange components of \( R \), \( = R_u, E, E \), \( \delta R = R_u, E, E, \delta a' \), where \( \delta R \) is a differentiation with respect to \( a' \). The right-hand side in eqn (25) is rewritten, in terms of \( a' \), as

\[
C(\delta R \cdot R^T) = C(R_u, E, E) \delta a'.
\]

(27)

The AMB condition for \( \delta \phi \) is represented by

\[
C = C^r
\]

(28)

while the AMB condition for \( \delta a' \) is expressed as

\[
C(R_u, E, E) = 0.
\]

(29)

Since \( R_u, E, E \) is a skew-symmetric tensor, eqn (29) is equivalent to eqn (28); the AMB conditions for \( \delta \phi \) and \( \delta a' \) are equivalent to each other.

In a similar fashion, the tensor equations for the boundary conditions are given by

\[
(M - \vec{m}) = (M - \vec{m})^T
\]

(for arbitrary \( \delta \phi \)) on \( S_u \),

\[
(M - \vec{m}) : (R_u, E, E) = 0
\]

(for arbitrary \( \delta a' \))

(30a, b)

\[
\delta R \cdot R^T = 0 \quad (for \ \phi)
\]

(30c, d)

\[
\delta a' = 0 \quad (for \ \alpha')
\]

on \( S_u \),

where

\[
M = M^1 e_1 e_1 + M^2 e_1 e_2 + M^3 e_1 e_3,
\]

(31a)

\[
\vec{m} = \vec{m}_u E e_3,
\]

(31b)

As seen in eqns (22), the equations of motion for \( \delta u \) and \( \delta \phi \) are the same as those derived \textit{a priori} from the so-called ‘static method’ (i.e. using the first principles of force and moment balance). Noting that the LMB conditions remain unchanged under the exchange of rotation parameters, the basic equations for \( \delta u \) and \( \delta a' \) take on different forms, but equivalent to those for \( \delta u \) and \( \delta \phi \).

3.2. Hamilton’s principle

When the potential energy \( \pi \) is obtained, Hamilton’s principle for elastodynamic problems is expressed as [16]

\[
\delta \int_{t_1}^{t_2} [T - \pi] \, dt = 0,
\]

(32)

where the subsidiary conditions are the geometrical boundary conditions and the conventional conditions at \( t_1 \) and \( t_2 \) cited before. It is not always possible, however, to construct the potential energy, especially in a finite rotation beam theory. When the external moments defined in eqns (19c, d) are applied on the beam, the use of \( \phi \) as a rotational variable makes it difficult to construct the potential energy (Iura and Atluri [1]). Vu-Quoc [10] has also indicated that the potential energy does not exist even at the equilibrium configurations as long as externally distributed moments exist. Note that the variation of rotational variable used by Vu-Quoc [10] is the same as that used by Atluri [14].

To obtain a well-defined functional, Iura and Atluri [1] have introduced the three components \( a' \) of \( R \) as rotational variables. As shown in Sec. 3.1, the resulting equations of motion are equivalent to those associated with another variable \( \phi \). When using \( a' \) as rotational variables, the potential energy is obtained as [1]

\[
\pi = \int \left[ W(u, \alpha^a) - g \cdot u - m \pi R_u(\alpha^a) \right] dL
\]

\[
- \int_{S_u} \left[ \dot{a} - \vec{m}_u R_u(\alpha^a) \right] \cdot ds.
\]

(33)

Introducing eqn (21a) and eqn (33) into eqn (32), Hamilton’s principle yields the LMB condition in eqn (22a), the AMB condition in eqn (29) and the mechanical boundary condition in eqn (30b).

4. FINITE ELEMENT FORMULATION

4.1. Rotation parameters

As discussed in the previous section, the significant advantage of the use of \( a' \) is that a well-defined functional is obtained for Hamilton’s principle. In this section, emphasis is placed on the definition of the present rotation parameters \( a' \) to avoid the singularity associated with finite rotation representations. It is well known that no three-parameter representation of \( R \) can be both global and nonsingular [17]; for this reason the four quaternion or Euler par-
ameters have been introduced to describe the large overall motions\[10, 18, 19\]. To avoid using the four parameters, the conformal rotation vector has been introduced\[11\]. The modification of the Rodrigues vector leads to the conformal rotation vector defined by

$$\theta^* = 4 \tan \frac{\theta}{4},$$

(34)

where $\epsilon$ is a unit vector satisfying $R \cdot \epsilon = \epsilon$ and $\omega$ a magnitude of rotation about the axis of rotation defined by $\epsilon$.

Using the conformal rotation vector, we define the present rotation parameters $\alpha'$ such that

$$\theta^* = \alpha' E_i,$$

(35)

Then the Langrangian components of $R$ are expressed by

$$R_d = \frac{1}{(4 - \alpha)^3} \left[ \left( (\alpha - \alpha') \right) \right]_d$$

$$+ 2 \left( \alpha' \alpha' - \epsilon \alpha \alpha' \right),$$

(36)

where

$$\alpha_0 = (16 - \alpha' \alpha')/8.$$  

(37)

Because of singularity, the Rodrigues vector, defined by $\theta = 2 \tan(\omega/2)\epsilon$, is valid only in the range of $|\omega| < \pi$. As shown in eqn (34), however, the conformal rotation vector is valid even at $|\omega| = \pi$. Therefore, with this simple manipulation, the finite rotations are described in terms of the only three rotation parameters.

The main idea to avoid the singularity of the conformal rotation vector is the following\[11\]: when the angle $\omega$ reaches a value such that

$$\omega = \pi + \epsilon,$$

(38)

with $\epsilon$ being a small positive value, we introduce a new rotation parameter $\alpha'$ defined by

$$\alpha' = -16 \alpha'/(\alpha' \alpha').$$

(39)

The corresponding velocity and acceleration are also defined by

$$\dot{\alpha}' = -16(\dot{\alpha}' - \dot{\alpha}' \alpha_0)/(\alpha' \alpha'),$$

(40a)

$$\ddot{\alpha}' = -16(\ddot{\alpha}' - \ddot{\alpha}' \alpha_0)/(\alpha' \alpha').$$

(40b)

4.2. Semi-discrete equations of motion

As a standard finite element discretization, the displacement and rotation components are interpolated by

$$u' = u_j N^j,$$

(41a)

$$\alpha' = \alpha_j N^j,$$

(41b)

where $u_j'$ and $\alpha_j'$ denote the nodal displacement and rotation components, respectively, and $N^j$ are the shape functions defined by

$$N' = 1 - L/l, \quad N^2 = L/l,$$

(42a, b)

where $l$ is an element length before the deformation. For later convenience, the following notations are introduced:

$$d = \{u_0\}, \quad r = \{x_0\}.$$  

(43a, b)

Introducing eqns (41) into Hamilton's principle and performing partial integrations with respect to time, we have

$$\int_0^t \left[ A_p \ddot{u}_j N^j \delta u_j + J_{ac}(R_{ac}, \dot{x}) \ddot{a}^4 \right.$$

$$+ R_{ac} \dot{\delta}^2 \right] R_{ac} N^j \delta a^j + GA_{ac} h_1 \delta h_1$$

$$+ GA_{ac} h_1 \delta h_2 + EAh_1 \delta h_1 + EI_{ac} \delta \delta a_c$$

$$+ E I_{ac} \delta^- \delta \delta a_c$$

$$+ (EI_{ac} - E \delta^2) \delta \delta a_c$$

$$- (E I_{ac} - E \delta^2) \delta \delta a_c$$

$$- (G I_{ac} - G \delta^2) \delta \delta a_c$$

$$- (G I_{ac} - G \delta^2) \delta \delta a_c$$

$$+ G I_{ac} \delta \delta a_c$$

$$- (EI_{ac} - E I_{ac}) \delta \delta a_c$$

$$- q^t N^j \delta u_j + m_j R_{ac} N^j \delta a^j \right] dL$$

$$= 0,$$

(44)

where

$$\delta h_j = R_{ac} (\delta^j + u_j N^j) N^j \delta a^j,$$

(45a)

$$\delta \delta a_c = \frac{1}{2} \epsilon_{ac} \left( R_{ac} R_{ac} N^j N^j + R_{ac} R_{ac} N^j N^j \right) \delta a^j.$$  

(45b)

Integrating eqn (44) over the beam length and noting that $\delta d$ and $\delta r$ are arbitrary, we obtain the following semi-discrete equations of motion:

$$M(\ddot{d}, \dot{r}, r) + C(\dot{r}, r) + K(\dot{d}, r) = f(r),$$

(46)

where $M$ is linear with respect to (w.r.t.) $\ddot{d}$ but
4.3. Time-integration scheme

The Newmark algorithm is employed herein to integrate the semi-discrete equations of motion in eqn (46). In a linear problem, this algorithm has received a wide attention because of its unconditional stability.

Let \( \dot{r}_n \) be the value at time \( t = t_n \). We postulate that the solution \( \{d_{n+1}, r_{n+1}\} \) satisfies the semi-discrete eqn (45), i.e.,

\[
M(\ddot{d}_{n+1}, \dot{r}_{n+1}) + C(\dot{r}_{n+1}, r_{n+1}) + K(d_{n+1}, r_{n+1}) = f(r_{n+1}).
\]  

(47)

According to the Newmark algorithm, the acceleration and velocity at time \( t = t_{n+1} \), are approximated by

\[
\dot{r}_{n+1} = \frac{1}{\beta(\Delta t)} \{ \dot{r}_{n+1} - \dot{r}_n - \Delta t \dot{r}_n \} - \frac{1 - 2\beta}{2\beta} \dot{r}_n,
\]

(48a)

\[
\ddot{r}_{n+1} = \frac{1}{\beta(\Delta t)} \{ \ddot{r}_{n+1} - \dot{r}_n - \Delta t \ddot{r}_n \} - \frac{1 - 2\beta}{2\beta} \ddot{r}_n.
\]

(48b)

where \( \dot{r} \) or \( \ddot{r} \) and \( \Delta(\cdot) \) is an incremental value, and \( \beta \) and \( \gamma \) are constants. Substituting eqns (48) into eqn (47), we obtain the nonlinear algebraic equations in terms of \( d_{n+1} \) and \( r_{n+1} \).

To solve the resulting nonlinear algebraic equations, we utilize the Newton–Raphson method. Then the \( 0 \) and \( i + 1 \) iterative solutions are given, by means of the converged solutions at time \( t = t_n \), as

\[
\dot{r}_{n+1}^{(0)} = \dot{r}_n,
\]

(49a)

\[
\dot{r}_{n+1}^{(i+1)} = \dot{r}_{n+1}^{(i)} + \Delta(\dot{r}_{n+1}^{(i)}),
\]

(49b)

where a superscript in parentheses denotes the iteration number. Substituting eqns (49) into the nonlinear algebraic equations and linearizing them with respect to the incremental values, we have the following linear equations with respect to the incremental values:

\[
[DM(r_{n+1}^{(i)}) + DC(r_{n+1}^{(i)}) + DK(d_{n+1}^{(i)}, r_{n+1}^{(i)})] \Delta d_{n+1}^{(i)} + \Delta r_{n+1}^{(i)} = -f(r_{n+1}^{(i)}) - M(\ddot{d}_{n+1}^{(i)}, \dot{r}_{n+1}^{(i)}, r_{n+1}^{(i)})\Delta \dot{r}_{n+1}^{(i)} - C(r_{n+1}^{(i)}, r_{n+1}^{(i)}) - K(d_{n+1}^{(i)}, r_{n+1}^{(i)}).
\]

(50)

where \( DK \) is defined by

\[
K(d_{n+1}^{(0)} + \Delta d_{n+1}^{(i)}, r_{n+1}^{(0)} + \Delta r_{n+1}^{(i)}) = K(d_{n+1}^{(0)}, r_{n+1}^{(0)})
\]

\[
+ DK(d_{n+1}^{(0)}, r_{n+1}^{(0)}) \times (\Delta d_{n+1}^{(i)}, \Delta r_{n+1}^{(i)}) + O(\Delta^2).
\]

(51)

In a similar way, \( DM, DC \) and \( Df \) are obtained from a consistent linearization. Note that \( DM \) and \( DC \) are nonsymmetric matrices, while \( DK \) and \( Df \) are symmetric matrices.

The initial values of acceleration and velocity at each time step follow from eqns (48):

\[
\dot{r}_{n+1}^{(0)} = -\frac{1}{\beta \Delta t} \dot{r}_n - \frac{1 - 2\beta}{2\beta} \dot{r}_n,
\]

(52a)

\[
\ddot{r}_{n+1}^{(0)} = \frac{1}{\beta \Delta t} ((1 - \gamma) \ddot{r}_n + \gamma \ddot{r}_n).
\]

(52b)

The \( i + 1 \) iterative values are also evaluated from eqns (48) as follows:

\[
\dot{r}_{n+1}^{(i+1)} = \dot{r}_{n+1}^{(i)} + \frac{1}{\beta \Delta t} \Delta(\dot{r}_{n+1}^{(i)}),
\]

(53a)

\[
\ddot{r}_{n+1}^{(i+1)} = \ddot{r}_{n+1}^{(i)} + \frac{\gamma \Delta(\dot{r}_{n+1}^{(i)})}{\beta \Delta t}.
\]

(53b)

These procedures expressed in eqns (52) and (53) are the same as those of Vu-Quoc [10].

The iterations continue until the appropriate convergence criterion is satisfied.

5. NUMERICAL EXAMPLES

Several numerical examples are considered in this section to demonstrate the validity and applicability of the present study. The considered structures consist of straight members. Therefore the origin of the coordinates \( Y^* \) in each element is so chosen that \( I_x = 0, I_y = I_z = 0, J_x = 0 \) and \( J_y = J_z = 0 \).

All solutions presented in the following have been obtained using \( \beta = 1/4 \) and \( \gamma = 1/2 \) in the Newmark algorithms. The tangent stiffness matrix and the residual forces are integrated by using a uniformly reduced one-point Gauss quadrature to avoid the shear locking [20]. The matrices associated with the inertia terms are integrated with two-point Gauss quadrature.

The iteration at each time step is terminated if the Euclidean norm of residual forces is less than the prescribed value.

5.1. Flexible beam in free flight, subject to constant force and constant moment

Vu-Quoc [10] has first solved this in-plane problem by using a linear shape function. The beam is subject
to a force and a torque simultaneously at one end, as shown in Fig. 3. The direction and the magnitude of the force and the torque are assumed to be constant during the deformation. In this example we use the definition for a torque introduced by Vu-Quoc [10]. Note, however, that a constant torque is not generated by conservative forces as long as the definition for moments introduced herein or by Argyris et al. [6] is employed. Figure 4 shows the present numerical results. Good agreement is obtained between the present results and the results of Vu-Quoc [10].

5.2. Right angle cantilever beam
This out-of-plane problem has been simulated first by Vu-Quoc [10] using a quadratic shape function. Figure 5 shows the material properties and the load condition. The dynamic responses are shown in Fig. 6. Although the present results are obtained with the use of a linear shape function, an excellent agreement is obtained between the present results with eight elements and the results of Vu-Quoc [10] using 10 elements. The results, obtained using four elements, are also shown in Fig. 6. The results with four elements provide a good fit to those with eight elements.

5.3. Flexible beam in free flight, subject to conservative force
We consider, once again, the problem discussed in Sec. 5.1, where the constant force and the constant moment are applied at the end of beam. As described earlier in this paper, the external moments generated by conservative forces are generally dependent on the deformation. Therefore, we investigate numerically the 'configuration dependency' of external moments. In this example, as shown in Fig. 7, only the external force is applied at one end so that the initial conditions are the same as those of the example in Sec. 5.1. Cases arise in which the point where the conservative force is applied does not lie in the cross-section of the beam. In such a case, the point may be imagined to be attached to a fictitious wall, which is fixed at the beam axis and moves rigidly with the beam.

As illustrated in Fig. 2, the external conservative force causes a configuration dependent moment as the beam deforms. In this example, the magnitude of external moment at the beam axis decreases due to the observed deformation. Consequently, the distinct difference in overall motions between the present example and that in Sec. 5.1 is observed in Fig. 8.

5.4. Right angle beam in free flight
This out-of-plane problem is solved for the first time in this paper. The material properties and the load conditions used are shown in Fig. 9. The forces $F_1$ and $F_2$ are applied at the beam axis, while the force $F_3$ is applied at a point away from the beam axis. As

--- Present
--- Vu-Quoc

Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$. 

--- Present
--- Vu-Quoc

Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$. 

--- Present
--- Vu-Quoc

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--- Vu-Quoc

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--- Present
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--- Vu-Quoc

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--- Vu-Quoc

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--- Vu-Quoc

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--- Vu-Quoc

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Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$. 

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--- Present
--- Vu-Quoc

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--- Present
--- Vu-Quoc

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--- Present
--- Vu-Quoc

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--- Present
--- Vu-Quoc

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--- Present
--- Vu-Quoc

Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$. 

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--- Vu-Quoc

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--- Present
--- Vu-Quoc

Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$. 

--- Present
--- Vu-Quoc

Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$. 

--- Present
--- Vu-Quoc

Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$. 

--- Present
--- Vu-Quoc

Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$. 

--- Present
--- Vu-Quoc

Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$. 

--- Present
--- Vu-Quoc

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--- Present
--- Vu-Quoc

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Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$. 

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Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$. 

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--- Vu-Quoc

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--- Present
--- Vu-Quoc

Fig. 4. Flexible beam in free flight, subject to constant force and constant moment. Comparison of the present results and those of Vu-Quoc. Time step $\Delta t = 0.1$.
mentioned in Sec. 5.3, if the point of application of $F_3$ does not lie in the cross-section of the beam, it may be imagined to lie on a rigid fictitious wall fixed at the beam axis.

We analyze the three examples in which the bending and the torsional rigidities are altered so that the behavior of the beam changes from a rigid to a highly flexible body. The overall deformations obtained using 10 elements are shown in Figs 10a-10c. At time $t = 0.0$, the transverse force and the torsional moment are applied at point A, as shown in Fig. 9. As the beam deforms, the bending moments, in addition to the loads described above, are applied at point A. To show the bending deformations due to these moments, the projections on the $Y' - Y^2$ plane, of deformations of beams, with lower rigidities, are shown in Fig. 11. Even after removal of the conservative forces, remarkable bending deformations due to the $F_3$, especially for the lowest rigidity, are observed.

As seen in this example, the total angle $\omega$ at each node exceeds $\pi$ rad. Although we do not employ the four quaternion, the large deformations with finite rotations can be simulated using the conformal rotation vector or the three rotation parameters.

Table 1 shows the Euclidean norm of residual forces in the case of the beam with the lowest rigidities at time $t = 2.0$. In the numerical results
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Fig. 7. Flexible beam in free flight, subject to conservative force. Problem data.

presented in this paper the convergence rate is quadratic. This is consistent with the basic characteristic of the Newton-Raphson method.

6. CONCLUSIONS

In nonlinear dynamic analysis of beams, a number of important problems remain to be resolved. In this paper, attention has been paid to develop the nonlinear elastodynamic theory of beams and to derive the consistent linearized forms of the discrete equations. With an emphasis on the definition of the external moments, we have shown the configuration dependency of external moments. In most of the structures encountered, the external forces do not act on the beam axis itself, but on the surface of the beam. This is the case in which the present beam theory is particularly applicable.

The rotation parameters \( \alpha' \) introduced herein lead to a symmetric tangent stiffness matrix and also to a symmetric load-stiffness matrix. The AMB conditions associated with \( \alpha' \) are different, but have forms equivalent to those derived from the static method. In the finite element formulation, the rotation parameters \( \alpha' \) have been defined as Lagrangian components of the conformal rotation vector. As a result, as shown in the numerical results, only three

Material Properties:

\[ \begin{align*}
EA &= G_0 = 10^5 \\
A \rho &= 1, J_{11} = J_{22} = 10 \\
\text{Case 1: } &E_{11} = E_{22} = GJ = 1000 \\
\text{Case 2: } &E_{11} = E_{22} = GJ = 200 \\
\text{Case 3: } &E_{11} = E_{22} = GJ = 100
\end{align*} \]

F.E. Mesh: 10 elements

Time history of loading:

\[ F_2 = F_0 \]

\[ F_1 = F_3 = F_0 / 5 \]

Fig. 8. Flexible beam in free flight, subject to conservative force. Comparison of the present problem and that in Sec. 5.1. Time step \( \Delta t = 0.1 \).

Table 1. Euclidean norm of residual forces

<table>
<thead>
<tr>
<th>Iteration number</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclidean norm of residual</td>
<td>( 0.21505 \times 10^4 )</td>
<td>( 0.46917 \times 10^3 )</td>
<td>( 0.20434 \times 10^1 )</td>
<td>( 0.33963 \times 10^{-3} )</td>
</tr>
</tbody>
</table>

Fig. 9. Right angle beam in free flight. Problem data.
Fig. 10a. Right angle beam in free flight. Sequence of motion in Case 1.

Fig. 10b. Right angle beam in free flight. Sequence of motion in Case 2.
Time step $\Delta t = 0.1$. 
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Fig. 10c. Right angle beam in free flight. Sequence of motion in Case 3. Time step $\Delta t = 0.1$. 
(parameters are enough to describe the finite rotations with a simple manipulation. The numerical results presented herein show the validity and the applicability of the present beam theory.

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9. H. M. Koh and R. B. Haber, Elastodynamic formu-
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Multi-Body Dynamics by the Finite Element Method in Time Domain

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Summary

This paper is an extension of [1], to the case of multiple rigid bodies undergoing large overall motion. The emphasis here, as in [1], is on direct formulation of weak solutions, consistent linearizations, and appropriate tangent matrices for multiple rigid body systems.

Multi-body dynamics can be formulated in two different ways. In the first the connectivities among the bodies can be taken into account by appropriate constraint equations via the Lagrangian multiplier technique. The use of this technique, obviously increases the number of the unknowns of the dynamics system; hence, for large problems, a second technique is sometimes preferred in order to minimize the number of the degrees of freedom.

This second formulation is only valid for systems with holonomic constraints and tree configurations. In this case one orders the tree. On the basis of this order each rigid body is a master of its follower and a slave of its predecessor. Then the absolute motion of the slave is resolved into the entailed motion with the master and into the relative motion with respect to it. Obviously this approach is more involved than the Lagrangian multipliers technique, but it is sometimes preferred not only because it involves fewer degrees of freedom, but also because the coordinates of the relative motion are very often closer to the physics of the system.

As to which technique is the best in a given situation, it clearly depends on the specific nature of the problem we are dealing with and on a general basis we can anticipate that the best performance can be obtained by a third formulation which is a convenient mixture of the previous two. In [2] the dynamics of a single rigid body with constraints has been formulated, and it is quite easy to extend it to the multiple-rigid body system. In this paper we focus the attention on the analytical developments involved in the second formulation.

Kinematics of Motion of Multiple Rigid Bodies

We consider two bodies, one a slave and the other its master. We ignore for the moment any constraints on the motion of either body, and resolve the kinematics of motion of the two bodies in the following way.

We use an absolute frame of reference at \( t = 0 \). Let the absolute position and orientation of the slave and the master, at time \( t \), be referred to the configuration at \( t = 0 \), be respectively denoted by \( (\mathbf{a}', \alpha') \) and \( (\mathbf{a}''', \alpha''') \) where \( \mathbf{a}' \) and \( \mathbf{a}''' \) are vectors, and \( \alpha' \) and \( \alpha''' \) are tensors. If \( b' \) and \( b'' \) are, respectively the relative position and orientation of the slave with respect to the master in the solution state at time \( t \), we have:

\[
\mathbf{a}' = b' + \mathbf{a'}_{m} \quad ; \quad \alpha' = b'' \cdot \alpha'''.
\]

In the context of a time-finite-element solution, we introduce a set of known or otherwise specified reference configurations, for the slave and for the master, which continually change with time. Let the specified configuration of the slave and the master, at a generic time \( t \), be referred to the time \( t = 0 \), be respectively denoted by the pairs \( (\mathbf{a}, \alpha) \) and \( (\mathbf{a}'', \alpha''') \). Note that \( t \), may in general be also equal \[1\]The authors gratefully acknowledge the support of this work by the USAFOR and SDIO/IST, as well as the encouragement of Dr. A. K. Amos.


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to t. If \( \theta \) and \( \beta \) are, respectively, the relative position and orientation of the slave with respect to the master in the specified configuration at time \( t \), we have:

\[
\begin{align*}
\alpha_s &= \theta + \alpha_m ; & \sigma_s &= \beta \cdot \alpha_m .
\end{align*}
\]

(2)

In the present time-finite-element-method, the solution state at time \( t \) is computed by proceeding from the known or specified state at \( t \), through a piecewise-linear Newton-Raphson method. In this sense, let \((u_s, \gamma_s)\) and \((u_m, \gamma_m)\) be the incremental displacement and rotation fields of the slave and of the master respectively, from the specified configuration at \( t \), to the solution state at \( t \). Thus we have:

\[
\begin{align*}
\alpha_s' &= \alpha_s + u_s ; & \sigma_s' &= \gamma_s \cdot \sigma_m ; & \alpha_m' &= \alpha_m + u_m ; & \sigma_m' &= \gamma_m \cdot \sigma_m .
\end{align*}
\]

(3)

Note that \( \alpha_s, \alpha_m, \sigma_s, \) and \( \sigma_m \) are, in general, functions of time, so that their time derivatives do not vanish; while, in as much they are specified, their variations are zero.

If, for the moment, the connectivity of the slave to the master is ignored, the weak form of the (linear and angular) momenta balance relations for the slave can be written as [1]:

\[
\int_0^t \delta \dot{\gamma}'_s : \mathcal{K}(\dot{X}') dt = \delta \gamma'_s : Q'_s (t),
\]

(4)

where \( \delta \dot{\gamma}'_s \) is the vector of appropriate generalized virtual displacements \( \delta A'_s \) and its time derivative \( \delta \dot{A}'_s \), for the slave, \( \mathcal{K} \) is in general a nonlinear function of generalized displacements and velocities \( X \), for the slave, and \( Q'_s \) are generalized momenta for the slave.

In order to account for the connectivity of the slave to the master, in the present approach, the generalized absolute velocities of the slave in the solution state at time \( t \) are referred, in a physically consistent fashion, to the configuration of the master in the specified (time-varying) configuration at time \( t \). Towards this end, the increments of the relative position (denoted by \( \delta \alpha \)) and rotation (denoted by \( \delta \theta \)) of the slave with respect to the master in the solution state at time \( t \), from those in the specified state at time \( t \), are defined such that:

\[
\begin{align*}
\alpha_s' &= \alpha_s + u_s ; & \sigma_s' &= \gamma_s \cdot \sigma_m ; & \alpha_m' &= \alpha_m + u_m ; & \sigma_m' &= \gamma_m \cdot \sigma_m .
\end{align*}
\]

(5)

and:

\[
\gamma_s' = \gamma_s' \cdot \gamma_m \cdot \gamma_s ; & \quad \gamma_m' = \gamma_m' \cdot (\gamma_s \cdot \beta) \cdot \gamma_m .
\]

(6)

It may be seen that \( u_s \) and \( \gamma_s \), may be viewed as the relative increments or "pull-back" increments of the relative position and orientation of the slave to the master in the solution state, as compared to the specified state, and they reduce respectively to zero and identity, whenever the slave is rigidly entrained by the master.

Henceforth, the rotation tensors \( \gamma_m \) and \( \gamma_s \) are given by the exponential representations, through the corresponding finite rotation vectors \( \phi_m \) and \( \phi_s \), i.e.

\[
\begin{align*}
\phi_m &= \exp(\phi_m \times I) ; & \phi_s &= \exp(\phi_s \times I).
\end{align*}
\]

(7)

In summary, the kinematics of the master and slave are represented by the generalized vectors \( U_m \) and \( U_s \), such that:

\[
\begin{align*}
U_m &= (u_m, \phi_m) ; & U_s &= (u_s, \phi_s).
\end{align*}
\]

(8)

We now turn to the absolute generalized velocities of the master and the slave. Let \( D \) denote the time derivative, then if:

\[
\begin{align*}
\alpha'_m &= (D \alpha'_m) ; & \sigma'_m &= \sigma_m \cdot (\gamma_s \cdot \beta) .
\end{align*}
\]

(9)

the absolute linear and angular velocities of the master are given by:

\[
\begin{align*}
\dot{V}_m &= (V'_m, \phi'_m) ;
\end{align*}
\]

(10)

where \( V'_m = D \alpha'_m \). The absolute linear and angular velocities of the slave are given by:

\[
\begin{align*}
\dot{V}_s &= \dot{V}_m + D' \equiv \dot{V}_m + V'_s + \phi'_s \times \dot{b} .
\end{align*}
\]

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where: \( \mathbf{V}' = D' \mathbf{V} \equiv D \mathbf{V} - \mathbf{V}_m \times \mathbf{V}' \), and:
\[
\mathbf{V}' = \mathbf{V}_m + \mathbf{V}'
\]
where: \( \mathbf{R} \times \mathbf{I} = (D^* \mathbf{R}) \mathbf{R}'^T \) and \( D' \mathbf{R} \equiv D \mathbf{R} - \mathbf{R}_m \times \mathbf{R}' + \mathbf{R} \times \mathbf{R}'_m \times \mathbf{I} \).
Finally, the generalized virtual displacements of the master and slave, from the solution state at \( t \), are considered as follows. For the master, the generalized virtual displacement vector is:
\[
\delta \mathbf{A}_m = (\delta \mathbf{u}_m, \delta \mathbf{q}_m) ; \delta \mathbf{q}'_m \times \mathbf{I} \mathbf{\Delta} \delta \mathbf{q}_m \cdot \mathbf{\Delta} \mathbf{q}_m^T
\]
since \( \delta \mathbf{u}_m \equiv 0 \). For the slave, the generalized virtual displacement vector is:
\[
\delta \mathbf{A}'_m = (\delta \mathbf{u}', \delta \mathbf{q}'_m) ; \delta \mathbf{q}'_m \times \mathbf{I} \mathbf{\Delta} \delta \mathbf{q}_m \cdot \mathbf{\Delta} \mathbf{q}_m^T
\]
while the variations of the corotational or "pull-back" increments of the position and orientation of the slave relative to the master in the solution state from those in the specified state, are represented as:
\[
\delta \mathbf{u}_m = \delta \mathbf{u}_m + \delta \mathbf{v}' \equiv \delta \mathbf{u}_m + \delta \mathbf{v}_m' + \delta \mathbf{v}'_m \times \mathbf{v}'
\]
Moreover we have:
\[
\delta \mathbf{v}_m' = \mathbf{\Delta} \delta \mathbf{u}_m \; ; \; \delta \mathbf{q}_m = \mathbf{\Delta} \delta \mathbf{q}_m
\]
where: \( \delta \mathbf{v}_m' \equiv \mathbf{\Delta} \mathbf{q}_m \mathbf{\Delta} \mathbf{v}_m' \) and:
\[
\delta \mathbf{q}_m = \mathbf{\Delta} \delta \mathbf{q}_m
\]
where: \( \delta \mathbf{v}_m' \times \mathbf{I} \mathbf{\Delta} \delta \mathbf{q}_m \mathbf{\Delta} \mathbf{v}_m' \) and: \( \delta \mathbf{q}' = \delta \mathbf{q}_m - \mathbf{\Delta} \delta \mathbf{q}_m \times \mathbf{\Delta} \mathbf{q}_m \times \mathbf{I} \).
It is very easy to show that the following relations hold:
\[
\delta \mathbf{v}_m' = \mathbf{\Delta} \delta \mathbf{u}_m \; ; \; \delta \mathbf{q}_m' = \mathbf{\Delta} \delta \mathbf{q}_m
\]
The equations of motions for the slave are pulled back to the specified reference configuration at \( t \), using:
\[
\delta \mathbf{a}_m = (\mathbf{\Delta} \delta \mathbf{u}_m, \mathbf{\Delta} \delta \mathbf{q}_m)
\]
as a primary virtual displacement and rotation in the variational formulation. This process along with the displacement resolution, entails the use of the following virtual displacement:
\[
\delta \mathbf{a}_m = (\mathbf{\Delta} \delta \mathbf{u}_m, \mathbf{\Delta} \delta \mathbf{q}_m)
\]
for the master, and:
\[
\delta \mathbf{a}_m = (\mathbf{\Delta} \delta \mathbf{u}_m, \mathbf{\Delta} \delta \mathbf{q}_m)
\]
Analysis of Traveling Wave Responses of Structures

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Summary

The prediction of transient response of structures, in the form of traveling waves, is very important for controlling the dynamic behavior of structures. It is well known that the standard semi-discrete form of the finite element method is not suitable for predicting the wave propagation, due to the inherent dispersion involved. In this paper, an application of space-time finite element method to the wave propagation problem is discussed. The main concerns in such problems consist of developing a consistent and stable scheme and also of capturing a shock wave, without wiggling. We discuss, at first, a weak form of the wave propagation problem, taking into account the jump condition associated with velocity and stress. A mixed finite element formulation plays an important role in evaluating the velocity explicitly. The application of present formulation to the linear wave equation shows that the present numerical results at the discontinuity give the mean values of jump. In the case of flexural wave propagations in Timoshenko beam, the present method captures the wave front easily rather than the semi-discrete method.

Space-Time Finite Element Formulation

Finite element methods have been used in the past to analyze wave propagation problems. Wellford and Oden (1976) have developed a finite element method to capture the shock front. This method is very powerful for an analysis of one-dimensional problems. It seems difficult, however, to apply this method to two- or three-dimensional problems. The "discontinuous" finite elements, developed by Lasaint and Raviart (1974), have been shown to be useful for discontinuous problems, such as those involving shock wave propagation. The drawback of this method is the increase of number of unknowns. One of the objectives of this paper is to develop a finite element formulation which predicts the propagation of wave fronts in beam or shell structures without increasing the computational cost.

A space-time finite element method is used for an analysis of shock wave propagation problems because of its accuracy and stability. In most of the existing space-time finite elements, the velocity field is evaluated by using a higher-order shape function (Riff and Baruch, 1984) or using a finite difference method (Cella, Lucchesi, and Pasquinelli 1980). As a result, the computational cost increases. It is one of the key issues for the space-time finite element method to evaluate the velocity accurately. In this paper,
A mixed finite element formulation is introduced to improve the accuracy of velocity obtained. In the case of the shock wave propagation problems, we have to note that the derivatives of displacement with respect to space and time may be discontinuous, while that the displacement itself remain continuous. This fact motivates us to introduce the strain and the velocity as independent values which are discontinuous across elements.

We consider, at first, the linear equation governing longitudinal waves in rods:

\[ N_x - \rho A u_t + f = 0, \quad N = E A \varepsilon, \quad \varepsilon = u_x \text{ in } \Omega \]
\[ u = 0 \text{ on } S_u, \quad N = N_0 \text{ on } S_n, \quad \rho A u_t = p \text{ at } t = t_0. \tag{1} \]

where \( N \) is the axial stress, \( \rho \) the mass density, \( f \) the body force, \( u \) the axial displacement, \( A \) the area of cross section, \( \varepsilon \) the axial strain and \( E \) the Young modulus; \( u \) and \( N \) denote the values on the boundaries \( S_u \) and \( S_n \) and \( p \) indicates the momentum at \( t = t_0 \). The derivatives with respect to space "x" and time "t" are denoted by \( (\cdot)_x \) and \( (\cdot)_t \), respectively. The jump condition for this problem is written as

\[ \rho A V || u || + || N || = 0 \text{ on } \Sigma \tag{2} \]

where \( V \) is the velocity of the wave front and \( || \cdot || \) is the jump of \( \cdot \) at \( \Sigma \). Let \( v \) be the velocity defined by \( v = u \). We consider a rectangular element with discontinuities on \( u \) and \( u_x \), along \( \Sigma \) as shown in Fig. 1. We assume that the boundary condition at \( S_u \) is satisfied a priori. Then the weak form for this problem is expressed by

\[
\int (E A u_x - \rho A u_t + f) u_x - \rho A (v - u) v + EA (\varepsilon - u_x) \varepsilon \, d\Omega - \int (E A N) u_x \, d\Sigma = 0
\]

\[
+ \int (\rho A v - p) u_t \, dx - \int_c (\rho A || v || u_t dx + EA || \varepsilon || dt) = 0 \tag{3}
\]

where \( (\cdot) \) denotes the test function. Integrating by parts the terms marked in Eq. (3) leads to the following weak form:

\[
\int (-E A u_x + \rho A v_t + f v - \rho A (v - u) v + EA (\varepsilon - u_x) \varepsilon \, d\Omega
\]

\[
+ \int [N u_x]_t \, dt - \int [p u]_t \, dx = 0 \tag{4}
\]

We note that the velocity \( v \) and the strain \( \varepsilon \) appear only in the domain integral. Therefore these values may be chosen to be discontinuous across finite elements. This choice is advantageous for an analysis of shock wave problems. In this formulation, unlike the conventional method, we assume that the test function \( \tilde{u} \) at \( t = t_0 \), and \( t = t_f \) do not vanish (see Borri et al. 1985 and references cited there). As a result, the number of equations is the same as that of unknowns; the unknown tractions and momenta at the boundaries are obtained from Eq. (4) directly. It is easily shown from Eq. (4) that the velocity and the strain are expressed in terms of the displacements since the matrices associated with \( v \) and \( \varepsilon \) are positive definite. As a result the size of the matrix in the formulation is the same as that in the standard displacement based formulation.

As numerical examples, we consider the cases of wave propagation in which stresses and velocities are discontinuous. A Galerkin method with exact integration is used to formulate the present finite elements. The space-time element is rectangular in shape.
The axial displacement $u$ is assumed to be a bilinear function of $x$ and $t$. The strain $e$ and the velocity $v$ are assumed constant; the tractions $N$ and the momentum $p$ are assumed as linear functions of $t$ and $x$, respectively. Simple calculations show that the strain and the velocity chosen above present the approximated values at the center of element. The responses of the rod subjected to the initial deflection are shown in Fig. 2. The same problem has been solved with the use of the space-time finite elements developed by Riff and Baruch (1984). They have pointed out that the ratio between the space increment and the time increment should be equal to the velocity of propagation of the longitudinal wave. Their numerical results for displacements, which are always continuous, have shown some wiggles. The present results shown in Fig. 2, however, give a good agreement with exact solutions even for the discontinuous stresses. Fig. 3 shows the responses of the rod subject to a step force at the tip. A good agreement between the present results and the exact ones has been obtained. Both of these examples show that, at the discontinuity, the present finite element predicts the mean values of jump.

Next we consider the problem of transverse wave propagation in a Timoshenko beam. At first, the static problems are solved using the present finite element formulation with exact integration. Table 1 shows the displacements of slender beams subjected to the point load. In both cases of a cantilevered beam and a fixed-fixed beam, no shear locking is observed. The dynamic response of beams subjected to a step load are shown in Fig. 4. The numerical results obtained from the semi-discrete method using the Newmark $\beta$ method are also shown in Fig. 4. The wave fronts are captured by the present finite element method more easily than by the semi-discrete method.

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References


Dynamic Analysis of Finitely Stretched and Rotated 3-D Space-Curved Beams

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Summary
A novel theory and its computational implementation are presented for the analysis of strongly nonlinear dynamic response of highly-flexible space-beams that undergo large overall motions as well as elastic motions with arbitrarily large rotations and stretches. The case of conservative force loading, which may also lead to configuration-dependent moments on the beam, is treated. A symmetric tangent stiffness matrix is derived at all times even if the distributed external moments exist. An example of transient dynamic response of the beam is presented to illustrate the validity of the theoretical methodology developed herein.

Transient Analysis of Space Beams
In most of the existing papers, so far as the conservative loading is concerned, the existence of constant external moments has been assumed a priori. Argyris, Dunne and Scharpf [1] and Iura and Atluri [3], however, have made the point that the external moments generated by the conservative forces are generally configuration dependent. Therefore, the external virtual work associated with the moments does not, on first sight, appear to correspond to the first variation of an external energy functional. Argyris, Dunne and Scharpf [1] have derived a nonsymmetric tangent stiffness matrix at the element level using the rotational degrees of freedom referred to fixed axes of a global cartesian system. Simo and Vu-Quoc [5] have concluded that, using the variation of rotational variable introduced by Atluri [2], the tangent stiffness matrix becomes symmetric at only an equilibrium configuration, provided that no distributed external moments exist. This lack of symmetry [1] and the recovery of symmetry at only an equilibrium configuration [5] have been attributed to the fact that the finite rotation field is noncommutative. It is shown herein, on the other hand, that the use of the present rotational variables leads to the symmetric tangent stiffness matrix at not only the equilibrium but also at the nonequilibrium configuration, even if the distributed external moments exist. The rotational variables employed herein are defined simply as any three independent components of a finite rotation tensor. It should be emphasized that the present rotation field remains noncommutative.

The kinematic relations of the present beam model have been derived on the basis of the following assumptions: (1) The plane cross sections of the beam remain plane and do not change any shape-change during the deformation; (2) The cross sections
are constant along the beam axis, which remains a smooth space-curve through the deformation. The equilibrium equations and the associated boundary conditions are obtained by employing the principle of virtual work for the elastodynamic problem. It is well known that the displacement field and the variations of variables determine whether the energy method yields the same equilibrium equations as those derived by the static method. When a tensor $\delta \mathbf{R} \cdot \mathbf{R}^T (= \delta \phi \mathbf{I})$, in which $\mathbf{R}$ denotes a rotation tensor and $\mathbf{I}$ an identity tensor, is introduced as a rotational variation (Atluri [2]), the angular momentum balance (AMB) condition takes the same form as that derived by the static method. The use of $\delta \mathbf{R} \cdot \mathbf{R}^T$ or $\delta \phi$, however, may yield the nonsymmetric tangent stiffness matrix except at the equilibrium configuration (Simo and Vu-Quoc [5]). Furthermore, as long as $\delta \mathbf{R} \cdot \mathbf{R}^T$ or $\delta \phi$ are used as a rotational variation, the tangent stiffness matrix becomes nonsymmetric when a prescribed moment vector generated by conservative forces is applied to a beam. On the other hand, as discussed by Iura and Atluri [3], the rotational variables $\alpha'$, defined as the Lagrangian components of $\mathbf{R}$, lead to a symmetric tangent stiffness matrix at all times, even if the prescribed moment vectors exist. Depending on the nature of rotation parameters, the AMB condition takes on different but equivalent forms. To show the equivalence, we consider the tensor equation for the AMB condition. The AMB condition associated with $\delta \mathbf{R} \cdot \mathbf{R}^T$ or $\delta \phi$ is written as (Iura and Atluri [4])

$$\{ \mathbf{M}_s + [\mathbf{z} + \mathbf{u}]_s \times \mathbf{T} + m - \mathbf{H}_s \} \cdot \delta \phi = C : (\delta \mathbf{R} \cdot \mathbf{R}^T) = 0$$

where $\mathbf{T}$ and $\mathbf{M}$ denote the stress resultant and moment vector, respectively, and $\mathbf{z}$ and $\mathbf{u}$ denote the undeformed position and the displacement vector at the beam axis, respectively; $m$ is the external moment vector, $\mathbf{H}$ the inertia term and $(\ldots)_s$ the differentiation with respect to the arc length along the beam axis. When using $\delta \alpha'$ in place of $\delta \mathbf{R} \cdot \mathbf{R}^T$, the resulting AMB condition takes the following form: $C : (\mathbf{R}_s \cdot \mathbf{R}^T) \delta \alpha' = 0$, where $(\ldots)_s$ denotes the differentiation with respect to $\alpha'$. Since both of $\delta \mathbf{R} \cdot \mathbf{R}^T$ and $\mathbf{R}_s \cdot \mathbf{R}^T$ are the skew-symmetric tensors, the corresponding AMB conditions are shown equivalent each other.

With the help of the rotational parameters $\alpha'$, the Hamilton principle, as a basis for the finite element formulation, is written as

$$\delta \int_0^L [T - \Pi] dt = 0,$$

$$\Pi = \int_0^L [W_s - q \cdot u - m_{\alpha'} R_{\alpha}] dL - m_{\alpha'} \mathbf{E}$$

where $T$ is the kinetic energy, $\Pi$ the potential energy, $W_s$ the strain energy function, $R_{\alpha}$ the Lagrangian component of $\mathbf{R}(\alpha')$; $q$ and $\mathbf{u}$ are external force vectors, and $m_{\alpha'}$ and $m_{\alpha'}$ the components of external moment vectors. The virtual work of external moments per unit length is written as $m \cdot \delta \phi$ where $m$ is a configuration dependent moment vector defined by $m_{\alpha'} \mathbf{E}_x \mathbf{E}_y$ in which $\mathbf{E}_x$ and $\mathbf{E}_y$ are the undeformed and deformed base vectors, respectively, at the beam axis. In a similar way, the virtual work for external moments at end cross section is expressed by $\mathbf{m} \cdot \delta \phi$. Since $\mathbf{m}$ is configuration dependent, the variation of inner product $m \cdot \phi$ does not correspond to $m \cdot \delta \phi$. It is shown, after some manipulations, that the inner product $m \cdot \delta \phi$ is rewritten as $m_{\alpha'} \delta R_{\alpha}$. Since $m_{\alpha'}$ is constant, the well-defined functional described above is derived through the use of $\alpha'$. 
Since no 3-parameter representation of \( \mathbb{R} \) can be both global and nonsingular, the "four-quaternion" has been introduced (Simo and Vu-Quoc [5]). Consequently the total number of degrees of freedom increases. In this paper, to avoid using the four rotation parameters per node, the rotation variables \( \theta \) are taken as the Lagrangian components of conformal rotation vector. The singularity associated with the finite rotation vector is avoidable with the simple manipulation.

Following a standard finite element discretization, we obtain the following semidiscrete equations of motion:

\[
M(\ddot{d}, \dot{r}, \tau) + C(\dot{r}, \tau) + K(d, \tau) = f(\tau)
\]

where \( d \) and \( r \) are the vectors of nodal displacement and rotational variables respectively; \( M \) is linear w. r. t. \( \dot{d} \) but nonlinear w. r. t \( \ddot{d} \) and \( r \), and \( C \), \( K \) and \( f \) are nonlinear w. r. t. their variables. Note that the vector \( C \) is derived not from the damping effects but from the nonlinear effects of finite rotations, and that no simplification is made in the present formulation in the sense that Coriolis and centrifugal effects as well as the inertia effects due to rotation are accounted for. The semidiscrete equations are integrated by the Newmark algorithm \((\beta = \frac{1}{4} \text{ and } \gamma = \frac{1}{4})\). Consistent linearization procedures are employed to obtain linearized forms of the balance equations. A full Newton-Raphson method is used in the present calculations. As a numerical example, a flexible right-angle beam in free flight is simulated with the use of a linear shape function. As shown in Fig. 1, the large deformations with finite rotations can be simulated with the help of the three rotation parameters per node.

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Large Deformation, Post-Buckling Analyses of Large Space Frame Structures, Using Explicitly Derived Tangent Stiffness Matrices

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Introduction

Large deformation and post-buckling analyses of structures have been studied by many researchers as an important subject in structural mechanics in the past decade or so. In all their studies, an incremental approach, either of the total Lagrange type or the updated Lagrange type, is employed. As the incremental approach is often based on the so-called tangent stiffness matrix, which reflects all the non-linear geometrical and mechanical effects, the majority of non-linear analyses of typical engineering structures, and especially truss- and frame-type large space structures, will be vastly simplified if an explicit expression (i.e., without involving assumed basis functions for displacements/stresses, and without involving element-wise numerical integrations) for the tangent stiffness matrix of an element can be derived.

Toward this end, the authors have recently proposed a method for explicitly deriving the tangent stiffness matrix of the truss- and the frame-type structures, and have demonstrated that this procedure is not only inexpensive but also highly accurate in a wide variety of the problem involving very large deformations and highly non-linear pre- and post-buckling responses1)-3).

In this paper, we present simplified procedures for the large deformation and post-buckling analysis of space frame structures.

Three-dimensional kinematics of deformation of a element

Consider a typical frame member, modeled here as a three-dimensional beam element, that spans between nodes 1 and 2 as shown in Fig.1. This element is assumed to have a uniform cross-section and to be of length $l$ before deformation. The coordinates $x_i$ are
the local coordinates, and \( u_i \) and \( \theta_i \) \((i = 1, 2, 3)\) denote the displacements at the centroidal axis of a element along the axes of \( x_i \) and the rotations about the axes of \( x_i \), respectively. After deformation, two other coordinate systems are introduced to represent the rigid and relative rotations of the element.

One is the coordinate system \( x_i^* \), which are locally tangential and normal to the deformed centroidal axis, and the other is \( x_i^\dagger \) which characterize the rigid motions of the element. Furthermore, unit vectors along the axes of \( x_i \), \( x_i^* \) and \( x_i^\dagger \) are described as \( e_i \), \( e_i^* \) and \( e_i^\dagger \) \((i = 1, 2, 3)\), respectively.

Considering each rotation as a semi-tangential rotation, we can treat rotations as vectors and obtain the following relations:

\[
\begin{align*}
\alpha_i^* &= \left( 1 - \alpha_i^* \right) e_i + 2 (\alpha_i^* : e_i) e_i + 2 (\alpha_i^* : e_i) \right) / (1 + \alpha_i^*)(1) \\
\alpha_i^\dagger &= \left( \alpha_i^* : e_3^\dagger - \alpha_i^* : e_2^\dagger \right) e_1^\dagger + \left( \alpha_i^* : e_1^\dagger - \alpha_i^* : e_3^\dagger \right) e_2^\dagger + \left( \alpha_i^* : e_2^\dagger - \alpha_i^* : e_1^\dagger \right) e_3^\dagger / (1 + \alpha_i^* (1) \\
\end{align*}
\]

where

\[
\begin{align*}
\alpha_i^* &= \sum_{i=1}^{3} \frac{\alpha_i^*}{\tan(\alpha_i^* / 2)} e_i \quad (3.a,b) \\
\end{align*}
\]

and \( \alpha_i^\dagger \) \((i = 1, 2, 3)\) are relative rotations about the axes of \( x_i^\dagger \).

Also, ' : ' and ' x ' denote inner and outer products, respectively, and ' \( \alpha() \) ' the value of ( ) at node \( \alpha (\alpha = 1, 2) \).

\( e_i \) may be, \( e_j \) being set to be zero, represented as \(3), 6)\):

\[
\begin{align*}
\hat{e}_3 &= (\hat{u}_1 / t^\dagger) e_1 + (\hat{u}_2 / t^\dagger) e_2 + ((1 + \hat{u}_3) / t^\dagger) e_3 \\
\hat{e}_j &= ((1 - \hat{\theta} : \hat{\theta}) e_j^* - 2 (\hat{\theta} : e_j^*) \hat{\theta} - 2 (\hat{\theta} x e_j^*) \hat{\theta}) / (1 + \hat{\theta} : \hat{\theta}) \\
\end{align*}
\]

where

\[
\begin{align*}
\hat{u}_1 &= 2 u_1 - u_1, \quad t^\dagger = \left[ \hat{u}_1^2 + \hat{u}_2^2 + (1 + \hat{u}_3)^2 \right]^{1/2} \quad (5.a,b) \\
\end{align*}
\]

and

\[
\text{Eqs. ( which are the element) }
\]

On the

Relation force as:

The position of the element axis, \( \alpha \)

Also, \( \alpha \)

where \( \alpha \), \( GJ \) and \( \alpha \) note the element about the axis, \( \alpha \)

Also,

Using

obtain

and

\[
\text{6/1)
\]

where
Eqs. (1)-(6) may offer the relations between \( \dot{\theta}_1 \) and \( \dot{\theta}_2 \). which are valid in the presence of arbitrarily large motions of the element.

On the other hand, total axial stretch of the beam \( \delta \) is given by

\[
\delta = \ell^* - \ell
\]

Relation between (stretch and relative rotations) and (axial force and bending moments)

The equilibrium equations and the boundary conditions of the beam, as shown in Fig. 1, may be, under the assumption of \( \dot{\theta}_1 \) to be pretty small, written as:

\[
EI_j \left( \frac{d^2 \delta_j}{dx^2} \right) - \left( \frac{^2}{2} \right) \frac{\delta_j}{\ell} - N \delta_j = 0
\]

\[
GJ \left( \frac{d^2 \delta_j}{dx^2} \right) = 0 \quad (j = 1, 2)
\]

and

\[
EI_j \left( \frac{d^2 \delta_j}{dx^2} \right) \mid_{x_j = \ell} = \frac{\dot{\theta}_j}{\ell} \quad (j = 1, 2)
\]

where \( (EI_1, EI_2) \) are the bending stiffness about \((x_1, x_2)\) axes and \( GJ \) is the torsional stiffness. Also, \( (\dot{\nu}_1, \dot{\nu}_2), \dot{\nu}_3 \) and \( \dot{N} \) denote the bending moments about \((x_1, x_2)\) axes, the twisting moment about \( x_3 \) axis and the axial force in the direction of \( x_3 \) axis, respectively.

Also, the total stretch of the element becomes to be

\[
\delta = -(1/2) \int_0^\ell (\dot{\delta}_1^* + \dot{\delta}_2^*) dx_3 + \frac{N}{EA} \ell
\]

where \( EA \) is the axial stiffness.

Using the solutions of Eqs. (8.a,b) and (9.a-c), one might obtain the following relations.

\[
\dot{\theta}_j = \left( \frac{1}{\dot{\nu}_j} \right) \dot{\delta}_j^* \quad \dot{\nu}_j = -\left( \frac{1}{\dot{\nu}_j} \right) \dot{\delta}_j^*
\]

\[
\dot{\nu}_1 = \left( GJ/\ell \right) \dot{\delta}_1^* \quad (j = 1, 2)
\]

and

\[
\delta = \left( \frac{1}{2} \right) \sum_{j=1}^{2} \left[ \left( \dot{\delta}_j^* \right)^2 \left( \frac{d^2 \delta_j}{dx^2} \right) + \left( \dot{\nu}_j^* \dot{\nu}_j \right) \left( \frac{d^2 \nu_j}{dx^2} \right) \right]
\]

\[
+ \left( \frac{1}{EA} \right) \dot{N}
\]

where

\[
\dot{\delta}_j^* = \left( \dot{\delta}_j^* + \dot{\delta}_j^* \right)/2, \quad \dot{\nu}_j^* = \left( \dot{\nu}_j^* - \dot{\nu}_j^* \right)/2
\]
\[ a_{m_j} = 2m_j - l_m^2, \quad b_{m_j} = 2m_j + l_m^2 \quad (13.a-d) \]

and
\[ h_j = (t/E_{1j})N, \quad m_j = (t/E_{1j})M_{j} (j=1,2) \quad (14.a,b) \]

and \((a_{h_1}, s_{h_1})\) and \((a_{h_2}, s_{h_2})\), which are flexural coefficients and their full descriptions are given in Refs. 2), 3) and 7), are the highly non-linear functions of only the axial force.

It should be noted that the present relations for each element account, as in the Von Karman plate theory, the non-linear coupling between the bending and the stretching deformations.

**Tangent stiffness matrix of a element**

Using the well-known concept of a Legendre contact transformation\(^8\) for the strain energy due to stretching, the internal energy in the element becomes to, in mixed form, be:

\[ n^m = \left( \frac{1}{2} \right) \sum_{j=1}^{n} \left( (\frac{a_{h_j}}{s_{h_j}})^2 + (\frac{b_{h_j}}{s_{h_j}})^2 \right) + \left( \frac{1}{2} \right) (G/E_A) \frac{d^2}{}^2 \]

\[ + \left[ N \cdot 6 \left( \frac{1}{2} \right) (4/EA) N^2 \right] \quad (15) \]

Note that the condition of vanishing of the first variation of \(n^m\) due to a variation in \(N\) leads to Eq. (12).

It should be emphasized that \(n^m\) is explicitly expressed and required no numerical integrations, and this might lead to be distinctly advantage for saving the computing time.

Re-forming Eq. (15) into the incremental form, neglecting the terms of higher than the second-order in the usual fashion, and furthermore, eliminating the incremental axial force \(AN\) from the incremental functional, wherein the stationary condition with respect to \(AN\) is employed, we may obtain the tangent stiffness equation of the element.

**References**

Time-Finite Element Method for the Constrained Dynamics of a Rigid Body

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Summary

Currently, there is a renewed interest in the study of multi-body-dynamics and its application in many fields of engineering. The mathematical model of a rigid body is useful whenever the overall motion, involving large rigid rotation, is of interest. The nonlinear dynamic equations of motion, in their explicit form, appear quite complex due to the expression for the absolute accelerations.

In this paper weak formulations of linear and angular momentum balance laws of a rigid body undergoing large overall motion are stated a priori. Holonomic as well nonholonomic constraints, that may exist on the motion of the rigid body, are introduced into this weak form in a fundamentally novel fashion here. Comments are made on the incremental form [and consistent linearization] of the weak formulation (with constraints), and the time-finite-element solutions thereof.

A Free Rigid Body

In order to define the configuration of the rigid body, we consider two right orthogonal frames of reference \((O, E)\) and \((P, e)\), the first of which is considered as fixed, while the second is embedded in the rigid body. At a given time instant the configuration is defined by the position vector \(r' = P' - O\) and by the rotation tensor \(a'\) so that \(e' = a' \cdot E\).

Let \(Q'\) and \(A'\) be, respectively, the linear and angular momenta, and let \(F'\) and \(M'\) denote, respectively, the external forces and moment resultants. Let \(D\) be the time derivative, and \(V'\) the velocity of \(P'\), so that: \(V' = D\mathbf{a'}\). Moreover let \(D''v = Dv - \dot{v} \times v\) be the corotational time derivative for any vector \(v\). Where \(\dot{v}''\) is the vector of angular velocity of the rigid body, defined through the relation \(D\mathbf{a'} \cdot \dot{\mathbf{a'}} = -\mathbf{a'} \cdot D\mathbf{a'} + \dot{v}'' \times I\), \(I\) being the identity. Let \(\delta\mathbf{a'}\) represent a virtual translation of the rigid body and \(\delta\mathbf{a'}\) a vector of virtual rotation defined through the relation \(\delta\mathbf{a'} \cdot \mathbf{a'} = -\mathbf{a'} \cdot \delta\mathbf{a'} + \delta\mathbf{a'} \times I\).

It can be easily shown that a variational form of the linear and angular momentum balance laws can be written as:

\[
\delta\mathbf{a'} \cdot (D\mathbf{Q}' - F') + \delta \mathbf{\dot{Q}}' \cdot (D\mathbf{a'} + V' \times Q' - M') = 0,
\]

or:

\[
\delta\mathbf{a'} \cdot (D''\mathbf{Q}' + \dot{\mathbf{a'}} \times Q' - F') + \delta \mathbf{\dot{Q}}' \cdot (D''\mathbf{a'} + \dot{\mathbf{a'}} \times a' + V' \times Q' - M') = 0.
\]

Equations 1 and 2 can also be rearranged in the following form:

\[
\delta''\mathbf{V}' \cdot \mathbf{Q}' + \delta''\mathbf{\dot{a'}} \cdot \mathbf{a'} + \delta\mathbf{a'} \cdot \mathbf{F}' + \delta \mathbf{\dot{a'}} \cdot \mathbf{M'} = D(\delta\mathbf{a'} \cdot \mathbf{Q}' + \delta \mathbf{\dot{a'}} \cdot \mathbf{a'}),
\]

where \(\delta''\mathbf{V}'\) and \(\delta''\mathbf{\dot{a'}}\) are defined as:

\[
\delta''\mathbf{V}' = \delta\mathbf{a'} (\overline{\mathbf{a'}} \cdot \mathbf{V}') = (D\mathbf{a'} + V' \times \mathbf{\dot{a'}}) \delta\mathbf{a'} + \mathbf{\dot{a'}} \times \delta\mathbf{a'},
\]

\[
\delta''\mathbf{\dot{a'}} = \mathbf{a'} \delta (\mathbf{a'} \cdot \mathbf{\dot{a'}}) = D\mathbf{\dot{a'}} = D''\mathbf{\dot{a'}} + \mathbf{\dot{a'}} \times \delta\mathbf{\dot{a'}}.
\]

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to eq. 7 the following variational terms:

$$\delta A' \cdot \frac{\partial \Psi'_t}{\partial V} \cdot D\lambda'_t - \Psi'_t \cdot \delta \lambda'_t = D(\delta A' \cdot \frac{\partial \Psi'_t}{\partial V} \cdot \lambda'_t) - \delta(\Psi'_t \cdot \lambda'_t) - \delta A' \cdot \mathcal{F},$$

where \( \lambda'_t \) are Lagrangian multipliers. These additional terms can be rearranged in the following way:

$$\delta A' \cdot \frac{\partial \Psi'_t}{\partial V} \cdot D\lambda'_t - \Psi'_t \cdot \delta \lambda'_t = \delta(\Psi'_t \cdot \lambda'_t) - \delta A' \cdot \mathcal{F}.$$  

The term \( \mathcal{F} = \{ F'_t, M'_t \} \) constitutes the generalised constraint reactions which enforce the integrability conditions of the constraint equations. Then, adding eq. 7 and 13, the variational principle for the constrained motion can then be written as:

$$\delta A' \cdot \frac{\partial \Psi'_t}{\partial V} \cdot D\lambda'_t + \Psi'_t \cdot \delta \lambda'_t + \delta A' \cdot \{ F' + \mathcal{F} \} = D(\delta A' \cdot \mathcal{Q}'),$$

or:

$$D''\delta A' \cdot \mathcal{Q}' + \Psi'_t \delta \lambda'_t + \delta A' \cdot \{ F' + (D''\delta A' \frac{\partial \Psi'_t}{\partial V}) \lambda'_t + \mathcal{K}' \mathcal{Q}' \} = D(\delta A' \cdot \mathcal{Q}'),$$

where \( \mathcal{Q}' \) are the new generalised momenta with the expressions:

$$\mathcal{Q}' = \mathcal{Q}' + \mathcal{Q}'_v \quad ; \quad \mathcal{Q}'_v = \frac{\partial \Psi'_t}{\partial V} \lambda'_t,$$

and the quantities \( \mathcal{Q}_v = \{ Q_v, L_v \} \) can be interpreted as the generalised momenta associated with the constraint equations. The term \( \mathcal{F} = \{ F'_t, M'_t \} \) is typical of non-holonomic constraints, since it is not identically for holonomic constraints. In fact, for holonomic constraint we have:

$$\delta \mathcal{F} = \delta \mathcal{F}_v + \delta \mathcal{F}_s,$$

and the virtual work of the constraint reactions becomes:

$$\delta A' \cdot \mathcal{F} = \delta(\delta A' \cdot \mathcal{Q}') = D(\delta A' \cdot \mathcal{Q}'),$$

The above quantity is obviously zero because the operations of time derivative and virtual variation commute. In the general case, however, \( \mathcal{F} \) is different from zero, and in order to derive its expression we write the constraint equations in the following way:

$$\Psi'_v = \mathcal{V}' \cdot C'_v + c'_v,$$

The corotational time derivative and virtual change of the vector \( C'_v \) are:

$$D''\mathcal{C}'_v = B'_v \mathcal{V}' + k'_v \quad ; \quad \delta \mathcal{C}'_v = B'_v \delta \mathcal{V}' + \delta k'_v.$$

It can be demonstrated that \( \mathcal{F} \) can be expressed as:

$$\mathcal{F} = \mathcal{K}_v \mathcal{V}' + \mathcal{H}_v,$$

where:

$$\mathcal{V}' = \mathcal{V}' - \mathcal{K}_v (B'_v - B'_v \cdot B'_v) \cdot \mathcal{V}' \quad ; \quad h_v = (B'_v - B'_v \cdot B'_v) \lambda'_v,$$

and:

$$\mathcal{K}_v = -\mathcal{K}' = \begin{bmatrix} 0 & Q'_v \times I \\ Q'_v \times I & A'_v \times I \end{bmatrix}.$$
We note that the matrix $\mathbf{N}$ is skew-symmetric so that the power of the non-holonomic constraint reactions is simply given by:

$$\mathbf{V'} \cdot \mathbf{F} = \mathbf{V'} \cdot \mathbf{N},$$

and it is zero when the constraint equations do not depend explicitly on time.

One of the most important aspects of the present formulation is that the new momenta $\mathbf{Q}$ can be considered as independent quantities, while the momenta $\mathbf{Q'}$ are obviously constrained by the constraint equations. In order to show this fact, let us write the constraint equation in the form:

$$(\mathbf{V'} - \mathbf{V'}') \cdot \mathbf{C'} = 0,$$

where the matrix $\mathbf{C'} = C'_{1} \ldots C'_{n}$ and $\mathbf{V'}_{i} \cdot \mathbf{C'} = - \mathbf{c}'$. Since the constraint equations are independent, the Jacobian matrix $\mathbf{C'}$ has full column rank, so that we can set: $\mathbf{V'}_{i} = - \mathbf{C'}(\mathbf{C'}^{-1} \mathbf{C'}')^{-1} \mathbf{c}'$ where: $\mathbf{c}' = c'_{1} \ldots c'_{n}$. Moreover let $\lambda'$ denote the vector of the Lagrangian multipliers. The constraint equations in terms of the new momenta $\mathbf{Q}$ can be written as:

$$\mathbf{C'} \cdot \mathbf{M}^{-1} \cdot (\mathbf{Q'} - \mathbf{Q'}) = 0 \quad ; \quad Q'_{i} = \mathbf{c}'_{i} \mathbf{V'}_{i}.$$  

This equation can be solved for $\lambda$ and $\mathbf{Q}'$, and we obtain:

$$\lambda = R' \cdot (\mathbf{Q'} - \mathbf{Q'}') \quad ; \quad Q'_{i} - Q'_{j} = (I - P') \cdot (\mathbf{Q'} - \mathbf{Q'}'),$$

where $R'$ and $P'$ are defined as:

$$R' = (C' \cdot \mathbf{M}^{-1} \cdot C')^{-1} \cdot C' \cdot R' = C' \cdot (C' \cdot \mathbf{M}^{-1} \cdot C')^{-1} \cdot C' \cdot \mathbf{M}^{-1}.$$  

We note that $P'$ is a projection: $P'^{2} = P'$. Moreover $P' \cdot C' = C'$ and, since $\mathbf{M}$ is symmetric, it is easy to check the following property: $\mathbf{M}^{-1} \cdot P' = P' \cdot \mathbf{M}$. The generalized velocity $\mathbf{V}'$ compatible with the constraints can then be obtained by:

$$\mathbf{V'} - \mathbf{V'}' = \mathbf{M}^{-1} \cdot (\mathbf{Q'} - \mathbf{Q'}') = \mathbf{M}^{-1} \cdot (I - P') \cdot (\mathbf{Q'} - \mathbf{Q'}') = (I - P') \cdot \mathbf{M}^{-1} \cdot (\mathbf{Q'} - \mathbf{Q'}')$$

and it is very easy to demonstrate that $\mathbf{V}'$ verify the constraint equations irrespectively of $\mathbf{Q'}$.

**Linearization and Finite Element Approximation**

The variational formulation involves the configuration parameters in a nonlinear form, hence an incremental approach, such as Newton or quasi-Newton, is required in order to solve a specific problem. Let $\alpha$ and $\beta$ specify a known time-variant reference state and $\alpha$ and $\beta$ represent respectively the relative displacement and (finite) rotation vectors of the actual configuration with respect to the reference one, so that:

$$\alpha' = \alpha + \alpha \quad ; \quad \beta' = \beta = \delta \phi \quad ; \quad \delta \alpha' = \delta \beta = \delta \phi,$$

and $\phi = \exp \phi$. The vector $\mathbf{U} = (u, \phi)$ constitutes the set of the Lagrangian coordinates for the incremental formulation. The equations of motion are pulled back to the reference configuration using $\beta' = \gamma' \cdot \beta$ and $\delta \phi = \gamma' \cdot \delta \phi$ as the primary virtual displacement and rotation in the variational formulation. The integration of eq. 15 over the time interval of interest gives:

$$\int_{t_{0}}^{t_{1}} |D'u \cdot \mathbf{N} + \Phi'_{i} \delta u_{i} + \delta \mathbf{u} \cdot \mathbf{N} + (D'u \cdot \Phi')_{i} \delta \mathbf{u}_{i} - \mathbf{X}' \cdot \mathbf{Q}'||dt = \delta \mathbf{u} \cdot \mathbf{N},$$

which can be rewritten more compactly in the following way:

$$\int_{t_{0}}^{t_{1}} \delta \mathbf{Y} \cdot \mathbf{N}(X)dt = \delta \mathbf{u} \cdot \mathbf{N},$$

where:

$$X = (D' \mathbf{u}, \Phi', \Delta \alpha) \quad ; \quad \delta \mathbf{Y} = (D' \mathbf{u}, \delta \mathbf{u}, \delta \phi) \quad ; \quad \delta \alpha = (\delta' \alpha, \delta' \phi),$$
41.5

and \( \mathcal{H} \) is a nonlinear function of \( X \). A consistent linearization gives:

\[
\int_{t_i}^{t_{i+1}} \delta y \cdot L \cdot X \, dt = \delta A^T \cdot [\mathcal{E}] \, X_i + \int_{t_i}^{t_{i+1}} \delta y \cdot R \, dt,
\]

where \( R \) means the residual term and \( \mathcal{L} \) is the tangent matrix depending on the specified, time-variant reference state.

The finite element approximation adopted for a time element \( t_0 - t_1 \), consists of a piecewise approximation where only \( C^0 \) continuity both in the relative displacement \( \delta u_i^j \) and in the virtual displacement \( \delta^* \) is required. Since finite rotations may occur in \( t_0 - t_1 \), the time integration for rotations and rotation dependent quantities must be objective [7]. As the Lagrangian multipliers \( \lambda \) do not require any continuity across the boundaries of finite elements in time, no continuity is imposed there and they are eliminated at element level. This is allowed by the present formulation, in which the momenta \( \mathcal{Q} \) appear on the boundary. The Newton Raphson approach is finally used in order to solve the non linear algebraic equation set corresponding to the dynamical equations.

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Introduction

This article deals with nonlinearities that arise in the study of dynamics and control of high flexible large-space-structures. Broadly speaking, these nonlinearities have various origins: (i) geometrical: due to large deformations and large rotations of these structures and their members; (ii) inertia: depending on the coordinate systems used in characterizing the overall dynamic motion as well as elastic deformations; (iii) damping: due to nonlinear hysteretic in flexible joints, viscoelastic coatings etc., and (iv) material: due to the nonlinear behavior of the structural material. The geometrical and material nonlinearities affect the "tangent stiffness operator" of the structure; the inertia nonlinearities affect the "tangent inertia operator".

To study the nonlinear transient dynamic response and control of flexible space-structures, one may think of: (i) semi-discrete approximation methods, and (ii) space-time methods. In the former class of methods, an appropriate spatial discretization is employed through weak formulations (finite-element and field/boundary element) in space, and thus a set of coupled nonlinear ordinary differential equations (O. D. E.) is derived. These O. D. E.'s are solved often through temporal integration techniques of the finite difference-type. The semi-discrete methods are not ideally suited for travelling-wave type propagating disturbances. The second category of methods, viz., the space-time methods, wherein weak formulations in both space and time are employed, are somewhat better suited for wave-propagation type problems. In this article, attention is primarily focused on semi-discrete methods, while some results recently obtained on space-time methods are deferred to a later publication.

Depending on the scale of the response that is required to be studied, a large-space-structure may either be modeled as an equivalent continuum, or as a lattice structure with the detail of each member being accounted for. The spatial discretization in either case is required to be of the least-order possible so that the control algorithms may be meaningfully implemented. The reduced-order-modelling of the "tangent stiffness" operator of either a continuum model or a lattice-model of a space-structure is treated in some detail in this chapter, for structures undergoing large dynamic deformations.

The control of dynamic motion of space-structures is currently envisaged to be through either active processes, passive processes, or some combinations there of. One of the concepts of active control that is considered in details here, and by other authors, is the use of piezo-ceramic actuators that are bonded to the truss and frame members of the space-structure in various locations. The controlling shear stress transmitted by the actuator to the truss by frame member depends on the axial force, transverse shear forces, and bending and twisting moments, in the member itself, as well as the excitation voltage applied to the piezo-actuator. This problem...
Dynamics of Continuum Models of Highly Flexible Space-Structures Undergoing Large Deformations

In this section we deal with strategies for reduced-order structural dynamic modeling of beam and shell-type space structures which undergo large deformations. The space-structures are assumed to be represented by equivalent elastic continua [see Noor and Mikulas, 1987].

The continuum model for a 3-D space-curved beam, that is employed here, is one wherein the effects of stretching, bending, torsion, and transverse shear deformations are accounted for. However, the cross-sectional warping is ignored. The case of conservative force loading, which may also lead to configuration-dependent moments on the beam, is treated. The beam is assumed to undergo arbitrarily large rotations and stretches. Using the three parameters associated with a conformal rotation vector representation of finite rotations, a well-defined Hamiltonian functional is established for the dynamic problem of the beam undergoing large rotational motion. The present approach leads to a symmetric tangent stiffness matrix at all times. In the present total Lagrangian description of motion, the mass-matrix of a finite element of the beam depends linearly on the linear acceleration, but nonlinearly on the rotational parameter and the attendant angular accelerations; the stiffness matrix depends nonlinearly on the deformation; and an "apparent" damping matrix depends nonlinearly on the rotations and the attendant velocities. A Newmark time-integration scheme is used to integrate the semi-discrete equations of motion. Examples of transient dynamic response of highly flexible beam-like structures in free-flight are presented to illustrate the presented methodologies.

Earlier notable contributions to 3-D beam theories undergoing finite deformations are to Reissner (1973, 1981) and Simo and Vu-Quoc (1986). In these references, the existence of prescribed moments has been postulated a priori. The present consistent total Lagrangian approach, leading to a symmetric tangent stiffness, even when distributed external moments (which are configuration-dependent) are present, is due to Iura and Atluri (1986, 1987).

Let \( Y^m \) be the convected orthogonal curvilinear coordinate system. The coordinates \( Y^m \) taken in the cross-section of the beam, while the coordinate \( Y^m \) is taken along the beam axis shown in Fig. 1. The unit base vectors associated with the coordinates \( Y^m \) are denoted by \( E_m \).

The problem of control of nonlinear dynamic motion is addressed in this article. The problem posed in the form of determining the feed-back gain matrix and the attendant control force vector, such that the response as predicted by a semi-discrete system of coupled nonlinear ordinary differential equations, subject to a set of arbitrary initial conditions, is damped out in a pre-set time.

In the first part of the article, continuum models of space-structures are analyzed. These include models of the space-beam type as well as the shallow shell type. In the case of space-beams, the problem of nonlinear dynamic response, when the beam undergoes large overall rigid as well as elastic motion, is discussed. The beam is assumed to undergo large rotations as well as stretches. A simple finite element algorithm to predict the response is presented. When a shallow-shell type continuum model is used, the field-boundary element approach based on nonlinear integral equations is presented as a means to create a reduced-order dynamic model of the semi-discrete type. A simple algorithm to control the response predicted by these nonlinear semi-discrete equations is discussed.

In the second part of the article, detailed models of the lattice-type space-structures are discussed. Each member of the structural lattice is assumed to be either a "truss member", as a "frame member". The "truss member" is assumed to carry only an axial load, and has three displacement degrees of freedom at each node. The "frame member" is assumed to carry an axial force, transverse shear forces, bending moments, and a twisting moment; and is assumed to have three displacement and three rotational degrees of freedom at each node. Explicit expressions for the tangent stiffness matrices of both "truss" type and "frame" type members, which undergo arbitrarily large displacements, arbitrarily large overall rigid rotations, and moderate local (relative) rotations, are derived. In all cases, each member (truss or frame type) is modeled by a single finite element, in the entire range of large deformations. Several examples are presented to illustrate the efficiency and on-board computational feasibility of these reduced-order models for lattice structures. In each instance, remarks on needs for future research are made.
The Equations of Motion

With the help of the Green strain tensor \( \varepsilon(\varepsilon, \Delta A) \) and the second Piola-Kirchhoff stress tensor \( S_1(S_1^i \Delta A_i) \), the internal virtual work is written as

\[
IVW = \int S_1^i \delta \varepsilon \delta \nu dV, \quad dV = g_d Y^i Y^j dL \]

The stress resultants and moments are defined, following Atluri (1984), as:

\[
\mathcal{T} = \int g_d A^2 \cdot (S_1 \cdot F^T) dA, \quad \mathcal{M} = \int Y^i g_d A^2 \cdot (S_1 \cdot F^T) dA \]

where \( F \) is the deformation gradient tensor, \( F^T \) its transpose and \( dA = dY^i Y^j dL \). By using the component representation, we obtain the stress resultants and moments in the form

\[
\mathcal{T} = T^i \varepsilon_{i}, \quad M = M^i \varepsilon_{i}, \quad M^i = \int t^{ij} Y^j g_d A, \quad M^i = \int (t^{ij} Y^j - t^{ij} Y^j) g_d A \]

where \( t^{ij} = S_1^{il} a_{lj} \cdot g_d \). The \( (\cdot) \) in the contravariant tensor is used to emphasize that these are not components in covected coordinates \( Y^m \).

As a rotational variation, we introduce, at first, a tensor \( \delta R \cdot R^T \) introduced by Atluri (1984). Since \( R \cdot R^T = I \) where \( I \) is an identity tensor, \( \delta R \cdot R^T \) is a skewsymmetric tensor. There exists, therefore, a vector \( \delta \theta \) satisfying \( \delta R \cdot R^T = \delta \theta \times I \). It follows from Eqs. (7) to (9) that the \( IVW \) is rewritten, after some manipulations, as

\[
IVW = -\int \mathcal{M} \cdot \delta \theta + (\mathcal{M} + (z + \mu) S) \cdot \delta \theta dL + \int \delta \theta dL + \mathcal{M} \cdot \delta \theta \]

where \( z \) is the undeformed position vector of a point at the beam axis before the deformation; \( S_0 \) and \( S \) are parts of boundary on which geometrical and mechanical boundary conditions are prescribed respectively.

Let \( P_1 (= P_1^i E_i) \) be the vector of body force defined per unit volume of the undeformed beam; \( P_2 (= P_2^i E_i) \) the vector of distributed surface traction defined per unit area of the undeformed cylindrical surface of the beam, denoted as \( S_t \); and \( P_3 (= P_3^i E_i) \) the vector of distributed surface tractions at the end cross sections denoted as \( S_e \). Then the external virtual work is written as

\[
EVW = \int \mathcal{P} \cdot \delta \theta dL + \int \mathcal{L} \cdot \delta \theta dS + \int \mathcal{M} \cdot \delta \theta dL + \int \mathcal{M} \cdot \delta \theta dS \]

The kinetic energy of the beam is written as

\[
T = \frac{1}{2} \int \rho \dot{\vec{u}} \cdot \vec{\dot{u}} dV \]

where \( \rho \) is the density in the reference state. The principle of virtual work for the elastodynamic problem is represented as

\[
\int_0^t [\delta T - IVW + EVW] dt = 0 \]

Using the conventional condition that the variations of displacements at \( t = t_1 \) and \( t = t_2 \) are equal to zero, we obtain the LMB and the AMD conditions, expressed as:

\[
T + q = \frac{L_0}{u}, \quad M_0 + (x + \mu) S + M = \frac{L_0}{\dot{u}} \quad \text{for arbitrary} \quad \delta \theta, \]

\[
M_0 + (x + \mu) S + M = \frac{L_0}{\dot{u}} \quad \text{for arbitrary} \quad \delta \theta, \]

The associated boundary conditions are written as

\[
T - \mathcal{Q} = \mathcal{M} = \mathcal{M} \quad \text{on} \quad S_0, \quad u = \mathcal{U}, \quad \dot{u} = \dot{\mathcal{U}} \quad \text{on} \quad S \]

where \( \mathcal{U} \) and \( \dot{\mathcal{U}} \) denote the prescribed value on \( S_0 \). It should be emphasized that the external moment vectors \( \mathcal{M} \) and \( \mathcal{M} \), which are generated by the conservative forces, are configuration dependent as shown in Eqs. (12c,d).

In the above development of equations of motion, the vector \( \delta \theta \) is used as rotational variable. Simo and Vu-Quoc (1986) have pointed out, using the \( \delta \theta \) as a variation of rotational variable that a well-defined functional exists at only an equilibrium configuration provided that no di- tributed external moments exist. Since the external moment vector \( m_d \) defined by Eq. (12c), configuration dependent, the variation of inner product \( (m \cdot \delta \theta) \) does not yield \( m \cdot \delta \theta \). Therefore, the \( EVW \), especially of the moments, does not, on first sight, appear to correspond to the first variation of an external energy functional.

In order to express \( (m \cdot \delta \theta) \) of Eq. (11b) as the first variation of an energy functional, we adopt a strategy wherein \( m \cdot \delta \theta \) can be expressed in terms of components of \( m \) and \( \delta \theta \) in an undeformed basis. It follows from Eqs. (2b) and (12c) that \( m \cdot \delta \theta = m_{a_0} (\varepsilon_{a_0} \times E_i) \cdot \delta \theta = m_{a_0} (E_i \cdot \delta \theta \cdot \varepsilon_{a_0}) = \delta (m_{a_0} E_{a_0}) \)
Note that the Lagrangian components \( R_{ii} \) of \( R \) are expressed in terms of three arbitrary parameters \( a' \), such that \( (\delta R = R_{ii} \delta \vec{e}_i \vec{e}_i) \), where \( (\vec{e}_i) \) denotes the differentiation with respect to \( a' \). Equation (18) indicates the possibility of constructing a well-defined functional for the present problem with the use of \( a' \).

To show the equivalence of the \( AMB \) condition for \( \delta a' \) and that for \( \delta \vec{f} \) we consider, at first, the tensor equation of \( AMB \) condition for \( \delta \vec{f} \). The inner product between the \( AMB \) condition and the variation \( \delta \vec{f} \) is expressed as

\[
(M_3 + (\alpha + \nu)_3 \cdot \gamma + m - \hat{H}_1) \cdot \delta \vec{f} = C : (\delta \vec{R} \cdot \vec{R} T),
\]

(19)

where

\[
C = Q^i \delta t^i + Q^i \delta t^i + Q^i \delta t^i + m + m_0 \vec{e}_0 - J_0 \vec{e}_0 - J_0 \vec{e}_0
\]

\[
Q^i \delta t^i = M_3 + (\alpha + \nu)_3 \cdot \gamma
\]

When \( R \) is expressed in terms of \( a' \), Eq. (19) is rewritten as

\[
C : (\delta \vec{R} \cdot \vec{R} T) = C : (\delta \vec{R} \cdot \vec{R} T) \delta a'.
\]

(20)

Since \( \delta \vec{R} \cdot \vec{R} T \) is a skew-symmetric tensor, the \( AMB \) condition for \( \delta a' \) is expressed in terms of three arbitrary parameters \( a' \), such that (6R a)

Since \( \delta \vec{R} \cdot \vec{R} T \) is a skew-symmetric tensor, the \( AMB \) condition for \( \delta a' \) is shown to be equivalent to that for \( \delta \vec{f} \).

To complete a beam theory, we consider a stress-strain relationships. Equations (9) indicate that the use of the stress tensor \( \sigma \) and the conjugate strain tensors \( T_i \) to construct the constitutive equation. The conjugate strain tensors \( T_{in} \) are defined as

\[
T_{in} = a_m \cdot \vec{e}_m - A_m \cdot \vec{e}_m
\]

For one-dimensional beams, we assume the following constitutive equations:

\[
T^2 = GT_{12}, \quad T^3 = ET_{13}
\]

(23)

where \( G \) is the shearing modulus and \( E \) the Young modulus. Substituting Eqs. (23) into Eqs. (9) and using Eqs. (5) and (22) lead to

\[
T_i = GA \vec{k}_3 - GL \vec{k}_3, \quad T^3 = GA \vec{k}_3 + GL \vec{k}_3,
\]

\[
T^3 = EA \vec{k}_3 + EI \vec{k}_3 - EI \vec{k}_3, \quad M^1 = EI \vec{k}_3 + EI \vec{k}_3 - EI \vec{k}_3,
\]

\[
M^1 = EI \vec{k}_3 - EI \vec{k}_3 - EI \vec{k}_3, \quad M^3 = GJ \vec{k}_3 - GJ \vec{k}_3 + GJ \vec{k}_3
\]

(24)

where

\[
\vec{k}_3 = \vec{k}_3, \quad h_a = g \sin \beta_a, \quad h_3 = g \beta_a - 1, \quad A = \int g_a dA,
\]

\[
A_a = h_a A, \quad I_0 = \int Y \vec{e}_0 dA, \quad I_1 = \int (Y^2 \vec{e}_0 dA, \quad I_1 = \int (Y^2 \vec{e}_0 dA, \quad J = \int ((Y^3)^2 + (Y^3)^2) g_a dA
\]

(25)

The factor \( k_a \) is the shear-correction factor [6].

The strain energy function \( W_s \) per unit length is expressed as

\[
W_s = \frac{1}{2} GA(h_3)^2 + \frac{1}{2} GA(h_3)^2 + \frac{1}{2} EA(h_3)^2 - \frac{1}{2} EI \vec{k}_3^2 + \frac{1}{2} EI \vec{k}_3^2
\]

\[
+ \frac{1}{2} GJ \vec{k}_3^2 + EI \vec{k}_3 \vec{k}_3 - EI \vec{k}_3 \vec{k}_3 - EI \vec{k}_3 \vec{k}_3 - GJ \vec{k}_3 \vec{k}_3 + GJ \vec{k}_3 \vec{k}_3.
\]

(26)
Displacement and the rotational components are interpolated by:

\[ u^i = u^i_0 N^p, \quad \alpha^i = \alpha^i_0 N^p \]  

(31)

where \( u^i_0 \) and \( \alpha^i_0 \) denote the nodal displacement and rotational components, respectively, and \( N^p \) be shape functions defined by \( N^1 = 1 - L/L, \) and \( N^2 = L/L, \) where \( L \) the element length. For later convenience, the following notations are introduced: \( \mathbf{d} = (u^i_0) \) and \( \mathbf{e} = (\alpha^i_0). \)

Following a standard finite element discretisation, we obtain the following semidiscrete equations of motion:

\[ \mathbf{M}(\mathbf{\ddot{d}}, \mathbf{t}) + \mathbf{C}(\mathbf{\dot{d}}, \mathbf{t}) + \mathbf{K}(\mathbf{d}, \mathbf{t}) = \mathbf{f}(\mathbf{t}) \]  

(32)

where \( \mathbf{M} \) depends linearly on \( \mathbf{d} \) but nonlinearly on \( \mathbf{\dot{d}} \) and \( \mathbf{\ddot{d}}, \) the \( \mathbf{C}, \mathbf{K} \) and \( \mathbf{f} \) depend nonlinearly on their variables. Note that the vector \( \mathbf{C} \) is derived not from the damping effects but from the nonlinear effects of finite rotations, and that no simplification is made in this formulation in the sense that Coriolis and centrifugal effects as well as the inertia effects due to rotation are accounted for. The resulting semidiscrete equations are integrated by the Newmark algorithm. Consistent linearization procedures are employed to obtain linearised forms of the balance equations. A full Newton-Raphson method is used in the present calculations.

To illustrate the validity of the present theory in simulating large rotations, we analyze the case of a highly flexible right-angle beam in free flight, as shown in Fig. 2. Although we have used only three rotational parameters per node, the large deformations with finite rotations can be simulated without singularities. Other numerical examples and more detailed development of transient dynamic analysis of highly-flexible space-curved beams undergoing finite rotations and stretches may be found in Iura and Atluri (1987).

We now turn to the analysis of shell-type continuum models of space-structures such as antennas. Here the problem is one of creating a reliable reduced-order structural model. In the traditional finite element method, for thin shells: (i) either a fourth-order theory with \( C^4 \) continuous trial and test functions; or (ii) a continuum based shell theory with \( C^0 \) continuous trial and test functions along with selective reduced integration schemes to alleviate shear/membrane locking, are used. In the present approach, a field-boundary-element method, based on unsymmetric variational statements and Petrov-Galerkin techniques, is developed for shallow shells undergoing large deformation (moderate rotation) dynamics. It is seen that the number of degrees of freedom in the current approach is reduced from that in the popular Galerkin finite-element-approach.

Consider a shallow shell of an isotropic elastic material with the mid-surface being designated by \( z = z(x, y). \) When finite deformations (only moderate rotations) are considered, the momentum balance laws may be written as [Zhang and Atluri, 1987]:

\[ \begin{align*}
\text{(inplane):} & \quad N_{a,a} + b_a - \rho \ddot{u}_a = 0 \quad (a = 1, 2) \\
\text{(transverse):} & \quad D \nabla^4 w + \frac{N_{a, a}}{R_{a, a}} - (b_3 - \rho \ddot{w}) = f_3 + (N_{a, a} w_a)_{, a}
\end{align*} \]  

(33)

(34)

where: \( N_{a, a} \) are membrane forces; \( (\cdot)_{, a} = \partial(\cdot) / \partial x_a; \) \( w \) is the transverse displacement of the mid-surface of the shell; \( b_i(i = 1, 2, 3) \) are body forces; \( f_3 \) is the load normal to the shell mid-surface; \( D \) is the bending rigidity; \( \nabla^4 \) is the bi-harmonic operator in \( x_a; \) \( \rho \) is the mass-density; \( (\cdot)_{, a} = \partial(\cdot) / \partial x_a; \) and

\[ R_{a, a} = 1/(x_{a, a}) \]  

(35)

The non-linear stress-displacement relations are:

\[ \begin{align*}
N_{11} &= C(\varepsilon_{11} + \varepsilon_{22}), \quad N_{22} = C(\varepsilon_{22} + \nu \varepsilon_{11}), \quad N_{12} = C(1 - \nu)\varepsilon_{11}
\end{align*} \]  

(36)

where \( C \) is the inplane rigidity, \( \nu \) the Poisson ratio and

\[ \varepsilon_{a, a} = \frac{1}{2}[u_{a, a} + u_{a, a} + \frac{2w}{R_{a, a}} + w_{a, a}] \]  

(37)

Fig. 2 Flexible right angle beam in free flight. Time step \( \Delta t = 0.1. \)
Using (36) we write:

\[ N_{ab} = N_{a} + Cx_{ab} w + N_{ab} \]  

where

\[ N'_{ab} = C(w_{ab} + v w_{ab}); \quad N_{ab} = C(w_{ab} + v w_{ab}); \quad N_{ab} = \frac{1}{2} C(1 - \nu) (u_{ab} + u_{ab}) \]  

The weak form of Eq. (33) and (34) are taken to be the "fundamental solutions" in

\[ \delta (x_{a} - \xi_{a}) \delta (x_{b} - \xi_{b}) = 0 \]  

where \( \delta (x_{a} - \xi_{a}) \) is a Dirac delta function at \( x_{a} = \xi_{a} \); \( \delta (x_{b} - \xi_{b}) \) denotes the direction of the applied point load in the \( x_{a} \) direction. This "fundamental solution" will be denoted as \( u_{\eta a}(x_{a}, \xi_{a}) \) i.e., the displacement along the \( x_{a} \) direction, in a plane infinite body, at the location \( x_{a} \) due to a unit load in the \( x_{a} \) direction at the location \( \xi_{a} \). Likewise, \( P_{\eta a}(x_{a}, \xi_{a}) \) will denote the traction along the \( x_{a} \) direction on an oriented surface at \( x_{a} \) (with a unit normal \( \eta_{a} \)) due to a unit load along \( x_{a} \) at the location \( \xi_{a} \). Thus, \( \delta (x_{a} - \xi_{a}) \) is a Dirac delta function at \( x_{a} = \xi_{a} \); \( \delta (x_{b} - \xi_{b}) \) denotes the direction of the applied point load in the \( x_{a} \) direction. This "fundamental solution" will be denoted as \( u_{\eta a}(x_{a}, \xi_{a}) \) i.e., the displacement along the \( x_{a} \) direction, in a plane infinite body, at the location \( x_{a} \) due to a unit load in the \( x_{a} \) direction at the location \( \xi_{a} \). Likewise, \( P_{\eta a}(x_{a}, \xi_{a}) \) will denote the traction along the \( x_{a} \) direction on an oriented surface at \( x_{a} \) (with a unit normal \( \eta_{a} \)) due to a unit load along \( x_{a} \) at the location \( \xi_{a} \). Thus,
seen that \( w \) and \( u \) need only be piecewise differentiable and need not even be \( C^0 \)-continuous at element boundaries in \( \Omega \).

