Project Title: Research Initiation - Dipole Radiation in a Linear Anisotropic Compressible Plasma

Project No: B-715 (E-27813)

Principal Investigator: Dr. John D. Norgard

Sponsor: National Science Foundation

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Sponsor Contact Person(s): National Science Foundation Division of Engineering Attn: Royal E. Rostenbach - Engineering Energetics Program Washington, D.C. 20550

Assigned to: School of Electrical Engineering

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RA-3 (2-71)
GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF RESEARCH ADMINISTRATION
RESEARCH PROJECT TERMINATION

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Project Title: Research Initiation - Dipole Radiation in a Linear Anisotropic Compressible Plasma
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Assigned to: School of Electrical Engineering

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RA-4 (6-71)
March 31, 1972

National Science Foundation
Division of Engineering
Attn: Royal E. Rostenback
Engineering Energetics Program
Washington, D. C. 20550

Dear Mr. Rostenback:

This letter is an informal report on the progress made and the current status of the NSF Grant No. GK-27813 entitled "Axial and Transverse Dipole Radiation in a Cylindrically Stratified, Moving, Linear, Anisotropic, Compressible (Hot) Plasma".

The work being performed under this grant has been outlined on the attached sheets.

Referring to those sheets, parts I, II, III, and IV have been completed along with the pertinent appendixes. After parts V and VI are completed, the various operations indicated in the development of the theory will be coded in Fortran V for a numerical solution on a UNIVAC 1108 digital computer. A parametric study will then be undertaken.

I am confident that the work is proceeding on schedule and will be completed within the allotted time.

Thank you for this opportunity to undertake this project.

Sincerely,

[Signature]

John D. Norgard
OUTLINE

I. postulate the new model of a dipole antenna in a high-velocity wind of ionized gases
   A. consider the hydrodynamic interaction of the antenna structure with the streaming fluid
   B. consider the electromagnetic interaction of the emitted radiation with the moving media

II. consider the dielectric regions (rest frame)
   A. introduce the Hertzian vector potentials (electric and magnetic)
      1. find the governing equations (frequency domain)
      2. find the relationships between the fields and the potentials
   B. uncouple the potential equations
      1. source currents zero
      2. source currents non-zero
   C. solve the uncoupled potential equations (seek hybrid modes)
      1. find the axial components of the fields (proportional to the potentials)
      2. find the transverse components of the fields (ohms law)

III. consider the plasma regions (rest frame)
   A. hydrodynamical considerations
      1. linearize and combine the following:
         \[ \begin{align*}
         & \text{hydrodynamic equation of motion} \\
         & \text{equation of continuity} \\
         & \text{equation of state}
         \end{align*} \]
      2. find the average drift velocity of the mobile particles
      3. find the macroscopic convection current density (invert the dyadic velocity operator)
      4. identify the collision, gyro, and plasma frequencies
B. electromagnetic considerations

1. determine the effective constitutive relationships
   a. find the permittivity dyad
   b. find the permeability dyad
   c. find the compressivity dyad

2. find the divergence relations from the curl relations

3. find the Helmholtz wave equations
   a. find the governing equations (frequency domain)
   b. find the relationships between the fields, the pressure deviations, and the potentials

4. uncouple the Helmholtz wave equations (source free region only)
   a. find the eigenvalues
   b. find the eigenfunctions
   c. find the eigenmatrices

5. solve the uncoupled Helmholtz wave equations (seek hybrid modes)
   a. find the axial components of the fields
   b. find the transverse components of the fields (ohms law)

IV. consider the motional effects

A. introduce the Lorentz transformation

1. find the spacetime relationships

B. introduce the 4-tensors

1. 4-scalar (pressure)

2. 4-vector (wave number/phase)

3. 4-dyad (fields)

C. determine the constitutive relations

1. in the rest frame

2. in the moving frame

V. solve the boundary value problem (moving frame)

1. state the boundary conditions (at each interface)

2. identify the matrices
   a. the field matrix
   b. the boundary matrix
3. solve the matrix equations (invert the boundary matrix)

VI. find the radiation fields
   A. perform a spherical transformation
   B. identify the radiating components
   C. introduce the asymptotic Hankel expansions
   D. transform to the complex plane
   E. find the path of steepest descent

Appendix:
1. dyadic projection operators
   a. dot product
   b. cross product
   c. divergence
   d. curl
   e. laplacian

2. non-zero source current solution for the axial components of the cylindrical Hertz vector potential
   a. the constant of proportionality
   b. the Bessel/Hankel function Wronskian