(6) At each point on the boundary, two of the in-plane variables \( M_\alpha (\alpha = 1,2) \) and \( P_\alpha (\alpha = 1,2) \) are specified and the other two are unknown. Likewise, two of the out-of-plane variables \( V_\alpha, M_\alpha, \) \( \Psi_\alpha \) and \( w \) are specified, and the other two are unknown. At each point in the interior, the three displacements, \( u \) and \( w \), are unknown. Thus, from Eqs. (44), (47), and (49), one obtains exactly as many equations as the number of unknowns, so that the problem is well-posed.

The above approach based on integral equations has been used in the control of linear or nonlinear dynamic response of continuum plate models of large-space-structures in O’Donoghue and Atluri (1986, 1987), and Atluri, Zhang, and O’Donoghue (1986). It has also been applied in creating reduced-order-dynamic-models for shallow shells undergoing small as well as large deformations in Zhang and Atluri (1986a, 1986b, and 1987).

To illustrate the advantageous factors of the present field/ boundary element method, we present here the results of the analysis of linear vibration and transient response of a simply supported shallow spherical shell. In these computations, the basis for the trial functions are assumed as follows: (i) over each boundary element at \( \Gamma \), the variables \( u, w, u_\alpha, w_\alpha, P_\alpha, M_\alpha, \) and \( V_\alpha \) are interpolated linearly, (ii) over each domain element, \( u \) and \( w \) are interpolated bilinearly. Only a quadrant of the shell is modeled, to consider the doubly-symmetric modes of vibration. The domain is discretized into interior elements such that the nodes are equidistant from each other in the radial as well as angular coordinates. The first mesh consists of nodes at \( \Delta R = R/4 \) and \( \Delta \theta = \pi/8 \); the second one with \( \Delta R = R/2 \) and \( \Delta \theta = \pi/4 \); the third with \( \Delta R = R/3 \) and \( \Delta \theta = \pi/6 \); and the fourth with \( \Delta R = R/4 \) and \( \Delta \theta = \pi/8 \); with this last mesh being illustrated in Fig. (3). Note that in the present field/boundary element approach, the total number of degrees of freedom are 11 (for mesh 1); 24 (mesh 2); 53 (mesh 3); 81 (mesh 4). The computed eigenvalues for various values of \( (r_1/R) \) [where \( r_1 \) is the base-radius of the shallow shell, and \( R \) is the principal radius of curvature of the sphere] and \( E = 1000 \), \( E_1 = 0.25 \) are shown in Table 1.

Table 1

<table>
<thead>
<tr>
<th>( r_1/R )</th>
<th>.05</th>
<th>.083</th>
<th>.111</th>
<th>.143</th>
<th>.2</th>
<th>.333</th>
<th>.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reissner[6]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mode Mesh 2</td>
<td>7.0871</td>
<td>7.1094</td>
<td>7.1364</td>
<td>7.1764</td>
<td>7.2729</td>
<td>7.6126</td>
<td>8.2359</td>
</tr>
<tr>
<td>Mesh 3</td>
<td>7.0871</td>
<td>7.1094</td>
<td>7.1364</td>
<td>7.1764</td>
<td>7.2729</td>
<td>7.6126</td>
<td>8.2359</td>
</tr>
</tbody>
</table>

Table 1 indicates that even mesh 2 with only 24 degrees of freedom gives the eigenvalues for the first 6 (doubly symmetric) modes with acceptable accuracy.

The above results indicate that the present reduced-order structural modeling strategy for shallow-shell type antenna structure is of considerable benefit in implementing algorithms for controlling linear or nonlinear dynamic response of large-space-structures, wherein the primary limiting factor, in a computational sense, is the size of the system of equations in the Riccati equation governing the feedback gain matrix.

Using the mesh with 24 degrees of freedom, results for undamped nonlinear, axisymmetric transient response are obtained for two initial conditions: (i) at \( t = 0 \), the crown of the spherical shell has an outward normal velocity of \( v_\omega = 15.0 \), which represents an outward push applied at the crown; (ii) at \( t = 0 \), \( v_\omega = 3 \cos(r/r_0) \) where \( r \) is the distance from the center of the base plate. The material density is taken as \( p = 0.8 \), and the time step is taken as \( \Delta t = 0.002 \) in the time integration of the semi-discrete equations. The results for the former case are shown in Fig. (4), while those for the latter are shown in Fig. (5).
The system of matrix equations describing the nonlinear transient response of a shallow shell derived from the present field/boundary element approach, are of the form (Zhang 1987; Zha and Hadiri 1987):

$$
\Delta \ddot{W} + C \dot{W} + K \Delta W = \Delta F + \Delta f,
$$

where \( M, C, \) and \( K \) are tangent inertia, damping, and stiffness matrices which, in general, depend on the prior state of deformation.

The control forces \( f \) must be designed so as to damp out the response of the system for a disturbance of the type:

$$
W(t = 0) = W_0; \quad \dot{W}(t = 0) = \dot{W}_0;
$$

Towards a simple solution for the nonlinear control problem, we consider first the problem of the linear system:

$$
M \ddot{W} + C \dot{W} + K W = f,
$$

where \( M, C, \) and \( K \) are the mass, damping, and stiffness matrices, respectively, of the linear system, and are thus assumed to be independent of deformation. Eqn. (52) may be recast in state variable form, as:

$$
\dot{S} = AS + Bf; \quad S(0) = S_0
$$

where

$$
S^t = [W, \dot{W}]
$$

$$
A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}
$$

$$
B = \begin{bmatrix} 0 \\ -M^{-1}f \end{bmatrix}
$$

For a linear regulator problem, a typical quadratic performance index may be defined as:

$$
J = \frac{1}{2} S^t(t_f)P S(t_f) + \frac{1}{2} \int_{t_i}^{t_f} (S^t Q S + L^t R L) dt
$$

where \( t_i \) is the initial time, \( t_f \) is the final time, and the non-unique constant matrices \( Q \) and \( R \) are positive definite weighting matrices which determine the magnitudes of the control forces and the quantitative decay of the controlled responses. The minimization of \( J \) in (57) subject to the constraint equation (53), using the Hamiltonian-Jacobi-Bellman equation (Bryson and Ho 1974), leads to the following relation for feedback gain matrix \( G \):

$$
L = -R^{-1}B^t G S
$$

where \( G \) is the solution of the Riccati equation:

$$
\dot{G} = -GA - A^t G + GBR^{-1} B^t G - Q; \quad G(t_f) = Q
$$

where the matrix \( A \) for the linear system is defined in Eq. (55). Once the Riccati equation (56) is solved for \( G \), and (58) is used in (53), the thus modified Eq. (53) becomes a standard initial value problem for the controlled response.

An efficient algorithm, based on the Schur vector approach (Laub, 1979), has been devised for solving Eq. (59) for systems of up to 100 degrees of freedom, in O'Donoghue (1986) and Zhang (1987).

A simple technique for controlling the nonlinear motion of structural systems considered for this purpose is described below. The feedback gain matrix \( G \) is assumed to be determined from Eq. (59), using the parameters of the system.
the linear system i.e., the matrices A and B for the system, assuming that the system behaves linearly. However, the feedback control forces are determined by applying this gain-matrix G to the sensed actual i.e., nonlinear state \( \{W, \dot{W}\} \) of the nonlinear system. Splitting the gain-matrix G, as determined for a pseudo-linear system, relating to the displacement and velocity vectors of the nodes, give the control forces on the nonlinear system as:

\[
\Delta F = -R^{-1}B'[G_1 \Delta w + G_2 \Delta \dot{w}]
\]

where \( \Delta w \) and \( \Delta \dot{w} \) refer to the components of the incremental state vector of the actual nonlinear system. The controlled nonlinear response is computed by using the following equation, which results from using Eq. (60) in Eq. (50):

\[
MA_\Delta w + [C + BR^{-1}B'^T] \Delta \dot{w} + [K + BR^{-1}B'^T] \Delta w = \Delta F
\]

where, it should be recalled that, M, C, and K are the tangent mass, damping, and stiffness matrices of the nonlinear system.

The asymptotic stability of such controlled nonlinear systems as described through Eq. (61) are discussed in O'Donoghue (1985) and Zhang (1987).

In the examples to follow, for simplicity, the final time \( t_f \) is taken to be infinity, and only the steady state Riccati equation is solved. The control actuators are assumed to be located at the interior nodal points. The requirement of controllability [O'Donoghue (1986) and Zhang (1987)] has been satisfied in choosing the number and locations of these controllers. In the present simply-supported shell problem, the transverse displacement \( w \) is much bigger than the in-plane displacements \( u \); only \( w \) is taken as the response quantity to be controlled. The field boundary element method consists of nodes at \( r = (r_i, \theta) = (r_i, \pi/8) \) in the circular baseplane. In the nonlinear response control problem, the weighting matrix \( \mathbf{R} \) is taken to be \( \mathbf{R} = \text{diagonal} [1, 1, \ldots, 1] \); and the matrix \( \mathbf{Q} \) is taken to be: \( \mathbf{Q} = \text{diagonal} [0.2, 0.2, \ldots, 0.2] \); thus satisfying the asymptotic stability criteria [Zhang, 1987].

The initial disturbance is taken to be an upward point impulse at the crown of the shell, resulting in an initial upward crown velocity of \( v_0 = 15.0 \). The controlled as well as uncontrolled nonlinear responses of the shell at the crown node, and the nodes corresponding to \( (r_i/4); (r_i/2), \) and \( (3r_i/4), \) are shown respectively, in Figs. (6), (7), (8), and (9). The variations of the corresponding control-actuator forces are shown in Figs. (10), (11), (12), and (13), respectively.
Fig. 11 Control force output by the actuator located at node 2

Fig. 12 Control force output by the actuator located at node 3

Fig. 13 Control force output by the actuator located at node 4

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Fig. 9 Controlled nonlinear response at node 4

Fig. 10 Control force output by the actuator located at node 1
2 Reduced-Order Structural Modelling of Truss-and-Frame Type Lattice Structures Undergoing Large Deformations

While continuum models of space-structures are convenient in analyzing the standing-wave type global responses, often times it is necessary to develop detailed local models of the lattice type structure, wherein each member may be modeled as a 'truss' member or a 'frame' member, depending on the design of the lattice joints. A 'truss' member is assumed to carry only an axial force, and the kinematics of deformation is characterized by the three displacements of each 'truss' joint. The 'frame' member carries bending moments, twisting moments, and lateral forces, in addition to the axial force; and the deformation is characterized by the displacements as well as rotations at each node.

In the non-linear transient response analyses, it is usual to employ the semidiscrete method, with tangent inertia, damping, and stiffness matrices as in Eq. (50), for instance. In the usual finite element analyses, much of the effort is expended in evaluating these tangent matrices. For instance, the tangent stiffness matrix $\mathbf{K}_t$ (at the $N$th increment of deformation) accounts for the effects of initial displacements and initial stresses. Usually, one assumes basis functions over each finite element, and integrates the appropriate strain energy terms (that depend nonlinearly on displacements) to obtain $\mathbf{K}_t$.

Recently, research had been performed to obtain explicit expressions for $\mathbf{K}_t$ for lattice truss and frame structures, without the use of the assumed element displacement basis functions and without the use of element integrations. In the case of a truss, each member is assumed to undergo arbitrarily large rigid rotations, moderate axial stretches, and local buckling. In its buckled state, the axial contraction of each member may be about 20% of the initial length. In the case of a beam member, each member is assumed to undergo arbitrarily large rigid rotations, moderate relative rotations, and have nonlinear bending-stretching coupling. Furthermore, for the range of deformations considered, each member of the space-truss or space-frame was sought to be modeled by a single finite element.

Such explicit tangent stiffness matrices for the space-truss members were derived by Kondoh and Atluri (1985a, b), and Tanaka, Kondoh, and Atluri (1985). As for frame members, Kondoh and Atluri (1985b) first presented a displacement-based approach for plane frames, where the exact solution for the displacement of each member, modeled as a beam-column, was employed. Later, Kondoh, Tanaka, and Atluri (1985, 1986) extended the displacement approach to the three-dimensional case, using the concept of semi-tangential rotations. However, these displacement approaches did not consider the effect of plastic deformations, nor of nonconservative loading. Later work by Kondoh and Atluri (1987) showed that a complementary energy approach, based on assumed stress, is a much simpler alternative to derive explicit expressions for tangent stiffness of plane frames undergoing large elasto-plastic deformations. This was later extended to the case of space-frames by Shi and Atluri (1987).

Consider an initially straight slender truss member spanning two nodes $(x_1, y)$ in space (where 1 and 2 denote the nodes; and 1, 2, and 3 denote the Cartesian directions). The initial length of the member is $\ell = (x_2 - x_1)(y_2 - y_1) = (z_2 - z_1)(x_2 - x_1)|$, and it is considered to have a uniform cross-section. Let $u_1$ and $u_2$ be the arbitrary displacements at the centroidal axis of nodes 1 and 2 respectively. From the polar-decomposition theorem, the total axial stretch of the member is given by:

$$\delta = \ell' - \ell = \left\{ \left[ (z_1 + u_1) - (x_1 + u_1) \right] \cdot \left[ (z_2 + u_2) - (x_2 + u_2) \right] \right\} - \ell$$

$$= (Dz + Du)(Dz + Du)\ell - (Dz \cdot Dz)\ell$$

$$= (Dz + Du)(Dz + Du)\ell - (Dz \cdot Dz)\ell$$

where $Dz = z_2 - z_1$, and $Du = u_2 - u_1$.

By taking the Taylor Series expansions of (62), the relation between the incremental axial stretch $\Delta \delta$, and the incremental nodal displacements $\Delta (Du_j) = \Delta u_j - \Delta u_{ij}$ may be derived as:

$$\Delta \delta = \frac{\partial \delta}{\partial (Du)} \Delta (Du) + \frac{\partial^2 \delta}{\partial (Du) \partial (Dz)} \Delta (Du) + \Delta (Dz) + \text{higher terms}$$

The truss member is assumed to transmit only an axial force $N$ while the material is assumed to remain elastic, the truss member may undergo local buckling. Since large deformations and local buckling are considered, $N$ is in general a nonlinear function of $\delta$. The tangent spring-constant of the truss member is assumed to be $k$ i.e.

$$k = \frac{\partial N}{\partial \delta}$$

or, in a linearized form,

$$\Delta N = k \Delta \delta$$

When the truss member remains straight and unbuckled, from linear elasticity and has:

$$k = EA/\ell; \quad N < N_{cr}$$

where $N_{cr}$ is the Euler buckling load of the truss-member treated as a simply supported beam-column.

On the other hand, when $N \geq N_{cr}$, the truss member undergoes local-buckling, and the well-known elastica solution [Timoshenko and Gere 1961] leads to:

$$N = N_{cr} \left[ 1 + \left( \delta/\ell \right) \right]$$

Eq. (68) is valid for $\delta/\ell$ in the range of 0 to 0.2 [See Tanaka, Kondoh, and Atluri 1987]. This range of values of $(\delta/\ell)$ is adequate for considerations of realistic large deformations of truss-member as a part of an overall large space structure.

Thus, in the post-buckling range, the axial stiffness of a truss member is given from Eq. (6) by:

$$k = \frac{\partial N}{\partial \ell} = -\frac{1}{2\ell} N_{cr} = \frac{\pi^2 EI}{2\ell^2} N \geq N_{cr}$$

(6)

Thus:

$$k = \frac{EA}{\ell}; \quad N < N_{cr}; \quad k = \frac{\pi^2 EI}{\ell^2}, \quad N \geq N_{cr}$$

which indicates a substantial loss of stiffness of a truss member due to local buckling.

If $N$ is the current axial force (either before or after local buckling) in a member that undergoes an incremental deformation resulting in an incremental axial stretch $\Delta \delta$, the incremental strain energy in the member is given by:

$$\Delta U = N \Delta \delta + \frac{1}{2} k (\Delta \delta)^2$$

(6)
where $k$ is given by Eqs. (70) depending on whether $N < N_{cr}$ or $N \geq N_{cr}$. Using (63) in (71), one has:

$$
\Delta U = N \left( \frac{\partial \Delta(Du)}{\partial(Du)} \Delta(Du) + \frac{\partial^2 \Delta(Du)}{\partial(Du)\partial(Du)} \Delta(Du) \Delta(Du) \right) + \frac{1}{2} \left( \frac{\partial \Delta(Du)}{\partial(Du)} \right) \Delta(Du) \Delta(Du) + h \cdot \sigma \cdot t
$$

(72)

where $\Delta(Du)$ is the increment of nodal displacements (the coefficients of the $(6 \times 6)$ tangent stiffness matrix $K$ of the 2 noded truss member, with incremental nodal displacements $\Delta u_n$ ($n = 1, 2$; and $i = 1, 2, 3$ Cartesian directions), are derived from evaluating the second-order derivatives as below:

$$
\frac{\partial^2 \Delta U}{\partial \Delta u_i \partial \Delta u_j} \quad (i, j = 1, 2, 3)
$$

(73)

The vector of residual nodal forces on the element, at the current stage of incremental deformation, is given by:

$$
R = \frac{\partial \Delta U}{\partial \Delta u_i} \quad (i = 1, 2, 3)
$$

(74)

When the tangent stiffness matrix of the element is derived in an explicit fashion as in Eq. (73), the tangent stiffness of the lattice structure as a whole is also obtained in an explicit form, through the usual element-assembly process.

In the present work, to determine the quasi-static finite deformations of a lattice structure, an "arc-length" method [Craighead, 1983; Bika 1979; and Kondoh and Aturi 1989] is used.

An example of a beam-like space-truss (the PACOSS truss), subjected to axial and bending loads is shown in Fig. (14). The structure is that of a twelve-bay truss whose member properties are indicated in Fig. (14). The load is applied in the transverse direction at one of the end-nodes, as shown in Fig. (14). For this predominately axial-load case, Fig. (15) shows the relation between the magnitudes of the axial load and that of the transversal displacement at the loaded end for two scenarios: (i) when local (member) buckling is suppressed through an active control mechanism (such as piezo-electric ceramics bonded to the member along segments of its length) and thus each member is assumed to remain straight and stable; and (ii) when each member is allowed to undergo local buckling. Fig. (15) clearly demonstrates the advantages of controlling the local buckling deformations of individual members and forcing them to remain straight and stable.

Fig. (16) shows the case when the PACOSS truss is subjected primarily to bending loads. This figure shows the relation between the magnitudes of transverse (bending) load, and transverse displacement, respectively, once again for two scenarios: (i) when local member buckling is suppressed, and (ii) when member buckling is allowed. Fig. (16) shows that a nearly linear-field increase in the load-carrying capacity of the truss results when active control mechanisms are employed to keep each individual truss member from buckling.

It should be remarked that in Figs. (15) and (16), the letters A, B, C etc. indicate the stages at which the respective members, whose numbers are identified in Figs. (15) and (16), respectively, undergo local buckling.

We now consider the case of lattice structures whose joints are designed such that each of the members may be modelled as an initially straight beam. Let $x_i$ and $c_i$ represent a fixed set of Cartesian coordinates and unit bases along the arbitrarily oriented undeformed member. Let $x_{i1}$ be the global coordinates ($i = 1, 2, 3$) of the node $a$ ($\alpha = 1, 2$) of the beam in space. The base vector $e_i$ are chosen such that:

$$
\begin{align*}
\text{if } & x_{11} = x_{21} = 0, \\
\text{then } & e_i = \frac{(Dx_i) c_i}{\|Dx_i\|}; \\
\text{if } & x_{11} \neq x_{21}, \\
\text{then } & e_i = \frac{(c_i \times e_1)/|c_i \times e_1|}{c_1} = c_1 \times e_1
\end{align*}
$$

(75)

where $Dx_i = (x_{11} - x_{21})$, and subscripts 1 and 2 denote the nodes. Let $\delta z_i$ and $\delta e_i$ be the local coordinates and basis as shown in Fig. (17), where $\delta z_i$ is along the straight line joining nodes 1 and 2 after deformation. Another basis system is $e_i'$ which is locally tangential and normal to the deformed centroidal axis of the beam.

Let $u_{n\alpha}$ ($\alpha = 1, 2; i = 1, 2, 3$) be the displacements of nodes 1 and 2 in the global Cartesian system. Defining the quantities:

$$
Du_i = u_{n\alpha} - u_{n\alpha} \quad \text{and} \quad D\delta z_i = D\delta z_i + D\delta e_i
$$

(76)
s, m, Is

Ig. IS Deflections at free end under axial loads with and without the influence of local buckling of truss members.

Fig. 10 Deflection at free end under bending loads with and without the influence of local buckling of truss members.

Fig. 15 Deflections at free end under axial loads with and without the influence of local buckling of truss members.

Fig. 16 Deflection at free end under bending loads with and without the influence of local buckling of truss members.

Eq. (77) may be written as:

\[ \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12}/L & D_{13}/L \\ D_{21}/L & D_{22}/L & D_{23}/L \\ -D_{31}/S & -D_{32}/S & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} \]

where \( L^2 = D_{31} \cdot D_{32} \); or Eq. (77) may be written as:

\[ \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} = \begin{pmatrix} D_{11} \varepsilon_1 + D_{12}/L \varepsilon_2 + D_{13}/L \varepsilon_3 \\ D_{21}/L \varepsilon_1 + D_{22}/L \varepsilon_2 + D_{23}/L \varepsilon_3 \\ -D_{31}/S \varepsilon_1 - D_{32}/S \varepsilon_2 + S/L \varepsilon_3 \end{pmatrix} \]

Equation (75) may likewise be written as:

\[ \varepsilon'_1 = R_1 \varepsilon_1 \]

Thus, \( R \) characterizes the orientation of the undeformed beam in a fixed Cartesian system \( \xi_1 \); and \( R \cdot R' \) characterizes the arbitrary rigid rotation of the deformed beam relative to its undeformed configuration. Note, however, \( R \cdot R' \) locates only the line joining the two nodes in the deformed configuration of the beam.

The relative rotations \( \theta_i \) of the deformed centroidal axis of the beam, with respect to the line joining the nodes 1 and 2 in the deformed configuration, are assumed to be small. Under these assumptions, the change in the length of the beam is given by:

\[ H = L - \ell = \left[ (D_{11} \varepsilon_1 + D_{12}/L \varepsilon_2 + D_{13}/L \varepsilon_3) - \ell \right] = (D_{11} \varepsilon_1 + D_{12}/L \varepsilon_2 + D_{13}/L \varepsilon_3)^2 \]

In view of (81), the finite rotation matrix \( R \) of Eq. (78) may be written as:

\[ R = \begin{pmatrix} \partial H / \partial (D_{11}) & \partial H / \partial (D_{12}) & \partial H / \partial (D_{13}) \\ \partial H / \partial (D_{21}) & \partial H / \partial (D_{22}) & \partial H / \partial (D_{23}) \\ -\partial H / \partial (D_{31}) \cdot \partial H / \partial (D_{32}) \cdot S/L \end{pmatrix} \]

When the element is parallel to the \( x_3 \) axis, \( \varepsilon^3 = (D_{11})^3 + (D_{12})^3 = 0 \) and hence Eqs. (78) are not valid. In this special case, we may take:

\[ \varepsilon'_1 = \varepsilon_1; \quad \varepsilon'_2 = \varepsilon_2; \quad \varepsilon'_3 = -\varepsilon_1 \]

The displacements of the nodes 1 and 2, in the basis vector system \( \varepsilon' \) are taken to \( u_{\alpha k} \) (\( \alpha = 1, 2; k = 1, 2, 3 \)). It is seen that:

\[ u_{\alpha k} = (\varepsilon R)_{\alpha k} \varepsilon_{ab} \]

where \( \varepsilon R \) characterizes the initial configuration of the beam, as in Eq. (80).

As explained in detail in Kondoh and Atluri (1987), the consistent forms of linear and angular momentum balance relations for the Jaumann Stress resultant and stress couples, defined in Atluri (1983), may be simply written as:

\[ \frac{\partial N}{\partial x_1} + \phi_1 = 0 \]

\[ \frac{\partial \phi_1}{\partial x_1} + \frac{\partial}{\partial x_1} (N \phi_1) + \phi_2 = 0 \]
where \( q_i \) are the distributed loads along the \( i \) directions per unit length of the undeformed element.

The load acting on the member may be considered to be both "dead-type" and "follower-type" (i.e. always along the \( i \) axis). Thus,

\[
\delta N = N + R_l \delta q_i + q_{\text{w}}
\]

where 'c' denotes conservative and 'n' denotes non-conservative. We assume trial functions for \( N \) and \( M_i \), which satisfy Eqs. (84) and (85a) identically; but trial functions for \( M_3 \) and \( M_4 \) satisfy only the linear parts of Eqs. (85b) and (86) respectively.

Thus, the chosen trial stress fields in the member are:

\[
N = n + \int_0^1 q_i dz_i + \frac{1}{2} \int_0^1 \int_0^1 q_{ij} dz_i dz_j
\]

\[
\frac{\partial \delta N}{\partial N} = \mu_i
\]

\[
\frac{\partial \delta M_i}{\partial M_i} = \mu_i (1 - \frac{2}{l}) + \mu_i \frac{2}{l}
\]

and

\[
\frac{\partial \delta M_3}{\partial M_3} = -m_{12} + m_{13}
\]

\[
\frac{\partial \delta M_4}{\partial M_4} = -m_{12} + m_{13}
\]

The corresponding test functions (or variations in \( N \) and \( M_i \)) are taken to be:

\[
\delta N = \nu
\]

\[
\delta M_i = \mu_i
\]

Assuming that the material is linear elastic, we define a complementary energy density \( W_e \) such that:

\[
\delta W_e / \delta N = h; \quad \delta V_e / \delta M_i = \kappa_i
\]

where \( h \) is the axial strain, and \( \kappa_i \) the curvature strains.

Rather than considering the point-wise compatibility conditions, we now write the weak form of the compatibility conditions for the beam as a whole. These are:

\[
\int_0^1 \frac{\partial W_e}{\partial N} \nu dz_i = \int_0^1 u_i \nu dz_i = \nu H = \nu \int (D_{ij} \cdot D_{ij}) - \nu
\]

\[
\int_0^1 \frac{\partial W_e}{\partial M_i} \mu_i dz_i = \int_0^1 \Phi_i \mu_i dz_i = (\Phi_i - \Phi_i) \mu_i
\]

\[
\int_0^1 \frac{\partial W_e}{\partial M_3} \mu_3 dz_i = -\int_0^1 (\Phi_3 \mu_3) dz_i = -\int_0^1 \mu_3 dz_i
\]

\[
\int_0^1 \frac{\partial W_e}{\partial M_4} \mu_4 dz_i = -\int_0^1 (\Phi_4 \mu_4) dz_i = -\int_0^1 \mu_4 dz_i
\]

Likewise,

\[
\int_0^1 \frac{\partial W_e}{\partial M_3} \beta_3 dz_i = \int_0^1 \frac{\partial W_e}{\partial M_4} \beta_4 dz_i = 0
\]

where \( \beta_\alpha (\alpha = 1, 2; i = 1, 2, 3) \) is the \( i \)th relative rotation (assumed small) at the \( \alpha \) node.

Note that the balance relations (84 and 85a) are satisfied identically. The weak forms of Eqs. (85b) and (86) may be written for each member, as:

\[
\int_0^1 \frac{\partial W_e}{\partial M_3} \beta_3 dz_i = \int_0^1 \frac{\partial W_e}{\partial M_4} \beta_4 dz_i = 0
\]

Finally, when 2 or more beam-members are joined at a lattice-joint in the space structure, the joint equilibrium must be satisfied. The forces acting on any joint are: (i) the given external forces and moments, if any, at the joint; and (ii) the internal forces and moments in all members meeting at the joint.

Let \( Q^* \) be the generalized internal nodal force vector:

\[
Q^* = (N_1; Q_{12}; Q_{13}; M_1; M_2; N_2; Q_{23}; M_3; M_{12}; M_{13}; M_{23})
\]

where \( N, Q, \text{etc.} \) are nodal forces along \( i \) axes. Let the external nodal forces (and moment) be specified along (and about) the global \( \xi_i \) axes. Let

\[
\mathbb{I}^* = [F_{11}; M_{11}; F_{11}; M_{11}]
\]

where \( n \) is the 3 x 3 rotation matrix defined in Eq. (82).

Then, the joint equilibrium may be written as:

\[
\sum_{\text{elem}} R^* \mathbb{Q} - \mathbb{I}^* = 0
\]

where

\[
\sum_{\text{elem}} \mathbb{v}^* \mathbb{Q}^* - \mathbb{v}^* \cdot \mathbb{I}^* = 0
\]

where \( \mathbb{v} \) is a (12 x 1) vector of trial functions, which may be interpreted as the virtual displacements (along \( \xi_i \)) and virtual rotations (around \( \xi_i \)) at the two nodes of each member. Thus,

\[
[y] = [\delta u_{11}; \delta u_{12}; \delta u_{13}; \delta \theta_{11}; \delta \theta_{12}; \delta \theta_{13}; \delta u_{21}; \delta u_{22}; \delta u_{23}; \delta \theta_{21}; \delta \theta_{22}; \delta \theta_{23}]
\]
Since the relative rotations $\phi_{ai}$ are assumed to be small, one may define a generalized member deformation vector:

$$
Q' = [H; (\phi_{11} - \phi_{12}); (\phi_{21} - \phi_{22}); (\phi_{31} - \phi_{32})]
$$

(108)

Let $\mathbf{g}$ denote the generalized member internal force vector which satisfies only the homogeneous linear part of the equilibrium equations (84-86), i.e.,

$$
\mathbf{g} = [n; m_1; m_2; m_3; m_4]
$$

(109)

and

$$
[\delta \mathbf{g}] = [v; \mu_1; \mu_2; \mu_3; \mu_4]
$$

(110)

From Eqs. (88-91), we see that the homogeneous parts of the trial functions for generalized member forces are:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & (1 - \tilde{z}_1/\ell) (\tilde{z}_1/\ell) & 0 & 0 & 0 \\
0 & 0 & 0 & (1 - \tilde{z}_1/\ell) (\tilde{z}_1/\ell) & 0 & 0
\end{bmatrix}
$$

(111)

and

$$
\begin{bmatrix}
\nu \\
\mu_1 \\
\mu_2 \\
\mu_3
\end{bmatrix} = \mathbf{F} \cdot [\delta \mathbf{g}]
$$

(112)

Finally, we define a vector:

$$
\mathbf{w}' = \begin{bmatrix}
\partial W_1/\partial \mathbf{u} \\
\partial W_2/\partial \mathbf{u} \\
\partial W_3/\partial \mathbf{u} \\
\partial W_4/\partial \mathbf{u}
\end{bmatrix}
$$

(113)

Using the notations in Eqs. (108-114), the combined weak forms of member compatibility equations (96 and 97); the member equilibrium equations (98 and 99); and the joint equilibrium equations (106); may be written as:

$$
\sum_{\text{members}} \left[ \int_0^1 \mathbf{w}' \cdot \mathbf{F} \cdot \delta \mathbf{g} \, d\tilde{z} + \Delta \cdot \delta \mathbf{g} + y' \cdot \mathbf{R} \cdot \mathbf{Q}' - y' \cdot \mathbf{Z} + \int_0^1 N \theta_1' \beta_1' \, d\tilde{z} + \int_0^1 N \theta_2' \beta_2' \, d\tilde{z} \right] = 0
$$

(115)

Thus, the only integrals to be evaluated over the element are those involving the integrand $\mathbf{w}' \cdot \mathbf{F}$. Since the integrands are simple, they can be trivially evaluated in closed form. On the other hand, omitting the terms $N \theta_1' \beta_1'$ and $N \theta_2' \beta_2'$ from Eq. (115) will lead to some errors in the tangent stiffness matrix, but it is entirely permissible in the context of the iterative arc-length method used in solving the present finite-deformation problem, as discussed in Kondoh and Atluri (1987). The details of the algebraic derivation of the explicit expression for the tangent stiffness matrix of the beam undergoing arbitrarily large over-all rotations as a part of a deforming lattice-structure, may be found in Kondoh and Atluri (1987) and Shi and Atluri (1987).

When plastic hinges form at any location in the member, only the member compatibility conditions, Eqs. (96 and 97), are changed accordingly. Even when such plastic hinges form in a member, the tangent stiffness matrix of the member may be evaluated explicitly [see Kondoh and Atluri (1987) and Shi and Atluri (1987) for further details].
As for active control of deformations of a truss or a beam member of a lattice structure, the concept of piezo-electric layers bonded to the surfaces of the members, in segments along the length of a member, appears promising. These actuators transfer shear forces to the underlying structural surfaces depending on the magnitude of the excitation voltage applied. Crawley and de Luis (1984) presented a static model for the shear transfer from a piezo-liner to a beam member which was assumed to be subjected to a pure bending moment, when the deformation of the beam is infinitesimal. Im and Atluri (1987) presented a more complete solution which considers axial forces, transverse shear forces, as well as bending moments in the structural member itself, in solving for the shear stress transfer from the piezo-liner. Further, Im and Atluri (1987) consider the deformation of the beam to be arbitrarily large.

Consider a beam with several segments of a piezo-liner at its bottom and top surface as shown in Fig. (20). Let the member be so deformed that a segment of the member, lined with the piezo-actuator, is as shown in Fig. (21) after deformation. We consider the member internal forces to $M$ (bending moment); $V$ (transverse shear) and $N$ (axial force). The detailed stress patterns in the beam-column as well as the liners are shown in Fig. (22).
Let $t_a$ be the thickness of the adhesive, $t_p$ be the thickness of the piezo-electric liner; $t$ the thickness of the beam; $L$ is the length of the segment of the beam-column; $G_a$ the shear modulus of the adhesive; $E_p$ the Young's modulus of the piezo material; and $E$ the Young's modulus of the beam column. We introduce the non-dimensional variables:

$$
\xi = \frac{x}{L}; \quad \zeta = \frac{L}{t_a}; \quad m = \left(\frac{LM}{EI}\right) \quad (116)
$$

$$
\alpha = \left(6h + h_p\right); \quad h = G_a L^2/\left(t_p E_p\right); \quad h_p = G_a L^2/\left(t_p E_p\right) \quad (117)
$$

$$
\eta = \frac{t}{t_a}; \quad \bar{C} = G_a/E_p \quad (118)
$$

$$
p^+ = \frac{t_p^e}{(t_p^e)}; \quad p^+ = \frac{t_p^e}{(t_p^e)} \quad (119)
$$

where $t^e$ and $t^e$ are the shear stresses transmitted by the 'upper' and 'lower' piezo-liners [see Fig. 22].

It has been shown by Im and Atluri (1987) that the solutions for $p^+$ and $p^-$ may be written as:

$$
p^+ = \frac{\eta \bar{C} \left[m_1 - m_1 - \xi (A' + A') \right.}{(t_p^e) \left(0 \sin h \alpha \right)} - \frac{m_1 - m_1 \cos h \alpha - \xi (A' - A') (1 - \cos h \alpha)}{(t_p^e) \sin h \alpha} \quad (120)
$$

and

$$
p^- = \frac{\eta \bar{C} \left(A' + A' - \frac{1}{2L} \right.}{(t_p^e) \left(0 \cos h \beta \right) \left(1 - \cos h \beta \right) + \sin h \beta} \quad (121)
$$

where $\beta = \left(2h + h_p\right), m_1$ and $m_2$ are the non-dimensional bending moments in the beam at the end points of the segment lined with a piezo-actuator; and $A'$ and $A''$ are the mechanical strains induced by the piezo-liners:

$$
A' = \frac{CV'}{t_p^e}; \quad A'' = \frac{CV''}{t_p^e} \quad (122)
$$

where $V'$ and $V''$ are excitation voltages applied to the upper and lower actuators, and $c$ is the piezo-electric constant.

For example, we consider the case when $A' = -A''$, and when the axial forces in the member are zero. Then $p^f = 0$ and $r^f/E_p = r^f/E_p = p^f/2$. We assume the values:

$$
\xi = 10; \quad \eta = 10; \quad G = 1/63; \quad h = 57
\quad h_p = 423; \quad A' = -A'' = 10^{-3}; \quad m_1 = 10^{-3} \quad (123)
$$

which corresponds roughly to that of an aluminum column, epoxy adhesive, and ceramic piezoelectric actuator.

We first plot the shear stress distribution as in Fig. 23 when there is no axial force in the beam-column. The present results agree with those in Crawley and de Luis (1984) when the transverse shear in the beam is taken to be zero ($V = 0$). On the other hand, when $V L/M_1 = 0.8$, their result is not different from the result for zero shear force over the range $0 \leq x/L \leq 0.8$; and such is not the case in the present results. It is also seen that the transverse shear force in the beam contributes a significant change in the distribution of the shear stress exerted by the actuator; and that the degrees of the localization of the transmitted shear stress at the two ends of the actuator segment may be very different depending upon both the magnitudes of the transverse shear force in the beam.

Because of the assumption that the piezo-actuator segment is short, the flexural deformation of the actuator segment is decoupled with its axial deformation, and the effect of the axial force

Fig. 23 Effect of shear force in the beam upon the shear stress transmitted to the beam by the actuator when there is no axial force in the beam; Crawley and de Luis' result agrees with the result of the present study when there is no shear force in the beam. However, for $V L/M_1 = 0.8$, their result is not different from the result for zero shear force over the range $0 \leq x/L \leq 0.8$ under the current scale. Such is not the case with the present results.

Fig. 24 Effect of axial force in the beam upon the distribution of shear stress exerted by the actuator when there is no shear force in the beam.
in the beam upon its flexural deformation is negligible. However, the axial force in the beam has a significant effect on the shear stress transmitted by the actuator to the beam-column. For numerical illustration, the distribution of the shear stress exerted by the actuator is plotted in Fig. (24) for a case when there is no transverse shear force, but nonzero axial force in the beam. It is seen that, for this case, the shear stress exerted by the upper actuator has a totally different distribution as compared to the shear stress exerted by the lower actuator. As another example, we consider the case when the transverse shear force as well as the axial force in the beam are nonzero. As seen from Fig. (25), for this case, the distribution of the transmitted shear stress to the beam may be more complex as compared to the two earlier cases.

Finally it is recalled that in the present study, only the rotation of one end of the beam-column, relative to the other, is assumed to be small, because the beam-column segment (with the attached actuator) is assumed to be small. Further, we imposed an approximate rigid rotation, which can be finite, to bring the deformed beam-column segment to the configuration in Fig. (21), so that the line connecting the two-nodes is horizontal. Therefore, the present results are applicable to the case of slender flexible structures undergoing large deformations and rotations. When these controlling shear stresses are included in the special space-truss and space-frame analysis methods described earlier in the paper [Kondoh and Atluri (1987); Shi and Atluri (1987), Tanaka, Kondoh, and Atluri (1985)], the controlled dynamic transient responses may be determined.

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References


ELASTO-PLASTIC LARGE DEFORMATION ANALYSIS OF SPACE-FRAMES: A PLASTIC-HINGE AND STRESS-BASED EXPLICIT DERIVATION OF TANGENT STIFFNESSES

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SUMMARY

This paper deals with elasto-plastic large deformation analysis of space-frames. It is based on a complementary energy approach. A methodology is presented wherein: (i) each member of the frame, modelled as an initially straight space-beam, is sought to be represented by a single finite element, (ii) each member can undergo arbitrarily large rigid rotations, but only moderately large relative rotations; (iii) a plastic-hinge method, with arbitrary locations of the hinges along the beam, is used to account for plasticity, (iv) the non-linear bending–stretching coupling is accounted for in each member, (v) the applied loading may be non-conservative and (vi) an explicit expression for the tangent stiffness matrix of each element is given under conditions (i) to (v). Several examples, with both quasi-static and dynamic loading, are given to illustrate the accuracy and efficiency of the approaches presented.

1. INTRODUCTION

Efficient and simplified procedures for geometrically non-linear as well as elasto-plastic analysis of space-frames are of continued interest to the off-shore technology and space exploration industries.

The subject of finite deformations (finite rotations and stretches) of beams is receiving a renewed scrutiny (see for instance Besseling1 and Geradin and Cardona2). Also, there have been attempts to derive an explicit expression for the tangent stiffness matrix of a beam-element, accounting for large rigid rotations, moderate relative rotations, and the bending–stretching coupling. Further, for ranges of deformation of each member of a space-frame of practical interest, each member was sought to be represented by a single finite element. Towards these ends, Kondoh and Atluri3 first presented a displacement-based approach for plane-frames, wherein the exact solution for the displacement of each member, modelled as a beam column, was employed. Later, Kondoh et al.4,5 extended the displacement approach to the three-dimensional case, using the concept of semi-tangential rotations. However, these displacement approaches did not consider the effect of plastic deformations, nor of non-conservative loading. Later work by Kondoh and Atluri6 showed that a complementary energy approach, based on assumed stresses, is a much simpler alternative to treat elasto-plastic large deformations of plane-frames. It was shown in Kondoh and Atluri that the plastic-hinge concept, and the complementary energy approach, permitted the derivation of an explicit expression (without numerical integration) for the tangent stiffness of each finite element, which was sufficient to model each member of a frame undergoing large elasto-plastic deformations.

In this paper, a complementary energy and plastic-hinge approach is presented for the large...
deformation elasto-plastic analysis of three-dimensional space-frames, under general non-conservative loading. The present formulation is based on assumed stress resultants and stress couples, which satisfy the linearized momentum balance conditions of the space-beam, a priori. The beam is assumed to undergo arbitrarily large deformations, which are decomposed into (i) an arbitrarily large three-dimensional rigid rotation of the beam as a whole and (ii) moderately large, non-rigid, point wise rotations. The non-linear bending–stretching coupling is accounted for exactly in each element. A plastic-hinge method, wherein the hinges may form at arbitrary locations along the beam, is used to account for plasticity. It is shown to be possible to derive an explicit expression for the tangent stiffness matrix of each beam (which is unsymmetric under non-conservative loading).

The paper is organized as follows. Section 2 deals with the kinematics of large deformations as presently assumed for each element; Section 3 deals with momentum balance relations; Section 4 deals with a weak formulation of the problem; Section 5 deals with plasticity effects; Section 6 deals with equation-solving algorithms; Section 7 deals with numerical examples; Section 8 deals with concluding remarks; and the Appendix gives the explicit expressions for the tangent stiffness matrix.

Notation. Second-order tensors are indicated by bold italic characters, and vectors by bold roman characters. If $A(A_i e_i e_i)$ and $B(B_i e_i e_i)$ are two second-order tensors, then $A + B = A_{ab} B_{bc} e_i e_j$ and $A : B = A_{ab} B_{bc} e_i e_j$.

2. KINEMATICS OF LARGE DEFORMATION OF A SPACE-FRAME MEMBER

Consider a typical frame member, modelled here as a beam-element that spans between nodes 1 and 2, in three-dimensional space, as shown in Figure 1. The element has a uniform cross section and is of length $l$ before deformation. The co-ordinates $X_i (i=1, 2, 3)$ are the global co-ordinates with the unit basis vectors $e_i$. $X'_i$ and $e'_i$ are the local co-ordinates for the undeformed element. Let $X_i$ denote the global co-ordinates of node $a$ of the member. The basis vector $e_i$ are chosen such that

$$e'_i = (DX_1 e_i + DX_2 e_2 + DX_3 e_3)/l$$
$$e'_2 = (e_3 \times e'_i)/|e_3 \times e'_i|$$
$$e'_3 = e'_i \times e'_2$$

where

$$DX_i = X'_i - X_i$$
$$l = ((DX_1)^2 + (DX_2)^2 + (DX_3)^2)^{1/2}$$

Let $\hat{x}_i$ and $\hat{e}_i$ be the local co-ordinates for the deformed element where $\hat{e}_i$ are determined in the same way as $e_i$. Another co-ordinate system is $e^*_i$, which is locally tangential and normal to the deformed centroidal axis of the element (See Figure 1). Let $x_i (i=1, 2, 3)$ denote the displacements at the centroidal axis of the element along the direction $e_i$ at node $a$, $\Theta_i$ the rotations about axis of $e_i$ at node $a$ and $\Theta^*_i$ the rotations about axis of $\hat{e}_i$. Then $e_i$ and $\hat{e}_i$ have the following relations:

$$\begin{bmatrix} e'_1 \\ e'_2 \\ e'_3 \end{bmatrix} = \begin{bmatrix} DX_1/l & DX_2/l & DX_3/l \\ -DX_2/s & DX_1/s & 0 \\ -DX_1 DX_3/s & -DX_2 DX_3/s & s/l \end{bmatrix} \begin{bmatrix} e_i \\ e_i \\ e_i \end{bmatrix}$$

$$= [CA_0] [e_1 e_2 e_3]^T$$

$$e_i = \begin{bmatrix} DX_1 \\ DX_2 \\ DX_3 \end{bmatrix}$$
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\[ \gamma_1 \times x_3 \cdot u_3 \]

where

By letting

By letting

one derives the following relations between \( e_i \) and \( \dot{e}_j \):

\[
\begin{pmatrix}
\dot{e}_1 \\
\dot{e}_2 \\
\dot{e}_3
\end{pmatrix} =
\begin{bmatrix}
\frac{\dot{D}X_1}{L} & \frac{\dot{D}X_2}{L} & \frac{\dot{D}X_3}{L} \\
-\frac{\dot{D}X_2}{S} & \frac{\dot{D}X_1}{S} & 0 \\
-\frac{\dot{D}X_2}{LS} & -\frac{\dot{D}X_1}{LS} & \frac{S}{L}
\end{bmatrix}
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}
\]

\[ \text{where} \]

\[ L = \left( (\dot{D}X_1)^2 + (\dot{D}X_2)^2 + (\dot{D}X_3)^2 \right)^{1/2} \]

\[ S = \left( (\dot{D}X_1)^2 + (\dot{D}X_2)^2 \right)^{1/2} \]

Under the assumption of small relative rotations \( (\Theta^* \ll 1) \), the change of the length of the element,
denoted here as $H$, is given by

$$H = \left( (D\dot{X}_1)^2 + (D\dot{X}_2)^2 + (D\dot{X}_3)^2 \right)^{1/2} - 1 \quad (13)$$

From equations (8) to (12), $CA$ can be written as

$$CA = \begin{bmatrix}
\frac{\partial H}{\partial \bar{u}_1} & \frac{\partial H}{\partial \bar{u}_2} & \frac{\partial H}{\partial \bar{u}_3} \\
-\frac{\partial S}{\partial \bar{u}_2} & \frac{\partial S}{\partial \bar{u}_1} & 0 \\
-\frac{\partial H}{\partial \bar{u}_3} & -\frac{\partial H}{\partial \bar{u}_3} & \frac{S}{L}
\end{bmatrix} \quad (14)$$

When the element is parallel to the $X_3$ axis, $S = [(D\dot{X}_1)^2 + (D\dot{X}_2)^2]^{1/2} = 0$ and equation (6) is not valid. In this case, the local co-ordinates can be determined by setting

$$e'_1 = e_3$$
$$e'_2 = e_2$$
$$e'_3 = -e_1 \quad (15)$$

From the definitions given above, the displacements along the local co-ordinates $X'_i$ at node $a$ are

$$\begin{bmatrix}
\bar{u}'_1 \\
\bar{u}'_2 \\
\bar{u}'_3
\end{bmatrix} = \begin{bmatrix}
\bar{u}_1 \\
\bar{u}_2 \\
\bar{u}_3
\end{bmatrix} = CA_0 \begin{bmatrix}
\bar{u}_1 \\
\bar{u}_2 \\
\bar{u}_3
\end{bmatrix} \quad \alpha = 1, 2 \quad (16)$$

By letting

$$\bar{v}_i = \bar{u}'_i - \bar{u}_i \quad i = 1, 2, 3 \quad (17)$$

the relative rotations about axis of $\bar{e}_i$ at node $a$ are

$$\Theta^*_i = (CA_0)_{ij} \Theta_j$$

$$\Theta^*_i = (CA_0)_{ij} \Theta_j + \tan^{-1} \left[ \frac{\bar{v}_3}{\bar{v}_1} \right] \quad \alpha = 1, 2 \quad j = 1, 2, 3 \quad (18)$$

The curvature strains in the deformed beam are given by

$$\kappa_i = \Theta^*_i,_{i,1}$$
$$\kappa_2 = -\Theta^*_2,_{i,1}$$
$$\kappa_3 = -\Theta^*_3,_{i,1} \quad (19)$$

3. MOMENTUM BALANCE RELATIONS

The nomenclature for nodal forces in a space-frame member is shown in Figure 2. The general and consistent forms of linear and angular momentum balance are
Figure 2. Nomenclature for nodal forces on a space framed member

\[ \frac{\partial N}{\partial x_1} + q_1 = 0 \]  
\[ \frac{\partial M_1}{\partial x_1} = 0 \]  
\[ \frac{\partial^2 M_2}{\partial x_1^2} + \frac{\partial}{\partial x_1} (N \Theta_1) + q_2 = 0 \]  
\[ \frac{\partial^2 M_3}{\partial x_1^2} + \frac{\partial}{\partial x_1} (N \Theta_1) - q_3 = 0 \]  

Where \( q_i \) \((i = 1, 2, 3)\) are the distributed loads along \( e_i \) directions per unit length of the undeformed element.

The load acting on the element can be considered to be of both a 'conservative' and of a 'non-conservative' type. Let \( q_{ei} \) be the conservative type loading along \( e_i \) direction per unit length of the undeformed beam. Let \( q_{ni} \), be the 'non-conservative' type loads which always remain, respectively, tangential and normal to the rigidly rotated axis of the element, i.e. along \( e_i \). Therefore, we have

\[ q_i = (CA)_i q_{ei} + q_{ni} \quad i = 1, 2, 3, \quad j = 1, 2, 3 \]  

In general, we shall assume, for the purposes of a discrete solution of the problem of a frame, trial functions for \( N \) and \( M_1 \) which satisfy \( a \) priori equations (20) and (21); and trial functions for \( M_2 \) and \( M_3 \) which only satisfy the linear parts of equations (22) and (23), namely,

\[ \frac{\partial^2 M_2}{\partial x_1^2} + q_2 = 0 \]  
\[ \frac{\partial^2 M_3}{\partial x_1^2} - q_3 = 0 \]
The admissible trial functions are thus
\[ N = n + N_p \]
\[ M_1 = m_1 \]
\[ M_2 = \left(1 - \frac{X_1}{l}\right)^{m_2 + \frac{X_1}{l} m_2 + M_{p2}} \]
\[ M_3 = \left(1 - \frac{X_1}{l}\right)^{m_3 + \frac{X_1}{l} m_3 - M_{p3}} \]
where
\[ N_p = (CA)_{ij}N_{pej} + N_{pe} \quad j = 1, 2, 3 \]
\[ M_{p2} = (CA)_{ij}M_{pej} + M_{pe} \quad j = 1, 2, 3 \]
\[ M_{p3} = (CA)_{ij}M_{pej} + M_{pe} \quad j = 1, 2, 3 \]
and
\[ N_{pej} = - \int_0^{\hat{x}_1} q_{ej} d\hat{x}_1 + \frac{1}{l} \int_0^{\hat{x}_1} \left[ \int_0^{\hat{x}_1} q_{ej} d\hat{x}_1 \right] d\hat{x}_1 \quad j = 1, 2, 3 \]
\[ N_{pe} = - \int_0^{\hat{x}_1} q_{n1} d\hat{x}_1 + \frac{1}{l} \int_0^{\hat{x}_1} \left[ \int_0^{\hat{x}_1} q_{n1} d\hat{x}_1 \right] d\hat{x}_1 \]
\[ M_{pej} = - \int_0^{\hat{x}_1} \left[ \int_0^{\hat{x}_1} q_{ej} d\hat{x}_1 \right] d\hat{x}_1 + \frac{1}{l} \int_0^{\hat{x}_1} \left[ \int_0^{\hat{x}_1} q_{ej} d\hat{x}_1 \right] d\hat{x}_1 \quad j = 1, 2, 3 \]
\[ M_{pe} = - \int_0^{\hat{x}_1} \left[ \int_0^{\hat{x}_1} q_{n1} d\hat{x}_1 \right] d\hat{x}_1 + \frac{1}{l} \int_0^{\hat{x}_1} \left[ \int_0^{\hat{x}_1} q_{n1} d\hat{x}_1 \right] d\hat{x}_1 \quad i = 2, 3 \]

The corresponding test functions (or variations in \( N \) and \( M_i \)) are taken to satisfy the homogenous form of equations (20) to (23), i.e.
\[ \delta N = v \]
\[ \delta M_1 = \mu_1 \]
\[ \delta M_i = \mu_i = \mu_i \left(1 - \frac{X_1}{l}\right) + 2\mu_i \left(\frac{X_1}{l}\right) \quad i = 2, 3 \]

4. A FORMULATION FOR OBTAINING A WEAK SOLUTION

The stress–strain relations between the conjugate pairs of mechanical and kinematic variables are assumed to be of the form
\[ \frac{\partial W_c}{\partial N} = h \]
\[ \frac{\partial W_c}{\partial M_i} = \kappa_i \quad i = 1, 2, 3 \]
where \( W_c \) is the complementary energy density, and \( h \) is the strain in the \( \epsilon_1 \) direction. For a linear
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elastic material, when the reference axis of the element is at mid-thickness, one has

\[ W_e = \frac{1}{2} \left( \frac{N^2}{EA} + \frac{M_1^2}{EI_1} + \frac{M_2^2}{EI_2} + \frac{M_3^2}{EI_3} \right) \]  

(40)

where \( A \) is the cross section area of the element, \( E \) is the Young's modulus and \( I_i (i = 1, 2, 3) \) are the moments of inertia of the cross section about \( X_i \) directions. Thus, it remains to enforce:

(i) 'compatibility' of deformation within each beam; (ii) momentum balance conditions within each beam; and (iii) the joint equilibrium (or 'inter-element' traction reciprocity).

The weak forms of these conditions may be written respectively as follows.

(i) Compatibility

\[ \int_0^l \frac{\partial W_e}{\partial \hat{x}_1} \, v \, d \hat{x}_1 = \int_0^l \frac{\partial W_e}{\partial \mu_1} \, u_1 \, v \, d \hat{x}_1 = vH = \nu \{ [D \hat{X}_1^2 + D \hat{X}_2^2 + D \hat{X}_3^2]^{1/2} - l \} \]  

(41)

\[ \int_0^l \frac{\partial W_e}{\partial M_1} \, \mu_1 \, d \hat{x}_1 = \int_0^l \Theta_1^\ast \mu_1 \, d \hat{x}_1 = [\Theta_1^\ast \mu_1]_0 = (\Theta_1^\ast - \Theta_1^\ast \mu_1) \mu_1 \]  

(42)

\[ \int_0^l \frac{\partial W_e}{\partial M_2} \, \mu_2 \, d \hat{x}_1 = - \int_0^l \Theta_2^\ast \mu_2 \, d \hat{x}_1 = - \int_0^l (\Theta_2^\ast \mu_2) \, d \hat{x}_1 + \int_0^l \Theta_2^\ast \frac{\partial \mu_2}{\partial \hat{x}_1} \, d \hat{x}_1 \]

Since \( \partial \mu_2/\partial \hat{X}_1 = (- \mu_1^2 + \mu_2)/l = \text{constant and} l', \Theta_2^\ast \, d \hat{x}_1 = 0 \), we have

\[ \int_0^l \frac{\partial W_e}{\partial M_2} \, \mu_2 \, d \hat{x}_1 = - \int_0^l [\Theta_2^\ast \mu_2] \, d \hat{x}_1 = - 2 \Theta_2^\ast \mu_2 + 1 \Theta_2^\ast \mu_2 \]  

(43)

Similarly, we have

\[ \int_0^l \frac{\partial W_e}{\partial M_3} \, \mu_3 \, d \hat{x}_1 = - \int_0^l [\Theta_3^\ast \mu_3] \, d \hat{x}_1 = - 2 \Theta_3^\ast \mu_3 + 1 \Theta_3^\ast \mu_3 \]  

(44)

Note that, unlike in traditional continuum mechanics, the above weak compatibility conditions are for the beam as a whole rather than at each point along the beam.

(ii) Interior momentum balance

Consider the variations along the beam (or test functions) of generalized displacements, such that \( \delta u_1 = v \), and \( \delta u_2 = \mu_2 \). We rewrite the momentum balance relations, equations (22) and (23), in two parts, as

\[ \frac{\partial \dot{Q}_2}{\partial \hat{X}_1} + 4_2 = 0 \]  

(45)

\[ \frac{\partial \dot{Q}_3}{\partial \hat{X}_1} + 4_3 = 0 \]  

(46)

\[ \frac{\partial \ddot{X}_2}{\partial \hat{X}_1} - Q_3 + N \Theta_2^\ast = 0 \]  

(47)

\[ \frac{\partial \ddot{X}_3}{\partial \hat{X}_1} + Q_2 + N \Theta_3^\ast = 0 \]  

(48)

Now equations (45) and (46) are assumed to be satisfied identically, a priori. The weak form of the
remaining balance conditions may be written as

\[
\int_0^l \left[ \frac{\partial M_2}{\partial X_1} - Q_3 + N \Theta_3 \right] \beta_3^* dX_1 = 0
\] (49)

\[
\int_0^l \left[ \frac{\partial M_3}{\partial X_1} + Q_4 + N \Theta_4 \right] \beta_4^* dX_1 = 0
\] (50)

Recall the trial functions for \( M_2 \) and \( M_3 \), assumed in each element, satisfy only the linear parts of equations (22) and (23). Hence, we have

\[
Q_2 = -\frac{\partial M_3}{\partial X_1} = -\frac{m_3 - m_1}{l} + \frac{\partial M_{p3}}{\partial X_1}
\] (51)

\[
Q_3 = \frac{\partial M_2}{\partial X_1} = \frac{m_3 - m_2}{l} + \frac{\partial M_{p3}}{\partial X_1}
\] (52)

(iii) Joint equilibrium equation

For each node which is the end point of the element, there are internal axial forces \( N(x=1, 2) \), transverse point forces \( Q(i=2, 3) \), bending moments \( \dot{M}_1 (i=1, 2, 3) \); and external forces \( F(x=1, 2, 3) \) (along \( e \) directions) and moments \( \dot{M}_1 (i=1, 2, 3) \) (around \( e \) axes), respectively.

Let \( (NM) \) denote the internal nodal force vector

\[
(NM) = (1N; 1 \dot{Q}_1; 2 \dot{M}_1; 1 \dot{M}_2; 1 \dot{M}_3; 2N; 2 \dot{Q}_1; 2 \dot{Q}_2; 2 \dot{M}_1; 2 \dot{M}_2; 2 \dot{M}_3)^T
\] (53)

and \( \bar{F} \) denote the external nodal force vector at nodes 1 and 2

\[
F = (1F_1; 1F_2; 1F_3; 1 \dot{M}_1; 1 \dot{M}_2; 1 \dot{M}_3; 2F_1; 2F_2; 2F_3; 2 \dot{M}_1; 2 \dot{M}_2; 2 \dot{M}_3)^T
\] (54)

and define a transformation matrix \( CAT \):

\[
CAT = \begin{bmatrix}
-CAT^T & 0 \\
0 & -CAT^T \\
0 & CAT^T
\end{bmatrix}
\] (55)

where \( CAT \) are defined by equation (10)

The joint equilibrium conditions at nodes 1 and 2 of the element can then be written as

\[
\sum_{\text{elem}} (CAT)(NM) = \bar{F}
\] (56)

where the summation extends over the elements meeting at each of the nodes. In developing the individual element stiffness matrix, load vector, etc., the externally applied nodal loads will henceforth be omitted. They will be treated as global nodal loads once the system stiffness and loads are assembled in the usual fashion. Let \( \{d\} \) represent the vector of nodal displacement of an element in the global co-ordinates, i.e.

\[
d = \{u_1; u_2; 1^\Theta_1; 1^\Theta_2; 1^\Theta_3; 2u_1; 2u_2; 2^\Theta_1; 2^\Theta_2; 2^\Theta_3\}^T
\] (57)

and \( v \) represent the variation of \( d \) (or test function) i.e.

\[
v = \delta d
\] (58)
Then we have the weak form of equation (56) (omitting $F$) as

$$\sum_{\text{elem}} (v)^T (CA T) (NM) = 0$$  \hfill (59)

(iv) The combined weak form of compatibility, and element as well as joint equilibrium conditions

Adding equations (41)-(44), (49), (50) and (59) together, we obtain the combined weak form as

$$\sum_{\text{elem}} \left\{ - \int_0^1 \frac{\partial W_c}{\partial N} v d\xi_1 - \int_0^1 \frac{\partial W_c}{\partial M_1} \mu_1 d\xi_1 - \int_0^1 \frac{\partial W_c}{\partial M_2} \mu_2 d\xi_1 - \int_0^1 \frac{\partial W_c}{\partial M_3} \mu_3 d\xi_1 ight\}$$

$$+ [vH + (\Theta^1_1 - \Theta^1_1) \mu_1 + \Theta^2_1 \mu_2 - \Theta^2_2 \mu_2 + \Theta^3_1 \mu_3 - \Theta^3_2 \mu_3]$$

$$+ \delta d^T (CA T) (NM) + \int_0^1 N \Theta^2_1 \beta_1 d\xi_1 + \int_0^1 N \Theta^3_2 \beta_2 d\xi_1 \} = 0 \hfill (60)$$

We let $D$ denote the relative displacement vector in the local co-ordinates $\xi$, for the deformed element, i.e.

$$D = [H; (\Theta^1_1 - \Theta^1_1); \Theta^2_2; \Theta^3_2] \hfill (61)$$

and let $\sigma$ denotes the internal force vector which only satisfies the homogeneous forms of the equilibrium equations (20) to (23), i.e.

$$\sigma = \{n; m_1; m_2; m_3; m_3\}^T \hfill (62)$$

The corresponding test functions are taken to be

$$\delta \sigma = \{v; \mu_1; \mu_2; \mu_3; \mu_3\}^T \hfill (63)$$

From equations (25)-(28), we see that the 'homogeneous' parts of the trial functions are

$$F = \begin{bmatrix}
  n & m_1 & 0 & 0 & 0 \\
  0 & m_2 & 0 & 0 & 0 \\
  0 & 0 & 1 & X_1/l & X_1/l \\
  0 & 0 & 0 & 1 & X_1/l \\
  0 & 0 & 0 & 2 & X_1/l \\
  0 & 2 & m_2 & 0 & 0 \\
  0 & 0 & 2 & m_2 & 0 \\
  0 & 0 & 0 & 2 & m_2 \\
  0 & 0 & 0 & 0 & 0
\end{bmatrix} \hfill (64)$$

Correspondingly, we have

$$\begin{bmatrix}
  v \\
  \mu_1 \\
  \mu_2 \\
  \mu_3
\end{bmatrix} = F \delta \sigma \hfill (65)$$

We define the vector $W$ as

$$W = \left[ \frac{\partial W_e \partial W_e \partial W_e \partial W_e}{\partial N \partial M_1 \partial M_2 \partial M_3} \right]^T \hfill (66)$$
Substituting equations (61) to (66) into equation (60), and neglecting terms \((N\Theta_2^p \beta_2^p)\) and \((N\Theta_3^p \beta_3^p)\), equation (60) can be written in the matrix form.

\[
\sum_{\text{elem}} \left\{ - \int_0^l W_{i} \dot{H} \sigma d\delta \dot{h}_1 + \dot{D} \dot{\sigma} + \dot{\delta} (C^T A^T)(NM) \right\} = 0
\]  

(67)

Equation (67) will play a key role in the present finite element development. While omitting the terms \((N\Theta_2^p \beta_2^p)\) and \((N\Theta_3^p \beta_3^p)\) may lead to errors in the tangent stiffness matrix, it is entirely consistent in the context of the present iterative approach to solve large deformation problems, as discussed in Reference 6. The details of the algebraic formulation of the stiffness matrix resulting from (67) are given in the Appendix. However, it should be noted that the only integrals to be evaluated over the length of the beam, i.e. the first term on the left-hand side of (67) can be evaluated trivially in closed form. This fact, and the nature of the plastic-hinge method discussed below, enable us to derive an explicit expression for the tangent stiffness of the beam undergoing large deformations and plasticity.

(v) Plasticity effects in the large deformation behaviour of frames

In the present approach, a plastic-hinge method is employed to derive an explicit expression for tangent stiffness, in the presence of plastic deformation.

For simplicity, we assume that the material is elastic–perfect plastic and that the yield condition has the form

\[ f(N, \dot{M}_1, \dot{M}_2, \dot{M}_3) = 0 \quad \text{at} \quad \dot{x}_1 = l_p \]  

(68)

where \(\dot{x}_1 = l_p\) is the location of the plastic hinge. The incremental plastic flow condition at the plastic hinge may be written as

\[
\frac{\partial f}{\partial N} \Delta N + \frac{\partial f}{\partial M_1} \Delta M_1 + \frac{\partial f}{\partial M_2} \Delta M_2 + \frac{\partial f}{\partial M_3} \Delta M_3 = 0 \quad \text{at} \quad \dot{x}_1 = l_p
\]  

(69)

and the incremental plastic deformations at the hinge are given by

\[
(\Delta H_p)_p = \lambda \frac{\partial f}{\partial N} \bigg|_{l_p} \]  

(70)

\[
(\Delta \Theta_1^p)_p = (\Delta \lambda) \frac{\partial f}{\partial M_1} \bigg|_{l_p}, \quad i = 1, 2, 3
\]  

(71)

where \((\Delta H_p)_p\) is the increment of plastic elongation \(H_p\), and \(\Delta \Theta_1^p\) is the increment of plastic rotation \(\Theta_1^p\).

If we let

\[
H_p = \Sigma (\Delta H_p)_p = \Sigma \Delta \lambda \frac{\partial f}{\partial N} \bigg|_{l_p} \]  

(72)

\[
\Theta_1^p = \Sigma (\Delta \Theta_1^p)_p = \Sigma \Delta \lambda \frac{\partial f}{\partial M_1} \bigg|_{l_p} \quad i = 1, 2, 3
\]  

(73)

then the compatibility conditions in the presence of plastic deformation, are given by

\[
\int \frac{\partial W_c}{\partial N} v dX_1 + H_p v = H_v
\]  

(74)
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\[ \int \frac{\partial W_e^s}{\partial M_1} \mu_1 dX_1 + \Theta_p^s \mu_1 = (2\Theta^s - \Theta^s_1) \mu_1 \]  
(75)

\[ \int \frac{\partial W_e^s}{\partial M_2} \mu_2 dX_1 + \Theta_p^s \mu_2 \right|_p = -\Theta^s_2 \mu_2 - 2\Theta^s_2 \mu_2 \]  
(76)

\[ \int \frac{\partial W_e^s}{\partial M_3} \mu_3 dX_1 + \Theta_p^s \mu_3 \right|_p = -\Theta^s_3 \mu_3 - 2\Theta^s_3 \mu_3 \]  
(77)

Therefore, the combined weak form for plastic analysis is obtained by adding the following to equation (67), i.e.

\[ \sum_{\text{elem}} - \left\{ H_v + \Theta_p^s \mu_1 + \Theta_p^s \mu_2 + \delta \left( \frac{\partial f}{\partial N_1} \Delta N + \frac{\partial f}{\partial M_1} \Delta M_1 + \frac{\partial f}{\partial M_2} \Delta M_2 + \frac{\partial f}{\partial M_3} \Delta M_3 \right) \right\}_p \]  
(78)

6. SOLUTION ALGORITHMS

The Newton-Raphson or the modified Newton-Raphson procedures are employed to solve the incremental tangent stiffness equations. A variant of the arc-length method (as discussed in detail in Kondoh and Alturi3 and the references cited therein) is used to control the load-increment for static analysis. The Newmark time integration algorithm, with \( \alpha = 1/4 \) and \( \delta = 1/2 \), is used for transient non-linear dynamic response, to obtain

\[ \mathbf{\dot{R}} \Delta u^{(\kappa)} = \mathbf{\dot{R}} \Delta p - \mathbf{\dot{R}} \Delta \mathbf{u}^{(\kappa-1)} \]  
(79)

where

\[ \mathbf{\dot{R}} = \mathbf{K} + \beta a_1 a_2 \mathbf{K} + (a_2 + \gamma a_1 a_2) \mathbf{M} \]

\[ \mathbf{\dot{R}} \Delta F^{(\kappa-1)} = \mathbf{M} \left\{ \mathbf{\dot{R}} \Delta \mathbf{u}^{(\kappa-1)} + \mathbf{\dot{R}} \Delta \mathbf{u}^{(\kappa-1)} \right\} + \beta \mathbf{K} \mathbf{\dot{R}} \Delta \mathbf{u}^{(\kappa-1)} \]  
(80)

and \( a_1 = \Delta t/2; a_2 = 4/(\Delta t)^2 \); \( \beta \) and \( \gamma \) are coefficients of proportional damping [i.e. \( \mathbf{C} = \gamma \mathbf{M} + \beta \mathbf{K} \)]; \( \mathbf{\dot{R}} \Delta \mathbf{F}^{(\kappa-1)} \) is the internal nodal-force vector at time \( (t+\Delta t) \) and iteration \( (\kappa-1) \).

7. NUMERICAL EXAMPLES

Several numerical examples are presented in this section to demonstrate the efficiency of the method presented in this paper.

Example 1

This is a three-member rigid-knee frame as shown in Figure 3. In the analysis model, each of the shorter members is modelled as an element. In order to compute the critical loading, the longer member is divided into two elements and a transverse perturbation loading \( \Delta p = 0.001p \) is applied at the mid point of the member. The force-displacement curve is plotted in Figure 4. Berke et al.7 studied the same problem. The computed critical load here is a little higher than that obtained by Berke et al., because the present method is based on assumed stresses, while that in Berke et al. is based on a potential energy approach.
Figure 3. Three-member right-angled knee-frame

Figure 4. Load-deflection curve for knee-frame
Example 2

The second example is a framed dome, the data for which are shown in Figure 5. Here, each member is modelled by a single element. We consider two systems of loadings. The first system of loading consists of concentrated vertical loads of equal magnitude $p$ at the crown point and at the end points of the horizontal members. The force—displacement curve at the crown point is shown in Figure 6. The second type of loading is that of a single concentrated load at the crown point. The force—displacement relation is shown in Figure 7. The results for case 1, in Figure 6 agree well with the independent results of Chu and Rampetsreiter. The second case was also studied by Remseth; however, his results appear to different significantly from the present ones shown in Figure 7. The second case was re-analysed using the displacement formulation of Kondoh et al., and the ensuing results, shown in Figure 7, agree quite well with the present complementary energy approach. Thus, it appears that Remseth’s results may be in error.

Figure 5. Framed dome (the unit of length is metre)
Example 3

Now we consider some examples of dynamic response. First we consider a simple example: a fixed-fixed beam. When a static load is applied at the centre of the beam, the linear and non-linear load-deflection curves are shown in Figure 8, using two elements to model the beam. For the dynamic problem, \( m = 5.63 \times 10^{-5} \text{ kg-sec}^2/\text{cm}^4 \), and the load \( F(t) \) acting at the centre of the beam is suddenly applied and remains constant in time. The dynamic responses, without damping and with damping (\( C/m = F/m/5000 \)), are shown in Figure 9. These results agree excellently with those of Weeks\(^{10}\) and Nickell.\(^{11}\)

Example 4

This concerns the same frame shown in Figure 3 with the mass density of all the members being 2400 kg/m\(^3\); and with a harmonically varying load, \( P(t) = 68.8 \text{ MN sin}(41.88t) \) being applied at the crown. In the computational model, each member is modelled by a single element. The dynamic response is shown in Figure 10. The present dynamic response results differ significantly from those
of Remseth. However, in a static analysis, the results from the present procedure were also found to differ significantly from those of Remseth, but agree excellently with results for the same problem by Argyris. Thus, the results of Remseth appear to be in error. However, the number of degrees of freedom in the present method is much less (by a factor of 2) as compared to that in Remseth.
Figure 8. Static deflection of a fixed-fixed beam (One element on half span)

Figure 9. Dynamic response of a fixed-fixed beam
Figure 10. Non-linear response due to a harmonic variation of concentrated load at crown point of a framed dome

Figure 11. Plastic hinge development in the right-angle bent
Example 5

We now consider an elastic—plastic problem. This right-angle bent as shown in Figure 11 is subjected to both bending and twisting. For a perfectly plastic material, the yielding condition is

$$\left(\frac{M}{M_0}\right)^2 + \left(\frac{T}{T_0}\right)^2 = 1$$

where $M_0$ and $T_0$ are fully plastic bending and twisting moments, respectively. Each of the members is modelled by a single element. The progressive development of plastic hinges as the load increases is shown in Figure 12. The results agree excellently with those in Hodge (1959).

Example 6

This concerns a four-legged jacket type of platform structure often used in the off-shore industries. The geometry and dimension of the structure, similar to the one treated in Soreide.
et al.\textsuperscript{13} are shown in Figure 13. Two different structural systems, denoted here as S1 and S2, respectively, are considered, the dimensions of S2 being given in parenthesis in Figure 13. In the present study, each member is modelled by a single finite element. The load versus horizontal displacement at node E, for systems S1 and S2, is shown in Figure 14. The results are in good qualitative agreement with those of Soreide et al.\textsuperscript{13}

8. CLOSURE

For space-frames undergoing elasto-plastic large deformations under non-conservative loading, an assumed stress approach and a plastic-hinge method are employed to obtain explicit expressions for the tangent stiffness matrix. For large deformations of practical interest it is sufficient to use a single element to model each member of the space-frame. Several numerical studies suggest that the present development is very economical and accurate in analysing large-deformation inelastic response of frames. The modification of the present procedures to include joint flexibility is straightforward, and will be discussed in a follow-on paper.
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APPENDIX

Explicit expressions for tangent stiffness matrix, load vectors and compatibility correction vectors for a space-frame member, undergoing large elasto-plastic deformation

The combined weak form of the incremental compatibility conditions and joint equilibrium condition can be derived from equation (67) by retaining only terms of the first order in the increments of the parameters in the test functions, and terms of order one, as well as of first order in
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the increments of the parameters, in the trial functions; i.e.

\[
\sum_{\text{elem}} \{ \delta \Delta d \cdot (\text{CAT})(\text{NM}) + \delta \Delta d' \cdot (\text{CAT})(\text{NM}) + \delta \Delta d'' \cdot (\text{CAT}) \Delta (\text{NM}) \}
\]

\[+ \delta \Delta \sigma \left( D - \int_0^1 F^t \text{W} d \xi_1 \right) + \delta \Delta \sigma' \left( \Delta D - \int_0^1 F^t \Delta \text{W} d \xi_1 \right) \} = 0. \quad (A1)\]

Let us first consider only the first three rows in \( \text{DCAT.(NM)} \).

\[
\Delta \text{CAT} \begin{bmatrix} 1^N \\ 1^Q_2 \\ 1^Q_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \Delta \vec{u}_1 \\ \Delta \vec{u}_2 \\ \Delta \vec{u}_3 \end{bmatrix} \quad (A2)
\]

where

\[
A_{11} = \frac{\delta^2 H}{\delta \vec{u}_1 \delta \vec{u}_2} 1^N - \frac{\delta^2 S}{\delta \vec{u}_1 \delta \vec{u}_2} 1^Q_2 - \left( \frac{\partial S}{\partial \vec{u}_1} \frac{\partial^2 H}{\partial \vec{u}_2^2} + \frac{\partial^2 S}{\partial \vec{u}_1 \partial \vec{u}_2} \right) 1^Q_3
\]

\[
A_{12} = \frac{\delta^3 H}{\delta \vec{u}_1 \delta \vec{u}_2 \delta \vec{u}_3} 1^N - \frac{\delta^2 S}{\delta \vec{u}_1 \delta \vec{u}_3} 1^Q_2 - \left( \frac{\partial S}{\partial \vec{u}_2} \frac{\partial^2 H}{\partial \vec{u}_3^2} + \frac{\partial^2 S}{\partial \vec{u}_2 \partial \vec{u}_3} \right) 1^Q_3
\]

\[
A_{13} = \frac{\delta^2 H}{\delta \vec{u}_1 \delta \vec{u}_3} 1^N - \left( \frac{\partial S}{\partial \vec{u}_1} \frac{\partial^2 H}{\partial \vec{u}_3^2} \right) 1^Q_3
\]

\[
A_{21} = \frac{\delta^2 H}{\delta \vec{u}_1 \delta \vec{u}_2} 1^N - \frac{\delta^2 S}{\delta \vec{u}_1 \delta \vec{u}_2} 1^Q_2 - \left( \frac{\partial S}{\partial \vec{u}_2} \frac{\partial^2 H}{\partial \vec{u}_3^2} + \frac{\partial^2 S}{\partial \vec{u}_2 \partial \vec{u}_3} \right) 1^Q_3
\]

\[
A_{22} = \frac{\delta^3 H}{\delta \vec{u}_2 \delta \vec{u}_3} 1^N - \frac{\delta^2 S}{\delta \vec{u}_2 \delta \vec{u}_3} 1^Q_2 - \left( \frac{\partial S}{\partial \vec{u}_3} \frac{\partial^2 H}{\partial \vec{u}_2^2} + \frac{\partial^2 S}{\partial \vec{u}_2 \partial \vec{u}_3} \right) 1^Q_3
\]

\[
A_{23} = \frac{\delta^2 H}{\delta \vec{u}_2 \delta \vec{u}_3} 1^N - \left( \frac{\partial S}{\partial \vec{u}_2} \frac{\partial^2 H}{\partial \vec{u}_3^2} \right) 1^Q_3
\]

\[
A_{31} = \frac{\delta^2 H}{\delta \vec{u}_1 \delta \vec{u}_3} 1^N + \left( \frac{1}{L} \frac{\partial S}{\partial \vec{u}_1} - \frac{S}{L^2} \frac{\partial L}{\partial \vec{u}_1} \right) 1^Q_3
\]

\[
A_{32} = \frac{\delta^2 H}{\delta \vec{u}_2 \delta \vec{u}_3} 1^N + \left( \frac{1}{L} \frac{\partial S}{\partial \vec{u}_2} - \frac{S}{L^2} \frac{\partial L}{\partial \vec{u}_2} \right) 1^Q_3
\]

\[
A_{33} = \frac{\delta^3 H}{\delta \vec{u}_3^3} 1^N - \frac{S}{L^2} \frac{\partial L}{\partial \vec{u}_3} 1^Q_3
\]

By replacing \( \{1^N 1^Q_2 1^Q_3\} \) by \( \{\bar{M}_1, \bar{M}_2, \bar{M}_3\} \), \( \{1^N 2^Q_2 2^Q_3\} \) and \( \{\bar{M}_1, \bar{M}_2, \bar{M}_3\} \), and recalling \( \bar{u}_i = u_i - \bar{u}_i \), we have

\[
\Delta(\text{CAT})(\text{NM}) = A_{\Delta d} \Delta d \quad (A3)
\]
Since the internal nodal force vector $NM$ can be decomposed into two parts

$$\begin{align*}
NM &= \begin{pmatrix}
^1N \\
^2N \\
^1Q_2 \\
^2Q_1 \\
^1Q_3 \\
^2Q_3 \\
^1\dot{\mathbf{m}}_2 \\
^2\dot{\mathbf{m}}_2 \\
^1M_2 \\
^2M_3
\end{pmatrix} = \begin{pmatrix}
n \\
\cdot \\
^1m_2 \\
^2m_2 \\
^1m_3 \\
^2m_3 \\
^1m_1 \\
^2m_1 \\
^1M_1 \\
^2M_1
\end{pmatrix} + \begin{pmatrix}
N_p | \dot{z}_1 = 0 \\
\frac{\partial M_p}{\partial X_1} | \dot{z}_1 = 0 \\
\frac{\partial M_p}{\partial X_1} | \dot{z}_1 = 0 \\
0 \\
M_p | \dot{z}_1 = 0 \\
M_p | \dot{z}_1 = 0 \\
0 \\
M_p | \dot{z}_1 = 0 \\
M_p | \dot{z}_1 = 0
\end{pmatrix} = T\sigma + NP \quad (A4)
\end{align*}$$

Then

$$(CAT) (NM) = CAT(T\sigma + NP) = A'_{\sigma \delta} \sigma + R_d \quad (A5)$$

where

$$A'_{\sigma \delta} = CAT \cdot T \quad (A6)$$

$$R_d = CAT \cdot NP \quad (A7)$$

Similarly

$$CAT\Delta NM = CAT(T\Delta\sigma + \Delta NP) = A_{\sigma \delta} \cdot \Delta\sigma + \Delta R_d \quad (A8)$$

where

$$\Delta R_d = CAT \cdot \Delta NP \quad (A9)$$

Letting

$$R_d = \int_0^1 F' W d\dot{X}_1 - D$$

by equation (61), we can get

$$\Delta D = H\Theta \cdot \Delta d \quad (A10)$$
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\[
\Delta W = \begin{bmatrix}
\frac{\partial^2 W_e}{\partial N^2} & \frac{\partial^2 W_e}{\partial N \partial M_1} & \frac{\partial^2 W_e}{\partial N \partial M_2} & \frac{\partial^2 W_e}{\partial N \partial M_3} \\
\frac{\partial^2 W_e}{\partial M_1 \partial N} & \frac{\partial^2 W_e}{\partial M_1 \partial M_2} & \frac{\partial^2 W_e}{\partial M_1 \partial M_3} & \\
\frac{\partial^2 W_e}{\partial M_2 \partial N} & \frac{\partial^2 W_e}{\partial M_2 \partial M_1} & \frac{\partial^2 W_e}{\partial M_2 \partial M_3} & \\
\frac{\partial^2 W_e}{\partial M_3 \partial N} & \frac{\partial^2 W_e}{\partial M_3 \partial M_1} & \frac{\partial^2 W_e}{\partial M_3 \partial M_2} & \frac{\partial^2 W_e}{\partial M_3 \partial M_3}
\end{bmatrix}
\begin{bmatrix}
\Delta N \\
\Delta M_1 \\
\Delta M_2 \\
\Delta M_3
\end{bmatrix} = C
\begin{bmatrix}
\Delta N \\
\Delta M_1 \\
\Delta M_2 \\
\Delta M_3
\end{bmatrix}
\]  

(A11)

and

\[
\begin{bmatrix}
\Delta N \\
\Delta M_1 \\
\Delta M_2 \\
\Delta M_3
\end{bmatrix} = \mathbf{F} \Delta \sigma + \Delta \mathbf{q}_4 + \Delta \mathbf{q}_6
\]  

(A12)

where \( F \) is defined by equation (64) and \( \Delta \sigma \) is the increment of \( \sigma \).

\[
\Delta \mathbf{q}_4 = \begin{bmatrix}
\Delta N_{pc1}(CA)_{11} + \Delta N_{pc3} \\
0 \\
\Delta M_{pc1}(CA)_{31} + \Delta M_{pc2} \\
\Delta M_{pc1}(CA)_{21} + \Delta M_{pc3}
\end{bmatrix}
\]  

(i = 1, 2, 3)

(A13)

where \( CA \) is defined in equation (10), \( N_{pc1}, N_{pc3}, M_{pc1}, \) and \( M_{pc3} \) are defined by equation (32) to (35).

We can write

\[
\Delta \mathbf{q}_6 = B \Delta d
\]  

(A14)

\( B \) is a \( 4 \times 12 \) matrix. By noting the sequence of \( d \) in equation (57) and recalling \( \Delta u_i = \Delta^2 u_i - \Delta u_i \), \( B \) has the following entries:

\[
\begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33} \\
B_{41} & B_{42} & B_{43}
\end{bmatrix} = 
\begin{bmatrix}
B_{17} & B_{18} & B_{19} \\
B_{27} & B_{28} & B_{29} \\
B_{37} & B_{38} & B_{39} \\
B_{47} & B_{48} & B_{49}
\end{bmatrix}
\]  

(A15)

where

\[
B_{17} = \left\{ \frac{\partial^3 H}{\partial u_1^2} N_{pc1} + \frac{\partial^3 H}{\partial u_1 \partial u_2} N_{pc2} + \frac{\partial^3 H}{\partial u_1 \partial u_3} N_{pc3} \right\}
\]

\[
B_{18} = \left\{ \frac{\partial^3 H}{\partial u_1} N_{pc1} + \frac{\partial^3 H}{\partial u_2} N_{pc2} + \frac{\partial^3 H}{\partial u_3} N_{pc3} \right\}
\]

\[
B_{19} = \left\{ \frac{\partial^3 H}{\partial u_2} N_{pc2} + \frac{\partial^3 H}{\partial u_3} N_{pc3} \right\}
\]

\[
\frac{\partial^3 H}{\partial u_3^2} N_{pc3} \right\}
\]
\[ B_{19} = \left\{ \frac{\partial^2 H}{\partial u_1 \partial u_3} N_{pc1} + \frac{\partial^2 H}{\partial u_2 \partial u_3} N_{pc2} + \frac{\partial^2 H}{\partial u_3^2} N_{pc3} \right\} \]

\[ B_{27} = \{0\} \]

\[ B_{28} = \{0\} \]

\[ B_{29} = \{0\} \]

\[ B_{37} = \left\{ -\left( \frac{\partial S}{\partial u_1} \frac{\partial^2 H}{\partial u_3^2} + \frac{\partial^2 S}{\partial u_2 \partial u_3} \right) N_{pc1} - \left( \frac{\partial S}{\partial u_2} \frac{\partial^2 H}{\partial u_3} \right) N_{pc2} + \left( \frac{\partial S}{\partial u_3} \frac{\partial^2 H}{\partial u_3^2} \right) N_{pc3} \right\} \]

\[ B_{38} = \left\{ -\left( \frac{\partial S}{\partial u_1} \frac{\partial^2 H}{\partial u_3^2} + \frac{\partial^2 S}{\partial u_2 \partial u_3} \right) N_{pc1} + \left( \frac{\partial S}{\partial u_2} \frac{\partial^2 H}{\partial u_3^2} + \frac{\partial^2 S}{\partial u_3 \partial u_2} \right) N_{pc2} + \left( \frac{\partial S}{\partial u_3} \frac{\partial^2 H}{\partial u_3} \right) N_{pc3} \right\} \]

\[ B_{39} = \left\{ -\left( \frac{\partial^2 S}{\partial u_1 \partial u_3} N_{pc1} - \frac{\partial^2 S}{\partial u_2 \partial u_3} N_{pc2} + \frac{\partial S}{\partial u_3^2} N_{pc3} \right) \right\} \]

\[ B_{47} = \left\{ -\frac{\partial^2 S}{\partial u_2 \partial u_3} N_{pc1} + \frac{\partial^2 S}{\partial u_1 \partial u_3} N_{pc2} \right\} \]

\[ B_{48} = \left\{ -\frac{\partial^2 S}{\partial u_2 \partial u_3} N_{pc1} + \frac{\partial^2 S}{\partial u_1 \partial u_3} N_{pc2} \right\} \]

\[ B_{49} = \{0\} \]

and others are zero.

So, we have

\[ \int_0^1 F^I DW \, d\tilde{X} = \int_0^1 F^I C(F \Delta \sigma + \Delta q_k + B \Delta d) \, d\tilde{X} \]  \hspace{1cm} (A16)

Let

\[ \int_0^1 F^I C F \, d\tilde{X} = A_{ee} \]  \hspace{1cm} (A17)

\[ \int_0^1 F^I C \Delta q_k \, d\tilde{X} = \Delta R_e \]  \hspace{1cm} (A18)

\[ \int_0^1 F^I C B \, d\tilde{X} = A_{ed} \]  \hspace{1cm} (A19)

then

\[ \int_0^1 F^I \Delta W \, d\tilde{X} = A_{ee} \Delta \sigma + \Delta R_e + A_{ed} \Delta d \]  \hspace{1cm} (A20)
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Substituting equations (A3) to (A20) into equation (A1), one obtains

\[\sum \{\delta \Delta \sigma^i[-A_{ee} \cdot \Delta \sigma - \Delta R_e + (H \Theta - A_{edc}) \cdot \Delta d - R_e] + \delta d[A_{ed} \cdot \Delta \sigma + A_{dd} \cdot \Delta d + R_d + \Delta R_d]\}

\[= \sum \{\delta \Delta \sigma^i[-A_{ee} \cdot \Delta \sigma - \Delta R_e + A_{ed} \cdot \Delta d - R_e] + \delta d[A_{ed} \cdot \Delta \sigma + A_{dd} \cdot \Delta d + R_d + \Delta R_d]\}\]

\[= 0 \quad (A21)\]

Since the parameters (δΔσ^i) in equation (A21) are independent and arbitrary in each element, one obtains from (A21)

\[\Delta \sigma = A_{ee}^{-1} (A_{ed} \cdot \Delta d - R_e - \Delta R_e) \quad (A22)\]

and

\[\sum \delta d\{K \cdot \Delta d - \Delta Q + R\} = 0 \quad (A23)\]

where K is the element tangent stiffness matrix and

\[K = A_{ed} A_{ee}^{-1} A_{dd} \quad (A24)\]

\[\Delta Q = A_{ed} A_{ee}^{-1} \Delta R_e - \Delta R_e \quad (A25)\]

\[R = -A_{dd}^{\prime} A_{ee}^{-1} R_d + R_d \quad (A26)\]

When there is no non-conservative distributed loading, the element stiffness matrix should be symmetric. In that case one has

\[A_{ee} = A_{dd}^{\prime}, \quad A_{ee}^{-1} = A_{ed}^{-1} \quad (A27)\]

We have

\[A_{ed}^{\prime} = A_{ed} = (H \Theta - A_{edc}) \quad (A28)\]

Because A_{edc} = 0 for this case, we have

\[H \Theta = A_{edc} \quad (A29)\]

We now turn to the plastic analysis. By equations (72) and (73), the incremental form of equation (78) is

\[\sum_{elem} \left\{H \mu \Delta v + \Theta^{p3}_{\alpha_1} \Delta \mu_1 + \Theta^{p3}_{\alpha_2} \Delta \mu_2 + \Theta^{p3}_{\alpha_3} \Delta \mu_3 + \Delta \lambda \left(\frac{\partial f}{\partial M_1} \Delta N_1 + \frac{\partial f}{\partial M_2} \Delta M_1 + \frac{\partial f}{\partial M_3} \Delta M_3\right)ight\}_{\delta \varepsilon = \varepsilon_p} \quad (A28)\]

The modification to equation (A21) can be written as

\[\sum_{elem} \{\delta \Delta d^i[-A_{ee} \cdot \Delta \sigma - \Delta R_e - \Delta R_e + A_{ed} \cdot \Delta d]

\[+ \delta (\Delta d^i) \{A_{ed} \cdot \Delta \sigma + A_{dd} \cdot \Delta d + \Delta R_d + R_d\} = 0 \quad (A29)\]
where

\[ \Delta \delta^i = [\Delta \sigma^i \Delta \lambda] \]  
(A30)

\[ \lambda_{ee} = \begin{bmatrix} A_{ee} & \lambda_{12} \\ A_{12} & 0 \end{bmatrix} \]  
(A31)

\[ A_{12} = \begin{bmatrix} \frac{\partial f}{\partial M_1} & \frac{\partial f}{\partial M_2} & \frac{\partial f}{\partial M_3} \end{bmatrix} \begin{bmatrix} 1 - \frac{l_p}{l} \\ \frac{l_p}{l} \end{bmatrix} \]  
(A32)

\[ \Delta R = \begin{bmatrix} \Delta N_p + \frac{\partial f}{\partial N} \Delta M_{p1} + \frac{\partial f}{\partial M_1} \Delta M_{p2} + \frac{\partial f}{\partial M_2} \Delta M_{p3} \\ 0 \end{bmatrix} \]  
(A33)

\[ R_e = R_e + \begin{bmatrix} H_{p1} \lambda_{21} = \lambda_{21} = \lambda_{21} \\ \Theta_{p1} \lambda_{21} = \lambda_{21} = \lambda_{21} \end{bmatrix} \]  
(A34)

\[ \lambda_{ee} = \begin{bmatrix} A_{ee} \\ 0 \end{bmatrix} \]  
(A35)

By letting

\[ C = (A_{12} A_{ee}^{-1} A_{12})^{-1} A_{12} \]  
(A36)

we have

\[ \tilde{R} = K - A_{ee} A_{ee}^{-1} A_{12} C A_{ee} \]  
(A37)

\[ \tilde{R} = -A_{ee} A_{ee}^{-1} (R_e - A_{12} C R_e) + R_e \]  
(A38)

\[ \Delta \tilde{Q} = A_{ee} A_{ee}^{-1} (\Delta R_e - A_{12} C \Delta R_e) - \Delta R_e \]  
(A39)

Thus, even in the presence of plasticity, the stiffness matrix is derived explicitly.

REFERENCES

POST-BUCKLING ANALYSIS OF SHALLOW SHELLS BY THE FIELD-BOUNDARY-ELEMENT METHOD

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SUMMARY
The non-linear field-boundary-element technique is applied to the analysis of snap-through phenomena in thin shallow shells. The equilibrium path is traced by using the arc-length method and the solution strategy is discussed in detail. The results show that, as compared to the approaches based on the popular symmetric-variational Galerkin finite element formulation, the current approach based on an unsymmetric variational Petrov-Galerkin field-boundary-element formulation gives a faster convergence while using fewer degrees of freedom. The illustrative numerical examples deal with post-buckling responses of several shallow shells with different geometries.

1. INTRODUCTION
The boundary-element method is increasingly considered as an efficient technique in solving the boundary-value/initial-value problems of mechanics. However, it has not been fully developed in dealing with the analysis of the shell problems, especially the non-linear static and dynamic problems such as of snap-through and post-buckling. As is well known, due to the curvature of the shell, the in-plane displacements and the transverse displacement in a shell are inherently coupled in the kinematics of deformation as well as in the momentum balance relations. Since it is difficult to establish a fundamental solution in infinite space for the entire differential operator in the shell equations, the simple boundary integral representations for displacements, with the mathematical forms as in the case of linear isotropic elasticity, cannot be obtained even for the linear theory of shells. Thus, even in the linear case, the integral representations for displacements involve field (domain) integrals involving displacements, their derivatives and accelerations. In the non-linear case, in general, all the non-linear terms also enter the field integrals. The non-linear case of post-buckling or snap-through analysis is an important challenge to the various techniques of computational mechanics. To analyse the response beyond the limit points in the equilibrium path, the arc-length method proposed by Riks, Crisfield is well known as an efficient technique. Some interesting results have been obtained based on the popular finite element formulation which requires usually large number of degrees of freedom to maintain the accuracy and the rate of convergence.

By considering the fundamental solutions, in the infinite space, to the highest-order linear differential operators of the shell equations as test functions, an unsymmetric variational
formulation can be established which gives the integral representations for displacements. In contrast to the so-called boundary-element method based on the boundary-integral equation, however, such integral representations inevitably include the field integrals of the trial solutions and their derivatives. A Petrov–Galerkin discretization of such integral equations with test function spaces different from those of the trial functions would lead to the presently developed field-boundary-element method. It will be shown later in this paper that, because the test functions are infinitely differentiable, the trial functions in the present Petrov–Galerkin approach need not even be continuous across the field-element boundaries. In contrast, in the symmetric-variational Galerkin finite element method, both the test as well as trial functions need to be \( C^1 \) continuous in the Kirchhoff–Love theory of shells while selective/reduced integration/hour-glass control methods may be needed for \( C^0 \) elements in the Reissner–Mindlin theories.

In the following, we present the results of snap-through and post-buckling analyses of shallow shells, based on the field-boundary-element formulation in conjunction with the arc-length method. The degrees of freedom in the field-boundary-element formulation can be reduced without losing accuracy, so that the full Newton–Raphson technique may be employed to solve the non-linear equations.

In Sections 2 and 3, the general theory of shallow shells and the general formulation of the field-boundary-element are presented; Section 4 discusses the arc-length method and its implementation based on the field-boundary-element formulation; Section 5 gives several numerical examples; and Section 6 some concluding remarks.

### 2. NON-LINEAR FIELD-BOUNDARY-ELEMENT FORMULATION

Consider a shallow shell of an isotropic elastic material with the mid-surface described by \( z = z(x, y), x, y \in \Omega, z = 1, 2 \). The von Karman equations of large deformation for the shell may be written for the in-plane equilibrium (ignoring inertia effects):

\[
N_{\alpha \beta} + b_{\alpha} = 0 \quad (\alpha, \beta = 1, 2)
\]  

(1a)

and for the out-of-plane equilibrium

\[
DV^4 w + \frac{N_{\alpha \beta}}{R_{\alpha \beta}} - b_3 = f_3 + (N_{\alpha \beta} w_{, \beta})_{, \alpha}
\]  

(1b)

where \( N_{\alpha \beta} \) are membrane forces; \( \gamma = \partial(x_{, \beta}) / \partial x_{, \alpha} \); \( w \) is the transverse deflection of the mid-surface of the shell; \( b_i \) \((i = 1, 2, 3)\) are body forces; \( f_3 \) is the load normal to the shell mid-surface; and \( D = \frac{E t^3}{12(1 - \nu^2)} \) where \( t \) is the thickness and \( E \) and \( \nu \) are the elastic constants; \( V^4 \) is the biharmonic operator in the variables \( x_{, \alpha} \) and

\[
R_{\alpha \beta} = \frac{1}{z_{, \alpha \beta}}
\]

are the radii of curvature of the undeformed shell. Along the boundary \( \Gamma \), the boundary conditions for the in-plane variables are

\[
u_u = \bar{u}, \quad N_{\alpha \beta} n_{, \beta} = \bar{P}_\alpha, \quad \Gamma = \Gamma_1 \cup \Gamma_2
\]  

(3)

where \( n_{, \beta} \) are the direction cosines of the unit outward normal to \( \Gamma \) in the base plane of the shell. The out-of-plane boundary conditions are

\[
w = \bar{w}, \quad v_a = \bar{P}_a, \quad \Psi_a = \bar{\Psi}_a, \quad M_a = \bar{M}_a
\]  

(4)

where \( \Psi_a = \partial w / \partial n \) is the rotation around the tangent to \( \Gamma \).
The non-linear in-plane strain–displacement relations are

\[ \varepsilon_{ab} = \frac{1}{2} \left[ u_{a,b} + u_{b,a} + \frac{2w}{R_{ab}} \right] \]

where \( u_a \) are the in-plane displacements at the shell mid-surface. The in-plane stress-resultant strain relations are

\[ N_{11} = \varepsilon_{11} + \nu \epsilon_{12} \quad N_{22} = \varepsilon_{22} + \nu \epsilon_{11} \quad N_{12} = N_{21} = C(1 - \nu) \epsilon_{12} \]

where \( C = E_t/(1 - \nu^2) \). The moment–curvature relations are

\[ M_{11} = -D(w_{11} + \nu w_{22}) \quad M_{22} = -D(w_{22} + \nu w_{11}) \quad M_{12} = -D(1 - \nu) w_{12} \]

In order to derive the integral representations for displacements, we shall consider a general weighted-residual formulation. Let \( u_t \) and \( w \) be the assumed trial solutions, and \( u^*_t \) and \( w^* \) be the corresponding test functions. The combined weak forms of the equilibrium equations and boundary conditions for the in-plane (equations (1a) and (3)) and out-of-plane (equations (1b) and (4)) deformations, respectively, may be written as

\[ (N_0 . P + b.)u^* = \int_{\Gamma_u} (P_0 - P)u^* d\Gamma + \int_{\Gamma_v} (\bar{u}_a - u_t)P^*_t(u^*_t) d\Gamma \]

and

\[ \int_{\Omega} \left\{ D \psi^*_t w + \frac{N_{ab}}{R_{ab}} - b_3 - f_3 - (N_{ab} w_{ab}) \right\} w^* d\Omega = \int_{\Gamma_u} (P_0 - P)w^* d\Gamma + \int_{\Gamma_w} (M_0 - M)\psi^*_t d\Gamma + \int_{\Gamma_v} (\bar{u}_a - \bar{u}_a)M^*_t d\Gamma \]

To make a specific choice for the test functions that results in convenient integral representations for the shell-displacements \( u_t \) and \( w \), we rewrite the in-plane equilibrium equations in a slightly different form, as follows. From the relations between \( (N_{ab}) \) and \( (u_{a,b}) \) as given in equations (5) and (6), we may write

\[ N_{ab} = N'_{ab} + Ck_{ab}w + N^{(0)}_{ab} \]

where

\[ N'_{11} = C(u_{1,1} + \nu u_{2,2}) \quad N'_{22} = C(u_{2,2} + \nu u_{1,1}) \quad N'_{12} = 1/2C(1 - \nu)(u_{1,2} + u_{2,1}) \]

or

\[ \kappa_{ij} = \frac{1}{R_{ij}} + \frac{v}{R_{ij}} \quad \kappa_{11} = \frac{1}{R_{11}} + \frac{v}{R_{11}} \quad \kappa_{12} = \frac{1}{R_{12}} \]

\[ \kappa_{21} = \frac{1}{R_{21}} + \frac{v}{R_{21}} \quad \kappa_{22} = \frac{1}{R_{22}} + \frac{v}{R_{22}} \quad \kappa_{12} = \frac{1}{R_{12}} \]
and the non-linear parts

\[ N^{(e)}_{11} = \frac{C}{2} [(w_{,1})^2 + v(w_{,2})^2]; \quad N^{(e)}_{12} = \frac{C}{2} [(w_{,2})^2 + v(w_{,1})^2] \]

\[ N^{(e)}_{12} = \frac{C}{2} (1 - v) w_{,1} w_{,2} \]

Use of (10) in (8) results in

\[ \int_{\Omega} \left[ N'_{e,\beta} + C(K_{e,\beta}) \right] u_{*} d\Omega \]

\[ = \int_{\Gamma_e} (P_e - \bar{P}_e) u_{*} d\Gamma + \int_{\Gamma_e} (\ddot{u}_e - u_e) P_{*}^e (u_{*}) d\Gamma \]

Use of the divergence theorem in equation (14a) results in

\[ \int_{\Gamma_e} N'_{e,\beta} u_{*} d\Gamma - \int_{\Omega} N'_{e,\beta} u_{*} d\Omega + \int_{\Omega} C(K_{e,\beta}) u_{*} d\Omega + \int_{\Omega} N^{(e)}_{11} u_{*} d\Omega + \int_{\Omega} (b_e - \rho \ddot{u}_e) u_{*} d\Omega \]

\[ = \int_{\Gamma_e} (P_e - \bar{P}_e) u_{*} d\Gamma + \int_{\Gamma_e} (\ddot{u}_e - u_e) P_{*}^e (u_{*}) d\Gamma \]

Since the material is linear elastic and isotropic, we have

\[ N'_{e,\beta} u_{*} = C_{e,\beta} u_{*,\beta} u_{*,\beta} = N'_{e,\beta} u_{*,\beta} \]

where the definition of \( N'_{e,\beta} \) is apparent. Now note that

\[ P_e = N_{e,\beta} n_{\beta} \]

or

\[ P_e = N_{e,\beta} n_{\beta} + C_{e,\beta} w n_{\beta} + N^{(e)}_{e,\beta} n_{\beta} \]

Using (15, 16b) in equation (14b) and applying the divergence theorem, it is easy to obtain

\[ \int_{\Omega} \left[ N'_{e,\beta} (u_{*})_{,\beta} \right] u_{\delta} d\Omega - \int_{\Delta} (b_e - \rho \ddot{u}_e) u_{*} d\Omega + \int_{\Gamma_e} \bar{P}_e u_{*} d\Gamma - \int_{\Gamma_e} P_{e} \ddot{u}_{e} d\Gamma \]

\[ = \int_{\Omega} C_{e,\beta} w u_{*} d\Omega - \int_{\Omega} N^{(e)}_{e,\beta} u_{*} d\Omega = 0 \]

where

\[ \bar{P}_e = P_e \text{ at } \Gamma_e; \quad\text{and}\quad \bar{P}_e = P_e \text{ at } \Gamma_e \]

and

\[ \ddot{u}_e = \ddot{u}_e \text{ at } \Gamma_e; \quad\text{and}\quad \ddot{u}_e = u_e \text{ at } \Gamma_e \]

Now we chose \( u_{*} \) to be the 'fundamental solution' in infinite space of the equation

\[ [N'_{e,\beta} (u_{*})]_{,\beta} + \delta(x_e - \xi_e) \delta_{e,\beta} e_{\beta} = 0 \]

where \( \delta(x_e - \xi_e) \) is the Dirac delta function at \( x_e = \xi_e \); \( \delta_{e,\beta} \) is the Kronecker delta; and \( e_{\beta} \) denotes that the direction of the application of the point load is along the \( x_{e,\beta} \) direction. The 'fundamental solution' of (18) will be denoted as \( u_{e,\beta} \), where \( u_{e,\beta} \) is the displacement along the \( x_{e,\beta} \) direction in a plane infinite body at any point \( x_{e,\beta} \) due to a unit load along the \( x_{e,\beta} \) direction, applied at the location \( x_{e,\beta} = \xi_{e,\beta} \). Likewise, \( P_{e,\beta} (x_{e,\beta}, \xi_{e,\beta}) \) will be considered to be the traction along the \( x_{e,\beta} \) direction on an oriented surface at \( x_{e,\beta} \) with a unit normal \( n_{\beta} \), due to a unit load along \( x_{e,\beta} \) at the location \( \xi_{e,\beta} \). These
solutions are well known and may be written as

\[ u^0_{0,1}(x, \xi) = \frac{1}{8 \pi G} \left[ (v - 3) \ln \rho \delta_{x_1} + (1 + v) \frac{\partial \rho}{\partial x_1} \frac{\partial \rho}{\partial x_2} \right] \quad (19a) \]

and

\[ P^a_{0,1}(x, \xi) = -\frac{t}{4 \pi \rho} \left\{ \frac{\partial \rho}{\partial n} \left[ (1 - v) \delta_{x_1} + 2(1 + v) \frac{\partial \rho}{\partial x_1} \frac{\partial \rho}{\partial x_2} \right] \right\} \]

\[ - (1 - v) \left( n_a \frac{\partial \rho}{\partial x_1} - n_1 \frac{\partial \rho}{\partial x_2} \right) \quad (19b) \]

where \( \rho = |x_1 - \xi_1| \) is the radius vector \( x_1 \) to \( \xi_1 \), and

\[ G = E t /[2(1 + v)]. \]

Due to the property of integrals involving Dirac functions, we have

\[ \int_{\Omega} (N^a_{a,1}) u_a d\Omega = - \int_{\Omega} \delta(x_1 - \xi_1) \delta_{x_1} u_a(x_1) d\Omega = - u_0(\xi_1) \quad (20) \]

Using (19) and (20) in equation (17a), we have

\[ \gamma u_0(\xi_1) = \int_{\Omega} \left[ \bar{b}_a(x_1) - \rho \bar{a}_a(x_1) \right] u^0_{0,1}(x_1, \xi_1) d\Omega + \int_{\Gamma} \bar{P}_a(x_1) u^0_{0,1}(x_1, \xi_1) d\Gamma - \int_{\Omega} \bar{a}_a(x_1) P^a_{0,1}(x_1, \xi_1) d\Omega - \int_{\Omega} C_{\xi_1} w_0(x_1) u^0_{0,1}(x_1, \xi_1) d\Omega \]

\[ - \int_{\Omega} N^a_{a,1}(x_1) u^0_{0,1}(x_1, \xi_1) d\Omega \quad (21) \]

It can be shown that, while the coefficient \( \gamma \) in the left-hand side of (21) is unity when \( \xi_1 \) is in the interior of \( \Omega \), the value of \( \gamma \) is (0.5) when \( \xi_1 \) falls on the 'smooth' boundary \( \Gamma. \) Equation (21) is the sought-after integral equation for \( u_a \) in a shallow shell.

We now choose the test function \( w^*(x_1) \) to be the 'fundamental solution' in an infinite plate corresponding to a unit point load at the location \( \xi_1 \) in the linear theory of Kirchhoff plates. Thus, \( w \) corresponds to the solution of the linear equation

\[ D V w^* = \delta(x_1 - \xi_1) \quad (22) \]

in an infinite domain in the base-plane of the shallow shell. It is well known that the solution for \( w^* \) is given by

\[ w^*(x_1, \xi_1) = \frac{1}{8 \pi} \rho^2 \ln \rho \quad (23) \]

where \( \rho = |x_1 - \xi_1| \).

Using equations (23) and (10) in equation (9) and employing repeated integrations by parts in the resulting equation, one easily obtains the integral equation:

\[ \gamma D w(\xi_1) = \int_{\Gamma} \bar{b}_a(x_1) w^*(x_1, \xi_1) d\Gamma - \int_{\Gamma} \bar{M}_a(x_1) \psi^*(x_1, \xi_1) d\Gamma \]

\[ + \int_{\Gamma} \bar{\psi}_a(x_1) M^a_*(x_1, \xi_1) d\Gamma - \int_{\Gamma} \bar{w}(x_1) V^a_*(x_1, \xi_1) d\Gamma \]
In equation (24) the terms with the superposed symbol 'A' should be taken to imply the respective prescribed values, if any, at \( \Gamma \); otherwise, they are to be treated as the unknown solution variables. Also, the symbol \( \langle (\ ) \rangle \) denotes the jump in the quantity \( (\ ) \) at a corner at \( \Gamma \), in the direction of the increasing arc length along \( \Gamma \); and the summation (1 to \( J \)) extends to all the \( J \) such corners.

Using equation (11) and the divergence theorem, it is easy to see that

\[
- \int_{\Omega} N_{sg}^{\infty}(x_\mu) w^*(x_\mu, \xi_\mu) d\Omega = \int_{\Gamma} C_{sg} n_p u_s(x_\mu) w^*(x_\mu, \xi_\mu) d\Gamma
\]

Use of (25) in (24) results in the final integral equation for \( w \) as follows:

\[
\gamma_w Dw(\xi_\mu) = \int_{\Gamma} \partial_\mu w(\xi_\mu) d\Gamma - \int_{\Gamma} \tilde{M}_\mu w(\xi_\mu) d\Gamma
\]

Since \( (\partial w/\partial n) \) is also an independent variable at \( \Gamma \), an integral relation for \( (\partial w/\partial n) \) should be derived. Towards this purpose, consider a second fundamental solution,

\[
w^*_s = \frac{1}{2\pi} \rho \ln \rho \cos \phi
\]
where $\phi$ is the angle between the outward normal to $\Gamma$ and the radius $\rho$ (see Stern$^{13}$). The resulting integral equation is

$$
\gamma_r \frac{\partial w}{\partial n} = \int_{\Gamma} \rho_r(x_r)w^{*}_{r}(x_r, \xi_r) d\Gamma - \int_{\Gamma} \tilde{N}_r(x_r)\psi_r(x_r, \xi_r) d\Gamma
$$

$$
+ \int_{\Gamma} \tilde{\psi}_r(x_r)M^{*}_{r}(x_r, \xi_r) d\Gamma - \int_{\Gamma} [\tilde{\psi}(x_r) - \tilde{\psi}(\xi_r)] V^{*}_{r}(x_r, \xi_r) d\Gamma
$$

$$
- \int_{\Omega} C_{k_{s}g_{s}}u_{s}(x_s)w^{*}_{s}(x_s, \xi_s) d\Omega + \int_{\Omega} C_{k_{s}g_{s}}w^{*}_{s}(x_s, \xi_s) d\Omega
$$

$$
- \int_{\Omega} \frac{C_{k_{s}g_{s}}}{R_{s}}w(x_s)\psi^{*}_{s}(x_s, \xi_s) d\Omega - \int_{\Omega} \frac{N^{(s)}}{R_{s}}(x_s)w^{*}_{s}(x_s, \xi_s) d\Omega
$$

$$
+ \int_{\Omega} \left[ b_3 - \rho \tilde{w} + f_3 + (N_{s}w_{s}g_{s})_{s}\right](x_s)w^{*}_{s}(x_s, \xi_s) d\Omega
$$

$$
+ \sum_{r} \left[ \langle M_{r} \rangle w^{*}_{r} - \langle M^{*}_{r} \rangle w \right]
$$

(27)

Remarks

In summary, equations (21), (26) and (27) represent the complete set of integral equations for $u_r$, $w$ and $\partial w/\partial n$. An examination of these equations reveals the following features.

(i) For given body forces $b_r$, the integral relations for $u_r$ (equation 21) involves the trial functions $u_r$ only at the boundary $\Gamma$. On the other hand, due to the curvature induced coupling of the trial functions ($u_r$ and $w$) in the shallow-shell problem, the integral relations for $u_{s}$ contain a domain-integral (over $\Omega$) involving the trial function for $w$. If the in-plane inertia forces ($\rho \tilde{u}_s$) appear in the problem, then the integral relations for $u_r$ involve a domain-integral (over $\Omega$) of $\tilde{u}_r$ as well.

(ii) Again, due to the curvature-induced coupling of the in-plane and out-of-plane displacements in the presently considered shallow shell, the integral equations for $w$ and $\partial w/\partial n$, equations (26) and (27), respectively, contain domain-integrals (over $\Omega$) involving trial functions for both $w$ and $u_r$.

(iii) In the non-linear problem, the non-linear terms $N_{s}^{(s)}$ and $(N_{s}w_{s}g_{s})_{s}$ involving trial functions for both $w$ and $u_r$ inevitably bring the domain-integrals (over $\Omega$) into the equations.

(iv) For reasons (i) to (iii) above, unlike the classical homogeneous isotropic elasto-statics$^1$ wherein a discretization of the relevant integral equations requires the use of basis functions for the displacements at the boundary alone, the present non-linear shallow-shell formulation requires the assumption of basis functions for the trial solutions $u_r$ and $w$, at the boundary $\Gamma$ as well as in the interior $\Omega$. Thus, the present solution methodology may, strictly speaking, be classified as a hybrid boundary-element/field-(finite)-element method based on a direct discretization of integral equations. We name this method the field-boundary-element method.

(v) Suppose now that in equations (21), (26) and (27) we let $\xi_r$ tend to a point on the boundary, i.e. $\xi_r \in \Gamma$. Thus, we obtain three integral relations for the boundary values of $u_r$, $w$ and $\partial w/\partial n$. An examination of these relations reveals that, in order to discretize these equations, one needs to assume only very simple trial functions $u_r$, $w$, $\partial w/\partial n$ not only at the boundary but also in the interior of $\Omega$. For instance, $\Omega$ may be discretized into a number of finite elements and $\Gamma$ into a number of boundary elements. As $\xi_r$ tends to $\Gamma$, the integral relations (21), (26) and (27) clearly show that $w$ and $u_r$ need only be piecewise differentiable and need not even be $C^0$ continuous at the element boundaries. In contrast, it is recalled that in the Galerkin finite element method, $u_r$ need be $C^0$
continuous and w be $C^1$ continuous in each element. The difficulties with such a finite element approach are too well documented in the literature to warrant further comment here.

(vii) At each point on the boundary, two of the in-plane variables $u_a (a = 1, 2)$, $P_a (a = 1, 2)$ are specified; and the other two are unknown. Likewise, two of the out-of-plane variables, $V_a$, $M_a$, $\psi_a$, and $w$ are specified; and the other two are unknown. At each point in $\Omega$, as seen from equations (21), (26) and (27), the three displacements $u_a$ and $w$ are unknown. Thus, if equations (2), (26) and (27) are discretized through finite as well as boundary elements and the nodal values of the variables are appropriately understood to be either specified or unknown, one would easily obtain exactly as many equations as the number of unknowns, so that the problem may be considered as well posed.

Note that due to the appearance of the non-linear terms in the obtained integral equations (21), (26) and (27), the discretized interior/boundary-element equations may be solved through an incremental approach. This will be discussed in detail in Section 3.

3. INCREMENTAL APPROACH AND SOLUTION STRATEGY

In the incremental approach, the load and the prescribed boundary conditions are applied in small but finite increments.

Consider that the shell is at the end of the $k$th load increment. All quantities which have been known during the previous $k$ steps of analysis are denoted by a superscript $'k'$. The displacement increments are denoted as

$$\Delta u_a = u_a^{k+1} - u_a^k; \quad \Delta w = w^{k+1} - w^k$$

With these notations, the integral equations in incremental form may be written as

$$\gamma(u_a^{k} + \Delta u_a) = \int_{\Omega} [b_a^{k} + \Delta b_a - \rho(\tilde{u}_a^{k} + \Delta \tilde{u}_a)]^T u_{a,\text{in}}^{k} \, d\Omega$$

$$+ \int_{\Gamma} (P_a + \Delta P_a) u_{a,\text{on}}^{k} \, d\Gamma - \int_{\Gamma} (P_{\theta a} + \Delta P_{\theta a}) u_{\theta a} \, d\Gamma - \int_{\Omega} C_{ka}(w^{k} + \Delta w) u_{\theta a} \, d\Omega$$

$$- \int_{\Omega} (N_{sa}^{(k)} + \Delta N_{sa}^{(n)} + \text{higher-order terms}) u_{a,\text{in},s} \, d\Omega$$

$$\gamma_D(w^{k} + \Delta w) = \int_{\Gamma} (P_a + \Delta P_a) w^{k} \, d\Gamma - \int_{\Gamma} (P_{\theta a} + \Delta P_{\theta a}) w^{k} \, d\Gamma$$

$$+ \int_{\Gamma} (\tilde{V}_a^{k} + \Delta \tilde{V}_a) M_a^{k} \, d\Gamma - \int_{\Gamma} (\tilde{V}_{\theta a}^{k} + \Delta \tilde{V}_{\theta a}) V_a^{k} \, d\Gamma$$

$$- \int_{\Omega} \left[ \frac{N_{sa}^{(k)}}{R_{sa}} + \Delta N_{sa}^{(n)} + C_{ka}(w^{k} + \Delta w) \right] w^{k} \, d\Omega$$

$$- \int_{\Omega} \left[ \frac{N_{sa}^{(k)}}{R_{sa}} + \Delta N_{sa}^{(n)} + \text{higher-order terms} \right] w^{k} \, d\Omega$$

$$+ \int_{\Omega} \left[ b_3^{k} + \Delta b_3 - \rho(\tilde{u}_3^{k} + \Delta \tilde{u}_3) + f_3^{k} + \Delta f_3 \right] w^{k} \, d\Omega$$

$$+ \int_{\Omega} \left[ (N_{sa}^{k} + \Delta N_{sa})(w^{k} + \Delta w)_a \right] w^{k} \, d\Omega$$

$$+ \sum_{T} \left[ \langle M_{\text{in}}^{k} + \Delta M_{\text{in}} \rangle w^{k} - \langle M_{\text{in}}^{k} \rangle (w^{k} + \Delta w) \right]$$

(29b)
where the increments of the non-linear parts of in-plane forces are
\[
\Delta N_{11}^{(n)} = C[w_{11}^i \Delta w_{11} + vw_{12}^i \Delta w_{12}],
\]
\[
\Delta N_{22}^{(n)} = C[w_{12}^i \Delta w_{12} + vw_{11}^i \Delta w_{11}],
\]
\[
\Delta N_{12}^{(n)} = C\frac{1-v}{2}[w_{11}^i \Delta w_{12} + w_{12}^i \Delta w_{11}],
\]
and
\[
\Delta N_{ab} = \Delta N_{ab}^s + CK_{ab} \Delta \psi + \Delta N_{ab}^{(q)} + \text{higher-order terms}
\]

where the definition of the increments of the linear part is apparent. Here, the incremental form of equation (27) is similar to that of equation (29b), and its treatment follows the same routine.

In equations (29a, b), the higher-order terms involve the products of the incremental displacements. In solving for these unknown incremental displacements, those higher-order terms are ignored. In the considered incremental equations, the terms with the superscript 'k' should have satisfied the equilibrium conditions at the end of the kth load increment; however, the equilibrium conditions are in fact not exactly satisfied because of the absence of the higher-order terms. Therefore, an 'equilibrium correction' iteration is employed at each step.

Note that in equation (29b), the non-linear term \((N_{ab}^s + \Delta N_{ab})(w^k + \Delta w)\) can be written as
\[
(N_{ab}^s + \Delta N_{ab})(w^k + \Delta w)_s = N_{ab}^s w_{ab}^k + N_{ab}^s \Delta w_{ab} + \Delta N_{ab} w_{ab}^k + \text{higher-order terms}
\]

Ignoring the higher-order terms and examining (30) and (31), we may see that those non-linear terms are linearized with respect to the displacement increments. Using (30) and (31) in equations (29a, b) and applying the divergence theorem, we may obtain the final integral equations in terms of unknown displacement increments:

\[
\gamma (u^k + \Delta u) = \int_\Omega \left[ b^k + \Delta b_s - \rho (\tilde{u}^k_s + \Delta \tilde{u}_s) \right] u^*_{0\theta \zeta} d\Omega
\]
\[
+ \int_{\Gamma} (\tilde{\phi}_{\theta}^s + \Delta \tilde{\phi}_s) u^*_{0\theta \zeta} d\Gamma - \int_{\Gamma} (\tilde{u}_s^* + \Delta \tilde{u}_s) P^*_{\theta \zeta} d\Gamma
\]
\[
- \int_{\Omega} CK_{ab} (w^k + \Delta w) u^*_{0\theta \zeta, s} d\Omega + \int_{\Gamma} N_{ab\theta} u^*_{0\theta \zeta, s} \Delta \psi d\Gamma
\]
\[
- \int_{\Omega} N_{ab\theta} u^*_{0\theta \zeta, s} \Delta w d\Omega + w_{ab}\bigg|_c
\]

\[
\gamma_D (w^k + \Delta w) = \int_{\Gamma} (P_{\theta}^s + \Delta P_{\theta}) w^* d\Gamma - \int_{\Gamma} (\tilde{M}_{\theta} + \Delta \tilde{M}_{\theta}) \psi^* d\Gamma
\]
\[
+ \int_{\Gamma} (\tilde{\psi}_{\theta}^s + \Delta \tilde{\psi}_s) M_{\theta}^* d\Gamma - \int_{\Gamma} (\tilde{\psi}_s^* + \Delta \tilde{\psi}_s) V^* d\Gamma
\]
\[
- \int_{\Omega} \left[ (C_{ab} \eta_{\theta} w^* + A_{\theta}) \Delta \tilde{u}_s d\Omega + \int_{\Omega} \left[ (C_{ab\theta} \eta_{\theta} w^*)_{\theta} + B_{ab\theta} \right] \Delta \tilde{u}_s d\Omega
\]
\[
- \int_{\Omega} \left[ (C_{ab} \eta_{\theta} w^* + N_{ab} \eta_{\theta} w^*)_{\theta} + A_{\theta} \right] \Delta \tilde{w} d\Gamma
\]
\[
+ \int_{\Gamma} \left[ C_{ab} \eta_{\theta} w^* + N_{ab} \eta_{\theta} w^* - C \frac{K_{ab}}{R_{ab}} w^* - B_{ab} \right] \Delta \psi d\Omega
\]
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\[ N_i \mathbf{k} + (\mathbf{M}_i + \mathbf{\Delta M}_i) \mathbf{w} - (\mathbf{M}_i) (\mathbf{w}^\bullet + \mathbf{\Delta w}) \]

where the constants \( A \) and \( B \) may be given as follows:

\[ A_1 = C \left[ \mathbf{w}_1 \mathbf{v}_1 + \mathbf{w}_2 \mathbf{v}_2 \right] \mathbf{m}_1 + \frac{1-v}{2} \left[ \mathbf{w}_1 \mathbf{v}_1 \mathbf{m}_2 + \mathbf{w}_2 \mathbf{v}_2 \mathbf{m}_2 \right] \]

\[ B_1 = C \left[ \mathbf{w}_2 \mathbf{v}_1 \right] \mathbf{n}_2 + \frac{1-v}{2} \left[ \mathbf{w}_1 \mathbf{v}_1 \mathbf{n}_1 + \mathbf{w}_2 \mathbf{v}_2 \mathbf{n}_1 \right] \]

and \( A_2, B_2 \) can be similarly written by cycling the subscripts; and the constants \( A_w \) and \( B_w \) may be written as

\[ A_w = C \left( \mathbf{w}_1^2 \mathbf{m}_1 + \mathbf{w}_2^2 \mathbf{m}_2 + \mathbf{w}_1 \mathbf{w}_2 (\mathbf{w}_1 \mathbf{n}_1 + \mathbf{w}_2 \mathbf{n}_2) \right) \]

\[ B_w = C \left( 2 \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_1 \mathbf{n}_2 + \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_2 \mathbf{n}_1 + \mathbf{w}_1 \mathbf{w}_2 (\mathbf{w}_1 \mathbf{n}_2 + \mathbf{w}_2 \mathbf{n}_1) \right) \]

We discretize the domain \( \Omega \) as well as the boundary \( \Gamma \) by using some appropriate interpolation functions for the unknown displacement increments. By carrying out the indicated integrations, we may obtain the field-boundary-element equations with the displacement increments as unknowns. For each load increment, these equations have to be solved iteratively because the equilibrium conditions are only approximately satisfied due to the absence of the higher-order terms.

The full Newton–Raphson algorithm is used to obtain the solution. This involves domain integrations for constructing the coefficient matrix, and reduction of the matrix in each iteration. Since, in the present field-boundary-element method, the coefficient matrix is not as large as in the usual finite element approach, the reduction of this matrix is not as critical, and moreover, it is found that the solution converges very rapidly.

In the present numerical implementation, at the beginning of each load increment, the equilibrium is checked first by subtracting the integrals of other terms with the superscript \( k \) from the total load integration; the residual load vector is used to solve the displacement increments; the obtained displacement increments are then used to update the coefficient matrix and check the equilibrium again and so on.
4. THE ARC-LENGTH METHOD FOR THE SOLUTION OF TANGENT STIFFNESS EQUATIONS OF THE FIELD-BOUNDARY-ELEMENT APPROACH

Let the system be at the beginning of the nth load increment. The equilibrium equation can be written as

\[ K_0^* \Delta p_0 = \lambda_0 q - r_0 \]  

(34)

where \( K_0^* \) is the coefficient matrix obtained by carrying out the integrations associated with the unknown trial solutions in the field-boundary-element equations (32a), (32b) (and a simpler equation for \( \partial W / \partial n \)) (for details, see Zhang and Atluri). Here, in this incremental equation, \( K_0^* \) is constructed based on the previous solution vector \( p_0 \) (up to the \((n-1)\)th load increment). As a matter of fact, \( K_0^* \) is equivalent to the tangent stiffness matrix in the finite element method; \( \Delta p_0 \) is the incremental unknown vector. In the field-boundary-element method, however, the vector \( p_0 \) includes both boundary forces and displacements as well as the displacements in the interior of \( \Omega \); \( \lambda_0 \) is the current load parameter; \( q \) is the fixed load vector; \( r_0 \) is the internal force vector which should have been equilibrated with the external load up to the \((n-1)\)th load increment.

Equation (34) is an approximate one because the higher-order non-linear terms have been ignored. Therefore, the unknown incremental vector should be decided by iterations, i.e. we have

\[ K_1^* \Delta p_1 = \lambda_1 q - r_1 \]  

(35)

where \( r_1 \) is obtained by using the previous solution vector \( p_1 \) up to the \((i-1)\)th iteration; and, if the modified Newton-Raphson solution technique is used, \( K_1^* \) is the same as \( K_0^* \) during the current increment; or it should be updated after each iteration if the full Newton-Raphson technique is used; \( \Delta p \) is the current displacement increment and we have

\[ \Delta p_i = \Delta p_{i-1} + \delta p_{i-1} \]

(36)

\[ P_i = P_{i-1} + \Delta p_{i-1} \]

The central concept of the arc-length method is to decide the value of the load parameter \( \lambda \), in other words, to give the suitable step length as well as the direction of the load increment when tracing the equilibrium path. A general formula to decide \( \lambda \) can be given as

\[ \Delta P_i^T \Delta P_i + b \Delta \lambda_i^2 q^T q = \Delta l^2 \]  

(37)

where \( b \) is a scaling parameter and \( \Delta l \) is a prescribed incremental length. Equation (37) is the so-called displacement-load control. Some experiences have shown that it is preferable to set \( b = 0 \), that is, to use the displacement control only. From equation (35), we have

\[ \delta p = \lambda_i P_i - P_r \]  

(38)

where \( P_i = (K_i^*)^{-1} q \) and \( P_r = (K_r^*)^{-1} r_i \), and from equation (36) one has

\[ \Delta P_i = \lambda_i P_i + \Delta P_{i-1} - P_r \]  

(39)

Substituting (39) into equation (37) and letting \( b = 0 \), we have

\[ a\lambda_i^2 + b\lambda_i + c = 0 \]  

(40)

where

\[ a = P_i^T P_i; \quad b = 2P_i^T (\Delta P_{i-1} - P_r); \]

\[ c = P_r^T P_r - 2P_r^T \Delta P_{i-1} + [\Delta P_{i-1}^T \Delta P_{i-1} - \Delta l^2] \]

If we assume an exact satisfaction of equation (37) for each increment, the terms in brackets in the formula for \( c \) may be ignored.
Taking a proper value of $\Delta l$ and solving equation (40), we may obtain $\lambda_i$ for each iteration which should make the solution point closer and closer to the true equilibrium path. As a matter of fact, equation (40) gives a constraint sphere and usually it cuts the equilibrium path at two points, i.e. equation (40) has two different roots. To choose an appropriate one, we may consider the angle $\theta$ between $\Delta P_1$ and $\Delta P_{i-1}$ where

$$\cos \theta = \frac{\Delta P_i^T \Delta P_{i-1}}{\Delta l^2} = 1 + \frac{1}{\Delta l^2} (\lambda_i \Delta P_{i-1}^T P_i - \Delta P_{i-1}^T P_r)$$

(41)

To avoid the so-called ‘double back’, i.e. to avoid reversing along the equilibrium path, one should choose the root that makes the value of $\cos \theta$ positive. If both values of $\cos \theta$ are positive, we may take the root as the one which is closest to the linear solution of equation (40).7

The incremental length $\Delta l$ is usually decided by trial in the first load increment. Using some guessed load parameter $\lambda_1$, $\Delta l$ can be chosen as

$$\Delta l = \lambda_1 / \sqrt{dP_i^T dP_i}$$

(42)

and it can be kept fixed, unless in some increments the number of iterations becomes more than a desired number (usually 4), when $\Delta l$ may be reduced. In our experience for the shell problem, $\lambda_i$ can be taken such that the largest deflection will be around 10 percent of the thickness, that is, to let deformation remain in the linear scope.

In the current work, based on the field-boundary-element formulation, the full Newton-Raphson technique is used to solve the non-linear system equations. Because of the advantage of having fewer degrees of freedom by the present formulation, the cost of refactorizing the tangent matrix after each iteration is not significant.

The difference between the current formulation and the popular displacement finite element method in applying the arc-length method is that in the field-boundary-element method, the solution vector is a mixture of displacement and boundary force components. To implement the displacement control, equation (37) should be changed to

$$\Delta P_i^T \Delta P_i = \Delta l^2$$

(43)

and equation (39) to

$$\Delta P_i = \lambda P_i + \Delta P_{i-1} - P_r$$

(44)

where the terms with the superposed bar are the vectors which have the non-linearly varying displacement components only.

In the literature, one may find discussions on some techniques such as line search to accelerate the convergence. In the current work by the full Newton-Raphson technique, the rate of convergence is quite satisfactory (3 to 4 iterations on average for each load increment), so that these accelerating techniques were found to be not necessary.

5. NUMERICAL RESULTS

The first problem consists of a shallow spherical shell, with a square base, as shown in the inset of Figure 1. The shell is loaded by a point load at the crown. The relevant geometrical and material-property data are given in Figure 1. In the present field-boundary-element method, a $3 \times 3$ mesh over a quarter of the shell (see the inset in Figure 1) with a total of 64 degrees of freedom is used. As shown in Figure 1, the present solution agrees excellently with:

(i) the analytical solution by Licester;11
(ii) the numerical solutions obtained by Bathe and Lo\textsuperscript{3} using triangular finite elements based on the usual Galerkin displacement formulation, with 206 degrees of freedom (results from Bathe and Lo\textsuperscript{3} being close to those of Leicester,\textsuperscript{11} are not shown in Figure 1 for the sake of clarity);

(iii) the numerical results of Dvorkin and Bathe,\textsuperscript{9} using continuum-based quadrilateral Galerkin finite elements, with 80 degrees of freedom (not shown in Figure 1, for clarity);

(iv) the results obtained by Bergan \textit{et al.},\textsuperscript{4} using triangular finite elements based on a mixed variational Galerkin formulation, with 180 degrees of freedom;

(v) the results obtained by Dhatt,\textsuperscript{8} using a triangular finite element based on a discrete-Kirchhoff type Galerkin formulation, with 134 degrees of freedom.

From this set of results, the potential of the present field-boundary-element method in shell analysis is evident. Also earlier results\textsuperscript{15} concerning free vibration and transient response of shallow shells indicate that the present method can yield a larger number of eigenvalues of vibration with a greater accuracy, while using a fewer number of degrees of freedom as compared to the Galerkin finite element method.

The second example concerns a shallow spherical shell with a circular base (shown schematically in Figure 2). The shell is subject to a concentrated load at the crown. Note that while the problem is axisymmetric, the formulation that is employed is in a Cartesian reference frame for the sake of generality. Thus, one quadrant of the problem is analysed by using different meshes, as shown in Figure 3, to study the convergence properties of the present approach. The three meshes in Figure 3 consist of 26, 51 and 61 degrees of freedom, respectively. The load versus crown-deflection curves, obtained for each of the 3 meshes respectively, are shown in Figure 4. The variation of the
Figure 2. Schematic of a circular-based shallow spherical shell

Figure 3. Different field-boundary-element (FBE) meshes for the analysis of the circular-based shallow spherical shell

Figure 4. Convergence of the load-deformation curve for various FBE meshes shown in Figure 3
computed bending moments $M_r$ and $M_\theta$ at cross sections along the radial length $r$ from the apex of the shell, after the fifth load increment, are shown in Figure 5. These results were obtained by using mesh (b) in Figure 3. For each load increment, convergence was achieved with 3 or 5 iterations. These results indicate that the present method is fully capable of accounting for stress concentrations/singularities near concentrated loads. Also using the mesh of Figure 3, several shells with varying radii of principal curvature were analysed, and the corresponding load–deflection diagrams are shown in Figure 6. As seen from Figure 6, as the radii of curvature increase, the load value at snap-through decreases, and the load–deflection response curve becomes much smoother in the post-critical range.

6. CONCLUSIONS

This paper presented an exploratory study concerning the use of the field-boundary-element method in the analysis of large deformations and post-buckling response of shallow shells. In this method, the fundamental solutions in infinite space to the highest-order linear differential operator of the problem are used as test functions. The results were found to be highly encouraging. The reduced-order modelling capability of the present method appears to make it attractive for purposes of implementing ‘deformation–control’ algorithms, wherein the limiting factor is the order of the Riccati equations that must be solved. The present study points to a need for further exploration of the mathematical convergence aspects of the field-boundary-element method.

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![Figure 5. Bending moments $M_r$ and $M_\theta$ along the radial direction of a point-loaded shallow spherical shell of Figure 2. (Results shown are for FBE mesh shown in Figure 3(b))](image-url)
Figure 6. Load-deformation diagrams for shells of various radii of principal curvature

REFERENCES


DYNAMICS OF 3-D SPACE-CURVED BEAMS
UNDERGOING FINITE ROTATIONS AND FINITE STRAINS:
A VARIATIONAL THEORY AND NUMERICAL STUDIES

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ABSTRACT

The problem of transient dynamics of highly-flexible 3-Dimensional space-beams, undergoing large rotations and stretches is treated. The case of conservative force loading, which may also lead to configuration-dependent moments on the beam, is treated. Based on the present governing equations, a general mixed variational principle for the static problem is presented. Furthermore, using the three parameters associated with a conformal rotation vector representation of finite rotations, a well-defined Hamilton functional is established for the dynamic problem of a flexible beam undergoing finite rotations and stretches. This is shown to lead to a symmetric tangent stiffness matrix at all times. In the present total Lagrangean description of motion, the mass-matrix of a finite element depends linearly on the linear accelerations, but nonlinearly on the rotational parameters and attendant angular accelerations; the stiffness matrix depends nonlinearly on the deformation; and an "apparent" damping matrix depends nonlinearly on the rotations and attendant velocities. A Newmark time-integration scheme is used to integrate the semi-discrete finite element equations in time. An example of transient dynamic response of highly flexible beam-like structures in free-flight is presented to illustrate the validity of the theoretical methodology developed in this paper.

INTRODUCTION

In this paper, using plausible and consistent kinematic hypotheses, a large deformation (and large rotation) theory for the dynamic analysis of beams is developed. Based on Timoshenko's hypotheses, the effects of stretching, bending, torsion and transverse
shear, are taken into account. For simplicity, however, the cross-sectional warping effects are neglected. These kinematic assumptions have been introduced also in [1], [12], [13] to develop a 3-D beam theory. In these papers, the existence of prescribed external moments has been postulated a priori. Argyris, Dunne and Scharpf [2], and Iura and Atluri [8], however, have pointed out that the external moments generated even by the conservative forces are generally configuration dependent. Therefore, the external virtual work associated with these moments does not, on first sight, appear to correspond to the first variation of an external energy functional.

Argyris, Dunne and Scharpf [2] have derived a nonsymmetric tangent stiffness matrix at the element level using the rotational degrees of freedom referred to fixed axes of a global cartesian system. Simo and Vu-Quoc [13] have concluded that, using the variation of rotational variables introduced by Atluri [5], the tangent stiffness matrix becomes a symmetric at only an equilibrium configuration, provided that no distributed external moments exist. This lack of symmetry [2] and the recovery of symmetry only at an equilibrium configuration [13] have been attributed to the fact that the finite rotation field is noncommutative. One of the objectives of this paper is to present a well-defined variational functional, and an associated 'principle' corresponding to the vanishing of the first variation of such a functional, when a conservative system of external forces and moments act on the finitely deformed beam. It should be emphasized that even though the present rotation field remains noncommutative, the functional presented leads to the symmetric tangent stiffness matrix not only at the equilibrium but also at the nonequilibrium configuration.

At first, we consider a general mixed variational principle for the static problem, wherein inertia terms are neglected in the equations of motion. The variational functional, in the presence of a conservative system of external forces and moments, which are presented in this paper, forms the basis of general mixed-hybrid finite element methods for finitely strained and rotated space-curved beams. The modus operandi for such finite element methods, involving finite rotation kinematics, has been discussed earlier by Atluri and Murakawa [3], Murakawa and Atluri [10], Atluri [5] for three-dimensional and shell problems; by Murakawa and Atluri [10] for plane elasticity problems; and by Punch and Atluri [11] for plate problems. For incremental functionals of the present problem, see Iura and Atluri [8].

For dynamic problems, we establish a well-defined Hamilton functional for the flexible beam undergoing finite rotations and stretches. The finite element formulation is utilized for deriving the semidiscrete equations of motion. A Newmark family of algorithms is used to integrate the semi-discrete finite element equation in time. A numerical example is presented to demonstrate the validity and applicability of the present study.

Throughout this paper, the summation convention is adopted; and the Latin indices will have the range 1,2, and 3, and the Greek indices the range 1 and 2.

THE GEOMETRY OF THE UNDEFORMED AND THE DEFORMED BEAM
Let $Y_m$ be the convected orthogonal curvilinear coordinate system. The coordinates $Y^m_\alpha$ are taken in the cross-section of the beam, while the coordinate $Y^3_\alpha$ is taken along the beam axis, as shown in Fig. 1. The unit base vectors associated with the coordinates $Y^m_\alpha$ are denoted by $E^m_\alpha$. The well-known Frenet-Serret formulae lead to the following relations:

$$E^m_\alpha, 3 = K \times E^m_\alpha, \quad K = k^m E^m_\alpha \quad \text{(1a,b)}$$

where $( )_3 = d( )/dL$ where $L$ is the arc length parameter of the line of origin of the coordinate system $Y^m_\alpha$ in the reference configuration; $K_\alpha$ are the components of initial curvature, and $k_3$ is the initial twist.

The undeformed base vectors at an arbitrary material point are given by

$$- \alpha = E^a - \alpha, \quad 3 = -Y^2 K_3 E^a_1 + Y^1 K_3 E^a_2 + g_o E^a_3, \quad g_o = 1 - Y^2 k_2^1 + Y^1 k_2^1. \quad \text{(2a,b)}$$

Let $\bar{8}_3$ be the unit vector tangential to the deformed beam axis. Without loss of any generality, the base vectors $E_\alpha$ and $E_3$ are assumed to be the maps of the base vectors $E^a_\alpha$ and $E^a_3$ after a purely rigid rotation, denoted by the tensor $R$, alone. In general, because of the transverse shear deformation, $E_3 \neq \bar{8}_3$. The relationship between the unit orthogonal vectors $E^m_\alpha$ and $E^m_\alpha$ is written as

$$E^m_\alpha = R^a E^a_\alpha, \quad R = R^a E^a_3 E^a_\alpha. \quad \text{(3a,b)}$$

From the definition of covariant base vectors, the vector $\bar{8}_3$ takes the natural form as:

$$\bar{8}_3 = (\delta^m_3 + u^m_3) E^m_\alpha / g, \quad g = \sqrt{(u^1_3)^2 + (u^2_3)^2 + (1 + u^3_3)^2} \quad \text{(4a,b)}$$

where $u (=u^m E^m_\alpha)$ is the displacement vector at the beam axis, $\delta^m_3$ the Kronecker delta, and $( )_3$ the covariant differentiation by using the metric tensor $E^a_\alpha E^a_\beta = E_\alpha E_\beta$. The angles of shear deformations, denoted by $\beta_\alpha$, are defined by $\sin \beta_\alpha = E^a_\alpha \bar{8}_3$.

According to the hypotheses used, the displacement vector at an arbitrary material point is expressed as $U = u + Y^a(E^a_\alpha - E^m_\alpha)$. The covariant base vectors at an arbitrary material point after the deformation are given by

$$- \alpha = E^a - \alpha, \quad 3 = (g \sin \beta_1 - Y^2 k_3^1) E^a_1 + (g \sin \beta_2 + Y^1 k_3^1) E^a_2$$

$$+ (g \beta_3 - Y^1 k_2^1 + k_2 k_1^1) E^a_3, \quad \text{(5a,b)}$$

where

$$\beta_3 = 1 - \sin^2 \beta_1 - \sin^2 \beta_2, \quad k_i = \frac{1}{2} E^a_{ijk} [(R^a E^a_\gamma), j] \cdot [R^a E^a_\gamma], \quad \text{(6a,b)}$$

in which $E^a_{ijk}$ is the permutation symbol. The vector $k$, defined by $K = k m E^m_\alpha$, satisfies the following differential relation: $E^m_\alpha, 3 = k \times E^m_\alpha$. 

- 3 -
THE EQUATIONS OF MOTION

With the help of the Green strain tensor \( \varepsilon_{ij} \) and the second Piola-Kirchhoff stress tensor \( S_j(S^2_{ij} A^i A^j) \), the internal virtual work is written as

\[
IVW = \int S^i_j \delta \varepsilon_{ij} dV, \quad dV = g_o dY^1 dY^2 dL. \tag{7a,b}
\]

The stress resultants and moments are defined, following Atluri [5], as:

\[
T = \int g_o ^3 \cdot (S_1 \cdot F^T) dA, \quad M = \int \gamma^{\alpha} A^\alpha \times [g_o ^3 \cdot (S_1 \cdot F^T)] dA, \tag{8a,b}
\]

where \( F \) is the deformation gradient tensor, \( \cdot \) a transpose and \( dA = dY^1 dY^2 \). By using the component representation, we obtain the stress resultants and moments in the form

\[
T = T^i_j e^i_j, \quad M = M^i_j = \int t^3 g_o dA, \quad M^2 = \int t^3 Y^1 g_o dA, \tag{9a-c}
\]

\[
M^3 = \int (t^3 Y^1 - t^3 Y^2) g_o dA, \quad \tag{9d,e}
\]

\[
M^3 = \int (t^3 Y^1 - t^3 Y^2) g_o dA, \quad \tag{9f}
\]

where \( t^{mn} = S^m_{ij} \cdot e^l_n \). The \( (\cdot) \) in the contravariant tensor is used to emphasize that these are not components in convected coordinates \( Y^m \).

As a rotational variation, we introduce, at first, a tensor \( \delta R \cdot R^T \) introduced by Atluri [5]. Since \( \delta R \cdot R = I \), \( \delta R \cdot R^T \) is a skewsymmetric tensor. There exists, therefore, a vector \( \delta \phi \) satisfying \( \delta R \cdot R^T = \delta \phi \times I \). It follows from Eqs (7) to (9) that the IVW is rewritten, after some manipulations, as

\[
IVW = - \int \left[ T^i_j \delta \varepsilon_{ij} + M^i_j + (x + u) \cdot x \cdot T \right] \cdot \delta \phi \, dL \tag{10}
\]

\[
\text{EVW} = \int P_b \cdot \delta u dV + \int P_c \cdot \delta u dS_c + \int \left[ P_e \cdot \delta u dS_e \right] \quad \text{L=1}
\]

\[
\left. \begin{array}{c}
S_u \quad L=0 \\
S_e \quad L=0 \\
S_c \quad L=0
\end{array} \right\} \tag{11a,b}
\]

where \( x \) is the undeformed position vector of a point at the beam axis and \( L \) the length of the beam axis before the deformation; \( S_u \) and \( S_e \) are parts of boundary on which geometrical and mechanical boundary conditions are prescribed respectively.

Let \( P_b (=P_b E_j) \) be the vector of body force defined per unit volume of the undeformed beam, \( P_c (=P_c E_j) \) the vector of distributed surface traction defined per unit area of the undeformed cylindrical surface of the beam, denoted as \( S_c \); and \( P_e (=P_e E_j) \) the vector of distributed surface tractions at the end cross sections denoted as \( S_e \). Then the external virtual work is written as

\[
\text{EVW} = \int P_b \cdot \delta u dV + \int P_c \cdot \delta u dS_c + \left[ \int P_e \cdot \delta u dS_e \right] \quad \text{L=1}
\]

\[
\left. \begin{array}{c}
S_u \quad L=0 \\
S_e \quad L=0 \\
S_c \quad L=0
\end{array} \right\} \tag{11a,b}
\]

where
The kinetic energy of the beam is written as
\[ T = \frac{1}{2} \int p u^* v^* \, \text{d}V \]
where \( p \) is the density in the reference state. The principle of virtual work for the elastodynamic problem is represented as
\[ \int_{t_1}^{t_2} [\delta T - IVW + EVW] \, \text{d}t = 0. \]
Using the conventional condition that the variations of displacements at \( t = t_1 \) and \( t = t_2 \) are equal zero, we obtain the LMB and the AMB conditions, expressed as:
\[ \begin{align*}
T, 3 + \delta u &= \frac{L}{t} \quad (\text{for arbitrary } \delta u), \\
M, 3 + (x+u), 3 \times T + \delta \phi &= \frac{H}{t} \quad (\text{for arbitrary } \delta \phi),
\end{align*} \]
where
\[ \begin{align*}
L_t &= \rho \dot{u} + J_{\alpha \alpha} \dot{e}_{\alpha}, \\
H_t &= J_{\alpha \alpha} e_{\alpha} \times \dot{u} + I_p \dot{W}, \\
I_p &= J_{\alpha \beta} (e_{\alpha} \times e_{\beta}) I - J_{\alpha \beta} e_{\alpha} e_{\beta}, \\
W &= I = R^{R^T}, \\
A_p &= \int p g_o \, \text{d}A, \quad J_{\alpha} = \int \rho g_o \, \text{d}A, \quad J_{\alpha \beta} = \int \rho g_\alpha g_\beta \, \text{d}A.
\end{align*} \]
The associated boundary conditions are written as
\[ \begin{align*}
T &= q, & M &= \bar{m} \quad \text{on } S_o, & u &= \bar{u}, & \delta \phi &= \bar{\theta} \quad \text{on } S_u
\end{align*} \]
where \( \bar{u} \) and \( \bar{\theta} \) denote the prescribed value on \( S_u \). It should be emphasized that the external moment vectors \( m \) and \( \bar{m} \), which are generated by the conservative forces, are configuration dependent, as shown in Eqs. (12c,d).
In the above development of equations of motion, the vector \( \delta \phi \) is used as rotational variables. Simo and Vu-Quoc [13] have pointed out, using the \( \delta \phi \) as a variation of rotational variables, that a well-defined functional exists at only an equilibrium configuration provided that no distributed external moments exist. Since the external moment vector \( m \), defined by Eq. (12c), is configuration dependent, the variation of inner product (\( m \cdot \delta \phi \)) does not yield \( m \cdot \delta \phi \). Therefore, the EVW, especially of the moments, does not, on first
sight, appear to correspond to the first variation of an external energy functional.

In order to express \( m \delta \phi \) of Eq. (11.b) as the first variation of an energy functional, we adopt a strategy wherein \( m \delta \phi \) can be expressed in terms of components of \( m \) and \( \delta \phi \) in the undeformed basis. It follows from Eqs. (3b) and (12c) and the definition of \( \delta \phi \) that

\[
m \delta \phi = m_{a_j} (e_a \times E_j) \delta \phi = m_{a_j} (E_j \delta R) = \delta(m_{a_j} R_j) \tag{18}
\]

Note that the Lagrangian components \( R_{i,a} \) of \( R \) are expressed in terms of three arbitrary parameters \( a_j \), such that \( \delta R = R_{i,k} E_k \delta a_i \), where \( ( ) \) denotes the differentiation with respect to \( a_i \). Equation (18) indicates the possibility of constructing a well-defined functional for the present problem with the use of \( a_i \).

To show the equivalence between the AMB condition for \( \delta a_i \) and that for \( \delta \phi \), we consider, at first, the tensor equation of AMB condition for \( \delta \phi \). The inner product between the AMB condition and the variation \( \delta \phi \) is expressed as

\[
\{m_{,3} + (x + u)_{,3} \times T + m_{,3} \} \cdot \delta \phi = C:(\delta R\cdot R^T) \tag{19}
\]

where

\[
C = Q^1_{a_3 a_2} + Q^2_{a_1 a_3} + Q^3_{a_2 a_1} + m_{a_j} E_j a - J_{a_i a - a_j} E_j a - J_{a_j a - a_i} E_i a,
\]

(20a,b)

When \( R \) is expressed in terms of \( a_i \), Eq. (19) is rewritten as

\[
C:(\delta R\cdot R^T) = C:(R_{a_j} R^a_{,i} \delta a_i) \tag{21}
\]

Since \( \delta R R^T \) is a skewsymmetric tensor, the AMB condition for \( \delta \phi \) is represented from Eq. (19) as \( C = C^T \). The AMB condition for \( \delta a_i \) is expressed from Eq. (21) as \( C:(R_{j,i} R^a_{,i} \delta a_i) = 0 \). Since \( R_{j,i} R^a_{,i} \) is a skewsymmetric tensor, the AMB condition for \( \delta a_i \) is shown to be equivalent to that for \( \delta \phi \).

To complete a beam theory, we consider a stress-strain relationships. Equations (9) indicate that the use of the stress tensor \( t^{mn} \) yield the compact definition for the stress resultants and moments. Therefore, we use the stress tensor \( t^{mn} \) and the conjugate strain tensors \( Y_{mn} \) to construct the constitutive equation. The conjugate strain tensors \( Y_{mn} \) are defined as

\[
Y_{mn} = a_m e_n - a_n e_m \tag{22}
\]

For one-dimensional beams, we assume the following constitutive equations:

\[
t_{3a} = G Y_{3a} \tag{23a,b}
\]

where \( G \) is the shearing modulus and \( E \) the Young modulus. Substituting Eqs. (23) into Eqs. (9) and using Eqs. (5) and (22) lead to

\[
T^1 = G A h_1 - GI \tilde{k}_3 \tag{24a,b}
\]

\[
T^2 = G A h_2 + GI \tilde{k}_3 ,
\]

\[
T^3 = EAh_3 + EI \tilde{k}_1 - EI \tilde{k}_2 \tag{24c,d}
\]

\[
M^1 = EI \tilde{k}_1 + EI \tilde{k}_3 - EI \tilde{k}_2
\]
\[ M^2 = EI_{22} \hat{k}_2 - EI_{2}h_3 - EI_{12} \hat{k}_1, \quad M^3 = GJk_3 - GI_1h_1 + GI_2h_2, \] \quad (24e,f)

where
\[ \hat{k}_i = k_i - K_i, \quad h_a = g \sin \theta_a, \quad h_3 = g \theta_3 - 1, \quad A = \int g_0 dA, \quad A = k_0 A, \] \quad (25a-e)
\[ I_a = \int Y \frac{g_0}{g_0} dA, \quad I_{12} = \int (Y^1 Y^2) g_0 dA, \quad I_{11} = \int (Y^1)^2 g_0 dA, \] \quad (25f-h)
\[ I_{22} = \int (Y^1)^2 g_0 dA, \quad J = \int (Y^1)^2 + (Y^2)^2 g_0 dA. \] \quad (25i,j)

The factor \( k_o \) is the shear-correction factor [6].

The strain energy function \( W_s \) per unit length is expressed as
\[ W_s = \frac{1}{2} G A_0 (h_1)^2 + \frac{1}{2} G A_0 (h_2)^2 + \frac{1}{2} E A (h_3)^2 + \frac{1}{2} E I_1 (k_1)^2 \]
\[ + \frac{1}{2} E I_{22} (k_2)^2 + \frac{1}{2} G J (k_3)^2 + E I_{1} h_3 \hat{k}_1 - EI_{12} \hat{k}_2 \]
\[ + \frac{1}{2} E I_{12} \hat{k}_1 \hat{k}_2 - GI_1 h_1 \hat{k}_3 + GI_2 h_2 \hat{k}_3. \] \quad (26)

It should be noted that the well-known and commonly used expression, given by Eq. (26), for strain energy function is derived from the strain-stress relationships given by Eqs. (23).

GENERAL MIXED VARIATIONAL PRINCIPLE FOR THE STATIC PROBLEM

As a basis of a numerical method, a variational principle often plays an important role. For many reasons, especially its flexibility in application, the mixed variational formulations are receiving a wide attention [4]. On the basis of the governing equations given above, we will derive the functional for general mixed variational principle for elastostatic beams.

As first shown by Fraeijes de Veubeke [7], and later generalized by Atluri and Murakawa [3], a general mixed principle, for a 3-dimensional elastic material, and involving the first Piola-Kirchhoff stress tensor \( \tilde{\sigma} \), the right stretch tensor \( U \), the finite rotation tensor \( \tilde{R} \) and the displacement vector \( v \) as variables, can be stated as the stationary condition of the functional \( F_1 \).
\[ F_1 (\tilde{\sigma}, U, R, V) = \int_V \left[ W_0 (U) + \tilde{\sigma}^T \{(1 + V_0 \cdot V)^T - R \cdot U\} \right] \]
\[ - \rho_o \frac{\partial}{\partial V} \int_S \left[ \tilde{\sigma} \cdot V ds - \int_S \frac{\partial}{\partial V} (\tilde{V} - V) ds, \right. \] \quad (27)

where \( W_0 \) is a strain energy function, \( \rho_o \) the mass density in the undeformed state, \( b \) the body force vector per unit mass, \( \tilde{F} \) the traction on the boundary per unit undeformed area and \( V_0 \) the gradient operator in the undeformed state.
The functional $F_1$ for a finitely deformed shell has been derived by Atluri [5]. Based on the resulting modified functional, some numerical results have been obtained by Punch and Atluri [11]. However, to the best of the authors' knowledge, no studies exist on the functional $F_1$ for a finitely deformed beam.

Employing $\alpha^i$ as rotational variables and after some manipulations, the functional for a finitely deformed beam is expressed as

$$G_1(T, M, h, \bar{k}, u, \alpha^i, L_j^+) = \left[ W_s(h, \bar{k}) + T \cdot \{(x+u), 3 - R \cdot (h+\bar{E} \cdot \bar{3})\}ight]_{L=1}$$

$$+ M \cdot \left\{1_3 - R \cdot \bar{k}\right\} \cdot g \cdot u - \alpha^j R \cdot \alpha^j \right\} \cdot dL - \left\{T \cdot (u - \bar{u}) + L_j(\alpha^j - \bar{\alpha}^j)\right\}_{L=0}$$

where $1_3$ is the vector satisfying $R \cdot 1_3 = 1_3 \times 1$, $L_j$ a Lagrangian multiplier. The stationary condition, $\delta G_1 = 0$, yields the constitutive equations, the compatibility equations, the LMB and the AMB conditions, the mechanical and geometrical boundary conditions, and the physical meaning of the Lagrangean multiplier. For incremental functionals, see Iura and Atluri [8].

HAMILTON'S PRINCIPLE AND NUMERICAL IMPLEMENTATION

When a potential energy $\pi_p$ is obtained, Hamilton's principle for elastodynamic problems is expressed as

$$\delta \int_{t_1}^{t_2} [T - \pi_p] dt = 0 \quad (29)$$

where the subsidiary conditions are the geometrical boundary conditions and the conventional conditions that the variations of displacements at $t=t_1$ and $t=t_2$ are equal to zero. For the present problem, we obtain the $\pi_p$ from Eq. (28) neglecting the compatibility conditions and the geometrical boundary conditions.

In the paper, to avoid using the four rotational parameters, the rotational variables $\alpha^i$ are defined as the Lagrangian components of the conformal rotation vector $\theta^*$ in the following way:

$$\theta^* = 4 \tan \frac{\omega}{4} \cdot \hat{e} = \alpha^i \bar{E}_i,$$  

where $\hat{e}$ is a unit vector satisfying $R \cdot \hat{e} = \hat{e}$ and $\omega$ a magnitude of rotation about the axis of rotation defined by $\hat{e}$. Because of singularity, the Rodrigues vector, defined by $\theta = 2 \tan \frac{\omega}{2} \cdot \hat{e}$, is valid only in the range of $|\omega| < \pi$. As shown in Eq. (30), however, the conformal rotation vector is valid even at $|\omega| = \pi$. Therefore, with the simple manipulation, the finite rotations are described in terms of only three rotation parameters.
The finite element formulation is used to derive the semidiscrete equations of motion. The displacement and the rotational components are interpolated by:

\[ u^i = u^i_N \theta^8, \quad \alpha^i = \alpha^i_N \theta^8, \] (31a,b)

where \( u^i_N \) and \( \alpha^i_N \) denote the nodal displacement and rotational components, respectively, and \( N^8 \) the shape functions defined by \( N^1 = 1 - L/L_e \) and \( N^2 = L/L_e \) where \( L_e \) is the element length. For later convenience, the following notations are introduced: \( \mathbf{d} = \{ u^i_N \} \) and \( \mathbf{r} = \{ \alpha^i_N \} \).

Following a standard finite element discretization, we obtain the following semidiscrete equations of motion:

\[ M(\mathbf{d}, \mathbf{r}, \dot{\mathbf{r}}) + C(\mathbf{r}, \dot{\mathbf{r}}) + k(\mathbf{d}, \mathbf{r}) = f(\mathbf{r}) \] (32)

where \( M \) depends linearly on \( \mathbf{d} \) but nonlinearly on \( \mathbf{r} \) and \( \dot{\mathbf{r}} \), and \( c, k \) and \( f \) depend nonlinearly on their variables. Note that the vector \( c \) is derived not from the damping effects but from the nonlinear effects of finite rotations, and that no simplification is made in this formulation in the sense that Coriolis and centrifugal effects as well as the inertia effects due to rotation are accounted for. The resulting semidiscrete equation is integrated by the Newmark algorithm. Consistent linearization procedures are employed to obtain linearized forms of the balance equations. A full Newton--Raphson method is used in the present calculations.

To illustrate the validity of the present theory in simulating large rotations, we analyze the case of a highly flexible right-angle beam in free flight, as shown in Fig. 2. Although we have used only three rotational parameters per node, the large deformations with finite rotations can be simulated without singularities. Another numerical examples and more detailed development of transient dynamic analysis of highly-flexible space-curved beams undergoing finite rotations and stretches may be found in Iura and Atluri [9].

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REFERENCES


Material Properties:

\( E A = G A_o = 10^5 \)
\( A p = 1, \ J_{11} = J_{22} = 10 \)
\( E I_{11} = E I_{22} = G J = 100 \)

F.E. Mesh: 10 elements

Time history of loading:

\( F_2 = F_0 \)
\( F_1 = F_3 = F_0 / 5 \)

Fig. 2 Flexible right angle beam in free flight.

Time step \( \Delta t = 0.1 \).
FIELD/BOUNDARY ELEMENT APPROACH TO THE LARGE DEFLECTION OF THIN FLAT PLATES

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Abstract—The problem of large deflections of thin flat plates is rederived here using a novel integral equation approach. These plate deformations are governed by the von Karman plate theory. The numerical solution that is implemented combines both boundary and interior elements in the discretization of the continuum. The formulation also illustrates the adaptability of the boundary element technique to nonlinear problems. Included in the examples here are static, dynamic and buckling applications.

INTRODUCTION

The boundary integral equation approach for the solution of initial and boundary value problems in continuum mechanics has proved to be an extremely efficient and effective technique. This has received considerable attention from many in the research community over the last 20 years. Initially, the method was restricted to linear problems, but recently extensions have been made to include nonlinear problems that have encompassed both the geometric and material varieties. The central feature of all integral equation approaches lies in the existence of a fundamental solution to the governing differential equation. The difficulty in the nonlinear case is that this solution, even if it exists, is unknown, and consequently alternative schemes must be pursued. One viable alternative is to use the fundamental solution in infinite space of just the linear portion of the equations. The price that must be paid here is that the equations will now contain domain integrals in addition to the boundary integrals. These domain integrals will primarily be associated with the nonlinearities in the system. Thus, the apparent advantage in reducing the dimensionality, as is the case in linear boundary element problems, is now lost in the nonlinear case.

As shall be seen, however, the integral equation approach possesses a number of benefits over other available numerical schemes. The modeling of the interior unknowns may not face such stringent inter-element continuity requirements as are found in the finite element method. Additionally, an incremental procedure most likely will be used and in such a scheme the interior integrals may be evaluated in terms of the results of the previous increment and so the only unknowns in this instance would be the boundary incremental unknowns. In general, a singular (fundamental) solution approach to nonlinear problems may best be viewed as a 'mixed boundary element/interior element method'.
restrictions. Another drawback that the finite element method suffers from is that many elements are required when the in-plane dimensions of the plate are large and consequently there will be many unknowns involved. It is advantageous to use the boundary element technique because it will achieve a given accuracy with fewer degrees of freedom and this has been illustrated in [6, 7] for the case of linear elastic plate vibrations. This results in an increased efficiency when the boundary element scheme is used. These were some of the factors that motivated the use of integral equation approach in the present analysis.

The weighted residual method is frequently used in finite element formulations, but in the boundary element method Betti's reciprocal theorem is very often used. However, it is found to be convenient also to express boundary element approaches in a weighted residual form also. This has been discussed in a recent publication by Atluri [8] and the suitability of the approach to a wide variety of nonlinear problems has been illustrated. Indeed it is correct to say that both the boundary element and finite element methods may be classified as members of the weighted residual family. The distinction between the two lies in the choice of the test functions. This will be addressed in greater detail later.

The remainder of this paper will concentrate on an integral equation approach for large deformations of plates. The technique has been well documented for linear static problems by Stern [9] and for linear dynamic analyses by Bezine [6]. An extension of these techniques is now proposed for the nonlinear problem. Several examples will also be presented to illustrate the versatility of the present approach. These include dynamic and buckling problems in addition to some static applications.

**BOUNDARY ELEMENT TECHNIQUE FOR LARGE DEFLECTIONS OF PLATES**

The formulation of the von Karman equations has been well documented [10] and here the dynamic analog of these equations will now be introduced [11].

\[ N_{n, \beta} = m u_n \quad (\alpha, \beta = 1, 2) \]  
\[ D \nabla^4 u_3 = (N_{ab} u_{3, \beta}), + p - m u_3, \]  

where \( m \) is the distributed mass, \( D \) is the bending rigidity of the plate and \( p \) is the transverse load. The in-plane displacement is denoted by \( u_n \) while the transverse displacement component is \( u_3 \). The in-plane stress resultants are represented by \( N_{ab} \). A comma is used to indicate differentiation with respect to position while a dot indicates temporal differentiation. It is important to remember that the above equations are nonlinear and this is also evidenced by the kinematic relationship

\[ e_{ab} = \frac{1}{2}(u_{a, b} + u_{b, a} + u_{a} u_{b, a}) - x_{1} u_{a, 3}. \]  

The above governing differential equations (1a), (1b) are complemented by the appropriate mechanical and geometric boundary conditions as outlined in [12]. If the inertia in the plane of the plate can be neglected, then it is possible to introduce a stress function [13] for the in-plane stresses \( N_{ab} \). This assumption is valid when the transverse modes under consideration are widely separated from the fundamental in-plane mode, and this assumption will be invoked here. This stress function approach results in the two equations (1a) being replaced by a single fourth order equation for the stress function. This approach will not be pursued here since the form of eqn (1) permits a more general range of applications. Also, with the use of stress functions, there is considerable difficulty in the implementation of the in-plane displacement boundary conditions. The apparent advantages in two fairly similar fourth order equations is lost and this will be discussed in more detail later.

Therefore, in deriving the integral constraints, eqn (1) will be used (with the in-plane inertias neglected). It may be noted that these equations are identical to the static set if the negative of the transverse load is considered to be the transverse inertia.

Integral equations representations will now be formulated for the three displacement components and also the normal slope on the boundary. Firstly, eqn (1b) will be considered so as to establish an expression for the transverse displacement.

In the classical boundary element approaches [14], the Betti reciprocal theorem was used to establish the integral constraints. This theorem is limited to linear elasticity, however particular forms of the reciprocal work theorem may be used in nonlinear problems [15]. A more versatile technique will be employed here; namely, a weighted residual scheme and at the same time a very careful choice of the test function is made. In the nonlinear problem there is no singular solution to the nonlinear differential equation as is the case for linear problems. The alternative scheme that is adopted here is to use the singular solution of the static equation which in this case is the biharmonic operator. The implication of this is that a number of the integrals will involve some interior components which relate to the nonlinear and time dependent portions [6].

Using the weighted residual approach, eqn (1b) may be recast as follows:

\[ \int_{A} (D \nabla^4 u_3 - (N_{ab} u_{3, \beta}), - p + m u_3) u^*_\eta \, dA = 0. \]  

Here, the weighting (test) function \( u^*_\eta \) is the aforementioned singular solution and this well known solution is given by:

\[ u^*_\eta (p, q) = 1/8 \pi r \ln r^2; \quad r = \|p, q\|. \]  

Physically, \( u^*_\eta \) is the transverse displacement at \( q \) due to a unit transverse load at a point \( p \) in an infinite
Large deflections of thin flat plates

Here the δ notation is used for the Dirac delta function and using the special properties of this function under the integral sign, the following well-known result may be established:

\[ \int_A u_3(q) \nabla^4 u_3^* (p, q) \, dA = u_3(p). \]  

(8)

This equation may now be substituted into eqn (6) and so an integral relationship has been developed for the transverse displacement \( u_3(p) \). In the interest of clarity, the distinction between \( S_a \) and \( S_f \) is ignored in eqn (6) and the ‘bar’ denoting the prescribed quantity has also been dropped. This equation is for the interior point \( p \) and it is now necessary to transform this to an expression for the displacement at a boundary point \( P \). This must be done with caution due to the discontinuous nature of the kernels when the source point coincides with the field point.

When this is done, the following relationship may be derived at the boundary:

\[ \frac{1}{D} \left[ \kappa u_3(P) + \int_S \left\{ u_3 V_3^* - u_3 M_{m,n}^* + u_3 M_{f,n}^* \right\} \, dS \right] - \frac{1}{D} \int_S \left\{ \frac{2}{\pi} \sum_{k=1}^{K} \left\{ u_3 \langle M_{m,n}^* \rangle - u_3 \langle M_{f,n}^* \rangle \right\} \right\} \]

(9)

where \( \kappa \pi \) is the included angle at the boundary.

A number of points need to be discussed in relation to the above equations. It is evident that this equation contains some volume integrals and these are specifically related to the nonlinear and time dependent portions of the differential equation. Unlike a finite element discretization, the interpolating functions will not be constrained by the \( C^1 \) continuity condition. In the case of a linear statics problem, there will no longer be interior integrals containing unknown terms and consequently the equation may then be termed a boundary constraint equation. However, even in this case, a single equation at each boundary point will be insufficient to solve a well posed engineering problem. This is because two quantities are specified on the boundary but there are also two unknowns. Thus a second integral equation is sought at the boundary and this will result in a pair of coupled integral equations. This may be achieved conveniently by considering another appropriate fundamental solution of the biharmonic equation and this is basically a directional derivative of the earlier solution and is given by

\[ u_3^* = \frac{1}{2\pi} r \ln r \cos \phi. \]  

(10)
The quantities \( \kappa_1 \) and \( \kappa_2 \) depend on the particular coordinate system \((\xi_1, \xi_2)\) at the boundary.

Equations (9) and (11) are now sufficient to solve a linear static problem with transverse normal loading. The only volume integral will not contain any unknowns, being a term of the form \( \int_\Omega p u^*_s \, d\Omega \), and this integration may be easily performed by splitting the interior into cells. Here, two of the four quantities, \( u_3 \), \( u^*_3 \), \( M_{n_3} \), and \( V_3 \), will be prescribed on the boundary and two will be unknown. Once these boundary quantities have been calculated, the required interior values may be estimated.

The formulation for the nonlinear problem still remains incomplete due to the presence of the in-plane quantities. The appropriate differential equations here are the homogeneous forms of eqn (1a) since the in-phase inertia is being neglected. Here again the differential equations are recast in weighted residual form along with suitable forms of the boundary conditions

\[
\int_\Omega N^{*,s}_{n_1} u_{n_1} \, d\Omega = 0 \quad \text{and} \quad \int_\Sigma (T - N^{*,s}_{n_2}) u^*_s \, d\Sigma = 0.
\]

where \( T^* = T^* (u^*_s) \) and \( n_2 \) represents the components of the normal vector at the boundary. For convenience in the subsequent analysis, the in-plane stress component \( N_{n_2} \) is written as the sum of the linear and nonlinear factors, where \( N_{n_2}^* \) denotes the linear part and \( N_{n_2}^* \) denotes the nonlinear part

\[
N_{n_2} = N_{n_2}^* + N_{n_2}^*.
\]

As in the transverse displacement case, the divergence theorem is again applied to eqn (12) and, on combination with the boundary condition, this leads to the expression

\[
\int_\Omega N^{*,s}_{n_1} u_{n_1} \, d\Omega - \int_\Sigma N^{*,s}_{n_2} u^*_s \, d\Sigma + \int_\Sigma (T - T^* ) u_s \, d\Sigma = 0. \quad (15)
\]

In this equation, the following notation has been added:

\[
u^*_s = \frac{1}{2} (u^*_3 + u^*_n).
\]

Since the in-plane stress resultants \( N_{n_2} \) contain nonlinear terms, there is no singular solution available to the homogeneous form of eqn (1a). Consequently, the choice for \( u^*_s \) is the fundamental solution of the (linear) plane stress problem. The displacement in the \( n_1 \) direction at \( q \) due to a unit load in the \( y \) direction at \( p \) is given by \( u^*_n (p, q) \) and this fundamental solution is well known to be [14]

\[
u^*_n = \frac{1}{8\pi\mu(1-v)} [(3-4v) \delta_{rn} \ln r - r_{m}]. \quad (17)
\]

Using this solution and again exploiting the properties of the Dirac delta function on the first term of eqn (15), an integral equation is found for the in-plane displacement components at the boundary:

\[
c_{n_2} u_{n_2} (P) = - \int_\Omega N^{*,s}_{n_2} u^*_s \, d\Omega + \int_\Sigma (T - T^* ) u_s \, d\Sigma, \quad (19)
\]

where the constant \( c_{n_2} \) depends on the included angle on the boundary and \( c_{n_2} = \delta_{n_2} \) in the interior.

Equation (19) consists of two equations at the boundary for the in-plane displacements \( u_{n_2} \) and \( u_s \). Again the interior term is evident and this is of course due to the nonlinear portion of the stress resultants. In the linear plane stress problem, this term will not be present and the system of equations will be entirely in terms of boundary unknowns.

The stress function approach for the in-plane deformation, that was mentioned earlier, initially appears to be an attractive alternative to the equilibrium eqns (1a) when the in-plane inertias are neglected. This is due to the presence of the fourth order operator governing both the stress function and the transverse displacement, facilitating the use of the same singular solution in both instances. However, the imposition of prescribed displacement boundary conditions presents extreme difficulties and this may
not be accomplished in a direct fashion [16]. The mathematical analogies that exist between linear plate bending and a stress function approach to the in-plane problem break down when it comes to the imposition of the boundary conditions since the stress function equation is merely a compatibility constraint. This was the basis of an earlier formulation by Tanaka [17]. As has been demonstrated the equilibrium equation approach is much simpler and both the in-plane displacement and stress resultant boundary conditions may be applied with little difficulty. Hence the adoption of this scheme is favored here. This formulation also allows for the in-plane inertias to be included if so desired.

**NUMERICAL IMPLEMENTATION**

In the preceding section, four integral constraints equations have been derived [eqns (9), (11), (19)]. These are basically four coupled equations which completely determine the large deformation problem of a thin flat plate, since at a general boundary point there will be four specified quantities and four unknowns. Formulating the problem in terms of total deformation is perfectly acceptable in the linear case. However, in the nonlinear situation, an incremental formulation, which leads to a piecewise linear incremental solution, must be pursued. In this approach, the prescribed boundary conditions are applied in small but finite increments and in a time dependent problem, these increments correspond to the time steps. Each state of deformation is labeled and in the notation employed here, the state before the addition of the (N+1)th load is referred to as C N, and the next state is labeled C N+1. Here all quantities will be referred to the initial or C0 configuration. This is commonly known as the 'Total Lagrangian' formulation. It differs from the 'Updated Lagrangian' formulation where the quantities are referred to the previous (C N) configuration.

The subscript N will be used to indicate a quantity in the Nth state, while the additional increment will be denoted with a Δ as follows:

\[ \Delta u = u^{N+1} - u^N. \]  

Equation (19) is now written in incremental form at state \( C_{N+1} \):

\[ c_{et}(u^N + \Delta u) = \int_s [(N_N) + \Delta(N_N)] u_{i,j} f_i dA \]

\[ + \int_s [(T_x(u^N + \Delta u) - T_x(u^N + \Delta u)) dS. \]  

Equation (21) may now be used to develop a further system of matrix equations; two at each boundary point and two at each interior point:

\[ P_i \Delta u_{i+} + R_i^N + \Delta E_i + H_i \Delta u_{+} + C_i \Delta u_{+} = 0. \]  

The next step is to discretize the boundary and the interior into elements in terms of the incremental variables. On the boundary the terms \( u_x, u_y, M_x, V_x, u, \) and \( T_x \) are interpolated in terms of nodal values in each one dimensional element. In the interior, the transverse displacement and its time derivatives must be interpolated over each element in addition to the two in-plane deformation variables. For convenience, the terms relating to the stress resultants \( N_{et} \) will be interpolated in terms of the corresponding kinematic values. Again it is significant to point out that the interior interpolating functions need not satisfy the \( C^1 \) continuity requirement.

Using eqns (A.1) and (A.2) with higher order terms neglected, two equations may be set up at each boundary node implementing the above described discretizations.

\[ G_i \Delta u_{i+} + J_i \Delta u_{+} + P_i \Delta u_{+} + R_i^N \]

\[ + \Delta E_i + C_i \Delta u_{+} = 0. \]  

where

\[ \Delta u_{i+}: \text{set of incremental nodal boundary unknowns from the set \{\Delta u_{i+}, \Delta M_{i+}, \Delta V_{i+}\}} \]

\[ \Delta E_i: \text{vector of forces due to prescribed boundary and external transverse loading conditions} \]

\[ \Delta u_{+}: \text{incremental nodal transverse displacements both on the boundary and in the interior} \]

\[ \Delta u_{+}: \text{incremental nodal in-plane displacement unknowns both on the boundary and in the interior} \]

\[ R_i^N: \text{correction vector from equilibrium at the Nth state} \]

The definitions of the matrices \( G_i, J_i, P_i, \) and \( C_i \) may be deduced immediately from the appropriate integrals in eqns (A.1) and (A.2). These equations must be complemented by a second system set up at each of the interior nodes (single equation per node).

\[ G_i \Delta u_{i+} + J_i \Delta u_{+} + P_i \Delta u_{+} + R_i^N \]

\[ + \Delta E_i + C_i \Delta u_{+} = 0. \]  

\[ \Delta u_{i+} \] are the unknown nodal traction values and the definitions of the matrices again follow from eqn (21). The three matrix systems of equations (22)-(24) are sufficient for solving the general large deformation problem. The number of equations equals the total number of unknowns in the vectors \( \Delta u_{i+}, \Delta M_{i+}, \Delta V_{i+} \) and \( \Delta u_{+} \). Equations (22) and (24) may then be solved for \( \Delta u_{i+}, \Delta u_{i+} \) and \( \Delta u_{+} \) in terms of the remaining unknowns to yield the following incremental system:

\[ MA \Delta u_{i+} + K \Delta u_{i+} = \Delta f. \]
where $\Delta f$ includes the incremental force vector and the equilibrium correction vector $R_0$. In a static problem the dynamic term in the above equation will no longer be present.

The $K$ matrix in eqn (25) must not be confused with the stiffness matrix in a conventional finite element approach. Also, because of the nonlinear nature of the problem, it must be remembered that $K$ is no longer constant and it is a function of the current state of deformation. It is perhaps best to label this matrix here as a coefficient matrix and in general it is fully populated and unsymmetric. Likewise the matrix $M$ in dynamic problems will have a similar structure to the $K$ matrix.

Because the coefficient matrix $K$ depends on the state of deformation, it must be updated periodically during the incremental process. The available procedures include such schemes as the Newton—Raphson or the modified Newton—Raphson techniques and these have been well documented in the literature. The particular choice usually governs the convergence rate in each step. A variant of these was discussed earlier in the introduction [8] wherein the interior terms are approximated by their values at the previous load step. This is similar to a linear boundary element scheme with a complicated system of body forces that are due to the nonlinear deformations. This is analogous to a finite element method where the stiffness matrix corresponding to linear elasticity is used throughout.

Several alternatives were pursued here with varying degrees of success. The linear elastic stiffness approach was found to converge extremely slowly and consequently it was discarded. The modified Newton—Raphson scheme behaved somewhat better but the full Newton—Raphson algorithm was the favored approach in this analysis. Since the coefficient matrix in the boundary element scheme is not as large as in the usual finite element approach, the reduction of this matrix at each iteration cycle is not as critical and, moreover, it was found that the solution converged very rapidly in this instance.

**RESULTS**

Several classes of problems in the domain of large plate deformations were analyzed to examine the effectiveness of the singular solution approach developed here. Included are static, dynamic and buckling problems, and comparisons will be made with some of the more classical series solution test cases that are available in the literature.

The earliest solution to the static von Karman thin plate equation was provided by Levy [2] from extensive investigations into aircraft sheet stringer panels. While these were basically textbook problems for rectangular plates with a restricted class of boundary conditions, they have become the benchmark solutions over the years for the large array of numerical techniques that have followed [4, 5]. These modern methods, the singular solution approach being one, are no longer confined to simplified geometries and idealistic boundary conditions and can be readily applied to more meaningful problems that arise for the design engineer.

The thin plate analyzed here was assumed to be simply supported, and a uniformly distributed transverse load $p$ was applied. The plate was square and, due to symmetry, just one quadrant was modeled (see Fig. 1). It was also assumed that there was no displacement perpendicular to the side of the plate, i.e. on $x_1 = 0$, then $u_2 = 0$. This was one of the boundary conditions imposed by Levy in his series solution. The thickness ratio was $h/a = 1/160$, Poisson’s ratio was assumed as 0.316 and Young’s modulus was $30 \times 10^6$ psi. Linear interpolation functions are used for the boundary unknowns. In addition here the in-plane displacements and tractions must also be discretized on the boundary. A linear interpolation field is also used in this instance, while in the interior, the unknowns (which comprise the three incremental displacement components) are also linearly interpolated. Since the incremental values in the interior include partial derivatives of these displacement components, the interpolation must be linear at least so that there will be a contribution from the nonlinear terms. The central deflection, using a $4 \times 4$ mesh to model the quadrant, is presented in Fig. 2 and is compared with the results of Levy [2], and the excellent agreement is obvious. The membrane stresses were also calculated at various locations in the plate and these are plotted in Fig. 3 where a comparison is again made with Levy and also with the finite element analysis of Bergan and Clough [4]. Here, the field/boundary element scheme contained 95 nodal unknowns while the finite element method [4] had 125 unknowns. Despite the smaller number of unknowns,
Large deflections of thin flat plates

Fig. 2. Central deflection of a square plate under uniform transverse load.

The present approach appears to give results that are closer to the analytical solution as shown in Fig. 3. These stresses were calculated using numerical differentiation, and extremely accurate solutions were obtained. Explicit expressions may also be derived for these stresses and also for the bending moments and the shears, but this was not carried out here due to the excessive algebra involved.

Buckling problems were also investigated by subjecting the plate to a uniaxial compression in one direction, and buckling was then induced by applying a small load in the transverse direction. Again the plate was simply supported and the two edges had displacements prescribed, and the other two sides were free in the sense that self-equilibrating loads were applied to maintain the edges straight. A mesh similar to that used in the normal loading case is used and the central deflection is plotted in Fig. 4 as a function of the average compressive stress on the sides with prescribed displacements. Again the correlation of the post-buckling behavior with the series solution is very good.

A dynamic response analysis was also conducted, permitting the influence of the large amplitudes on the period of free vibration to be investigated. The incremental acceleration term $\Delta \ddot{u}_i$ must also be discretized in the interior, and again a linear interpolation similar to the other interior unknowns is used. The period and amplitude were calculated from the dynamic response plot of the initial value problem with the Newmark Beta method used for the time integration. The results for a square plate, with the
placement and stress function were expressed in terms of in-plane inertia, and then both the transverse displacement and stress function were used for the in-plane portion, thus neglecting the transverse inertia. It is seen that the period decreases rapidly with increasing amplitude, and this is an example where the boundary integral equation procedure carried out by Chu and Herrmann [18]. In this reference, the stress function approach was used for the in-plane portion, thus neglecting the in-plane inertia, and then both the transverse displacement and stress function were expressed in terms of a trigonometric series. It is seen that the period decreases rapidly with increasing amplitude, and this feature will also be evident later in the nonlinear control examples.

The preceding three examples all demonstrate that the 'mixed boundary element/finite element' singular solution scheme gives accurate results for relatively few degrees of freedom and illustrate that it is a viable alternative to some of the other available numerical schemes. A somewhat similar procedure has also been developed recently for the nonlinear shallow shell problem [19] and this also shows considerable promise.

CONCLUSIONS

A singular solution approach for large deformations of a thin flat plate has been outlined in this paper and it is apparent that the results obtained in both the static, dynamic and buckling cases are extremely accurate and compare favorably and efficiently with other available numerical procedures. This is an example where the boundary integral equation procedure has been extended to nonlinear problems yielding a so-called mixed boundary element/interior element procedure. Advantages of this approach over finite element techniques already have been enumerated and these include the reduction in number of degrees of freedom and simplified interpolating functions for the plate bending problem. The extension of these ideas to shell problems also has been alluded to and this is currently the subject of further investigation.