3. the Lorentz transformation for a cylindrical coordinate system

4. saddle point integration
DIPOLE RADIATION
IN A
CYLINDRICALLY STRATIFIED
MOVING
LINEAR
ANISOTROPIC
CONDUCTING
COMPRESSIBLE
PLASMA:
AXIAL AND TRANSVERSE CASES

by
John D. Nordgard

September 1972

NSF
Final Technical Report

Research Initiation Grant GK-27813
Proposal No. P1K1052
DIPOLE RADIATION
IN A
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Research Initiation Grant GK-27813
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I. Abstract

The radiation characteristics of a point Hertzian dipole antenna are calculated in a high-velocity wind of ionized gases. The problem has two interdependent parts, one being the hydrodynamic interaction of the antenna and the support structure with the streaming fluid, and the other being the electromagnetic interaction of the emitted radiation with the moving medium. In this study, the antenna and the support structure are assumed to give rise to a cylindrically stratified plasma flow field, which is allowed to possess a radial distribution in the electron concentration, collision frequency, and temperature. Each cylindrical layer of the plasma shell moves at a uniform axial velocity with respect to the antenna and is biased magnetically along the axis of the cylinder. The antenna is mounted off-axis and is oriented either axially or transverse to the axis of the cylinder. Integral expressions for the cylindrical components of the electric and magnetic field intensity vectors and for the scalar pressure deviation are obtained by a relativistic solution of the uncoupled wave equations in the wave number domain, and are evaluated in the rest frame of the antenna using the techniques of asymptotic expansions. The resulting far field power density and intensity distributions are given.
II. Introduction

The problem of communicating with an instrumented space probe as it passes through an extraterrestrial planetary atmosphere has not been fully analyzed. Due to the passage of the probe through the atmosphere of the planet, a shock induced envelope of high temperature ionized gases forms around, and trails, the probe; and communications to and from the probe, via electromagnetic signals propagating through the wake region of the probe, are seriously disrupted. Several previous studies have attempted to analyze the communication "blackout" problem but have failed to use a sufficiently realistic model of the plasma in which the probe is embedded. This study removes several of the severe restrictions on the model of the plasma which are present in the previous studies.

The radiation characteristics of a point Hertzian dipole antenna are calculated in a high-velocity wind of ionized gases. The problem has two interdependent parts, one being the hydrodynamic interaction of the antenna and the support structure with the streaming fluid, and the other being the electromagnetic interaction of the emitted radiation with the moving medium. In this study, the antenna and the support structure are assumed to give rise to a cylindrically stratified plasma flow field, which is allowed to possess a radial distribution in the electron concentration, collision frequency, and temperature. Each cylindrical layer of the plasma shell moves at a uniform axial velocity with respect to the antenna and is biased magnetically along the axis of the cylinder. The antenna is mounted off-axis and is oriented either axially or transverse to the axis of the cylinder. Integral expressions for the cylindrical components of the
electric and magnetic field intensity vectors and for the scalar pressure deviation are obtained by a relativistic solution of the uncoupled wave equations in the wave number domain, and are evaluated in the rest frame of the antenna using the techniques of asymptotic expansions. The resulting far field power density and intensity distributions are given.
III. Model

The plasma in its most general state is assumed to be moving, linear, inhomogeneous, anisotropic*, conducting, and compressible.

The electron concentration profile in the wake region of the probe possesses axial symmetry about the center line of the probe. The electron concentration is also a slowly varying continuous function of the radial and the axial distances away from the probe. For increasing axial distance behind the probe, there is a decay in the peak electron concentration and an expansion of the wake radius. Since the axial changes are much slower than the radial changes, they are not expected to significantly alter the results obtained from assuming that the inhomogeneity is a function of the radius only.

Therefore, the wake region of the probe is approximated by a cylindrically stratified plasma shell consisting of $n$ homogeneous plasma layers. The $i$th layer of the plasma shell ($1 \leq i \leq n$), which is described by the electron concentration $N_i$, the collision frequency $f_{ci}$, and the temperature $T_i$, is biased in the z direction by an applied static magnetic field $B_{0i}$ and is moving in the z direction with a velocity $V_i$. The geometry of the problem is shown in Figure 1.

It is assumed that the electron concentration inside the first layer of the plasma shell, i.e., directly behind the probe in the recirculation region, and the electron concentration outside the $n$th layer of the plasma shell, i.e., in the undisturbed atmospheric region, are negligibly small in comparison to the electron concentration within any of the layers.

*The origin of the magnetic bias that gives rise to the anisotropy of the plasma is considered to be outside the scope of this study.
Fig. 1. Wake Geometry (behind probe)
of the plasma shell. Therefore, the interior and exterior regions surrounding the plasma shell are assumed to be simple stationary conducting dielectrics.