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For completeness, the incremental and its normal derivative at the boundary are given by

\begin{align*}
\frac{\partial}{\partial n}(\lambda + \delta) - \frac{\partial}{\partial n}(\lambda_0 + \delta_0) &= 0 \\
\frac{\partial}{\partial n}(\mu + \delta) - \frac{\partial}{\partial n}(\mu_0 + \delta_0) &= 0
\end{align*}

For large deflections of thin flat plates, the equations for the incremental and derivative at the boundary are given by

\begin{align*}
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} &= \frac{\partial^2}{\partial x^2} \left( \lambda + \delta \right) - \frac{\partial^2}{\partial x^2} \left( \lambda_0 + \delta_0 \right) \\
\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} &= \frac{\partial^2}{\partial y^2} \left( \mu + \delta \right) - \frac{\partial^2}{\partial y^2} \left( \mu_0 + \delta_0 \right)
\end{align*}

APPENDIX A

Large deflections of thin flat plates.
Large-deformation, elasto-plastic analysis of frames under nonconservative loading, using explicitly derived tangent stiffnesses based on assumed stresses

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Abstract. Simple and economical procedures for large-deformation elasto-plastic analysis of frames, whose members can be characterized as beams, are presented. An assumed stress approach is employed to derive the tangent stiffness of the beam, subjected in general to non-conservative type distributed loading. The beam is assumed to undergo arbitrarily large rigid rotations but small axial stretch and relative (non-rigid) point-wise rotations. It is shown that if a plastic-hinge method (with allowance being made for the formation of the hinge at an arbitrary location or locations along the beam) is employed, the tangent stiffness matrix may be derived in an explicit fashion, without numerical integration. Several examples are given to illustrate the relative economy and efficiency of the method in solving large-deformation elasto-plastic problems. The method is of considerable utility in analysing off-shore structures and large structures that are likely to be deployed in outerspace.

1 Introduction

Space frames are versatile forms of structures that are widely used in diverse engineering applications such as off-shore structures in the petroleum industry or large structures that are intended to be deployed in outerspace for uses such as radio telescopes, space platforms, etc. The finite element method has an obvious natural appeal in solving such problems. Most such solutions are currently limited to small deformation, linear elastic analyses, using stiffness matrices based on assumed displacement approaches. An efficient design, as well as integrity analyses, of such structures often necessitates the study of large-deformation, elasto-plastic behavior in the post-buckling range.

At present, most nonlinear (either geometrically or materially nonlinear) analyses of frames are based on assumed-displacement type formulations, based on the variants of a Lagrangean (updated, total, corotational, or combinations thereof) approach, wherein the stiffness matrix is evaluated numerically several times during the analysis. Such displacement formulations were given, for instance, by Archer (1965); Argyris, Hilpert, Malejannakis and Scharpf (1979), and Saran (1984). In such formulations, polynomial type displacement fields (say, cubic for transverse and linear for axial) are assumed along each beam and appropriate nonlinear-terms are retained in the total strain-displacement relations. The nature of these formulations is such that, in practice, several elements are needed to model each beam member of the frame in order to capture the effects of change in the axial length of the beam due to large deformations. This necessity to use a large number of elements, coupled with the need to numerically evaluate the (tangent) stiffness matrix of each element several times during the analysis, renders the simple displacement-based finite element analysis of geometrically nonlinear behavior of frames still not economically feasible. Symbolic manipulation procedures to explicitly evaluate the integrals in the stiffness matrix, based on assumed cubic or linear displacement fields, were used by Nedergaard and Pedersen (1985). On the other hand, Kondoh and Atluri (1984) and Kondoh, Tanaka and Atluri (1985a, b) presented procedures whereby the tangent
stiffness matrix of a beam element, undergoing large deformations, can be evaluated explicitly, without employing either numerical or symbolic integration and without using simple polynomial (linear or cubic) type basis functions for displacements of the beam. In these procedures, the polar-decomposition of the deformation is used to separate out the deformation of the beam into an arbitrarily large rigid rotation of the beam as a whole, on which are superposed only moderate axial stretches and moderate non-rigid (relative) point-wise rotations of the beam. The strongly nonlinear dependence of the total axial stretch of the beam on the arbitrarily large nodal displacements has been fully accounted for. The explicit expressions for tangent stiffness matrix of the beam, undergoing the above-described type large deformations, were obtained by Kondoh and Atluri (1984) and Kondoh, Tanaka and Atluri (1985a, b), using the exact solutions for the problem of a beam-column, subject to axial forces as well as bending moments, wherein the nonlinear coupling between the axial stretch and transverse deformation is accounted exactly and wherein the non-rigid (relative) point-wise rotations are only moderate in magnitude. Using an "arc-length" method for studying the response near and beyond critical points, Kondoh and Atluri (1984) and Kondoh, Tanaka and Atluri (1985a, b) have presented studies of nonlinear behavior of various 2- and 3-D space frames and demonstrated that, even in situations of large deformations far beyond those that may occur in practice, each member of the frame can be modeled by a single element.

It is important to point out that most of the above studies are restricted to conservative or dead-type loading that is independent of deformation. In practice, however, there is an important class of problems, wherein a non-conservative (deformation-dependent) distributed loading, on each member of the frame, plays an important role. A case in point is the currently studied concept of distributed control of deformations of space frames through using piezoelectric lining all along the members of the frame. These distributed control devices, which are feedback-control mechanisms, exert non-conservative type distributed loading on the beams. The stiffness matrix of each member, under these conditions, ceases to be symmetric. It is well known that it is cumbersome, at best, to deal with such non-conservative loads in a displacement-based formulation.

If any member of the frame undergoes plastic deformations, the effects of such deformations can be accounted for, either by a detailed study of the spread of plastic zones in the depth and length directions of the beam or by a simpler method such as the plastic-hinge method. The former method, in general, involves numerical integration through the depth and length of the beam in order to evaluate the tangent stiffness of the member and, hence, is computationally expensive. On the other hand, the plastic-hinge methods, as developed by Hodge (1959); Ueda, Matsuishi, Yamakawa and Akamatsu (1968); and Ueda and Yao (1982), are computationally very attractive.

It is the purpose of this paper to present a new, computationally attractive alternative to the analysis of large-deformation, elasto-plastic behaviour of frames (with beam-type members), under a general non-conservative loading. The present formulation is based on assumed stress resultants and stress couples, which satisfy the momentum balance conditions in the beam *a priori*. The beam is assumed to undergo arbitrarily large deformations, which are decomposed into (i) an arbitrarily large rigid rotation of each beam as a whole, which is superposed on (ii) moderately large, non-rigid, point-wise rotations and displacements gradients. The nonlinear stretching-bending coupling is accounted for exactly in each element. A plastic-hinge method, wherein the hinge may form at an arbitrary location or locations along the beam, is used to account for plasticity. Under these circumstances, it is shown to be possible to obtain the tangent stiffness matrix of the beam (which is unsymmetric under nonconservative loading) in an explicit form. Several examples are included to demonstrate that, in most problems of practical importance, it suffices to use a single element to model each member and thus, the present procedure may be economically viable to analyse large space structures.

The paper is organized as follows: Section 2 deals with the kinematics of large deformations as presently assumed for each element; Section 3 deals with momentum balance relations; Section 4 deals with a weak (symmetric variational) formulation of the problem; Section 5 deals with plasticity effects; Section 6 deals with a brief description of the equation-solving algorithm; Section 7 deals with numerical examples and Section 8 deals with some concluding remarks.

*Notation* Second-order tensors are indicated by capital bold italic and vectors by minor bold italic letters. For instance if \( A(A_{ij} e_i e_j) \) and \( B(B_{ij} e_i e_j) \) are two second-order tensors, then \( AB = A_{ik} B_{kj} e_i e_j \) and \( A:B = A_{ij} B_{ij} \). Matrices in general will be denoted by \( (A) \).
2 Kinematics of large deformation of a frame member

For simplicity, we consider, without loss of generality, a beam of initial length \(l\) lying along the \(x_1\) axis and consider its deformation to be confined to the \(x_1 x_2\) plane. We invoke the familiar Kirchhoff-hypotheses and thus assume that the deformation everywhere in the beam is known, once the deformation of the reference axis is described. The case of arbitrarily large mid-plane stretches and rotations of beams, plates and shells has been comprehensively discussed in Atluri (1983). In the present paper, however, as shown in Fig. 1, the deformation of the beam is divided into two stages: (i) moderate stretch \(h\) and moderate rotation \(\theta^*\) relative to the undeformed configuration and (ii) arbitrarily large rigid rotations of the beam as a whole superimposed on the deformation in stage (i).

Let \(dx = dx_1 e_1\) be a differential vector along the reference axis of the undeformed beam. Let \(dy^*\) be its map due to deformation in stage (i). Thus,

\[
dy^* = R_0^* U_0 dx = F_0^* dx,
\]

where

\[
R_0^* = (\cos \theta^* e_1 e_1 - \sin \theta^* e_1 e_2 + \sin \theta^* e_2 e_1 + \cos \theta^* e_2 e_2)
\]

and the subscript "0" denotes the value of the tensor at the reference axis of the beam. In (2.2), \(\theta^*\) is assumed to be \(\leq 1\). Under the present assumptions for deformation, the stretch tensor at the reference axis is expressed as

\[
U_0 = (1 + h) e_1 e_1 + e_2 e_2,
\]

where \(h\leq 1\). The displacements of the reference axis in stage (i) are \(u_1^*\) and \(u_2^*\) along \(x_1\) and \(x_2\), respectively and these are functions of \(x_1\). The deformation gradient \(F_0^*\), at the reference axis, in stage (i), is

\[
F_0^* = (1 + u_{1,1}^*) e_1 e_1 + u_{2,1}^* e_1 e_2 + n^* e_2,
\]

where \(n^*\) is the unit normal to the deformed reference axis after deformation in stage (i).

From Eqs. (2.1)—(2.4) it follows that

\[
(1 + u_{1,1}^*) = (1 + h) \cos \theta^* , \quad u_{2,1}^* = (1 + h) \sin \theta^*
\]

and

\[
h = [(1 + u_{1,1}^*)^2 + (u_{2,1}^*)^2]^{1/2} - 1 .
\]

Inasmuch as it is assumed that, in stage (i), \(h, u_{1,1}^*, u_{2,1}^* \leq 1\), we obtain from (2.6) the following expression for the total change in the length of the beam

\[
H = \int_0^l (1 + h) dx_1 - l \approx \int_0^l u_{1,1}^* dx_1 = (\tilde{u}_1^*) \equiv 2u_{1,1}^* - 1 u_{1,1}^*,
\]

where \(\tilde{u}_1^* (\alpha = 1, 2)\) are displacements of the two ends of the beam along \(x_1\) axis. The final deformation of the reference axis of the beam, as assumed, results from the superposition of stage (ii) on stage (i)

\[
dy = R_0 dy^* = \tilde{R}_0 F_0^* dx = \tilde{R}_0 R_0^* U_0 dx = F_0 dx,
\]

where \(F_0\) is the total deformation gradient at the reference axis and \(\tilde{R}_0\) is an arbitrarily large rigid rotation, expressed as

\[
\tilde{R}_0 = (\cos \beta e_1 e_1 - \sin \beta e_1 e_2 + \sin \beta e_2 e_1 + \cos \beta e_2 e_2).
\]

If the total displacements from stages (i) and (ii) are \(u_1\) and \(u_2\), we have, through the use of (2.2), (2.3) and (2.9) in (2.8), that

\[
dy = (1 + u_{1,1}) dx_1 e_1 + u_{2,1} dx_1 e_2 = (1 + h) \cos (\tilde{\beta} + \theta^*) e_1 dx_1 + (1 + h) \sin (\tilde{\beta} + \theta^*) e_2 dx_2 .
\]

1 The extension of the present work to a three-dimensional case is simple and follows the procedures outlined in Kondoh, Tanaka and Atluri (1985a, b)
Hence, we have

\[ H = \int_0^l (1 + h) \, dx_i - l = \int_0^l \{ (1 + u_{1,1})^2 + (u_{2,1})^2 \} \, dx_i - l \]  

(2.11)

Now, since \( \bar{R} \) is a rigid rotation of the beam as a whole, it is clear from Fig. 1 that \( H \) in (2.11) may be approximated as

\[ H \approx \{ (l + \bar{u}_1)^2 + (\bar{u}_2)^2 \}^{1/2} - l \]  

(2.12)

where \( \bar{u}_\beta \equiv u_\beta - u_\beta (\beta = 1, 2) \) and \( u_\beta \) are displacements along \( x_\beta \) axis of the end \( \alpha (\alpha = 1, 2) \) of the beam. From Fig. 1, it is apparent that the rigid rotation \( \bar{\theta} \) of the beam as a whole is such that

\[ \tan \bar{\theta} = \frac{\bar{u}_2}{(l + \bar{u}_1)} \]  

(2.13)

From (2.12) and (2.13), it is clear that

\[ \cos \bar{\theta} \equiv \frac{\partial H}{\partial \bar{u}_1} = \frac{\partial H}{\partial (l + u_1)} = \frac{\partial H}{\partial (l_1)} \]  

\[ \sin \bar{\theta} \equiv \frac{\partial H}{\partial \bar{u}_2} = \frac{\partial H}{\partial (l + u_2)} = \frac{\partial H}{\partial (l_2)} \]  

(2.14)

Note that \( \bar{\theta} \equiv (\bar{u}_1, \bar{u}_2) \) and that

\[ \frac{\partial \bar{\theta}}{\partial \bar{u}_1} \approx \frac{\sin \bar{\theta}}{l + H} \approx \frac{\sin \bar{\theta}}{l} ; \quad \frac{\partial \bar{\theta}}{\partial \bar{u}_2} \approx \frac{\cos \bar{\theta}}{(l + H)} \approx \frac{\cos \bar{\theta}}{l} \]  

(2.15)

Note that under the present assumptions, \( H \ll l \).

Let the total rotation of a differential segment of the beam be \( \theta \), i.e., \( \theta = \bar{\theta} + \theta^* \). Hence the relative rotations at the ends of the beam, \( (\theta^*) (\alpha = 1, 2) \), are related to the total nodal rotations of the beam, \( (\bar{\theta}) \), as

\[ \theta^* = \theta - \tan^{-1} \left( \frac{\bar{u}_2}{l + \bar{u}_1} \right) \]  

(2.16)

Finally, the normal to the undeformed reference axis of the beam is \( N \equiv e_2 \). Due to the straining deformations in stage (i), this normal is mapped into \( n^* \):

\[ n^* = R^* e_2 \equiv -\sin \theta^* e_1 + \cos \theta^* e_2 \approx -\theta^* e_1 + e_2 \]  

(2.17)

Since \( n^* \) is assumed to be normal to the deformed reference axis, it follows that \( \theta^* = u_{2,1}^* \) and that the curvature strain, \( \kappa \), induced in the stage (i) deformation, is
3 Momentum balance relations

If the coordinates $x_1$ and $x_2$ are assumed to be convected with the deformed beam, the base vector which is tangential to the deformed reference axis, denoted as $a_1$, is given by

$$a_1 = F_0 e_1 = \bar{R}_0 R_0^* U_0 e_1 \approx \bar{R}_0 R_0^* e_1,$$

(3.1)

since $U_0 \approx I$. Thus, $a_1$ is approximately a unit vector. Let $\tau$ be the tensor of Cauchy stress-resultants in the beam, which, for the present deformation hypotheses, is assumed to have the form

$$\tau = a_1 (Na_1 + Qn),$$

(3.2)

such that the force system on a cross section normal to the deformed reference axis, is given by $[a_1 (\tau)]$. Likewise, the Cauchy stress-couple resultant in the beam is given by

$$\mu = a_1 [Me_3],$$

where $e_3 = (e_2 \times e_1)$ as shown in Fig. 1. As discussed in detail in Atluri (1983), the second Piola-Kirchhoff stress-resultant tensor may be defined as

$$s = F_0^{-1} (\tau) F_0^{-T} (F_0) = U_0^{-1} R_0^* \bar{R}_0^* (\tau) \bar{R}_0 R_0^* U_0^{-1} (F_0) \approx R_0^* \bar{R}_0^* (\tau) \bar{R}_0 R_0^*$$

(3.4)

since $U_0 \approx I$. Also, for the same reason, the Jaumann stress resultant tensor may be written as (Atluri 1983)

$$r \approx R_0^* \bar{R}_0 (\tau) \bar{R}_0 R_0^*.$$ 

(3.5)

Using (2.18), (3.1) and (3.2) in (3.5), it is easy to see that

$$r = (Ne_1 e_1 + Qe_1 e_2).$$

(3.6)

Thus, in the present problem, the Jaumann stress-resultant tensor is such that it has the same components (i.e. $N$ and $Q$) in the basis system $e_1$ and $e_2$ as does the Cauchy stress-resultant tensor in the convected basis system $(a_1, n)$. A similar interpretation holds for the Jaumann stress-couple tensor in the present problem.

Under the assumptions of deformation ($R_0^* \approx I$ and $\bar{R}_0$ is a rigid motion of the beam as a whole, etc.) as in the present problem, the general and consistent forms of linear and angular momentum balance relations, for Jaumann stress resultants and stress couples, given in Atluri (1983), may be simplified as

$$\frac{\partial N}{\partial x_1} + \tilde{q}_1 = 0,$$

(3.7)

$$\frac{\partial^2 M}{\partial x_1^2} + \frac{\partial}{\partial x_1} \left( N \frac{\partial u_2}{\partial x_1} \right) + \tilde{q}_2 = 0,$$

(3.8a)

where $\tilde{q}_1$ and $\tilde{q}_2$ are forces along $\tilde{e}_1$ and $\tilde{e}_2$ directions, per unit of undeformed length, where

$$\tilde{e}_1 = \bar{R}_0 e_1, \quad \tilde{e}_2 = \bar{R}_0 e_2$$

(3.9)

e.i., the vectors which result upon rigidly rotating $e_1$ and $e_2$ by the rigid rotation $\bar{R}_0$ of the beam as a whole.

The forces applied on the beam can be considered to be of both the "conservative" as well as of the "non-conservative" type. Let $q_{1n}$ and $q_{2n}$ be the "conservative" type (or "dead") loads along $e_1$ and $e_2$ directions and per unit length of the undeformed beam. Let $q_{1n}$ and $q_{2n}$ be "non-conservative" type
loads per unit length of the beam which always remain tangential and normal to the rigidly rotated axis of the beam. Thus, it is seen that

\[ \dot{q}_1 = q_{1n} + q_{1c} \cos \vartheta + q_{2c} \sin \vartheta, \]
\[ \dot{q}_2 = q_{2n} - q_{1c} \sin \vartheta + q_{2c} \cos \vartheta. \]

In the sequel, we shall assume, for purposes of a discrete solution of the problem of a frame, trial functions for \( N \) and \( M \) that satisfy a priori Eqs. (3.7) and only the linear part of Eq. (3.8a), namely,

\[ \frac{\partial^2 M}{\partial x_1^2} + \dot{q}_2 = 0. \]

The corresponding test functions (or variations in \( N \) and \( M \)) are taken to satisfy the homogeneous forms (i.e., without \( \dot{q}_a \)) of Eqs. (3.7) and (3.8b). The trial functions for \( N \) and \( M \) are thus

\[ N = n + N_p, \]
\[ M = \left( 1 - \frac{x_1}{l} \right) m_1 + \frac{x_1}{l} m_2 + M_p, \]

where

\[ N_p = (N_{p1c}) \cos \vartheta + (N_{p2c}) \sin \vartheta + (N_{pn}), \]
\[ M_p = (-M_{p1c}) \sin \vartheta + (M_{p2c}) \cos \vartheta + (M_{pn}), \]
\[ N_{p1c} = -\int_0^{x_1} q_{1c} \, dx_1 + \frac{1}{l} \int_0^{x_1} \left[ \int_0^{x_1} q_{1c} \, dx_1 \right] \, dx_1, \]
\[ M_{p1c} = -\int_0^{x_1} \left[ \int_0^{x_1} q_{1c} \, dx_1 \right] \, dx_1 + \frac{x_1}{l} \int_0^{x_1} \left[ \int_0^{x_1} q_{1c} \, dx_1 \right] \, dx_1. \]

The quantities \((N_{p2c}) \) and \((N_{pn}) \) in (3.14) are obtained from (3.16) by replacing \((q_{1c}) \) by \((q_{2c}) \) and \((q_{1n}) \), respectively. Likewise, \((M_{p2c}) \) and \((M_{pn}) \) are obtained from (3.17) by replacing \((q_{1c}) \) by \((q_{2c}) \) and \((q_{1n}) \), respectively.

The variations in \( N \) and \( M \) are assumed as

\[ \delta N = v, \]
\[ \delta M = \mu = \mu_1 \left( 1 - \frac{x_1}{l} \right) + \mu_2 \left( \frac{x_1}{l} \right). \]

4 A formulation for obtaining a weak solution

In this section, we shall consider the beam to remain elastic. We shall consider the effects of plasticity in Sect. 5. The stress-strain relations between the conjugate pairs \((N \) and \( h \)) and \((M \) and \( \kappa \)) are assumed to be satisfied a priori as

\[ \frac{\partial W_c}{\partial N} = h, \quad \frac{\partial W_c}{\partial M} = \kappa, \]

where \( W_c \) is the complementary energy density. For a linear elastic material, when the reference axis of the beam is at mid-thickness, one has

\[ W_c = \frac{1}{2} \left( \frac{N^2}{EA} + \frac{M^2}{EI} \right), \]

where \( A \) is the area and \( I \) the moment of inertia of the cross section. Thus, it remains to enforce: (i) "compatibility" of deformation within each beam, (ii) momentum balance conditions, (3.7) and (3.8a), within each beam element and (iii) the joint equilibrium or "inter-element" traction reciprocity (Atluri 1975).
The “weak” forms of these conditions may be written, respectively, as follows

(i) **Compatibility**

\[
\int_0^1 \frac{\partial W_e}{\partial N} v dx_1 = \int_0^1 u_{x_1}^* v dx_1 = vH = v[(l + \bar{u}_1)^2 + (\bar{u}_2)^2]^{1/2} - l
\]

and

\[
\int_0^1 \frac{\partial W_e}{\partial M} \mu dx_1 = -\int_0^1 \frac{\partial \theta^*}{\partial x_1} \mu dx_1 = -\theta^* \mu|_0^1 + \int_0^1 \theta^* \frac{\partial \mu}{\partial x_1} dx_1.
\]

Since \(\frac{\partial \mu}{\partial x_1} = \frac{1}{l}(\mu_1 + \mu_2)\) and \(\int_0^1 \theta^* dx_1 = 0\), we have

\[
\int_0^1 \frac{\partial W_e}{\partial M} \mu dx_1 = -\theta^* \mu|_0^1 = -(\theta^* \mu_1 + \theta^* \mu_2).
\]

(ii) **Interior Momentum Balance.** Consider variations along the beam (or test functions) of generalized displacements, such that \(\delta u_{x_1}^* = u_{x_1}^*\) and \(\delta \theta^* = \theta^*\). Let the momentum balance relation (3.8a) be rewritten, in two parts, as

\[
\frac{\partial Q}{\partial x_1} + \bar{q}_2 = 0, \quad \frac{\partial M}{\partial x_1} = Q + N \frac{\partial u_{x_1}^*}{\partial x_1} = 0.
\]

We will assume that the linear momentum balance relations (3.7) and (4.5a) are satisfied identically a priori. The weak form of the remaining balance condition, (4.5b), may be written as

\[
\int_0^1 \left(\frac{\partial M}{\partial x_1} - Q + N \theta^*\right) \beta^* dx_1 = 0.
\]

Recall that the trial functions for \(M\) assumed in each element satisfy only the condition (3.8b), i.e.,

\[
(\partial M/\partial x_1) - Q = 0.
\]

Hence, in the present formulation,

\[
Q = \frac{(m_2 - m_1)}{l} + \frac{\partial M_e}{\partial x_1} = \frac{(m_2 - m_1)}{l} + \left[\frac{\partial (M_{p1})}{\partial x_1} \sin \bar{\theta} + \frac{\partial (M_{p2})}{\partial x_1} \cos \bar{\theta} + \frac{\partial (M_{p3})}{\partial x_1}\right].
\]

(iii) **Joint-Equilibrium Equation.** The axial force \(N\) in each member is assumed to be along the axis of the rigidly rotated beam, i.e., along the line at angle \(\bar{\theta}\) to \(e_1\) axis. The joint equilibrium conditions at nodes 1 and 2 of the beam in question may be written as

\[
\{\sum (N \cos -Q \sin \bar{\theta}) + \bar{F}_1 = 0 \quad (x = 1, 2) \quad \text{along } e_1.
\]

\[
\{\sum (N \sin + Q \cos \bar{\theta}) + \bar{F}_2 = 0 \quad (x = 1, 2) \quad \text{along } e_2.
\]

\[
\{\sum \bar{M} = 0 \quad (x = 1, 2) \quad \text{about } e_3.
\]

where the summation extends over all elements meeting at each of the nodes. In the above, \(N, Q, M\) \((x = 1, 2)\) are, respectively, the internal axial force, transverse force, and bending moment at the ends of each member joined at the node; \(\bar{F}_1, \bar{F}_2\) are the externally prescribed forces along \(e_1\) and \(e_2\) directions and moment around \(e_3\) axis, respectively. In developing the individual element stiffness matrix, load vector, etc., the externally applied nodal loads \(\bar{F}_1\), etc. will henceforth be omitted. They will be treated as global nodal loads once the system stiffness and loads are assembled in the usual fashion. One may also consider the externally prescribed nodal forces to be non-conservative, i.e., \(\bar{F}_1\) and \(\bar{F}_2\) in (4.8) and (4.9) to depend on nodal rotations, such as \(\bar{F}_1 = \bar{F}_1 \cos \bar{\theta} - \bar{F}_2 \sin \bar{\theta}\), etc., where \(\bar{F}_1\) and \(\bar{F}_2\) are non-conservative nodal forces. The algebraic details of such cases are omitted here for simplicity.

Taking variations (or test functions) \(\delta u_{x_1}^* = u_{x_1}^*\) and \(\delta \theta^* = \theta^*\) at each node \((x = 1, 2)\), one may write the weak forms of (4.8)–(4.10) as
\[
\sum \left\{ 2N \cos \theta (v_1) - 1N \cos \theta (v_1) + 2N \sin \theta (v_2) - 1N \sin \theta (v_2) - 2Q \sin \theta (v_1) + 1Q \sin \theta (v_1) + 2Q \cos \theta (v_2) - 1Q \cos \theta (v_2) - 2M(\gamma + 1\beta^*) + 1M(\gamma + 1\beta^*) \right\} = 0 \quad (4.11)
\]

where $Q$, viz., the values of $Q$ at the two nodes, are given by (4.7). It is of interest to note that when the distributed loading (both conservative and non-conservative) is absent, one has

\[
^*Q = \frac{(m_2 - m_1)}{l} \quad (4.12)
\]

and

\[
M = \left( \frac{1 - x_1}{l} \right) m_1 + \frac{x_1}{l} m_2 , \quad 2M = m_2 , \quad 1M = m_1 . \quad (4.13)
\]

Thus, for this special case, through the use of Eqs. (2.15), (4.12), and (4.13), it can easily be shown that

\[
-2Q \sin \theta (v_1) + 1Q \sin \theta (v_1) + 2\cos \theta (v_2) - 1Q \cos \theta (v_2) - 2M(\gamma + 1\beta^*) + 1M(\gamma + 1\beta^*) = 0 . \quad (4.14)
\]

(iv) The combined weak form of compatibility and element as well as joint equilibrium. By combining (i) to (iii) above, one may write

\[
\sum_{\text{elem}} \left\{ vH - \left[ \frac{1}{\partial} \frac{\partial W}{\partial N} \right] v dx_1 - 2\theta* \mu_2 + 1\theta* \mu_1 - \frac{1}{\partial} \frac{\partial W}{\partial M} \mu dx_1 + \int_0^l N\theta* \beta^* dx_1 + (2N \cos \theta - 2Q \sin \theta) v_1 - (1N \cos \theta - 1Q \cos \theta) v_1 + (2N \sin \theta + 2Q \cos \theta) v_2 - (1N \sin \theta + 1Q \cos \theta) v_2 - 2M(\gamma + 1\beta^*) + 1M(\gamma + 1\beta^*) \right\} = 0 , \quad (4.15)
\]

where, in the general case of distributed (conservative as well as non-conservative) loading in the beam, $N$ is given by (3.12), $M$ by (3.13) and $Q$ by (4.7). In the special case when there is no distributed loading, Eqs. (4.12) and (4.13) hold and $N=n$. In this case, the use of (2.14), (3.12), (3.13) and (4.7) simplifies (4.15) to

\[
\sum_{\text{elem}} \left\{ vH - \left[ \frac{1}{\partial} \frac{\partial W}{\partial N} \right] v dx_1 - 2\theta* \mu_2 + 1\theta* \mu_1 - \frac{1}{\partial} \frac{\partial W}{\partial M} \mu dx_1 + \int_0^l n\theta* \beta^* dx_1 + n(\delta H) - m_2(2\beta^*) + m_1(1\beta^*) \right\} = 0 \quad (4.16)
\]

where $(\delta H)$ is the variation in ($H$).

Equation (4.15) will now form the basis of the present finite element development. It is worthwhile to comment on some aspects of the weak form (4.15). The trial functions in (4.15) are: $N$ as in (3.12); $M$ as in (3.13); $Q$ as in (4.7); the end values of $N$, $M$, and $Q$; the nodal displacements $^u_1$, $^u_2$ which determine $H$ as in (2.12); $\theta$ as in (2.13) and the non-rigid nodal rotations $\theta^*$. The trial function $\theta^*$ along the beam enters into only one term, i.e. the one involving the integral of $(N\theta^* \beta^*)$. The unknown parameters in the trial functions are $n, m_1, m_2, x_1, x_2$, and $\theta^*$. The test functions in (4.15) are: $v$, which is a constant, as in (3.18); $\mu$ as in (3.19); the variations in the rigid rotation of the beam as a whole, i.e. $\gamma$; the variations in the generalized nodal displacements, $^v_{v_1}$ ($x = 1, 2$) and $^v_{v_2}$ ($x = 1, 2$). The test function $\beta^*$ along the beam enters only in the integral of $(N\theta^* \beta^*)$. The unknown parameters in the test functions are $v, \mu_1, \mu_2, x_1, x_2$, and $\theta^*$. It can be seen that the only geometric quantity that must be assumed along the beam in (4.15) is $\theta^*$. Further, as discussed comprehensively in Karamanlidis and Atluri (1984), the term involving $N\theta^* \beta^*$ contributes the so-called "initial-stress" stiffness correction to the tangent stiffness matrix. As also discussed in detail in Karamanlidis and Atluri (1984); Atluri and Murakawa (1977), and Atluri (1980), neglecting the term $N\theta^* \beta^*$ in the formulation, while resulting in a slightly incorrect tangent stiffness matrix, is entirely consistent in the context of an iterative solution of a nonlinear problem, as in the present work. Henceforth, the term $(N\theta^* \beta^*)$ will be omitted.
Some details of the algebraic formulation of the stiffness matrix resulting from (4.15) are given in Appendix A. However, it should be noted that only two integrals over the length of the beam need to be evaluated in (4.15). When the material is linearly elastic, and the reference axis is at mid-thickness, one has

\[ \int_0^L \frac{\partial W_c}{\partial N} v \, dx_1 = \int_0^L \frac{N}{EA} v \, dx_1 = \int_0^L \frac{ny}{EA} \, dx_1 + \int_0^L \frac{N_{\tau}}{EA} v \, dx_1, \]

\[ \int_0^L \frac{\partial W_c}{\partial M} \mu \, dx_1 = \int_0^L \frac{M}{EI} \mu \, dx_1 = \int_0^L \frac{1}{EA} \left[ \left( 1 - \frac{x_1}{I} \right) m_1 + \left( \frac{x_1}{I} \right) m_2 \right] \left[ \left( 1 - \frac{x_1}{I} \right) \mu_1 + \left( \frac{x_1}{I} \right) \mu_2 \right] \, dx_1 + \int_0^L \frac{1}{EA} M_{\tau} \left[ \left( 1 - \frac{x_1}{I} \right) \mu_1 + \left( \frac{x_1}{I} \right) \mu_2 \right] \, dx_1. \]  

Thus, it is seen that all quadratic functionals (or bilinear forms) involving trial function parameters and test function parameters can be evaluated trivially in an explicit form. Even for a nonlinearly elastic material, i.e., when \( \partial W_c/\partial N \) and \( \partial W_c/\partial M \) are nonlinear in \( N \) or \( M \), the integrals in (4.17) and (4.18) may be evaluated explicitly without much difficulty. Thus, it is easy to see (also from Appendix A) that the tangent stiffness matrix based on (4.15) can be evaluated explicitly, i.e., (i) without recourse to any numerical quadrature in each element and (ii) with recourse to assuming any shape functions for element displacement fields, \( u_1, u_2, \) and \( \theta^* \), as in the conventional assumed displacement finite element approach.

5 Plasticity effects in the large deformation behavior of frames

When the material undergoes plasticity, the stress-to-strain transformation which was Hookean in (4.1) must be replaced by an elastic-plastic one, as derived from an appropirate flow theory of plasticity. Such an accounting of plasticity effects may be done in two ways: (i) a detailed procedure for the tracking of the plastic-zone development along the depth and length of the beam, (ii) a simplified procedure which accounts for the overall plasticity effects through the concepts of "plastic nodes" and "plastic hinges" (Hodge 1959; Ueda and Yao 1982). In the former case, assuming that an appropriate flow theory of plasticity is used, the incremental stress-strain relation may be written as

\[ \Delta \sigma_{11} (x_2) = \Delta \varepsilon_{11} (x_2) E_i (x_2) = (\Delta h + x_2 \Delta \kappa) E_i (x_2), \]  

(5.1)

where \( E_i (x_2) \) is the current tangent modulus in the uniaxial stress-strain curve, depending on the stress level as well as a loading/unloading criterion. From (5.1) it follows that

\[ \Delta N = D_{11} \Delta h + D_{12} \Delta \kappa, \quad \Delta M = D_{21} \Delta h + D_{22} \Delta \kappa, \]  

(5.2)

where

\[ D_{11} = \int E_i (x_2) \, dx_2, \quad D_{12} = \int E_i (x_2) x_2 \, dx_2, \quad D_{22} = \int E_i (x_2) x_2^2 \, dx_2. \]  

(5.3)

Thus \( D_{ij} (i, j = 1, 2) \) depend on the current load level and a loading/unloading criterion. The inverse of (5.2) may be written as

\[ \Delta h = C_{11} \Delta N + C_{12} \Delta M = \Delta h (N, M, \Delta N, \Delta M), \quad \Delta \kappa = C_{21} \Delta N + C_{22} \Delta M = \Delta \kappa (N, M, \Delta N, \Delta M). \]  

(5.4)

Thus, in the elastic-plastic case, the compatibility conditions (4.2) and (4.4) must be replaced by an incremental one, in general, in the form

\[ \int_0^L \Delta h \Delta v \, dx_1 = \int_0^L (\Delta u_{1,1} \Delta v) \, dx_1 = \Delta H \Delta v. \]  

(5.5)
\[ \int_0^l \Delta \kappa \Delta \mu \, dx_1 = - \int_0^l \Delta \theta_*^1 \Delta \mu \, dx_1 = - \Delta(\theta^*) \Delta \mu_2 + \Delta(\theta^*) \Delta \mu_1 . \]  

(5.6)

Since plastic flow will, in general, develop progressively from layer to layer along the depth at each section of the beam, and from section to section along the length of the beam, the integrals in (5.3), (5.5) and (5.6) will need to be evaluated through numerical quadrature. Thus, in the presence of plasticity, as may be seen from (4.15) as well as Appendix A, the "tangent-stiffness matrix" of the element can no longer be evaluated explicitly.

In this context, the "plastic-hinge" or "plastic-node" method has several advantages. Ueda, Matsuishi, Yamakawa and Akamatsu (1968), Ueda and Yao (1982) and Argyris, Boni, Hindenlang and Kleiver (1982) have earlier presented applications of the plastic-hinge concept in the context of the finite element method. However, in these studies, which are based on the assumed displacement method, the locations of the plastic hinges are restricted to be at the ends (nodes) of the element and the relations between the stress resultants and plastic deformations at the plastic hinge are replaced by those between the generalized nodal forces and the generalized nodal displacements.

In the present approach, the plastic hinge is assumed to be located at an arbitrary point along the length of the beam. We first assume that the yield condition of the elastic-perfect-plastic type is given by

\[ f(N, M) = 0 \quad \text{at} \quad x_1 = l_p , \]  

(5.7)

the specific forms of which, for various cross sections, are given in detail by Hodge (1959). In the above, \( x_1 = l_p \) is the location of the plastic hinge. The incremental plastic flow condition at the plastic hinge may be written as

\[ \frac{\partial f}{\partial N} \Delta N + \frac{\partial f}{\partial M} \Delta M = 0 \quad \text{at} \quad (x_1 = l_p) . \]  

(5.8)

The incremental "plastic deformations" at the hinge are assumed to be given by

\[ (\Delta H_p)_{x_1 = l_p} = (\Delta \lambda) \left. \frac{\partial f}{\partial N} \right|_{x_1 = l_p} , \]  

(5.9)

\[ (\Delta \theta_p^*)_{x_1 = l_p} = (\Delta \lambda) \left. \frac{\partial f}{\partial M} \right|_{x_1 = l_p} , \]  

(5.10)

where \((\Delta H_p)\) is the increment of "plastic" elongation \(H_p\) and \((\Delta \theta_p^*)\) is the increment of plastic rotation \(\theta_p^*\).

Thus, in the present plastic hinge method, the incremental counterparts of the compatibility conditions (4.2) and (4.4) are replaced by

\[ \int_0^l \frac{\Delta N}{EA} \Delta v \, dx_1 + (\Delta \lambda) \left( \frac{\partial f}{\partial N} \Delta v \right)_{x_1 = l_p} = (\Delta H) \Delta v , \]  

(5.11)

\[ \int_0^l \frac{\Delta M}{EI} \Delta \mu \, dx_1 + (\Delta \lambda) \left( \frac{\partial f}{\partial M} \Delta \mu \right)_{x_1 = l_p} = - \Delta(\theta^*) (\Delta \mu_2) + \Delta(\theta^*) (\Delta \mu_1) . \]  

(5.12)

From the above discussion, as well as Appendix A, it may be seen that the combined weak form of the compatibility and equilibrium (local and joint) conditions of the elastic-plastic frame, using the present "hinge" method, may be stated through "adding" the following terms

\[ - \left( (\Delta \lambda) \left[ \frac{\partial f}{\partial N} \Delta v + \frac{\partial f}{\partial M} \Delta \mu \right] \right)_{x_1 = l_p} , \]  

(5.13)

corresponding to each member to the incremental counterpart of (4.15). Once again, since no integrations are involved in evaluating the terms in (5.13), the tangent-stiffness, as effected by plasticity, is still evaluated explicitly.
As discussed in Atluri and Murakawa (1977) and Atluri (1980), in an approach of the present type, iterations are performed in each increment of a nonlinear problem to "correct" the "compatibility" of total strains produced by the trial solutions for stresses. The weak forms of these "initial" compatibility conditions read as

\[ \int_{E_A} \frac{N}{E_A} \Delta v \, dx_1 + (H_p) \Delta v \bigg|_{x_1=l_p} = (H) \Delta v , \quad (5.14) \]

and

\[ \int_{E_I} M \Delta \mu \, dx_1 + (\theta^* \Delta \mu) \bigg|_{x_1=l_p} = -2 \theta^* \Delta \mu_2 + 1 \theta^* \Delta \mu_1 . \quad (5.15) \]

The above can be checked in an iterative scheme by retaining the "order one" terms as well as first-order terms in the increments of the parameters in the trial functions and the first-order terms in the increments of parameters in the test functions, as shown in Appendix A.

6 Strategy for solution of tangent stiffness equations

A large number of solution procedures is available for nonlinear structural analyses. Some of these may be characterized as the standard load-control method, the displacement-control method, the perturbation method, the method based on the current stiffness parameter and the so-called "arc-length" method. These have been summarized by Gallagher (1983). In the present paper, either the load-control method or the arc-length method of Ramm (1980) and Crisfield (1981), with modifications as described in Kondoh and Atluri (1984a, b), are employed. Further details are omitted here.

7 Numerical examples

Example 1. The problem is that of a cantilever bar which, in its undeformed state, is inclined at an angle \( \phi \) to the vertical line and is subjected to a vertical concentrated load at the tip (Fig. 2a). The bar is of a rectangular cross section and of an elastic-perfectly-plastic material. The structure is modeled by a single finite element, and five different cases of \( \phi = 0^\circ, 2.5^\circ, 10^\circ, 30^\circ, 60^\circ \) are analysed using the standard load control method.

While the theoretical development presented earlier and in Appendix A is valid for finite deformations, this incremental formulation is employed to solve the present example wherein deformations are small. A "detailed" elastic-plastic analysis, based on a flow theory, is employed and Eqs. (5.4)–(5.6) are used as the incremental compatibility conditions. For evaluating the integrals involved Numerically, the depth of the beam is divided into 20 layers and the length into 20 strips, and over each subdomain the stress and strain are assumed to be constant. Assuming that \( \Delta \sigma \) in the finite element (in this case, the entire beam), as defined in (A.5), is determined from (A.23); the plasticity algorithm proceeds as follows.

(i) From \( \Delta \sigma (\Delta \nu; \Delta m_1; \Delta m_2) \) and the known \( \Delta N_p \) and \( \Delta M_p \), determine \( \Delta N \) and \( \Delta M \).

(ii) Using the value of \( (C) \), as defined in (5.4), at the beginning of the increment and compute the strains from

\[ \{ \Delta h \} = \beta (C) \{ \Delta N \} \{ \Delta M \} \]

where \( \beta \) is a scaling parameter, \( 0 \leq \beta \leq 1 \).

(iii) Calculate \( \Delta \sigma_{11} = E_t (\Delta h + x_2 \Delta \kappa) \), where \( x_2 \) is the coordinate of the center of each layer in the depth, and \( E_t \) is the tangent modulus of the stress-strain curve at the current stage. Determine \( \beta \) in (ii) above such that it is the minimum value required to produce a new "plastic" layer at the section along length in question.

(iv) Let \( \Delta N^* = \Delta N - \int \Delta \sigma_{11} \, dx_2 \) and \( \Delta M^* = \Delta M - \int \Delta \sigma_{11} x_2 \, dx_2 \).

(v) If \( \Delta N^* \) and \( \Delta M^* \) are not zero, use them as the additional values in the iterative process, beginning with step (i). Also, at the beginning of iteration, the value of \( (C) \) is updated to correspond to the total stress at the beginning of iteration.
Let the yield stress of the present elastic-perfect-plastic material be $\sigma_y$. The axial force and moment at the root of the beam are

$$N = -P \cos \phi, \quad M = P \sin \phi L.$$  \hspace{1cm} (7.1)

Let the stress field in the beam be tensile in the region $\xi \leq x_2 \leq D/2$ ($D$ being the depth of the beam) and compressive in the region $-H/2 \leq x_2 \leq \xi$. If the width of the beam is $B$, it is then seen that

$$N = \frac{2\xi}{H} N_0, \quad M = \left[1 - \left(\frac{2\xi}{H}\right)^2\right] M_0,$$  \hspace{1cm} (7.2)

where $N_0 = BH \sigma_y$ and $M_0 = \frac{1}{2} BH^2 \sigma_y$. From (7.1) and (7.2), it follows that

$$\left(\frac{\cos \phi}{N_0}\right)^2 P^2 + \left(\frac{\sin \phi}{N_0}\eta\right) P - 1 = 0,$$  \hspace{1cm} (7.3)

where

$$\eta = \frac{N_0 L}{M_0}.$$  

Thus, the limit load on the beam is obtained as

$$P = \frac{N_0}{\cos \phi} \left[\left\{1 + \frac{1}{2} \eta^2 \left(\tan \phi\right)^2\right\}^{1/2} - \frac{1}{2} \eta \tan \phi - \eta\right].$$  \hspace{1cm} (7.4)

Equation (7.4) is shown plotted in Fig. 2a, along with the presently obtained numerical results for $P$, using a single element. The plastic zones at near the collapse load for each value of $\phi$ are shown in Fig. 2b.

**Example 2.** The second example is that of a uniformly loaded beam, of rectangular cross section and of elastic-perfectly-plastic material, subject to boundary conditions as shown in the inset of Fig. 3a. The variation of the transverse displacement at the center of the beam with applied load is shown in Fig. 3a, from which it can be seen that the present method predicts well the collapse load of $(PL/M_0) = 11.657$, where $M_0$ is the fully plastic bending moment, as determined analytically from classical limit analysis. Figure 3b shows the development of plastic zones for various values of $(PL/M_0)$.  

**Fig. 2a and b.** a Collapse of an end-loaded inclined beam. b Plastic zones, at near collapse load, in an end-loaded inclined beam.
Example 3. In this example, concerning a gable frame as shown in Fig. 4a, the plastic-hinge method is employed with the yield condition: 
\[(M/M_0)^2 - 1 = 0\]
where \(M_0\) is the fully plastic bending moment. This yield condition is continuously checked, in the present incremental analysis, at each of 20 sections along the length, including the end points of each beam. The loading consists of both a conservative-type distributed loading as well as concentrated nodal loading, as shown in Fig. 4a. Each of the four members in the frame is modeled by a single element. The variations of the displacements \(\delta_1, \delta_2, \delta_3\) (as defined through the insert in Fig. 4b) with applied load \((PL/M_0)\) are shown in Fig. 4b, which also shows that the presently determined collapse load agrees with that determined from a classical limit analysis of \((PL/M_0) = 1.9153\) (Hodge 1959). The progressive formation of plastic hinges at various locations, as the load increases, is shown in Fig. 4c.

Example 4. The fourth example is that of a two-bay, three-story frame, with geometrical details being given in Fig. 5a. The material is elastic-perfectly-plastic and a plastic-hinge method is employed with the yield condition \((M/M_0)^2 = 1\). The structure is modeled by 15 elements. The results obtained are summarized in Figs. 5b and 5c. Figure 5b shows that the present method predicts the collapse loads well, as compared to the results of classical limit analysis (Hodge 1959) which are: 
\[P = 2.667 (M_0^*/L)\] (upper bound), where \(M_0\) is the fully plastic bending moment of member 1 and \(P = 2.500 M^*/L\) (lower bound). The presently computed value is \(P = 2.6186 M_0^*/L\).

It should be noted that in Examples 3 and 4 only the effect of the bending moments is considered at the plastic hinge. If the effects of stress resultants were also considered, the collapse loads may have been found to be slightly lower.

The Examples 1 to 4 above may be considered to fall into the category of small-deformation elasto-plastic analysis. Now, three additional numerical examples falling into the category of finite deformations are presented, the solutions which are obtained through the arc-length method.
Example 5. This concerns a cantilever beam of uniform cross section, subjected to different types of loading as shown in Fig. 6a. The beams are assumed to remain elastic. The non-conservative loads are assumed to remain normal to the deformed axis. Large deformations of cantilevers under conservative loading were studied, for instance, by Barten (1945), Bishop and Drucker (1945), Rohde (1953), Wang, Lee and Zienkiewicz (1961) and Schmidt and Dadepo (1970); while those under non-conservative loadings were studied by Bathe, Ramm and Wilson (1975) and Argyris and Symeonidis (1981). To test the convergence of the present scheme, the beam was modeled successively by 1, 2, 4, and 8 elements. The present results for the four cases are summarized in Figs. 6b, c, d, e along with appropriate comparison results. In Fig. 6b the theoretical result for a cantilever subject to a dead-type tip load is due to Bishop and Drucker (1945) and the analytical solution for dead-type distributed loading is due to Schmidt and Dadepo (1970). Figures 6d and 6e show, respectively, the differences in the present results (obtained using eight elements) for the cases of conservative and non-conservative concentrated and distributed loadings, respectively. The deformed shapes of the beam at various load levels, for each of the four cases of loading, are shown in Fig. 6f.

From Figs. 6b and 6c, it can be seen that an idealization by two elements alone gives results of acceptable accuracy. While this example concerns a single beam, when the beam is considered to be a member of a frame structure, the present results using a single element in Figs. 6b and 6c, demonstrate that a single-element representation of each beam in a structure may be adequate to study the nonlinear behavior of the structure as a whole.
Example 6. This concerns a tip-loaded (dead-load) cantilever beam of uniform cross section and of a bi-linear stress-strain law, as shown in Fig. 7a. This problem, with identical material properties, has also been studied by Tang, Yeung, and Chon (1980) and by Yang and Saigal (1984). A linear kinematic hardening rule is employed, in accounting for reverse plasticity. To study the convergence, the beam was modeled successively by 1, 2, 4, and 8 elements. A detailed elastic-plastic analysis, based on an algorithm as described under Example 1, has been used.

Figures 7b and c show the relations between the applied load and displacements at the tip in the "small" deformation range and "large" deformation range, respectively. Also shown in Figs. 7b and c are comparison results based on a geometrically linear elastic-plastic analysis. The shapes of the deformed beam and the development of plastic zones, in tension as well as in compression, are shown in Figs. 7d and e, respectively.

The beam begins to yield at the extreme fibers along the depth, at the fixed end, at approximately \( P = 25 \text{ Lb} \). The results of the present large-deformation, kinematic-hardening analysis agree with those from a small-deformation, elastic-plastic analysis until a vertical displacement value at the tip
of $\delta_y = 1$ in Fig. 7b. At this point, the stiffening effects of axial tension, in the present (consistent) large-deformation analysis, begin to manifest themselves as do the effects of strain-hardening of the material in the plastic zone. Figures 7b and c show the influences of these hardening effects. Figures 7b and c also indicate that convergence may have been achieved with a four-element discretization, as in the elastic large-deformation case of Example 5. The deformed shapes of the beam at various load levels are shown in Fig. 7d and the development of plastic zones in tension and in compression are shown in Fig. 7e. The present results also indicate that when the beam in question is a member of a frame, a single element representation of each member may be adequate to study the large deformations of practical interest to the structure as a whole.

**Example 7.** This final example concerns a column, fixed at the base and subjected to a concentrated (dead-type) vertical load at the other end, as shown in Fig. 8a. Here, the material property is assumed to be specified directly as a relation between moment and curvature, as

$$M = (D_0 - D_1) \kappa_0 \tan (\kappa/\kappa_0) + D_1 \kappa,$$

where $D_0$ and $D_1$ are characteristic flexural rigidities as shown in Fig. 8a and $\kappa_0$ is a material constant. This relation, in which no unloading is considered, is analogous to that also employed by Oden and
Childs (1970) and by Yang and Wagner (1973) in their numerical analyses of the post-buckling of a cantilever bar. In this problem, the axial rigidity is assumed to remain constant.

In (7.5) the parameter \( (K_0L) \) may be regarded as a measure of the relative importance of linear elastic behavior and \( (D_1/D_0) \) indicates the relative stiffness of the material after it yields. Thus, \( (D_1/D_2) = 1 \) indicates linear elastic material behavior. The analyses were carried out for \( K_0L = 0.05 \) and 0.4 and \( (D_1/D_0) = 0.25, 0.5, 0.75, \) and 1, respectively. Also, an imperfection was introduced into the problem through the application of a small (dead-type) horizontal load of magnitude \( (P/10^4) \) (where \( P \) is the vertical dead load) at the tip. To study convergence, analyses were performed with 2, 4, and 8 elements over the length of the column, in the two cases of \( (D_1/D_0) = 1 \) and 0.25 and \( K_0L = 0.05 \), while all other problems were solved using four elements. A detailed elastic-plastic analysis, based on an algorithm similar to that in Example 1, was employed, except for the difference that no integration through the depth of the beam is needed in the present problem as compared to Example 1 since the material constitutive law is prescribed directly in terms of \( M \) and \( \kappa \) as in (7.5).

Figure 8b shows the results for large deformation, linear elastic material behavior \( [(D_1/D_0) = 1] \), along with the classical elastica solution (Timoshenko and Gere 1961). Converged results appear to have been obtained with four elements. Figure 8c shows the results for the large-deformation, post-
Fig. 7a–e. a A tip-loaded cantilever of a bilinear elastic-plastic material. b, c Load-deflection diagram for the tip-loaded (dead-load) cantilever of Fig. 7a, c a continuation of results in Fig. 7b. d Deformed profiles of the cantilever of Fig. 7a. e Plastic regions, at various load levels, in the cantilever of Fig. 7a
buckling behavior, with $\kappa_b L = 0.05$ and $(D_i/D_o) = 0.25$, along with comparison results of Oden and Childs (1970), who directly solve the governing differential equations using the Newton-Kantorovich method. Figure 8d shows results for the cases $\kappa_b L = 0.05$ and $(D_i/D_o) = 0.25, 0.50, 0.75$ and 1.00, while those in Fig. 8e correspond to $\kappa_b L = 0.4$ and the same four values of $(D_i/D_o)$. It should be remarked that the present results in Figs. 8b, c were based on using four elements (with the results based on using only two elements being also of acceptable accuracy).

The deformed shapes of the bar for the two cases of $(D_i/D_o) = 1$ and of $(D_i/D_o) = 0.25$ and $\kappa_b L = 0.05$ are shown in Fig. 8f.

The above examples effectively serve to illustrate the accuracy, versatility and efficiency of the presently developed simplified procedures for finite element analysis of large-deformation, inelastic analysis of structures consisting of beam-type members.

8 Closure

The salient features of the developments in the present study may be summarized as follows:

(1) Arbitrarily large rigid rotations are accounted for while the non-rigid relative rotation and axial stretch of the beam are assumed to be small. The axial stretch, however, is a highly nonlinear function of the nodal displacements of the beam.

(2) As long as the material remains elastic, or when a plastic-hinge method is used to account for plasticity, the tangent stiffness matrix of the beam element can be developed explicitly for all deformations of the type described under (1).

(3) The assumed-stress approach as in the present problem makes it very simple and convenient to treat non-conservative loading, either distributed along the length of the beam or concentrated at the joints, even when the beam undergoes arbitrarily large deformations of the type described in (1).

(4) Available evidence, based on several numerical examples, suggests that the present development is a very economical and accurate procedure for analysing large-deformation, elasto-plastic behaviour of frames, as compared to the standard finite element displacement method based on assumed displacements, over each beam, of a simple cubic type.

The extensions of the present approach to treat elasto-plastic, large-deformation dynamic response of space frames is underway and will be reported on soon. The method has also a potential application in treating arbitrary shell structures as a network of space beams.

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References


Fig. 8a–c. a A tip-loaded column (dead-load), with a nonlinear moment-curvature relation. b Convergence of results for $P$ vs. $\theta$ for the elastica. c Convergence of results for $P$ vs. $\theta$ for the case of nonlinear moment-curvature relation [$K_0L=0.05; D_1/D_2=0.25$].

Atluri, S.N. (1983): Alternate stress and conjugate strain measures and mixed variational formulations involving rigid rotations, for computational analyses of finitely deformed solids, with application to plates and shells. Theory, Comp. & Struct. 18/1, 93–106


Fig. 8d-f. d Results for $P$ vs. $\theta$ for various values of $(D_i/D_0)$ and $K_{OL} = 0.05$. e Results for $P$ vs. $\theta$ for various values of $(D_i/D_0)$ and $K_{OL} = 0.4$. f Deformed shapes of the cantilever of Fig. 8a at various load levels.
Appendix A

Explicit expressions for tangent stiffness matrix and load, and compatibility correction vectors, for a beam undergoing large deformations and plastic-hinge formations are given here.

First, we treat the case when the frame remains elastic. The combined weak form of the incremental compatibility conditions, joint equilibrium conditions, as well as "checks" on these conditions at the beginning of an increment/iteration, may be derived from Eq. (4.15) by retaining only terms of the first order in the increments of the parameters in the test functions and terms of order one, as well as of first order in the increments, of the parameters in the trial functions. While this process involves rather straightforward algebra, it is of interest to note here some special features of considering appropriate increments in a term of the type, for instance,

\[(2N \cos \bar{\vartheta} - 2 \bar{Q} \sin \bar{\vartheta}) \Delta (v_1)\]  

(A.1)

in Eq. (4.15). Using relations (2.14), (2.15), (3.12), (3.13) and (4.7) in (4.15), it can be seen that the "incremental" form of (A.1) may be written as

\[\left[(2N) \frac{\partial^2 H}{\partial (u_1)^2} \Delta u_1 + (2N) \frac{\partial^2 H}{\partial u_1 \partial u_2} \Delta u_2 - (2Q) \frac{2 \Delta u_2}{\partial u_2} \frac{\partial^2 H}{\partial (u_1)^2} \Delta u_1\right] \Delta (v_1),\]  

(A.2)

where

\[\Delta Q = \frac{\Delta (m_2 - m_1)}{l} + \frac{\partial (\Delta M_p)}{\partial x_1} = \frac{\Delta (m_2 - m_1)}{l} + \Delta Q_p,\]  

(A.3)

\[\Delta N = \Delta n + \Delta N_p \quad \text{etc.} \]  

(A.4)

Now, we introduce the notations for the following vectors in each element

\[\Delta \sigma^T = [\Delta n; \Delta m_1; \Delta m_2],\]  

(A.5)

\[\delta \Delta \sigma^T = [\Delta v; \Delta \mu_1; \Delta \mu_2],\]  

(A.6)

\[d = [v_1; \mu_1; v_2; \mu_2; \theta; \bar{\vartheta}]\]  

(A.7)

with \(\Delta d\) being the increment of the vector in (A.7):

\[R_{dc} = \left[ - N_{p1} : N_{p1} : - \frac{d}{dx_1} (M_{p2}) \right]_{x_1 = 0} ; \left[ \frac{d}{dx_1} (M_{p2}) \right]_{x_1 = l} ; 0 ; 0 \]  

(A.8)

with \(\Delta R_{dc}\) being the increment of the vector in (A.8):

\[R_{dn} = \left[ \left\{ - N_{m1} \frac{\partial H}{\partial u_1} \right\} - \left( \frac{d M_{pm}}{dx_1} \right) \frac{\partial \bar{\vartheta}}{\partial u_1} \right] ; \left\{ 2 N_{m1} \frac{\partial H}{\partial u_1} + \left( \frac{d M_{pm}}{dx_1} \right) \frac{\partial \bar{\vartheta}}{\partial u_1} \right\} ; \left\{ - N_{m2} \frac{\partial H}{\partial u_2} \right\} - \left( \frac{d M_{pm}}{dx_1} \right) \frac{\partial \bar{\vartheta}}{\partial u_2} \right] ; \left\{ 2 N_{m2} \frac{\partial H}{\partial u_2} + \left( \frac{d M_{pm}}{dx_1} \right) \frac{\partial \bar{\vartheta}}{\partial u_2} \right\} ; 0 ; 0 \]  

(A.9)
with \( \Delta R^T \) being defined from (A.9) by replacing
\[
\begin{bmatrix}
\frac{1}{2} N_{pn} ; \frac{1}{2} \left( \frac{dM_{pn}}{dx_1} \right) ; \frac{1}{2} \left( \frac{dM_{pn}}{dx_1} \right)
\end{bmatrix}
\text{by}
\begin{bmatrix}
\Delta^1 N_{pn} ; \Delta^2 N_{pn} ; \frac{1}{2} \left( \frac{dM_{pn}}{dx_1} \right) ; \frac{1}{2} \left( \frac{dM_{pn}}{dx_1} \right)
\end{bmatrix},
\]
respectively;
\[
\begin{align*}
R^T_a &= \left\{ \begin{bmatrix}
\frac{\partial W_c}{\partial N} d x_1 - H \\
\frac{\partial W_c}{\partial M} \left( 1 - \frac{x_1}{l} \right) d x_1 - \theta_1^* \\
\frac{\partial W_c}{\partial M} x_1 d x_1 + \theta_2^*
\end{bmatrix} \right\}, \\
R^T_{\alpha_0} &= \left\{ \begin{bmatrix}
\partial H + \left( m_2 - m_1 \right) \frac{\partial \theta}{\partial u_1} \\
\partial H + \left( m_2 - m_1 \right) \frac{\partial \theta}{\partial u_2}
\end{bmatrix} \right\}
\end{align*}
\tag{A.10}
\]

With these notations, the incremental form of (4.15) may be written as
\[
0 = \sum_{\text{elem}} \left\{ \delta \Delta \sigma^T \left\langle - (A)_{\alpha 0} \Delta \sigma - (\Delta R_{\alpha c} + \Delta R_{\alpha \theta}) - R_a + [(A)_{\alpha 0} - (A)_{\alpha \theta}] \Delta d \right\>ight.
+ \delta (\Delta d) \langle (A)_{\alpha 0} \Delta \sigma + [(A)_{\alpha 0} + (A)_{\alpha \theta}] \Delta d + (\Delta R_{\alpha c} + \Delta R_{\alpha \theta}) + (R_{\alpha 0} + R_{\alpha \theta} + R_{\alpha d}) \rangle \\
= \sum_{\text{elem}} \left\{ \delta \Delta \sigma^T \left\langle - (A)_{\alpha 0} \Delta \sigma - \Delta R_a - (A)_{\alpha \theta} \Delta d \right\>ight.
+ \delta (\Delta d) \langle (A)_{\alpha 0} \Delta \sigma + (A)_{\alpha \theta} \Delta d + \Delta R_d + R_d \rangle \right\}, \tag{A.12a, b}
\]

where
\[
(A)_{\alpha 0} = \int_0^1 (H)^T (C) (H) d x_1 , \quad (H)_\sigma = \begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{1-x_1}{l} & \frac{x_1}{l}
\end{bmatrix}, \tag{A.13}
\]
\[
(C) = \begin{bmatrix}
\frac{\partial W_c}{\partial N} & \frac{\partial^2 W_c}{\partial N \partial M} \\
\frac{\partial^2 W_c}{\partial N \partial M} & \frac{\partial^2 W_c}{\partial M^2}
\end{bmatrix}, \tag{A.14a}
\]

which, for a linearly elastic material, and when the reference axis of the beam is at mid-thickness, may be written as
\[
C_{11} = 1/EA , \quad C_{12} = 0 , \quad C_{22} = 1/EI , \tag{A.14b}
\]
\[
\Delta R_{\alpha c} = \int_0^1 (H)^T (C) \begin{bmatrix}
\Delta N_{pc1} \cos \theta + \Delta N_{pc2} \sin \theta \\
- \Delta M_{pc1} \sin \theta + \Delta M_{pc2} \cos \theta
\end{bmatrix} d x_1 , \tag{A.15}
\]
\[
\Delta R_{\alpha \theta} = \int_0^1 (H)^T (C) \begin{bmatrix}
\Delta N_{pn} \\
\Delta M_{pn}
\end{bmatrix} d x_1 , \tag{A.16}
\]
\[
(A)_{\alpha 0} = \begin{bmatrix}
\frac{\partial H}{\partial u_1} I^T & \frac{\partial H}{\partial u_2} I^T & 0 & 0 \\
\frac{\partial \theta}{\partial u_1} I^T & - \frac{\partial \theta}{\partial u_2} I^T & 1 & 0 \\
\frac{\partial \theta}{\partial u_1} I^T & \frac{\partial \theta}{\partial u_2} I^T & 0 & -1
\end{bmatrix}, \tag{A.17}
\]
\[
I^T = [-1, 1] , \tag{A.18}
\]
\( (A)_{\sigma d} = \left( \int_\delta (H)^T_s(C) \left\{ \begin{array}{l} -N_{p1} \sin \theta + N_{p2} \cos \theta \\ -M_{p1} \cos \theta - M_{p2} \sin \theta \end{array} \right\} \, dx_1 \right) \left[ \begin{array}{c} \frac{\partial \bar{\theta}}{\partial \bar{u}_1} \\ \frac{\partial \bar{\theta}}{\partial \bar{u}_2} \end{array} \right] I^T; \left[ \begin{array}{c} \frac{\partial \bar{\theta}}{\partial \bar{u}_1} \\ \frac{\partial \bar{\theta}}{\partial \bar{u}_2} \end{array} \right] I^T; 0^T \right], \quad \text{(A.19)}

where \( \theta \) is a null vector with two components.

\[
(A)_{\sigma 0} = \begin{bmatrix}
  \left(n \frac{\partial^2 H}{\partial (u_1)^2} + (m_2 - m_1) \frac{\partial^2 \bar{\theta}}{\partial (u_1)^2} \right) & \left(n \frac{\partial^2 H}{\partial u_1 \partial u_2} + (m_2 - m_1) \frac{\partial^2 \bar{\theta}}{\partial u_1 \partial u_2} \right) \\
  \left(n \frac{\partial^2 H}{\partial (u_1)^2} + (m_2 - m_1) \frac{\partial^2 \bar{\theta}}{\partial (u_2)^2} \right) & \left(n \frac{\partial^2 H}{\partial u_1 \partial u_2} + (m_2 - m_1) \frac{\partial^2 \bar{\theta}}{\partial u_1 \partial u_2} \right)
\end{bmatrix} (E) \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix}, \quad \text{(A.20)}
\]


where

\[
(E) = \begin{bmatrix}
  1 & -1 \\
  -1 & 1
\end{bmatrix}, \quad \text{(A.21)}
\]

Since, in (A.12b), the parameters \( (\delta \Delta \sigma)^T \) are independent and arbitrary in each element, it follows

\[
\Delta \sigma = (A)^{-1}_{\sigma 0} [(A)_{\sigma d} \Delta d - R_\sigma - \Delta R_\sigma]. \quad \text{(A.23)}
\]

Upon substituting (A.23) in (A.26), one obtains

\[
\sum_{\text{elem}} \delta(\Delta d) [(K)^T \Delta d - \Delta Q + R],
\]

where

\[
(K)^T = \text{element tangent stiffness} = (A)^T_{\sigma 0} (A)^{-1}_{\sigma d} A_{\sigma d} + A_{\sigma d}, \quad \text{(A.24)}
\]

\[
\Delta Q = (A)^T_{\sigma 0} (A)^{-1}_{\sigma d} \Delta R_\sigma - \Delta R_d, \quad \text{(A.25)}
\]

\[
R = -(A)^T_{\sigma 0} (A)^{-1}_{\sigma d} R_\sigma + R_d, \quad \text{(A.26)}
\]

It is clear that, in the presence of non-conservative distributed loading, the stiffness matrix \( (K)^T \) in (A.24) is unsymmetric.

One can easily extend the development of the element-load vector as in (A.25) and (A.26) to the case wherein the externally specified nodal loads in the frame are also non-conservative, viz., those that depend on the nodal rotation. Such details are omitted here for the sake of brevity.
We now turn to the case of development of plastic hinges as discussed earlier in Sect. 5.

In the elastic-plastic case, as explained in Sect. 5, the incremental compatibility conditions as well as the “check” on initial compatibility are appropriately altered by augmenting the incremental total “weak” form in (A.12) by terms

\[
\sum_{\text{elem}} \left\{ \Delta \lambda \left( \frac{\partial f}{\partial \lambda} \Delta \lambda + \frac{\partial f}{\partial \mu} \Delta \mu \right) + H_p \Delta \lambda + \theta_p \Delta \mu \right\} = 0 \quad \text{for all } \lambda \in \mathcal{I}_p
\]

A modification to (A.12b) may be best accommodated by writing

\[
0 = \sum_{\text{elem}} \left\{ \delta \Delta \sigma^T \left\{ -\left( \dot{A}_{\sigma \sigma} \right) \Delta \sigma - \Delta \dot{R}_\sigma - \dot{\dot{R}}_\sigma + \left( \dot{A}_{\sigma d} \right) \Delta d \right\} + \delta \Delta d \left( \dot{A}_{\sigma d0} \right) \Delta \sigma + \left( \dot{A}_{d d} \right) \Delta d + \Delta \dot{R}_d + R_d \right\}
\]

and, by redefining appropriate matrices as follows:

\[
\Delta \sigma = \begin{bmatrix} \Delta \sigma \end{bmatrix} + \dot{A}_{\sigma \sigma} \Delta \sigma
\]

\[
A_{\sigma \sigma} = \begin{bmatrix} A_{\sigma \sigma} & A_{\sigma d} \\ A_{\sigma d} & 0 \end{bmatrix}
\]

\[
A_{\sigma d} = \begin{bmatrix} \frac{\partial f}{\partial \lambda} \delta \lambda \\ \frac{\partial f}{\partial \mu} \delta \mu \end{bmatrix}_{\lambda p} \begin{bmatrix} \delta \lambda \Delta \lambda \delta \mu \Delta \mu \end{bmatrix}_{\lambda p}
\]

\[
\Delta R = \begin{bmatrix} \Delta R \\ \frac{\partial f}{\partial \lambda} \Delta N \delta \lambda + \frac{\partial f}{\partial \mu} \Delta M \delta \mu \end{bmatrix}
\]

\[
\dot{R}_\sigma = \begin{bmatrix} R_\sigma + \theta_\sigma \begin{bmatrix} \delta \lambda \Delta \lambda \delta \mu \Delta \mu \end{bmatrix}_{\lambda p} \\ \theta_\mu \begin{bmatrix} \delta \lambda \Delta \lambda \delta \mu \Delta \mu \end{bmatrix}_{\lambda p} \end{bmatrix}
\]

\[
\dot{A}_{\sigma d} = \begin{bmatrix} \dot{A}_{\sigma d} \\ 0 \end{bmatrix}, \quad \dot{A}_{\sigma d0} = \begin{bmatrix} \dot{A}_{\sigma d0} \\ 0 \end{bmatrix}
\]

From (A.28), since \( \Delta \sigma \) are arbitrary for each element, \( \Delta \sigma \) is eliminated at the element level and expressed in terms of \( \Delta d \). As a result, we obtain an expression for the stiffness matrix \( \hat{K} \) as

\[
(\hat{K}) = (K) - (A_{\sigma d0})^T (A_{\sigma d})^{-1} A_{\sigma d} C^T (A_{\sigma d})
\]

where \( (K) \) is given in (A.24), and

\[
C^T = \begin{bmatrix} A_{12}^T (A_{\sigma d})^{-1} A_{12} \end{bmatrix}^{-1} A_{12}^T (A_{\sigma d})^{-1}
\]

\[
\hat{R} = (A_{\sigma d0})^T (A_{\sigma d})^{-1} (\dot{R}_\sigma - A_{\sigma d} C^T \dot{R}_\sigma) + R_d
\]

\[
\hat{Q} = (A_{\sigma d0})^T (A_{\sigma d})^{-1} [\Delta \dot{R}_\sigma - A_{\sigma d} C^T \Delta \dot{R}_\sigma] - \Delta \dot{R}_d
\]

Once again, the tangent stiffness matrix is explicitly derived and may be unsymmetric in general.

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Mechanical sublayer model for elastic-plastic analyses

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Abstract. Strain-hardening behavior for plane stress problems is modeled by a panel with \( n \) layers, the first \((n-1)\) layers are elastic-perfectly-plastic under Mises-Hencky condition, each with different yield stress, and the \( n \)-th layer is elastic. Equivalent incremental stress-strain relations for the panel can be obtained. The resulting uniaxial stress-strain curve contains \( n \) segments. Those segments in the plastic range are not straight lines.

1 Introduction

An important ingredient in computational solid mechanics is the choice of constitutive relations. Such relations, on the one hand, should represent the stress-strain relations realistically and, on the other hand, should be easy to implement in computational algorithms. The present paper is concerning the modeling of time-independent elastic-plastic behavior with kinematic hardening. For a simple loading of uniaxial specimen the so called mechanical subelement model or overlay model (Besseling 1953; Leech et al. 1968; Zienkiewicz et al. 1973) consists in replacing the actual stress-strain curve by one with piecewise linear segments with the successive slopes equal to \( E_1, E_2, \ldots, E_n \). The sectional area is modeled by a parallel arrangement of \( n \) bar elements, \((n-1)\) of which are elastic-perfectly-plastic while the \( n \)-th one is elastic. The ratio of the area of individual element \( A_i \) to the total area \( A \) is given by

\[
\frac{A_i}{A} = \frac{(E_i - E_{i+1})}{E_1} - \sum_{k=1}^{i-1} \left( \frac{A_k}{A} \right).
\]  

If the transition points are at strain values \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \), the yield stress for the \( i \)-th element is \( \varepsilon_i E_1 \). The resulting model will account for the kinematic hardening behavior and the Bauschinger effect.

In constructing a subelement model for isotropic materials under general 3-D loading condition (Hunsaker et al. 1976; Yamada 1980), it was found that the relative distributions of the individual elements are not the same as those determined under the uniaxial loading condition, and a modification factor which depends on the Poisson's ratio of the material should be added to Eq. (1). A model so constructed will again yield uniaxial stress-strain curves with straight line segments. Thus, when a piecewise linear stress-strain diagram is given the mechanical subelement model for 3-D conditions can be readily constructed.

In a set of unpublished notes (Pian 1966), the author has found that when a solid under plane stress condition is represented by a two-layer model the resulting strain-hardening behavior under uniaxial loading can be determined only through the solution of two differential equations and the corresponding stress-strain relation is no longer linear. In fact, the modification factors that were obtained in (Pian 1984a, b), for the thickness ratios of the two layers for isotropic and transversely isotropic plastic behaviors were based only on the tangent modulus at the initial yielding.

This paper contains a redevelopment of such a mechanical sublayer model for plane stress problems and an extension to cases with more than two layers.
Chapter 6

Analysis and Control of Finite Deformations of Plates and Shells: Formulations and Interior/Boundary Element Algorithms

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Summary

The following topics are discussed in this paper: (i) some of the recent advances in formulating finite deformation (large rotations as well as stretches) plate and shell theories, and attendant mixed finite element formulations based on symmetric variational statements; (ii) finite element/boundary element formulations based on unsymmetric variational statements, Petrov-Galerkin methods, and the use of fundamental solutions in infinite space, for the highest-order differential operator of the problem, as test functions in solving nonlinear plate and shell problems; and (iii) algorithms for solving the problems of control of nonlinear dynamic motion of plates and shells.

1. Finite Deformation (Large Stretch and Large Rotation) Plate and Shell Theories and Mixed Finite Elements Based on Symmetric Variational Statements

Let $x^1$ and $y^1$ be fixed right-hand Cartesian coordinates in the undeformed and deformed structure (plate or shell), respectively and $\xi^1$ a convected curvilinear coordinate system. The "reference surface" $S$ of the undeformed structure is defined by $\xi^{\alpha}$ ($\alpha = 1,2$), the generic point on $S$ has a position vector $R_0$, and the thickness coordinate $\xi^3$ lies along the unit normal $N$ to $S$ at $P_0$. An arbitrary point $P$ in the plate or shell has the position vector:

$$R = R_0 + \xi^3 N$$  \hspace{1cm} (1.1)
where \( \mathbf{U} \) is the arbitrarily large "midplane" stretching.

Using (1.11a,b), two new "objective" measures of bending strains, \( b \) and \( b^* \), were defined in Atluri (1983):

\[
\begin{align*}
A_\alpha &= \frac{\partial \mathbf{R}^0}{\partial \xi^\alpha}; \quad N = \frac{1}{2} c_{ab} A_\alpha \times A_\beta; \quad A_3 = \mathbf{N} \tag{1.2}
\end{align*}
\]

where \( |N| = 1 \). The base vectors at \( P \) are:

\[
\mathbf{G}_\alpha = (I - \xi^3 \mathbf{B}) \cdot A_\alpha; \quad G_3 = \mathbf{N} \tag{1.3}
\]

where the symmetric undeformed curvature, \( \mathbf{B} \), is defined thus

\[
\mathbf{B} = \frac{\partial \mathbf{N}}{\partial \xi^\alpha} = \mathbf{B}^\alpha_{\alpha} A_\alpha A^\alpha \tag{1.4}
\]

Deformation carries \( P_0 \) to \( P_0 \) and \( P \) to \( p \), respectively. The midsurface displacement \( P_0 P_0 \) is \( \mathbf{u} \), but the displacement \( P P \) is defined through Love's approximation that material fibers originally straight and normal to \( S \) are mapped, without stretching, onto fibers straight and normal to \( s \). The position vectors of \( P_0 \) and \( p \) are, respectively:

\[
\begin{align*}
\mathbf{r}_0 &= \mathbf{R}_0 + \mathbf{u} \tag{1.5} \\
\mathbf{r} &= \mathbf{r}_0 + n \xi^3 = \mathbf{R}_0 + \mathbf{u} + n \xi^3 \tag{1.6}
\end{align*}
\]

where \( n \) is the unit normal to \( s \), and the total displacement of \( P \) is:

\[
\mathbf{u}_t = \mathbf{r} \cdot \mathbf{R} = \mathbf{u} + (n - N) \xi^3 \tag{1.7}
\]

Deformed base vectors \( \mathbf{G}_\alpha \) and \( \mathbf{g}_\alpha \) at \( P_0 \) and \( p \), and a symmetrical deformed curvature tensor \( \mathbf{B} \) of \( s \), can be defined in direct analogy with \( A_\alpha \), \( G_\alpha \), and \( \mathbf{B} \) [see Atluri (1983)].

The deformation gradient tensor \( \mathbf{F} \) relates an undeformed differential vector \( d\mathbf{R} \) at \( P \) to its image \( d\mathbf{r} \) at \( p \) in the deformed domain as follows:

\[
d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R} = d\mathbf{R} \cdot \mathbf{F}^t \tag{1.8}
\]

where \( \mathbf{F} = \mathbf{g}_\alpha \mathbf{G}^\alpha + n N \tag{1.9} \)

Provided \( \mathbf{F} \) is nonsingular, it has the decomposition

\[
\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R} \tag{1.10}
\]

Properties of \( \mathbf{F} \), \( \mathbf{R} \), \( \mathbf{U} \), and \( \mathbf{V} \) for a finitely deformed shell (large stretch and large rotation) have been discussed in detail in Atluri (1983). Confining attention to the right polar decomposition, \( \mathbf{U} \) is sometimes written as \( (I + \mathbf{b}) \), where

\[
\mathbf{b} = \mathbf{R}^{-1} \cdot \mathbf{F} \mathbf{U}^{-1} \tag{1.16a,b,c}
\]

Stress and moment resultants are defined as counterparts to these alternate stress measures [see Atluri (1983) for a detailed discussion]. Considering a differential element of the deformed plate or shell, the traction on a strip of height \( d\xi^3 \), width \( d\xi^3 \), and at distance \( \xi^3 \) from the reference surface, is given by:

\[
(1.81); \quad \text{and, for unit normals} \ N \ \text{and} \ \mathbf{n} \ \text{in a Kirchhoff-Love hypothesis:}
\]

\[
\mathbf{n} = F \cdot N = R \cdot U \cdot N = R \cdot N \tag{1.11}
\]

Through the use of a set of identities [Atluri (1983)], the deformed bases \( \mathbf{g}_\alpha \) can be represented in two ways:

\[
\mathbf{g}_\alpha = A_\alpha + u_\alpha + (R \cdot N) \xi^3 \tag{1.12}
\]

\[
\mathbf{g}_\alpha = A_\alpha + u_\alpha + (R \cdot N) \xi^3 \tag{1.13}
\]

and, accordingly, the following representations result for \( \mathbf{F} \)

\[
\begin{align*}
\mathbf{F} &= (R \cdot U) - \xi^3 b \cdot n \mathbf{U} \cdot (R \cdot N) N \tag{1.14a} \\
\mathbf{F} &= (A_\alpha + u_\alpha + (R \cdot N) \xi^3) \mathbf{G}^\alpha + (R \cdot N) N \tag{1.14b}
\end{align*}
\]

where \( U_0 \) is the arbitrarily large "midplane" stretching. Using (1.11a,b), two new "objective" measures of bending strains, \( b \) and \( b^* \), were defined in Atluri (1983):

\[
\begin{align*}
b^* &= (R \cdot U) \cdot b \cdot (R \cdot U) \tag{1.15a} \\
b^* &= b \cdot n \mathbf{U}^{-1} \tag{1.15b}
\end{align*}
\]

Bending strain measures \( b \) and \( b^* \) are symmetric, while \( b^* \), in general, is not.

A number of alternate finite deformation stress measures may be defined through Nansen's law [see Atluri (1983) for a comprehensive discussion]. The Cauchy stress is denoted as \( \mathbf{t} \). The nominal stress \( \mathbf{t} \) (or the transpose of the first Piola-Kirchhoff stress), the generally unsymmetric Biot-Lure-stress \( r^* \), and the symmetrized Biot-Lure (Jaumann) stress \( r \) prove to be more useful in the computational mechanics of finitely deformed plates and shells [Atluri (1983)]. These are defined through:

\[
\begin{align*}
t &= J F^{-1} \cdot \mathbf{F} \cdot \mathbf{U} \cdot \mathbf{n} \cdot (R \cdot N) N \tag{1.16a,b,c}
\end{align*}
\]
The Cauchy stress resultant per unit $\xi^\beta$ in the deformed state is defined as:

$$\mathbf{t}_r^\alpha = \int \sqrt{g} \tau^{LM} \mathbf{g}_M^L \mathbf{d}\xi^3 = \sqrt{g} \tau^{LM} \mathbf{g}_M^L \mathbf{d}\xi^3$$  (1.17)

The Cauchy stress resultant per unit $\xi^\beta$ in the deformed state is defined as:

$$\mathbf{t}_r^\alpha = \int \sqrt{g} \tau^{LM} \mathbf{g}_M^L \mathbf{d}\xi^3$$  (1.18)

and the Cauchy stress-resultant tensor is defined as:

$$\mathbf{t}_r = a_{\alpha} (\mathbf{t}_r^\alpha / a_{\alpha}) = \mathbf{a}_{\alpha} (\mathbf{t}_r^\alpha)$$  (1.19)

The Cauchy stress-couple, per unit of $\xi^\beta$, in the deformed configuration, is:

$$\mathbf{t}_r^{\alpha \beta} = \int \mathbf{n} \times (\sqrt{g} \tau^{LM} \mathbf{g}_M^L) \mathbf{d}\xi^3 = \mathbf{n} \times \mathbf{t}_r^\alpha = n \times \mathbf{t}_r^\alpha$$  (1.20)

The Cauchy stress-couple tensor is defined as:

$$\mathbf{t}_r = a_{\alpha} (\mathbf{t}_r^{\alpha \beta} / a_{\alpha}) ;$$  (1.21)

The Jaumann stress resultants and couples are defined [see Atluri (1983)], respectively, as:

$$\mathbf{r}_n = \mathbf{V}_2 (\mathbf{t}_r \cdot \mathbf{R} + \mathbf{R} \cdot \mathbf{t}_r \cdot \mathbf{t})$$  (1.22)

$$\mathbf{r}_c = \mathbf{V}_2 (\mathbf{t}_r \cdot \mathbf{R} + \mathbf{R} \cdot \mathbf{t}_r \cdot \mathbf{t})$$  (1.23)

Furthermore, it transpires [see Atluri (1983)] that $\mathbf{t}_r$ is most usefully written in components of the undeformed basis $\mathbf{A}_1$, thus:

$$\mathbf{t}_r^\alpha = t^{\alpha \beta} \mathbf{A}_\beta + t^{\alpha ^3} \mathbf{N}$$  (1.24)

Linear and angular momentum balance conditions for first Piola-Kirchhoff stress and moment resultants are [Atluri (1983)]:

$$\frac{\partial}{\partial \xi^\alpha} (\sqrt{\mathbf{A}} \mathbf{t}_r^\alpha) + \sqrt{\mathbf{A}} \mathbf{p} = 0$$  (1.25)

The linear form of (1.25) makes it ideally suitable for a mixed finite element formulation [Murakawa and Atluri (1981); Atluri and Murakawa (1977); Fukuchi and Atluri (1981); Murakawa, Reed, Atluri, and Rubinstein (1981); Atluri (1983)].

The strain-energy density (per unit undeformed reference area) for a finitely deformed shell (with large mid-plane stretches and large rotations), for semi-linear isotropically elastic behavior, may be written consistently [see Atluri (1983) for a comprehensive discussion] as:

$$\mathbf{W}_0 = \frac{E_h}{(1-v)^2} \left[ \left( \mathbf{U}_0 - \mathbf{I}_0 \right): \mathbf{I}_0 \right] + \frac{(1-v)}{2} \left[ \left( \mathbf{U}_0 - \mathbf{I}_0 \right): \left( \mathbf{U}_0 - \mathbf{I}_0 \right) \right]$$

$$+ \frac{E_h^3}{12(1-v^2)} \left[ \left( \mathbf{b}^* - \mathbf{b} \right): \mathbf{I}_0 \right] + \frac{(1-v)}{2} \left[ \left( \mathbf{b}^* - \mathbf{b} \right): \left( \mathbf{b}^* - \mathbf{b} \right) \right]$$

$$= \mathbf{W}_0 + \mathbf{W}_0 + \mathbf{W}_0$$  (1.27)

such that:

$$\frac{\partial \mathbf{W}_0}{\partial \mathbf{U}_0} = \mathbf{r}_n ; \frac{\partial \mathbf{W}_0}{\partial \mathbf{b^*}} = \mathbf{r}_c$$  (1.28)

The above constitutive theory can be consistently generalized to elasto-plasticity, under the presence of finite stretches and rotations in the shell, as discussed in Atluri (1984), Reed and Atluri (1985), and Atluri (1985a,b). Through Legendre contact transformations, one may define complementary energy densities (per unit of undeformed reference area) as:

$$\mathbf{W}_0 (\mathbf{R}) = \mathbf{r}_n : \mathbf{U}_0 - \mathbf{W}_0 (\mathbf{U}_0)$$  (1.29)

and

$$\mathbf{W}_0 (\mathbf{b}^*) = \mathbf{r}_c : \mathbf{b}^* - \mathbf{W}_0 (\mathbf{b}^*)$$  (1.30)

The most general mixed-hybrid variational principle, for a shell undergoing large stretches and rotations, has been stated by Atluri (1983) as the condition of stationarity of the functional:

$$F_1 (\mathbf{U}_0; \mathbf{b}^*; \mathbf{w}^*; G_1 ; \mathbf{t}^\alpha) = 0$$  (1.26)
The rigid rotation \( R \) is represented as:

\[
R = I + (\mathbf{u} \times \mathbf{l}) + [(\mathbf{u} \times \mathbf{l}) \cdot (\mathbf{u} \times \mathbf{l})]/2\cos^2(\omega/2)
\]

(1.35)

The details of finite-element development based on the above theories, and computational results for several examples of finite-deformation behavior of plates and shells, are given in Punch (1983), Punch and Atluri (1985), Fukuchi and Atluri (1981), and Murakawa and Atluri (1981).

2. FINITE/Boundary Element Analysis of Shells, Based on Fundamental Solutions, and Unsymmetric Variational Statements and Petrov-Galerkin Methods

The problems imposed by the requirement of \( C^1 \) interelement continuity of transverse displacement in thin-plate/shell finite elements, based on a symmetric variational formulation (i.e., using identical polynomial basis function spaces for trial as well as test functions), are now well known. While in current practice, only \( C^0 \) plate and shell bending elements are popular, the problem of spurious-mode control in such elements is still a subject of much research [see Hughes, Cohen, and Haroun (1979) and Belytschko, Ong, and Liu (1984)]. In the following, we present a finite/boundary element method for shallow shells, based on an unsymmetric variational formulation and the use of fundamental solutions to the highest-order differential operator of the problem as test functions. In this method, the problem of \( C^1 \) continuity of trial functions for transverse displacement becomes totally vacuous. In some instances, in this formulation, the trial function for transverse displacement in each element may simply be piecewise constant. We restrict our attention in the following to the case of linear elasto-statics of shells.

Consider a shallow shell of an isotropic elastic material with the mid-surface being described by \( z = z(x_0) \), \( \alpha = 1, 2 \). The base-plane of the shell is defined by a domain \( \Omega \) in the \( 0x_1x_2 \) plane, and \( \Omega \) is bounded by a smooth curve \( \Gamma \). Using the Reissner's (1946) linear theory of shallow shells, the pertinent equilibrium equations may be written as:

\[
N_{\alpha \beta} + b_{\alpha} = 0 \quad (\alpha, \beta = 1, 2)
\]

(2.1a)
where: \(N_{ab}\) are membrane forces; \(\varphi\) are body forces; \(w\) is the transverse deflection of the midsurface of the shell; \(b_1(1, 2, 3)\) are body forces; and \(f_3\) is the load normal to the shell mid-surface; and \(D = \frac{E t^3}{12(1 - v^2)}\); \(t\) is the thickness; and \(E\) and \(v\) are the elastic constants; \(V^3\) is the biharmonic operator in the variables \(x_i\); and

\[
R_{\alpha\beta} = -1/z_{\alpha\beta}
\]

are the radii of curvature of the undeformed shell. Along \(\Gamma\), the boundary conditions are:

\[
\begin{align*}
\text{u}_a &= \text{u}_a \quad \text{at} \quad \Gamma_u; \quad N_{a\beta}n_\beta = \overline{P}_{a} \quad \text{at} \quad \Gamma_0; \quad \Gamma = \Gamma_u \cup \Gamma_0
\end{align*}
\]

where \(n_\beta\) are the direction cosines of the unit outward normal to \(\Gamma\) in the base plane. The out-of-plane boundary conditions are:

\[
\begin{align*}
w &= \bar{w} \quad \text{or} \quad V_n &= \bar{V}_n \quad (2.3a) \\
\overline{V}_n &= \bar{V}_n \quad \text{or} \quad M_n &= \bar{M}_n \quad (2.3b)
\end{align*}
\]

where

\[
\begin{align*}
V_n &= -D \frac{\partial}{\partial n} (V^2 w) + \frac{\partial}{\partial t} M_t \\
M_n &= \frac{\partial}{\partial n} w
\end{align*}
\]

are the reduced Kirchhoff shear force; \(\overline{V}_n\) is the rotation around the tangent to \(\Gamma\); and \(n\) and \(t\) are directions normal and tangential, respectively, to \(\Gamma\) in the base plane.

It is well known that the equilibrium equations (2.1) can be written more concisely in terms of a stress function (for \(N_{ab}\)) and the transverse displacement \(w\). However, we leave the equations in the form (2.1), which is somewhat more general, so as to treat inplane inertia forces and to extend the development to the case of general non-shallow shells, and nonlinear kinematics, in forthcoming papers.