The antenna in this study is mounted off-axis a quarter wavelength above the aft part of the probe. The antenna is represented by a point Hertzian dipole mounted a quarter wavelength above an infinite ground plane. Although the aft part of the probe is finite, it is large compared with the wavelengths of interest in this study; consequently, the diffraction effects of the finite capsule are negligible and the assumption of an infinite ground plane is reasonable.

Since the radiation from an antenna with an infinite ground plane can be constructed from a knowledge of the radiation from the antenna and its image in the absence of the ground plane, the problem reduces to one of finding the radiation emitted by a single point Hertzian dipole located axially or transverse to the axis of the probe, as shown in Figure 1.

The compressible nature of the plasma is accounted for by introducing the hydrodynamic equations of motion, the equation of continuity, and the equation of state into the solution for the induced convection/conduction currents.

To analyze the effects of the moving plasma on the radiation from the antenna, the theories of Minkowski's phenomenological electrodynamics of a moving medium are used to derive the required field equations in the moving plasma. This approach is based on the covariance of the Maxwell equations and on the invariance of the constitutive parameters of the plasma when Lorentz-transforming the field equations between inertial reference frames.

Assume that there exists n moving reference frames, one for each of the n
moving layers of the plasma shell, and one stationary reference frame. One moving reference frame is chosen to be at rest with respect to the plasma in each of the \( n \) moving layers of the plasma shell, and the one stationary reference frame is chosen to be at rest with respect to the antenna. In the rest frame of each moving layer of the plasma shell, the problem is one of solving the inhomogeneous wave equations in a stationary, linear, homogeneous, anisotropic, conducting, compressible plasma. After the stationary wave equations in each of the \( n \) moving layers of the plasma shell are solved, the resulting integral expressions for the cylindrical components of the field intensity vectors and the scalar pressure deviations are Lorentz-transformed into the stationary rest frame of the antenna. Then, the boundary conditions at each common layer interface are satisfied. Finally, the complete integral expressions for the spherical components of the field vectors are evaluated in the far field of the antenna using the techniques of asymptotic expansions to yield the radiation patterns of the antenna for various combinations of the parameters of the plasma.
IV. Conventions

In this study, the following conventions are used.

A. Domains

A symmetric Fourier integral transformation is used to transform the various field vectors between the time, the frequency, and the wave number domains.

1. Time

In the time domain, let any vector \( \vec{\gamma} \) (denoted by a script letter) represent a function of the time \( t \) and the space \( \mathbf{r} \), i.e., \( \vec{\gamma} = \gamma(\mathbf{r}, t) \).

2. Frequency

In the frequency domain, let any vector \( \mathbf{V} \) (denoted by a capital letter) represent a function of the frequency \( \omega \) and the space \( \mathbf{r} \), i.e., \( \mathbf{V} = \mathbf{V}(\mathbf{r}, \omega) \).

Therefore,

\[
\gamma(\mathbf{r}, t) = \int_{-\infty}^{+\infty} \frac{dt}{2\pi} \mathbf{V}(\mathbf{r}, \omega) e^{-i\omega t}
\]

\[
\mathbf{V}(\mathbf{r}, \omega) = \int_{-\infty}^{+\infty} \frac{dt}{2\pi} \gamma(\mathbf{r}, t) e^{i\omega t}
\]

3. Wave Number

In the wave number domain, let any vector \( \mathbf{v} \) (denoted by a small letter) represent a function of the wave number \( k \) and the frequency \( \omega \), i.e., \( \mathbf{v} = \mathbf{v}(k, \omega) \).

Therefore,

\[
\mathbf{V}(\mathbf{r}, \omega) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} k \mathbf{v}(k, \omega) e^{-ik \cdot \mathbf{r}}
\]

\[
\mathbf{v}(k, \omega) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} r \mathbf{V}(\mathbf{r}, \omega) e^{-ik \cdot \mathbf{r}}
\]
For the cylindrical system of this study, in which an axial direction and a transverse plane can be identified, it is desirable to separate the wave number transformation into an axial and a transverse part, denoted, respectively, by a superscript $a$ and a superscript $t$.

Therefore, let $V$ represent a function of the transverse part of the position vector $\mathbf{r}_t$, the axial part of the wavenumber $k^a$, and the frequency $\omega$, i.e., $V = V(\mathbf{r}_t, k^a, \omega)$.

\[
V(\mathbf{r}_t, k^a, \omega) = \int_{-\infty}^{+\infty} \frac{dk^a}{\sqrt{2\pi}} V(\mathbf{r}_t, k^a, \omega) e^{+ik^a \mathbf{r}_a}.
\]

\[
V(\mathbf{r}_t, k^a, \omega) = \int_{-\infty}^{+\infty} \frac{dr^a}{\sqrt{2\pi}} V(\mathbf{r}_t, r^a, \omega) e^{-ik^a \mathbf{r}_a}
\]

and

\[
V(\mathbf{r}_t, k^a, \omega) = V(\mathbf{r}_t, k^a, \omega) + a V^a(\mathbf{r}_t, k^a, \omega)
\]

where

$a$ = unit vector in the axial direction.

B. Summation

In this study, the Einstein summation convention is used. That is, an upper and a lower repeated index are understood to be implicitly summed over, even though the summation sign is not explicitly written. If the repeated index is a roman letter, the sum is from 1 to 3; if the repeated index is a greek letter, the sum is from 0 to 3.
V. Stationary Media

Consider first, stationary media, both dielectric and plasma in characteristics. The results obtained in this section are used directly in the rest frames of each moving layer of the plasma shell and in the rest frames of the interior and exterior dielectrics surrounding the plasma shell.

A. Field/Sources

In the time domain, the Maxwell equations in a stationary medium

\[-\nabla \times E = \frac{\partial B}{\partial t}\]

\[\frac{1}{\mu_0} \nabla \times H = \varepsilon_0 \frac{\partial E}{\partial t} + J\]

\[\varepsilon_0 \nabla \cdot E = \rho_t\]

\[\nabla \cdot B = 0\]

and the conservation of charge

\[\nabla \cdot J_t = -\frac{\partial \rho_t}{\partial t}\]

describe the electromagnetic field by the vectors \(E\) and \(H\) and characterize the medium by the total charge density \(\rho_t\) and the total current density \(J_t\). The terms \(\mu_0\) and \(\varepsilon_0\) denote, respectively, the permeability and the permittivity of the vacuum.

The total charge and current density terms can be separated into two distinct parts

\[\rho_t = \rho_o + \rho_i\]

\[J_t = J_o + J_i\]

where

\[\rho_o = \rho_a \text{ (applied)}\]

\[\rho_i = -\nabla \phi + \frac{1}{2} \mu_0 \nabla \times \mathbf{B} + \ldots \text{ (induced)}\]
and

\[ \mathcal{E}_0 = \begin{cases} \mathcal{E}_a & \text{(applied)} \\ \mathcal{E}_{\text{cond}} & \text{(conduction)} \\ \mathcal{E}_{\text{conv}} & \text{(convection)} \end{cases} \]

\[ \mathcal{E}_i = \frac{\partial \mathcal{E}}{\partial t} - \frac{1}{c^2} \frac{\partial \mathcal{B}}{\partial t} \nabla \cdot \mathcal{E} + \nabla \times \mathcal{H} + \ldots \] (induced)

The terms \( \mathcal{P}, \mathcal{Q}, \mathcal{H}, \) etc. denote the volume densities of the induced multipole moments that are produced by the effect of the electromagnetic field on the neutral particles of the medium; and, therefore, are functions of the field vectors \( \mathcal{P} \) and \( \mathcal{Q} \).

The electric field vector \( \mathcal{E} \) is defined by

\[ \nabla \cdot \mathcal{E} = \rho_0 \]

or, after comparing this relationship with the above equations, by

\[ \mathcal{E} = \varepsilon_0 \mathcal{E} + \rho - \frac{1}{2} \mathcal{E} \nabla \mathcal{E} + \ldots \]

Similarly, the magnetic field vector \( \mathcal{H} \) is defined by

\[ \nabla \wedge \mathcal{H} = \frac{\partial \mathcal{B}}{\partial t} + \mathcal{E}_0 \]

or, after comparing this relationship with the above equations, by

\[ \mathcal{H} = \frac{\mathcal{B}}{\mu_0} - \mathcal{H} + \ldots \]

1. **Constitutive Relations**

It is assumed that linear relationships exist between the multipole moments and the field vectors in the frequency domain such that

\[ \mathcal{D} = \varepsilon \cdot \mathcal{E} = \varepsilon_0 \varepsilon_r \cdot \mathcal{E} \]

\[ \mathcal{B} = \mu \cdot \mathcal{H} = \mu_0 \mu_r \cdot \mathcal{H} \]
where $\varepsilon_r$ is the relative permittivity dyad of the medium and $\mu_r$ is the relative permeability dyad of the medium.