2.1 Integral Equations for Shell Displacements; and a Boundary Element Solution Strategy:

In an approximate analysis of the boundary/initial-value problem described above, let \(u_\alpha\) and \(w\) be the assumed trial solutions. We shall consider a general weighted-residual formulation, and let \(u_{a\beta}\) and \(w^*\) be the corresponding test functions. In the familiar Galerkin finite-element method, the trial functions \((u_\alpha, w)\) and the test functions \((u_{a\beta}^*\) and \(w^*)\) belong to the same category of function spaces. In the present formulation, however, as will be seen, the test functions \((u_{a\beta}^*\) and \(w^*)\) belong to an entirely different class of function space from that of the trial functions. With this in mind the combined weak forms of the equilibrium equations and boundary conditions for the
\[
\int_\Omega (N_{ab} + b_a)u_a^* \, d\Omega = \int_\Gamma_0 (\tilde{P}_a - \bar{P}_a)u_a^* \, dr
\]

\[
+ \int_{\Gamma_u} (\tilde{u}_a^* - u_a^*)P_a^* \, d\Gamma
\]

(2.7)

and

\[
\int_\Omega \left( \nabla^t w + \frac{N_{ab}}{R_{ab}} - b_3 - \gamma^3 \right) w^* \, d\Omega = \int_{\Gamma_0} (\tilde{v}_n - v_n) w^* \, dr
\]

\[
+ \int_{\Gamma_M} (\tilde{M}_n - M_n) w^* \, dr + \int_{\Gamma_{\nabla w}} (\tilde{v}_n - v_n) M_n \, dr + \int_{\Gamma_{\nabla v}} (w - \tilde{w}) v_n \, dr
\]

(2.8)

Use of the Divergence theorem in Eq. (2.12a) results in:

\[
\int_{\Gamma} N'_{ab} u_a^* \, d\Gamma = \int_{\Gamma} N'_{ab} u_a^* \, d\Gamma + \int_{\Omega} C(\kappa_{ab} w)_{,\beta} u_a^* \, d\Omega + \int_{\Omega} \bar{u}_a^* \, d\Omega
\]

\[
+ \int_{\Gamma_u} (\tilde{u}_a^* - u_a^*)P_a^* \, d\Gamma
\]

(2.12b)

Since the material is linear elastic and isotropic, we have:

\[
N'_{ab} u_a^* = C_{ab} Y_0 u_{ab} \text{, } u_{ab} = N'_{ab}^*(u_{ab})_0
\]

(2.13)

where the definitions of \( N'_{ab}^* \) are apparent. We now introduce the additional notations:

\[
P'_a = N'_{ab} n_b \text{, } P_a = N_{ab} n_b
\]

(2.14a)

or

\[
P_a = P'_a + C \kappa_{ab} w n_b
\]

(2.14b)

Using (2.13, 14a,b) in (2.12b) and applying the Divergence theorem, it is easy to obtain:

\[
\int_{\Omega} [N'_{ab}^* (u_{ab})_0]_{,\beta} u_a^* \, d\Omega + \int_{\Gamma} \hat{P}_a u_a^* \, d\Gamma - \int_{\Gamma} P_a \tilde{u}_a^* \, d\Gamma
\]

\[
- \int_{\Omega} C \kappa_{ab} w u_{ab}^* \, d\Omega = 0
\]

(2.15a)

where \( \hat{P}_a = P_a \) at \( \Gamma_0 \) ; and \( \hat{P}_a = P_a \) at \( \Gamma_u \)

(2.15b)

and \( \tilde{u}_a = \tilde{u}_a^* \) at \( \Gamma_u \) ; and \( \tilde{u}_a = u_a \) at \( \Gamma_0 \)

(2.15c)

Now, we choose \( u_{ab}^* \) to be the "fundamental solution" of the equation:

\[
[N'_{ab}^* (u_{ab}^*)]_{,\beta} + \delta(x_u - \xi_u) \delta_{\kappa_{ab} e_{ab}} = 0
\]

(2.16)
direction of the application of the point load is along the 
\( x_0 \) direction. The "singular solution" of (2.16) will be 
denoted as \( u_0(x) \); where \( u_0(x) \) is the displacement 
along the \( x_0 \) direction in a plane infinite body at any point 
\( x_0 \), due to a unit load along the \( x_0 \) direction, applied at 
the location \( x_0 = \xi_\mu \). Likewise, \( P_0(\theta)x_0(x_0,\xi_\mu) \) will 
be considered to be the traction along the \( x_0 \) direction on an 
oriented surface at \( x_0 \), with a unit normal \( n_0 \), due to a unit 
load along \( x_0 \) at the location \( \xi_\mu \). These solutions are well 
known and may be written as:

\[
u^*(x_0,\xi_\mu) = \frac{1}{8\pi} \left[ (\nu - 3)2np \left( \frac{\partial p}{\partial x_0} - (1 + \nu) \frac{\partial p}{\partial x_\alpha} \right) \right]
\]

and

\[
P_0(\theta)x_0(x_0,\xi_\mu) = -\frac{t}{4\pi\rho} \left[ (1 - \nu)\delta_{\theta\alpha} + 2(1 + \nu) \frac{\partial p}{\partial x_0} \frac{\partial p}{\partial x_\alpha} \right]
\]

where \( \rho = |x_0 - \xi_\mu| \) is the radius vector from \( x_0 \) to \( \xi_\mu \);

\( G = E\nu/[2(1+\nu)] \).

Due to the property of integrals involving Dirac functions, we have:

\[
\left( N_{\alpha\beta}^* \right)_{\alpha\beta} u_0 d\Omega = -\int_{\Omega} \delta(x_0 - \xi_\mu) \delta_{\alpha\beta} u(x_0) d\Omega = -u_0(\xi_\mu)
\]

Using (2.17 and 18) in (2.15a), we have:

\[
\nu_\theta(\xi_\mu) = \int_{\Omega} b(x_0)u_0^*(x_0,\xi_\mu) d\Omega + \int_{\Gamma} \hat{P}_0(\theta)x_0(x_0,\xi_\mu) d\Gamma
\]

\[
-\int_{\Omega} \hat{u}_0(\xi_\mu) P_0(\theta)x_0(x_0,\xi_\mu) d\Omega + \int_{\Gamma} Ck_{\alpha\beta} w(x_0)u_0^{(\alpha)}(x_0,\xi_\mu) d\Omega
\]

It can be shown that while the coefficient \( Y \) in the 
left-hand side of (2.19) is unity when \( \xi_\mu \) is in the interior 
of \( \Omega \), the value of \( Y \) is (0.5) when \( \xi_\mu \) falls on the "smooth" 
boundary \( \Gamma \) [Atluri and Grannell (1978)]. Equation (2.19) is 
the sought-after integral equation for \( u_\alpha \) in a shallow 
shell.

A unit point load at the location \( \xi_\mu \). Thus, \( u^* \) corresponds 
to the solution of the linear equation:

\[
D \frac{\partial^2 w^*}{\partial x_0^2} = \delta(x_0 - \xi_\mu),
\]

in an infinite domain in the base-plane of the shallow shell.

It is well known [Love (1900)] that the solution for \( w^* \) is 
given by

\[
w^*(x_0,\xi_\mu) = \frac{1}{8\pi} \rho^2 \ln \rho
\]

where \( \rho = |x_0 - \xi_\mu| \).

Using Eqs. (2.21) and (2.9) in Eq. (2.8) and employing repeated integrations by part in the resulting equation, one 
easily obtains the integral equation:

\[
Y_D \frac{\partial w(\xi_\mu)}{\partial x_0} = \int_{\Gamma} \hat{v}(x_0)w^*(x_0,\xi_\mu) d\Gamma - \int_{\Gamma} \hat{M}_n(x_0)\psi^*(x_0,\xi_\mu) d\Gamma
\]

\[
+ \int_{\Gamma} \hat{M}_n(x_0)\psi^*(x_0,\xi_\mu) d\Gamma - \int_{\Gamma} \hat{w}(x_0)\psi^*(x_0,\xi_\mu) d\Gamma
\]

\[
- \int_{\Omega} \left[ N_{\alpha\beta}^{*\alpha \beta} + C_{\alpha\beta} \frac{\partial w^*}{\partial x_\alpha} \right] u_0(x_0,\xi_\mu) d\Omega
\]

\[
+ \int_{\Omega} \left[ M_{\alpha\beta}^{*\alpha \beta} - M_{\alpha\beta}^{*\alpha \beta} \right] w^*(x_0,\xi_\mu) d\Omega
\]

where

\[
\nu_\alpha = -D \frac{\partial \nu}{\partial x_\alpha} (\nu^2 \omega) + \frac{3}{3\Delta} M_T
\]

\[
M_n = M_{11} n_1^2 + 2 M_{12} n_1 n_2 + M_{22} n_2^2
\]

\[
M_T = (M_{22} - M_{11}) n_1 n_2 + M_{12} (n_1^2 - n_2^2)
\]

\[
Y = 1 \text{ for } \xi_\mu \in \Omega \text{ ; } Y = \frac{1}{2} \text{ for } \xi_\mu \in \Gamma \text{ (smooth)}
\]

In Eq. (2.22) the terms with the superposed symbol "\( ^* \)" 
should be taken to imply the respective prescribed values, 
if any, at \( \xi_\mu \); otherwise, they are to be treated as the 
unknown solution variables. Also, the symbol \( \{ \} \) denotes
Using Eq. (2.10) and the Divergence theorem, it is easy to see that:

$$
\int_{\Omega} \frac{N^\alpha}{R^\alpha} (x) \delta(x, \xi) \, d\Omega = \int_{\Omega} C \kappa_{\alpha \beta} \mathbf{u}(x) \cdot \omega(x, \xi) \, d\Omega
$$

$$
- \int_{\Omega} C \kappa_{\alpha \beta} \mathbf{u}(x) [\omega(x, \xi)]_{\beta} \, d\Omega
$$

(2.23)

Use of (2.23) in (2.22) results in the final integral equation for \( w \) as follows:

$$
\mathbf{Y} \cdot \mathbf{D} \left[ \omega(\xi) \right] = \int_{\Gamma} \hat{\mathbf{U}}(x) \cdot \omega(x, \xi) \, d\Gamma - \int_{\Gamma} \mathbf{n}(x) \cdot \omega(x, \xi) \, d\Gamma
$$

$$
+ \int_{\Gamma} \hat{\mathbf{U}}(x) \cdot \mathbf{M}(x) \cdot \omega(x, \xi) \, d\Gamma - \int_{\Omega} \omega(x) \cdot \mathbf{M}(x) \cdot \omega(x, \xi) \, d\Omega
$$

$$
- \int_{\Gamma} C \kappa_{\alpha \beta} \mathbf{u}(x) \cdot \omega(x, \xi) \, d\Gamma + \int_{\Gamma} C \kappa_{\alpha \beta} [\omega(x, \xi)]_{\beta} \mathbf{u}(x) \, d\Omega
$$

$$
- \int_{\Omega} \kappa_{\alpha \beta} \mathbf{w}(x) \cdot \omega(x, \xi) \, d\Omega + \int_{\Omega} \mathbf{n}(x) \cdot \omega(x, \xi) \, d\Omega
$$

$$
+ \int_{\Omega} \mathbf{M}(x) \cdot \omega(x, \xi) \, d\Omega + \sum \frac{k}{1} \left( \mathbf{M}_p \omega_p - \mathbf{M}_p \right) \omega
$$

(2.24)

Since \((\partial \omega/\partial n)\) is also an independent variable at \( \Gamma \) in the present boundary-value problem of the shallow shell, an integral relation for \((\partial \omega/\partial n)\) should also be derived. Towards this purpose, consider a second fundamental solution,

$$
\omega_p = \frac{1}{2\pi} \varphi \ln r \cos \phi
$$

where \( \phi \) is the angle between the outward normal to \( \Gamma \) and the radius \( r \) [see Stern (1979)]. The resulting integral equation is:

$$
\mathbf{Y} \cdot \mathbf{D} \left[ \omega_p(\xi) \right] = \int_{\Gamma} \hat{\mathbf{U}}(x) \cdot \omega_p(x, \xi) \, d\Gamma - \int_{\Gamma} \mathbf{n}(x) \cdot \omega_p(x, \xi) \, d\Gamma
$$

$$
+ \int_{\Gamma} \hat{\mathbf{U}}(x) \cdot \mathbf{M}(x) \cdot \omega_p(x, \xi) \, d\Gamma - \int_{\Omega} \omega(x) \cdot \mathbf{M}(x) \cdot \omega_p(x, \xi) \, d\Omega
$$

$$
- \int_{\Gamma} C \kappa_{\alpha \beta} \mathbf{u}(x) \cdot \omega_p(x, \xi) \, d\Gamma + \int_{\Gamma} C \kappa_{\alpha \beta} [\omega_p(x, \xi)]_{\beta} \mathbf{u}(x) \, d\Omega
$$

$$
- \int_{\Omega} \kappa_{\alpha \beta} \mathbf{w}(x) \cdot \omega_p(x, \xi) \, d\Omega + \int_{\Omega} \mathbf{n}(x) \cdot \omega_p(x, \xi) \, d\Omega
$$

$$
+ \int_{\Omega} \mathbf{M}(x) \cdot \omega_p(x, \xi) \, d\Omega + \sum \frac{k}{1} \left( \mathbf{M}_p \omega_p - \mathbf{M}_p \right) \omega
$$

(2.25)

2.2 Remarks

In summary, Eqs. (2.19), (2.24), and (2.25) represent the complete set of integral equations for \( u, w, \) and \( \omega/\partial n \). An examination of Eqs. (2.19), (2.24), and (2.25) reveals the following features:

(i) For given body forces \( \mathbf{b}_D \), the integral relation for \( u_0 \) (Eq. (2.19)) involves the trial functions \( u_0 \) only at the boundary \( \Gamma \). On the other hand, due to the curvature induced coupling of the trial functions \( u_a \) and \( w \) in the shallow-shell problem, the integral relation for \( u_0 \) contains a domain-integral (over \( \Omega \)) involving the trial function for \( w \).

(ii) If the body forces \( \mathbf{b}_D \) include in-plane inertia forces \( \mathbf{b}_D \), then the integral relation for \( u_D \) involves a domain integral (over \( \Omega \)) of \( w \) as well.

(iii) Again, due to the curvature-induced coupling of the in-plane and out-of-plane displacements in the presently considered shallow shell, the integral equations for \( w \) and \( \omega/\partial n \), Eqs. (2.24) and (2.25) respectively, contain domain-integrals (over \( \Omega \)) involving trial functions for both \( w \) and \( u_n \). Also, in a transient dynamic analysis, the term \( \mathbf{Q} \) appears inside a domain-integral.

(iv) For reasons (i) to (iii) above, unlike the classical homogeneous isotropic elasto-statics [Atluri and Grannell (1978)] wherein a discretization of the relevant integral equations requires a use of basis functions for the displacements at the boundary alone, the present shallow-shell formulation requires the assumption of basis functions for the trial solutions \( u_0 \) and \( w_0 \) at the boundary \( \Gamma \) as well as in the interior \( \Omega \). Thus, the present solution methodology may, strictly speaking, be classified as a hybrid boundary-element/interior (finite) element method based on a direct discretization of integral equations.

(v) Unlike in the homogeneous isotropic elasto-statics [Atluri and Grannell (1978)], the present integral equations are no longer boundary-integral equations alone.
Thus, we obtain three integral relations for the boundary values of $u_\alpha$, $w$, and $\partial w/\partial n$. 

An examination of these relations reveals that, in order to discretize these equations, one needs to assume only very simple trial functions $u_\alpha$, $w$, $\partial w/\partial n$ not only at the boundary but also in the interior of $\Omega$. For instance, $\Omega$ may be discretized into a number of finite elements and $\Gamma$ into a number of boundary elements. As $\xi$, tends to $\Gamma$, the integral relations (2.19), (2.24), and (2.25) clearly show that $w$ and $u_\alpha$ need only be piecewise constant functions in each finite element in $\Omega$, in any discretization process. Utmost, one may need only consider $C^0$ continuous functions for $w$ and $u_\alpha$ in each element [to extend the formulations to finite deformation cases, see Zhang and Atluri (1986)].

In contrast, it is recalled that in the Galerkin finite element method, $u_\alpha$ need be $C^1$ continuous and $w$ be $C^0$ continuous in each element. The difficulties with such a finite element approach are too well documented in literature to warrant further comment here.

At each point on the boundary, two of the in-plane variables $u_\alpha (\alpha = 1, 2)$, $P_\alpha (\alpha = 1, 2)$ are specified; and the other two are unknown. Likewise, two of the out-of-plane variables, $V_n$, $M_n$, $\Phi_n$, and $w$ are specified; and the other two are unknown. At each point in $\Omega$, as seen from Eqs. (2.19), (2.24), and (2.25), the three displacements, $u_\alpha$ and $w$, are unknown. Thus, if Eqs. (2.19), (2.24), and (2.25) are discretized, through finite as well as boundary elements and the nodal values of the variables are appropriately understood to be either specified or unknown, one would easily obtain exactly as many equations as the number of unknowns, so that the problem may be considered as well posed.

Further details of the algebraic formulation of the finite-element/boundary-element method, and results for static stress analysis and vibration analysis of shallow spherical shells (representing continuum models of large space antennae), may be found in Zhang and Atluri (1985).

Here we present certain new results for shallow shells of arbitrary plan form.

The first example deals with a uniformly loaded and simply supported rhombic plate as shown in Fig. 1. This example was also analyzed by Hughes, Cohen, and Haroun (1978) using two types of elements: (a) four-node bilinear $C^0$ element with three degrees of freedom ($w$ and two rotations) per node with either uniform one-point reduced integration ("U1") or selective reduced integration (1 point shear and 2 x 2 bending) denoted as ("S1"); (b) a node biquadratic element with three degrees of freedom per node.

Results were presented by Hughes, Cohen, and Haroun (1978) using 96 four-node (- 350 total d.o.f.) and 24 nine-node elements (- 350 total d.o.f.). In contrast, the presently used mesh is shown in Fig. 2. The total number of degrees of freedom in the present mesh is only 81. The presently computed results for bending moments $M_x$ and $M_y$ along the X axis are shown in Fig. 3 and compared with the analytical solution of Morley (1963). That the present finite/boundary element method with only 81 degrees of freedom produces results in excellent agreement with the analytical solution, as well as with those of reduced-integrated elements (based on a symmetric variational formulation) of Hughes, Cohen, and Haroun (1978), points to the excellent accuracy and inherent simplicity of the present-type methodology based on unsymmetric variational formulations and test functions that are fundamental solutions to the highest-order operator in the problem.
The next example concerns a shallow shell of a square plan-form, with the base plane being \((8 \times 8)\) and the radius of curvature at the crown being 50. The shell is simply supported at the boundary and loaded by a concentrated force of \(P = 10\) at the center. Figure 4 shows the transverse deflection along the \(X\) axis (with \(x = 0\) being the center of the plate) for three different meshes, with 35, 91, and 171 degrees of freedom, respectively. It is seen that even the mesh with 35 d.o.f. yields results for both displacements and stresses that are in excellent agreement with the analytical solution.

The dynamic analogs of the von Karman equations for large displacements of an initially flat plate may be written as:

\[
N_{\alpha\beta,\alpha} = \rho \frac{\partial^2 w}{\partial t^2}
\]

\[
DV^\alpha w = \frac{\partial}{\partial x_1} \left( N_{11,\alpha} \frac{\partial w}{\partial x_1} + N_{12,\alpha} \frac{\partial w}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left( N_{22,\alpha} \frac{\partial w}{\partial x_2} + N_{12,\alpha} \frac{\partial w}{\partial x_1} \right)
\]

\[
\frac{\partial}{\partial t} - m \frac{\partial^2 w}{\partial t^2} - c \frac{\partial w}{\partial t}
\]

where \(w\) are inplane displacements and \(w\) the transverse displacement; \(N_{\alpha\beta,\alpha}\) inplane stress resultants, and \(T\) consists of both externally prescribed (known a priori) force \(f_e\) as well as control forces \(f_c\) to be determined from some active-control algorithms such that the dynamic response of the plate is damped out in a predetermined time.

Using procedures analogous to those in Section 2 of this paper, and discussed elsewhere in detail [O'Donoghue and Atluri (1985a,b); O'Donoghue (1985)], the nonlinear problem in Eqs. (3.1) and (3.2) is cast in an integral form, as follows:
where \( w^* = 1/8 \pi r^2 \ln r \) is a fundamental solution to the biharmonic operator; and \( V_n^\#, M_{nn}^\# \) are the corresponding Kirchhoff shear and normal moment, respectively. \( P \) is a point on the boundary \( \Gamma \) (see Fig. 6 for nomenclature at boundary).

![Fig. 6 Notation for Origin at Plate Corner](image)

A second boundary/interior integral equation concerning the derivative of \( w \) at the boundary is obtained [see O'Donoghue and Atluri (1985) and O'Donoghue (1985)] as:

\[
\begin{align*}
D K \frac{\partial w}{\partial x} |_P + D K \frac{\partial w}{\partial y} |_P + \int_\Gamma (\nabla w - w(p)) V_n^\# - \frac{\partial w}{\partial x} M_{nn}^\# + \frac{\partial w}{\partial y} M_{nn}^\# & = \int_\Omega \left( [w w(P)] - w w(P) \right) d\Omega \\
+ N_{12} \left[ \frac{\partial w}{\partial x} M_{nn}^\# - \frac{\partial w}{\partial y} M_{nn}^\# \right] d\Omega
\end{align*}
\]

(3.3)

where \( w^* = 1/2\pi r \ln r \) is the second fundamental solution to the biharmonic operator, and \( V_n^\#, M_{nn}^\# \) are the corresponding Kirchhoff shear and normal moment, respectively; \( k_c \) and \( k_n \) are constants at a boundary corner point, as defined in O'Donoghue and Atluri (1985) and O'Donoghue (1985).

The inplane stress resultants are related nonlinearly to inplane displacements through relations of the type:

\[
N_{ab} = E \frac{\partial u}{\partial x} \delta_{ab}^0
\]

(3.5)

\[
\varepsilon_{11} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2; \varepsilon_{22} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2
\]

(3.6a,b,c)

We write \( N_{ab} = N_{ab}^l + N_{ab}^n \) (3.7)

where \( N_{ab}^l \) denotes the linear part and \( N_{ab}^n \) the nonlinear part, as related to the inplane displacements \( u_\alpha \). Then, using procedures as in Section 2, one obtains an integral equation for inplane displacements [see O'Donoghue and Atluri (1986) and O'Donoghue (1985)] of the type:

\[
\int_\Gamma \left( \frac{\partial u}{\partial x} \right) \sigma \beta dA + \int_\Gamma \frac{\partial u}{\partial y} \sigma \alpha dA - \int_\Gamma T \sigma u_\alpha dA = 0
\]

(3.8)

where \( u_\alpha \), \( T \sigma \) are inplane fundamental solutions as defined in Section 2.

Equations (3.3), (3.4), and (3.8) are three integral equations governing the present nonlinear problem. These nonlinear integral equations are solved by the standard full Newton-Raphson incremental procedure [O'Donoghue and Atluri (1986) and O'Donoghue (1985)]. The matrix equations resulting from this procedure are:

\[
M (N+1)^{-1} + C (N+1) + (N) K \Delta q = (N+1) f_c + (N+1) f_c - N \beta (q)
\]

(3.9)

The response is calculated in the standard fashion of time integration, once \( f_c \) are known. The control force
Schur-vector type solution of the Riccati matrix differential equation arising out of the optimal control strategy for a nominally linear system. The details of these control algorithms are beyond the scope of this paper [see O'Donoghue and Atluri (1986) and O'Donoghue (1985)]. The Lyapunov stability of the system, under such control force action, has been verified in the cited references, along with a proof of controllability of the system.

In the following, we present some results for nonlinear bending, dynamic response, and control of initially flat plates, using the present Petrov-Galerkin type finite-element/boundary-element approaches. Figure 7 shows the central deflection of a square plate under transverse load (obtained with a total of 56 degrees of freedom) and its comparison with some analytical results of Levy for a simply supported, uniformly loaded, square plate. Figure 8 shows the comparison of results for post-buckling deflection of a square plate (again with about 50 d.o.f.).

We now present results for control of a simply supported square plate subjected to initial velocities of the type:

$$\dot{u}(t=0) = \frac{135}{a} \frac{\sin x}{b} \frac{\sin x}{2}$$

An actuator, exerting a feedback control force, is assumed to be located at the center of the plate.

Assuming no other passive damping, the actively controlled dynamic response of the system is shown in Fig. 9, for assumptions of linear and nonlinear systems, respectively. The actuator force, as determined from the earlier-mentioned control algorithms, required to attenuate the dynamic response as in Fig. 9 is shown in Fig. 10. These results indicate the viability of the present approaches for active control of nonlinear dynamic response of structures.
REFERENCES


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BOUNDARY ELEMENTS

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Nonlinear Quasi-static and Transient Response Analysis of Shallow Shells: Formulations and Interior/Boundary Element Algorithms

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ABSTRACT

Nonlinear integral equations are derived for the representation of displacements of shallow shells undergoing moderately large, quasi-static or dynamic deformations. A combined interior/boundary element method based on these integral equations, and its implementation for the shell problem are discussed in detail. A "tangent stiffness" iterative procedure is used to obtain the nonlinear solution. Numerical results are presented to demonstrate the efficiency and the accuracy by the present approach.

1. INTRODUCTION

For boundary-value/initial-value problems in solid mechanics, it is often possible to derive certain boundary integral representations for displacements [Atluri and Grannel (1978); Brabbia and Walker (1980); and Atluri (1984)]. A boundary-integral representation, when discretized, leads to the so-called boundary-element method. Such boundary-element methods are possible, for example, in linear, isotropic, elastostatics [see, for instance, Atluri and Grannel (1978)], and in problems of static bending of linear elastic isotropic plates [Stern (1979); Bezine (1981)] in which, the integral representation for displacements involves only boundary integrals of the unknown trial functions and their
2. THE BOUNDARY-VALUE/INITIAL-VALUE PROBLEM

Consider a shallow shell of an isotropic elastic material with the mid-surface being described by \( z = z(x) \), \( a = 1,2 \). The base-plane of the shell is defined by a domain \( \Omega \) which is bounded by a piecewisely smooth curve \( \Gamma \) in the \( Ox_1x_2 \) plane. Based on the Reissner's (1946) shallow shell theory, the von Karman equations of large deformation for the shell may be written as:

\[
N_{\alpha\beta} + b_{\alpha} = 0 \quad (\alpha, \beta = 1,2) \tag{2.1a}
\]

and

\[
Dv^4w + \frac{N_{\alpha\beta}}{R_{\alpha\beta}} - (b_3 - p\ddot{w}) = f_3 + (N_{\alpha\beta}w_{,\beta})_{,\alpha} \tag{2.1b}
\]

where: \( N_{\alpha\beta} \) are membrane forces; \( f_3 \) is the transverse deflection of the midsurface of the shell; \( b_i (i=1,2,3) \) are body forces; \( p \) is the load normal to the shell mid-surface; and \( D = \frac{Et^2}{12(1-v^2)} \) is the thickness; and \( E \) and \( v \) are the elastic constants; \( v^2 \) is the biharmonic operator in the variables \( x_1, x_2 \); \( \dot{u}_a \) and \( w \) are accelerations; and

\[
R_{\alpha\beta} = \frac{-1}{z_{,\alpha} z_{,\beta}} \tag{2.1c}
\]

are the radii of curvature of the undeformed shell. Along \( \Gamma \), the boundary conditions are:

\[
\dot{u}_a = \ddot{u}_a \text{ at } \Gamma_u; \quad N_{\alpha\beta} = \ddot{N}_{\alpha\beta} \text{ at } \Gamma_n; \quad \Gamma = \Gamma_u \cup \Gamma_n \tag{2.2a,b}
\]

where \( n \) are the direction cosines of the unit outward normal to \( \Gamma \) in the base plane. The out-of-plane boundary conditions are:

\[
w = \ddot{w} \text{ or } w = \ddot{w} \quad \text{(2.3a)}
\]

\[
\nabla w = \ddot{\nabla} w \text{ or } \nabla w = \ddot{\nabla} w \quad \text{(2.3b)}
\]

where

\[
\nabla w = -D \frac{\partial}{\partial n} (v^2w) + \frac{\partial}{\partial n} M_{\alpha\beta} + N_{\alpha\beta} \frac{\partial w}{\partial n} + N_{\alpha\gamma} \frac{\partial w}{\partial n} \quad (\text{red Kirchoff shear force})
\]

\[
\nabla w = \frac{\partial w}{\partial n} \quad \text{is the rotation around the tangent to } \Gamma;
\]

The remainder of the paper is organized as follows: Section 2 gives the statement of the boundary-value/initial-value problem, Section 3 deals with the integral representation for shell displacements; Section 4 discusses the solution strategy for the nonlinear problem; Section 5 gives several concluding comments.
and $n$ and $s$ are directions normal and tangential respectively, to $\Gamma$ in the base plane.

The nonlinear inplane strain-displacement relations are:

$$\varepsilon_{\alpha\beta} = \frac{1}{2} [u_{\alpha,\beta} + u_{\beta,\alpha} + \frac{2w}{R_{\alpha\beta}} + w_{,\alpha} w_{,\beta}] \quad (2.4a)$$

where $u_{\alpha}$ are the inplane displacements at the shell mid-surface. The inplane stress-resultant/strain relations are:

$$N_{11} = C(\varepsilon_{11} + \nu \varepsilon_{22}); \quad N_{22} = C(\varepsilon_{22} + \nu \varepsilon_{11});$$
$$N_{12} = C(1 - \nu)\varepsilon_{12} \quad (2.4b)$$

where $C = E\nu/(1 - \nu^2)$. The moment-curvature relations are:

$$M_{11} = -D(w_{,11} + \nu w_{,22}); \quad M_{22} = -D(w_{,22} + \nu w_{,11});$$
$$M_{12} = -D(1 - \nu)w_{,12} \quad (2.5)$$

Finally, the initial conditions on the shell may be written as:

$$u_{\alpha}(x_{\beta}, 0) = u_{\alpha 0}(x_{\beta}) \quad \text{at} \ t = 0$$
$$\dot{u}_{\alpha}(x_{\beta}, 0) = \dot{u}_{\alpha 0}(x_{\beta}) \quad \text{at} \ t = 0$$
$$w(x_{\beta}, 0) = w_{0}(x_{\beta}); \quad \dot{w}(x_{\beta}, 0) = \dot{w}_{0}(x_{\beta}) \quad \text{at} \ t = 0 \quad (2.6)$$

where ($^*$) $\equiv \frac{d(\cdot)}{dt}$.

Here, our attention is restricted to the case of moderately large deformation.
\[ N_{11} = C(u_{1,1} + vu_{2,2}); \quad N_{22} = C(u_{2,2} + vu_{1,1}); \]
\[ N_{12} = \frac{1}{2} C(1 - \nu)(u_{1,2} + u_{2,1}) \]

or

\[ N_{\alpha \beta} = C_{\alpha \beta \gamma} u_{\gamma,0}; \quad (3.4) \]

\[ \kappa_{11} = \frac{1}{F_{11}} + \nu; \quad \kappa_{22} = \frac{1}{F_{22}} + \nu; \quad \kappa_{12} = \frac{1}{F_{12}} \]

and the nonlinear parts:

\[ N^{(n)}_{11} = \frac{C}{2} \left( \frac{\partial w}{\partial x_1} \right)^2 + \nu \left( \frac{\partial w}{\partial x_1} \right)^2; \quad N^{(n)}_{22} = \frac{C}{2} \left( \frac{\partial w}{\partial x_2} \right)^2 + \nu \left( \frac{\partial w}{\partial x_2} \right)^2 \]

\[ N^{(n)}_{12} = \frac{C}{2} (1 - \nu) \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} \]

Use of (3.3) in (3.1) results in:

\[ \int_{\Omega} [N_{\alpha \beta} u_{\beta,0} + C(\kappa_{\alpha \beta} w)u_{\beta,0} + N^{(n)}_{\alpha \beta} + b_{\alpha} - \rho \tilde{u}_{\alpha} u_{\alpha,0} d\Omega \]
\[ = \int_{\Gamma_{0}} (P_{\alpha} - \tilde{P}_{\alpha}) u_{\alpha,0} d\Gamma + \int_{\Gamma_{u}} (\tilde{u}_{\alpha} - u_{\alpha}) P_{\alpha} u_{\alpha,0} d\Gamma \]

(3.7a)

Use of the Divergence theorem in Eq. (3.7a) results in:

\[ \int_{\Gamma} N_{\alpha \beta} u_{\beta,0} d\Gamma - \int_{\Omega} N^{(n)}_{\alpha \beta} u_{\beta,0} d\Omega + \int_{\Gamma} C(\kappa_{\alpha \beta} w)u_{\beta,0} d\Omega \]
\[ + \int_{\Omega} b_{\alpha} u_{\alpha,0} d\Omega + \int_{\Gamma} (b_{\alpha} - \rho \tilde{u}_{\alpha}) u_{\alpha,0} d\Omega = \int_{\Gamma_{0}} (P_{\alpha} - \tilde{P}_{\alpha}) u_{\alpha,0} d\Gamma \]
\[ + \int_{\Gamma_{u}} (\tilde{u}_{\alpha} - u_{\alpha}) P_{\alpha} u_{\alpha,0} d\Gamma \]

(3.7b)

\[ N_{\alpha \beta} u_{\alpha,0} = C_{\alpha \beta \gamma} u_{\gamma,0}, u_{\alpha,0} = N^{(n)}_{\alpha \gamma} u_{\gamma,0} \quad (3.8) \]

where the definition of \( N^{(n)}_{\alpha \beta} \) are apparent. Now note that

\[ P_{\alpha} = N_{\alpha \beta} n_{\beta} \quad (3.9a) \]

or

\[ P_{\alpha} = N_{\alpha \beta} n_{\beta} + C_{\alpha \beta \gamma} n_{\gamma} + N^{(n)}_{\alpha \beta} n_{\beta} \quad (3.9b) \]

Using (3.8, 3.9a,b) in Eq. (3.7b) and applying the Divergence theorem, it is easy to obtain:

\[ \int_{\Omega} [N^{(n)}_{\alpha \beta} u_{\alpha,0} + \int_{\Gamma} (b_{\alpha} - \rho \tilde{u}_{\alpha}) u_{\alpha,0} d\Gamma + \int_{\Gamma} P_{\alpha} u_{\alpha,0} d\Gamma \]
\[ - \int_{\Gamma} C_{\alpha \beta \gamma} \frac{\partial w}{\partial x_\gamma} u_{\alpha,0} d\Omega - \int_{\Omega} N^{(n)}_{\alpha \beta} u_{\alpha,0} d\Omega = 0 \]

(3.10a)

where \( \tilde{P}_{\alpha} = \tilde{P}_{\alpha} \) at \( \Gamma_{0} \); and \( P_{\alpha} = P_{\alpha} \) at \( \Gamma_{u} \)

(3.10b)

and \( \tilde{u}_{\alpha} = \tilde{u}_{\alpha} \) at \( \Gamma_{u} \); and \( u_{\alpha} = u_{\alpha} \) at \( \Gamma_{0} \)

(3.10c)

Now, we choose \( u_{\alpha,0} \) to be the "fundamental solution" in infinite space, of the equation:

\[ N^{(n)}_{\alpha \beta} u_{\alpha,0} + \delta(\chi_{\mu} - \xi_{\mu}) \delta_{\alpha \beta} e_{\theta} = 0 \quad (3.11) \]

where \( \delta(\chi_{\mu} - \xi_{\mu}) \) is the Dirac delta function at \( \chi_{\mu} = \xi_{\mu} \); \( \delta_{\alpha \beta} \) is the Kronecker delta; and \( e_{\theta} \) denotes that the direction of the application of the point load is along the \( x_{\theta} \) direction. The "fundamental solution" of (2.16) will be denoted as \( u_{\alpha,0} \); where \( u_{\alpha,0} = u_{\alpha} \) is the displacement along the \( x_{\alpha} \) direction in a plane infinite body at any point \( \chi_{\mu} \), due to a unit load along the \( x_{\mu} \) direction, applied at the location \( \chi_{\mu} = \xi_{\mu} \). Likewise, \( P_{\alpha,0}(\chi_{\mu}, \xi_{\mu}) \) will be considered to be the traction along the \( x_{\alpha} \) direction on an oriented surface at \( \chi_{\mu} \), with a unit normal along \( x_{\theta} \) due to a unit load along \( x_{\theta} \) at the location \( \xi_{\mu} \). These solutions are well known and may be written as:

\[ u_{\alpha,0}(\chi_{\mu}, \xi_{\mu}) = \frac{1}{8\pi G} [(\nu - 3)\delta_{\alpha \mu} + (1 + \nu) \frac{\partial p}{\partial x_{\mu}} \frac{\partial p}{\partial x_{\alpha}} ] \]

(3.12a)

and
\[ - (1 - v) \left( n_\theta \frac{\partial w}{\partial x_\theta} - n_\alpha \frac{\partial w}{\partial x_\alpha} \right) \quad (3.12b) \]

where \( p = |x_\mu - \xi_\mu| \) is the radius vector from \( x_\mu \) to \( \xi_\mu \);
\( G = E_t/[2(1+v)] \).

Due to the property of integrals involving Dirac functions, we have:
\[ \int \delta(x_\mu - \xi_\mu) \delta(x_\theta - \xi_\theta) u_\alpha(x_\mu) \, d\Omega \equiv - u_\theta(\xi_\mu) \quad (3.12a) \]

Using (3.12 and 3.13) in Eq. (3.10a), we have
\[ \nu_\theta(\xi_\mu) = \int [b_\alpha(x_\mu) - p u_\alpha(x_\mu)] u_\alpha^*(x_\mu, \xi_\mu) \, d\Omega \]

\[ + \int \beta_\alpha(x_\mu) u_\alpha^*(x_\mu, \xi_\mu) \, d\Gamma \]
\[ - \int \tilde{u}_\alpha(x_\mu) P_\alpha(\theta) u_\alpha^*(x_\mu, \xi_\mu) \, d\Gamma - \int C_{\alpha\beta} w(x_\mu) u_\beta^*(x_\mu, \xi_\mu) \, d\Omega \]
\[ - \int \delta(x_\mu - \xi_\mu) \delta(x_\theta - \xi_\theta) u_\alpha(x_\mu) \, d\Omega \equiv - u_\theta(\xi_\mu) \quad (3.13) \]

(3.14)

It can be shown that while the coefficient \( \gamma \) in the left-hand side of (3.14) is unity when \( \xi_\mu \) is in the interior of \( \Omega \), the value of \( \gamma \) is (0.5) when \( \xi_\mu \) falls on the "smooth" boundary \( \Gamma \) [Atluri and Grannell (1978)]. Equation (3.14) is the sought-after integral equation for \( u_\alpha \) in a shallow shell.

We now choose the test function \( w^*(x_\mu) \) to be the "fundamental solution" in an infinite plate corresponding to a unit point load at the location \( \xi_\mu \). Thus, \( w^* \) corresponds to the solution of the linear equation:

\[ w^*(x_\mu, \xi_\mu) = \frac{1}{\delta w} \rho^2 \eta \rho \]

where \( \rho = |x_\mu - \xi_\mu| \).

Using Eqs. (3.16) and (3.13) in Eq. (3.2) and employing repeated integrations by part in the resulting equation, one easily obtains the integral equation:

\[ \gamma_w D w(\xi_\mu) = \int \left[ \phi_\alpha(x_\mu) w^*(x_\mu, \xi_\mu) - \int \phi_\alpha(x_\mu) w^*(x_\mu, \xi_\mu) \, d\Gamma \right. \]
\[ + \int \phi_\alpha(x_\mu) M_n^w(x_\mu, \xi_\mu) \, d\Gamma - \int \phi_\alpha(x_\mu) V_n^w(x_\mu, \xi_\mu) \, d\Gamma \]
\[ \int \left. \frac{N_{n\beta}}{R_{\alpha\beta}} + \frac{K_{n\beta}}{R_{\alpha\beta}} \right] w^*(x_\mu, \xi_\mu) \, d\Omega \]

(3.17)

where

\[ \Phi_n = -D \frac{\partial}{\partial n} (V^2 w) + \frac{\partial}{\partial s} M_n + N_\eta \frac{\partial w}{\partial n} + N_{nt} \frac{\partial w}{\partial s} \]

\[ M_n = M_{11}n_1^2 + 2M_{12}n_1n_2 + M_{22}n_2^2 \]

\[ M_t = (M_{22} - M_{11})n_1n_2 + M_{12}(n_1^2 - n_2^2) \]
In Eq. (3.17) the terms with the superposed symbol "*" should be taken to imply the respective prescribed values, if any, at \( r \); otherwise, they are to be treated as the unknown solution variables. Also, the symbol \( \langle \rangle \) denotes the jump in the quantity \( (\bullet) \) at a corner at \( r \), in the direction of the increasing arc length along \( r \); and the summation \( (1 \text{ to } k) \) extends to all the \( k \) such corners.

Using Eq. (2.10) and the Divergence theorem, it is easy to see that:

\[
\int_{\Omega} \mathbf{a} \mathbf{b}(x) \mathbf{w}^\bullet(x, \xi, \mu) \, d\Omega = \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{w}^\bullet(x, \xi, \mu) \, d\Gamma
\]

\[
- \int_{\Omega} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{w}^\bullet(x, \xi, \mu) \, d\Omega
\]

Use of (3.18) in (3.17) results in the final integral equation for \( w \) as follows:

\[
\gamma_\nu \mathbf{D} \mathbf{w}^\bullet(\xi, \mu) = \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma - \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
+ \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma - \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
- \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{n} \mathbf{w} + \mathbf{n} \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma + \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{w}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
- \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{w}^\bullet(\xi, \mu) \, d\Gamma + \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{w}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
+ \int_{\Gamma} \mathbf{b}_3 - \rho \mathbf{w} + \mathbf{f}_3 + (\mathbf{N}_\alpha \mathbf{w}_\beta) \mathbf{w} + (\mathbf{N}_\alpha \mathbf{w}_\beta) \mathbf{w} + \mathbf{w} \, d\Gamma
\]

Since \( (\partial \mathbf{w}/\partial \mathbf{n}) \) is also an independent variable at \( \Gamma \), an integral relation for \( (\partial \mathbf{w}/\partial \mathbf{n}) \) should also be derived. Towards this purpose, consider a second fundamental solution.

Using Eq. (2.10) and the Divergence theorem, it is easy to see that:

\[
\gamma_\nu \mathbf{D} \mathbf{w}^\bullet(\xi, \mu) = \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma - \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
+ \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma - \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
- \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{n} \mathbf{w} + \mathbf{n} \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma + \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{w}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
- \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{w}^\bullet(\xi, \mu) \, d\Gamma + \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{w}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
+ \int_{\Gamma} \mathbf{b}_3 - \rho \mathbf{w} + \mathbf{f}_3 + (\mathbf{N}_\alpha \mathbf{w}_\beta) \mathbf{w} + (\mathbf{N}_\alpha \mathbf{w}_\beta) \mathbf{w} + \mathbf{w} \, d\Gamma
\]

\[
\int_{1} \left[ \langle \mathbf{K}_\nu \mathbf{w}^\bullet - \langle \mathbf{K}_\nu^* \mathbf{w}^\bullet \rangle \right]
\]

where \( \phi \) is the angle between the outward normal to \( \Gamma \) and the radius \( p \) [see Stern (1979)]. The resulting integral equation is:

\[
\gamma_\nu \mathbf{D} \mathbf{w}^\bullet(\xi, \mu) = \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma - \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
+ \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma - \int_{\Gamma} \mathbf{v}^\bullet(\xi, \mu) \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
- \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{n} \mathbf{w} + \mathbf{n} \mathbf{w} + \mathbf{v}^\bullet(\xi, \mu) \, d\Gamma + \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{w}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
- \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{w}^\bullet(\xi, \mu) \, d\Gamma + \int_{\Gamma} \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{w}^\bullet(\xi, \mu) \, d\Gamma
\]

\[
+ \int_{\Gamma} \mathbf{b}_3 - \rho \mathbf{w} + \mathbf{f}_3 + (\mathbf{N}_\alpha \mathbf{w}_\beta) \mathbf{w} + (\mathbf{N}_\alpha \mathbf{w}_\beta) \mathbf{w} + \mathbf{w} \, d\Gamma
\]

\[
\int_{1} \left[ \langle \mathbf{K}_\nu \mathbf{w}^\bullet - \langle \mathbf{K}_\nu^* \mathbf{w}^\bullet \rangle \right]
\]

Remarks

In summary, Eqs. (3.14), (3.19), and (3.20) represent the complete set of integral equations for \( u \), \( w \), and \( \partial \mathbf{w}/\partial \mathbf{n} \). An examination of these equations reveals the following features:
Aw \equiv w_{k+1} - w_k \quad (4.1)

(ii) Again, due to the curvature-induced coupling of the in-plane and out-of-plane displacements in the presently considered shallow shell, the integral equations for \( w \) and \( \partial w / \partial n \), Eqs. (3.19) and (3.20) respectively, contain domain-integrals (over \( \Omega \)) involving trial functions for both \( w \) and \( u \). Also, in a transient dynamic analysis, the term \( (w) \) appears inside a domain-integral.

(iii) In the nonlinear problem, the nonlinear terms \( N(n) \) and \( \left( N \ast \phi \ast , w \ast \phi \ast \right) \), involving trial functions for both \( w \) and \( u \) inevitably bring the domain-integrals (over \( \Omega \)) into the equations.

(iv) For reasons (i) to (iii) above, unlike the classical homogeneous isotropic elasto-statics (Atluri and Grannell (1978)) wherein a discretization of the relevant integral equations requires a use of basis functions for the displacements at the boundary alone, the present shallow-shell formulation requires the assumption of basis functions for the trial solutions \( u \) and \( w \) at the boundary \( \Gamma \) as well as in the interior \( \Omega \). Thus, the present solution methodology may, strictly speaking, be classified as a hybrid boundary-element/interior (finite) element method based on a direct discretization of integral equations.

(v) Suppose now that in Eqs. (3.14), (3.19) and (3.20), we let \( \xi \) tend to a point on the boundary, i.e., \( \xi \in \Gamma \). Thus, we obtain three integral relations for the boundary values of \( u \), \( w \), and \( \partial w / \partial n \). An examination of these relations reveals that, in order to discretize these equations, one needs to assume only very simple trial functions \( u \), \( w \), \( \partial w / \partial n \) not only at the boundary but also in the interior of \( \Omega \). For instance, \( \Omega \) may be discretized into a number of finite elements and \( \Gamma \) into a number of boundary elements. As \( \xi \) tends to \( \Gamma \), the integral relations (3.14), (3.19), and (3.20) clearly show that \( w \) and \( u \) need only be piecewise constant functions in each finite element in \( \Omega \), in any discretization process. Umost, one may need only consider \( C^0 \) continuous functions for \( w \) and \( u \) in each element (to extend the formulations to finite formation cases, see Zhang and Atluri (1986)). In contrast, it is recalled that in the Galerkin finite element method, \( u \) need be \( C^2 \) continuous and \( w \) be \( C^1 \) continuous in each element.

(vi) At each point on the boundary, two of the in-plane variables \( u(\alpha = 1, 2) \), \( P(\alpha = 1, 2) \) are specified; and the other two are unknown. Likewise, two of the out-of-plane variables, \( V \), \( M \), \( \psi \), and \( w \) are specified; and the other two are unknown. At each point in \( \Omega \), as seen from Eqs. (3.14), (3.19), and (3.20), the three displacements, \( u \) and \( w \) are unknown. Thus, if Eqs. (3.14), (3.19) and (3.20) are discretized, through finite as well as boundary elements and the nodal values of the variables are appropriately understood to be either specified or unknown, one would easily obtain exactly as many equations as the number of unknowns, so that the problem may be considered as well posed.

Note that due to the appearance of the nonlinear terms in the obtained integral equations (3.14), (3.19) and (3.20), the discretized interior/boundary-element equations can not be solved directly, that is, an incremental approach is necessary. This will be discussed in detail in Section 4.

4. INCREMENTAL APPROACH AND SOLUTION STRATEGY

In the incremental approach, the load and the prescribed boundary conditions are applied in small but finite increments and, in a time dependent problem, these increments correspond to the time steps.

Consider that the shell is at the end of the \( K \)th load increment. All the quantities which have been known during the previous \( K \) steps of analysis are denoted by a superscript \( "K" \). The displacement increments are denoted as

\[
\Delta u_{\alpha} = u_{\alpha}^{k+1} - u_{\alpha}^k \\
\Delta w = w^{k+1} - w^k \\
\]

(4.1)

With these notations the integral equations in incremental form may be written as:

\[
\gamma(u_{\Theta}^k + \Delta u_{\Theta}) + \int [b_{\alpha}^k + \Delta b_{\alpha} - \rho(\bar{u}_{\alpha} + \Delta \bar{u}_{\alpha})_u^*_{\Theta} \, d\Omega \\
+ \int (\hat{\beta}_{\alpha} + \Delta \hat{\beta}_{\alpha} u^*_{\Theta}) \, d\Gamma - \int (\bar{u}_{\alpha} + \Delta \bar{u}_{\alpha})_P(\Theta) \, d\Gamma - \int C_{\alpha \beta} (\bar{w}_{\alpha} + \Delta \bar{w})_U^*_{\Theta} \, d\Omega \\
- \int (N_{\alpha \beta}^k \Delta N_{\alpha \beta}^{(n)} + \text{higher order terms}) u_{\Theta \alpha \beta}^* \, d\Omega \quad (4.2a) \\
\]

\(\Theta\in \Gamma\)
Here, the incremental form of Eq.(3.20) is similar to Eq.(4.26), and its treatment follows the same routine.

In Eq.(4.2a,b), the higher order terms involve the products of the incremental displacements. In solving these unknown incremental displacements, those higher order terms are ignored. In those incremental equations, the terms with the superscript "K" should have satisfied the equilibrium conditions at the end of the Kth load increment, but, the equilibrium conditions are in fact not exactly satisfied because of the absence of the higher order terms, therefore, those terms are written in the incremental equations as the equilibrium constraints.

Note that in Eq.(4.2b), the nonlinear term \((N_{k}^{'k} + \Delta N_{a}^{'k}) (w^{k} + \Delta w)\) can be written as:

\[
(N_{k}^{'k} + \Delta N_{a}^{'k}) (w^{k} + \Delta w)_{\beta} = N_{k}^{\beta} w^{k}_{\beta} + N_{a}^{\beta} \Delta w^{k}_{\beta} + \Delta N_{a}^{\beta} w^{k}_{\beta} + \text{higher order terms}
\]

Ignoring the higher order terms and examining (4.3), (4.4), we may see that those nonlinear terms are linearized with respect to the displacement increments. Using (4.3), (4.4) in Eq. (4.2a, b) and applying the divergence theorem, we may obtain the final integral equations in terms of unknown displacement increments:

\[
\gamma(u^{k}_{\theta}, \Delta u_{\theta}) = \int_{\Omega} [b_{\alpha} + \Delta b_{\alpha} - \rho(\dddot{u}^{k}_{\alpha} \Delta u_{\alpha})] u^{*}_{\alpha} d\Omega
\]

\[
+ \int_{\Gamma} (P_{\alpha}^{\beta} \Delta \beta_{\alpha}) u^{*}_{\alpha} d\Gamma - \int_{\Gamma} (\dddot{u}_{\alpha} \Delta \beta_{\alpha}) p^{*}_{\alpha} d\Gamma
\]

\[
- \int_{\Omega} C_{\alpha} \beta_{\alpha} (w^{k} + \Delta w) u^{*}_{\alpha} d\Omega + \int_{\Gamma} N_{(\theta)\alpha\beta} w^{k}_{\alpha\beta} d\Omega
\]

\[
- \int_{\Omega} N_{(\theta)\alpha\beta} w^{k}_{\alpha\beta} d\Omega + w^{k}_{\theta|\mu}
\]
\[ + \int (\vec{\phi}_n^k + \Delta \vec{\phi}_n^k) \vec{w}_n^k \, d\Gamma - \int (\vec{v}_n^k + \Delta \vec{v}_n^k) \vec{v}_n^k \, d\Gamma \]

\[ - \int [C_{\alpha\beta} \vec{w}_\alpha^k \vec{w}_\beta^k + A_{\alpha}] \Delta \vec{u}_\alpha \, d\Omega + \int [C(\vec{\omega}_\alpha^k)]_{\beta} + B_{\alpha}] \Delta \vec{u}_\alpha \, d\Omega \]

\[ - \int [(C_{\alpha\beta} \vec{w}_\alpha^k + N_{\alpha\beta} \vec{w}_\alpha^k) n_\beta + A_w] \Delta \vec{w} \, d\Omega \]

\[ + \int [C(\vec{\omega}_\alpha^k \vec{w}_\alpha^k) \vec{w}_\beta^k - C_{\alpha\beta} \vec{w}_\alpha^k \vec{w}_\beta^k - B_w] \Delta \vec{w} \, d\Omega \]

\[ + \int [(N_{\alpha\beta} \vec{w}_\alpha^k \vec{w}_\beta^k) - \frac{N_{\alpha\beta}}{R_{\alpha\beta}} - C_{\alpha\beta} \vec{w}_\alpha^k \vec{w}_\beta^k + B_3 + \Delta \vec{b}_3] \]

\[ - \rho(\vec{w}_n^k + \Delta \vec{w}) + \psi_3 + \Delta \psi_3 \vec{w} \, d\Omega \]

\[ + \frac{1}{2} \left< \vec{w}_t^k \vec{\omega}_t^k \vec{w}_t^k \vec{\omega}_t^k \right> \left\{ \left( \vec{w}_t^k + \Delta \vec{w}_t \right) \right\} \] (4.5b)

where, A, B's may be given as follows:

\[ A_1 = C[(w^k_1 w'^*_1 + w^k_2 w'^*_2)n_1 + \frac{1-v}{2}(w^k_1 w'^*_1 + w^k_2 w'^*_2)n_2] \] (4.6a)

\[ B_1 = C[(w^k_1 w'^*_1 + w^k_2 w'^*_2)n_1 + \frac{1-v}{2}(w^k_1 w'^*_1 + w^k_2 w'^*_2)n_2] \] (4.6b)

\[ A_w = C[(w^k_1)^2 w'^*_1 n_1 + (w^k_2)^2 w'^*_2 n_2 + \psi_3 (w^k_1 w'^*_1 n_2 + w^k_2 w'^*_2 n_1)] \]

\[ B_w = C[(w^k_1 w'^*_1 n_1 + w^k_2 w'^*_2 n_2) + (w^k_1 w'^*_1 n_2 + w^k_2 w'^*_2 n_1)] \]

\[ + \frac{1-v}{2}(w^k_1 w'^*_1 n_1 + w^k_2 w'^*_2 n_2 + 2w^k_1 w'^*_1 n_1) + \frac{1-v}{2}(w^k_1 w'^*_1 n_2 + w^k_2 w'^*_2 n_2) \]

\[ + \frac{3-v}{2}(w^k_1 w'^*_1 n_1 + w^k_2 w'^*_2 n_2 + w^k_1 w'^*_1 n_2)] \] (4.6c)

We discretize the domain (\(\Omega\)) as well as the boundary \(\Gamma\) by using some appropriate interpolation functions for those unknown displacement increments. By carrying out the indicated integrations, we may obtain the boundary/interior element equations with the displacement increments as unknowns. For each load increment, these equations have to be solved iteratively because the equilibrium conditions are only approximately satisfied due to the absence of the higher order terms.

The full Newton-Raphson algorithm is used to obtain the solution. This involves domain integrations for constructing the coefficient matrix, and reduction of the matrix in each iteration. Since, in the present boundary/interior element method, the coefficient matrix is not as large as in the usual finite element approach, the reduction of this matrix is not as critical, and moreover, it is found that the solution converges very rapidly.

In the present numerical implementation, at the beginning of each load increment, the equilibrium is checked first by subtracting the integrals of the other terms with the superscript K from the total load integration; the residual load vector is used to solve the displacement increments; the obtained displacement increments are then used to update the coefficient matrix and check the equilibrium again and so on.

For the dynamic problem, the Newmark method is used to carry out the transient response analysis.
In the numerical examples, the basis functions for each of the trial functions are assumed as follows: (i) over each boundary element at $\Gamma$, the boundary variables are interpolated linearly; (ii) over each interior finite element, displacement components are interpolated bilinearly, that is, these functions are $C^0$ continuous at the element boundaries. The integrations over each boundary element are carried out numerically by the fifth order Gaussian quadrature, and the integrations over the domain element by $2 \times 2$ Gaussian quadrature.

Numerical results are obtained by applying the method to a circular spherical shallow shell (fig. 1) with the hinged edge. Under an axisymmetric load or initial condition, the shell undergoes an axisymmetric deformation so that only a quarter of the shell is examined. The mesh for the interior/boundary element discretization is shown in fig. 2 with the total number of degrees of freedom equal to 51, including the displacement, components and unknown boundary forces. Note that in fig. 2, the curved boundary of the circular shell is approximated by piecewise straight lines. In our numerical tests, we found that the boundary integrations should be carried out along the curved boundary; otherwise, the error would become significant.

Two quasi-static loading cases are examined. The linear solutions by current approach and the discussion about their accuracy may be found in Zhang and Atluri (1985). The nonlinear results for a concentrated load $P$ applied at the crown of the shell are shown in fig. 3. Figure 4 shows the result for the uniformly distributed load case. In these solutions, the load increment is reduced to 50% after the 5th loading step in order to avoid bad convergence in the iteration as the deformation is approaching the point where snap through may happen. From these figures, we may see that as deformation increases, the nonlinear effect due to the variation of the curvature becomes more and more significant.

Figure 5 shows the result of transient response of the shell with the following initial condition: at $t=0$, the crown of the shell has an upward velocity $v=3.0$ which corresponds to an upward point impulse applied at the crown of the shell. Figure 6 shows the transient response of the shell with the following initial condition: at $t=0$, the velocity distribution is $v=0.8\cos(\frac{\pi}{10}r)$, where $r$ is the radial distance from the center. In the dynamic problem, the density of the shell material is taken as $\rho=0.8$, and the time step size $\Delta t=0.01\sec$.

An efficient "interior/boundary element" method, based on an integral equation formulation for static and transient response analysis of shallow shells is presented. Due to the displacement coupling and the nonlinearity of the shell problem. The interior unknown must appear in the finally obtained equations, so that this method is named as interior/boundary element approach. By using the fundamental solution on infinite space as the test function, one may relax the requirement for the trial functions, as compared with usual Galerkin finite element method. The present method yields results of acceptable accuracy, with a discrete model of a much small number of degrees of freedom.

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References:


Fig. 3. The Crown Deflection due to a Concentrated Load $P$ Applied at the Crown of the Shell.

Fig. 4. The Crown Deflection due to Uniformly Distributed Load $q$

Fig. 5. The Time Variation of Crown Deflection due to a Point Impulse

Fig. 6. Time Variation of Crown Deflection for the Initial Velocity Distribution $v_0 = 0.8 \cos \left( \frac{t \pi}{10} \right)$
AN EXPLICIT EXPRESSION FOR THE TANGENT-STIFFNESS OF A FINITELY DEFORMED 3-D BEAM AND ITS USE IN THE ANALYSIS OF SPACE FRAMES

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Abstract—Simplified procedures for finite-deformation analyses of space frames, using one beam element to model each member of the frame, are presented. Each element can undergo three-dimensional, arbitrarily large, rigid motions as well as moderately large non-rigid rotations. Each element can withstand three moments and three forces. The nonlinear bending-stretching coupling in each element is accounted for. By obtaining exact solutions to the appropriate governing differential equations, an explicit expression for the tangent-stiffness matrix of each element, valid at any stage during a wide range of finite deformations, is derived. An arc length method is used to incrementally compute the large deformation behavior of space frames. Several examples which illustrate the efficiency and simplicity of the developed procedures are presented. While the finitely deformed frame is assumed to remain elastic in the present paper, a plastic hinge method, wherein a hinge is assumed to form at an arbitrary location in the element, is presented in a companion paper.

1. INTRODUCTION

There is a renewed interest in efficient and simple analyses of three-dimensional frame structures due to their increasing viability for use as both offshore structures as well as multipurpose structures in outer space. There are plans for deploying very large structures in outer space, for a variety of reasons, such as antennae, radio telescopes, etc. While the offshore structures are in general massive, the large space structures (LSS) are necessarily of low mass and very high flexibility. A technological problem in the operation of the LSS is the need for active or passive control of transient dynamic (traveling wave type) response. Since the LSS are high flexible, large deformation behavior needs to be considered. The transient dynamic response of LSS, modeled as space frames, may be written as

\[ M\ddot{q} + D(q, \dot{q}) + S(q) = f + Q_x, \]  

where \( M \) is the mass matrix; \( D \) is the vector of nonlinear structural (or other passive) damping which may depend nonlinearly on the velocity \( \dot{q} \) as well as displacement \( q \) (depending on the joint design); \( S \) is the vector of nodal restraining forces which, for large deformations, depend nonlinearly on the nodal displacements \( q \); \( f \) is the vector of control forces to be determined from a properly formulated active control algorithm; and \( Q_x \) is the vector of externally applied dynamic forces; and \( \ddot{q} \) is the acceleration vector. Assuming that the control forces \( f \) are determined from the control algorithm (which is a complicated problem and the object of a wide body of research in its own right), eqn (1.1) is a nonlinear initial value problem to be integrated by time-stepping algorithms. In such procedures, it is customary to write the displacement vector, \( q_{N+1} \) at time \( t_{N+1} \), as \( q_{N+1} = q_N + \Delta q. \) Thus, the internal restraining nodal-force vector \( S_{N+1} \) is often written as

\[ S_{N+1} = S(q_{N+1}) = (\varepsilon^N)K\Delta q + (\varepsilon^N)R, \]  

where \( (\varepsilon)K \) is the "tangent-stiffness matrix" at state \( t_N \) (accounting for geometric and material nonlinearities), and \( (\varepsilon)R \) are the internal restraining forces at \( t_N \).

In the usual finite element analysis, much effort is usually expended in evaluating \( (\varepsilon)K. \) To account for large deformations and material nonlinearities, the usual procedures for analyzing space frames involve:

(i) the use of several finite elements to model each member of the space frame;
(ii) the assumption of polynomial basis functions for each component of displacement/rotation of each element; and
(iii) the numerical (quadrature) integration, over each element, of appropriate strain energy terms.

One of the aims of the present paper is to present an explicit expression for \( (\varepsilon)K \) of a three-dimensional beam element undergoing arbitrarily large rigid motion and moderately large non-rigid rotations. It is sufficient to model each member of the space frame by a single beam element of the aforementioned type. The joint design of the LSS is assumed to be such that each beam element can carry three bending moments.
and three forces (axial and shear). The nonlinear bending–stretching coupling (and axial shortening of each beam element due to large rotations) in each beam element is accounted for. Under these conditions, an explicit expression for $K$ is derived, without the use of assumed polynomial basis functions for element deformation, and without the use of element-wise numerical quadrature. Analytical solutions for the appropriate differential equations are derived and used to derive explicit expressions for the stiffness coefficients. The present development for three-dimensional frame elements is an extension of that presented earlier for plane frames by Kondoh and Atluri [1].

The present paper is limited to a geometrically nonlinear quasistatic analysis of space frames. An arc-length method is used to generate the finite-deformation response solution. Several examples are given to illustrate the efficiency of the present approach. Simplified analyses accounting for material nonlinearities through a plastic-hinge method, wherein the hinge may form at an arbitrary location along the member, are being presented in a companion paper.

The organization of the remainder of the paper is as follows. In Section 2.1, the kinematics of three-dimensional deformation of a beam element is considered. The deformation includes arbitrarily large rotations, which are characterized by finite-rotation vectors [2–4]. The governing differential equation for a three-dimensional beam undergoing large displacements and rotations are treated in 2.2. By assuming that the relative or non-rigid rotations are only moderate, those differential equations are simplified and are of the beam-column type. The axial-stretch of the beam depends on the integral over length of the squares of relative rotations. The simplified differential equations are then solved exactly, and analytical relations are derived between the "axial stretch and relative rotations" on the one hand, and the "axial force and bending moments" on the other. Using the formalism of a mixed-variational method, a closed-form (explicit) expression for the $(12 \times 12)$ tangent stiffness matrix is derived in Section 2.3. The solution strategy is briefly discussed in Section 3; numerical examples are treated in Section 4. Section 5 gives some concluding remarks. Appendices and attendant tables list the explicit expressions for the coefficients of the present three-dimensional beam tangent stiffness matrix, so that they may be directly implemented by researchers and code developers.

2. DERIVATION OF AN EXPLICIT TANGENT STIFFNESS MATRIX FOR FINITE-DEFORMATION POST-BUCKLING ANALYSIS OF SPACE FRAMES

The frame-type structures discussed herein are assumed to remain elastic, and only a conservative system of concentrated loads are assumed to act at the nodes of the frame.

2.1. Three-dimensional kinematics of deformation member of a space-frame

Consider a typical frame member, modeled here as a three-dimensional beam element, that spans between nodes 1 and 2 as shown in Fig. 1. The element is considered to have a uniform cross-section and to be of length $l$ before deformation. The co-ordinates $x_j$ are the local co-ordinates at the node $j$ ($j = 1, 2$) of an undeformed element. Likewise, $u_j (j = 1, 2, 3)$ denote the displacements at the centroidal axis of the element along the coordinate directions $x_i$, $i = 1, 2, 3$ respectively. Also, as shown in Fig. 1, $\theta_j$ are the angles of rotation about the axes of $x_i$. After
between the tangent and the normal components: on the one hand, the tangent components introduce to represent the rigid and relative (non-rigid) rotations of the element. One is the semi-tangential system $x_i$ which is locally "tangential" and "normal" to the deformed centroidal axis; another is $\hat{x}$ which characterizes the rigid translations and rotations of the member (see Fig. 1).

Considering each rotation as a semi-tangential rotation, we can treat rotations as vectors [2-4]. Thus, the relation among the total, rigid and relative rotation vectors is given by

$$\gamma = \beta + \alpha \quad (i = 1, 2), \tag{2.1}$$

where $\gamma$ is the total rotation vector at the node $i$, $\beta$ the vector of rigid rotation of the beam as a whole, and $\alpha$ is the relative rotation vector at the node $i$. Using eqn (2.1), the total rotation vector at the node 2, $\gamma_2$, is represented as

$$\gamma_2 = \beta + \alpha + \alpha' \quad (i = 1, 2), \tag{2.2}$$

where

$$\alpha' = \gamma - \gamma. \tag{2.3}$$

Therefore, the relative rotation vector at the node 2 can be defined using eqns (2.1) and (2.2) as

$$\alpha = \alpha + \gamma. \tag{2.4}$$

On the other hand, the expressions of the rotation vectors may be written, by using their components in any co-ordinate system, as follows [2-4]. Using the local co-ordinate system, the total rotation vector at the node $i$ may be written [2-4] as

$$\gamma = \tan \frac{\theta_i}{2} \hat{e}_i, \quad (i = 1, 2), \tag{2.5}$$

$$\gamma_j = \tan \frac{\theta_j}{2} \hat{e}_j, \quad (j = 1, 2, 3). \tag{2.6}$$

The relative rotation vector at the node $i$ in the co-ordinate system $\hat{x}$, is given by

$$\alpha = \tan \frac{\theta_i}{2} \hat{e}_i, \quad (i = 1, 2), \tag{2.7}$$

$$\alpha_j = \tan \frac{\theta_j}{2} \hat{e}_j, \quad (j = 1, 2, 3). \tag{2.8}$$

Substituting eqn (2.5) into eqn (2.3), the difference between the rotation vectors at nodes 1 and 2 is given by

$$\alpha = \left( \tan \frac{\theta_i}{2} - \tan \frac{\theta_2}{2} \right) \hat{e}_i \tag{2.9}$$

$$= \left( \alpha - \hat{e}_i \right) \cdot \hat{e}_i \equiv \tan \frac{\theta_i}{2} \hat{e}_i. \tag{2.10}$$

Also substituting eqns (2.6) and (2.8) into eqn (2.4), the relative rotation at node 2 is represented as

$$\alpha = \tan \frac{\theta_i}{2} \hat{e}_i \tag{2.11}$$

Furthermore, the action of a rotation $R$, which transforms a vector $dX$ to $dX^R$, is represented by the relation [2-4]

$$dX^R \equiv \frac{1}{1 + R \cdot \hat{e}} \left[ (1 - R \cdot \hat{e}) \cdot dX \right. \right. \left. + 2(R \cdot dX) \cdot R + 2R \cdot dX \right]. \tag{2.12}$$

Using (2.11) and considering the action of the total rotation $\gamma$ on the unit vectors $e_i$, one obtains the following equations:

$$e_i = T_{\gamma} e_i, \quad (i = 1, 2), \tag{2.13}$$

where $e_i$ are the vectors $e_i$ at node $i$, and

$$T_{\gamma} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2.14}$$

With $A = \frac{1}{1 + \theta_i^2} \left[ 1 + \tan^2 \left( \frac{\theta_i}{2} \right) - \tan \frac{\theta_i}{2} \right] - \tan \frac{\theta_i}{2}, \tag{2.14a}$$

$B = \frac{2}{1 + \theta_i^2} \left[ \tan \frac{\theta_i}{2} \tan \frac{\theta_i}{2} + \tan \frac{\theta_i}{2} \right], \tag{2.14b}$

$C = \frac{2}{1 + \theta_i^2} \left[ \tan \frac{\theta_i}{2} \tan \frac{\theta_i}{2} - \tan \frac{\theta_i}{2} \right], \tag{2.14c}$

$D = \frac{2}{1 + \theta_i^2} \left[ \tan \frac{\theta_i}{2} \tan \frac{\theta_i}{2} - \tan \frac{\theta_i}{2} \right], \tag{2.14d}$

$E = \frac{1}{1 + \theta_i^2} \left[ 1 - \tan^2 \left( \frac{\theta_i}{2} \right) + \tan \frac{\theta_i}{2} \right] - \tan \frac{\theta_i}{2}, \tag{2.14e}$

$F = \frac{2}{1 + \theta_i^2} \left[ \tan \frac{\theta_i}{2} \tan \frac{\theta_i}{2} + \tan \frac{\theta_i}{2} \right], \tag{2.14f}$

$G = \frac{2}{1 + \theta_i^2} \left[ \tan \frac{\theta_i}{2} \tan \frac{\theta_i}{2} + \tan \frac{\theta_i}{2} \right], \tag{2.14g}$

$H = \frac{2}{1 + \theta_i^2} \left[ \tan \frac{\theta_i}{2} \tan \frac{\theta_i}{2} - \tan \frac{\theta_i}{2} \right]. \tag{2.14h}$
where
\[ \theta^2 = \tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_2}{2}\right) + \tan^2\left(\frac{\theta_3}{2}\right). \]  
(2.15)

On the other hand, \( \hat{e}_3 \), as a unit vector in the direction of the line joining node 1 to 2 in the deformed configuration, may be represented as
\[ \hat{e}_3 = r \cdot e_1 + s \cdot e_2 + t \cdot e_3, \]  
(2.16)

where
\[ r = \frac{\hat{u}_1}{I}, \quad s = \frac{\hat{u}_2}{I}, \quad t = \frac{I + \hat{u}_3}{I}, \]  
(2.17a, b)

and
\[ \hat{u}_i = u_i - I_i. \]  
(2.19)

Other unit vectors, \( \hat{e}_1, \hat{e}_2 \), corresponding to the coordinate system, \( \hat{x}_i \), may be written, using eqn (2.11) and the rotation vector, \( 'w \), at node 1, shown in Fig. 2, as
\[ \hat{e}_i = \frac{1}{1 + 'w \cdot 'e_i} \left[(1 - 'w \cdot 'e_i) \cdot 'e_i + 2('w \cdot 'e_i) \cdot 'w + 2('w \times 'e_i)\right], \quad (i = 1, 2), \]  
(2.20)

where
\[ 'w = \tan\frac{w}{2} \cdot \frac{('e_i \times \hat{e}_i)}{|('e_i \times \hat{e}_i)|} \]  
(2.21)

Substituting eqns (2.12)-(2.15) and eqns (2.21) (2.24) into eqn (2.20), the following equations are obtained:
\[ \hat{e}_1 = o \cdot e_1 + p \cdot e_2 + q \cdot e_3, \]  
(2.22)

\[ \hat{e}_2 = u \cdot e_1 + v \cdot e_2 + w \cdot e_3, \]  
(2.23)

where
\[ o = [C, A + 2h \cdot C_3 + 2(l \cdot C - m - B)]/C_3 \]  
(2.24)

\[ p = [C_1 \cdot B + 2l \cdot C_2 + 2(m \cdot A - m - C)]/C_3 \]  
(2.25)

\[ q = [C_1 \cdot C + 2m \cdot C_2 + 2(h \cdot B - l \cdot A)]/C_3 \]  
(2.26)

\[ u = [C_1 \cdot D + 2h \cdot C_4 + 2(l \cdot F - m \cdot E)]/C_3 \]  
(2.27)

\[ v = [C_1 \cdot E + 2l \cdot C_4 + 2(m \cdot D - h \cdot F)]/C_3 \]  
(2.28)

\[ w = [C_1 \cdot F + 2m \cdot C_4 + 2(h \cdot E - l \cdot D)]/C_3 \]  
(2.29)

We denote by \( \zeta \) the relative rotation at node 1. Thus, \( \zeta \) characterizes the transformation of the coordinate system \( \hat{x}_i \) to \( x_i \) at node 1. From eqn (2.22) one obtains
\[ \zeta = -'w = -(h \cdot e_1 + l \cdot e_2 + m \cdot e_3). \]  
(2.29)
Also, using eqns (2.16), (2.25) and (2.26),

\[
\frac{1}{2} \Delta \theta = \left(\frac{1}{2} \Delta \theta_0 + \frac{1}{2} \Delta \theta_1\right) \cdot \mathbf{v}
\]

(2.30)

Therefore, the components of the relative rotation at node 1, i.e. \( \Delta \theta \), are obtained from eqns (2.6), (2.16), (2.25) and (2.29) to (2.30), as

\[
\theta_0 = \left(\frac{1}{2} \Delta \theta - \frac{1}{2} \Delta \theta_0\right) \cdot \mathbf{v}
\]

(2.31a)

\[
\frac{1}{2} \Delta \theta - \frac{1}{2} \Delta \theta_0 = -\left(h \cdot r + l \cdot s + m \cdot t\right)
\]

(2.31b)

\[
\frac{1}{2} \Delta \theta - \frac{1}{2} \Delta \theta_0 = -\left(h \cdot u + l \cdot v + m \cdot w\right)
\]

(2.31c)

Also, the components of the relative rotation at node 2, i.e. \( \Delta \theta \), are obtained from eqns (2.7)-(2.10), (2.16), (2.25) and (2.29), as

\[
\frac{1}{2} \Delta \theta = \frac{1}{2} \Delta \theta_0 + \left(\frac{1}{2} \Delta \theta_1 - \frac{1}{2} \Delta \theta_0\right) \cdot \mathbf{v}
\]

(2.32a)

\[
\frac{1}{2} \Delta \theta = \frac{1}{2} \Delta \theta_0 + \left(\frac{1}{2} \Delta \theta_1 - \frac{1}{2} \Delta \theta_0\right) \cdot \mathbf{v}
\]

(2.32b)

It should be noted that the component \( \Delta \theta \) of the relative rotation at node 1 is zero due to the rotation \( \Delta \theta_0 \) being as in eqn (2.21).

Finally, the relation between the total axial stretch and displacements of the member is

\[
\delta = \left[\Delta \mathbf{u} + \Delta \mathbf{u}_1 + (l + \Delta l) \Delta \mathbf{u}_2\right] - l
\]

(2.33)

where \( \delta \) is the total axial stretch, and

\[
\Delta \mathbf{u}_i = \mathbf{u}_i - \mathbf{u}_0 \quad (i = 1, 2 \text{ and } 3)
\]
all of the rotations are semitangential rotations [2–4] and \( \theta_1 \) at node 1 is zero. Using the relative rotations, \( \theta_1, \theta_2 \) and \( \theta_3 \), the relation between unit vectors \( \mathbf{e}_1^* \) and \( \mathbf{e}_2^* \) at any point along the beam is written, using eqn (2.11), as

\[ e_1^* = S_{ij} e_j \]  

(2.34)

where

\[ S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \]  

(2.35)

\[ S_{11} = \frac{1}{1 + \beta_1^2} \left[ 1 + \tan^2 \left( \frac{\theta_1}{2} \right) - \tan^2 \left( \frac{\theta_2}{2} \right) \right] - \tan^2 \left( \frac{\theta_3}{2} \right) \]  

(2.36a)

\[ S_{12} = \frac{2}{1 + \beta_1^2} \left[ \tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} + \tan \frac{\theta_3}{2} \right] \]  

(2.36b)

\[ S_{13} = \frac{2}{1 + \beta_1^2} \left[ \tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} - \tan \frac{\theta_3}{2} \right] \]  

(2.36c)

\[ S_{21} = \frac{2}{1 + \beta_1^2} \left[ \tan \frac{\theta_2}{2} \tan \frac{\theta_1}{2} - \tan \frac{\theta_3}{2} \right] \]  

(2.36d)

\[ S_{22} = \frac{1}{1 + \beta_1^2} \left[ 1 - \tan^2 \left( \frac{\theta_1}{2} \right) + \tan^2 \left( \frac{\theta_2}{2} \right) \right] - \tan^2 \left( \frac{\theta_3}{2} \right) \]  

(2.36e)

\[ S_{23} = \frac{2}{1 + \beta_1^2} \left[ \tan \frac{\theta_2}{2} \tan \frac{\theta_1}{2} + \tan \frac{\theta_3}{2} \right] \]  

(2.36f)

\[ S_{31} = \frac{2}{1 + \beta_1^2} \left[ \tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} + \tan \frac{\theta_3}{2} \right] \]  

(2.36g)

\[ S_{32} = \frac{2}{1 + \beta_1^2} \left[ \tan \frac{\theta_2}{2} \tan \frac{\theta_3}{2} - \tan \frac{\theta_1}{2} \right] \]  

(2.36h)

\[ S_{33} = \frac{1}{1 + \beta_1^2} \left[ 1 - \tan^2 \left( \frac{\theta_1}{2} \right) - \tan^2 \left( \frac{\theta_2}{2} \right) \right] + \tan^2 \left( \frac{\theta_3}{2} \right) \]  

(2.36i)

\[ \theta^2 = \tan^2 \left( \frac{\theta_1}{2} \right) + \tan^2 \left( \frac{\theta_2}{2} \right) + \tan^2 \left( \frac{\theta_3}{2} \right) \]  

(2.37)

The curvatures along the centroidal axis of a deformed member are given by

\[ K_i^* = \frac{d e_i^*}{d x_i^*} \]  

(2.38a)

Substituting eqns (2.34) and (2.35) into eqn (2.38a) following equations are obtained:

\[ K_1^* = \frac{d S_{11}}{d x_1^*} + \frac{d S_{12}}{d x_1^*} + \frac{d S_{13}}{d x_1^*} \]  

(2.38b)

\[ K_2^* = \frac{d S_{21}}{d x_2^*} + \frac{d S_{22}}{d x_2^*} + \frac{d S_{23}}{d x_2^*} \]  

(2.38c)

\[ K_3^* = \frac{d S_{31}}{d x_3^*} + \frac{d S_{32}}{d x_3^*} + \frac{d S_{33}}{d x_3^*} \]  

(2.38d)

Also, the moments along the centroidal axis of a deformed member are given by

\[ M_i^* = E I_i \cdot K_i^* \]  

(2.40a)

\[ M_j^* = G J \cdot K_j^* \]  

(2.40b)

where \( E I_i \) is the bending stiffness about \( x_i \) axis, \( E J_i \) is the bending stiffness about \( x_j \) axis and \( G J \) is the torsional stiffness.

As shown in Fig. 4, the moments \( \dot{M}_1, \dot{M}_2 \) and \( \dot{M}_3 \) are represented, in terms of \( M_i^*, M_j^*, M_k^* \) and \( S_{ij} \),

\[ \dot{M}_1 = -S_{12} \cdot M_j^* + S_{22} \cdot M_i^* - S_{11} \cdot M_k^* \]  

(2.41a)

\[ \dot{M}_2 = S_{11} \cdot M_i^* - S_{21} \cdot M_j^* + S_{12} \cdot M_k^* \]  

(2.41b)

\[ \dot{M}_3 = S_{13} \cdot M_i^* - S_{23} \cdot M_j^* + S_{33} \cdot M_k^* \]  

(2.41c)
The equation of equilibrium in the two transverse directions of the beam may be written [5, 6] as

\[ -\frac{d\dot{M}_t}{dx_t^2} + \dot{Q}_t \cdot (e^t \cdot \xi_t) - \dot{N} \cdot (e^t \cdot \xi_t) = 0 \]  
(2.42a)

\[ -\frac{d\dot{M}_s}{dx_s^2} + \dot{Q}_s \cdot (e^s \cdot \xi_s) - \dot{N} \cdot (e^s \cdot \xi_s) = 0 \]  
(2.42b)

Also

\[ -\frac{d\dot{M}_d}{dx_d^2} = 0 \]  
(2.42c)

where

\[ \dot{Q}_t = -\frac{1}{I + \delta} (\dot{M}_t - \dot{M}_1) \]  
(2.43a)

\[ \dot{Q}_s = -\frac{1}{I + \delta} (\dot{M}_s - \dot{M}_2) \]  
(2.43b)

Substituting eqns (2.34)-(2.36) and (2.39)-(2.41) into eqn (2.42), the following equilibrium equations are obtained:

\[ E_l \frac{d}{dx_t^2} \left[ S_{11} \left( \frac{dS_{11}}{dx_t^2} S_{11} + \frac{dS_{11}}{dx_s^2} S_{12} + \frac{dS_{11}}{dx_d^2} S_{13} \right) \right] - \frac{dS_{11}}{dx_t^2} \cdot \dot{S}_{11} - GJ \]

\[ = 0 \]  
(2.44a)

\[ -E_l \frac{d}{dx_t^2} \left[ S_{12} \left( \frac{dS_{12}}{dx_t^2} S_{12} + \frac{dS_{12}}{dx_s^2} S_{12} + \frac{dS_{12}}{dx_d^2} S_{13} \right) \right] + GJ \frac{d}{dx_t^2} \left[ S_{12} \left( \frac{dS_{12}}{dx_t^2} S_{11} + \frac{dS_{12}}{dx_s^2} S_{12} + \frac{dS_{12}}{dx_d^2} S_{13} \right) \right] \]

\[ + \frac{dS_{12}}{dx_t^2} \cdot \dot{S}_{12} - \dot{N} \cdot S_{12} = 0 \]  
(2.44b)

On the other hand, the expression for the total axial stretch, \( \delta \), of the beam may be written as

\[ \delta = \int_0^l \left\{ 1 + \frac{1}{EA} \left[ (N \cdot (e^t \cdot \xi_t) + Q_t \cdot (e^t \cdot \xi_t) \right] + \dot{Q}_t \cdot (e^t \cdot \xi_t) \right\} \cdot (e^t \cdot \xi_t) \cdot dx_t^2 - l \]  
(2.45)

where \( A \) is the cross-sectional area of the member and \( E \) is Young’s modulus.

Using eqn (2.34), it is seen that

\[ \delta = \int_0^l \left\{ 1 + \frac{1}{EA} \left[ (N \cdot S_{11} + \dot{Q}_t \cdot S_{11} + Q_t \cdot S_{11}) \right] \right\} x \cdot S_{11} \cdot dx_t^2 - l \]  
(2.46)

For the type of problems contemplated, we assume that the deformation of the frame as a whole is such that the relative rotations, \( \theta_t, \theta_s \) and \( \theta_d \) (non-rigid rotations) in each individual member (its elements) of the frame may be considered as being small. Under this assumption, eqns (2.43), (2.44) and (2.46) may be approximated as follows.

\[ E_l \frac{d^2 \delta_t}{dx_t^2} \cdot \frac{1}{l} \dot{M}_t - \ddot{M}_t - \dot{N} \cdot \delta_t = 0 \]  
(2.47a)

\[ -E_l \frac{d^2 \delta_s}{dx_s^2} \cdot \frac{1}{l} \dot{M}_s - \ddot{M}_s + \dot{N} \cdot \delta_s = 0 \]  
(2.47b)

\[ -GJ \frac{d^2 \delta_d}{dx_d^2} = 0 \]  
(2.47c)

Also, the boundary conditions are given by

\[ -E_l \frac{d\delta_t}{dx_t^2} \bigg|_{x_t=0} = 1 \dot{M}_1 \]  
(2.48a, b)

\[ -E_l \frac{d\delta_s}{dx_s^2} \bigg|_{x_s=0} = 2 \dot{M}_2 \]  
(2.48c, d)

\[ \theta_t(x_t=0) = 0 \]  
(2.48e, f)
The total axial stretch becomes

\[
\delta = -\frac{1}{2} \int_0^1 (\delta_1^2 + \delta_3^2) \, dx + \frac{1}{EA} \frac{N}{l} \quad (2.49)
\]

Thus, the non-linear terms, \((\delta_1)^2\) and \((\delta_3)^2\), are retained in the axial stretch relation as, for instance, in the Von Karman plate theory. Eqns (2.47)–(2.49) form the basis of the present derivation of the relations between the generalized displacements and forces in the element.

The non-dimensional axial forces and bending moments, denoted \(n_1, n_2, m_1,\) and \(m_2\), may be defined, respectively, through the relations

\[
n_1 = \frac{N_l}{EI_1}, \quad m_1 = \frac{M_1}{EI_1} \quad (2.50a, b)
\]

\[
n_2 = \frac{N_l}{EI_2}, \quad m_2 = \frac{M_1}{EI_2} \quad (2.50c, d)
\]

The solutions of eqns (2.47a) and (2.48a, b) are given by:

\[
(1) \text{ For } n_1 < 0
\]

\[
\delta = \frac{l}{2} \left( \delta_1^2 + \delta_3^2 \right) \frac{N}{l} + \frac{1}{EA} \frac{N}{l} \quad (2.51)
\]

where

\[
d = \sqrt{-n_1} \quad (2.52)
\]

\[
(2) \text{ For } n_1 > 0
\]

\[
\delta = \frac{l}{2} \left( \delta_1^2 + \delta_3^2 \right) \frac{N}{l} - \frac{1}{EA} \frac{N}{l} \quad (2.53)
\]

where

\[
e = \sqrt{n_1} \quad (2.54)
\]

Equations (2.51)–(2.58) lead to the following relations between the relative rotations, \(\delta_1, \delta_2, \delta_3, \) \(\delta_1^2,\) at the ends of the member and the corresponding bending moments, \(m_1, m_1^2, m_2^2\) and \(m_2^2:\)

\[
(1) \text{ For } n_1 < 0
\]

\[
\delta_1 = \frac{1}{m_1} \left[ \frac{1}{d^2} \frac{1}{d} \tan \frac{e x^2}{l} + \frac{d x^2}{l} \right]
\]

\[
\delta_2 = \frac{1}{m_1} \left[ \frac{1}{e^2} \frac{1}{e} \cosh \frac{e x^2}{l} \right] \quad (2.55)
\]

\[
\delta_3 = \frac{1}{m_1} \left[ \frac{1}{e^2} \frac{1}{e} \sinh \frac{e x^2}{l} \right]
\]

\[
\delta_1^2 = \frac{1}{m_1^2} \left[ \frac{1}{d^2} \frac{1}{d} \csc \frac{d x^2}{l} + \frac{d x^2}{l} \right] \quad (2.56)
\]

\[
\delta_2^2 = \frac{1}{m_1^2} \left[ \frac{1}{e^2} \frac{1}{e} \cot \frac{e x^2}{l} \right] \quad (2.57)
\]

\[
\delta_3^2 = \frac{1}{m_1^2} \left[ \frac{1}{e^2} \frac{1}{e} \tan \frac{e x^2}{l} \right]
\]

\[
(2) \text{ For } n_1 > 0
\]

\[
\delta_1 = \frac{1}{m_1} \left[ \frac{1}{d^2} \frac{1}{d} \csc \frac{d x^2}{l} + \frac{d x^2}{l} \right] \quad (2.58)
\]

\[
\delta_2 = \frac{1}{m_1} \left[ \frac{1}{e^2} \frac{1}{e} \cot \frac{e x^2}{l} \right] \quad (2.59)
\]

\[
\delta_3 = \frac{1}{m_1} \left[ \frac{1}{e^2} \frac{1}{e} \cosec \frac{e x^2}{l} \right]
\]

\[
\delta_1^2 = \frac{1}{m_1^2} \left[ \frac{1}{d^2} \frac{1}{d} \csc \frac{d x^2}{l} + \frac{d x^2}{l} \right] \quad (2.60)
\]

\[
\delta_2^2 = \frac{1}{m_1^2} \left[ \frac{1}{e^2} \frac{1}{e} \cot \frac{e x^2}{l} \right] \quad (2.61)
\]

\[
\delta_3^2 = \frac{1}{m_1^2} \left[ \frac{1}{e^2} \frac{1}{e} \cosec \frac{e x^2}{l} \right]
\]

\[\text{Similar solutions for planar deformation of a beam-column were given previously in [5, 6].}\]
For $n_2 > 0$

$$
\cot \frac{\theta}{2} = \cot \frac{\theta}{2} + \left( \frac{1}{\cot \frac{\theta}{2}} - \frac{1}{\cot \frac{\theta}{2}} \right)
$$

$$
= \cot \frac{\theta}{2} + \frac{1}{\cot \frac{\theta}{2}} - \frac{1}{\cot \frac{\theta}{2}}
$$

(2.61a)

$$
(2.61b)
$$

Also, in terms of the new variables, $\theta_2$, $\theta_3$, $\theta_4$, and $\theta_5$, eqns (2.53a, 2.54a) are obtained as:

$$
\cot \frac{\theta}{2} = \cot \frac{\theta}{2} + \left( \frac{1}{\cot \frac{\theta}{2}} - \frac{1}{\cot \frac{\theta}{2}} \right)
$$

(2.62a)

$$
(2.62b)
$$

Also, using eqns (2.49) and (2.51)-(2.53), the following expressions concerning the total axial stretch, $\delta$, are obtained as:

$$
\delta = \sum_{i=1}^{3} \left[ \frac{1}{2(-n)^2} \csc^2 \left( \frac{-n}{2} \right) - \cot \frac{-n}{2} \right] \left( \theta_i^2 + \theta_i^2 \right)
$$

(2.56a)

$$
(2.56b)
$$

(1) For the case in which $n_1 < 0$ ($i = 1, 2$)

$$
\delta = \sum_{i=1}^{2} \left[ \frac{1}{2(-n)^2} \csc^2 \left( \frac{-n}{2} \right) - \cot \frac{-n}{2} \right] \left( \theta_i^2 + \theta_i^2 \right)
$$

(2.57a)

$$
(2.57b)
$$

(2) For the case in which $n_1 > 0$ ($i = 1, 2$)

$$
\delta = \sum_{i=1}^{2} \left[ \frac{1}{2n} \csc \left( \frac{n}{2} \right) + \cot \frac{n}{2} \right] \left( \theta_i^2 + \theta_i^2 \right)
$$

(2.58a)

$$
(2.58b)
$$

The set of eqns (2.53)-(2.57) may be written in a more convenient form by decomposing the kinematic and mechanical variables of the beam into "symmetric" and "antisymmetric" parts, as

$$
\theta_i = \frac{1}{4} \left( \theta_2 + \theta_3 \right), \quad \theta_i = \frac{1}{4} \left( \theta_2 - \theta_3 \right)
$$

(2.64a, b)

Also

$$
\theta_i = \frac{1}{4} \left( \theta_2 + \theta_3 \right), \quad \theta_i = \frac{1}{4} \left( \theta_2 - \theta_3 \right)
$$

(2.65a, b)

where the superscripts $s$ and $a$ refer to "antisymmetric" and "symmetric" parts, respectively.

Therefore, in terms of the variables, $\theta_i$, $\theta_i$, $\theta_i$, and $\theta_i$, eqns (2.53)-(2.57) may be written as

$$
\theta_i = \frac{1}{4} \left( \theta_2 + \theta_3 \right), \quad \theta_i = \frac{1}{4} \left( \theta_2 - \theta_3 \right)
$$

(2.66a, b)

$$
(2.66b)
$$

wherein:

$$
\theta_i = \frac{1}{4} \left( \theta_2 + \theta_3 \right), \quad \theta_i = \frac{1}{4} \left( \theta_2 - \theta_3 \right)
$$

(2.67a, b)

Also, in terms of the new variables, eqns (2.66a, b)
may be rewritten in a unified form as follows:

$$
\delta = \frac{1}{7} \sum_i \left[ \frac{m_i d^2 h_i}{2 d n_i} + \frac{m_i d^2 h_i}{2 d n_i} + \frac{N}{EA} \right] (2.72a)
$$

$$
= \frac{\delta_i^2 h_i}{2 h_1^2 d n_1} + \frac{\delta_i^2 h_i}{2 h_1^2 d n_1} + \frac{N}{EA},
$$

$$
+ \frac{\delta_i^2 h_i}{2 h_1^2 d n_1} + \frac{\delta_i^2 h_i}{2 h_1^2 d n_1} + \frac{N}{EA} (2.72b)
$$

where:

1. For \( n_1 < 0 \)

$$
d^2 h_1 \frac{d n_1}{d n_1} = \frac{1}{(-n_1)^2} \frac{1}{4(-n_1) \sqrt{-n_1}} - \frac{1}{8(-n_1)} \cdot \csc \left( \frac{\sqrt{-n_1}}{2} \right) \quad (2.73a)
$$

$$
\frac{d^2 h_1}{d n_1} = \frac{1}{4(-n_1) \sqrt{-n_1}} \cdot \cot \frac{\sqrt{-n_1}}{2} - \frac{1}{8(-n_1)} \cdot \csc \left( \frac{\sqrt{-n_1}}{2} \right) \quad (2.73b)
$$

2. For \( n_1 > 0 \)

$$
\frac{d^2 h_1}{d n_1} = \frac{1}{(-n_1)^2} \cdot \cot \left( \frac{\sqrt{-n_1}}{2} \right) - \frac{1}{8(-n_1)} \cdot \csc \left( \frac{\sqrt{-n_1}}{2} \right) \quad (2.74a)
$$

$$
\frac{d^2 h_1}{d n_1} = \frac{1}{4(-n_1) \sqrt{n_1}} \cdot \tanh \frac{\sqrt{n_1}}{2} + \frac{1}{8n_1} \cdot \sech^2 \left( \frac{\sqrt{n_1}}{2} \right) \quad (2.74b)
$$

3. For \( n_2 < 0 \)

$$
\frac{d^2 h_2}{d n_2} = \frac{1}{(-n_2)^2} \cdot \cot \left( \frac{\sqrt{n_2}}{2} \right) + \frac{1}{8(-n_2)} \cdot \csc \left( \frac{\sqrt{n_2}}{2} \right) \quad (2.75a)
$$

$$
\frac{d^2 h_2}{d n_2} = -\frac{1}{4(-n_2) \sqrt{-n_2}} \cdot \tanh \frac{\sqrt{-n_2}}{2} + \frac{1}{8(-n_2)} \cdot \sech^2 \left( \frac{\sqrt{-n_2}}{2} \right) \quad (2.75b)
$$

4. For \( n_2 > 0 \)

$$
\frac{d^2 h_2}{d n_2} = \frac{1}{n_2^2} \cdot \cot \left( \frac{\sqrt{n_2}}{2} \right) + \frac{1}{8n_2} \cdot \csc \left( \frac{\sqrt{n_2}}{2} \right) \quad (2.76a)
$$

Equations (2.66), (2.67) and (2.72) are the after relations between the generalized displacements and forces at the nodes of an individual member, for the range of deformations considered. In connection with eqns (2.66), (2.67) and (2.72), it is worthwhile to recall that:

1. \( N \) is in the direction of the straight line connecting the nodes of the frame member and deformation.