2. **Constitutive Parameters**

It is also assumed that linear relationships exist between the conduction and the convection currents and the field vectors in the frequency domain such that

\[
\begin{align*}
    \mathbf{i}_{\text{cond}} &= \sigma \cdot \mathbf{E} \\
    \mathbf{i}_{\text{conv}} &= \tau \cdot \mathbf{E}
\end{align*}
\]

where $\sigma$ is the conduction dyad of the medium and $\tau$ is the convection dyad of the medium.

Therefore, in the frequency domain, the Maxwell equations are

\[
\begin{align*}
    -\nabla \times \mathbf{E} &= -i\omega \mu_r \mathbf{H} \\
    \nabla \times \mathbf{H} &= -i\omega \varepsilon_r \sigma \cdot \mathbf{E} + \mathbf{j}_a + (\sigma + \tau) \cdot \mathbf{E} \\
    \varepsilon_r \nabla \cdot \mathbf{E} &= \rho_a \\
    \mu_r \nabla \cdot \mathbf{H} &= 0
\end{align*}
\]

In order to solve the Maxwell equations subject to the conservation of charge and the constitutive relations, the constitutive parameters $\mu_r$, $\varepsilon_r$, $\sigma$, and $\tau$ are determined for the media under consideration.
B. Dielectric (Conduction Currents)

The conduction current density in a stationary dielectric is given simply by the Ohm Law

\[ \mathbf{J} = \sigma \cdot \mathbf{E}_{\text{cond}} \]

where

\[ \sigma = \mathbb{I} \sigma \]

and

\[ \mathbb{I} = \text{unit identity dyad} \]
\[ \sigma = \text{scalar conductivity} \]

Also

\[ \varepsilon_r = \mathbb{I} \varepsilon_r \]
\[ \mu_r = \mathbb{I} \mu_r \]

where

\[ \varepsilon_r = \text{scalar relative permittivity} \]
\[ \mu_r = \text{scalar relative permeability} \]

1. Maxwell's Equations

Substituting the constitutive parameters into the independent Maxwell curl equations gives

\[ \nabla \times \mathbf{E} = -i \omega \mu_0 \varepsilon_r \mathbf{H} \]
\[ \nabla \times \mathbf{H} = -i \omega \varepsilon_0 (\varepsilon_r + \frac{\sigma}{\omega \varepsilon_0}) \mathbf{E} + \mathbf{J}_a \]

To put the above equations into a symmetric form, a complex relative permittivity \( \varepsilon_c \) and a complex relative permeability \( \mu_c \) are defined by

\[ \varepsilon_c = \varepsilon_r + i \frac{\sigma}{\omega \varepsilon_0} \]
\[ \mu_c = \mu_r \]
such that
\[-\nabla \mathbf{E} = -i \omega \mu_0 \mathbf{H} \]
\[\nabla \mathbf{H} = -i \omega \varepsilon_0 \mathbf{E} + \mathbf{J}_a \]

2. Helmholtz' Equations

a. Scalar/Vector Potentials

To uncouple the independent Maxwell curl equations, a scalar potential \( \phi \) and a vector potential \( \mathbf{A} \) are introduced by defining
\[\mathbf{E} = -\nabla \phi + i \omega \varepsilon_0 \mathbf{A} - \frac{1}{\varepsilon_0} \nabla \nabla \phi \]
\[\mathbf{H} = -\nabla \mathbf{A} + i \omega \mu_0 \mathbf{H} - \frac{1}{\mu_0} \nabla \nabla \mathbf{A} \]

The potentials are required to satisfy the Lorentz gauge
\[\nabla \cdot \mathbf{A} = \frac{i \omega}{c^2} \varepsilon_0 \mathbf{H} \phi \]

A system of uncoupled equations are obtained by substituting the above potentials into the Maxwell equations.

The resulting potential forms of the Helmholtz equations are
\[\Box^2 \phi_e = -\frac{1}{\varepsilon_0} \mathbf{J}_a \]
\[\Box^2 \phi_m = 0 \]
\[\Box^2 \mathbf{A}_e = -\mu_0 \mathbf{J}_a \]
\[\Box^2 \mathbf{A}_m = 0 \]

where
\[\Box^2 \equiv \nabla^2 k^2 \quad (d'alembertian)\]

and
\[k = k_o \kappa \quad (wave \ number)\]

with
\[k_o \equiv \frac{\omega}{c} \quad (free \ space \ wave \ number \ reference)\]
\[\kappa \equiv \sqrt{\varepsilon_0 \mu_0} \quad (index \ of \ refraction)\]
\[c \equiv \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad (free \ space \ speed \ of \ light)\]
b. Hertz Potential

The potential forms of the Helmholtz equations are simplified by introducing a vector Hertz potential \( \mathbf{\Pi} \), satisfying the Lorentz gauge, such that

\[
\begin{align*}
\phi_e &= -\nabla \cdot \mathbf{\Pi} _e \\
A_e &= -i\omega \frac{\epsilon_0}{c} \mathbf{\Pi} _e \\
A_m &= -i\omega \frac{\mu_0}{c} \mathbf{\Pi} _m
\end{align*}
\]

Then, the potential forms of the Helmholtz equations reduce to

\[
\begin{align*}
\square^2 \mathbf{\Pi}_e &= \frac{\mathbf{J}_a}{i\omega \epsilon_0 c} \\
\square^2 \mathbf{\Pi}_m &= 0
\end{align*}
\]

and the field/potential relationships reduce to

\[
\begin{align*}
\mathbf{E} &= \nabla \mathbf{\nabla} \mathbf{\Pi}_e + i\omega \mu_0 \nabla \mathbf{\nabla} \mathbf{\Pi}_m \\
\mathbf{H} &= \nabla \mathbf{\nabla} \mathbf{\Pi}_m - i\omega \epsilon_0 \nabla \mathbf{\nabla} \mathbf{\Pi}_e
\end{align*}
\]

3. Cylindrical Systems (Axial/Transverse)

For the cylindrical system of this study in which an axial direction \( z \) and a transverse plane \( t = (\rho, \phi) \) can be identified, it is desirable to separate the field/potential relationships and the Helmholtz equations into axial and transverse parts.

a. Field/Potential Relations

In the wave number domain, the axial and the transverse components of the field variables are

\[
\begin{align*}
\mathbf{E}_z &= - (\nabla \times \nabla \mathbf{\Pi}_e - ik_0 \mathbf{\nabla} \cdot \mathbf{\Pi}_e - i\omega \mu_0 \nabla \mathbf{\nabla} \mathbf{\Pi}_m) \\
\mathbf{E}_t &= \mathbf{\nabla} \cdot \mathbf{\nabla} \mathbf{\Pi}_e + ik_0 \mathbf{\nabla} \cdot \mathbf{\Pi}_e + k_0^2 \mathbf{\nabla} \times \mathbf{\nabla} \mathbf{\Pi}_e \\
&\quad - i\omega \mu_0 \nabla \mathbf{\nabla} \mathbf{\Pi}_m + (\nabla \times \nabla \mathbf{\Pi}_m - ik_0 \mathbf{\nabla} \cdot \mathbf{\Pi}_m)
\end{align*}
\]
and

\[ h^z = -(q^z t^2 z_m - i k_o \kappa^z t^z m_t + i \omega \epsilon \kappa c v^z c \cdot t^z m_e ) \]
\[ h^t = c^z v^z t^z c \cdot t^z m_t + i k_o \kappa^z t^z m + k_o^2 \kappa^2 z^2 t^z m \]
\[ + i \omega \epsilon \kappa c v^z c \cdot (v^z t^z m_e - i k_o \kappa t^z m_e ) \]

where \( v^z \) denotes the transverse component of the differential operator \( v \),
and \( k^z = k_o \kappa^z \) (axial wave number)
and \( \epsilon c = -\kappa^2 z \) (curl dyad)

Therefore, when the dyadic operators are expanded, the axial and the transverse cylindrical components of the field vectors become

\[ e_\rho = \frac{1}{\rho} \frac{\partial}{\partial \phi} \left[ - \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \kappa^\rho e) + \frac{1}{\rho} \frac{\partial \kappa^\rho e}{\partial \phi} \right] + i k_o \kappa^z \frac{\partial \kappa^z e}{\partial \phi} + k_o^2 \kappa^2 z^2 \frac{\partial e}{\partial \phi} \]
\[ - i \omega \kappa \epsilon \kappa \left( \frac{1}{\rho} \frac{\partial \kappa^z e}{\partial \phi} + i k_o \kappa^z \frac{\partial e}{\partial \phi} \right) \]
\[ e_\phi = \frac{\partial}{\partial \rho} \left[ - \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \kappa^\phi e) + \frac{1}{\rho} \frac{\partial \kappa^\phi e}{\partial \phi} \right] + i k_o \kappa^z \frac{1}{\rho} \frac{\partial \kappa^z e}{\partial \phi} + k_o^2 \kappa^2 z^2 \frac{\partial e}{\partial \phi} \]
\[ - i \omega \kappa \epsilon \kappa \left( \frac{1}{\rho} \frac{\partial \kappa^z e}{\partial \phi} - i k_o \kappa^z \frac{\partial e}{\partial \phi} \right) \]
\[ e_z = k_o^2 \kappa^2 t^2 z^e + i k_o \kappa^z \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \kappa^\rho e) + \frac{1}{\rho} \frac{\partial \kappa^\rho e}{\partial \phi} \right) \]
\[ - i \omega \kappa \epsilon \kappa \left[ - \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \kappa^\phi e) + \frac{1}{\rho} \frac{\partial \kappa^\phi e}{\partial \phi} \right] \]