2. The parameters \( \delta_1, \delta_2, i_1, i_2 \) and \( \delta \) are related from eqns (2.31)-(2.33), which are valid in the presence of arbitrarily large rigid motions (translations and rotations) of the individual member.

Thus, while the local stretch (pure strain) and relative rotation (non-rigid) of a differential element of an individual frame-member may be small, the individual member as a whole (and as a part of the overall frame) may undergo arbitrarily large deformation. Hence, the generalized force-displacement relations embodied in eqns (2.66), (2.67) and (2.72) remain valid in the presence of arbitrarily large rigid motions of the individual member of the frame. It is important to note that the present relations for each element account, as in the Von Karmam theory, the non-linear coupling between the bending and stretching deformations, as seen from (2.66), (2.67) and (2.72).

2.3 Tangent stiffness matrix of a space member/element

Recall that, for the most part of the present subsection, each member of the frame is treated as a beam column; but in extreme cases, i.e. of "pathological" deformations, it may be modeled by two or three elements at most.

Now we consider the strain energy due to stretching of the member. Since the total axial stretch \( \xi \) is related in a highly non-linear fashion to the axial force, \( N \), as well as the bending moments, \( M_i \) and \( M_j \) (i = 1, 2), from eqn (2.72), the inverse of the relation in an explicit form, which expresses the axial force \( N \) as a function of \( \xi \), appears impossible. With a view towards carrying out this inversion of the \( N - \xi \) relation incrementally, the strain energy due to stretching, which is denoted as \( \pi \), needs to be expressed in a "mixed" form using the well-known concept of a Legendre contact transformation [7].

$$
\pi = N \cdot \delta - \frac{1}{2} N^2 \frac{1}{2E_A} \quad (2.77)
$$
The internal energy in the member due to combined bending, stretching, and torsion is represented as

\[ \pi = \frac{E I_1}{2l} \left[ \frac{\delta_1}{h_1} + \frac{\delta_2}{h_2} \right] + \frac{E l_2}{2l} \left[ \frac{\delta_1}{h_1} + \frac{\delta_2}{h_2} \right] + \frac{G J}{2l} \cdot \delta_3 - \frac{l \cdot N^*}{2E A} \]  

(2.82)

The condition of vanishing of the first variation of \( \pi \), which is denoted here as \( \pi^* \), in eqn (2.82) due to a variation in \( N^* \), which is denoted here as \( \pi^* \), is given by

\[ \frac{\pi^*}{l} = - \frac{1}{2} \left[ \frac{\delta_1^2}{h_1^2} \frac{d^2 h_1}{d n_1} + \frac{\delta_2^2}{h_2^2} \frac{d^2 h_2}{d n_2} \right] \]

(2.83)

However, unless the flexibility coefficients are equal to zero, one may invert eqns (2.66) and (2.67) to write the "force-displacement" relations as

\[ \delta_1 = \frac{d_1^2}{h_1} m_1, \quad \delta_2 = \frac{d_2^2}{h_2} m_2, \quad \delta_3 = \frac{d_3^2}{h_3} m_3 \]  

(2.78a, b)

\[ \delta_1 = \frac{d_1^2}{h_1} m_1, \quad \delta_2 = \frac{d_2^2}{h_2} m_2, \quad \delta_3 = \frac{d_3^2}{h_3} m_3 \]  

(2.79a, b)

The internal energy in the member due to combined bending, stretching, and torsion is represented as

\[ \pi = \frac{E I_1}{2l} \left[ \frac{\delta_1}{h_1} + \frac{\delta_2}{h_2} \right] + \frac{E l_2}{2l} \left[ \frac{\delta_1}{h_1} + \frac{\delta_2}{h_2} \right] + \frac{G J}{2l} \cdot \delta_3 - \frac{l \cdot N^*}{2E A} \]  

(2.82)

The condition of vanishing of the first variation of \( \pi \), which is denoted here as \( \pi^* \), in eqn (2.82) due to a variation in \( N^* \), which is denoted here as \( \pi^* \), is given by

\[ \frac{\pi^*}{l} = - \frac{1}{2} \left[ \frac{\delta_1^2}{h_1^2} \frac{d^2 h_1}{d n_1} + \frac{\delta_2^2}{h_2^2} \frac{d^2 h_2}{d n_2} \right] \]

(2.83)

Equation (2.83) leads clearly to the relation between \( \delta \) and the generalized forces as given in eqn (2.72).

The reason for using the "mixed" form for the stretching energy in eqn (2.77) is now clear from the above result. By using a similar mixed form for the increment of stretching energy, the incremental axial stretch vs incremental generalized force relation can be derived in a manner analogous to that used in obtaining eqn (2.83) from eqn (2.82). This incremental relation, which is, by definition, piecewise linear, may easily be inverted, as demonstrated in the following. Also, it is shown in the following that eqn (2.82) forms the basis for generating an explicit form for the "tangent-stiffness" of the member.

The increment of the internal energy of the member, which is denoted as \( \Delta \pi \), involving terms up to second order in the "incremental" variables, \( \Delta \delta_1 \), \( \Delta \delta_2 \), \( \Delta \delta_3 \), \( \Delta \delta_4 \), \( \Delta N \) and \( \Delta \delta \) can be seen from eqn (2.82) as

\[ \Delta \pi = \frac{E I_1}{l} \left[ \frac{\delta_1}{h_1} \Delta \delta_1 + \frac{\delta_2}{h_2} \Delta \delta_2 + \frac{\delta_3}{h_3} \Delta \delta_3 \right] \]

(2.84)
\[ \Delta \delta \text{ may be expressed in terms of } \hat{u}_i \text{ and } \hat{\theta}_j \text{ (} i = 1, 2, 3, \\
\text{ } j = 1, 2 \text{) and/or their increments. Henceforth, we used the notation for the vector } d'' \text{ that } \]
\[ d'' = \begin{bmatrix}
\hat{u}_1; \hat{u}_i; \hat{u}_2; \hat{u}_3; \hat{u}_j; \hat{u}_j; \\
\hat{\theta}_1; \hat{\theta}_i; \hat{\theta}_2; \hat{\theta}_3; \hat{\theta}_3; \hat{\theta}_3 \end{bmatrix} 
\]
\[ (2.85) \]
as shown in Fig. 1.

In terms of the increment \( \Delta d'' \), eqn (2.84) may be written as
\[ \Delta \pi = \frac{1}{2} \Delta d'' \cdot A'' \Delta d'' + \Delta \hat{N} \cdot A'' \Delta d'' \\
+ \frac{1}{2} A'' \Delta \hat{N}^2 + B' \Delta d'' + B' \Delta \hat{N} \]  
\[ (2.86) \]
The details of \( A', A'', A', B' \), and \( B' \) are as shown in Appendix A.

By setting to zero the variation of \( \Delta \pi \) in eqn (2.86) with respect to \( \Delta \hat{N} \), one obtains the following relation as
\[ A'' \Delta d'' + B' = -A'' \Delta \hat{N}. \]  
\[ (2.87) \]
Thus, the above equation is the incremental counterpart of \( \delta \) vs the generalized force relation obtained in eqn (2.83). Unlike the non-linear relation in eqn (2.83), the piecewise linear relation, eqn (2.87), can be inverted to express \( \Delta \hat{N} \) in terms of the generalized displacements as
\[ \Delta \hat{N} = -\frac{1}{A''} [A'' \Delta d'' + B']. \]  
\[ (2.88) \]
Substituting eqn (2.88) into eqn (2.86), one obtains the internal energy expression as
\[ \Delta \pi = \frac{1}{2} \Delta d'' \cdot K'' \Delta d'' + \Delta \hat{N} \cdot R'' - \frac{B'}{2A''} \]  
\[ (2.89) \]
where \( K'' \) is the tangent stiffness matrix of member/element,
\[ K'' = A'' - \frac{1}{A''} A'' A'', \]  
\[ (2.90) \]
and \( R'' \) is the internal generalized force vector for member/element,
\[ R'' = B' - \frac{B'}{A''} A''. \]  
\[ (2.91) \]

Recall that the tangent stiffness matrix and the internal force vector are written in the member co-ordinate system as shown in Fig. 1. Thus, it is necessary to transform \( d'' \) from a member co-ordinate system to a global co-ordinate system.

It should be emphasized once again that the tangent stiffness matrix \( K'' \) of eqn (2.90) is given an explicit expression, as in Appendix A; and likewise, the internal generalized force vector \( R'' \) is also given explicitly. No member-wise numerical integrations are involved. During the course of deformation of the frame, once the nodal displacements of the frame at a stage \( C_n \) are known, the tangent stiffness of each member and hence of the frame structure, which governs the deformation of the frame from stage \( C_n \) to an incrementally close neighboring stage \( C_{n+1} \), can be easily evaluated from eqn (2.90). This is a distinguishing feature of the present formulation reported in Williams [15], the large deformation analysis of framed structures performed by a much more computationally inexpensive than the incremental method be standard incremental (updated or total Lagrangian) formulations. However, finite element formulations reported in current literature [8]. Numerical examples illustrating this solution use are given later.

3. SOLUTION STRATEGY

Although a number of solution procedures are available for non-linear structural analyses, a reliable approach to trace the structural response near limit points, and in a post-buckled range, is the arc-length method which was proposed by Ricks [9], Wempner [10] and modified by Crisfield [11,12] and Ramm [13]. This method is the incremental/iterative procedure which represents a generalization of the displacement control approach. The arc-length method, in which the Euclidean norm of the increment in the displacement and load space is adopted as the prescribed increment, allows one to trace the equilibrium path beyond limit points such as in snap-through and snap-back phenomena.

A full description of the arc-length method presently adopted, is given in Ref. [14] and repeated here.

4. NUMERICAL EXAMPLES

Several numerical examples are considered in this section, to demonstrate the validity of the present study.

The first example is that of the so-called Williams' toggle frame, which was first treated by Williams [15] and later analyzed by Wood and Zienkiewicz [16] and Karamanlidis et al. [17]. A schematic of the structure is shown in Fig. 5. The structure has a semispan of 0.243 (in.), a height of 12.943 (in.), a width of 0.366 (in.) and a thickness of 0.103 x 10 (in.), and is modeled by a frame. Figure 6 is a schematic diagram of the structure.

Fig. 5. Schematic diagram of Williams' toggle.
and likewise, a beam subject to a transverse load at the tip, as shown in Fig. 7. It is seen that the present results, using just two elements, agree excellently with those of Bathe and Bolourchi [18]. The relative rotation at tip, as computed from the present procedure, is shown in Fig. 8 and is found to agree excellently with an independent analytical solution.

We now consider the example of a space frame, whose geometry and pertinent material properties are shown in Fig. 9.

The results for the case of axial loading are shown in Fig. 10. In this case, to trigger global buckling, a loading imperfection of magnitude \( P/1000 \) is considered in the transverse direction (where \( 4P \) is the axial load) as shown in the inset of Fig. 10. Also shown in Fig. 10 is the comparison response of the structure when modeled as a space-truss with and without local buckling [19]. An examination of Fig. 10 shows that the response of the space frame under an axial load system indicated in Fig. 10 is nearly the same as that predicted when a space-truss-type model is employed and when the local (member) buckling is accounted for. (Note that both the responses, i.e., those predicted by a space-frame modeling as well as in the present analysis, are reliable and limit design of structures are achieved through an approximation of the length of the admissible solution space in this approach, as is not possible with a space-truss-type model.)

Prior to consideration of space frames, we consider the case of large-deformation bending response of a single member, through the example of a cantilever beam like the one shown in Fig. 3.1. Prior to consideration of space frames, we consider the case of large-deformation bending response of a single member, through the example of a cantilever beam like the one shown in Fig. 3.1.

![Fig. 6. Variations of load-point displacement and support reaction with load, for Williams’ toggle in the post-buckling range.](image-url)
Fig. 7. Deflections of a cantilever under a concentrated load.

Fig. 8. Rotations of a cantilever under a concentrated load.
Tangent-stiffness of a finitely deformed 3-D beam

<table>
<thead>
<tr>
<th>Material Property</th>
<th>Longerons</th>
<th>Diagonal Batters</th>
<th>Short Longerons</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_A$</td>
<td>$7.08 \times 10^6$</td>
<td>$2.70 \times 10^5$</td>
<td>$1.65 \times 10^6$ lb</td>
</tr>
<tr>
<td>$E_I$</td>
<td>$2.16 \times 10^6$</td>
<td>$6.43 \times 10^4$</td>
<td>$2.20 \times 10^5$ lb in$^2$</td>
</tr>
<tr>
<td>$G_J$</td>
<td>$1.63 \times 10^6$</td>
<td>$4.77 \times 10^4$</td>
<td>$1.40 \times 10^6$ lb in$^2$</td>
</tr>
</tbody>
</table>

Fig. 9. Schematic of a 12-bay space frame.

Fig. 10. Deflections at free end under axial loads.
as a space-truss modeling with member buckling, are considerably more flexible than that predicted by a space-truss modeling without local buckling being considered.) This points to the potential use of space-truss-type modeling with local buckling being accounted for.

The results for the case of transverse (bending) loading are shown in Fig. 11, when the structure is modeled as a space frame. Also included in Fig. 11 are the comparison results [19], when the structure was modeled as a space truss when local buckling was suppressed or accounted for. Figure 11 reveals that the bending response of the structure, when modeled as a space frame, is nearly similar to that of a space truss when local (individual) buckling is properly accounted for.

5. CLOSURE

In this paper, simple and effective procedures of explicitly determining the tangent stiffness matrix, and an arc-length method, have been presented for analyzing the large deformation and post-buckling response of (three-dimensional) space frames. Certain salient features of the present methodology are indicated below.

(1) An explicit expression (i.e. requiring no further element-numerical integration) is given for the "tangent-stiffness" matrix of an individual element (which may then be assembled in the usual fashion to form the "tangent-stiffness" matrix of the frame structure). The formulation that is employed accounts for (a) arbitrarily large rigid rotations and translations of the individual element, (b) the nonlinear coupling between the bending and axial stretching motions of the element. Each element can withstand bending moments, a twisting moment, transverse shear forces, and an axial force.

(2) The presently proposed simplified methodology has excellent accuracy in that only one element may be sufficient, in most cases (of practical interest in the behavior of structural frames), to model each member of the frame structure. Inasmuch as the relative (non-rigid) rotation of a differential segment of the present element is restricted to be small, a single element alone is not enough to model the post-buckling response of an entire beam column undergoing excessively large deformations as in an elastica. However, when considered as a part of a practical frame structure, the situation of each member of the frame undergoing abnormally large deformations, as in an elastica, represents a pathological case.

(3) Because of (1) and (2), the present method is by far the most computationally inexpensive method to analyze three-dimensional (space) frame structures and, therefore, is of considerable potential applicability in analyzing large practical space-structures.

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REFERENCES

APPENDIX A

Representations of matrices forming tangent stiffness matrix of a frame member

The vectors for representing eqn (2.86), herein, are defined as

\[ M' = \left[ *\delta \tilde{M}_i, \ast \delta \tilde{M}_i, \ast \delta \tilde{M}_i, \ast \delta \tilde{M}_i \right] \]  
(A.1)

\[ T' = \left[ *\delta \tilde{B}_i, \ast \delta \tilde{B}_i, \ast \delta \tilde{B}_i, \ast \delta \tilde{B}_i \right] \]  
(A.2)

\[ C' = \begin{bmatrix} EI_1, EI_2, EI_3, GJ_1, J_1 \end{bmatrix} \left( \begin{array}{c} \delta \tilde{P}_i \\delta \tilde{P}_i \\delta \tilde{P}_i \\delta \tilde{P}_i \end{array} \right) \]  
(A.3)

\[ \left( \begin{array}{c} \delta \tilde{P}_i \\delta \tilde{P}_i \\delta \tilde{P}_i \\delta \tilde{P}_i \end{array} \right) = \left( \begin{array}{c} -1 \\ast \\ast \ast \end{array} \right) \]  
(A.4)

\[ \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \left( \begin{array}{c} -1 \\ast \\ast \ast \end{array} \right) \]  
(A.5)

\[ E = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \]  
(A.6)

\[ D' = \left[ \frac{d}{da} \left( \frac{1}{\tilde{h}_i} \right) + \frac{d}{da} \left( \frac{1}{\tilde{h}_i} \right) + \frac{d}{da} \left( \frac{1}{\tilde{h}_i} \right) \right] \]  
(A.7)

\[ \left[ \begin{array}{c} \delta \tilde{P}_i \\delta \tilde{P}_i \\delta \tilde{P}_i \\delta \tilde{P}_i \end{array} \right] = \left( \begin{array}{c} -1 \\ast \\ast \ast \end{array} \right) \]  
(A.8)

\[ J = \text{unit vector (5 \times 1)} \]  
(A.9)

\[ A_{w} \]  
(A.10)

\[ G_{w} = \frac{d^{2} T_{s}}{d \theta^{2}} + \frac{d T_{s}}{d \theta} \]  
(A.11)

\[ H_{w} = \frac{d^{2} T_{s}}{d \theta^{2}} + \frac{d T_{s}}{d \theta} \]  
(A.12)

where \( L, J = 1, 2, 3; \ m, n = 1, 2 \).

\[ A_{w} \]  
(A.13)

\[ \lambda_{w} = \frac{\delta \tilde{B}_i}{\delta \theta} \]  
(A.14)

where \( i = 1, 2, 3; j = 1, 2 \).

\[ A_{w} \]  
(A.15)

\[ \lambda_{w} = \frac{\delta \tilde{B}_i}{\delta \theta} \]  
(A.16)

\[ R_{w} = \frac{\delta T_{s}}{\delta \theta} + \frac{\delta}{\delta \theta} \]  
(A.17)

where \( i = 1, 2, 3; j = 1, 2 \).

\[ B_{w} \]  
(A.18)
Approximations of relation between total and relative rotations of a frame member

It is necessary that eqns (2.31) and (2.32) are approximated to form the tangent stiffness matrix for frame-type elements because eqns (2.31) and (2.32) have high order terms and are too complicated to formulate. To keep formulations simple and yet to achieve the intended purpose, the following approximations to eqns (2.24), (2.26) and (2.28) are made and used in eqns (2.31) and (2.32) to obtain

\[
\tan \frac{\theta_2}{2} = -\left(h^{-1}A + l^{-1}B + m^{-1}C\right)
\]

\[
\tan \frac{\theta_3}{2} = -\left(h^{-1}D + l^{-1}E + m^{-1}F\right)
\]

\[
\tan \frac{\theta_4}{2} = -\left(h \cdot r + l \cdot s + m \cdot t\right)
\]

\[
\tan \frac{\theta_1}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_3}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_5}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_1}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_6}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_3}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_7}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_4}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_8}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_1}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_9}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_3}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_{10}}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_4}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_{11}}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_1}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_{12}}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_3}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_{13}}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_4}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_{14}}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_1}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_{15}}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_3}{2} - \frac{\theta_2}{2}\right)
\]

\[
\tan \frac{\theta_{16}}{2} = \frac{\theta_2}{2} + \left(\frac{\theta_4}{2} - \frac{\theta_2}{2}\right)
\]
Tangent-stiffness of a finitely deformed 3-D beam

\[ + \left( \tan \frac{\theta_1}{2} - \tan \frac{\theta_2}{2} \right) s \]
\[ + \left( \tan \frac{\theta_2}{2} - \tan \frac{\theta_3}{2} \right) t. \]  

(B.6)

Substituting eqns (2.14), (2.15) and (2.24) into eqns (B.1) to (B.3), one obtains the following equations as

\[ \tan \frac{\theta_1}{2} = \frac{-1}{1 + e} \left\{ t \left( 1 + \tan^2 \frac{\theta_1}{2} - \tan^2 \frac{\theta_2}{2} - \tan^2 \frac{\theta_3}{2} \right) \right. \]
\[ + 2t \left( \tan \frac{\theta_1}{2} - \tan \frac{\theta_2}{2} + \tan \frac{\theta_3}{2} \right) \]  

(B.7)

\[ \tan \frac{\theta_2}{2} = 0, \]  

where \( e = G \cdot r + H \cdot s + I \cdot t. \)

Equations (B.4)–(B.9) are the approximated relations between the relative and total (rigid plus relative) rotations for forming the tangent stiffness matrix of the element.
Substituting eqns (2.14), (2.15) and (2.24) into eqns (B.1) (B.3), one obtains the following equations as

\[
\tan \frac{\theta_2}{2} = -\frac{1}{1+e} \left\{ \tan \frac{\theta_1}{2} + 2s \left( \tan \frac{\theta_1}{2} + \tan \frac{\theta_2}{2} \right) \right\}
\]

(B.6)

\[
\tan \frac{\theta_1}{2} = -\frac{1}{1+e} \left\{ \tan \frac{\theta_1}{2} + 2s \left( \tan \frac{\theta_1}{2} - \tan \frac{\theta_2}{2} \right) \right\}
\]

(B.7)

where \( e = \frac{1}{1+e} \).

(B.9)

Equations (B.4)–(B.9) are the approximated relations between the relative and total (rigid plus relative) rotations for forming the tangent stiffness matrix of the element.
A BOUNDARY/INTERIOR ELEMENT METHOD FOR QUASI-STATIC AND TRANSIENT RESPONSE ANALYSES OF SHALLOW SHELLS

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Abstract—Integral equations are derived for the representation of in-plane as well as transverse displacements of shallow shells undergoing small, quasi-static or dynamic deformations. A combined boundary/interior element method for static stress, free-vibration, and transient response analyses of shallow shells, based on these integral equations, is derived. Numerical results are presented to demonstrate not only the computational economy but also the superior accuracy realizable from the present approach, in contrast to the popular Galerkin finite element approach.

1. INTRODUCTION
For boundary-value/initial-value problems in (linear or nonlinear) solid mechanics, it is often possible to derive certain integral representations for displacements [1-3]. A key ingredient, which makes such derivations possible, is the singular-solution, in an infinite space, of the corresponding differential equation (in the fully linear case) or of the highest-order differential operator (in the nonlinear case, or even in the linear case when the full linear equation cannot be conveniently solved) for a "unit" load applied at a generic point in the infinite space. When the problem is linear, and the singular solution can be established for the complete linear differential equation of the problem, the aforementioned integral representation for displacements involves only boundary integrals of the unknown trial functions and their appropriate derivatives. Such an integral representation, when discretized, leads to the so-called boundary-element method [1, 2]. Such boundary-element methods are possible, for example, in linear, isotropic, elastostatics (see, for instance, ref. [1]) and in problems of static-bending of linear elastic isotropic plates [4, 5]. On the other hand:

(1) even for linear problems wherein the singular solution cannot be established for the entire linear differential equations,
(2) for anisotropic materials, and
(3) for problems of large deformation and material inelasticity, the integral representation, if any, for displacements would involve not only boundary integrals but also interior integrals (i.e. integrals over the domain) of the trial functions and/or their derivatives [3]. A discretization of such integral equations would lead not to a simple boundary-element method, but to a sort of hybrid boundary/interior element method [3].

The literature on the static or dynamic analysis of shells by the boundary element method is rather sparse. The primary objectives of this paper are:

(1) to explore the integral equation formulations for static and dynamic analysis of shallow shells;
(2) to consider the discretization of these integral equations; and
(3) to explore the advantages of the present approach as compared to the usual Galerkin finite element approaches wherein higher-order continuities of trial solution for displacements, such as C1 continuity of transverse displacement, has long plagued the successful development of shell finite elements.

As is well known, due to the curvature of the shell, the in-plane displacements and the transverse displacement in a shell are inherently coupled in the kinematics of deformation as well as in the momentum balance relations for the shell. In the present integral equation formulation, the test functions are chosen to be the singular solutions, in infinite space, of parts of the relevant momentum balance equations. However, (1) when the dynamic motion of the shell is considered and (2) since the in-plane displacements w, are coupled to the transverse displacement w in the shell equilibrium equations, the presently considered test functions represent singular solutions to only the highest-order differential operator occurring in the elastostatic equilibrium equations of the shell. Due to this reason, the integral representations for the shell displacements involve not only boundary integrals, but also domain integrals involving the trial solutions for displacements. Thus, when discretized, the present integral equation approach leads to a hybrid boundary/domain element method. However, unlike in the Galerkin finite element method, in the present approach, the trial functions for transverse displacement need only be piecewise constant or
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where \( n_i \) are the direction cosines of the unit outward normal to \( \Gamma \) in the base plane. The out-of-plane boundary conditions are

\[
\frac{\partial w}{\partial n} = 0 \quad \text{or} \quad V_n = V_n^0
\]

\[
\Psi_n = \Psi_n^0 \quad \text{or} \quad M_n = M_n^0
\]

and

\[
V_n = -D \frac{\partial}{\partial n} (\nabla^2 w) + \frac{\partial}{\partial t} M_n
\]

is the reduced Kirchhoff shear force;

\[
\Psi_n = \frac{\partial w}{\partial n}
\]

is the rotation around the tangent to \( \Gamma \);

\[
M_n = -D \left\{ (n_1^2 + n_2^2) \frac{\partial^2 w}{\partial x_1^2} + 2(1 - v)n_1n_2 \right\}
\]

\[
\times \frac{\partial^2 w}{\partial x_1 \partial x_2} + (n_1^2 + n_2^2) \frac{\partial^2 w}{\partial x_2^2} + 2n_1 \frac{\partial^2 w}{\partial x_1 \partial n} + 2n_2 \frac{\partial^2 w}{\partial x_2 \partial n}
\]

and \( n \) and \( t \) are directions normal and tangent respectively, to \( \Gamma \) in the base plane.

It is well known that the equilibrium eqns (2.1) can be written more concisely in terms of a stress function \( \phi \) (for \( N_{x,y} \)) and the transverse displacement \( w \). However, we leave the equations in the form (2.1), which is somewhat more general, so as to treat in-plane in-plane forces and to extend the development to the case of general non-shallow shells, and nonlinear kinematics in forthcoming papers.

The in-plane strain–displacement relations are

\[
\epsilon_{i,j} = \frac{1}{2} \left[ u_{i,j} + w_{i,j} + \frac{2w}{R_{i,j}} \right]
\]

where \( u_i \) are the in-plane displacements at the mid-surface. The in-plane stress-resultant/stress relations are

\[
N_{i,j} = C(\epsilon_{i,j} + w_{i,j});
\]

\[
N_{i,2} = C(\epsilon_{i,2} + w_{i,2});
\]

\[
N_{i,1} = C(1 - v)\epsilon_{i,1};
\]

where \( C = E(1 - v^2) \). The moment-curvature relations are

\[
M_{i,1} = -D(w_{i,1} + w_{i,1}^2);
\]

\[
M_{i,2} = -D(w_{i,2} + w_{i,2}^2);
\]

\[
M_{i,3} = -D(1 - v)w_{i,2}.
\]
3. INTEGRAL EQUATIONS FOR SHELL DISPLACEMENTS AND A BOUNDARY ELEMENT SOLUTION STRATEGY

In an approximate analysis of the boundary/initial-value problem described in Section 2, let \( u \) and \( w \) be the assumed trial solutions. We shall consider a general weighted-residual formulation, and let \( u^* \) and \( w^* \) be the corresponding test functions. In the familiar Galerkin finite-element method, the trial functions \( (u, w) \) and the test functions \( (u^*, w^*) \) belong to the same category of function spaces. In the present formulation, however, as will be seen, the test functions \( (u^*, w^*) \) belong to an entirely different class of function space from that of the trial functions. With this in mind the combined weak forms of the equilibrium equations and boundary conditions for the in-plane \((eqns (2.1a) and (2.2a,b))\) and out-of-plane \((eqns (2.1b) and (2.3a,b))\) deformations, respectively, may be written (see, for instance, ref. [8]) as

\[
\begin{align*}
\int_{\Omega} \left[ (N_{1d} + b_{12})u^* + b_{12} \right] d\Omega &= \int_{\Gamma_r} \left( P_\gamma - P_\gamma \right) u^* d\Gamma + \int_{\Gamma_r} \left( \bar{u}_r - u_\gamma \right) P_\gamma^* (u^*) d\Gamma \quad (3.1) \\
\int_{\Omega} \left( D\nabla^2 w + \frac{N_{1d}}{R_{1d}} - b_3 + f_3 \right) w^* d\Omega &= \int_{\Gamma_r} \left( P_\gamma - P_\gamma \right) w^* d\Gamma + \int_{\Gamma_r} \left( \bar{u}_r - u_\gamma \right) P_\gamma^* (u^*) d\Gamma \quad (3.2)
\end{align*}
\]

Use of (3.3) in (3.1) results in

\[
\int_{\Omega} \left[ N_{1d}^* + C (\kappa_{1d}^*) w + b_{12} \right] u_{1d}^* d\Omega = \int_{\Gamma_r} \left( P_\gamma - P_\gamma \right) u_{1d}^* d\Gamma + \int_{\Gamma_r} \left( \bar{u}_r - u_\gamma \right) P_\gamma^* (u_{1d}^*) d\Gamma.
\]

Use of the Divergence theorem in eqn (3.6) results in

\[
\int_{\Omega} N_{1d}^* u_{1d}^* d\Omega - \int_{\Omega} N_{1d}^{*'} u_{1d}^{*'} d\Omega
\]

\[
+ \int_{\Omega} C (\kappa_{1d}^*) w u_{1d}^* d\Omega + \int_{\Omega} b_{12} u_{1d}^* d\Omega
\]

\[
= \int_{\Gamma_r} \left( P_\gamma - P_\gamma \right) u_{1d}^* d\Gamma + \int_{\Gamma_r} \left( \bar{u}_r - u_\gamma \right) P_\gamma^* (u_{1d}^*) d\Gamma.
\]

Since the material is linear elastic and isotropic, we have

\[
N_{1d}^* u_{1d}^* = C_{1d} u_{1d}^* u_{1d}^* = N_{1d}^{*'} (u_{1d}^*) u_{1d}^*.
\]
where the definitions of $N_2$ are apparent. We now introduce the additional notations:

$$P_s = N_{3p} n_s; \quad P_n = N_{3q} n_s$$  \hspace{1cm} (3.9a)

or

$$P_n = P_s + C_{3k} w n_s.$$  \hspace{1cm} (3.9b)

Using (3.8, 3.9a, 3.9b) in (3.7) and applying the Divergence theorem, it is easy to obtain:

$$\int_0^1 [N_2(u^*)_y]_y u^* d\Omega + \int_0^1 b_s u^*_s d\Omega$$

$$+ \int_r \hat{P}_s u^*_s d\Gamma - \int_r P_s^* \hat{u}_s d\Gamma$$

$$- \int_n C_{3k} w u^*_s d\Omega = 0.$$  \hspace{1cm} (3.10a)

where

$$\hat{P}_s = P_s \text{ at } r_s; \quad \text{and} \quad \hat{P}_n = P_n \text{ at } r_n.$$  \hspace{1cm} (3.10b)

and

$$\hat{u}_s = \hat{u}_s \text{ at } r_s; \quad \text{and} \quad \hat{u}_n = u^*_n \text{ at } r_n.$$  \hspace{1cm} (3.10c)

Now, we choose $u^*_n$ to be the “singular solution” of the equation

$$[N_2(u^*)_y]_y + \delta(x_n - \xi_n) \delta w e = 0.$$  \hspace{1cm} (3.11)

where $\delta(x_n - \xi_n)$ is the Dirac delta function at $x_n = \xi_n; \delta e$ is the Kronecker delta; and $\varepsilon_0$ denotes that the direction of the application of the point load is along the $x_n$ direction. The “singular solution” of (3.11) will be denoted as $u^*_m$; where $u^*_m$ is the displacement along the $x_n$ direction in a plane infinite body at any point $x_n$ due to a unit load along the $x_n$ direction, applied at the location $x_n = \xi_n$. Likewise, $P_m^*(x_n, \xi_n)$ will be considered to be the traction along the $x_n$ direction on an oriented surface at $x_n$, with a unit normal $n_s$, due to a unit load along $x_n$ at the location $x_n = \xi_n$. These solutions are well known [9] and may be written as

$$u^*_m(x_n, \xi_n) = \frac{1}{8\pi G} \left[ (v - 3) \ln \rho \delta_n + (1 + v) \left( \frac{\partial \rho}{\partial x_n} \frac{\partial \rho}{\partial x_s} \right) \right]$$  \hspace{1cm} (3.12a)

and

$$P_m^*(x_n, \xi_n) = -\frac{1}{4\pi \rho} \left[ \frac{\partial \rho}{\rho n_s} \left( 1 - v \right) \delta_n \right]$$

$$+ 2(1 + v) \left( \frac{\partial \rho}{\partial x_n} \frac{\partial \rho}{\partial x_s} \right) - (1 - v)$$

where $\rho = |x_n - \xi_n|$ is the radius vector from $x_n$ to $\xi_n$ and $G = E/(2(1 + v))$. Due to the property of integrals involving Dirac functions, we have

$$\int_0^1 (N_2^*)_y u^*_m d\Omega = -\int_0^1 \delta(x_n - \xi_n)$$

$$\delta w e \hat{u}_s(x_n) d\Omega = u^*_m(\xi_n).$$  \hspace{1cm} (3.13)

Using eqns (3.12) and (3.13) in (3.10a), we have

$$vy^*_m(\xi_n) = \int_0^1 b_s(x_n) u^*_m(x_n, \xi_n) d\Omega$$

$$+ \int_r \hat{P}_s(x_n) u^*_m(x_n, \xi_n) d\Gamma$$

$$- \int_n C_{3k} w(x_n) u^*_m(x_n, \xi_n) d\Omega.$$  \hspace{1cm} (3.14)

It can be shown that while the coefficient $\gamma$ in the left-hand side of (3.14) is the unity when $\xi_n$ is in the interior of $\Omega$, the value of $\gamma$ is $(0.5)$ when $\xi_n$ falls on the “smooth” boundary $r$ [1]. Equation (3.14) is the sought-after integral equation for $u^*$ in a shallow shell.

We now choose the test function $w^*(x_n)$ to be the “singular solution” in an infinite plate corresponding to a unit point load at the location $\xi_n$. Thus, $w^*$ corresponds to the solution of the linear equation

$$D^{\alpha} w^* = \delta(x_n - \xi_n)$$  \hspace{1cm} (3.15)

in an infinite domain in the base-plane of the shallow shell. It is well known [9] that the solution for $w^*$ is given by

$$w^*(x_n, \xi_n) = \frac{1}{\pi} \rho \rho \text{ ln } \rho - \frac{1}{8\pi \rho}$$  \hspace{1cm} (3.16)

where $\rho = |x_n - \xi_n|$. Using eqn (3.16) and (3.3) in eqn (3.2) and employing repeated integrations by parts in the resulting equation, one easily obtains the integral equation

$$\gamma D^{\alpha} w^*(\xi_n) = \int_r \hat{V}_s(x_n) w^*(x_n, \xi_n) d\Gamma$$

$$- \int_r \hat{M}_s(x_n) \psi^*(x_n, \xi_n) d\Gamma$$

$$+ \int_r \hat{\psi}_s(x_n) M^*_s(x_n, \xi_n) d\Gamma$$

$$- \int_r \hat{w}(x_n) V^*_s(x_n, \xi_n) d\Gamma.$$
Quasi-static and transient response analyses of shallow shells

\[ -\int_0^1 \left[ N_{\theta} + C_{\theta \theta} w - b - f_i \right] \times (x_\theta)\omega^*(x_\theta, \xi_\theta) d\Omega \]
\[ + \sum_i \left( \left[ M_i \omega^* - [M_i^*]^* \right] \right) \]
\[ \times (x_\theta)\omega^*(x_\theta, \xi_\theta) d\Omega \]
\[ + \sum_i \left( \left[ M_i \omega^* - [M_i^*]^* \right] \right) \]
\[ \times (x_\theta)\omega^*(x_\theta, \xi_\theta) d\Omega \]
\[ \times (x_\theta)\omega^*(x_\theta, \xi_\theta) d\Omega \]
\[ \times (x_\theta)\omega^*(x_\theta, \xi_\theta) d\Omega \]
\[ \times (x_\theta)\omega^*(x_\theta, \xi_\theta) d\Omega \]
\[ \times (x_\theta)\omega^*(x_\theta, \xi_\theta) d\Omega \]
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\[ \times (x_\theta)\omega^*(x_\theta, \xi_\theta) d\Omega \]
\[ \times (x_\theta)\omega^*(x_\theta, \xi_\theta) d\Omega \]

In eqn (3.17) the terms with the superposed symbol \(\omega^*\) should be taken to imply the respective prescribed values, if any, at \(\Gamma\); otherwise, they are to be treated as the unknown solution variables. Also, the symbol \([\cdot]\) denotes the jump in the quantity \(\cdot\) at a corner at \(\Gamma\), in the direction of the increasing arc length along \(\Gamma\); and the summation \((1 \rightarrow k)\) extends to all the \(k\) such corners.

Using eqn (3.4) and the Divergence theorem, it is easy to see that

\[ \gamma D \frac{\partial w}{\partial n}(x_\theta, \xi_\theta) = \int_\Gamma \rho_* (x_\theta) \frac{\partial w^*}{\partial n}(x_\theta, \xi_\theta) d\Gamma \]
\[ - \int_\Gamma \gamma D \frac{\partial w^*}{\partial n}(x_\theta, \xi_\theta) d\Gamma \]
\[ + \int_\Gamma \gamma D \frac{\partial w^*}{\partial n}(x_\theta, \xi_\theta) d\Gamma \]
\[ - \int_\Gamma \gamma D \frac{\partial w^*}{\partial n}(x_\theta, \xi_\theta) d\Gamma \]
\[ + \int_\Gamma \gamma D \frac{\partial w^*}{\partial n}(x_\theta, \xi_\theta) d\Gamma \]
\[ + \int_\Gamma \gamma D \frac{\partial w^*}{\partial n}(x_\theta, \xi_\theta) d\Gamma \]

In summary, eqns (3.14), (3.19) and (3.20) represent the complete set of integral equations for \(u_\theta, w,\) and \(\partial w/\partial n\). An examination of eqns (3.14), (3.19), and (3.20) reveals the following features:

(1) For given body forces \(b\), the integral relation for \(u_\theta\) [eqn (3.14)] involves the trial functions \(u_\theta\) only at the boundary \(\Gamma\). On the other hand, due to the curvature induced coupling of the trial functions \((u, w)\) in the shallow-shell problem, the integral relation for \(u_\theta\) contains a domain-integral (over \(\Omega\)) involving the trial function for \(w\).
2. If the body forces $b_\alpha$ include in-plane inertia forces ($\rho \ddot{u}_\alpha$), then the integral relation for $u_\alpha$ involves a domain integral (over $\Omega$) of $\ddot{u}_\alpha$, as well.

3. Again, due to the curvature-induced coupling of the in-plane and out-of-plane displacements in the presently considered shallow shell, the integral equations for $w$ and $\partial w/\partial n$, eqns (3.19) and (3.20) respectively, contain domain-integrals (over $\Omega$) involving trial functions for both $w$ and $u_\alpha$. Also, in a transient dynamic analysis, the term $(w)$ appears inside a domain-integral.

4. For reasons (1) to (3) above, unlike the classical homogeneous isotropic elastostatics [1] wherein a discretization of the relevant integral equations requires a use of basis functions for the displacements at the boundary alone, the present shallow-shell formulation requires the assumption of basis functions for the trial solutions $u_\alpha$ and $w$, at the boundary $\Gamma$ as well as in the interior $\Omega$. Thus, the present solution methodology may, strictly speaking, be classified as hybrid boundary-element/interior (finite) element method based on a direct discretization of integral equations.

5. Unlike in the homogeneous isotropic elastostatics [1], the present integral equations are no longer boundary-integral equations alone.

6. Suppose now that in eqns (3.14), (3.19), and (3.20) we let $\xi_\alpha$ tend to a point on the boundary, i.e. $\xi_\alpha \in \Gamma$. Thus, we obtain three integral relations for the boundary values of $u_\alpha$, $w$, and $\partial w/\partial n$. An examination of these relations reveals that, in order to discretize these equations, one needs to assume only very simple trial functions $u_\alpha$, $w$, $\partial w/\partial n$ not only at the boundary but also in the interior of $\Omega$. For instance, $\Omega$ may be discretized into a number of finite elements and $\Gamma$ into a number of boundary elements. As $\xi_\alpha$ tends to $\Gamma$, the integral relations (3.14), (3.19), and (3.20) clearly show that $w$ and $u_\alpha$ need only be piecewise constant functions in each finite element in $\Omega$, in any discretization process. Finally, one may need only consider $C^0$ continuous functions for $w$ and $u_\alpha$ in each element (to extend the formulations, later, to finite deformation cases).

7. In contrast, it is recalled that in the Galerkin finite element method, $u_\alpha$ need be $C^{\infty}$ continuous and $w$ be $C^1$ continuous in each element. The difficulties with such a finite element approach are too well documented in literature to warrant further comment here.

8. At each point on the boundary, two of the in-plane variables $u_\alpha (\alpha = 1, 2)$, $P_\alpha (\alpha = 1, 2)$ are specified; and the other two are unknown. Likewise, two of the out-of-plane variables $\psi_\alpha$, $M_\alpha$, and $w$ are specified; and the other two are unknown. At each point in $\Omega$, as seen from eqns (3.14), (3.19), and (3.20), the three displacements, $u_\alpha$ and $w$, are unknown. Thus, if eqns (3.14), (3.19), and (3.20) are discretized, through finite as well as boundary elements and the nodal values of the variables are appropriately understood to be either specified or unknown, one would easily obtain exactly as many equations as the number of unknowns, so that the problem may be considered as well posed.

4. ALGEBRAIC FORMULATION

We discretize the boundary into $M$ segments,

$$\Gamma = 1, \ldots, M,$$ such that the corner points, if any, $\Delta$ coincide with the nodes of $\Gamma$. The domain $\Omega$ is discretized into $L$ finite elements $\Omega_l$, $l = 1, \ldots, L$. Altogether, one has $N$ nodes, of which $N_\Gamma$ are on $\Gamma$ and $N_\Omega$ inside $\Omega$. Of course, the discretization is such that each $\Gamma_l$ is a boundary segment of some $\Omega_l$. At each nodal point on the boundary, $u_\alpha$, $w$, $\partial w/\partial n$, $V_\alpha$, and $M_\alpha$ are treated as nodal variables; and at a point in the interior of $\Omega$, only $u_\alpha$ and $w$ are treated as nodal variables. While a $C^0$ continuity is necessary in the present formulation for $u_\alpha$ and $w$ inside $\Omega$, such a continuity is used in the present algebraic convenience (as well as for reasons of extending the present formulation to a finite deformation case).

Let the numerals 1, 2, 3, 4, 5 denote respectively the vectors $u_\alpha$ (values of in-plane displacements), nodes on $\Gamma$; $u_\alpha$ (values of in-plane displacements), nodes in $\Omega$; $w$ (values of out-of-plane displacements) as well as rotations $\partial w/\partial n$ at nodes on $\Gamma$; $w$ (values of $w$ at nodes in $\Omega$); and $P_\alpha$ (values of in-plane tractions $\mathbf{N}_\alpha \tau_\alpha$ as well as out-of-plane generalized forces, viz. the reduced Kirchhoff shear and bending moments, at nodes on $\Gamma$). Using the above discussed interpolations, for the various variables in $\Omega$ and $\Gamma$, in eqns (3.14), (3.19), and (3.20), it is easy to generate the algebraic equations:

$$G_{11} u_1 + G_{12} w_1 + G_{13} w_0 + G_{14} P_1 = F_1$$

$$G_{i1} u_1 - G_{i0} + G_{i2} w_1 + G_{i3} w_0 + G_{i4} P_1 = F_i$$

$$G_{31} u_1 + G_{32} u_0 + G_{33} w_1 + G_{34} w_0 + G_{35} P_1 = F_3$$

$$G_{41} u_1 + G_{42} u_0 + G_{43} w_1 + G_{44} w_0 + G_{45} P_1 = F_4$$

$$G_{51} u_1 + G_{52} u_0 + G_{53} w_1 + G_{54} w_0 + G_{55} P_1 = F_5$$

The notation in eqns (4.1)-(4.4) is as follows:

1. The numerals 1, 2, 3, 4, 5 should be identified as explained before;

2. The matrices are defined such that, for instance, $G_2$ implies that it has as many rows as the dimension of the (2) vector, i.e. $u_0$, and has as many columns as the (1) vector, i.e. $u_1$;

3. Likewise, for instance, $F_3$ is a vector whose dimension is the same as the (3) vector, i.e. $w$.

Equation (4.2) can be trivially solved for the (2), vector, i.e. $u_0$ as

$$u_0 = G_{21} u_1 + G_{22} w_1 + G_{23} w_0 + G_{24} P_1 - F_2$$

Use of (4.5) in (4.1), (4.3), and (4.4) results in equations in the only unknown variables, $u_1$, $w_1$, and $P_1$, thereby reducing the dimensionality of algebraic equations yet to be solved.
For dynamic response problems, one may consider body forces \( b_1 \) and \( b_2 \) to include the inertia forces \(-\ddot{u}_i\) and \(-\ddot{w}_i\), respectively. Thus, in eqns (4.1)-(4.4), for dynamic response analyses, one may define \( F_1, \ldots, F_4 \) as

\[
\begin{align*}
F_1 &= F_1^* - M_{11}\ddot{u}_1; \\
F_2 &= F_2^* - M_{22}\ddot{w}_2; \\
F_3 &= F_3^* - M_{11}\ddot{w}_1; \\
F_4 &= F_4^* - M_{22}\ddot{u}_2. \\
\end{align*}
\]

Equations (4.6a)-(4.6d) are now used in eqns (4.1)-(4.4), and the resulting equations are easily rearranged to read as

\[
\begin{bmatrix}
M_{11}
\begin{bmatrix}
\ddot{u}_1
\ddot{w}_2
\end{bmatrix} +
G_{11}\begin{bmatrix}
u_r
\end{bmatrix} +
G_{12}\begin{bmatrix}
u_r
\end{bmatrix} P_r &=
\begin{bmatrix}
F_1
F_3
\end{bmatrix} \\
M_{22}
\begin{bmatrix}
\ddot{w}_2
\ddot{u}_1
\end{bmatrix} +
G_{22}\begin{bmatrix}
u_r
\end{bmatrix} +
G_{21}\begin{bmatrix}
u_r
\end{bmatrix} P_r &=
\begin{bmatrix}
F_2
F_4
\end{bmatrix}
\end{bmatrix}
\]

(4.7)

Recall that the number (1) denotes \( u_i \) and (3) denotes \( w_r \) (including boundary rotations) and (5) denotes \( P_r \) (boundary nodal tractions). It is clear that, at the boundary nodes, the number of nodal generalized displacements is exactly the same as that of the generalized nodal tractions. Thus, the matrix

\[
\begin{bmatrix}
G_{13}
\end{bmatrix}
\]

is a square matrix. From (4.7), one may thus eliminate the unknown nodal generalized forces and write an equation of the type

\[
P_r =
\begin{bmatrix}
G_{11}
G_{12}
\end{bmatrix}^{-1}
\begin{bmatrix}
F_1
F_3
\end{bmatrix} -
\begin{bmatrix}
G_{11}
G_{12}
\end{bmatrix}\begin{bmatrix}
\ddot{u}_1
\ddot{w}_2
\end{bmatrix} -
\begin{bmatrix}
G_{12}
G_{21}
\end{bmatrix}\begin{bmatrix}
\ddot{w}_2
\ddot{u}_1
\end{bmatrix} \]

(4.10)

When eqn (4.10) is used in (4.8), the resulting equations may be rearranged in the standard form, as

\[
Gq + Mr = F_r.
\]

(4.11)

In the present series of computations, basis functions for each of the trial functions are assumed as follows:

(1) over each boundary element at \( r \), the boundary variables \( u_i, w, \partial w/\partial n, P_r, M_r, \) and \( V_r \) were interpolated linearly, and

(2) over each interior finite element, \( u, w \) were interpolated bilinearly such that each of these functions is \( C^0 \) continuous at the element boundaries.

The boundary elements and interior finite elements are shown in Fig. 3. Results were obtained for four different meshes, as shown in Fig. 4. The three-noded elements, each with a node at the center of the shell, are generated from the four-noded element shown in Fig. 3 by collapsing two of the nodes together. The boundary conditions at \( r \) are: (1) \( w = 0, \partial w/\partial n = 0 \) and (2) the shell is free to move in the in-plane directions (in-plane traction-free).

The deflection at the center of the shell (with the geometry: \( R = 100, r_o = 5.0 \)), due to an applied concentrated load at the center of the shell, is shown in Fig. 5. It is seen that the numerical results for all the meshes shown in Fig. 4 agree excellently with the
exact solution [11]. Note that the total number of
degrees of freedom for each of the meshes is: 3
(Mesh 1); 27 (Mesh 2); 99 (Mesh 3); 183 (Mesh 4).
The shell, in this example, is very shallow and
approaches the geometry of a circular plate.

To examine the accuracy of the presently employed
simple theory of shallow shells, problems of shells
with various values of $r_o/R$ ($r_o/R = 1$ denotes a hemi-
spherical shell) were analyzed under a uniformly
applied transverse pressure loading (see Fig. 6). From
the results for the crown deflection plotted in Fig. 6,
it may be seen that for $(r_o/R) \leq 0.3$, the shallow-shell
theory is a very good approximation (i.e. errors of
less than 10%) to the deep-shell theory.

The stresses in the present integral equation
approach may be computed in several alternate ways.
The first approach is to use eqns (3.14), (3.19), and
(3.20) for $u_x$, $w$, and $\partial w/\partial n$, respectively, and analyti-
cally derive integral equations for the in-plane
stress resultants $N_x$, bending moments $M_x$, and the
reduced Kirchhoff shear, directly. However, this pro-
cedure involves rather tedious algebra. The second
and simpler, approach is to numerically differen-
tiate the computed displacement field to derive the stress
field. Both approaches are used in the present work.

For the case of $(r_o/R) = (1/20)$, the obtained nume-ical results for $M_x$ and $M_y$ are compared with the
analytical results in Fig. 7. At the boundary, the exact
solution for $M_x$ is $3.09 (KN/m)$, whereas that obtained
through the direct integral representation is $3.18
(KN/m)$ and that obtained through numerical
differentiation of the displacement field is $3.24 (KN/m)$.

Next, the eigenvalues of free vibration of the shell
were computed for various values of $(r_o/R)$.
Quasi-static and transient response analyses of shallow shells

Fig. 6. Crown deflection for shells of various depth ratios.

Fig. 7. Results for radial and tangential bending moments.

Fig. 8. Time variation of crown deflection due to a concentrated pulse at the crown.
results for the first six eigenmodes are shown in Table 1, for Mesh 2 (27 d.o.f) and Mesh 3 (81 d.o.f).
For the first mode, even Mesh 2 gives an eigenvalue of acceptable accuracy, for all the cases, as compared to the analytical solution of [12]. From a comparison of the results for the six eigenvalues from the two meshes, it may be seen that even the 27 d.o.f. mesh gives results of acceptable accuracy for all the six modes. Also, Table 1 indicates the increased stiffening effect in the shell as the depth of the gel \((r_0/R)\) increases.

The transient dynamic response of the shell subjected to a time-varying concentrated load at the center of the shell is analyzed for different types of time variation of the load. These results, computed from using the Mesh 3, are shown in Figs. 8 and 9.

![Fig. 9. Time variation of crown deflection due to a concentrated force \(P(T)\) at the crown.](image)
with the time variation of the applied load being depicted in the inset of each of these figures. These results indicate that the frequency content of the response of the shell, as of course, altered by the nature of the applied load.

6. CONCLUSION

A simple "boundary-element/interior-element" method, based on an integral equation formulation for static and dynamic response analysis of shallow shells, is presented. Unlike in the Galerkin finite element method, the trial functions for displacements are not required to be C1 continuous in the present approach. Thus, the present approach is much simpler than the Galerkin FEM and yields results of acceptable accuracy with a discrete model of much smaller number of degrees of freedom.

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REFERENCES

CONTROL OF DYNAMIC RESPONSE OF A CONTINUUM MODEL OF A LARGE SPACE STRUCTURE

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Abstract—The problem of active control of the transient dynamic response of large space structures, modeled as equivalent continua, is investigated here. The effects of initial stresses, in the form of in-plane stress resultants in an equivalent plate model, on the controllability of transverse dynamic response, are studied. A singular solution approach is used to derive a fully coupled set of nodal equations of motion which also include nonproportional passive damping. One approach considers a direct attack on this system of nodal equations. An alternative scheme implements a reduced-order model of coupled ordinary differential equations which are obtained in terms of the amplitudes of the pseudomodes of the nominally undamped system. Optimal control techniques are employed to develop a feedback control law. Algorithms for the efficient solution of the Riccati equation are implemented. Several examples are presented which involve the suppression of vibration of the transient dynamic response of the structure using an arbitrary number of control force actuators.

1. INTRODUCTION

It is currently intended that very large, low-mass structures will play important roles in space missions, which include such activities as communications, earth resource surveillance, and multipurpose large space platforms. It is envisaged that these structures will be extremely large (perhaps occupying several square miles); and this, coupled with the minimum weight requirements, will result in a very high degree of structural flexibility and thus in low levels of dynamic frequencies. The central problems in the design of these large space structures (LSS) are vibration suppression and shape control. These undesirable deformations are due to disturbances such as unbalanced rotating machinery on board, thruster firings, slewing/pointing maneuvers, thermal transients, etc. Therefore, during the course of their intended use, these structures must be controllable: (i) attitude control, which involves the control of the spacecraft so as to maintain a given sun pointing or earth pointing accuracy, and (ii) shape or configuration control, which involves the suppression of vibration of critical members, such as flexible antennae, which degrades the overall pointing accuracy. Thus, the interaction of a highly flexible large space structure with control systems, whether they are active and/or passive, is one of the more challenging problems in large space structure technology.

A problem of the opposite variety arises in aeroelasticity, wherein the aerodynamic forces acting on the elastic body have the equivalent influence of negative damping, i.e. the surrounding fluid medium acts as an energy supplier to the vibrating elastic body. Therefore, in parallel with the subject of aeroelasticity, we have the emerging subject of servoelasticity—that of control of the dynamic motion of deformable structures. Similar problems also arise in the design of tall buildings and bridges, wherein it is required to control the dynamic motion to amplitudes within the bounds of human comfort levels, say under seismic loads or wind loads.

Some of the topics relevant to the issue of LSS controllability are now summarized: (i) In the usual free or forced vibration response analysis of structures, efficient algorithms based on many thousands of degrees of freedom exist; but in the optimal control problem, the algorithms are currently limited to just a few degrees of freedom. Thus, there is a need for innovative algorithms for the control of medium-sized and large-sized systems. (ii) In earlier studies of the LSS control problem, it was usually assumed that damping was either negligible or else proportional to the mass or stiffness matrices. The free-vibration modes of the system are then obtained. The orthogonality properties of these modes lead to a completely decoupled system of second-order differential equations. A control is then implemented for each mode individually and this is known as Independent Modal Space Control (IMSC)[1, 2]. One drawback of this relatively simple approach is that it requires as many control force actuators as the number of modes controlled, and this may not always be feasible. In addition, when damping is present, either due to material hysteresis or due to deliberate introduction of damping mechanisms or due to deliberate design of LSS joints[3], the concept of modal decomposition is not applicable. Therefore, it is desirable to implement a control based on the fully coupled system of equations of motion with an arbitrary number of

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control force actuators that is perhaps much less than the order of the system that is controlled.

The LSS, in general, may be viewed as three-dimensional frame- or truss-type structures, depending on the joint design such as sleeve-stiffened beam joints or truss-pin joints, respectively. The beam joints would induce bending moments at the nodes of each member, thus requiring the modeling of the LSS as a three-dimensional frame. However, truss-pin joints would allow the members to rotate freely about the joints, and so the members may be modeled as tension-compression members. However, it may be prohibitively expensive for the direct modeling of an entire LSS as a three-dimensional frame or truss. Thus, a combination of equivalent continuum models and detailed three-dimensional frame/truss models may be necessary. These equivalent continuum models involve the modeling of repetitive lattice grids by equivalent beams or plates.

In this paper, we restrict our attention to situations wherein an equivalent two-dimensional model is sufficient for the purposes of the control and suppression of the overall vibratory response of the LSS. The current state in modeling for purposes of studies of the control of the dynamic response, has been primarily one-dimensional and linear in nature[1, 2, 8], for example, axial motion of one-dimensional members[1] and the transverse motion of a beam[2]. In earlier studies by the authors, control of an equivalent continuum model of a plate has been investigated[9].

Cameron and his coworkers[10, 11] provide state-of-the-art studies on dynamics and control of large space antennae. A finite element model of the antenna, with 1240 degrees of freedom, was used to obtain the first twelve nondrig modes of vibration of the antenna assembly, of which only seven correspond to the distortional modes of the antenna dish. More than seven modes may be necessary for more precise shape control, in which case, the total number of degrees of freedom would have to be much higher. This underscores the need for alternate concepts of reduced-order structural modeling of the LSS. The issues to be addressed are: (i) for a system with a given number of degrees of freedom M what is the accuracy of that first N modes? and (ii) if the first N* modes are to be accurately evaluated, what is the minimum number of degrees of freedom M* that is needed?

In the present paper, the control problem of the equivalent continuum model of a plate governed by a fourth-order plate bending theory is investigated. A boundary element approach is used to discretize the equation of motion. Nonproportional passive damping is included, and this does not permit the decoupling of the equations of motion. Two separate approaches are considered, both of which employ optimal control techniques. The first involves a direct attack on the coupled nodal system of equations. The other approach involves the use of “global” basis functions, and these functions are the undamped normal modes of the system. In the presence of nonproportional damping, the equations of motion, even in these “global” basis functions, will still be coupled. However, it is usually necessary to control just a number of the predominant modes and so the order of the coupled system of equations can be reduced. This modal truncation may, however, lead to the undesirable spillover phenomenon[9]. The optimal control techniques require the solution of the Riccati matrix differential equation, efficient solution techniques for which are explored here. The contents of the paper are as follows. Section 2 deals with the discretization of the equation of motion using the boundary element method. Section 3 deals with algorithms for the control of the coupled system of equations. Some illustrative examples are presented in Sec. 4, and certain concluding remarks are given in Sec. 5.

2. BOUNDARY ELEMENT APPROACH TO THE EQUATION OF MOTION OF A PLATE

Here it is assumed that the LSS may be modeled by an equivalent plate model. Further, we restrict the amplitude of vibration to be small, so that the following linear dynamic equation of motion for a fourth-order plate applies:

\[ D \nabla^4 w(x, t) = -c(x) \frac{\partial^2 w(x, t)}{\partial t^2} + f(x, t) \]

where \( D \), \( c \), and \( m \) represent the equivalent distributed bending stiffness, distributed damping, and distributed mass of the system. In addition, we assume that initial stresses exist in the form of known in-plane stress resultants \( N_{11}, N_{12}, \) and \( N_{22} \), \( x_o \) are the in-plane coordinates of the plate and \( f \) is the force. The force may be expressed in two parts as follows:

\[ f(x, t) = f_e(x, t) + f_c(x, t), \]

where \( e \) and \( c \) denote, respectively, the “externally prescribed” and the “yet-to-be-solved control” forces. If eqn (2.1) is discretized using the finite element method, it is well known that (i) the trial and test functions must be \( C^1 \) continuous in and across each element, (ii) when the planar dimensions of the LSS are much larger than the equivalent thickness, a large number of elements are needed, (iii) even to obtain the first few global shape functions corresponding to undamped free vibration, a large number of finite element nodal equations have to be considered.
Another approach is to model the structure using the boundary element method. Let $w^*$ be a test function and $w$ a trial function, and therefore eqn (2.1) in weighted residual form is

$$
\int_{\Omega} L_1(w) w^* \, d\Omega = \int_{\Omega} L_2(w) w^* \, d\Omega, \tag{2.3}
$$

where $L_1$ and $L_2$ are the differential operators on the left- and right-hand sides of eqn (2.1), respectively. $w^*$ is taken to be the singular solution of the biharmonic operator. Using repeated integration by parts, it is easy to obtain the following integral relation for $w(x^*, \tau)$:

$$
\begin{align*}
&- \frac{1}{D} \int_{\Omega} \frac{\partial w^*}{\partial \tau} \left[ \frac{\partial^2 w^*}{\partial x_1^2} - \frac{\partial^2 w^*}{\partial x_2^2} \right] \, d\Omega \\
&+ \frac{1}{D} \int_{\Omega} \left[ \frac{\partial w^*}{\partial \tau} \left( f - c \frac{\partial w}{\partial \tau} - m \frac{\partial^2 w}{\partial \tau^2} \right) \right] \, d\Omega \\
&+ \frac{1}{D} \int_{\Omega} \left[ \frac{\partial w^*}{\partial \tau} \left( N_{11} \frac{\partial^2 w^*}{\partial x_1^2} - \frac{\partial w^*}{\partial \tau} \right) \right] \, d\Omega \\
&+ \frac{1}{D} \int_{\Omega} \left[ \frac{\partial w^*}{\partial \tau} \left( N_{12} \frac{\partial^2 w^*}{\partial x_1 \partial x_2} + N_{22} \frac{\partial^2 w^*}{\partial x_2^2} \right) \right] \, d\Omega. \tag{2.6}
\end{align*}
$$

The foregoing equations for $w(P)$, i.e. eqns (2.4) and (2.6), permit the solution of the general dynamic response boundary value problem wherein, in general, two of the four quantities $w$, $\partial w/\partial \tau$, $m_1(w)$, and $V_r(w)$ may be specified at $\partial \Omega$ and the other two are unknowns.

Note that there are no interior elements used to discretize the biharmonic operator in this scheme. Also, it is evident that for the integral in $\Omega$, there are no continuity requirements on the trial function $w$. Thus, in the present discretization, (i) $w$, $\partial w/\partial \tau$, $m_1(w)$, and $V_r(w)$ are interpolated in each (one-dimensional) element at the boundary $\partial \Omega$, and (ii) arbitrary interpolating functions are used for $w$, $\partial w/\partial \tau$, and $\partial^2 w/\partial \tau^2$ in each (two-dimensional) element within $\Omega$ to discretize the integral over $\Omega$ in eqns (2.4) and (2.6).

The use of eqns (2.4) and (2.6) at the boundary in conjunction with the above described discretizations leads to

$$
G_r w_\theta + L_r f - J_r \dot{w} - C_r \ddot{w} + P_r w = 0. \tag{2.7}
$$

Here it is assumed that the boundary conditions are homogeneous and so the vector $w_\theta$ is the vector of boundary unknowns, i.e. the unknowns in the set $w$, $\partial w/\partial \tau$, $m_1$, $V_r$. $G_r$ may be calculated from the appropriate boundary integrals. $f$, $w$, and $\dot{w}$ are, respectively, the vectors of force, velocity, and acceleration at the interior nodes in $\Omega$; and $L_r$, $J_r$, $C_r$, and $P_r$ are related to the appropriate interior integrals.

A second equation may be sought to express $w$, in terms of the nodal values of force, velocity, and acceleration in the interior. This second equation may be obtained by considering eqn (2.4) for various interior nodal displacements as

$$
w_i = G_i w_\theta + L_i f - J_i \dot{w} - C_i \ddot{w} + P_i w, \tag{2.8}
$$

where the matrices $G_i$, $L_i$, $J_i$, $C_i$, and $P_i$ are obtained through the appropriate integrations indicated in eqn (2.4). Using eqn (2.7), it is possible to eliminate $w_\theta$ and obtain

$$
w_\theta = -G_r^{-1} [L_r f - J_r \dot{w} - C_r \ddot{w} + P_r w]. \tag{2.9}
$$
Hence, use of eqn (2.9) in eqn (2.8) leads to
\[
\begin{align*}
\left( J_e - G_e G_r^{-1} J_r \right) \dot{w}_e + \left( C_e - G_e G_r^{-1} C_r \right) \ddot{w}_e \\
+ \left( I - P_e + G_e G_r^{-1} P_r \right) w_e = \left( L_e - G_e G_r^{-1} L_r \right) f.
\end{align*}
\tag{2.10}
\]

It is noted that a procedure analogous to the above has been used by Stern[12] and Bezine[13] in connection with problems of elastostatic response and calculation of frequencies of free vibration, respectively. Equation (2.10) may be solved directly to analyze the transient dynamic response problem. However, the central issue here is to design the control actuator forces so that the dynamic response of the system (2.10) is either damped out completely or damped to a predetermined level in a finite settling time \( t_f \).

The optimal control techniques that will be implemented for this task will be outlined below.

### 3. CONTROL OF DYNAMIC RESPONSE OF THE DISCRETIZED SYSTEM

Earlier it was assumed that the forces \( f(x_0, t) \) consisted of two parts—due to the external transverse loading and due to the actuator forces. In the following, the external loading is taken to be zero. The control forces are designed so as to minimize the response of the system due to an initial disturbance \( w(x_0, 0) = w_0(x_0) \) and \( \dot{w}(x_0, 0) = \dot{w}_0(x_0) \).
The control forces \( f_r \) are exerted by point force actuators located at discrete positions in the plate as follows:
\[
f_r(x_0, t) = \sum_{i=1}^{m} \delta(x_0 - x_0^0) f_i(t), \tag{3.1}
\]
where \( \delta(x_0 - x_0^0) \) denotes a Dirac delta function at \( x_0^0 \), where \( x_0^0 \) is the location of the \( i \)th actuator. If the dimension of \( w \) in eqn (2.17) is \( n \), we seek the number of actuators \( m \) to be such that \( m \leq n \).

Likewise, it is assumed that viscous-type passive dampers are located at discrete locations \( x_0^0 \) given by
\[
c(x_0) = \sum_{j=1}^{p} \delta(x_0 - x_0^0) C_j, \tag{3.2}
\]
where, in general, \( p \leq m \leq n \); and the locations of the dampers \( x_0^0 \) may be totally independent of the actuator locations. The resulting discretized damping matrix, in general, is not proportional either to the discretized "mass" matrix \( [J_e - G_e G_r^{-1} J_r] \) or to the discretized "stiffness" matrix \( [I - P_e + G_e G_r^{-1} P_r] \). Therefore, the passive damping described by eqn (3.2) will facilitate the investigation of nonproportional damping.

Equation (2.10) is now rewritten generically as
\[
\bar{M} \ddot{w} + \bar{C} \dot{w} + \bar{K} w = \bar{b} f, \tag{3.3}
\]

Since \( \bar{C} \) is not proportional to either \( \bar{M} \) or \( \bar{K} \), normal modes that would decouple eqn (3.3) do not exist. One method is to consider a direct attack on the nodal system of equations with an arbitrary number of actuators \( m \leq n \). Another approach is to use a set of "global shape functions", and this will be outlined later.

In the first approach, eqn (3.3) is recast in state variable form as follows:
\[
\dot{S} = \mathbf{A} S + \mathbf{B} F, \tag{3.4}
\]

where
\[
\begin{align*}
S' &= [w_1, w_2, \ldots, w_n; \dot{w}_1, \dot{w}_2, \ldots, \dot{w}_n], \tag{3.5} \\
F' &= [f_1, f_2, \ldots, f_m]. \tag{3.6}
\end{align*}
\]
\[
\mathbf{A} = \begin{bmatrix}
0 & I \\
-\bar{M}^{-1}\bar{K} & -\bar{M}^{-1}\bar{C}
\end{bmatrix}, \tag{3.7}
\]
\[
\mathbf{B} = \begin{bmatrix}
0 \\
-\bar{M}^{-1}\bar{b}
\end{bmatrix}. \tag{3.8}
\]

Thus, \( S \) is a \( 2n \times 1 \) vector, \( A \) is \( 2n \times 2n \), \( B \) is \( 2n \times m \). Further, the observations of \( P \) sensors are described by
\[
y = \mathbf{D} S. \tag{3.9}
\]

The control forces may be designed using the theory of linear optimal control[14]. Here \( F \) is chosen so as to minimize a quadratic performance index of the form
\[
J = \frac{1}{2} \int_0^{t_f} (S' Q S + F' R F) \, dt, \tag{3.10}
\]
where \( t_f \) is the final settling time, and \( Q \) and \( R \) are weighting matrices which will determine the magnitudes of the actuator forces and the quantitative decay of the vibration amplitudes.

From the minimization of eqn (3.10), we obtain[14] the equation for the feedback control forces,
\[
F = -R^{-1} B' K S, \tag{3.11}
\]
where \( K \) is the solution of the Riccati matrix differential equation
\[
\dot{K} = -KA - A'K + KB R^{-1} B' K - Q, \quad K(t_f) = Q. \tag{3.12}
\]
The solution to eqn (3.12) may be written as[15]
\[
K(t) = K_0 + Z^{-1}(t), \tag{3.13}
\]
where \( K_0 \) is the solution of the steady-state Riccati matrix differential equation.
Control of dynamic response of a continuum model

\[ -K_{tt} A - A' K_{tt} + K_{tt} B R^{-1} B' K_{tt} - Q = 0, \]

(3.14)

and \( Z(t) \) is the solution of

\[ Z(t) = A Z(t) + Z(t) A' - B R^{-1} B', \]

\[ Z(t_f) = (Q - K_{tt})^{-1}, \]

(3.15)

where

\[ A = A - B R^{-1} B' K_{tt}. \]

(3.16)

A closed form solution of eqn (3.15) exists and is given by [16]

\[ Z(t) = Z_{ss} + e^{\tilde{\alpha} t} \left[ Z(t_f) - Z_{ss} \right] e^{-\tilde{\alpha} t}, \]

(3.17)

with

\[ \tilde{\alpha} Z_{ss} + Z_{ss} \tilde{\alpha}' = B R^{-1} B'. \]

(3.18)

The solution of the nonlinear Riccati equation is one of the most limiting factors in the optimal control problem. An efficient technique that is used to solve the steady-state equation (3.14) is that based on Schur vectors which is at least an order of magnitude less expensive in computational time as compared to the simpler iterative methods. In the Schur vector approach, the Hamiltonian of the system (3.14) is defined as

\[ H = \begin{bmatrix} A & -B R^{-1} B' \\ -Q & -A' \end{bmatrix}. \]

(3.19)

To solve eqn (3.14), an orthogonal transformation \( U \) is found such that

\[ U' H U = T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \]

(3.20)

where \( T \) is a quasi-upper-triangular matrix with (1 \( \times \) 1) or (2 \( \times \) 2) blocks on the diagonal corresponding to real or complex eigenvalues. In addition, the real parts of the \( T_{11} \) eigenspectrum are negative while those of the \( T_{22} \) eigenspectrum are positive. The eigenvalues are also arranged in decreasing order. If one writes the matrix \( U \) from (3.20) as

\[ U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \]

(3.21)

then the solution to the steady-state Riccati equation may be written as [17]

\[ K_{tt} = U_{21} U_{11}'. \]

(3.22)

The above algorithm, along with certain modifications to the algorithm for finding and ordering the eigenvalues of an upper Hessenberg matrix has been implemented to solve eqn (3.14). For the number of equations considered (~100), the Schur vector algorithm has been found to be many times faster than standard iterative approaches.

3.1 Reduced order model

As mentioned earlier, another solution technique for the dynamic response equation (3.3) is to use a "global shape function" method. This has the advantage of reducing the dimensionality of the equations. As a first step, we seek global eigensolutions of the system

\[ \lambda^2 \bar{M} \bar{w} = \bar{K} \bar{w}. \]

(3.23)

The system (3.27) has eigenvalues \( \lambda_i \) and eigenvectors \( \phi_i, \) \( i = 1, 2, \ldots, n, \) which possess the usual orthogonal properties.

It is well known that the higher modes calculated through any discrete system tend to be inaccurate. Also, it is usually necessary to control just the first \( N \) modes \( (N \ll n) \). Thus, the vector \( \bar{w} \), expressed in terms of the first \( N \) global modes is

\[ \bar{w} = \xi. \]

(3.24)

where \( \xi \) is the matrix whose columns are \( N \) global eigenvectors of eqn (3.23), and \( \xi \) is a vector of undetermined coefficients. When eqn (3.24) is used, the system of equations (3.3) transform to

\[ \dot{\xi} + \bar{C}' \xi + A \xi = b' f, \]

(3.25)

where

\[ \bar{C}' = \bar{\Phi}' \bar{C} \bar{\Phi}, \]

(3.26)

\[ A = \text{diag}[\lambda_i], \quad i = 1, \ldots, N, \]

(3.27)

\[ b' = \bar{\Phi}' b. \]

(3.28)

Note that \( \bar{C}' \) is not diagonal and so eqn (3.25) still represents a coupled system of \( N \) equations with reduced dimensionality \( (N \ll n) \). This equation is similar in form to eqn (3.3), and likewise it is recast in state variable form as

\[ \dot{\eta}' = A' \eta' + B' F, \]

(3.29)

where

\[ \eta' = [\xi_1, \xi_2, \ldots, \xi_n], \quad \xi = [\xi_1, \xi_2, \ldots, \xi_n]. \]

(3.30)

and \( F, A', B' \) have analogous definitions as in eqns (3.6)-(3.8).

The control forces for this system (3.29) are designed using the procedures outlined earlier.
Table 1: Finite elements versus boundary elements. 1st Eigenvalue for S. S. plate. (The accuracy of both methods is compared with the analytical result[23].)

<table>
<thead>
<tr>
<th></th>
<th>FEM [19]</th>
<th>BEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of eqns</td>
<td>% diff.</td>
<td>No. of eqns</td>
</tr>
<tr>
<td>39</td>
<td>-2.9</td>
<td>25</td>
</tr>
<tr>
<td>95</td>
<td>1.4</td>
<td>36</td>
</tr>
<tr>
<td>175</td>
<td>-0.81</td>
<td>64</td>
</tr>
</tbody>
</table>

4. RESULTS

Numerical results are presented herein for the direct approach using the full set of nodal equations and also for the case involving the global shape functions. The domain integrals in the dynamic response equations involve the terms \( f_\omega, \omega, \partial \omega / \partial t, \) and \( \partial^2 \omega / \partial t^2; \) and interpolation of the displacement and its time derivatives assumes these quantities to be piecewise constants within each element. However, on the boundary, where the unknowns are \( w, \partial w / \partial n, m_n, \) and \( V_n, \) the interpolation is assumed to be linear in each element. Some examples are also considered where the displacement on the boundary is interpolated by a cubic polynomial. Recall that in the finite element method, the biharmonic operator is discretized in the interior requiring \( C^1 \) continuity of the trial and test functions for \( w. \) This has been a source of major difficulty in the development of the finite element method.

The effectiveness of this boundary element approach is investigated in a number of classical plate eigenvalue problems. The questions of interest are: (a) what is the number of degrees of freedom required in the two methods, i.e. the standard finite element and the present boundary element methods, to achieve a desired accuracy in the first few modes? and (b) how many fundamental modes can be computed with the desired accuracy for a given number of degrees of freedom in the two methods?

In an attempt to answer those two questions, both simply supported and cantilever plates were examined. For a simply supported square plate of length \( a, \) comparison of the boundary element and finite element methods is made in Table 1. It is evident that for comparable accuracy a far greater number of equations is required in the displacement based finite element method (a rectangular plate bending element in this case).

In the studies involving a cantilever plate of length \( a, \) cubic interpolation for the displacement was used in the boundary elements. Results for the cantilever plate are presented in Table 2 where the first four eigenvalues are given for various types of elements. The percentage difference for each of these from the Rayleigh–Ritz method is also given. It must be noted that the results from Ref. [20] were obtained using mixed elements; and so even though these results are better than the boundary element results, the cost will be much greater. However, it is seen that when compared to the conventional displacement based finite elements, the boundary element method gives superior results for a given number of degrees of freedom.

|                      | \( \omega = \frac{\lambda_1}{a^2} \sqrt{\frac{D}{m}} \) |
|----------------------|----------|----------|----------|----------|
|                      | \( \omega_1 \) | \( \omega_2 \) | \( \omega_3 \) | \( \omega_4 \) |
| Mixed elements[10]   | 3.450    | 8.642    | 21.68    | 27.89    |
|                      | ( -1.2)  | (+1.1)   | (+1.1)   | (+1.6)   |
| Displacement         | 3.424    | 8.291    | 20.37    | 25.26    |
| finite elements[11]  | ( -2.0)  | (-3.0)   | (-5.0)   | (-8.0)   |
| Boundary elements    | 3.500    | 8.812    | 22.51    | 29.47    |
|                      | (+0.2)   | (+3.1)   | (+5.0)   | (+7.3)   |
|                      | 3.491    | 8.720    | 22.13    | 28.74    |
|                      | (-0.1)   | (+2.0)   | (+3.2)   | (+4.6)   |

( ): % difference from Rayleigh–Ritz
To answer the second question, the first 17 eigenvalues for a simply supported plate, computed using the 8 \times 8 mesh of Fig. 1, are listed and compared with the analytical results of Timoshenko[23] in Table 3. As might be expected for a simply supported square plate, several of the eigenvalues are coalescent and the maximum error in the seventeenth eigenvalue is about 8%.

In the control problems considered here, the LSS is modeled as a free-free plate described by a fourth-order bending theory which has been outlined earlier. In the case of a free-free plate, the rigid motion was suppressed by supporting the plate at three corners as shown in Fig. 2. Also, in all the following control examples, the interpolation for displacement along the boundary is assumed to be linear. For the direct approach, two sets of meshes, (4 \times 4) and (5 \times 5) respectively, were employed over the entire plate as shown in Fig. 2. The displacements were assumed constant over each of these interior elements, and discrete actuators and dampers were considered to be located at the centers of these elements. Various forms of the weighting matrices Q and R of eqn (3.10) were assumed, and different combinations of point force actuators and viscous dampers were investigated. In the interest of simplicity, the final time \( t_f \) in eqn (3.10) was taken to be infinity; and hence, only the steady-state Riccati equation was solved, using the techniques outlined in Sec. 3.

The displacement at Node 16 of the plate (with (4 \times 4) mesh), with no passive damping and no initial stresses when an initial velocity \( \dot{\omega}(0) = 16.0 \) is applied at Node 16, is shown in Fig. 3. In addition,
point force actuators are at the center points of each of the 16 interior elements. Note that all quantities are expressed in a dimensionless form. The weighting matrices are taken to be

\[ \mathbf{Q} = \text{diag}\{0.2, 0.2, \ldots, 0.2\} \]

and

\[ \mathbf{R} = \text{diag}\{1.0, 1.0, \ldots, 1.0\}. \]

The corresponding force exerted by the actuator at Node 16 is shown in Fig. 4. The additional effects of passive damping are examined by assuming a value of \( C^* = 0.75 \) (eqn 3.2) at each of the sixteen interior nodes, with all other parameters remaining the same as in the previous case. The displacement for this situation is shown in Fig. 5, while Fig. 6 illustrates the effect of increasing \( C^* \) to 1.5. The beneficial effect of passive damping control on the actively controlled system is evidenced in a comparison of Figs. 3, 5, and 6. An example in which only three control force actuators (located at Nodes 9, 10, and 11) are employed to control the vibration, with no passive damping, is presented in Fig. 7. The maximum displacement is larger in this situation and the vibrations take longer to damp out. An example demonstrating the effect of the initial stresses is shown in Fig. 8, where all the other relevant parameters are similar to those in the first example (Fig. 3). The initial stresses here consisted of a tensile stress resultant in the \( x_1 \) direction of
magnitude \( N_{11}^* = (a^2/D) N_{11} = 2.0 \), while the other two stress resultants, i.e. \( N_{22} \) and \( N_{12} \), were taken to be zero. In a (frame/truss)-type structure, this corresponds to some of the members initially stressed. As might be expected, when these resultant forces are tensile in nature, the deflections will be less due to the extra stiffening effect. This is in fact the case, and it was found that the peak displacement amplitude is 13% less here as compared to when there are no initial stresses (Fig. 3). Some examples, based on a 5 x 5 mesh in the interior, are given in Figs. 9-11; and these results show similar trends to those for the 4 x 4 mesh. Here, an initial velocity \( \ddot{w}_{16} = 16.0 \) is applied at Node 25, and Fig. 9 shows the corresponding displacement at Node 25 with no passive damping or initial stresses and actuators at all the nodes. The force in the actuator at Node 25 is given in Fig. 10. Figure 11 demonstrates the effect of using just three actuators at Nodes 18, 19, and 20.

Now we discuss some results pertaining to the global basis function approach for the control problem. In these examples, we consider an 8 x 8 boundary element mesh shown in Fig. 1 and seek to control the first 17 undamped modes. Figure 12 shows the displacement at Node 16 of a free-free plate, subjected to an initial velocity at Node 16, when controllers are assumed to be located at each of the 16 nodes indicated in Fig. 1. The weighting matrix \( R \) in this case is again chosen to be the iden-
5. CONCLUDING REMARKS

In this paper a boundary element approach, based on the singular solution of the biharmonic operators, has been presented for controlling the transient dynamic response of a flat plate taken to represent the continuum model of a large space structure. Nonproportional damping is considered, and control of the response using a nodal system of equations in addition to a global shape function approach (with a corresponding reduction in dimensionality) is examined. An arbitrary number of active control force actuators as well as passive dampers is assumed to be present, and the form of damping considered leads to a fully coupled system of equations.

Implementation of the Schur vector approach in solving the steady-state Riccati equation has been carried out, and it is found to be much more economical than the standard iterative approach. Several example problems illustrating the various methodologies have been presented.

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Control of dynamic response of a continuum model

A SINGULAR-SOLUTION APPROACH FOR CONTROLLING THE NONLINEAR RESPONSE OF A CONTINUUM MODEL OF A LARGE SPACE STRUCTURE

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Abstract

The topic of vibration control of large space structures which, in this instance are modeled by equivalent continua, is addressed here. A "singular (or fundamental) solution" approach is utilized and the control algorithm is based on a fully coupled nodal system of equations which permits the effects of non-proportional damping to be monitored. Additionally, the structure will be allowed to undergo large deformations which will necessitate the implementation of a nonlinear control algorithm. In the scheme proposed here, the calculation of the feedback control forces will be based on the feedback gain matrix obtained from a solution of the linear optimal control problem. Examples will be presented to illustrate the controllability of the vibrations in both the linear and nonlinear cases.

1. Introduction

In the near future, very large low mass structures will play important roles in space missions and their size requirements, coupled with the minimum weight constraints will result in high degrees of flexibility and consequently in low levels of structural dynamic frequencies. Because these structures are so flimsy, they will collapse under their own weight if experimental tests are carried out on earth. While limited tests are presently being conducted in space, it is still necessary to develop precise mathematical models so that behavior will be predicted with a high degree of accuracy. From a materials standpoint, materials such as graphite reinforced composites exhibit mechanical deformation when an electric field is applied to them [1]. Bonding these materials to the surface would allow the application of localized stresses and strains which means the deformation could be controlled. Thus, one of the most important tasks is the need to develop a control law so as to calculate the appropriate feedback forces.

From a control viewpoint, the large space structure may be considered as a distributed parameter system which, in general, consists of a three-dimensional frame or truss. However, the direct analysis of such a structure may be too expensive and so it is feasible to employ equivalent continuum models [2]. These consist of modelling repetitive lattice grids as equivalent beams or plates and this technique is most suited to obtaining the overall response of the structure. This approach involves the determination of the equivalent elastic and dynamic properties of the continuum in terms of the geometric and material properties of the truss or frame type lattice structure and these may be obtained by equating the kinetic and strain energies of the two systems.

One technique that is often implemented as a means of control is referred to as Independent Modal Space Control (IMSC) [3]. Here, damping is either assumed negligible or is proportional to the mass or stiffness matrices and so it is possible to obtain the normal modes of the system. Based on the orthogonality of these normal modes, the system of linear differential equations is completely decoupled and the control of the response of each equation is carried out individually. Despite the fact that IMSC is mathematically simple, it is not without a number of major drawbacks which render it an ineffective means of structural control. It is likely that, due to the large complex structures and to the impulsive nature of the external disturbing forces, many modes would have to be considered. In a typical problem of practical interest, damping may also exist say due to material hysteresis or to the deliberate introduction of damping mechanisms or to the deliberate design of joints. Consequently, the concepts of modal decomposition and the decoupling of modal equations of motion are not applicable in view of the fact that the damping will generally be non-proportional in nature. Therefore, an alternative approach must be pursued here.

This method takes the form of a direct attack on the fully coupled nodal system of equations. This formulation accommodates non-proportional damping and allows the number of actuators to be much less than the order of the system that is controlled. Also, it facilitates the inclusion of nonlinearities into the problem. For a linear system, one of the most effective procedures for the design of the feedback forces is by the linear optimal control technique. An extension of this

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scheme is proposed here for the nonlinear problem with the forces calculated using the feedback gain matrix of the linear system. These control algorithms are outlined in Section 3 of this paper. Section 2 deals with the discretization of the equations of motion using a singular solution approach. Several examples that highlight the effectiveness of the proposed control scheme are presented in Section 4, while some concluding remarks are given in Section 5.

2. Integral Equation Approach to the Equations of Motion of a Plate

It was mentioned earlier that the large space structure may be conveniently described in terms of an equivalent continuum model. A thin flat plate is considered as an appropriate model for the structure and, in addition, the plate may undergo large deformations of the type where the transverse deflection may be greater than the thickness, but still small in comparison to the in-plane dimensions of the plate. These deformations are effectively described by the von Karman plate bending theory. The dynamic form of these three coupled nonlinear equations is given as follows

\[ N_{ij} = 0, \quad i, j = 1, 2 \]  
\[ D w = (N_{ij} w_{,j})_{,i} + f_{\text{ext}} \]
\[ + f_{c} - m(x_k) w - c(x_k) w \]

(2.1a)

(2.1b)

where \( m \) and \( c \) are the distributed mass and damping respectively and \( w \) is the transverse displacement. The force consists of two parts, \( f_{\text{ext}} \) is the externally applied loading and \( f_{c} \) is the control force which is yet to be calculated. The in-plane stress resultants and in all cases \( i, j \) range from 1 to 2. Note that von Karman further reduced these equations using a stress function with eq. (2.1a) replaced by a fourth order compatibility equation. However, the above form is more suitable for the present application.

Despite the versatility of the finite element method, it may not always be the most convenient approach to use. This is borne out in the applications that are proposed here. Accompanying the fourth order operator is the requirement that in the finite element method, the C' continuity condition must be satisfied in and across each element and this may be difficult to achieve. An alternative is to use the present "singular solution approach", which leads to a combined boundary/interior integral equation method which will be free of the above restrictions. Another drawback to the finite element method which comes from is that many elements are required when the in-plane dimensions of the plate are large and, consequently, there are many unknowns involved. For a given accuracy, fewer unknowns are required in an integral equation scheme [4]. Integral constraints will now be established for the above equilibrium equations and firstly the equation for transverse displacement will be considered.

Eq. (2.1b) is now expressed in weighted residual form [5] as follows with \( w^* \), an appropriate test function.

\[ \int_{A} \left( D w - (N_{ij} w_{,j})_{,i} - f_{\text{ext}} \right) \]
\[ - f_{c} - m w - c w \, w \, dA = 0 \]  

(2.2)

In order to establish a pure boundary-integral-equation type formulation, as shown in [5], the test function \( w^* \) must be carefully chosen, and in a linear problem, this test function would be taken as the fundamental (singular) solution, in infinite space, of the governing differential equation. In the nonlinear problem here, no such solution exists. The alternative proposal that is adopted is to use the singular (fundamental) solution, in infinite space, of the linear portion of the static equation which in this case is the biharmonic operator. Thus, the governing equation will not be satisfied exactly, and this will result in some integrals over the interior involving the nonlinear and time-dependent parts [6]. Therefore, the present formulation results in a combined boundary/interior integral equation method. For the biharmonic operator, the singular solution, in infinite space, is given by

\[ w^* = \frac{1}{8\pi} r^2 \ln r \]  

(2.3)

Combining with suitable weighted residual forms of the boundary conditions and using repeated integration by parts, the following relations may be derived for the transverse displacement [7]:

\[ w(P) + \frac{1}{2} \int_{S} \left( w_{*} - \frac{\partial w_{*}}{\partial n} - \frac{\partial w}{\partial n} - M_{nn} - \phi_{w_{*}} V_{n} \right) dS \]
\[ - \frac{1}{2} \int_{S} (w_{*} - M_{ns} - w) dS \]

(2.4)

\[ \frac{1}{B} \int_{A} N_{ij} w_{,i} w_{,j} dA \]
\[ - \frac{1}{B} \int_{A} \left( f_{\text{ext}} + f_{c} - m w - c w \right) dA \]

where \( x = 1, \ V_{P}A; \ x = \frac{3}{2} \ V_{P}S \) (\( \alpha \) is the included angle at \( P \)). \( M_{nn}, M_{ns} \) and \( V_{n} \) are the normal bending moment, twisting moment and effective shear respectively as functions of \( w \). \( <( \, )> \); denotes a jump in \( ( \, ) \) at a corner on \( S \) with \( K \) corners in all.

Quantities denoted by an asterisk are functions of the test function \( w^* \). Detailed expressions for each of these are given in ref. [7]. As expected, this integral equation also contains domain integrals due to the presence of the nonlinearities but these integrals do not have the continuity requirements that are inherent in the finite element method.
However, even for the linear problem, a single equation at each boundary point will be insufficient to solve a well-posed engineering problem. This is the case, because on the boundary there are two unknown quantities. Consequently, a second linear independent integral equation is sought and is achieved by considering a second fundamental solution of the biharmonic operator. A new polar coordinate system \((r, \phi)\) with respect to the \(z-n\) system of Figure 1 is formed and the singular solution \(w^{**}\) is given by

\[
 w^{**} = \frac{1}{2\pi} \ln r \cos \phi 
\]  
\tag{2.5}

and the second integral relation at the boundary point \(P\) is shown to be [7]

\[
 \frac{3w}{\delta t} \mid_{P} + \frac{w}{n_{nn}} \mid_{P} + \frac{1}{B} \int_{S} \left( ((\omega - w(P))w^{**} - \frac{\partial w}{\partial n}M^{**} \right) \mid_{n_{nn}} + \frac{\partial M}{\partial n}w^{**} \mid_{n_{nn}} ds - \frac{1}{B} \int_{A} N_{ij}w_{i}^{**}w_{j}^{**} dA 
\tag{2.6}
\]

\[
 = \frac{1}{B} \left( M_{ns} > 0 \right) \left( (\omega - w(P)) < M_{ns} \right) - \frac{1}{B} \int_{A} \left( f_{\text{ext}} + f_{c} - mw - cw \right) dA 
\]  

where \(\kappa_{s}\) and \(\kappa_{n}\) depend on \(s\) and \(n\).

\[\text{Figure 1 Details of Coordinate System}\]

While the transverse displacement portion is now complete, two further equations remain to be established for the nonlinear problem associated with the in-plane displacement. Therefore, in a similar fashion Eq. (2.1a) is recast in weighted residual form as follow.

\[
 \int_{A} N_{ij}u_{j}^{*} dA = 0 
\]  
\tag{2.7}

Here the test function \(u_{j}^{*}\) is the fundamental solution, in infinite space, of the plane stress problem [8]. Application of the divergence theorem leads to a pair of integral equations for the in-plane displacement components \(u_{i}\) [7]

\[
 \alpha_{ij}u_{j}(P) = - \int_{A} N_{ij}^{*} u_{k}(1,j) dA + \int_{S} (T_{i}^{*}u_{j}^{*} - T_{j}^{*}u_{j}^{*}) dS 
\]  
\tag{2.8}

where \(T_{i}\) is the traction and \(\alpha_{ij}\) depends on whether a boundary or interior point is under consideration. The notation \(N_{ij}^{*}\) denotes the nonlinear portion of the in-plane stress resultant and this again results in a domain integral.