and

\[ h^\rho = - \frac{1}{\rho} \frac{\partial}{\partial \phi} \left[ - \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \kappa^\rho e) + \frac{1}{\rho} \frac{\partial \kappa^\rho e}{\partial \phi} \right] + i k_o \kappa^z \frac{\partial \kappa^z e}{\partial \phi} + k_o^2 \kappa^2 z^2 \frac{\partial e}{\partial \phi} \]
\[ + i \omega \kappa \epsilon \kappa \left( \frac{1}{\rho} \frac{\partial \kappa^z e}{\partial \phi} + i k_o \kappa^z \frac{\partial e}{\partial \phi} \right) \]
b. Wave Equations
In the wave number domain, the reduced potential forms of the Helmholtz

\[
\begin{align*}
\mathbf{h}^o &= \frac{\partial}{\partial \rho} \left[ -\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \mathbf{r}^o_m) + \frac{1}{\rho} \frac{\partial \mathbf{r}^o_m}{\partial \rho} \right] + i k_0 \mathbf{z} \frac{1}{\rho} \frac{\partial \mathbf{r}^o_m}{\partial \phi} + k_0 z^2 \mathbf{r}^o_m \\
&+ i \omega \epsilon_0 c_0 \left( \frac{\partial \mathbf{r}^o_e}{\partial \rho} - i k_0 \mathbf{z} \frac{\partial \mathbf{r}^o_e}{\partial \phi} \right)
\end{align*}
\]

\[
\mathbf{h}^z = k_0^2 \mathbf{z} \frac{\partial^2 \mathbf{r}^o_m}{\partial \rho^2} + i k_0 \mathbf{z} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \mathbf{r}^o_m) + \frac{1}{\rho} \frac{\partial \mathbf{r}^o_m}{\partial \phi} \right] i \omega \epsilon_0 c_0 \left[ -\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \mathbf{r}^o_e) + \frac{1}{\rho} \frac{\partial \mathbf{r}^o_e}{\partial \phi} \right]
\]

4. Solutions (Transverse)
For a point Hertzian dipole located at \( \rho_s, \phi_s, z_s \), the source distri-

**\( \mathbf{q}_a = \hat{\delta}_s \)**

where

**\( \hat{d} = \hat{x} d^x + \hat{y} d^y + \hat{z} d^z \) (unit displacement vector)\**

and

**\( d^i = \text{csn} \delta^i \mathbf{i} \mathbf{x} = (x, y, z) \)**

where

\[
\sum_{i=1}^{3} \text{csn}^2 \delta^i = 1
\]
\[ j_s = i \frac{1}{s} \frac{\delta(\rho - \rho_s) \delta(\varphi - \varphi_s) \delta(z - z_s)}{\rho} \text{csn}(\omega t + \phi) \quad \text{(time)} \]

\[ J_s = i \frac{1}{s} \frac{\delta(\rho - \rho_s) \delta(\varphi - \varphi_s) \delta(z - z_s)}{\rho} e^{i\phi} \quad \text{(frequency)} \]

\[ j_s = \frac{i}{\sqrt{2\pi}} \frac{\delta(\rho - \rho_s) \delta(\varphi - \varphi_s)}{\rho} e^{i\phi - ik_0 n^2 z_s} \quad \text{(wave number)} \]

where the following transformation of the Dirac delta function in \( z \) has been used

\[ \delta(z - z_s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz e^{ikz}(z - z_s) \]

where

\[ \delta(\xi) = \begin{cases} 0 & \xi \neq 0 \\ \infty & \xi = 0 \end{cases} \]

and

\[ \int_{-\infty}^{+\infty} d\xi \delta(\xi) = 1 \]

Correspondingly, let

\[ \nabla \tau e = \nabla \tau m^x + \nabla \tau y^y + 2 \nabla \tau z^z \]

\[ \begin{align*}
\Box \tau e &= \frac{d^2 j_s}{i\omega_0 C_{\tau e}} \\
\Box \tau m &= 0
\end{align*} \]

\[ i = (x, y, z) \]

In a cylindrical coordinate system

\[ \nabla t^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{2} \frac{\partial^2}{\partial \phi^2} \quad \text{(transverse Laplacian)} \]
and
\[ \Box t^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho} \frac{\partial^2}{\partial \phi^2} + k_0^2 \kappa^2 t^2 \] (transverse d'alembertian)