In all, four integral constraints have been derived; Eqs. (2.4), (2.6) and (2.8) and these are basically four coupled nonlinear equations which completely solve the large deformation problem of a plate since at a general boundary point there will be four specified quantities and four unknowns. For the nonlinear case, an incremental formulation which leads to a piecewise linear incremental system of equations must be pursued. Space limitations in this paper prevent the detailed presentation of these equations, however, the incremental boundary quantities \(\omega_{i}, \omega_{n_{nn}}, \Delta \omega_{i}\) and \(\Delta \omega_{n_{nn}}\) are interpolated in terms of the nodal values of each one-dimensional element. In the interior, the transverse displacement along with its time derivatives will be interpolated over each element along with the two in-plane displacement variables. In conjunction with the four equations at each boundary point, it is necessary to establish three equations at each interior point corresponding to each of the displacement components. Upon performing a number of substitutions, it is possible to derive the following matrix system of equations for the incremental nodal values of the transverse displacement \(\Delta \omega_{i}\).

\[
 M_{\omega_{i}} + C_{\omega_{i}} + K_{\omega_{i}} = \Delta F + \Delta f_{c} 
\]  
\tag{2.9}

where \(\Delta F\) includes the incremental external force and the equilibrium correction vector. \(M, C\) and \(K\) are not the "usual" symmetric mass damping and stiffness matrices as in the finite element sense. In general, they will be fully populated and unsymmetric and are best labeled as coefficient matrices. Remember also the \(K\) is a function of the current deformation, i.e., \(K = K(\omega_{i})\).

3. Control of a Nonlinear Dynamical System

It is now desirable to design a control mechanism so that the dynamic response of the system governed by Eq. (2.9) is damped out to some appropriate level in a finite settling time. As was mentioned earlier, these control forces are, in the present work, assumed to be supplied by point force actuators located at various points on the structure and mathematically these are expressed as

\[
 f_{c}(x_{i}, t) = \sum_{a=1}^{m} \delta(x_{i} - x_{i}^{(a)}) f_{ca}(t) 
\]  
\tag{3.1}
where \( \delta(x_i - x_{i(c)}) \) denotes a Dirac delta function at \( x_{i(c)} \).

Also, for the purposes of simulating passive damping, discrete viscous type dampers are located as follows

\[
C(x_i) = \sum_{a=1}^{p} \delta(x_i - x_{i(a)})C_a
\]

where there are \( p \) dampers in all.

The design of the control forces for the nonlinear system will be based on the feedback gain matrix established for an 'optimal' control of linear system of equations which are repeated here for the sake of completeness

\[
\dot{w}_s + C \dot{w}_s + K^L w_s = BF_c
\]

and this is subject to the initial conditions

\[
w_s|_{t=0} = w_0 ; \quad \dot{w}_s|_{t=0} = \dot{w}_0
\]

Here, the "stiffness matrix" \( K^L \) relates to just the linear problem and the dimension of \( w_s \) is \( n \). It is usually desirable that the number of actuators and dampers is much less than \( n \) and also the locations of these actuators and dampers will be totally independent in the general case. The one requirement on the number of actuators is that the system is controllable. System controllability will be defined later. Remember also that generally \( C \) is not proportional to either \( K^L \) or \( M \) and the so the normal modes that would decouple Eq. (3.3) do not exist. Here, a direct attack on the coupled nodal system of equations is implemented and so Eq. (3.3) is recast in the standard state variable form as follows

\[
\dot{S} = AS + BF ; \quad S|_{t=0} = S_0
\]

where

\[
S^T = [w_{s1} \ w_{s2} \ldots \ w_{sn} \ \dot{w}_{s1} \ \dot{w}_{s2} \ldots \ \dot{w}_{sn}]
\]

\[
F^T = [f_{c1} f_{c2} \ldots f_{cm}]
\]

\[
A = \begin{bmatrix}
0 & 1 \\
-M^{-1}K^L & -M^{-1}C
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
-M^{-1}b
\end{bmatrix}
\]

Note that \( A \) is a \( 2n \times 2n \) matrix and \( B \) is a \( 2n \times m \) matrix.

A linear system is said to be controllable at \( t_0 \) if there exists a control vector \( f_c(t) \), \( t_c[t_0, t_f] \), which transfers the initial state \( S \) to the origin at some finite time \( t_f > t_0 \). If this is the case for all \( t_0 \) and all \( S_0 \), then the system is said to be controllable. A system is controllable if and only if the following \( 2n \times 2n \) matrix has rank \( 2n \) \([9]\).

\[
E_c = \begin{bmatrix}
B & A^2B & \ldots & A^{2n-1}B
\end{bmatrix}
\]

This condition must be verified for each particular example.

The linear control problem consists of finding a control \( f_c(t) \) that minimizes a performance index subject to a set of constraint equations. Typically a quadratic performance index may be used and here this will take the following form

\[
J = \frac{1}{2} \int_{0}^{t_f} (S^T QS + F^T RF) dt
\]

\( Q \) and \( R \) are referred to as weighting matrices and their selection is somewhat arbitrary although the existence of a solution is dependent on their positive definiteness. The quantity \( t_f \) is the final settling time and for simplicity it is taken as infinity. It has been shown in the optimal control literature \([10]\), using the Hamilton-Jacobi-Bellman equation, that the feedback control forces may be expressed as

\[
F = -R^{-1}B^T GS
\]

Hence, for the linear problem, Eq. (3.12) leads to the optimal closed loop system

\[
\dot{S} = A^* S ; \quad S|_{t=0} = S_0
\]

where

\[
A^* = A - GR^{-1}B^T G
\]

In Eq. (3.12) \( G \) is the solution of the steady state Riccati matrix equation.

\[
GA + A^T G - GBR^{-1}B^T G + Q = 0
\]

Thus, the nonlinear algebraic equation (3.15) must be solved for the \( (2n \times 2n) \) symmetric matrix \( G \). This is one of the more limiting factors in implementing the control algorithm. While iterative techniques have often been used for the solution of this equation, a method based on a Schur vector approach has been found here to be more efficient \([11]\). The Hamiltonian of the system
is defined as

\[ Maw + (C bR -1T G )aw s - 2.5 + (K + B T G I )aw s = aF \]  
(3.21)

An orthogonal similarity transformation \( U \) is found such that

\[ U^T M U = T \]

where each \( T_{ij} \) is a 2n x 2n matrix. This is known as real Schur canonical form. \( T \) is a quasi upper triangular matrix, i.e., upper triangular with the possibility of 2 x 2 blocks on the diagonal corresponding to complex conjugate pairs of eigenvectors. The columns of \( U \) are referred to as the Schur vectors corresponding to \( T \), and \( U \) can be chosen so that the diagonal blocks appear in any order. Moreover, it is possible to arrange that the real parts of the \( T_{11} \) eigenspectrum are negative while those of the \( T_{22} \) eigenspectrum are positive. If \( U \) is written as four 2n x 2n blocks

\[ U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \]

(3.18)

it has been shown that the solution to the steady state Riccati equation may be expressed as

\[ G = U_{22} U_{11}^{-1} \]

(3.19)

Thus, an algorithm has been developed for establishing the feedback gain matrix for the linear optimal control problem but such optimal methods may not be applied to nonlinear problems. A set of control forces must now be designed for the nonlinear system given by Eq. (2.9). The idea that will be promoted here is comprised of taking the feedback gain matrix from the linear control problem and using it to calculate the nonlinear controls, i.e., the Riccati equation solution for the linear problem is used. Splitting \( G \) into portions relating to the displacement and velocity vectors of the nodal values gives the control forces as

\[ aF_c = -R^{-1} T [G_1 \Delta w_s + G_2 \Delta \dot{w}_s] \]

(3.20)

where \( \Delta w_s \) is now the incremental displacement of the nonlinear system, and \( \Delta \dot{w}_s \) is the incremental velocity of the nonlinear system. Substituting

\[ -M \Delta \dot{w}_s + (C + BR^{-1} T G_2 \Delta w_s + (K + BR^{-1} T G_1) \Delta w_s = aF \]

(3.21)

This reduces the problem to a standard nonlinear dynamic response problem for the incremental displacement \( \Delta w_s \), which may be solved using regular nonlinear techniques. It is important to remember that this result will no longer provide a result which is optimal.

The question of stability of this nonlinear system also needs to be addressed. Using the second method of Liapunov [12], it is possible to find a Liapunov function and it may be shown that the condition for asymptotic stability is the same as that for the linear optimal control problem. This criterion is satisfied if the eigenvalues of \( A \) (given by Eq. (3.14)) occur in complex conjugate pairs with negative real parts and this may be checked for each geometrical configuration.

4. Results

A number of examples will now be presented that will illustrate the nonlinear control concepts which were introduced in the previous section. Here, the emphasis will be on the nonlinear problems as the linear case has already been treated in earlier publications [5, 13]. For the purposes of illustration, a simply supported plate, as shown in Figure 2, will be considered here. A 4x4 mesh is adopted in the interior and linear interpolations are used for both the boundary and interior incremental unknowns. In all cases, the plate was initially disturbed by a nondimensional velocity expressed as

\[ \sqrt{\frac{m a^4}{D h^2}} w_{t=0} = 135 \sin \frac{x_1}{a} \sin \frac{x_2}{a} \]

(4.1)

The first example (Figure 3) plots the nonlinear response at the center of the plate for the following forms of the weighting matrices.

\[ Q = \text{diag} [0.02 0.02 ... 0.02] \]

(4.2)

and

\[ R = \text{diag}[1.0 1.0 ... 1.0] \]

(4.3)

In this example, no passive damping is included, but point actuators are located at each of the nine internal nodes and it is seen that they effectively damp out the vibrations in a finite period of time. Note also that, in these examples, the results are expressed in a dimensionless form. The corresponding problem was also analyzed using just the linear portion of the equations and
these results are also included in Figure 3. The nonlinearities in the plate bending problem are strain hardening in nature and this should produce an additional damping on the structure which is evident in Figure 3 since the nonlinear displacements are less. The corresponding actuator forces at the plate center are illustrated in Figure 4.

![Figure 2 Sixteen Element Mesh for Nonlinear Problem](image)

To quantify the effects of the nonlinearities, the performance index (J) was calculated for both cases and this is illustrated in Figure 5 and, as expected, J is greater in the nonlinear problem. With reference to this plot, it is seen that the two curves coincide initially and this is when the deformations are in the linear range. When the nonlinear effects come into play and the curves diverge with the values for the linear case being greater. Then, at later times when the vibrations have diminished, the performance index levels off in both cases.

![Figure 3 Displacements at Node 5 for Linear and Nonlinear System with No Damping](image)

![Figure 4 Acuator Forces at Node 5 for Linear and Nonlinear Systems with No Damping](image)

![Figure 5 Comparison of Linear and Nonlinear Performance Indices (J)](image)
To illustrate the influence of passive damping, the viscous dampers described by Eq. (3.2) are located at each of the nine interior nodes. Figure 6 shows the results at the plate center with a comparison made with the undamped response. It is evident that the dampers have a beneficial effect on the vibration magnitude since the displacements have been reduced.

Figure 6: Comparison of Damped and Undamped Nonlinear Response at Node 5

$$Q = \text{diag}[0.02, 0.02, \ldots]$$
$$R = \text{diag}[1.00, 1.00, \ldots]$$

Figure 7 demonstrates the influence of varying the weighting matrix $Q$. Here, $Q$ is adjusted to

$$Q = \text{diag}[0.015, 0.015, \ldots, 0.015]$$

and as may be concluded from Figure 7, the central deflection increases due to the lesser weighting attached to the state vector.

The results presented here demonstrate the suitability of the proposed nonlinear control algorithm for suppressing the vibrations of the large space structure which is modeled here as a flat plate.

5. Conclusions

A singular solution approach using an innovative weighted residual form of the equilibrium equations has been presented here for controlling the dynamic response of a large space structure. The fundamental solutions of both the biharmonic operator and the Navier equations are used and since the equilibrium equations are nonlinear, a number of interior integrals will remain but the requirements on the interpolating functions are not as stringent as in the finite element method. This singular solution scheme has the advantage of achieving a given accuracy with fewer degrees of freedom. The direct attack on the nodal system of equations also permits the inclusion of a number of practical features such as non-proportional damping.

Acknowledgements

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References


INSTABILITY ANALYSIS OF SPACE TRUSSES USING EXACT TANGENT-STIFFNESS MATRICES

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Abstract. A simple (exact) expression for the tangent-stiffness matrix of a space truss undergoing arbitrarily large deformation, as well as member buckling, is given. An arc-length method is used to solve the tangent-stiffness equations in the post-buckling range of the structural deformation. Several examples to illustrate the viability of the present approaches in analyzing large space structures, simply, efficiently, and accurately, are given.

Introduction

Currently, there is an enormous interest in deploying large structures in outerspace for a variety of reasons. These structures are, in general, of very low mass and very high flexibility. A pressing technical problem in the design of these structures is the need for active or passive control of transient dynamic (traveling wave type) response. Since these structures are highly flexible, there is the inherent need to account for large deformations. The transient dynamic response equations for the space structure may be written as

\[ M \ddot{d} + C \dot{d} + S(d) = f + Q_E. \]  

where \( M \) is the mass matrix, \( C \) is the matrix of viscous damping which arises due to a deliberate design of the structural joints, among other reasons, \( S \) is the vector of nodal restraining forces which \( d \) and \( \dot{d} \) nonlinearly on the vector of nodal displacements \( d \). \( f \) is the vector of control forces to be determined from a properly formulated feedback control strategy. \( Q_E \) is the vector of externally applied dynamic nodal loads, and \( d, \dot{d}, \) and \( \ddot{d} \) are, respectively, the vectors of velocity and acceleration. In transient dynamic response calculations, it is customary to linearize equation (1) as

\[ M^{(N)} \ddot{d} + C^{(N)} \dot{d} + S^{(N)} \Delta d = f^{(N)} + Q_E^{(N)} - R^{(N)}. \]

where, now, \( K^{(N)} \) is the so-called tangent-stiffness matrix at state \( N \) (or at time \( t_N \) in a time-integration scheme). \( \Delta d \) is the incremental displacement vector, and \( R \) are internal restraining forces at \( t_N \).

In the usual finite element analysis, much of the effort is usually expended in evaluating the tangent-stiffness matrix \( K^{(N)} \), which accounts for the effects of initial displacements and initial stresses. Usually, one assumes basis functions and integrates over the element the appropriate strain energy terms (that depend nonlinearly on the displacements) to obtain \( K^{(N)} \). The

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objective of the present research is to obtain an explicit expression for \( (N \times K) \) for large space structures, without the use of assumed element basis functions and element integrations. The aim is to obtain analytical solutions to the appropriate nonlinear ordinary differential equations and to use them to derive exact (closed form) expressions for stiffness matrices. It should be noted that, while the concepts of tangent stiffness finite element method and arc-length solution method were not used, several interesting analytical studies of stability of space-trusses were earlier presented by Britvec [1].

Depending on the design of the joints, each member of a tree-dimensional (space) structure may be considered as a 'truss' member or a 'frame' member. The truss member carries only an axial force, and the kinematics of deformation is characterized by the three displacements of each node. The frame member carries bending moments, a twisting moment, and lateral forces, in addition to an axial force; and the deformation is characterized by the three displacements and three rotations at each node.

The present paper is limited to a large deformation, post-buckling analysis of large space trusses under quasi-static loading using explicit expressions for \( (N \times K) \). Thus, in (2), \( \delta \), \( \delta \), \( f \) are set to zero. The nonlinear tangent-stiffness equations are then solved by using an arc length method. Several examples are given to illustrate the simplicity of the present approach which renders the large deformation analysis of reasonably large-sized space trusses suitable for currently available personal computers.

In the next section, a derivation of the tangent-stiffness matrix is given, the third (short) section deals with a brief description of the arc length method, and the final section deals with several examples.

**Derivation of an explicit tangent-stiffness matrix for finite-deformation, post-buckling analysis of space trusses**

The space truss structures discussed herein are assumed to remain elastic. Also, only a conservative system of concentrated loads at the nodes of the space truss structures is considered.

**Relation between stretch and axial force in a truss member**

Consider a typical slender truss member spanning between nodes 1 and 2 as shown in Fig. 1. This member is considered to have a uniform cross section, and its length before deformation is \( l \). The coordinates \( x_1 \), \( x_2 \), and \( x_3 \) are the member's local coordinates, while \( u_1 \), \( u_2 \), and \( u_3 \) denote the displacements at the centroidal axis of a member along the coordinate directions \( x_1 \), \( x_2 \), and \( x_3 \), respectively.

From the polar decomposition theorem, the relation between the total axial stretch and displacements of the member is

\[
\delta = \left[ (\bar{u}_1)^2 + (\bar{u}_2)^2 + (1 + \bar{u}_3) \right]^{1/2} - 1, \tag{3}
\]

where \( \delta \) is the total axial stretch, \( \bar{u}_1 = \bar{u}_1 - u_1 \), \( \bar{u}_2 = \bar{u}_2 - u_2 \), and \( \bar{u}_3 = \bar{u}_3 - u_3 \). Equation (3) holds for both the pre- and post-buckled states of the member.

The incremental relation between the incremental total stretch and the incremental axial force in the member is written as

\[
\Delta N = k \cdot \Delta \delta. \tag{4}
\]
where

\[ \Delta N : \text{incremental axial force in the member}, \]
\[ \Delta \delta : \text{incremental total axial stretch in the member}, \]
\[ k = EA/I \text{ in the pre-buckled state}, \]
\[ = \pi^2 EI/2I^3 \text{ in the post-buckled state (for the range of deformations considered)}, \]
\[ E : \text{Young's modulus}, \]
\[ A : \text{cross sectional area of the member}, \]
\[ I : \text{moment of inertia}. \]

Equation (5a) simply follows from the linear-elastic (isotropic) stress-strain law of the material of the member. On the other hand, equation (5b) for the post-buckled state of the member is derived in Appendix A by simplifying and modifying the governing equations of the problem of the elastica, which is treated as a simply supported beam.

Here, one should note that \( N \) is in the direction of the straight line connecting node 1 and node 2 of the member after its deformation (see Fig. 1), and \( \delta \) is calculated from equation (3). Hence, equation (4) holds even when the rigid motion of the member is very large. Also, note that the stiffness-coefficient \( k \) is a constant in each of the two states, such as pre-buckled and post-buckled, of each member, of a space truss.

The condition for the buckling of a member, treated as a simply supported beam, is given by the following well-known equation:

\[ N = \sigma_{\text{crit}} N, \]

where

\[ \sigma_{\text{crit}} = -\pi^2 EI/I^3. \]

the negative sign being used to denote the compressive axial force.

\(^1\) While the material is assumed to be linear-elastic in the present, the subsequent derivations of the tangent stiffness matrix remain valid, with straightforward modifications, even when the material stress-strain law is of the Ramberg-Osgood type: \( \sigma = E\varepsilon - B\varepsilon^n \).
Tangent-stiffness matrix of a member for arbitrarily large deformation

The only force acting on a truss-member is considered to be the axial force. Hence, the strain energy of the member, \( U \), in either the pre- or post-buckled states of the member, is given by

\[
U = \frac{1}{2} \int_0^1 \left( EA \cdot \varepsilon^2 + EI \cdot \kappa^2 \right) \, dx = \int_0^1 N \, d\delta. 
\]  

(8)

where \( \varepsilon \) is the point-wise axial stretch, and \( \kappa \) the curvature. \( \kappa = 0 \) for the pre-buckled state, and \( \kappa \neq 0 \) for the post-buckled state.

The incremental form of equation (8) is represented, using equation (4), as

\[
\Delta U = N \cdot \Delta \delta = \frac{1}{2} k (\Delta \delta)^2. 
\]  

(9)

The incremental form of equation (3) is given by

\[
\Delta \delta = a \cdot \Delta \bar{u}_1 + b \cdot \Delta \bar{u}_2 + c \cdot \Delta \bar{u}_3 \\
+ \frac{1}{2l^*} \left[ (b^2 + c^2) \cdot \Delta \bar{u}_1^2 + (c^2 + a^2) \cdot \Delta \bar{u}_2^2 + (a^2 + b^2) \cdot \Delta \bar{u}_3^2 \right] \\
- \frac{1}{l^*} \left[ (a \cdot b) \cdot \Delta \bar{u}_1 \Delta \bar{u}_2 + (b \cdot c) \cdot \Delta \bar{u}_2 \Delta \bar{u}_3 + (c \cdot a) \cdot \Delta \bar{u}_3 \Delta \bar{u}_1 \right] \\
+ \text{higher order terms.} 
\]  

(10)

where

\[
l^* = \left[ \left( \bar{u}_1 \right)^2 + \left( \bar{u}_2 \right)^2 + \left( l + \bar{u}_3 \right)^2 \right]^{1/2}, \\
a = \bar{u}_1/l^*, \quad b = \bar{u}_2/l^*, \quad c = (l + \bar{u}_3)/l^*, \\
\Delta \bar{u}_1, \Delta \bar{u}_2, \text{ and } \Delta \bar{u}_3 \text{ represent the increments of } \bar{u}_1, \bar{u}_2, \text{ and } \bar{u}_3, \text{ respectively.}
\]

Substituting equation (10) into equation (9), one finds that

\[
\Delta U = N(a \cdot \Delta \bar{u}_1 + b \cdot \Delta \bar{u}_2 + c \cdot \Delta \bar{u}_3) + \frac{1}{2} \left[ (b^2 + c^2) \cdot \frac{N}{l^*} + k \cdot a^2 \right] \Delta \bar{u}_1^2 \\
+ \frac{1}{2} \left[ (c^2 + a^2) \cdot \frac{N}{l^*} + k \cdot b^2 \right] \Delta \bar{u}_2^2 + \frac{1}{2} \left[ (a^2 + b^2) \cdot \frac{N}{l^*} + k \cdot c^2 \right] \Delta \bar{u}_3^2 \\
+ \left( k - \frac{N}{l^*} \right) \left[ (a \cdot b) \cdot \Delta \bar{u}_1 \Delta \bar{u}_2 + (b \cdot c) \cdot \Delta \bar{u}_2 \Delta \bar{u}_3 + (c \cdot a) \cdot \Delta \bar{u}_3 \Delta \bar{u}_1 \right] \\
+ \text{higher order terms.} 
\]  

(11)

Furthermore, neglecting terms of higher than the second order, the variation in the incremental strain-energy may be derived from equation (11) as

\[
\delta(\Delta U) = \delta \Delta \bar{u}_1(\cdot \Delta \bar{u}_1) + \delta \Delta \bar{u}_2(\cdot \Delta \bar{u}_2) + \delta \Delta \bar{u}_3(\cdot \Delta \bar{u}_3) \\
+ \delta \Delta \bar{u}_1 \left\{ \left( b^2 + c^2 \right) \frac{N}{l^*} + k \cdot a^2 \right\} \Delta \bar{u}_1 \\
+ \left( k - \frac{N}{l^*} \right) \cdot a \cdot \Delta \bar{u}_2 + \left( k - \frac{N}{l^*} \right) \cdot c \cdot \Delta \bar{u}_3 \\
+ \delta \Delta \bar{u}_2 \left\{ \left( c^2 + a^2 \right) \frac{N}{l^*} + k \cdot b^2 \right\} \Delta \bar{u}_2 \\
+ \left( k - \frac{N}{l^*} \right) \cdot b \cdot \Delta \bar{u}_3 + \left( k - \frac{N}{l^*} \right) \cdot a \cdot \Delta \bar{u}_1 \\
+ \text{higher order terms.} 
\]
\[ + \delta \Delta \vec{u}_3 \left[ \left( a^2 + b^2 \right) \frac{N}{I_3} + k \cdot c^2 \right] \Delta \vec{u}_3 \]

\[ + \left( k - \frac{N}{I_3} \right) \cdot c \cdot a \cdot \Delta \vec{u}_1 + \left( k - \frac{N}{I_3} \right) \cdot b \cdot c \cdot \Delta \vec{u}_2 \]

\[ = \delta \Delta d^m \cdot R^m + \delta \Delta d^m \cdot K^m \cdot \Delta d^m. \tag{12} \]

where

- \( d^m \): vector of generalized nodal displacements.
- \( R^m \): vector of internal forces.
- \( K^m \): stiffness matrix of the element.

\[ \Delta d^m = [\Delta u_1; \Delta^2 u_1; \Delta^3 u_1; \Delta^2 u_2; \Delta^3 u_2]. \]

\[ \begin{align*}
R^m &= \begin{pmatrix} (N \cdot a) \cdot (I) \\ (N \cdot b) \cdot (I) \\ (N \cdot c) \cdot (I) \end{pmatrix}, \\
\end{align*} \]

\[ C_1 = (b^2 + c^2) \cdot \frac{N}{I_3} + k \cdot a^2, \quad C_4 = \left(k - \frac{N}{I_3}\right) \cdot a \cdot b. \]

\[ C_2 = (c^2 + a^2) \cdot \frac{N}{I_3} + k \cdot b^2, \quad C_5 = \left(k - \frac{N}{I_3}\right) \cdot b \cdot c. \]

\[ C_3 = (a^2 + b^2) \cdot \frac{N}{I_3} + k \cdot c^2, \quad C_6 = \left(k - \frac{N}{I_3}\right) \cdot c \cdot a. \]

\[ (I) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad [E] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{13} \]

One should note that equations (12) and (13) are written in the local coordinate system, so that it is necessary to transform the displacement vector from the local coordinate system to the global coordinate system in the usual fashion.

It should be emphasized again that equations (12) and (13) (and, thus, the tangent-stiffness matrix and the load vector) are applicable for both the pre- and post-buckled states of the member, and that \( k \) has a constant value in each of the two states as given in equation (5). Consequently, if a member buckles, it is only necessary for the value of \( k \) to be changed. In view of this, it is seen to be very simple to derive the tangent-stiffness of the member, and, thus, of the structure as a whole.

**Solution strategy: Arc-length method**

Although a number of solution procedures is available for nonlinear structural analysis, a reliable approach to trace the structural response near limit points, and in post-buckling range, is the 'arc-length' method [2,3,4]. This method is the incremental iterative procedure which represents a generalization of the displacement control approach. The arc-length method, in which the Euclidian norm of the increment in the displacement and load space is adopted as the prescribed increment, allows one to trace the equilibrium path beyond limit points such as in snap-through and snap-back phenomena.

A full description of the presently adopted procedure is already given in [4] and will not be repeated here.

**Example problems of space trusses**

The first example considered in this category is the shallow geodesic dome shown in Fig. 2. This structure, which exhibits a snap-through phenomenon, is subjected to one concentrated
Two initial configurations of the structure, one geometrically perfect and the other with slight imperfections, as specified in Table 1, are considered. This example was also analyzed, using a perturbation method, by Hangai and Kawamata [6] to study global stability. In the present study, however, the influence of member buckling on global stability is also examined.

Figs. 3 and 4, for the case of perfect geometry, show a typical snap-through phenomenon wherein the first limit point is reached at a load of $3.15 \times 10^{-4} \, EA$ (Kg). The present results are seen to be in good agreement with those of Hangai and Kawamata [6].

Table 1

<table>
<thead>
<tr>
<th>Node</th>
<th>(1) Perfect geometry</th>
<th>(2) Imperfect geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>25.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>12.5</td>
<td>-21.65</td>
</tr>
<tr>
<td>4</td>
<td>-12.5</td>
<td>-21.65</td>
</tr>
<tr>
<td>5</td>
<td>-25.0</td>
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<tr>
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<td>-12.5</td>
<td>21.65</td>
</tr>
<tr>
<td>7</td>
<td>12.5</td>
<td>21.65</td>
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<tr>
<td>8</td>
<td>43.30</td>
<td>-25.0</td>
</tr>
<tr>
<td>9</td>
<td>0.0</td>
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</tr>
<tr>
<td>10</td>
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<tr>
<td>11</td>
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<tr>
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<tr>
<td>13</td>
<td>43.30</td>
<td>25.0</td>
</tr>
</tbody>
</table>
The influence of member buckling on global instability is illustrated in Figs. 5, 6, and 9 which indicate that a global behavior strongly depends on the buckling of a single member. In a practical design of a three-dimensional truss structure, this understanding is very essential and useful. Also, the effects of slight geometric imperfection are illustrated in Figs. 7 and 8 wherein the comparison results of Hangai and Kawamata [6] are also included.

Example (2) is also that of a shallow geodesic dome, analyzed earlier by Noor and Peters [7] and shown in Fig. 10. Two types of loading systems are considered: the first loading system...
Fig. 5. Vertical displacements of central node with and without the influence of local buckling of truss members.

Fig. 6. Horizontal radial displacements of non-central nodes with and without the influence of local buckling of truss members.

consists of lateral concentrated loads $P_1$ over the entire dome; the second one, $P_2$, consists of concentrated lateral loads only over a quarter of the dome. An important difference between the present analysis and that of Noor and Peters [7] should be mentioned. Noor and Peters [7] ignore member buckling and assume each member of the truss to remain straight and stable. On the other hand, in the present analysis, local buckling of each member is allowed; and only for
comparison purposes, results are also obtained using the present procedure with local buckling being intentionally suppressed.

Fig. 11 provides a comparison of the vertical displacement of the central node in the present and Noor and Peters' solutions for various combinations of $P_1$ and $P_2$, when local (member) buckling is ignored. The present results agree well with those of [7] except beyond the limit.
point of $P_1 = -5.132 \times 10^{-5}$. The stability boundary, i.e., the combinations of the load parameters $P_1$ and $P_2$, which render the structure unstable when local member buckling is ignored, is shown in Fig. 12, from which an excellent correlation of the present results (with member-buckling being suppressed) with those of [7] may also be noted.

Figs. 13 to 16 show the present results when local (member) buckling is considered. Fig. 13 shows the variation of vertical displacement of the central node; Fig. 14 shows the stability boundary under the various combinations of $P_1$ and $P_2$, and Fig. 15 shows the equilibrium path under the load system $P_2 = 0$ and $P_1 \neq 0$.

From this numerical example (especially Fig. 14), it is clear that the decrease in the magnitude of critical loads for the structure, due to buckling of an individual member or members, i.e., the influence of local buckling on the response of the structure as a whole, is quite remarkable.

The third example of space trusses is that of a beam-like space truss (PACOSS Truss) subjected to axial and bending loads. The structure is that of a twelve-bay truss whose member properties
Equation of Surface
\[ x_1^2 + x_2^2 + (x_3 + 7.2)^2 = 60.84 \]

Loading System
- \( P_1 \): Concentrated Load in \( x_3 \) direction at all nodes
- \( P_2 \): Concentrated Load in \( x_3 \) direction at nodes having \( x_2 = 0, x_2 < 0 \)

Fig. 10. Schematic of shallow geodesic dome.

Fig. 11. Vertical displacements of central node under various combinations of loads, \( P_1 \) and \( P_2 \), without the influence of local buckling of truss members.

are shown in Figs. 17 and 19. In order to trigger the coupling between the axial and transverse displacements, which is characteristic of the buckling mode, in the case of only an axial-load application, a 'load imperfection' equal to \( P/1000 \) is added in the transverse direction at one of the end nodes, as shown in Fig. 17.
Fig. 12. Stability boundary under various combinations of loads, $P_1$ and $P_2$, without the influence of local buckling of truss members.

Fig. 13. Vertical displacements of central node under various combinations of loads, $P_1$ and $P_2$, with and without the influence of local buckling of truss members.
Fig. 14. Stability boundaries under various combinations of loads, $P_1$ and $P_2$, with and without the influence of local buckling of truss members.

Fig. 15. Vertical displacements of central node under load $P_1$ with and without the influence of local buckling of truss members.
For the above predominantly axial-load case, Fig. 18 shows the relation between the magnitudes of the axial load and that of the transverse displacement at the loaded end, for two scenarios: (i) when local (member) buckling is suppressed and each member is assumed to remain straight and stable, and (ii) when each member is allowed to undergo local buckling. Fig. 18 clearly demonstrates the advantageous effect of controlling the local buckling deformations of individual members and forcing them to remain straight and stable. This leads one to the concept of active/passive control of member deformations.

Fig. 19 shows the schematic of the PACOSS Truss subject to predominantly bending loads. Fig. 20 shows the relation between the magnitudes of transverse (bending) load and transverse displacement, respectively, once again for two scenarios: (i) when local member buckling is suppressed, and (ii) when member buckling is allowed. Fig. 20 again demonstrates the beneficial effects of control of deformations of each member. Fig. 21 shows a computer plot of the deformed shape of the PACOSS Truss under bending loads.

It should be noted that in Figs. 18 and 20, the letters A, B, C, etc. indicate the stages at which the respective members, whose numbers are identified in Figs. 18 and 20 respectively, undergo local buckling.
Closure

While simple methodologies for large deformation analyses of large space structures (LSS) modeled as trusses are treated in this paper, similar methodologies for analyses of LSS modeled as frames (with each member carrying three moments and three forces) are objects of a forthcoming paper.
Appendix A. Post-buckling behavior of a truss member

In this appendix, equation (5b) for the post-buckled state of the truss member is derived. Consider the truss member being subjected to the compressive force \((-N)\), as shown in Fig. 1. When \(N\) satisfies equation (6), this member undergoes bifurcation buckling. From the detailed treatment of the elastica problem given in [8], the post-buckling behavior of the member, treated as a simply-supported beam, is governed by the following equations:

\[
 l = \frac{1}{f} \cdot F(\beta) . \tag{A.1a}
\]

\[
 l + \delta = \frac{2}{f} \cdot E(\beta) - l . \tag{A.1b}
\]

\[
 \delta = \frac{2}{f} \cdot \beta . \tag{A.1c}
\]

where

\[
 F(\beta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\phi}{1 - \beta^2 \sin^2 \phi} . \quad E(\beta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - \beta^2 \sin^2 \phi}{1 - \beta^2 \sin^2 \phi} \cdot d\phi . \tag{A.2}
\]

\[
 f^2 = -\frac{N}{EI}, \quad \beta = \sin(\frac{1}{2} \alpha). \quad \alpha = \theta_{\mid_{u=0}} = -\theta_{\mid_{u=\lambda}} .
\]
and $F(\beta)$. $E(\beta)$ are the elliptic integrals of the first and the second kind, respectively. Also, $\hat{\delta}$ is the stretch after the buckling of the member, and $\hat{\delta}$ is the lateral deflection at the middle of the centroidal axis of the element. Note that the total stretch $\delta$ is given by the sum of $\hat{\delta}$ and the stretch, $\frac{1}{1/EA} \text{E} \cdot F(\beta)$, before the buckling of the member. Also, it should be noted that in the derivation of equations (A.1), the change in the length of the member due to the compressive force is neglected.
Equations (A.1) give the exact relations between $N_i$, $\delta_i$, and $\delta_i$ in the post-buckled range, except for the assumption concerning the length of the element. We now simplify and modify these relations to a form more useful for the present purposes of evaluating a tangent-stiffness matrix. To this end, we start by expanding $F(\beta)$, $E(\beta)$ in terms of $\beta$ (see [9]):

$$F(\beta) = 1 + \frac{1}{2} \beta^2 \cdot S_2 + \frac{1}{2} \cdot \beta^4 \cdot S_4 + \ldots.$$  \hspace{1cm} (A.3a)

$$E(\beta) = -\frac{1}{2} \beta^2 \cdot S_2 - \frac{1}{2} \cdot \beta^4 \cdot S_4 + \ldots.$$  \hspace{1cm} (A.3b)

where

$$S_4 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 \phi \cdot d\phi.$$  \hspace{1cm} (A.4)

We shall retain the terms of equations (A.3) up to the second order for the approximations of $F(\beta)$, $E(\beta)$:

$$F(\beta) = 1 + \frac{1}{2} \pi \cdot \beta^2.$$  \hspace{1cm} (A.5a)

$$E(\beta) = -\frac{1}{2} \pi \cdot \beta^2.$$  \hspace{1cm} (A.5b)

The range of validity of these approximations will be demonstrated momentarily.
Fig. 21. Deformed configurations of a twelve-bay space truss shown in Fig. 17.

Then, equations (A.1a) and (A.1b), respectively, become

\[ l = \frac{1}{f} (\pi + \frac{1}{2} \pi \beta^2) \]  \hspace{1cm} (A.6a)

\[ l + \delta = \frac{2}{f} (\pi - \frac{1}{2} \pi \beta^2) - l \]  \hspace{1cm} (A.6b)
From equations (A.6) one obtains
\[ 4l + \delta = \frac{4}{f} \cdot \pi. \]  
(A.7)

Noting that \( f^2 = (-N/EI) \), one sees from equation (A.7).
\[ N = N^{(cr)} \left[ 1 - \frac{1}{4} \left( \frac{\delta}{l} \right)^2 \right]. \]  
(A.8)

where \( N^{(cr)} \) is the critical axial force for bifurcation buckling as given in equation (7).

For small values of \(-\delta/l\), equation (A.8) may be approximated as
\[ N = N^{(cr)} \left[ 1 - \frac{1}{2} \left( \frac{\delta}{l} \right) \right]. \]  
(A.9)

The incremental form of equation (A.9) results in equations (5). The linear relation (A.9), and its incremental counterparts, are useful in tangent-stiffness evaluations.

We now derive the relation between \( \delta \) and \( \delta \). This relation is not necessary for the construction of the tangent stiffness, but it is useful for the determination of maximum and/or minimum stress in each of the members.

Noting that \( \beta \) is nonnegative except for \( \alpha > 2\pi \), one obtains from equation (A.6a)
\[ \beta = 2 \sqrt{\frac{1}{\pi} \cdot f \cdot 2 - 1}. \]  
(A.10)

Substituting equation (A.10) into equation (A.1c), it is seen that
\[ \delta = \frac{4}{\pi} \sqrt{\frac{1}{\pi} \cdot f \cdot l - 1}. \]  
(A.11)

**Fig. A.1.** Axial stretches and lateral deflections (at the center of the span of the member) in the post-buckled range under an axial force.
Substituting for \( f \) in terms of \( N \) and using equation (A.8), the following relation between \( \delta \) and \( \hat{\delta} \) is obtained:

\[
\frac{\delta}{l} = 4 - \sqrt{\frac{\delta}{4l}} \left( 1 + \frac{\delta}{4l} \right)
\]  

(A.12)

Thus, when the axial contraction \( \delta \) is solved from the finite element stiffness equation, equation (A.12) may be used to calculate the transverse displacement \( \hat{\delta} \) at midspan of the member, and from it, one may calculate the maximum or minimum stress in the member.

Fig. A.1 shows the relations between \( N, \delta, \) and \( \hat{\delta} \) as given by equations (A.9) and (A.12) and their comparisons with the exact solutions for the elastica problem. The dashed lines indicate the present solutions and the solid ones indicate the exact ones. From this figure it is seen that equations (A.9) and (A.12) are good approximations in the range of values for \( -\left( \hat{\delta}/l \right) \) and \( \left( \delta/l \right) \) being smaller than about 0.15 and 0.25, respectively. It is also seen that this range of values for \( -\left( \hat{\delta}/l \right) \) and \( \left( \delta/l \right) \) is typical in the problem of local (member buckling in a practical truss structure.

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References


A SIMPLIFIED FINITE ELEMENT METHOD FOR LARGE DEFORMATION, POST-BUCKLING ANALYSES OF LARGE FRAME STRUCTURES, USING EXPLICITLY DERIVED TANGENT STIFFNESS MATRICES

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SUMMARY

In this paper, a simplified, economical and highly accurate method for large deformation, post-buckling analyses of frame-type structures is presented. Using the concepts of polar-decomposition of deformation, an explicit expression for the tangent stiffness matrix of a member/element of the frame, that accounts for the nonlinear bending-stretching coupling, is derived in closed form, at any point in the load-deformation path. The ranges of validity of the present simplified approach are discussed. Several example problems to demonstrate the feasibility of the present approach, over ranges of deformation that are well beyond those likely to occur in practical large frame structures, are included.

1. INTRODUCTION

In the past few years there has been a renewed interest in the analysis of frame- and truss-type structures, due to the current dreams of many to deploy very large structures in outer space. These space-structures, generally of the frame type, are often envisioned to be as large as Manhattan Island, and probably just as unwieldy! One of the primary topics of current interest is to analyse parts of such structures, that may be subject to large local disturbances, with the ultimate goal of controlling the dynamic deformations through active or passive mechanisms. This paper, however, is limited, as a first step, to considerations of simple, yet highly accurate, methods of nonlinear analyses of frames under quasi-static loads. While plane frames are treated in the present paper, the extensions of the central methodology to space frames, and inelastic deformations, are subjects of forthcoming publications.

Large deformation and post-buckling analyses of structures have been the subjects of extensive research in the past decade or so (see, for instance, References 1-10). In all these studies, an incremental approach, either of the 'total Lagrangian type' or of the 'updated Lagrangian type' is employed. The incremental approach, in turn, is often based on the development, for each element, of the so-called 'tangent stiffness matrix', which reflects all the nonlinear geometric and mechanical effects. In most of the literature, the derivation of the 'tangent stiffness matrix' of an element (which may be based, alternatively, on potential energy, complementary energy or general mixed-hybrid formulations), in general, involves: (a) simple polynomial basis functions for displacements and/or stress and moment resultants; and (b) Numerical integration of matrices (dependent on the assumed basis functions and their spatial derivatives) over the domain of the element. The key factors that determine the economic feasibility of the routine use of the above nonlinear analysis

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methods are: (i) the computational time involved in forming the tangent stiffness matrix of each element (and, thus, of the entire structure) at each increment of external loading; (ii) the degree of refinement of the finite element grid, when elements with simple polynomial basis functions are used; and (iii) the techniques for solution of the system stiffness equations especially at or beyond the critical (buckling) points in the load-path.

A majority of nonlinear analyses of typical engineering structures, and especially the frame-type large space-structures, can be simplified if an explicit expression (i.e. without involving assumed basis functions for displacement/stress and without involving element-wise numerical integrations) for the tangent stiffness matrix of an element (incorporating the effects of initial displacements on the current stiffness) can be derived. Towards this end, the authors have recently presented\textsuperscript{11} a method for explicitly deriving the tangent stiffness matrix of a truss-type structure, wherein each of the members is assumed to carry only an axial load. This procedure for analysing trusses has been demonstrated\textsuperscript{11} to be not only inexpensive, but also highly accurate, in a wide variety of problems involving very large deformations and highly nonlinear pre- and post-buckling responses.

In this paper, we consider simplified procedures for large deformation and post-buckling analysis of frame-type structures, wherein each member undergoes bending as well as stretching deformations (which are coupled) and can carry axial loads as well as bending moments. Explicit expressions for the coefficients of the tangent stiffness matrix of an element (applicable over a wide range of deformations) are derived. By 'explicit' we mean that the procedure does not involve assumptions of basis functions for the element nor of numerical integrations over each element.

The concept of a 'polar-decomposition' of the deformation is employed to decompose the arbitrary deformation of the element into rigid rotations and pure stretches. The present work is based on the assumptions: (i) arbitrarily large rigid translations and rotations of each member/element of the frame are accounted for; (ii) the local relative (non-rigid) rotations of a differential segment of a member/element of the frame are moderately small, and that their squares enter to the expression for axial stretch, in a manner analogous to that in the well-known Von Karman theory for plates, and (iii) the nonlinear coupling between the bending and stretching motions of the member/element is inherently accounted for. The ranges of validity of these approximations are critically examined. Several numerical examples dealing with the nonlinear pre- and post-buckling responses of frames are presented. In these examples, the arc-length method\textsuperscript{12-14} is employed to solve the system stiffness equations. It is demonstrated that in most of the cases a single element, with the presently derived explicit stiffness matrix, is adequate to model each member of the frame; and that the methodology is not only inexpensive, but is also highly accurate even for ranges of deformations that are well beyond those likely to be encountered in practice.

The remainder of the paper is organized as follows: Section 2 gives a detailed account of the present procedure for an explicit evaluation of the tangent stiffness matrix of a frame-member, at any point in the nonlinear load-deformation path; the ranges of validity of the present simplified approach are delineated in Section 3; several (five) numerical examples are presented and discussed in Section 4; and concluding comments are made in Section 5.

2. EXPLICIT EVALUATION OF TANGENT STIFFNESS FOR A FRAME MEMBER

In this work, each member of a frame, over ranges of its deformation that are likely to occur in practice, shall be sought to be modelled, for the most part, by a single finite element. The frame-type structures discussed herein are assumed to remain elastic. Only a conservative system of concentrated loads are assumed to act at the nodes of the frame.
2.1. Kinematics of deformation of a member: decomposition to stretch and rotation

Consider a typical frame member, modelled here as a beam element, that spans between nodes 1 and 2 as shown in Figure 1. The element is considered to have a uniform cross-section and to be of length l before deformation. The co-ordinates z and x are the local co-ordinates for the undeformed element, as shown in Figure 1. The functions w(z) and u(x) denote the displacements, at the centroidal axis of the element, along the co-ordinate directions z and x, respectively. Also, as shown in Figure 1, (i) \( \theta \) is the angle between the straight line joining the nodes of the deformed element and the z axis (and thus, is the rigid rotation of the member/element), and (ii) \( \theta^* \) is the angle between the tangent to the deformed centroidal axis and the line joining the two nodes of the deformed element.

From the polar decomposition theorem, the relations between the point-wise stretch \( \varepsilon \), the point-wise rotation \( \theta \), and the displacements \( w(z) \) and \( u(x) \) in the element, are

\[
\begin{align*}
(1 + \varepsilon) \cos \theta &= 1 + \frac{dw}{dz} \\
(1 + \varepsilon) \sin \theta &= \frac{du}{dz}
\end{align*}
\]

where \( \theta = \theta + \theta^* \)

Integrating (1a, b) along the length of the element, one obtains

\[
\begin{align*}
\int_0^l (1 + \varepsilon) \cos \theta dz &= l + \tilde{w} \\
\int_0^l (1 + \varepsilon) \sin \theta dz &= \tilde{u}
\end{align*}
\]

where

\[
\tilde{w} = w|_{z=l} - w|_{z=0} = w_2 - w_1
\]
and
\[ \bar{u} = u|_{z=1} - u|_{z=0} = y_2 - u_1 \]  
(4b)

From Figure 1, it is easy to see that
\[ \int_0^l (1 + \epsilon) \cos \theta^* dz = l + \delta \]  
and
\[ \int_0^l (1 + \epsilon) \sin \theta^* dz = 0 \]
(5a)
(5b)

where \( \delta \) is the total 'stretch' of the element (see Figure 1). Substituting for \( \theta \) from (2) in (3a, b) and using (5a, b), one obtains
\[ (l + \delta) \cos \bar{\theta} = l + \bar{\omega} \]  
(6a)
and
\[ (l + \delta) \sin \bar{\theta} = \bar{u} \]
(6b)

From (6a, b) one obtains
\[ \delta = \left[ (l + \bar{\omega})^2 + (\bar{u})^2 \right]^{1/2} - l \quad \text{and} \quad \bar{u} = \tan^{-1} \left( \frac{\bar{u}}{l + \bar{\omega}} \right) \]
(7a)
(7b)

The non-rigid ('relative') rotations at nodes 1 and 2, denoted hereafter as \( \theta^*_1 \) and \( \theta^*_2 \), respectively, are defined, using (2) and (8) as
\[ \theta^*_1 = \theta^*|_{z=0} = \theta|_{z=0} - \bar{\theta} = \theta_1 - \tan^{-1} \left( \frac{\bar{u}}{l + \bar{\omega}} \right) \]
(9a)

and
\[ \theta^*_2 = \theta^*|_{z=1} = \theta|_{z=1} - \bar{\theta} = \theta_2 - \tan^{-1} \left( \frac{\bar{u}}{l + \bar{\omega}} \right) \]
(9b)

2.2 Relations between the stretch and relative rotation, and axial force and bending moment, at the nodes of an individual frame member or finite element

In preparation for the task of deriving an explicit expression for the tangent stiffness matrix that is valid for a wide range of deformations of a frame member, we first derive certain explicit relations between the kinematic variables of stretch and relative rotations, and the mechanical variables of

\[
\begin{pmatrix}
M \\
-N
\end{pmatrix}
\]

Figure 2. Sign convention for system of generalized forces on a member
axial force and bending moments of an individual frame member, or of a finite element if more than
one finite element is contemplated for modelling an individual member. These 'generalized' force
displacement relations for an individual member/(element) are also intended to be valid over a
range of deformations that may be considered as 'large'.

To achieve the above purpose, we consider a beam-column, as shown in Figure 2. The equation
of equilibrium in the transverse direction of the beam may be written\(^{16,17}\) as

\[
EI \frac{d^2 \theta^*}{dS^2} - \frac{1}{l+\delta}(M_1 - M_2) \cos \theta^* - N \sin \theta^* = 0
\]

where

\[
M = -EI \frac{d\theta^*}{dS}
\]

\(E = \text{Young's modulus, } l = \text{cross-sectional moment of inertia})

\(M_1 = M|_{S=a}, \quad M_2 = M|_{S=b}\)

In the above, \(S\) is the curvilinear co-ordinate along the centroidal axis of the deformed member, and
\(\theta^*\) has been defined earlier. For the type of problems contemplated, we assume that the
deformation of the frame as a whole is such that the non-rigid rotation \(\theta^*\) in each individual
member/(or its elements) of the frame may be considered as being small. Under this assumption,
(10) may be approximated as

\[
EI \frac{d^2 \theta^*}{dS^2} - \frac{1}{l}(M_1 - M_2) - N \theta^* = 0
\]

The boundary conditions are

\[
- EI \frac{d\theta^*}{dS} \bigg|_{S=a} = M_1; \quad - EI \frac{d\theta^*}{dS} \bigg|_{S=b} = M_2
\]

The total axial 'stretch' of the beam may be written\(^{16,17}\) as

\[
\delta = \left( \int_0^l \left[ 1 + \frac{1}{EA} (N \cos \theta^* + Q \sin \theta^*) \right] \cos \theta^* dS \right) - 1
\]

where

\[
Q = -\frac{1}{l+\delta}(M_1 - M_2)
\]

and \(Q\) is shown in Figure 2. For the type of deformations contemplated here, i.e. \(\theta^*\) being small,
(14) may be approximated as

\[
\delta = -\frac{1}{2} \int_0^l (\theta^*)^2 dS + \frac{1}{EA} N
\]

Thus, the nonlinear term \((\theta^*)^2\) is retained in the axial stretch relation as, for instance, in the Von
Karman plate theory. Equations (12), (13a, b) and (16) form the basis of the present derivation of the
relations between the generalized displacements and forces in the element.

We first define non-dimensional axial force, and bending moment, denoted as \(n\) and \(m\)
respectively, through the relations

\[
n = \frac{Nl^2}{EI}; \quad m = \frac{Ml}{EI}
\]
Now, when \( n \leq 0 \), the solution of equations (12), (13a, b) is

\[
\theta^* = \left\{ \frac{1}{(n)} - \frac{1}{(n)} \sin \left[ \frac{(n)}{2} \sin \left( \frac{(n)}{2} \right) \right] - \frac{\cot \left( \frac{(n)}{2} \right)}{(n)} \cos \left( \frac{(n)}{2} \right) \right\} m_1 \\
+ \left\{ \frac{1}{(n)} + \frac{\csc \left( \frac{(n)}{2} \right)}{(n)} \cos \left( \frac{(n)}{2} \right) \right\} m_2
\]

Equation (18) leads to the following relations between the relative rotations \( \theta^*_1 \) and \( \theta^*_2 \) at the ends of the member, and the corresponding bending moments \( m_1 \) and \( m_2 \):

\[
\theta^*_1 = \left\{ \frac{1}{(n)} - \frac{\cot \left( \frac{(n)}{2} \right)}{(n)} \right\} m_1 + \left\{ - \frac{1}{(n)} + \frac{\csc \left( \frac{(n)}{2} \right)}{(n)} \right\} m_2
\]

and

\[
\theta^*_2 = \left\{ \frac{1}{(n)} - \frac{\csc \left( \frac{(n)}{2} \right)}{(n)} \right\} m_1 + \left\{ - \frac{1}{(n)} + \frac{\cot \left( \frac{(n)}{2} \right)}{(n)} \right\} m_2
\]

Also, using (18) in (16), we obtain

\[
\delta = \left\{ \frac{1}{(2n)} - \frac{1}{(n)} \cosech^2 \left( \frac{(n)}{2} \right) - \frac{1}{(n)} \coth \left( \frac{(n)}{2} \right) \right\} \left( m_1 + m_2 \right)
\]

On the other hand, when \( n > 0 \), one may obtain, instead of equations (18), (19a, b) and (20), the following equations for \( \theta^*_1 \), \( \theta^*_2 \), \( \theta^*_3 \) and \( \delta \), respectively:

\[
\theta^*_1 = \left\{ \frac{1}{(n)} - \frac{1}{(n)} \sinh \left( \frac{(n)}{2} \right) \right\} \left( m_1 + m_2 \right)
\]

and

\[
\delta = \left\{ \frac{1}{(2n)} - \frac{1}{(n)} \coth^2 \left( \frac{(n)}{2} \right) - \frac{1}{(n)} \coth \left( \frac{(n)}{2} \right) \right\} \left( m_1 + m_2 \right)
\]

The sets of equations (19a, b) and (20) and (22a, b) and (23) may be written in a more convenient form by decomposing the kinematic and mechanical variables of the beam into 'symmetric' and 'antisymmetric' parts, as

\[
\theta^*_1 = \frac{1}{2} (\theta^*_1 + \theta^*_2); \quad \text{and} \quad \theta^*_2 = \frac{1}{2} (\theta^*_1 - \theta^*_2)
\]

and

\[
m_s = (m_1 - m_2); \quad \text{and} \quad m_a = (m_1 + m_2)
\]
where subscripts $a$ and $s$ refer to 'antisymmetric' and 'symmetric' parts, respectively (it is worth while recalling the sign conventions for $\theta^a$ and $\theta^s$ as shown in Figure 2). In terms of these variables, equations (19a, b) as well as equations (22a, b) may be written as

$$\theta^a = h_au \quad \text{and} \quad \theta^s = h_sm,$$

where

(i) for $n \leq 0$:

$$h_u = -\frac{1}{n} - \frac{1}{2(n)^{1/2}} \cot \left[ \frac{(n^{1/2})}{2} \right], \quad \text{and} \quad h_s = \frac{1}{2(n)^{1/2}} \tan \left[ \frac{(n^{1/2})}{2} \right].$$

(ii) for $n > 0$:

$$h_u = -\frac{1}{n} + \frac{1}{2(n)^{1/2}} \coth \left[ \frac{(n^{1/2})}{2} \right], \quad \text{and} \quad h_s = \frac{1}{2(n)^{1/2}} \tanh \left[ \frac{(n^{1/2})}{2} \right].$$

Also, in terms of the new variables, equations (20) and (23) may be rewritten, in a unified form, as

$$\delta = 2 \left( \frac{d\theta_u}{dn} \right) m^2 + 2 \left( \frac{d\theta_s}{dn} \right) m^2 + \frac{N}{E\alpha},$$

$$= \frac{1}{2} \left( \frac{d^2\theta_u}{dn^2} \right) h_u^2 + \frac{1}{2} \left( \frac{d^2\theta_s}{dn^2} \right) h_s^2 + \frac{N}{E\alpha}.$$ \hspace{1cm} (29a)

$$where, \quad \frac{d\theta_u}{dn} = \frac{1}{n} \sec^2 \left[ \frac{(n^{1/2})}{2} \right] - \frac{1}{4(n)^{1/2}} \cot \left[ \frac{(n^{1/2})}{2} \right].$$ \hspace{1cm} (29c)

$$\frac{d\theta_s}{dn} = -\frac{1}{n} \sec^2 \left[ \frac{(n^{1/2})}{2} \right] + \frac{1}{4(n)^{1/2}} \tan \left[ \frac{(n^{1/2})}{2} \right].$$ \hspace{1cm} (29d)

and, for $n > 0$,

$$\frac{d\theta_u}{dn} = \frac{1}{n^2} - \frac{1}{8n} \coth^2 \left[ \frac{(n^{1/2})}{2} \right] - \frac{1}{4n^3} \coth \left[ \frac{(n^{1/2})}{2} \right].$$ \hspace{1cm} (29e)

$$\frac{d\theta_s}{dn} = \frac{1}{8n} \sech^2 \left[ \frac{(n^{1/2})}{2} \right] - \frac{1}{4n^3} \tanh \left[ \frac{(n^{1/2})}{2} \right].$$ \hspace{1cm} (29f)

Equations (26a, b) and (29) are the sought-after relations between the generalized displacements and forces at the nodes of an individual frame member, for the range of deformations considered. In connection with equations (26a, b) and (29), it is worth while to recall that (i) $N$ is in the direction of the straight line connecting the nodes of the frame-member after its deformation (see Figure 2), and (ii) the parameters $\delta, \theta^a$ and $\theta^s$ are calculated from equations (7), (9a) and (9b) which are valid in the presence of arbitrarily large rigid motions (translations and rotations) of the individual member. Thus, while the local stretch (pure strain) and relative rotation (non-rigid) of a differential-element of an individual frame-member may be small, the individual member as a whole (and as a part of the overall frame) may undergo arbitrarily large rigid motion. Hence, the 'generalized' force displacement relations embodied in equations (26a, b) and (29) remain valid in the presence of arbitrarily large rigid motion of the individual member of the frame. Also, it is important to note that the present relations for each element account, as in the Von Karman plate theory, for the nonlinear coupling between the bending and stretching deformations, as seen from equations (26a, b) and (29).
The limits of validity of the relations (26a, b) and (29) are discussed and illustrated in Section 3 of the present paper. However, certain special features of these relations are briefly discussed here. First, note that when $N = 0$, the solution of (12) for the boundary conditions shown in Figure 2 may be written as

$$\theta^* = \frac{1}{2}(m_1 - m_2) \frac{S^2}{l^2} - m_1 \frac{S}{l} + \frac{1}{2}(2m_1 + m_2)$$

and

$$\delta = -\frac{1}{2} \int_0^l (\theta^*)^2 dS$$

From (30) it is seen that

$$\theta^* = \frac{1}{12} m_s; \quad \theta^*_s = \frac{1}{4} m_s$$

(or, $h_s = \frac{1}{12}$ and $h_s = \frac{1}{4}$)

and

$$\delta/l = -\frac{1}{12} \int_0^l m_s^2 - \frac{1}{6} m^2_s \equiv -\frac{1}{12} (\theta^*)^2 - \frac{1}{6} (\theta^*_s)^2$$

(or, $\frac{dh_s}{dn} = -\frac{1}{720}$ and $\frac{dh_s}{dn} = -\frac{1}{48}$)

Also, when the axial force $N$ is of magnitude $(-4\pi^2 EI/l^2)$ (i.e. the Euler buckling load of 'fixed-fixed' column), $h_s \rightarrow 0$. However, this situation is unlikely to arise in a practical frame structure.

2.3. Tangent stiffness matrix of a frame member/element

Recall that each member of the frame is sought, for the most part, to be treated herein as a beam column as discussed in the previous subsection. In the case of extreme, and what may from a practical viewpoint be considered as 'pathological', deformations, each member of the frame may be modelled, say, by two or three elements utmost.

Note from equations (26a, b), the 'flexibility' coefficients $h_s$ and $h_s$ are highly nonlinear functions of the axial force (as given in (27a, b) for $n \leq 0$, and in (28a, b) for $n > 0$). However, unless $h_s$ and $h_s$ are equal to zero, one may invert (26a, b) to write the 'force—displacement' relations

$$m_s = \left(\frac{1}{h_s}\right) \theta^*_s; \quad m_s = \left(\frac{1}{h_s}\right) \theta^*_s$$

Recalling the definition of non-dimensional moments as in (17b), one may express the strain energy due to bending as

$$\frac{1}{2} EI \left[ \int \frac{1}{h_s} (\theta^*)^2 \right]$$

On the other hand, as explained earlier, in the limit as $N \rightarrow [(-4\pi^2 EI/l^2)]$, $h_s \rightarrow 0$. Thus, the inversion of (26b) to obtain (33b) is not meaningful. As in the classical problem of incompressibility, in such a case one may use a 'mixed' form for the bending energy of the symmetric mode, through the well-known Legendre contact transformation, as

$$\frac{1}{h_s} (\theta^*)^2 \equiv \frac{EI}{l} \left[ m_s \theta^*_s - \frac{1}{2} h_s m_s^2 \right]$$

and treat both $m_s$ and $\theta^*_s$ as variables. However, the case of $N \rightarrow [(-4\pi^2 EI/l^2)$, i.e. the Euler buckling
load of a 'fixed-fixed' beam is unlikely to occur in a practical framed-structure. Thus, without loss of generality, we restrict ourselves to considering the strain energy in the form (34).

Now we consider the strain-energy due to axial stretch of the member. Note from (29) that \( \delta \) is related in a highly nonlinear fashion to the axial force \( N \) as well as the moments \( m_a \) and \( m_b \). The inversion of this relation in explicit form, in order to express the axial force \( N \) as a function of \( \delta \), appears impossible. With a view towards carrying out this inversion of the \( \delta \) vs. \( \{N, m\} \) relation incrementally, later on, we express the strain energy due to stretching, denoted here as \( \pi_* \), in a 'mixed' form, using the well-known concept of a Legendre contact transformation [18], as

\[
\pi_* = N\delta - \frac{1}{2} \frac{1}{E_A} N^2
\]  

(36)

Note, however, that in view of the dependence of \( h_a \) and \( h_b \) on \( n \) as in (26, 28), there is coupling between 'bending' and stretching variables. The internal energy in the member due to combined bending and stretching, is

\[
\pi = \frac{1}{2} \left( \frac{EI}{L} \right) \left[ \frac{1}{h_a} (\theta_s^*)^2 + \frac{1}{h_b} (\theta_s^*)^2 \right] + \left[ N\delta - \frac{1}{2} \frac{1}{E_A} N^2 \right]
\]

(37)

The condition of vanishing of the first variation of \( \pi \) (denoted here as \( \pi^* \)) in (37) due to a variation in \( N \) (denoted as \( N^* \)), i.e., the condition

\[
\pi^* = 0 = -\frac{1}{2} \left( \frac{dh_a}{dn} \frac{1}{h_a^2} (\theta_s^*)^2 + \frac{dh_b}{dn} \frac{1}{h_b^2} (\theta_s^*)^2 \right) N^* + \left( \frac{\delta - N}{E_A} \right) N^*
\]

(38)

leads, clearly, to the relation between \( \delta \) and the generalized forces as given (29b). The reason for using the 'mixed form' for the stretching energy in (36) is now clear from the result in (38). By using a similar mixed form for the increment of stretching energy, the incremental axial stretch-vs.-incremental generalized force relation can be derived in a manner analogous to that used in obtaining (38) from (37). This incremental relation, which is, by definition, piecewise linear, may easily be inverted, as demonstrated in the following. Also, it is shown in the following that equation (37) forms the basis for generating an explicit form for the 'tangent-stiffness' of the member.

The increment of the internal energy of the member, denoted as \( \Delta \pi \), involving terms up to second order in the 'incremental' variables \( \Delta \theta_s^*, \Delta \theta_s^*, \Delta N \) and \( \Delta \delta \) can be seen, from (37), to be

\[
\Delta \pi = \frac{EI}{L} \left[ \frac{1}{h_a} \theta_s^* \Delta \theta_s^* + \frac{1}{2} \frac{1}{h_a} (\Delta \theta_s^*)^2 + \frac{1}{2} \frac{1}{h_b} (\Delta \theta_s^*)^2 \right] + \Delta \left( \frac{1}{h_a} (\theta_s^*)^2 \right)
+ \frac{1}{2} \Delta \left( \frac{1}{h_a} \right) \Delta \theta_s^* \Delta \theta_s^* + \frac{1}{2} \Delta \left( \frac{1}{h_b} \right) \Delta \theta_s^* \Delta \theta_s^* + \Delta \left( \frac{1}{h_b} \right) \Delta \theta_s^* \Delta \theta_s^*
+ \frac{1}{2} \Delta \left( \frac{1}{h_a} \right) \Delta (\theta_s^*)^2 - \frac{1}{E_A} N \Delta N - \frac{1}{2} \left( \frac{1}{E_A} \right) (\Delta N)^2 + \Delta N \delta + N \Delta \delta + \Delta N \Delta \delta
\]

(39)

In the above, it should be recalled that \( h_a \) and \( h_b \) are functions of \( N \).

Now, using equations (4a, b), (7), (8), (9a, b) and (24a, b), the incremental quantities \( \Delta \theta_s^*, \Delta \theta_s^* \) and \( \Delta \delta \) may be expressed in terms of \( \{w_1, w_2, u_1, u_2, \theta_1, \theta_2\} \) and/or their increments. Henceforth, we use the notation for vector \( \{d^m\} \) that

\[
[d^m] = [w_1, w_2, u_1, u_2, \theta_1, \theta_2]_{\text{member}}
\]

(40)
as shown in Figure 1. In terms of the increment $\{\Delta d^m\}$, equation (39) may be written as

$$\Delta \pi = \frac{1}{2} \{\Delta d^m\} [A_{uu}] \{\Delta d^m\} + \Delta N \{A_{ud}\} \{\Delta d^m\} + \frac{1}{2} A_{mm} \{\Delta N\}^2 + [B_d] \{\Delta d^m\} + B_\pi \Delta \pi$$

where

$$[A_{uu}] = \begin{bmatrix}
\left( \frac{\partial^2 \delta}{\partial \nu^2} + M_s \frac{\partial^2 \theta^*_s}{\partial \nu^2} \right) & \left( \frac{\partial^2 \delta}{\partial \nu \partial \lambda} + M_s \frac{\partial \theta^*_s}{\partial \lambda} \right) & 1 \cdot \frac{E I_1 \theta^*_s}{2 / h_u \partial \lambda} [I] \\
\frac{E I_1 \left( \frac{\partial^2 \theta^*_s}{\partial \nu \partial \lambda} \right)}{2 / h_u \partial \lambda} & \left( \frac{\partial^2 \delta}{\partial \nu^2} + M_s \frac{\partial^2 \theta^*_s}{\partial \nu^2} \right) & 1 \cdot \frac{E I_1 \theta^*_s}{2 / h_u \partial \lambda} [I] \\
\frac{E I_1 \left( \frac{\partial^2 \theta^*_s}{\partial \nu \partial \lambda} \right)}{2 / h_u \partial \lambda} & \frac{E I_1 \left( \frac{\partial^2 \theta^*_s}{\partial \nu \partial \lambda} \right)}{2 / h_u \partial \lambda} & \frac{E I_1 \left( \frac{\partial^2 \theta^*_s}{\partial \nu \partial \lambda} \right)}{2 / h_u \partial \lambda} [I]
\end{bmatrix}
$$

$$[A_{ud}] = \begin{bmatrix}
\left( \frac{\partial^2 \delta}{\partial \nu^2} + M_s \frac{\partial^2 \theta^*_s}{\partial \nu^2} \right) \frac{1}{h_u} \\
\frac{E I_1 \left( \frac{\partial^2 \theta^*_s}{\partial \nu \partial \lambda} \right)}{2 / h_u \partial \lambda} & \frac{E I_1 \left( \frac{\partial^2 \theta^*_s}{\partial \nu \partial \lambda} \right)}{2 / h_u \partial \lambda} [I]
\end{bmatrix}
$$

$$[B_d] = \begin{bmatrix}
\frac{1}{E} \left( \frac{\partial^2 \delta}{\partial \nu^2} + M_s \frac{\partial^2 \theta^*_s}{\partial \nu^2} \right) & \frac{1}{E} \left( \frac{\partial^2 \delta}{\partial \nu \partial \lambda} + M_s \frac{\partial \theta^*_s}{\partial \lambda} \right) \\
\frac{1}{E} \left( \frac{\partial^2 \delta}{\partial \nu^2} + M_s \frac{\partial^2 \theta^*_s}{\partial \nu^2} \right) & \frac{1}{E} \left( \frac{\partial^2 \delta}{\partial \nu \partial \lambda} + M_s \frac{\partial \theta^*_s}{\partial \lambda} \right)
\end{bmatrix}
$$

$$A_{uu} = \frac{1}{2 E I} \left( \frac{\partial^2 \delta}{\partial \nu^2} + \frac{1}{h_u} \frac{\partial \theta^*_s}{\partial \nu} \right) \frac{1}{E} \left( \frac{\partial^2 \delta}{\partial \nu^2} + M_s \frac{\partial^2 \theta^*_s}{\partial \nu^2} \right) - \frac{1}{E A}$$

$$B_d = \frac{1}{2 E} \left( \frac{\partial^2 \delta}{\partial \nu^2} + M_s \frac{\partial^2 \theta^*_s}{\partial \nu^2} \right) - \frac{1}{E A} N$$

$$\{ I \} = \begin{bmatrix}
-1 \\
1
\end{bmatrix}, \quad [F] = \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}$$
By setting to zero the variation of $\Delta \pi$ in (41) with respect to $\Delta N$, one obtains the relation
\[ [A_{nn}] \{\Delta d^m\} + B_n + A_{nm} \Delta N = 0 \] (48)

Thus, (48) is the incremental counterpart of the $\delta$ vs. the generalized force relation obtained in (38). Unlike the nonlinear relation in (38), however, the piecewise linear relation (48) can be inverted to express $\Delta N$ in terms of generalized displacements, as
\[ \Delta N = -\frac{1}{A_{nn}} [A_{nn}] \{\Delta d^m\} - \frac{1}{A_{nn}} B_n \] (49)

Substituting (49) into (41), one obtains the internal energy expression
\[ \Delta \pi = \frac{1}{2} [\Delta d^m] [K^m] \{\Delta d^m\} + [\Delta d^m] \{R^m\} \] (50)

where
\[ [K^m] = \text{tangent stiffness matrix of member/element} \]
\[ = [A_{nn}] - \frac{1}{A_{nn}} \{A_{nn}\} \{A_{nn}\} \] (51)
\[ \{R^m\} = \text{internal generalized force vector for member/element} \]
\[ = \{B_n\} - \frac{1}{A_{nn}} B_n \{A_{nn}\} \] (52)

Recall that the tangent stiffness matrix and the internal force vector are written in the member coordinate system as shown in Figure 1. Thus, it is necessary to transform $\{d^m\}$ from a member coordinate system to a system (global) coordinate system in the usual fashion.

It should be emphasized once again that equations (42)–(44) give an explicit expression for the tangent stiffness matrix $[K^m]$ of (51); and, likewise, equations (43)–(46) give an explicit expression for the internal generalized force vector $\{R^m\}$.

The assembled tangent stiffness equations for the frame are solved using the now well-known 'arc-length' method.\(^{12-14}\)

3. RANGES OF VALIDITY OF THE PRESENT SIMPLIFIED APPROACH

Recall that the principal assumptions underlying the present development of an explicit expression for the tangent-stiffness matrix of an element are as follows: (i) arbitrarily large rigid translations and rigid rotations of the element are accounted for; (ii) however, the local (non-rigid) rotation $\theta^*$ is restricted to be small such that $\sin \theta^* \approx \theta^*$ and $\cos \theta^* = 1$; (iii) the local axial stretch $\varepsilon$ is restricted to be small; (iv) the nonlinear coupling between the bending deformation (characterized by $\theta^*$) and stretching deformation (characterized by total axial stretch $\varepsilon$) is accounted for, as seen from (a) the dependence of $h_u$ and $k_1$ in (26a, b) and (27a, b) on the axial force $n$ and likewise, (b) the dependence of axial stretch $\varepsilon$ on the moments (rotations) as seen from equation (29).

We now critically examine the ranges of validity of the above assumptions underlying the present developments. The most severe test is to see if a single element, based on the present formulation as given in Section 2, can model the entire history of post-buckling behaviour of a column subjected to axial compressive load. Clearly, the ability of a single element to characterize the entire post-buckling behaviour can be seen to be limited by assumptions (ii) and (iii) above. These two assumptions clearly show that a single element can predict only the initial post-buckling response, i.e. as long as $\theta^* \ll 1$ and $\varepsilon/l \ll 1$. This can be seen by considering, for instance, a 'pinned-pinned' (but axially movable) column subject to axial load $N$ alone. The Euler buckling load for
Figure 3(a). Problem definition for an eccentrically loaded column.

Figure 3(b). Dependence of axial and transverse displacements on axial load for a buckled column under axial eccentric load: convergence of solution.

Figure 3(c). Results similar to those in Figure 3(b), except for a wider range of deformations.
We consider the initial post-buckling response. Let \( \Delta_p \) denote the increment of the quantity ( ) immediately after the Euler buckling load is reached. The boundary conditions of the problem are

\[
\Delta_p m_a = \Delta_p m_t = 0
\]

From equations (26a, b), (27a, b) and (29b), it may be seen that in the immediate vicinity of the Euler buckling load,

\[
\Delta_p \theta^*_e = 0
\]

and

\[
\Delta_p \left( \frac{\delta}{I} \right) = -\frac{1}{4}(\Delta_p \theta^*)^2
\]

even while \( \Delta_p n = 0 \). Thus, when a single element is used to model post-buckling response of a beam, while the nonlinear coupling between bending and stretching is accounted for, the slope of the post-buckling response curve \([\delta/l] vs. n\) is not correctly accounted for. This is the inherent limitation of assumptions (ii) and (iii) above and is further illustrated in the numerical example below.

We consider the problem of an 'elastica'—a simply-supported (but axially movable) beam, of length \( l \), that is subjected to an axial compressive load, with a load eccentricity of \((l/l_0)\) from the undeformed axis of the beam, as shown schematically in Figure 3(a). The beam has a square cross-section of area \(1.0\) (in.\(^2\)), \( l = 100.0\) (in.), \( E = 10^9\) (psi) and \( I = 0.8333\) (in.\(^4\)). For testing the range of deformations, over which the present explicit expression for tangent stiffness matrix of an element arc valid, the beam is idealized, successively, by 1, 2, 4 and 8 elements over its length, respectively. As shown in Figure 3(a), \( \delta_r \) is the total axial 'stretch' of the beam, while \( \delta_t \) is the transverse displacement at midspan, in the post-buckling range. Figure 3(b) shows the dependence of \( \delta_r/l \) and \( \delta_t/l \) on the axial load \( N \), for the range of deformations \([\delta_r/l] and [\delta_t/l] \approx 0.30\). On the other hand, Figure 3(c) shows the dependence of \( \delta_r/l \) and \( \delta_t/l \) on the axial load \( N \), for a much wider range of deformations, viz. \( [\delta_r/l] \approx 1.0 \) and \( [\delta_t/l] \approx 0.5 \).

From Figures 3(b) and 3(c), it is seen that while a 'single element' representation of the entire beam does account for the nonlinear coupling between bending and stretching (as seen from the large values of \( \delta_t \) and \( \delta_t \) at \( N \approx N_{ge} \)), the slope of the post-buckling response curve for \( N \) vs. \( \delta_r/l \) or \( \delta_t/l \) is not accurately represented. On the other hand, Figures 3(b) and 3(c) clearly indicate that even a two-element representation of the beam yields results for post-buckling response, that are in close agreement with the classical 'elastica' solution, even for very large deformations of the order, \( \delta_r/l \approx 0.4 \) and \( \delta_t/l \approx 0.5 \). It is also seen from Figures 3(b) and 3(c) that a four-element representation produces solutions that are in exact agreement with the 'elastica' solutions for the entire range of deformations considered. The reason for this excellent behaviour of the 'two-' or 'multi-'element models of the elastica is due to the fact that the present element development can account for arbitrarily large rigid motions, even if for element-wise small-strain motions, as discussed under assumptions (i) to (iv) at the beginning of this section.

It is worth pointing out that while arbitrarily large deformations (such as the straight beam folding into a circle) have been considered in the present example of a single beam, when a practical frame-structure is considered, it is unlikely that each of its members will undergo such gross deformations. Thus, inasmuch as the present element development accounts for nonlinear bending–stretching coupling, it may be sufficient to model each member of the frame by only one or two of the present elements, whose stiffness matrices are given explicitly in Section 2. The following five numerical examples illustrate this assertion.
$E = 0.03 \times 10^7$ (lb/in$^2$)

Figure 4(a). Schematic of Williams' toggle

Figure 4(b). Variations of load-point displacement and support reaction with load, for Williams' toggle in the post-buckling range
4. ASSORTED NUMERICAL EXAMPLES

The first example is that of the so-called Williams’ toggle frame, which was first treated by Williams and later analysed by Wood and Zienkiewicz and Karamanlidis, Honecker and Knothe. A schematic of the structure is shown in Figure 4(a). The structure has a semi-span of 12.943 (in.), a raise of 0.386 (in.), and is composed of two identical members, each with a rectangular cross-section of width 0.753 (in.), depth of 0.243 (in.), and \( E = 1.03 \times 10^7 \) (psi). Each member of the frame is modelled by a single element of the type derived in Section 2. Figure 4(b) shows the presently computed relation between the external load \( P \) and the conjugate displacement \( \delta \), and also that between \( P \) and the horizontal reaction \( R \) at the fixed end. Also, shown in Figure 4(b) are the comparison experimental results of Williams as well as the numerical solutions obtained by Wood and Zienkiewicz. Excellent agreement between all the three sets of results may be noted. However, the efficiency of the present method is clearly borne out by the facts that: (a) the present solution uses one element to model each member, while Reference 5 uses five elements to model each member; and (b) no numerical integrations are used to derive the tangent stiffness of the element during each step of deformation, since an explicit expression for such is given in Section 2.

The next two examples concern frames, with two and three members, respectively, which bring out rather fascinating features of responses of frames. For these examples, experimental results were reported by Britvec and Chilver, while theoretical solutions for the buckling load and post-buckling responses were also reported in References 16 and 17. In these two examples, each of the members has a rectangular cross-section of width 1.0 (in.), depth 0.0625 (in.) and \( l = 20.0 \) (in.). Also, the buckling load of each member, when considered individually as a pinned-pinned column, is 8.1 (lb). The schematics of the two examples are given in Figures 5(a) and 6(a), respectively. In both these examples, each member of the frames is modelled by a single element. Each of the structures shown in Figures 5(a) and 6(a), respectively, has two distinctly different post-buckling load-displacement curves, corresponding to the two types of buckling modes, designated, respectively, as (a) and (b) in the insets of Figures 5(b) and 6(b). Each of the modes (a) and (b) may be excited by considering a corresponding type of load eccentricity, designated also as cases (a) and (b), respectively, in Figures 5(a) and 6(a).

![Figure 5(a). Schematic of Britvec and Chilver's two-bar frame](attachment:image.png)

- Cross-sectional Area of Members = 0.0625 (in²)
- Euler Buckling Load of Each Member (Treated as a Pinned-Pinned Beam) = 8.1 (lb)
- Case 1: \( e/l = 0.001 \) (Mode (a))
- Case 2: \( e/l = 0.001 \) (Mode (b))
Figure 5(b) shows the presently computed post-buckling $P$ vs. $\delta$ relation for the two-member frame of Figure 5(a), along with the experimental and analytical results reported in References 16 and 17. The present results agree excellently with those in References 16 and 17, except for the mode (f) deformation, in which case the present results are close to the analytical results of References 16 and 17, while experiment appears to predict a much stiffer response than either the present results or the analytical results of References 16 and 17. Similar observations apply to the results given in Figure 6(b) for the post-buckling response of the three-member frame of Figure 6(a).

The fourth example is that of a right-angled frame, shown schematically in Figure 7(a). This structure was first studied experimentally and analytically by Roorda$^{20}$ and Koiter,$^{21}$ and later analysed by Argyris and Dunne$^{22}$ to demonstrate the imperfection sensitivities of structures. Recently, this problem was also analysed in Reference 9. The dimensions and material properties of the members are identical to those used in Reference 9 and are indicated in Figure 7(a). Based on the experience with the example of a beam considered in Section 3 (see Figure 3(a)), each of the members in Figure 7(a) is modelled by two elements of the present type, derived in Section 2. Five different cases of load eccentricity, with $e$ as marked in Figure 7(a) being given the values $(e/l) = 0.0001; 0.01; 0.05; -0.001; -0.01$, respectively, are considered. Figure 7(b) shows the presently
Cross-sectional Area of Members = 0.0625 (in²)
Euler Buckling Load of Each Member (Treated as a Pinned-Pinned Beam) = 8.1 (lb)
Case 1: e/l = -0.001 (Mode (a))
Case 2: e/l = 0.001 (Mode (b))

Figure 6(a). Schematic of Britvec and Chilver's three-bar frame

Figure 6(b). Two modes of deformation of the three-bar frame
Figure 7(a). Eccentrically loaded right-angled frame

Figure 7(b). Variation of corner rotation of a right-angled frame for various values of load eccentricity, (e/I), in the post-buckling range

Figure 7(c). Variation of vertical displacement (of the corner) with load, for various values of load eccentricity (e/I), in the post-buckling range
Figure 8(a). Schematic of a four-member square frame

Figure 8(b). Variation of displacements $\delta_1$ and $\delta_3$ (see Figure 8a) with load for a square frame
Figure 8(c). Variation of moments $M_1$ and $M_2$ (at points 1, 2 in Figure 8a) with load, for a square frame.

Figure 8(d). Deformation profiles at various load levels for a square frame.
computed results (for each of the five e values) for the P vs. $\theta$ (the rotation at the corner, as defined in Figure 7a) relations, along with the available numerical results of Karamanlidis, Honecker and Knothe (i.e. for $e/l = 0.01$ and $0.05$) and the analytical results of Koiter for the case of zero eccentricity ($e = 0$) of the load. From Figure 7(b), it is seen that when the imperfection ($e$) is very small (i.e. $e \approx 0.001$), the present solutions agree excellently with those of Koiter (e = 0), in the range of small deformations ($\theta \approx 10$ degrees). However, the present numerical results indicate that the structure stiffens gradually, as the post-buckling deformation progresses. This apparent effect of very large deformations is also confirmed by Roorda's experimental results. Thus, the present results appear to be accurate over a wide range of deformations. Moreover, for the values of ($e/l = 0.001$ and $e/l = 0.05$), the present results are in excellent agreement with those of Karamanlidis el al. However, it should be remarked that the present solutions are based on using four elements to model this two-member frame, while Reference 9 uses 18 elements to model the same frame. To provide a further insight into the post-buckling response, and imperfection sensitivity, of this simple frame, the presently computed variations of the displacement $\delta$ (see Figure 7a) with load $P$, for each of the five values of load eccentricity, $e$, are shown in Figure 7(c).

The final example concerns a four-member frame subjected to point loads, as sketched in Figure 8(a). The geometric and material data of the members, which are identical, is given in Figure 8(a). Because of symmetry, one-quarter of the frame (the rectangle 1–2–3) alone is modelled by using four elements (two each in segments 1–2 and 2–3, respectively, as in Figure 8a). The presently computed variations of displacements $\delta_1$ and $\delta_3$ (as defined in Figure 8a) with the applied load $P$ are shown in Figure 8(b), along with comparison results of Lee et al. and the theoretical results. The variations of the presently computed moments $M_1$ and $M_2$ (at points 1 and 2, respectively) with the applied load are shown in Figure 8(c), along with the theoretical results. Figures 8(b) and 8(c) illustrate the excellent accuracy of the present simplified method. Lastly, the profiles of deformation of the frame at various levels of applied load $P$ are sketched in Figure 8(d).

5. CONCLUSIONS

In this paper, a simple and effective way of forming the tangent stiffness matrix has been presented for finding the large-deformation and post-buckling response of large frame-structures of the type that are contemplated for use in outer space.

The salient features of the present methodology are:

1. An explicit expression is given for the 'tangent stiffness' matrix of an individual element (which may then be assembled in the usual fashion to form the 'tangent stiffness matrix' of the frame structure). The formulation that is employed accounts for (a) arbitrarily large rigid rotations and translations of the individual element, and (b) the nonlinear coupling between the bending and axial stretching motions of the element.

2. The presently proposed simplified methodology has excellent accuracy in that only one element may be sufficient, in most cases (of practical interest in the behaviour of structural frames), to model each member of the frame structure. Inasmuch as the relative (non-rigid) rotation of a differential segment of the present element is restricted to be small, a single element alone is not enough to model the post-buckling response of an entire beam column undergoing excessively large deformations as in an elastica. However, when considered as a part of a practical frame structure, the situation of each member of the frame undergoing abnormally large deformations, as in elastica, represents a pathological case.

3. Because of features 1 and 2, the present method is a computationally inexpensive method to
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REFERENCES

INFLUENCE OF LOCAL BUCKLING ON GLOBAL INSTABILITY: SIMPLIFIED, LARGE DEFORMATION, POST-BUCKLING ANALYSES OF PLANE TRUSSES

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Abstract—In this paper, the influences of local (individual member) buckling and minor variations in member properties on the global response of truss-type structures are studied. A simple and effective way of forming the tangent stiffness matrix of the structure and a modified arc length method are devised to trace the nonlinear response of the structure beyond limit points, etc. Several examples are presented to indicate: (i) the broad range of validity of the simple procedure for evaluating the tangent stiffness, (ii) the effect of buckling of individual members on global instability and post-buckling response and (iii) the interactive effects of member buckling and global imperfections.

1. INTRODUCTION

The phenomena of structural instability are generally classified as: (i) the bifurcation phenomenon, such as the response of elastic columns and plates subject to compressive loads in the axial and inplane directions, respectively, or the response of an elastic-plastic bar in tension (also often referred to as the necking phenomenon) and (ii) limit phenomenon, such as the response of laterally loaded shallow arches and shells. A detailed discussion of the classification of these instability phenomena may be found, for instance, in [1-3].

In a structural assembly, such as a truss, frame, stiffened plates, etc., the response may involve both local buckling as well as global buckling. In the present context, local buckling implies the buckling of a discrete member in the structure under consideration. The local buckling is often of the bifurcation type. The influences of local buckling on subsequent load transfer in the structure and on the overall response of the structure are subjects of prime concern in this paper.

An extensive literature exists concerning computational methods for structural stability [4-10]. Most of these works deal with either global stability or local stability separately. In a majority of these works dealing with elastic stability, the onset of instability is treated as a linear or nonlinear eigenvalue problem. Also, a number of "incremental" solution methods to find the response in the "post-buckling" range, i.e. beyond a bifurcation or a limit point, have been proposed [11-21], and these include the standard load control method, the displacement control method, the artificial spring method, the perturbation method, the "current-stiffness-parameter" method and the "arc length method". In calculating the nonlinear pre-buckling as well as post-buckling response, an incremental finite element approach, which results in a "tangent stiffness matrix" (which includes all the nonlinear geometric as well as mechanical effects), is often employed. It is now well recognized that one of the major time-consuming aspects of these nonlinear analyses is the computation of this tangent stiffness matrix when such a matrix is based on rigorous continuum-mechanics considerations.

On the other hand, the literature that deals with the effect of local instability (or instability of one or a few members of the structure) on the overall buckling and post-buckling response of the structure is rather sparse. Rosen and Schmit [22] present an interesting study of such phenomena. However, their study pertains to the effects of interaction of local and global imperfections on the overall response of the structure. Further, the methodology employed by Rosen and Schmit [22] makes it difficult to compute the post-buckling response of the structure.

The objective of the present work is to study the effect of local (member) buckling and the subsequent load transfer and overall structural response of both geometrically perfect as well as imperfect structures. Here, plane trusses are treated exclusively as the example structures. While these problems may be considered as clearly within the reach of standard "incremental" finite element methods (wherein, in each element, a tangent stiffness matrix for each member, and thus for the whole structure, may be routinely evaluated from appropriate variational principles or weak forms), it is known that such methods become prohibitively expensive to treat realistic structures. Examples of such structures of current interest include the large-space-structures and antennae that may be deployed in space. Here, a simple method is proposed to derive consistently, explicitly, the stiffness matrix of a truss member in both its pre-buckled and post-buckled ranges of behavior. This simplifies the formation of the tangent stiffness matrix of the structure considerably and thus renders the pre- and post-buckling analyses of the structure feasible for...
the above mentioned types of structures. This simplified procedure of forming the stiffness matrix, in conjunction with the arc length method [15-21], which is appropriately modified herein to account for an individual member's buckling, is used to study the post-buckling behavior of the truss structures. The effects of local buckling on the pre-buckling and post-buckling response of the structure as a whole are critically examined.

Six examples, each characterizing different features of structural response, are analyzed. The first two deal with simple truss structures for which analytical, as well as experimental, results were reported in [23]. These two examples characterize the structural response in which global buckling is caused by local buckling of one or several of the members. The third example brings out the effect of local (member) buckling on structural response of the snap-through type. The fourth example is that of the so-called Thompson's Strut [5, 22], wherein the effect of small variations in cross-sectional properties of individual members is critically evaluated. The fifth example deals with an idealized truss model of the plane arch shape. Finally, in the sixth example, the effect of structural imperfections, in addition to that of local buckling, is explored in the case of the Thompson's Strut [5, 22]. All these examples also serve to effectively bring out the ranges of applicability and advantages of the presently proposed simplified procedures for forming the tangent stiffness of the members as well as that of the structure.

2. EVALUATION OF TANGENT STIFFNESS FOR A TRUSS

The plane truss structures to be discussed in this paper are assumed to remain elastic and are properly supported. Only a conservative system of concentrated loads at the nodes is considered.

2.1 Stretch and rotation of members

Consider a typical slender truss member spanning between nodes 1 and 2 as shown in Fig. 1. This member is considered to have a uniform cross section, and its length before deformation is \( l \). The coordinates \( z \) and \( x \) are the member's local coordinates. \( w(z) \) and \( u(z) \) denote the displacements at the centroidal axis of a member along the coordinate directions \( z \) and \( x \), respectively. Also the two angular coordinates, \( \theta \) and \( \bar{\theta} \), in the deformed configuration of a member, as shown in Fig. 1, characterize the angle between the straight-line joining the nodes of the deformed member and the \( z \) axis, and that between the tangent to the deformed centroidal axis and the \( z \) axis, respectively. It should be noted that \( \theta \) is constant for both the pre- and post-buckled configurations of the member, but \( \bar{\theta} \) varies along the member's axis in its buckled state.

From the polar decomposition theorem [24], the relations between the point-wise stretch, rotation and displacements are

\[
(1 + \epsilon) \cos \theta = 1 + \frac{dw}{dz} \tag{1a}
\]
\[
(1 + \epsilon) \sin \theta = \frac{du}{dz} \tag{1b}
\]

where \( \epsilon \) is the point-wise axial stretch of the member, and \( \theta = \bar{\theta} + \tilde{\theta} \) (see Fig. 1). Eliminating \( \theta \) from eqns (1), \( \epsilon \) is represented as

\[
\epsilon = \left[ \left( 1 + \frac{dw}{dz} \right)^2 + \left( \frac{du}{dz} \right)^2 \right]^{1/2} - 1. \tag{2}
\]

Substituting into eqns (1) the relation

\[
\theta = \bar{\theta} + \tilde{\theta}, \tag{3}
\]

noting that \( \bar{\theta} \) is a constant in both the pre- and post-buckled states of the member, and integrating the resulting equations along the length of the member, one obtains the following equations:

\[
(l + \delta) \cos \bar{\theta} = l + \tilde{w} \tag{4a}
\]
\[
(l + \delta) \sin \bar{\theta} = \tilde{u}, \tag{4b}
\]

where

\[
\tilde{w} = w_2 - w_1, \quad \tilde{u} = u_2 - u_1, \tag{5}
\]

and \( \delta \) is the total axial stretch of the member (see Fig. 1). In the above derivation, the following relations are used:

\[
\int_0^l (1 + \epsilon) \sin \bar{\theta} \, dz = 0 \tag{6a}
\]
\[
\int_0^l (1 + \epsilon) \cos \bar{\theta} \, dz = l + \delta \tag{6b}
\]

and

\[
\int_0^l \frac{dw}{dz} \, dz = \tilde{w}, \quad \int_0^l \frac{du}{dz} \, dz = \tilde{u}. \tag{7a,b}
\]
From eqns (4), it is seen that:

\[ \delta = [(l + \delta w)^2 + \delta a^2]^{1/2} - l. \]  

Equation (8) holds for both the pre- and post-buckled states of the member. It should be noted that in the pre-buckled state of the member, of course, \( \cos \delta = 1 \) and \( \varepsilon = \) constant, so that the following relation between \( \varepsilon \) and \( \delta \) holds:

\[ \delta = (\varepsilon l). \]  

2.2 Relation between stretch and axial force in a member

The relation between the total stretch of and the axial force in the member is written as

\[ \Delta N = k \Delta \delta \]  

and \( \Delta \) denotes an increment.

Equation (11a) simply follows from eqn (9) and the linear-elastic (isotropic) stress–strain law of the material of the member. On the other hand, eqn (11b) for the post-buckled state of the member is derived in this paper by simplifying and modifying the governing equations of the elastica problem with each member being treated as a simply supported elastica. The details of the derivation of eqn (11b) along with a verification of its accuracy are given in the next section.

Here, it should be noted that \( N \) is in the direction of the straight line connecting node 1 and node 2 of the member after its deformation (see Fig. 1), and \( \delta \) is calculated from eqn (8). Hence, eqn (10) holds even when the rigid motion of the member is very large. Also, note that \( k \) is a constant in each of the two states, i.e. pre-buckled and post-buckled.

Furthermore, the condition for buckling of the member, treated as a simply supported beam, is given by the following well-known equation:

\[ N = N^{(cr)} \]  

where

\[ N^{(cr)} = -\frac{\pi^2 EI}{l^2} \]

the negative sign being used to denote the compressive axial force.

2.3 Strain energy in, and stiffness matrix of, a member

The only force acting on a truss-member is considered to be the axial force. Hence, the strain energy of the member, \( U \), in either the pre- or post-buckled states of the member, is given by:

\[ U = \frac{1}{2} \int_0^l (EA\varepsilon^2 + EL\chi^2)dz \]

\[ = \frac{1}{2} \int_0^l N\delta \delta, \]  

where \( \chi \) is the curvature which is zero for the pre-buckled state, and non-zero in the post-buckled state, of the member.

The incremental form of eqn (14) is represented, using eqn (10), as

\[ \Delta U = N\Delta \delta + \frac{1}{2}k(\Delta \delta)^2. \]  

Substituting the incremental form of eqn (8) into eqn (15), one finds that

\[ \Delta U = N\mathbf{c} \cdot \Delta \mathbf{w} + N\mathbf{s} \cdot \Delta \mathbf{u} \]

\[ + \frac{1}{2} \left( N\frac{1}{l^*}c^2 + k \mathbf{c}^2 \right) \cdot \Delta \mathbf{w}^2 \]

\[ + \left( - N\frac{1}{l^*}c \mathbf{s} + k \mathbf{c} \mathbf{s} \right) \Delta \mathbf{u} \cdot \Delta \mathbf{w} \]

\[ + \frac{1}{2} \left( N\frac{1}{l^*}c^2 + k \mathbf{s}^2 \right) \cdot \Delta \mathbf{u}^2 \]

\[ + \text{Higher order terms.} \]  

where

\[ \mathbf{c} = \frac{1}{l^*}(l + \mathbf{v}), \quad \mathbf{s} = \frac{1}{l^*}\mathbf{u} \]  

\[ l^* = \sqrt{(l + \mathbf{w})^2 + \mathbf{u}^2}. \]  

Furthermore, neglecting terms of higher than the second-order, the variation in the strain-energy may be derived, from eqn (16), as

\[ \delta \Delta U = \delta \Delta \mathbf{w} \cdot (N\mathbf{c}) + \delta \Delta \mathbf{u} (N\mathbf{s}) \]

\[ + \delta \Delta \mathbf{w} \left[ \left( N\frac{1}{l^*}c^2 + k \mathbf{c}^2 \right) \Delta \mathbf{w} \right] \]

\[ + \left( - N\frac{1}{l^*}c \mathbf{s} + k \mathbf{c} \mathbf{s} \right) \Delta \mathbf{u} \cdot \Delta \mathbf{w} \]

\[ + \delta \Delta \mathbf{u} \left[ \left( - N\frac{1}{l^*}c \mathbf{s} + k \mathbf{c} \mathbf{s} \right) \Delta \mathbf{w} \right] \]

\[ + \left( N\frac{1}{l^*}c^2 + k \mathbf{s}^2 \right) \cdot \Delta \mathbf{u} \]

\[ = [\delta \Delta \mathbf{w}]^T [R^n] + [\delta \Delta \mathbf{u}]^T [K^n] [\Delta \mathbf{d}^m], \]  

where
where

\[
\{d^m\} = \begin{bmatrix} w_1 \\ w_2 \\ u_1 \\ u_2 \end{bmatrix}, \quad \{R^m\} = \begin{bmatrix} (N-c)\{l\} \\ (N-s)\{l\} \end{bmatrix}
\]

\[
[K^m] = \begin{bmatrix}
\left( N - \frac{1}{s^2} c^2 \right) \left( -N - \frac{1}{s^2} c^2 \right) & \left( N - \frac{1}{s^2} c^2 \right) \\
\frac{\text{Sym}}{2} & \left( N - \frac{1}{s^2} c^2 \right) \left( -N - \frac{1}{s^2} c^2 \right)
\end{bmatrix} \{l\} = \begin{bmatrix} -1 \\ 1 \\ -1 \\
1 \end{bmatrix} \quad \{E\} = \begin{bmatrix} -1 & -1 & 1 \\
1 & 1 & 0 \end{bmatrix}
\]

and \(\{d^m\}, \{R^m\}, \{K^m\}\) are the vector of generalized displacements, the vector of internal forces and the stiffness matrix of the element, respectively. However, note that eqns (18) and (19) are written in the local coordinate description, so that it is necessary to transform the displacement vector from the local coordinate system to the global coordinate system in the usual fashion.

It should be emphasized again that eqns (18) and (19) are applicable for both the pre- and post-buckled states of the member; and, moreover, \(k\) has a constant value, in each of the two states, as given in eqns (10, 11). Consequently, if a member buckles, it is only necessary for the value of \(k\) to be changed. In view of this, it is seen that it is very simple to derive the tangent stiffness of the member and thus of the structure; and this, in turn, leads to distinct advantages of numerical solution, as will be demonstrated later in this paper.

3. POST-BUCKLING BEHAVIOR OF A MEMBER

In this section, eqn (11b) for the post-buckled state of the member is derived.

Consider the truss member being subjected to the compressive force \((-N)\), as shown in Fig. 1. When \(N\) satisfies eqn (12), this member undergoes bifurcation buckling. From the detailed treatment of the elastica problem given in Ref. [25], the post-buckling behavior of this member, treated as a simply-supported beam, is governed by the following equations:

\[
l = \frac{1}{f} F(\beta)
\]

\[
l + \delta = \frac{2}{f} E(\beta) - l
\]

\[
\delta = \frac{2}{f} \beta
\]

where

\[
F(\beta) = \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\sqrt{1 - \beta^2 \sin^2 \phi}}
\]

\[
E(\beta) = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \beta^2 \sin^2 \phi} \cdot d\phi
\]

and \(F(\beta), E(\beta)\) are the elliptic integrals of the first and the second kind, respectively. Also, \(\hat{\delta}\) is the stretch after the buckling of the member, and \(\delta\) is the lateral deflection at the middle of the centroidal axis of the element. Note that the total stretch \(\hat{\delta}\) is given by the sum of \(\delta\) and the stretch, \(N''''/EA\), before the buckling of the member. Also, it should be noted that in the derivation of eqns (20) the change in the length of the member due to the compressive force is neglected.

Equations (20) give the exact relations between \(N, \delta\) and \(\delta\) in the post-buckled range, except for the assumption concerning the length of the element. We now simplify and modify these relations to a form more useful for the present purposes of evaluating a tangent stiffness matrix. To this end, we start by expanding \(F(\beta), E(\beta)\) in terms of \(\beta\) [26].

\[
F(\beta) = \pi + \frac{1}{2} \beta^2 S_2 + \frac{1}{2} \beta^4 S_4 + \ldots
\]

\[
E(\beta) = \pi - \frac{1}{2} \beta^2 S_2 - \frac{1}{2} \beta^4 S_4 + \ldots
\]

where

\[
S_n = \int_{-\pi/2}^{\pi/2} \sin^n \phi \cdot d\phi.
\]

We shall retain the terms of eqns (22) up to the second order for the approximations of \(F(\beta), E(\beta)\):

\[
F(\beta) = \pi + \frac{\pi}{4} \beta^2
\]

\[
E(\beta) = \pi - \frac{\pi}{4} \beta^2.
\]

The range of validity of these approximations will be demonstrated momentarily.

Then, eqns (20a) and (20b), respectively, become

\[
l = \frac{1}{f} \left( \pi + \frac{\pi}{4} \beta^2 \right)
\]

\[
l + \delta = \frac{2}{f} \left( \pi - \frac{\pi}{4} \beta^2 \right) - l.
\]

From eqns (25a,b) one obtains

\[
4l + \hat{\delta} = \frac{4}{f} \pi
\]
Noting that \( f^2 = (-N/EI) \), one sees from eqn (26) that

\[
N = N^{(cr)} \frac{1}{1 + \left(\frac{6}{4l}\right)^2}, \tag{27}
\]

where \( N^{(cr)} \) is the critical axial force for bifurcation buckling as given in eqn (13).

For small values of \(-\delta/l\), eqn (27) may be approximated as follows:

\[
N = N^{(cr)} \left[ 1 - \frac{1}{2} \left(\frac{\delta}{l}\right) \right]. \tag{28}
\]

The incremental form of eqn (28) results in eqn (11b). The linear relation (28) and its incremental counterpart are useful in tangent stiffness evaluations.

We now derive the relation between \( \delta \) and \( \delta \). This relation is not necessary for the construction of the tangent stiffness, but it is useful for the determination of maximum and/or minimum stress in each of the members.

Noting that \( \beta \) is non-negative except for \( \alpha > 2\pi \), one obtains from eqn (25a)

\[
\beta = 2 \sqrt{\frac{1}{\pi} f/4l - 1}. \tag{29}
\]

Substituting eqn (29) into eqn (20c), it is seen that

\[
\delta = \frac{4}{f} \sqrt{\frac{1}{\pi} f/4l - 1}. \tag{30}
\]

Substituting for \( f \) in terms of \( N \) and using eqn (27), the following relation between \( \delta \) and \( \delta \) is obtained:

\[
\frac{\delta}{l} = \frac{4}{\pi} \sqrt{\frac{\delta}{4l} \left(1 - \frac{\delta}{4l}\right)}. \tag{31}
\]

Thus, when the axial contraction \( \delta \) is solved for from the finite element stiffness equation, eqn (31) may be used to calculate the transverse displacement \( \delta \) at midspan of the member, and from it one may calculate the maximum or minimum stress in the member.

Figure 2 shows the relations between \( N \), \( \delta \) and \( \delta \), as given by eqns (28) and (31), and their comparisons with the exact solutions for the elastica problem. The dotted lines indicate the present solutions, and the solid ones indicate the exact. From this figure, it is seen that eqns (28) and (31) are good approximations in the range of values for \(-\delta/l\) and \(\delta/l\) being smaller than about 0.15 and 0.25, respectively. It is also seen that this range of values for \(-\delta/l\) and \(\delta/l\) is typical in the problem of local (member) buckling in a practical truss structure.

4. SOLUTION STRATEGY

A number of solution procedures are available for nonlinear structure analyses. These are the standard load control method, the displacement control method [11, 12], the artificial spring method [20], the perturbation method [3], the method using the current stiffness parameter [13, 14], etc. Another promising approach to trace the structural response near limit points, etc., is the arc length method proposed by Riks [15, 16] and Wempner [17] and modified by Chrisfield [18, 19] and Ramm [20]. This method is the incremental/iterative procedure, which represents a generalization of the displacement control approach. Also, this approach, wherein the Euclidian norm of the increment in the displacement and load space (the arc length) is adopted as the prescribed increment, allows one to trace the equilibrium path beyond limit points, such as in snap-through and snap-back phenomena.

We adopt this arc length method as a basis of the solution procedure in the present study. However, if any individual member buckles in a smaller than the prescribed incremental arc length for the structure, we adopt the smaller value as the incremental arc length.

The \( i \)th incremental/iterative stiffness equation can be written as follows:

\[
(p_{-1} - \{R_{-1}\}) + \Delta P = [K] \Delta d, \tag{32}
\]
where

\[
\begin{align*}
\{P\} & \text{: Load vector; } p : \text{Load parameter} \\
\{R\} & \text{: Internal force vector [see eqn (18)]} \\
[K] & \text{: Stiffness matrix} \\
\{d\} & \text{: Displacement parameter vector} \\
p_i, \{R_i\}, \{d\}_i : \text{Total values after the } i\text{th iteration} \\
\Delta p, \Delta d : \text{Incremental values during this iteration} \\
\{\Delta d\}_i : \text{Incremental displacement vector after the } i\text{th iteration} \\
\end{align*}
\]

and \(\{\Delta d\}_i\) is given by

\[
\{\Delta d\}_i = \{\Delta d\}_{i-1} + \{\Delta d\} \tag{33}
\]

It should be noted that in eqn (32), \(i = 1\) represents the incremental process and \(i > 1\) the iterative process. Also, in the numerical implementation of eqn (32), the standard Newton-Raphson procedure, or the modified Newton-Raphson procedure, etc. may be employed.

We decompose \(\{\Delta d\}\) into two parts:

\[
\{\Delta d\} = \{\Delta d^*\} + \Delta p \{\Delta d^{**}\}, \tag{34}
\]

where

\[
\begin{align*}
\{\Delta d^*\} &= [K]^{-1}(p_{i-1}(\bar{P}) - \{R\}_{i-1}) \tag{35a} \\
\{\Delta d^{**}\} &= [K]^{-1}\{\Delta P\} \tag{35b}
\end{align*}
\]

Thus, \(\{\Delta d^*\}\) and \(\{\Delta d^{**}\}\) represent the responses to the unbalanced force and the external load, respectively. From eqns (33) and (34), it is seen:

\[
\{\Delta d\}_i = \{\Delta d\}_{i-1} + \{\Delta d^*\} + \Delta p \{\Delta d^{**}\}. \tag{36}
\]

The incremental arc length \(\Delta \eta\), after the \(i\)th iteration is defined as [19]:

\[
\Delta \eta_i = \left(\langle \{\Delta d\}_i | \{\Delta d\}_i \rangle + \gamma \Delta p^2 |\bar{P}|^2 |\bar{P}| \right)^{1/2} \tag{37}
\]

where \(\Delta \eta_i\) is the incremental value of \(\eta\) after the \(i\)th iteration. On the other hand, \(\gamma\) is a scaling parameter that represents the contribution of the load term to the arc length. If a larger value for \(\gamma\) is adopted, \(\Delta \eta\) tends to be proportional to the incremental load, and the method tends toward the standard load control method. However, numerical experience has shown that it is preferable to ignore this contribution [16]. In the present study, \(\gamma\) is set to be zero, following Refs. [18, 20]. Consequently, using eqn (36), \(\Delta \eta_i\) becomes:

\[
\Delta \eta_i = \left(\langle \{\Delta d\}_i | \{\Delta d\}_i \rangle \right)^{1/2}
\]

\[
= \left[|\{\Delta d^{**}\}|^2 |\{\Delta d^{**}\}|^2 + 2|\{\Delta d^{**}\}| \langle \{\Delta d\}_{i-1} \rangle \{\Delta d^*\} \right]
\]

\[
= \left(\{\Delta d\}_{i-1} + \{\Delta d^*\}\right)^2 \times \left(\{\Delta d\}_{i-1} + \{\Delta d^*\}\right)^{1/2}. \tag{38}
\]

In the arc length method, \(\Delta \eta_i\) is decided by the equation, (see Fig. 3):
Influence of local buckling on global instability

We have discussed the situation of the progressive buckling of an individual member. It might happen that after a member has buckled, during the continued deformation of the structure as whole, the same member may be forced to undergo a "restraightening". Thus, if a previously buckled member undergoes restraightening in an incremental arc length smaller than the prescribed arc length for the structure, then the smaller value is used as the incremental arc length for the structure itself.

The numerical process to handle this situation is entirely analogous to the one treated above wherein a member begins to undergo buckling. Thus, instead of eqn (43) we use

\[ N_i = k \delta_i + N^{(er)}, \]

change eqn (45) accordingly and reverse the inequality signs in eqn (49). In eqn (50), \( \delta_i \) is the value of the stretch in the post-buckled range after the \( i \)th iteration in the increment, \( k \) is the value of "stiffness" in the post-buckled range.

5. NUMERICAL EXAMPLE

In this section, six numerical examples are given to test the validity of the present procedures. As a criterion for the convergence of the iteration, the following equation, using the modified Euclidean norm, is adopted:

\[ \left( \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\Delta d_i[\Delta d]}{\Delta d} \right] \right)^{1/2} < \epsilon \]

where \( n \) is the total number of degrees of freedom, and \( \epsilon \) is set to be \( 10^{-3} \) for all the numerical examples. In the present numerical solution, based on eqn (32), the standard Newton-Raphson procedure is employed.

Examples 1 and 2 are those of simple truss structures, for which theoretical solutions for the buckling load and the initial slope of the post-buckling load-displacement curve are given by Britvec [23]. For these structures, experiments were also carried out by Britvec, who found good correlation between his theoretical and experimental solutions. These structures are composed of two and three members, respectively. All the members have a rectangular cross-section of the width 154 cm, depth of 0.16 cm and length of 38.1 cm. Also, the buckling loads of the individual members are 13.26 kg for Example 1, 13.15 kg for Example 2, respectively.

The schematics of the structures and the results obtained are summarized in Figs 4. Both of these structures have a special type of structural behavior in which the global buckling is caused by the buck-
The theoretical solution for the buckling load of the members is 13.15 kg.

Fig. 5(b). Load–displacement relation for the simple truss structure of Fig. 5(a).

Example 3 is that of a simple structural model [Fig. 5(a)], which exhibits a snap-through phenomenon and is chosen here to study the effect of member buckling on such phenomena. In this example, the range of deformations is much larger than in the earlier examples. The structure is composed of two members, each of which has a local buckling process. The stiffness of the structure gradually increases as the post-buckling deformations progress. This phenomenon is brought out to be the effect of the geometrical nonlinearity, and the results of Britvec’s experiments also show the same tendencies in the post-buckling range. Thus, the present results appear to be reasonably accurate.

Fig. 4(a). Britvec’s truss structure (Example 1).

Fig. 4(b). Britvec’s truss structure (Example 2).

All members have solid circular sections, each of area 96.77 cm².

Young’s modulus is 7.03 × 10⁵ kg/cm².

Case 1 — Local buckling of each of the two members is ignored

Case 2 — Local buckling of only one of the two members is considered

Case 3 — Local buckling of both the members is considered

Fig. 5(a). A simple truss structure (Example 3).
Influence of local buckling on global instability

Table 1. Cross-sectional areas of the members of Thompson’s strut structure

<table>
<thead>
<tr>
<th>Member’s Number</th>
<th>Case 1, 2</th>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-13, 17-21</td>
<td>54.84 cm²</td>
<td>54.84</td>
<td>54.84</td>
</tr>
<tr>
<td>15</td>
<td>54.84</td>
<td>51.61</td>
<td>54.84</td>
</tr>
<tr>
<td>14, 16</td>
<td>54.84</td>
<td>54.84</td>
<td>51.61</td>
</tr>
<tr>
<td>22 - 35</td>
<td>51.61</td>
<td>51.61</td>
<td>51.61</td>
</tr>
</tbody>
</table>

All members have solid circular cross sections.

Young’s modulus is 7.03 x 10⁵ (kg/cm²).

In Case 1, the local buckling of individual is not considered.

identical members, which have a solid circular cross section of area 96.77 cm², a length of 38.1 cm and a Young’s modulus of 7.03 x 10⁵ kg/cm². To study the influence of the member’s buckling, three different cases are investigated. In Case 1, the buckling of both the members is ignored; in Case 2, the buckling of only one of the members is considered; and in Case 3, the buckling of both the members is considered.

Figure 5(b) shows the relations between the applied load and the vertical displacement of the center. Case 1 exhibits a typical snap-through phenomenon and reaches the limit point at a load of 3.76 x 10⁶ kg. In Cases 2 and 3, the individual members buckle at a load of 2.93 x 10⁶ kg and cause the structure to be in the unstable region just after this load. There is little difference to be found between Cases 2 and 3.

The fourth example is that of a strut structure, which was first suggested by Thompson and Hunt [15] and later analyzed by Rosen and Schmit [22] to study the influence of local as well as global geometric imperfections on global stability.

The outline of this structure is shown in Fig. 6(a) and Table 1. The structure is composed of 35 members, all of which have a solid circular cross section and an identical Young’s modulus of 7.03 x 10⁵ kg/cm². As in the case of Example 3, four different cases are dealt with in this example, also, to investigate the influence of the member’s buckling and a slight difference of the cross-sectional area of individual members on global buckling. The cross-sectional areas of the member for each case are shown in Table 1. Note that the structure of this example is not strictly symmetric about the z-axis, and this unsymmetry causes the effective neutral axis of the strut to be slightly above the z-axis for Cases 1–3 or slightly below the z-axis for Case 4. In Case 1, the buckling of all of the individual members is ignored. In the other cases, the buckling of all of the members is considered: however, the cross-sectional area of the members is slightly different for each case, as shown in Table 1. The results obtained are shown in Figs 6(b–d).

Case 1 exhibits an entirely stable equilibrium path in the load-displacement space. At a load of about 7.2 x 10⁵ kg, the global buckling occurs; the stiffness of the structure goes down and tends to zero after that. However, the equilibrium path is still stable.

The difference between Cases 1 and 2 is that member buckling is considered only in the latter, while the cross-sectional areas of the members are the same in both the cases. Thus, the structure of Case 2 exhibits exactly the same behavior as that of Case 1 until a load of 6.916 x 10⁵ kg, when the member of No. 15 buckle.

In Case 3, the cross-sectional area of the member of No. 15 is set to be about 5.89% smaller than the corresponding area in Case 3. However, the structural behavior is almost the same as that in Case 2. With this slight reduction in cross-sectional area of one member, the stiffness of the structure as well as the load level when the member of No. 15 buckles are reduced as compared with Case 2.

In Case 4, the cross-sectional areas of the members 14 and 16 are set to be 94.11% of the corresponding areas in Case 1. This reduction of the

Fig. 6(a). Thompson’s strut structure (Example 4).
cross-sectional areas causes the effective neutral axis of the strut to be slightly below the z-axis. Also, the members 14 and 16 buckle at an external load \( P \) of 6.323 kg.

It is interesting to see that even in a fairly complicated structure such as in Fig. 6(a), the buckling of only one or a few members renders the structure to be unstable. It is also noted that even a slight difference of the cross-sectional area of the members has a great influence on the overall behavior of the structure. In Fig. 6(c), the z-displacement of node 19 [Fig. 6(a)] is shown as a function of the load \( P \) for each of the four cases. The variations of axial forces (directed along the undeformed axes of the members) in members 14 and 15 as a function of the external load \( P \) are shown in Fig. 6(d) for each of the four cases. It is instructive, while examining Fig. 6(d), to remember that Case 1 precludes buckling of any member; in Cases 2 and 3, member 15 buckles (this load is lower in Case 3 than in Case 2); and in Case 4, member 14 buckles first. Figure 6(d) indicates that the load transfer mechanism in a structure after the buckling of an individual member is rather complicated.

Example 5 is an idealized model of a truss of the plane arch shape. This structure was also analyzed by Rosen and Schmit [22] to investigate the influence of geometric imperfections. This thin, shallow arch is made up of 35 truss members, all of which have a solid circular cross section and a Young modulus of \( 7.03 \times 10^6 \) kg/cm². It is shown sche-
Influence of local buckling on global instability

matically in Fig. 7(a) and in Tables 2(a,b). Again, three cases are considered for this example. In Case 1, the buckling of any member is entirely ignored, while it is considered in Cases 2 and 3. The difference between Cases 2 and 3 is only that the cross-sectional areas of members 27 and 28 in Case 3 are 25.00% smaller than the corresponding areas in Case 2. The results obtained are given in Figs. 7(b—d).

Case 1 indicates the snap-through phenomenon similar to that of the behavior of thin shallow arches made of homogeneous isotropic elastic materials. The limit point is reached at a load of about $2.64 \times 10^3$ kg.

In Case 2, members 11 and 12 buckle slightly after the whole structure passes the limit point. As seen from Fig. 7(b), the global structural response in Case 2 is markedly different from that in Case 1. In Case 3, the cross-sectional areas of two members (i.e., Nos. 27, 28) are smaller than the corresponding areas in the other cases. Thus, the overall response in Case 3 is slightly different from the other two cases, until buckling occurs first in members 27 and 28, after passing the limit point of the structure as a whole. However, in spite of the buckling of members 27 and 28, there is little change in the overall behavior of the structure as compared with the former cases. However, when the deformation progresses further, the members 21 and 22 buckle, and this alters the load-carrying capacity of the structure more decisively.

The sixth and final example deals with the interactive effects of imperfections of the structure at the global level and the possibility of local buckling of individual members. The structure considered is identical to that in Example 4 and shown in Fig. 7(a).
6(a). While Example 4 treated a perfect structure, now two cases of global imperfections are considered. The imperfection is of a half-sine-wave form. Two different values of the amplitude of this imperfection mode, 1.32 cm and 2.64 cm, respectively, are considered. In both the cases of imperfection, individual member buckling is considered; and the cross-sectional areas of members are identical to those in Cases 1 and 2 of Example 4, as shown in Table 1. The present example is summarized in Table 3. The results are shown in Figs. 8(a–c). Cases 1 and 2 as marked in Fig. 8(a) are identical to Cases 1 and 2 as marked in Fig. 6(b) for a perfect structure. Comparing these cases with Cases 3 and 4 in Fig. 8(a), the dramatic combined effects of small global imperfections of the structure and the buckling of individual members on global response may be noted. The variations of z-displacement of Node No 3 in the X Direction

Table 2(a). Nodal coordinates of the arch-truss structures

<table>
<thead>
<tr>
<th>Nodal Number</th>
<th>∇ Coordinate</th>
<th>x Coordinate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 19</td>
<td>3429.0</td>
<td>0.00</td>
</tr>
<tr>
<td>2, 18</td>
<td>3048.0</td>
<td>50.65</td>
</tr>
<tr>
<td>3, 17</td>
<td>2667.0</td>
<td>34.75</td>
</tr>
<tr>
<td>4, 16</td>
<td>2286.0</td>
<td>83.82</td>
</tr>
<tr>
<td>5, 15</td>
<td>1905.0</td>
<td>65.30</td>
</tr>
<tr>
<td>6, 14</td>
<td>1524.0</td>
<td>110.85</td>
</tr>
<tr>
<td>7, 13</td>
<td>1143.0</td>
<td>87.99</td>
</tr>
<tr>
<td>8, 12</td>
<td>782.0</td>
<td>128.50</td>
</tr>
<tr>
<td>9, 11</td>
<td>381.0</td>
<td>100.05</td>
</tr>
</tbody>
</table>

In the second column, (-) and (+) respectively indicate the z-coordinates of the first and second members identified in the first column.
Influence of local buckling on global instability

Table 2(b). Cross-sectional areas of the members of the arch–truss structure

<table>
<thead>
<tr>
<th>Member's Number</th>
<th>Case 1, 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 10, 35</td>
<td>51.61(cm²)</td>
<td>51.61</td>
</tr>
<tr>
<td>11, 12</td>
<td>64.52</td>
<td>64.52</td>
</tr>
<tr>
<td>13 - 16</td>
<td>83.87</td>
<td>83.87</td>
</tr>
<tr>
<td>17, 18</td>
<td>96.77</td>
<td>96.77</td>
</tr>
<tr>
<td>19 - 22</td>
<td>103.23</td>
<td>103.23</td>
</tr>
<tr>
<td>23, 24</td>
<td>161.29</td>
<td>161.29</td>
</tr>
<tr>
<td>25, 26</td>
<td>193.55</td>
<td>193.55</td>
</tr>
<tr>
<td>27, 28</td>
<td>258.06</td>
<td>193.55</td>
</tr>
<tr>
<td>29 - 32</td>
<td>290.32</td>
<td>290.32</td>
</tr>
<tr>
<td>33, 34</td>
<td>309.68</td>
<td>309.68</td>
</tr>
</tbody>
</table>

All members have solid circular cross sections.

Young's modulus is 7.03 x 10⁵(kg/cm²).

In Case 1, the local buckling of individual members is not considered.

placement, at node 19, with the external load is shown in Fig. 8(b). The complicated nature of load-transfer in the structure after an individual member's buckling in an imperfect structure may be seen from Fig. 8(c).

The present numerical examples thus delineate: (i) the effect of buckling of an individual member or members on the response of the structure as a whole and on the subsequent load-distribution in the structure, (ii) the effects of even minor variations in the cross-sectional areas of individual members and (iii) the effects of imperfections at the global level, while imperfections at the local level, in each member, may be expected to have similar effects. The present numerical examples also serve to point out the relative efficiency of simple pro-

Fig. 8(b). Load–displacement relation for Thompson's strut with initial global imperfections.

Fig. 8(c). Relation between external load and member forces for Thompson's strut with initial global imperfections.
Table 3. Thompson's strut structure with global imperfections

All members have solid circular cross sections, with areas as follows:

No. 1 — No. 21 54.84\,(cm^2)
No. 22 — No. 35 51.61\,(cm^2)

Young's modulus is 7.03 \times 10^5\,(kg/cm^2).

<table>
<thead>
<tr>
<th>Member's Buckling</th>
<th>System Imperfection*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>No</td>
</tr>
<tr>
<td>Case 2</td>
<td>Yes</td>
</tr>
<tr>
<td>Case 3</td>
<td>Yes</td>
</tr>
<tr>
<td>Case 4</td>
<td>Yes</td>
</tr>
</tbody>
</table>

* Imperfection mode is of a half sine wave shape, from node no. 1 to node no. 19; and the initial x positions of the nodes are located along the half sine wave.

The method proposed in the paper for obtaining tangent stiffnesses.

**CLOSURE**

In this paper, a simple and effective way of forming the tangent stiffness matrix and a modified arc length method have been presented for finding the nonlinear response of truss-type structures, in which the possibility of local (member) buckling is accounted for. The salient features of the present methodology are the following:

(i) The stiffness matrix of an individual member is written down explicitly, both for the pre-buckled and post-buckled states of the member.

(ii) The stiffness coefficient \( k \), that relates the member axial force to the member axial displacement, has constant values in the pre- and post-buckled states, respectively. The range of validity of this approximation has been demonstrated to cover most practical situations of truss-type structures.

(iii) Because of (i) and (ii), it is very simple matter to evaluate the tangent stiffness matrix of the structure as a whole.

(iv) The arc length method, modified to account for member buckling, is efficient in tracing the response of the structure as a whole beyond limit points, if any.

The methodology proposed in the paper is thus useful in analyzing structures of the type that are being considered for use as deployable large-space-structures, space antennae, etc.

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Control of transient dynamic response of structures

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ABSTRACT: In this paper, the problem of active control of transient dynamic response of large space structures, using the fully coupled nodal equations of motion, taking account of non-proportional passive damping, is treated. Equivalent continuum plate-models of LSS are considered. A reduced-order modeling of the structure using a boundary element method is presented. Efficient algorithms for solving the matrix algebraic Riccati equation are discussed. Several examples are presented to illustrate the controllability of the transient dynamic response of the structure, using an arbitrary number of control-force actuators.

1 INTRODUCTION

Very large, low-mass structures are currently intended to play important roles in space missions, which include such activities as communications, earth resource surveillance, and multipurpose large space platforms, etc. The large space structures (LSS) are to be deployed, erected, assembled, or fabricated in space. The requirement of minimum weight coupled with very large physical dimensions (perhaps of the size of Manhattan Island) will result in a very high degree of structural flexibility and, thus, in low levels of structural dynamic frequencies. These large space structures, during the course of their intended use, must be controllable: (i) altitude control, which involves the control of the spacecraft so as to maintain a given sun-pointing or earth-pointing accuracy, and (ii) shape or configuration control, which involves the suppression of vibration of critical members, such as flexible antennae, which degrade the overall pointing accuracy. Also, the uncontrolled surface modes of the antenna reflector cause defocus and deviations of the surface from the desired shape. Thus, the interaction of a highly flexible, large space structure (LSS) with (active and/or passive) control systems is one of the challenging problems of large space structure technology.

From a control viewpoint, the LSS may be treated as distributed parameter systems. The LSS, in general, may be viewed as three-dimensional frame- or truss-type structures, depending on the joint design such as sleeve-stiffened beam joints or truss-pin joints, respectively. The sleeve-stiffened joint would induce bending moments at the nodes of each (one-dimensional) member of the joint, thus necessitating the modeling of LSS as a three-dimensional frame. On the other hand, the use of truss-pin joints allows the members to rotate freely about the joint, and the members may be modeled as tension-compression members. However, the direct modeling of an entire LSS as a three-dimensional frame/truss is not only prohibitively expensive even for large computers stationed on ground but is also entirely inconceivable for purposes of implementing control algorithms using onboard computers. Thus, a combination of equivalent continuum models (Noor 1978; Aswani 1982) and detailed 3-D frame/truss models (Kondoh 1984a,b) may be necessary, depending on the nature of the disturbing forces on the LSS. Also, depending on the spatial distribution and time variation (including the time-span of an impulse) of disturbances and forcing functions on the LSS, the dynamic response of the LSS may be of a standing-wave or of a traveling-wave type. An equivalent continuum model may be reasonably used to predict the overall modes and frequencies and standing-wave type of response of an LSS;
while, for the control of transient response due to highly localized impulses (such as due to thruster firings), a transient wave-propagation analysis using a detailed 3-D frame/truss model may be necessary.

In this paper, we restrict our attention to situations wherein an equivalent two-dimensional continuum model (Noor 1978; Aswani 1982) may be sufficient for the purposes of the control and suppression of the overall vibratory response of the LSS. The current state in modeling, for purposes of studies of control of dynamic response, has been primarily one-dimensional and linear in nature (Meirovitch 1983, 1984; Venkayya 1984; Von Flotow 1984), i.e., axial motion of one-dimensional members (Meirovitch 1983, 1984; Venkayya 1984) and either continuum or periodic models of LSS as one-dimensional members in torsion, as Timoshenko beams and as micropolar beams for inplane deformations (Von Flotow 1984). In this paper, the LSS is to be represented by an equivalent continuum model of a plate, governed by a 4th-order bending theory.

Wang (1984) and Rodriguez (1982) provide state-of-the-art studies on dynamics and control of large space antennae. Wang (1984) used a finite element model of the antenna, with 1240 degrees of freedom, to obtain the first 12 non-rigid modes of vibration of the antenna assembly, of which only seven correspond to the distortional modes of the antenna dish. For a more precise shape control, more than seven modes of the reflector may be necessary — in which case the total number of degrees of freedom would have to be much higher. This underscores the need for alternate concepts of reduced-order structural modeling of LSS; and the issues to be addressed are: (i) for a system with a given number of degrees of freedom \( M \), what is the accuracy of the first \( N \) modes? and (ii) if the first \( N^* \) modes are to be accurately evaluated, what is the minimum number of degrees of freedom, say \( M^* \), that is needed? Also, while the current state-of-the-art algorithms for vibration analysis may effectively handle very large systems, those for implementation of control are still limited to very small-order systems. Thus, there is a need for innovative algorithms for control of medium-sized systems (of the order of 100 degrees of freedom).

The state-of-the-art in LSS control is: (i) to assume that damping is either negligible or is proportional to mass and stiffness, (ii) to obtain the free-vibration modes and thus obtain a decoupled system of second-order equations in modal coordinates, and (iii) to implement a control for each individual mode separately. Thus, in this approach — the so-called Independent Modal Space Control (IMSC) (Meirovitch 1983, 1984) — one needs as many control-force-actuators as the number of modes controlled. However, when damping is present, either due to material hysteresis or due to deliberate introduction of damping mechanisms or due to deliberate design of LSS joints (Hertz 1984), the concepts of modal decomposition and decoupling of modal equations of motion are not possible, especially when such damping is of a non-proportional type. Thus, a direct attack on the coupled equations of motion of the discretized continuum model of the LSS and implementation of control with an arbitrary number (much smaller than the number of degrees of freedom) of actuators is desirable.

In view of the above discussions, in the present paper we deal with the following topics. In Section 2 we present a boundary-element model for the transient dynamic response of an equivalent continuum model of an LSS, viz., a "free-free" plate governed by a 4th-order plate theory. We consider the passive damping to be of a non-proportional type. In Section 3 we deal with the topic of direct nodal control of the system, through the coupled system of nodal equations of motion as obtained from the boundary-element model of the "free-free" plate. Here, we also discuss efficient algorithms for solving the Riccati matrix equations arising out of the implementation of "linear optimal active control" of the system. In Section 4 we present several numerical results for the force output of control actuators as well as the dynamic transient response of the controlled system. Section 5 ends with a set of concluding remarks.

2 BOUNDARY-ELEMENT MODEL FOR TRANSIENT RESPONSE OF LSS MODELED AS A 4TH-ORDER PLATE

Here we assume that the lattice structure of the LSS is reduced to an equivalent plate model using procedures as outlined, for instance, by Noor (1978) and Aswani (1982). Further, we restrict the amplitude of vibration to be small, so that the following linear dynamic equations of motion for a 4th-order plate
applies:

\[ \Delta^4 \omega(x,t) = -c(x) \frac{\partial^2 \omega}{\partial t^2}(x,t) \]

\[ - m(x) \frac{\partial^2 \omega}{\partial t^2}(x,t) + f(x,t) \]

(2.1)

where \( D, c, \) and \( m \) represent the equivalent distributed bending stiffness, distributed damping, and distributed mass of the LSS modeled as a plate. If a finite element model is used to discretize (2.1), it is well known that (i) one needs \( C \) elements, (ii) when the planar dimensions of LSS are much larger than its equivalent thickness, a large number of elements are needed, and (iii) to obtain the first \( N \) modes accurately, one, in general, needs a finite element of equations of order \( M \), where, in general \( M \gg N \).

As an alternative, we use a boundary-element modeling. Let \( w^* \) be a test function, and \( w \) be a trial function. Thus, the weak form of (2.1) becomes:

\[ \int_{\Gamma} (\Delta^4 \omega)w^* \, d\Omega = \int_{\Gamma} \left[ -c \frac{\partial^2 \omega}{\partial t^2} - m \frac{\partial^2 \omega}{\partial t^2} \right] w^* \, d\Omega \]

(2.2)

Let \( w^* \) be the singular solution of the biharmonic equation (Stern 1979). Using this property of \( w^* \), and by repeated integration by parts, one may obtain an integral relation for \( w(x) \):

\[ \int_{\Gamma} \frac{\partial^2 \omega}{\partial t^2} \, d\Omega = \frac{1}{D} \int_{\Gamma} \left[ w^* \left[ \frac{\partial^2 \omega}{\partial n^2}(w) - N(w^*)m_n(w) \right] - M(w^*)m_t(w) \right] \, d\Omega - \int_{\Gamma} \left[ w^* f - \frac{\partial^2 \omega}{\partial t^2} \left[ c \frac{\partial^2 \omega}{\partial t^2} \right] \right] \, d\Omega \]

(2.3)

where:

- \( \beta \subseteq \Gamma; \beta \subseteq \frac{\partial \Omega}{2} \subseteq \partial \Omega \) (the included angle at \( P \); \( \partial \Omega \) is boundary of \( \Omega \))
- \( n, \) \( m_n \): normal and tangential directions to \( \partial \Omega \)
- \( V_n(w) \): Kirchhoff shear as a function of \( w \)
- \( m_t(w) \) and \( m_n(w) \): tangential and normal moments as functions of \( w \)

\[ \left[ \begin{array}{c} \omega \end{array} \right] \] denotes a jump in \( ( ) \) at a corner on \( \partial \Omega \), with \( K \) corners

\[ w^*(P,Q) = \frac{1}{\delta} \frac{1}{r^2} \ln r \cdot r = \| PQ \| \]

\[ N(w^*); M(w^*); T(w^*), \text{ and } V(w^*): \text{ normal slope, normal bending moment, and Kirchhoff shear, respectively, as functions of } w^* \]

A second equation, that for \( \delta \omega/\delta n \) where \( n_p \) is a unit outward normal at the boundary, \( \partial \Omega \), may be written as:

\[ \frac{\partial^2 \omega}{\partial n^2} = \frac{1}{D} \int_{\Gamma} \left[ w^* \left[ \frac{\partial^2 \omega}{\partial n^2}(w) - N(w^*)m_n(w) \right] - M(w^*)m_t(w) \right] \, d\Omega \]

(2.4)

The above two equations permit the solution of the general dynamic response boundary value problem wherein, in general, two of the four quantities \( \omega, \delta \omega/\delta n, m_n(w), \) and \( V_n(w) \) may be specified at \( \partial \Omega \) and the other two are unknowns.

Note that there are no interior elements used to discretize the biharmonic operator. The only interior unknowns in Eqs. (2.3) and (2.4) are: (i) \( \omega \) and its time derivatives and (ii) the component of the force \( f(x,t) \) which is exerted by the control actuator. (Thus, we may set \( f = f_e + f_c \) where \( e \) and \( c \) denote, respectively, the "externally prescribed" and the "yet-to-be-solved control" forces.) Thus, in the present discretization, (i) \( \omega \), \( \delta \omega/\delta n, m_n(w), \) and \( V_n(w) \) are interpolated in each element at the boundary (we use linear interpolation here), and (ii) \( \omega \), and its derivatives are interpolated in the interior elements (we use simple piecewise-constant interpolation for \( \omega, \delta \omega/\delta n, \) and \( \delta^2 \omega/\delta t^2 \) in the interior).

Omitting further algebraic details, the use of the Eqs. (2.3) and (2.4) at the boundary \( \partial \Omega \), in conjunction with the above described discretizations, leads to:

\[ \int_{\Gamma} \omega_s + L \int_{\Gamma} \omega_s - C \int_{\Gamma} \dot{\omega}_s = 0 \]

(2.5)

wherein, it has been assumed that the given boundary conditions are of the homogeneous type. In the above, \( \omega_s \) is the
vector of unknown nodal quantities at \( \partial \Omega \), i.e., the unknowns in the set \( w, \partial w/\partial n, m, \) and \( V \) at each boundary node; \( \dot{w} \) and \( \ddot{w} \) are, respectively, the velocity and acceleration of nodes in \( \Omega \); \( C_r \) is related to appropriate integrals at \( \partial \Omega \); \( J_r \) and \( C_r \) are related to appropriate integrals in \( \Omega \). A second equation may be sought to express \( \dot{w}_s \) in terms of the nodal values of force, velocity, and acceleration in the interior. This second equation may be obtained by considering Eq. (2.3) for various interior-nodal-displacements, as:

\[
\mathbf{w}_s = \mathbf{G}_s \mathbf{w}_s + J_s \ddot{w}_s - C_s \mathbf{w}_s
\]

where the matrices are obtained through appropriate integrations indicated in (2.3). We use (2.5) to eliminate \( \mathbf{w}_s \) and write:

\[
\mathbf{w}_s = \mathbf{G}_s^{-1} \{ J_s \ddot{w}_s - G_s \mathbf{w}_s \} \quad (2.7)
\]

Use of (2.7) in (2.6) results in the equation:

\[
(\mathbf{J}_s - \mathbf{G}_s \mathbf{G}_s^{-1} \mathbf{J}_r) \ddot{\mathbf{w}}_s + \left[ \mathbf{G}_s - \mathbf{G}_s \mathbf{G}_s^{-1} \mathbf{C}_r \right] \mathbf{w}_s + \mathbf{I} \mathbf{w}_s = \left[ \begin{array}{c} 1 \\ \mathbf{C}_r \end{array} \right] \mathbf{w}_s
\]

Thus, \( \mathbf{G}_s \mathbf{G}_s^{-1} \) may be computed only once, in order to form (2.8). Note that by analogy to standard dynamic response equation, the stiffness of the system is identity, \( \mathbf{I} \). It is noted that a procedure analogous to that above has been used by Stern (1979) and Bezine (1980) in connection with problems of elasto-static response and calculation of frequencies of free-vibration, respectively. Here, Eq. (2.8) would be solved directly to analyze the transient dynamic response. It is important to note, however, that the central problem here is to design the control-actuator force \( f \) such that the transient response of the system (2.8) is either damped out completely or to a predetermined level in a finite settling-time, \( t_f \). To this end, optimal control techniques may be employed and are briefly sketched below.

3 CONTROL OF TRANSIENT RESPONSE OF THE DISCRETIZED SYSTEM

As noted earlier, we consider \( f(x,t) \) in (2.1) to consist of two parts:

\[
f(x,t) = f_c(x,t) + f_e(x,t) \quad (3.1)
\]

where \( f_c \) is the control force, and \( f_e \) is the external loading. In the following, we let \( f_c(x,t) \) to be zero and consider the design of control force \( f_c(x,t) \), such that the response of the system to initial disturbances, viz., \( \mathbf{w}_0(x) \equiv \mathbf{w}_0 \), and \( \dot{\mathbf{w}}(x) = \mathbf{w}_0(x) \). Further, we let the control force to be discrete in nature:

\[
f_c(x,t) = \sum_{i=1}^{m} \delta(x - x_i) f_{ci}(t)
\]

Note that in (2.8), the dimension of the vector \( \mathbf{w}_s \) is \( n \). We seek the number of actuators, \( m \), to be such that \( m \leq n \). Thus, we may rewrite (2.8) as:

\[
(\mathbf{J}_s - \mathbf{G}_s \mathbf{G}_s^{-1} \mathbf{J}_r) \mathbf{w}_s = \mathbf{b} \mathbf{f}_c
\]

Note that \( \mathbf{I} \) is a unit matrix; and \( \mathbf{G}_s \), the damping matrix, is not proportional to either \( \mathbf{M} \) or \( \mathbf{I} \). Thus, there do not exist normal modes, in the usual sense, that would decouple Eq. (3.3). We now consider a direct attack on the nodal equations (3.3), to control the dynamic response using an arbitrary number of actuators, \( m \leq n \). Eq. (3.3) is recast in the state variable form, as:

\[
\dot{\mathbf{S}} = \mathbf{A} \mathbf{S} + \mathbf{B} \mathbf{F}
\]

where, \( \mathbf{S}^T = [w_1, w_2, \ldots, w_n; \dot{w}_1, \dot{w}_2, \ldots, \dot{w}_n] \)

\[
\mathbf{F}^T = [f_{1c}, f_{2c}, \ldots, f_{mc}]
\]

Thus, \( \mathbf{S} \) is a \((2n \times 1)\) vector, \( \mathbf{A} \) is \((2n \times 2n)\), \( \mathbf{B} \) is \((2n \times m)\). Further, the observations of \( P \) sensors may be written as:

\[
\mathbf{Y} = \mathbf{D} \mathbf{S}
\]

The control forces \( f_c \) may be designed using the theory of linear optimal control (Bryson 1975). Here, \( \mathbf{F} \) in (3.4) may be chosen so as to minimize a quadratic objective function of the type:

\[
1 = \frac{1}{2} \int_0^{t_f} (\mathbf{S}^T \mathbf{Q} \mathbf{S} + \mathbf{F}^T \mathbf{R} \mathbf{F}) \, dt
\]
where $t_f$ is the final settling time, and $Q$ and $R$ are weighting matrices that will influence the magnitudes of the actual forces and the quantitative decay of the response amplitude. Assuming, for simplicity, that $t_f \rightarrow \infty$, we obtain (Bryson 1975) from the minimization of (3.9) the equation for the feedback-control forces:

$$F = -S^{-1}S^tK\cdot S$$

where $K$ is the solution to the so-called "steady-state" Riccati matrix algebraic equation:

$$KA + A^tK - KBR^{-1}B^tK + Q = 0$$

Thus, the nonlinear algebraic equation (3.11) has to be solved for the $(2n \times 2n)$ symmetric matrix $K$ (where $n$ is the number of nodes in the discretized system).

Assuming that (3.11) can be solved, the use of (3.10) in (3.4) leads to the optimal closed-loop system:

$$S = A^*S$$

where $A^* = A - BR^{-1}B^tK$ (3.13)

The integration of Eq. (3.12), subject to given initial conditions $S(0) \equiv S_0$, leads to the dynamic response of the system, in the presence of feedback-control forces.

The solution of the steady-state nonlinear Riccati equation (3.11) is in fact, however, the single most limiting factor in the implementation of optimal control, especially, using on-board computers. The simplest method to solve (3.11) is through iteration (Venkayya 1984). However, the computational time involved will be prohibitive even for moderate size systems, say $n \approx 50$. There exists a large body of literature pertaining to the Riccati equation in the electrical engineering literature (Jamshidi 1980). A promising and efficient technique appears to be that based on the use of Schur vectors (Laub 1979). In this approach, the Hamiltonian, $H$ of the system (3.11) is defined as:

$$H = \begin{bmatrix} A & B \cdot R^{-1} \cdot B^t \\ -Q & A^t \end{bmatrix}$$

For solving (3.11), an orthogonal transformation, $U$, is found such that:

$$U^t \cdot H \cdot U = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$$

where $T$ is a quasi-upper-triangular matrix with $(1 \times 1)$ or $(2 \times 2)$ blocks on the diagonal, corresponding to real or complex eigenvalues. In addition, the real parts of the $T_{11}$ eigen spectrum are negative, while those of the $T_{22}$ eigen spectrum are positive. The eigenvalues are arranged in decreasing order. If one writes, correspondingly, the matrix $U$ from (3.15), as:

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

then the solution $K$ to the Riccati equation (3.14), may be written (Laub 1979) as:

$$K = U_{21}U_{11}^{-1}$$

The above algorithm, along with certain modifications to the algorithm for finding and ordering the eigenvalues of an upper Hessenberg matrix, has been presently implemented to solve (3.11). For the number of nodes considered, $n \approx 25$, [i.e. $K$ is of dimension $50 \times 50$], the present Schur vector algorithm has been found to be about 40 times faster than the standard iterative approach for solving (3.11).

4 RESULTS

Here we present results for the direct nodal-response control of an LSS, modeled here as a free-free plate described by a 4th-order bending theory, as in Eq. (2.1). For illustrative purposes, direct nodal control of a simply supported plate is also considered.

In the analyses carried out here, the discretization on the boundary of the plate consisted of linear interpolations for $w$, $\frac{\partial w}{\partial n}$, $m_n$, and $V_n$ in each boundary-element, while in the interior of the domain, $w$, $\frac{\partial w}{\partial n}$, and $\frac{\partial^2 w}{\partial n^2}$ were assumed to be piecewise constant in each element. On the other hand, in the usual displacement-based finite element analysis of plates, the fourth-order plate operator presents a number of difficulties, since it requires a $C^1$ interpolation in each element. However, when a boundary integral equation approach is used, terms related to the fourth-order operator have been transformed to boundary (line) integrals. In addition, the inertia
operator is represented by piecewise constant interpolants. The effectiveness of this scheme was evident (from results not included here), in that, for a given size of the system of equations, the present boundary element methods gave better results than the usual finite element method with the same number of degrees of freedom, for the lowest eigenvalues. Also, conversely, in order to achieve acceptable degree of accuracy in the first \( N^* \) modes, fewer degrees of freedom were needed in the boundary-element method than in the finite element method.

In the case of a free-free plate, the rigid motion was suppressed by supporting the plate at three corners as shown in Fig. 1. Two sets of meshes (4 x 4) and (5 x 5), respectively, were considered over the entire plate. Control was implemented by considering the entire set of coupled nodal equations as in (2.8) and (3.3). Various forms of the weighting matrices \( Q \) and \( R \) of Eq. (3.9) were assumed; and different combinations of point-force actuators were investigated. In the interest of simplicity, the final time \( t_f \) in (3.9) was assumed to be infinity; and hence, only the steady-state Riccati equation was solved using techniques outlined in Section 3.

In this work, it is assumed that passive viscous damping mechanisms exist at discrete points in the structure. Thus, the damping function \( c(x) \) of (2.1) is assumed as:

\[
c(x) = \sum_{i=1}^{m} \delta(x - x_i)C_i \tag{4.1}
\]

where \( C_i \) are constants; and the number of dampers, \( m \), is such that \( m \leq n \) [\( n \) is the dimension of \( \omega \) in (2.8)]. For convenience, \( x_i \) in (4.1) is assumed to be the coordinates of the interior nodes. Note that when (4.1) is used in (2.3) and (2.4), the resultant damping matrix, viz., \( (\Gamma - \Gamma_0 \Gamma^{-1} \Gamma_0^T) \) in (2.8) is fully populated and, in general, not proportional to the stiffness or mass matrices in (2.8).

Fig. 2 shows the displacement at Node 16 of the plate (with 4 x 4 mesh), with no passive damping and with actuators at each of the nodes of the 16 interior elements, when a unit initial impulse is applied at Node 16. The weighting matrices \( Q \) and \( R \) are taken to be:

\[
Q = \text{diag}[0.1, 0.1, \ldots, 0.1]
\]

and

\[
R = \text{diag}[1.0, 1.0, \ldots, 1.0].
\]

The control force exerted by the actuator at Node 16 is shown in Fig. 3. For the case of a constant value of \( C_i = 1.0 \) at each of the 16 interior nodes, and all the other parameters being the same as in the previous case, the displacement at Node 16 is shown in Fig. 4. The effect of increasing the value of each \( C_i \) from 1.0 to 1.5 is illustrated in Fig. 5. A comparison of Figs. 2, 4, and 5 indicates the effect of passive damping control on the response of the actively controlled system. Fig. 6 illustrates the effect of using just four control-force actuators at the indicated nodes, with no passive damping.

The above examples (except the case with each \( C_i = 1.5 \)) are repeated in Figs. 7-10 for the 5 x 5 mesh; and the results are shown for the displacement and actuator-force at Node 25, due to an initial impulse of unit magnitude at Node 25.

For a simply supported plate using 25 nodes, the displacement at the center due to a unit impulse at the center is plotted in Fig. 11 with all the other parameters being the same as the first example. As expected, the deflections in this case are much smaller. The corresponding actuator-force at the center-node (Node 13) is plotted in Fig. 12. If the weighting matrix \( Q \) is changed to

\[
Q = \text{diag}[0.05, 0.05, \ldots, 0.05],
\]

the displacement and actuator force at Node 13 are plotted as in Figs. 13-14.
Fig. 2 Displacement at Node 16 in Free-Free Case with Actuators at All Nodes and No Damping (16 Nodes)

Fig. 4 Displacement at Node 16 in Free-Free Case with Actuators and Damping at All Nodes ($C^* = 1.0$)

Fig. 3 Force in Actuator 16 in Free-Free Case with Actuators at All Nodes and No Damping (16 Nodes)

Fig. 5 Displacement at Node 16 in Free-Free Case with Actuators and Damping at All Nodes ($C^* = 1.5$)
Fig. 6 Displacement at Node 16 in Free-Free Case with Actuators at Nodes 9, 11, 13, 15 and No Damping (16 Nodes)

Fig. 7 Displacement at Node 25 in Free-Free Case with Actuators at All Nodes and No Damping (25 Nodes)

Fig. 8 Force in Actuator 25 in Free-Free Case with Actuators at All Nodes and No Damping (25 Nodes)

Fig. 9 Displacement at Node 25 in Free-Free Case with Actuators and Damping at All Nodes (25 Nodes)
Fig. 10  Displacement at Node 25 in Free-Free Case with Four Actuators at Nodes 19, 20, 23, 24 and No Damping (25 Nodes)

Fig. 11  Displacement at Node 13 in Simply Supported Case with $Q = 0.1$ and Actuators at All Nodes and No Damping (25 Nodes)

Fig. 12  Force in Actuator 13 in Simply Supported Case with $Q = 0.1$ and Actuators at All Nodes and No Damping (25 Nodes)

Fig. 13  Displacement at Node 13 in Simply Supported Case with $Q = 0.05$ and Actuators at All Nodes and No Damping (25 Nodes)
The above examples serve to illustrate the feasibility of the present algorithms for full nodal control of non-proportionally damped discretized continuum models of an LSS, using as many actuators as desired (with the number of actuators, in general, being less than the number of nodal degrees of freedom).

5 CONCLUDING REMARKS

In this paper, a boundary-element approach, based on the singular solution of the biharmonic operator, is presented for a reduced-order structural modeling of LSS via an equivalent continuum plate model. Non-proportional damping is considered; and control of transient dynamic response using a full, uncoupled, nodal system of equations is presented. Implementation of the Schur vector approach in solving the steady-state Riccati equation has been carried out, and it is found to be much more economical than the standard iterative approach. The examples presented shown that the system is controllable when an arbitrary number of actuators are used. Further refinements of the methodology are under current scrutiny.

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SERVO-ELASTIC OSCILLATIONS:
CONTROL OF TRANSIENT DYNAMIC MOTION OF A PLATE

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ABSTRACT

A singular-solution approach is used to derive n discrete coupled ordinary differential equations governing the transient dynamic response of an initially stressed flat plate that is assumed to be a continuum model of a large space structure. Non-proportional damping is assumed. A reduced-order model of N (≤ m) coupled ordinary differential equations are derived in terms of the amplitudes of pseudo-modes of the nominally undamped system. Optimal control is implemented using m (≤ N) control-force actuators, in addition to a possible number p (≤ m) passive viscous dampers. Algorithms for efficient solution of Riccati equations are implemented. The problem of control spillover is discussed. Several example problems involving suppression of vibration of "free-free" and "simply-supported" plates are presented and discussed.

INTRODUCTION

The subject of "aeroelasticity" — a study of the interaction of flexible elastic bodies with the surrounding fluid medium — has blossomed in the late 1940's and early 1950's and played an important role in the design of modern aircraft. In aeroelasticity the aerodynamic forces acting on the elastic body have an equivalent influence of "negative damping", i.e., the surrounding fluid medium acts as an energy supplier to the vibrating elastic body. A problem of the opposite variety arises in the design of large space structures (LSS) intended for a variety of operations in outerspace. These structures are envisaged to be as large as Manhattan Island, for instance, and to be very flexible. Other examples of LSS are the large space antennae. The central problems in the design of these LSS are vibration suppression and shape control, when the LSS are subject to disturbances such as due to unbalanced rotating machinery on board, thruster firings, slewing/pointing maneuvers, thermal transients, etc. Vibration suppression and shape control of LSS are sought through either active or passive (or a combination of the two) types of control mechanisms. Thus, in parallel with the subject of aeroelasticity, we have the emerging subject of servoelasticity — that of dynamic motion of deformable structures. In addition to LSS, similar problems arise in the design of tall buildings on earth, wherein it is required to control the dynamic motion, say under seismic loads, to amplitudes within the bounds of human comfort levels. The topics germane to the issue of LSS controllability are as follows: (i) While for the usual free or forced vibration response analysis of structures, efficient algorithms based on many thousands of degrees of freedom exist, the algorithms for optimal control are currently limited to but a few degrees of freedom. In the traditional finite element modelling of a structure, several hundreds or thousands of element-modal degrees of freedom may have to be used to obtain even the first few (say, 10 or 20) fundamental frequencies and global mode shapes. Thus, there is a need for alternative approaches for reduced-order modelling of the structure — i.e., its stiffness and inertia; (ii) Design of algorithms for implementation of optimal control of systems of moderate dimension of, say, 50-100 (i.e., 50-100 global modes); (iii) The more prevalent concepts (i,2) have been to either ignore damping or consider it being proportional to mass or stiffness and obtain the "normal modes" of the (linear) structure. Based on the orthogonality of these normal modes, the system of (linear) ordinary differential equations are completely decoupled, and control of response of each decoupled equation is attempted individually. This concept is labelled the so-called "Independent Modal Space Control". This approach, while mathematically simple, depends on the use of as many control-force actuators as the number of decoupled modes being controlled. On the other hand, if damping exists due to deliberate design of passive dampers or due to deliberate design of joints, it may be of "non-proportional" type; and thus, the (linear ordinary differential) equations of motion cannot be decoupled. Thus, one needs to implement control based on the system of coupled equations of motion and with an arbitrary number of active control-force actuators that is perhaps much smaller than the order of the
system being controlled; (iv) The effect of nonlinearities in the system, such that the discretized equations of motion are nonlinear ordinary differential equations in time. Depending on the spatial and temporal variations of the disturbances, the LSS may be modeled as a three-dimensional network of beams and bars (3,4); or, alternatively, an equivalent continuum model such as a plate may be used. The authors have recently (5) presented a methodology wherein: (i) an equivalent continuum model is used for the in-plane stress resultant, (ii) the linear transient dynamic response of the plate is modeled by a boundary-element technique based on the singular solution of a biharmonic operator, (iii) a nodal-control, i.e., control of nodal response based on the completely coupled nodal system of equations is used, (iv) non-proportional damping is considered, and (v) the final time in the control algorithm is set to infinity so that a steady-state Riccati equation is solved.

In the present paper, we consider the generic model problem of control of simply-supported plates using a fourth-order theory. We include the effects of initial in-plane stresses in the formulation of the problem of control of dynamic motion of the plate in the transverse direction. We consider non-proportional passive damping, so that the discrete equations of motion cannot be completely decoupled. We solve the undamped equations of motion to obtain the undamped normal modes of the system, which will be used as "global" basis functions. In the presence of non-proportional damping, the equations of motion even in these "global"-basis-function amplitudes will still be coupled. However, by considering the problem of control of only a certain number of predominant global shape functions, the order of the coupled system of equations can be made much smaller than that of the (local) "nodal" system of equations. The response of this reduced system of "global equations" is sought to be controlled by an arbitrary number of control-force-actuators. Thus, once again, the concept of "independent-modal-space-control" (1-2) is not employed. The settling time for the controlled response is taken to be finite, so that a non-steady state Riccati matrix equation as to be solved. The contents of the paper are as follows. Section 2 deals with algorithms for control of coupled "global-shape-function" system of equations. Section 4 deals with illustrative examples, and certain concluding comments are given in Section 5.

**DISCRETIZATION OF EQUATION OF MOTION OF A PLATE USING A SINGULAR SOLUTION APPROACH**

We assume that initial stresses exist in the plate, in the form of known in-plane stress resultants \( N_{11}, N_{22}, \) and \( N_{12} \). As a first approximation, we assume that the motion of the plate is governed by a fourth-order theory. Under these assumptions, the equation of motion is (6):

\[
\begin{align*}
\ddot{w}(x, t) &= - c(x) \frac{\partial^2 w}{\partial s^2} + m(x) \frac{\partial^2 w}{\partial t^2} \\
&+ f_{ext}(x, t) + f_c(x, u^*t) + N_{11} \frac{\partial^2 w}{\partial x^2} + N_{12} \frac{\partial^2 w}{\partial x \partial y} + N_{22} \frac{\partial^2 w}{\partial y^2}
\end{align*}
\]

\[ \text{where: } x, y \text{ are the in-plane coordinates of the plate; } t \text{ the time; } D \text{ the bending stiffness; } c \text{ the distributed viscous damping; } m \text{ the distributed mass, } f_{ext} \text{ the external transverse loading, and } f_c \text{ the control-actor forces exerted in the transverse direction. If } (2.1) \text{ is discretized by the finite-element method, with trial and test functions for } w, \text{ it is well known that: (i) the trial and test functions must be } \mathbb{C}^1 \text{ continuous in and across each element, (ii) a large number of elements are needed when the in-plane dimensions of the plate are very large (as in an LSS) as compared to the thickness, and (iii) even to obtain the first few global shape functions corresponding to undamped free-vibration, a large number of finite-element nodal equations have to be considered. As an alternative, we use a singular-solution approach to the discretization of (2.1). Let } w^* \text{ be a test function and } w \text{ the trial function. The weighted residual form of Eq. (2.1) becomes:}
\]

\[
\int_{\Omega} L_1(w) \, w^* \, d\Omega = \int_{\Omega} L_2(w) \, w^* \, d\Omega
\]

\[ \text{where: } L_1(w) \text{ and } L_2(w) \text{ are the differential operators on the right- and left-hand sides of Eq. (2.1), respectively. We take } w^* \text{ to be the well-known (2) singular solution of the biharmonic operator } \nabla^4. \text{ Using this special property of } w^*, \text{ and by repeated integration by parts, we obtain an integral relation for } w(x,t): \]

\[
\beta w(P) = \frac{1}{d} \int_{\partial \Omega} (w v_n(P) - L(w) m(w) \, dS + M(w) \frac{\partial w}{\partial n} - V(w) w \, dS) + \frac{1}{d} \int_{\partial \Omega} \{w v^*(P) - w v^*(P)\} + \frac{1}{d} \int_{\partial \Omega} \{w^* v_n(P) - w v^*(P)\} + \frac{1}{d} \int_{\partial \Omega} \{w^* v_n(P) - w v^*(P)\} + \frac{1}{d} \int_{\partial \Omega} \{w^* v_n(P) - w v^*(P)\} + \frac{1}{d} \int_{\partial \Omega} \{w^* v_n(P) - w v^*(P)\} + \frac{1}{d} \int_{\partial \Omega} \{w^* v_n(P) - w v^*(P)\}
\]

\[ \text{where: } \beta = 1, \nabla v^*(P) \text{ is the included angle at } P; \partial \Omega \text{ is the boundary of } P, n, s: \text{ normal and tangential directions to } \partial \Omega \]

\[
V_n(w): \text{ Kirchhoff shear as a function of } w
\]

\[
m_c(w) \text{ and } m_n(w): \text{ tangential and normal moments derived from } w
\]

\[
\langle\rangle: \text{ denotes a jump in ( ) at a corner on } \partial \Omega \text{ with } K \text{ corners}
\]

\[
w^*(P, Q) = \frac{1}{8\pi} \int_{P} \langle x^{2n} \rangle r = (PQ)
\]

\[
L(w^*), M(w^*), T(w^*), \text{ and } V(w^*): \text{ normal slope, normal bending moment, tangential moment, and Kirchhoff shear, respectively, derived from } w^*
\]

A second equation, that for \(\nabla w/\nabla n\), where \(n_0\) is a unit outward normal at the boundary, \(\partial \Omega\), may be written as:
The above two equations permit the solution of the general dynamic response boundary value problem wherein, in general, one specifies two out of the four quantities \(w, \partial w/\partial n, m_w(w),\) and \(V_p(w)\) at each point in the boundary and the other two are the unknowns.

It is seen that the biharmonic operator is not discretized within \(\Omega\). Also, from the integral on \(\Omega\) in Eq. (2.4), it is seen that there are no continuity requirements on the trial function \(w\). Thus, in the present discretization, (i) \(w, \partial w/\partial n, m_w(w),\) and \(V_p(w)\) are interpolated in each (one-dimensional) element at the boundary \(\partial \Omega\) and (ii) an arbitrary interpolating function is used for \(w\) in each (two-dimensional) element within \(\Omega\) to discretize the integral over \(\Omega\) in (2.4).

The use of Eqs. (2.3) and (2.4) at the boundary \(\partial \Omega\), in conjunction with the above-mentioned interpolations, leads to:

\[
G_s w_B + L_f w_B - j_1 w_B - C_G w_s + P_G w_s = 0
\]

The definitions of the matrices \(G_r, L_f, j_1, C_G,\) and \(P_G\) may be immediately obtained from the appropriate integrals in (2.3) and (2.4) and are omitted here for want of space. We note, however, that \(w_B\) is the vector of boundary unknowns at \(\partial \Omega, \) and \(w_s, \partial w_s/\partial n,\) and \(w_s\) are, respectively, the vectors of displacement, velocity, and acceleration at the interior nodes in \(\Omega;\) and further it has been assumed that the prescribed conditions at \(\partial \Omega\) are of the "homogeneous" type. From (2.5) we obtain:

\[
W_B = G_s^{-1}(G_s + L_f - j_1)w_s - C_G w_s + P_G w_s = 0
\]

Likewise, using (2.3) at interior nodes in \(\Omega,\) we obtain the equation:

\[
W_s = G_s w_s + L_f w_s - j_1 w_s - C_G w_s + P_G w_s = 0
\]

Use of (2.6) in (2.7) results in:

\[
(\lambda_2 m_w^f + j_2 m_w^s) + \left( G_s - G_s^{-1} L_f \right) \delta w_s + \left( j_1 - P_s + G_s G_s^{-1} \right) w_s = (j_1 - G_s G_s^{-1} L_f) f
\]

Note that an analogous procedure for transient response of a plate with zero initial-stresses has been used in (5). Let the dimension of \(w_s\) in (2.8) be, say, \(n.\) We assume that \(f(x,t)\) in (2.1) may be a distributed external force. On the other hand, we assume that \(f_c(x,t),\) i.e., the control force, is exerted by point-force actuators located at discrete locations in the plate. Thus,

\[
f_c(x_0,t) = \sum_{i=1}^m \delta(x_0 - x_0^{(i)}) f_c(t)
\]

where, in general, \(m < n,\) and \(\delta(x_0 - x_0^{(i)})\) denotes a Dirac delta function at \(x_0^{(i)}\) where \(x_0^{(i)}\) is the location of the \(i\)th actuator. Likewise, we assume that viscous-type passive dampers are located at discrete locations \(x_0^{(i)},\) i.e.,

\[
c(x_0) = P_s - \sum_{i=1}^m \delta(x_0 - x_0^{(i)}) c_f
\]

where, in general, \(p < m < n;\) and the locations of the dampers \(x_0^{(i)}\) may be totally independent of the locations of the actuators, \(x_0^{(i)},\) it is seen that the discretized damping matrix, i.e.,

\[
\begin{bmatrix}
g_s - s^{-1} c_r
\end{bmatrix}
\]

corresponding to the discrete passive dampers as described in (2.10), is not proportional, in general, to the discretized mass matrix (i.e., \(G_s - c_s^{-1} G_t\)) or to the discretized "stiffness" matrix, \(G_s - G_t c_t\).

Equation (2.8) may now be written, generically, as:

\[
\begin{bmatrix}
\bar{M} \bar{w}_s + \bar{C} \bar{w}_s + \bar{K} \bar{w}_s = \bar{b} f
\end{bmatrix}
\]

where, the definitions of the mass, damping, and stiffness matrices \(\bar{M}, \bar{C},\) and \(\bar{K}\) are apparent from (2.8). Note that none of the matrices \(\bar{M}, \bar{C},\) and \(\bar{K}\) is symmetric. The system of equations (2.12) is subjected to initial conditions: \(\bar{w}_s(t=0) = \bar{w}_s(0)\) and \(\bar{w}_s(t=0) = \bar{w}_s(0).\) Assuming that the external forces, \(f_e,\) are zero, the central problem here is to design the control forces \(f_c\) on the right-hand side of (2.12) such that the solution of (2.12) subject to the stated initial conditions is damped out in a finite settling time, \(t_f.\)

CONTROL OF DYNAMIC RESPONSE

Control of a Reduced-Order System

Recall that the discrete equation of motion under the present approach are:

\[
\begin{bmatrix}
\bar{M} \bar{w}_s + \bar{C} \bar{w}_s + \bar{K} \bar{w}_s = \bar{b} f_c
\end{bmatrix}
\]

where \(\bar{C}\) is not proportional to \(\bar{M}\) or \(\bar{K};\) and moreover, \(\bar{M}, \bar{C},\) and \(\bar{K}\) are, in general, unsymmetric. Recall that \(\bar{w}_s\) is the vector of nodal displacements. Implementation of direct nodal control, based on a direct attack on the fully coupled nodal equations, (3.1), has been discussed in (3). However, to reduce the dimensionality, we use a "global-shape-function" approach here. Thus, first we seek global eigensolutions of the system:

\[
\begin{bmatrix}
\bar{M} \bar{w}_s + \bar{K} \bar{w}_s = 0
\end{bmatrix}
\]

or

\[
\lambda^2 \bar{M} \bar{w}_s = \bar{K} \bar{w}_s
\]

The system (3.2b) has eigenvalues \(\lambda_2\) and eigenvectors...
\( \Phi, \mathbf{\Phi} \), \( i = 1 \ldots n \). The orthonormal properties of the eigenvectors are such that:

\[
\Phi^t \mathbf{\Phi} = \mathbf{I} \quad \mathbf{\Phi}^t \Phi = \operatorname{diag} \left[ \lambda_i^2 \right]
\]  

(3.3)

where \( \mathbf{\Phi} \) is the matrix whose columns are the eigenvectors. The higher modes calculated through any discrete system tend to be inaccurate. Also, it is usually necessary to control only the first \( N \) modes, say. Thus, the vector \( \mathbf{\Phi} \) expressed in terms of the first \( N \) \( [N \times n] \) global modes, is:

\[
\mathbf{w} = \Phi \xi
\]  

(3.4)

where \( \mathbf{\Phi} \) is the matrix whose columns are the \( N \) global eigenvectors of (3.2), and \( \xi \) is a vector of undetermined coefficients. When (3.4) is used, the system of equations (3.1) transform to:

\[
\mathbf{I} \dot{\xi} + \mathbf{A} \xi + \mathbf{B} \xi = \mathbf{b} f_c
\]  

(3.5)

where \( \mathbf{I} = \text{identity matrix (NxN)} \)

\[
\mathbf{\xi}^t = \begin{bmatrix} \xi_1^t & \xi_2^t & \cdots & \xi_N^t \end{bmatrix}
\]  

\( \mathbf{A} = \operatorname{diag} \left[ \lambda_i^2 \right] \quad i = 1 \ldots N \)

and

\[
\mathbf{b}^t = \begin{bmatrix} b_1^t & b_2^t & \cdots & b_m^t \end{bmatrix}
\]

Note that \( \mathbf{\xi}^t \) is not diagonal. Thus, (3.5) still represents a coupled system of \( N \) equations \( (N \leq n) \). We seek to control the response of the system (3.5), using \( m \) control-force actuators, such that \( m < N \leq n \). Note that we also use \( p \) passive dampers, such that \( p < m \).

Eq. (3.5) is recast in the state variable form as:

\[
\mathbf{\dot{U}} = \mathbf{AU} + \mathbf{BF}
\]  

(2Nx1)

\( \mathbf{F} = \left( \begin{array}{c} \mathbf{f}_c^t \mathbf{F}^t \end{array} \right) \)  

(3.6)

where

\[
\mathbf{U}^t = \begin{bmatrix} \mathbf{\xi}_1 & \mathbf{\xi}_2 & \cdots & \mathbf{\xi}_N \end{bmatrix}
\]

\( \mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{A} & -\mathbf{B} \end{bmatrix} \)  

where each block is \( (\text{NxN}) \)

\( \mathbf{B}^t = \mathbf{f}_c \ldots \mathbf{f}_m \)

\( \mathbf{F} = \left[ \begin{array}{c} 0 \\ \mathbf{b}^t \end{array} \right] \)  

(3.7)

The observations of \( P \) sensors are given by:

\[
\mathbf{y} = \mathbf{D} \mathbf{U}
\]  

(3.8)

The control forces \( \mathbf{F} \) are chosen so as to minimize a quadratic performance index \( (\xi) \) of the form:

\[
J = \frac{1}{2} \int_0^{\infty} \left[ \mathbf{y}^t \mathbf{R} \mathbf{y} + \mathbf{F}^t \mathbf{F} \right] dt
\]  

(3.9)

where \( \mathbf{R} \) and \( \mathbf{B} \) are arbitrary weighting matrices which determine the magnitude of the actuator forces and the quantitative decay of the vibration amplitudes, and \( t_f \) is the settling time.

The well-known linear optimality criterion \( (\xi) \) leads to the following feedback law for the actuator control forces, \( \mathbf{F} \):

\[
\mathbf{F} = -\mathbf{R}^{-1} \mathbf{F}^t \mathbf{S} \mathbf{U}
\]  

(3.10)

where \( \mathbf{S} \) is the solution of the Riccati matrix differential equation:

\[
\dot{\mathbf{S}} = -\mathbf{S} \mathbf{A} + \mathbf{A}^t \mathbf{S} + \mathbf{S} \mathbf{B} \mathbf{R}^{-1} \mathbf{F}^t \mathbf{S} - \mathbf{Q} \quad \mathbf{S}(t_f) = \mathbf{Q}
\]  

(3.11)

The solution of (3.11) may be written in the form (9):

\[
\mathbf{S}(t) = \mathbf{S}_{ss} + \mathbf{Z}^{-1}(t)
\]  

(3.12)

where \( \mathbf{S}_{ss} \) is the solution of the steady-state Riccati equation:

\[
-\mathbf{S}_{ss} \mathbf{A} - \mathbf{A}^t \mathbf{S}_{ss} + \mathbf{S}_{ss} \mathbf{B} \mathbf{R}^{-1} \mathbf{F}^t \mathbf{S}_{ss} = \mathbf{Q} = 0
\]  

(3.13)

and \( \mathbf{Z}(t) \) is the solution of

\[
\dot{\mathbf{Z}}(t) = \mathbf{A} \mathbf{Z}(t) + \mathbf{Z}(t) \mathbf{A}^t - \mathbf{B} \mathbf{R}^{-1} \mathbf{F}^t
\]

(3.14)

where

\[
\mathbf{A} = \mathbf{A} - \mathbf{B} \mathbf{R}^{-1} \mathbf{F}^t \mathbf{S}_{ss}
\]

A closed form solution of (3.14) exists and is given (10) by

\[
\mathbf{Z}(t) = \mathbf{Z}_{ss} + \mathbf{A}(t-t_f)^{\mathbf{A}(t-t_f)} \mathbf{Z}(t_f) \mathbf{A}(t-t_f)^{\mathbf{A}(t-t_f)} - \mathbf{S}_{ss} \mathbf{F}^t(t-t_f)
\]  

(3.15)

with

\[
\mathbf{A} \mathbf{Z}_{ss} + \mathbf{A} \mathbf{S}_{ss} \mathbf{A}^t = \mathbf{B} \mathbf{R}^{-1} \mathbf{F}^t
\]

To solve Eq. (3.13) on the other hand, an efficient technique appears to be that based on Schur vectors (11), which is at least an order of magnitude inexpensive, in computational time, as compared to the simpler iterative approach. In this approach, the Hamiltonian \( \mathbf{H} \) of the system (3.13) is defined as:

\[
\mathbf{H} = \begin{bmatrix} \mathbf{A} & -\mathbf{B} \mathbf{R}^{-1} \mathbf{F}^t \\ -\mathbf{B} \mathbf{R}^{-1} \mathbf{F}^t & \mathbf{A}^t \end{bmatrix}
\]  

(3.16)

For solving (3.13), an orthogonal transformation, \( \mathbf{U} \), is found such that:

\[
\mathbf{y}^t \mathbf{H} \mathbf{U} = \mathbf{T} = \begin{bmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} \\ \mathbf{0} & \mathbf{I}_{22} \end{bmatrix}
\]  

(3.17)

where \( \mathbf{T} \) is a quasi-upper-triangular matrix with \( (1 \times 1) \) or \( (2 \times 2) \) blocks on the diagonal, corresponding to real or complex eigenvalues. In addition, the real parts of the \( \mathbf{I}_{11} \) eigenspectrum are negative, while those of the \( \mathbf{I}_{22} \) eigenspectrum are positive. The eigenvalues are arranged in decreasing order. If one writes, correspondingly, the matrix \( \mathbf{U} \) from (3.17) as:

\[
\mathbf{U} = \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{bmatrix}
\]  

(3.18)

then the solution \( \mathbf{S}_{ss} \) to the steady-state Riccati equation may be written as (11):

\[
\mathbf{S}_{ss} = \mathbf{U}_{21} \mathbf{U}_{11}^{-1}
\]  

(3.19)

The above algorithms, along with certain modifications to the algorithm for finding and ordering the eigenvalues of an upper Hessenberg matrix, have been presently implemented to solve (3.13). For the number of global modes considered, \( N \sim 25 \) (i.e., \( \mathbf{S} \) of dimension \( 50 \times 50 \)), the present Schur vector algorithm
has been found to be about 40 times faster than the standard iterative approach for solving (3.13).

CONTROL SPILLOVER

Recall that in the present approach n nodal equations (3.1) are considered and are reduced to N (N < n) global-modal coupled second-order differential equations in time. The response of these N global-modal coupled equations is controlled by m control-force actuators (m < N). The residual uncontrolled global modes (n - N) may, however, be excited, inadvertently, causing the so-called "spillover". The spillover phenomenon can be one of two forms. Observer spillover occurs when the sensor outputs contain terms associated with the residual modes. Control spillover, on the other hand, occurs when the feedback control forces excite the residual modes (12). The displacement \( w(x, t) \) is given in two parts:

\[
w(x, t) = w_c(x, t) + w_R(x, t)
\]  (3.20)

where \( w_c \) are controlled displacements and \( w_R \) the residual displacements.

Note that if (3.2) is solved for all the n eigenvectors, the global shape function \( w_b \) can be expressed as:

\[
w_b = \Phi \beta
\]  (3.21)

where \( \Phi \) is the matrix whose columns are the n eigenvectors and \( \beta \) are n undetermined parameters. Eq. (3.21) may be written as:

\[
w_b = \Theta \xi + \Theta R \xi R
\]  (3.22)

(\( \Phi \)) \( (n \times n) (N \times n) \)

where \( \Theta \) represents a matrix whose columns are the first N global-modes that are controlled as in (3.4); and \( \Theta R \) represents the matrix of the remaining (n-N) modes that are uncontrolled. Using the state vector \( \xi R \) to denote the quantities:

\[
\xi R = [ \xi R_1 \ldots \xi R_{(n-N)} ]
\]  (3.23)

1 x (n-N)

Corresponding to the uncontrolled modes, one obtains [using procedures similar to those used in (3.6) and (3.5)], the following equation:

\[
\xi R = A_R \xi R + B_R F
\]  (3.24)

\([2(n-N)\times 1] = [2(n-N)\times 2(n-N)] [2(n-N)\times 1] + [2(n-N)\times m]

where \( F \) is the same as defined in (3.6). Also, the observation may, in general, be defined as:

\[
F = D \xi + D R \xi R
\]  (3.25)

Observer spillover is zero when \( D R = 0 \). When control forces \( F \) are designed as in (3.10) to control the first N modes, we have [see (3.10)]:

\[
F = - E^{-1} \dot{\xi} S \xi
\]  (3.26)

The excitation of the uncontrolled modes due to the control forces \( F \) is determined by using (3.26) in (3.24), to yield:

\[
\hat{\xi} R = A_R \hat{\xi} R + B_R E^{-1} \dot{S} \xi
\]  (3.27)

Thus, control spillover occurs when \( B_R \neq 0 \).

RESULTS

Numerical results are presented herein for the cases when the initial stresses \( N_{11} = N_{12} = N_{22} = 0 \). Thus, the domain integrals in Eqs. (3.2) and (2.4) involve only the terms \( f_c, \partial w/\partial t, \) and \( \partial^2 w/\partial t^2 \). Thus, in the discrete approximations, interpolation of the unknowns \( w, \partial w/\partial t, m, \) and \( v_c \) in each boundary-element is linear, while the interpolation of \( \partial w/\partial t, \) and \( \partial^2 w/\partial t^2 \) is piecewise constant in each element in the interior. Recall that in the finite element method, the biharmonic operator is discretized in the interior, requiring \( C^1 \) continuous trial and test functions for \( w \). This has been a source of many difficulties in the development of the finite element method. The effectiveness of the present approach is demonstrated through an eigenvalue analysis of free vibration analysis of a simply-supported plate sketched in Fig. 1, wherein, for instance, an \((8 \times 8)\) mesh is indicated. The questions of interest are:

(a) what is the number of degrees of freedom required in the two methods, i.e., the standard finite-element and the present boundary-element methods, to achieve a desired accuracy in the first few modes? and (b) how many fundamental modes can be computed, with the desired accuracy, for a given number of degrees of freedom in the two methods? Table 1 provides a partial answer to the first question (it appears difficult to ascertain the finite-element literature in this regard), when only the first eigenvalue is considered.

### Table 1

<table>
<thead>
<tr>
<th>Finite Elements vs Boundary Elements</th>
<th>1st Eigenvalue for S. S. Plate</th>
</tr>
</thead>
<tbody>
<tr>
<td>[The accuracy of both methods is compared with the analytical result (12).]</td>
<td></td>
</tr>
<tr>
<td>FEM (13)</td>
<td>BEM</td>
</tr>
<tr>
<td>No. of Eqs.</td>
<td>% Diff.</td>
</tr>
<tr>
<td>39</td>
<td>- 2.92%</td>
</tr>
<tr>
<td>95</td>
<td>- 1.42%</td>
</tr>
<tr>
<td>175</td>
<td>- 0.81%</td>
</tr>
</tbody>
</table>

The finite element referred to in Table 1 is a rectangular plate-bending element (13).

To answer the second question, the first 17 eigenvalues, computed by using an \((8 \times 8)\) mesh of Fig. 1 in the present boundary-element method, are listed and compared with the analytical results of Timoshenko (14) in Table 2. A similar comparison of the finite-element-computed eigenvalues is not included here for such data does not appear to be readily available. Note that out of the 17 eigenvalues in Table 2, several are coalescent for the considered square plate. Note that the maximum error in the 17th eigenvalue is about less than 82%.
Table 2
Comparison of the First 17 Eigenvalues from a (8 x 8) Boundary-Element Mesh (Simply-Supported Plate)

<table>
<thead>
<tr>
<th>S.S. Plate Eigenvalues - BEM</th>
<th>Exact S.S. Plate Eigenvalues (Timoshenko)</th>
</tr>
</thead>
<tbody>
<tr>
<td>19.92</td>
<td>19.74</td>
</tr>
<tr>
<td>50.31</td>
<td>49.34</td>
</tr>
<tr>
<td>50.31</td>
<td>49.34</td>
</tr>
<tr>
<td>81.53</td>
<td>78.96</td>
</tr>
<tr>
<td>102.14</td>
<td>98.70</td>
</tr>
<tr>
<td>102.14</td>
<td>98.70</td>
</tr>
<tr>
<td>134.63</td>
<td>128.30</td>
</tr>
<tr>
<td>134.63</td>
<td>128.30</td>
</tr>
<tr>
<td>177.29</td>
<td>167.78</td>
</tr>
<tr>
<td>177.29</td>
<td>167.78</td>
</tr>
<tr>
<td>189.67</td>
<td>177.65</td>
</tr>
<tr>
<td>211.17</td>
<td>197.39</td>
</tr>
<tr>
<td>211.17</td>
<td>197.39</td>
</tr>
<tr>
<td>268.44</td>
<td>246.74</td>
</tr>
<tr>
<td>268.44</td>
<td>246.74</td>
</tr>
<tr>
<td>277.15</td>
<td>256.60</td>
</tr>
<tr>
<td>277.15</td>
<td>256.60</td>
</tr>
</tbody>
</table>

Now we discuss some results pertaining to the control problem. Here we consider the (8 x 8) boundary-element mesh shown in Fig. 1 and seek to control the first 17 undamped modes, using active control-force actuators as well as non-proportional passive damping. In the several cases to be discussed below, the number of control-force actuators is chosen to be either 16 or 4. The question of control spillover is examined by considering the excitation of the 18th mode due to spillover from the control of the first 17 modes. By comparing Fig. 5 with Fig. 2 or Fig. 4, it is seen that amplitude of the 18th node due to control spillover is at least two orders of magnitude smaller compared to the first 17 controlled modes. The affect of additional passive damping may be judged by comparing Figs. 2 and 6, respectively, wherein only the passive damping parameter is different with all other parameters being the same.

Finally, Figs. 7 and 8 show the results for the displacement at the center of the plate when the first 17 undamped nodes are controlled by 16 actuators and when additional passive damping is absent and present, respectively.

Q = diag $20.0 \lambda_1^2, 2 \lambda_2^2, 2 \lambda_3^2, \ldots$

while in Cases 6-7 it is chosen as:

Q = diag $0.02 \lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2, \ldots$

Figure 2 shows the displacement at Node 16 (see Fig. 1) of a free-free plate subjected to an initial velocity at Node 16 when controllers are assumed to be present at each of the 16 nodes, and Fig. 3 shows the force generated by the actuator at Node 16. Figure 4 shows results similar to those in Fig. 2 when only four actuators are used to control the free-free plate; and the actuators are located at Nodes 10, 12, 14, and 16.

Figure 5 shows that fraction of displacement at the 16th node of a free-free plate due to the excitation of the 18th mode due to spillover from the control of the first 17 nodes. By comparing Fig. 5 with Fig. 2 or Fig. 4, it is seen that amplitude of the 18th node due to control spillover is at least two orders of magnitude smaller compared to the first 17 controlled modes. The affect of additional passive damping may be judged by comparing Figs. 2 and 6, respectively, wherein only the passive damping parameter is different with all other parameters being the same.

Finally, Figs. 7 and 8 show the results for the displacement at the center of the plate when the first 17 undamped nodes are controlled by 16 actuators and when additional passive damping is absent and present, respectively.

Table 3
Summary of Examples Presented

<table>
<thead>
<tr>
<th>Case</th>
<th>Boundary Condition</th>
<th>Initial Condition</th>
<th>Damping</th>
<th>No. of Actuators</th>
<th>Output</th>
<th>Fig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Free-Free</td>
<td>$w_i^{*16}$</td>
<td>$C_i = 0$</td>
<td>16</td>
<td>$w_{16}$</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$w_{16}$</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>$w_{16}$</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>16</td>
<td>$w_{16}^{(due \ to \ 18th \ Mode)}$</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td>5.0</td>
<td></td>
<td>$w_{16}$</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>Simp. Sup.</td>
<td>$w_i^{*16}$</td>
<td>0</td>
<td></td>
<td>$w_{7}$</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td>2.0</td>
<td></td>
<td>$w_{7}$</td>
<td>8</td>
</tr>
</tbody>
</table>

(All the (*) quantities are nondimensional quantities)

Note that, in all the cases indicated in Table 3, an (8 x 8) mesh was used. In Cases 1-5, the weighting matrix Q of (3.9) was chosen as:

Figure 1 64 Element Mesh Used in the Analysis
Fig. 2 Displacement at Node 16 in Free-Free Case with 16 Actuators

Fig. 3 Force in Actuator 16 in Free-Free Case with Actuators at 16 Nodes and No Damping

Fig. 4 Displacement at Node 16 Due to Spillover From 18th Mode

Fig. 5 Displacement at Node 16 in Free-Free Case with Actuators and Dampers at 16 Nodes
CONCLUDING COMMENTS

In this paper, a boundary-element approach has been presented for controlling the transient dynamic response of an initially stressed flat plate, taken to represent the continuum model of a large space structure. An arbitrary number of active control-force actuators as well as passive dampers are assumed to be present. The damping is of non-proportional type. A reduced-order model of the transient dynamic response is created in terms of the amplitudes of pseudo-modes, (i.e., based on assuming zero damping). Even these equations are coupled. Control of the reduced-order system and algorithms for solving the attendant non-steady Riccati matrix differential equation have been discussed. The problem of control spillover has been addressed. Several example problems illustrating the developed methodologies have been presented.

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