Since the solution is periodic in \( \phi \) with a period of \( 2\pi \), it is desirable to expand the solution in the form

\[ s(\rho, \phi, k^2, \omega) \cdot \int_{-\infty}^{+\infty} = \sum_{\nu=-\infty}^{+\infty} s(\nu, k^2, \omega) e^{-i
\nu \phi} \]

which is just a Fourier series expansion in \( \phi \). The index \( \nu \) is assumed to be an integer. In what follows, let any transformed scalar \( s \), represent a function of \( \rho, k^2, \) and \( \omega \), i.e. \( s = s(\rho, k^2, \omega) \).

Consequently, the differential equations in terms of the transformed variables are

\[ \left[ \rho \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + (k_0^2 \kappa^2) - \nu^2 \right] \int_{-\infty}^{+\infty} = \frac{i}{3} \chi_{\nu \zeta} \phi \delta(\rho - \rho_s) e^{i \nu \phi} e^{-i \nu \phi} \]

where the following transformation of the Dirac delta function in \( \phi \) has been used

\[ \delta(\phi - \phi_s) = \frac{1}{2\pi} \sum_{\nu=-\infty}^{+\infty} e^{-i \nu \phi} \delta(\nu - \nu_s) \]

For \( \rho \neq \rho_s \) the differential equations reduce to

\[ \left[ \rho \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + (k_0^2 \kappa^2) - \nu^2 \right] \int_{-\infty}^{+\infty} = 0 \]

The complementary solutions of these equations are

\[ \pi_{\nu, \zeta} = c_i \left[ \frac{Z_{\nu}^{(x)}(k_0 \kappa \rho)}{Z_{\zeta}^{(x)}} \right] \int_{-\infty}^{+\infty} \]

where

\[ c_i \left[ \frac{Z_{\nu}^{(x)}(k_0 \kappa \rho)}{Z_{\zeta}^{(x)}} \right] \int_{-\infty}^{+\infty} \]

represents any linear combination of the Bessel functions of the complex
argument $k_o \nu^t \rho$ and the integer order $\nu$. This is valid for each index $i$, each order $\nu$, and for both the electric and magnetic solutions.

For $\rho < \rho_s$ the particular solutions of these equations are finite at the origin; and, therefore,
\[
i \nu _{\text{le}} = c < \nu \text{ } J (k_o \nu \rho) \quad (\rho < \rho_s) \quad i = (x,y,z)
\]
\[
i \nu _{\text{lm}} = 0
\]

For $\rho > \rho_s$ the particular solutions of these equations are outwardly traveling waves; and, therefore,
\[
i \nu _{\text{le}} = c > \nu \text{ } H(1) (k_o \nu \rho) \quad (\rho > \rho_s) \quad i = (x,y,z)
\]
\[
i \nu _{\text{lm}} = 0
\]

The constants $c < \nu$ and $c > \nu$ are determined by the boundary conditions on $\nu _{\text{le}}$ at $\rho = \rho_s$. The continuity of $\nu _{\text{le}}$ across the cylinder $\rho = \rho_s$ requires that
\[
i \nu _{\text{le}} = c < \nu \text{ } J (k_o \nu \rho <) H(1) (k_o \nu \rho >) \quad i = (x,y,z)
\]
\[
i \nu _{\text{lm}} = 0
\]

where $\rho <$ is the lesser of $\rho$ and $\rho_s$ and $\rho >$ is the greater of $\rho$ and $\rho_s$. The restrictions on the first derivatives of $\nu _{\text{le}}$ on the cylinder $\rho = \rho_s$ are determined by substituting the above expressions for $\nu _{\text{le}}$ into the differential equations and integrating the resulting equations over a small interval containing $\rho_s$ as an interior point. The results of the integrations are

\[
c _{\text{ls}} = d ^i c _{\text{ls}} \quad i = (x,y,z)
\]

where
\[
c _{\text{ls}} = -i \frac{1}{4 \sqrt{2 \pi}} \frac{i s _{\text{ls}}}{i \phi \nu \rho o c} e ^{i \phi - i k_o z s} e ^{-i \nu s}
\]
where use has been made of the Wronskian relation

\[ J_z(k \sqrt{t}, \rho_s)H_1^{(1)}(k \sqrt{t}, \rho_s) - J'_z(k \sqrt{t}, \rho_s)H_1^{(1)}(k \sqrt{t}, \rho_s) = \frac{2}{\pi} \frac{1}{k \sqrt{t} \rho_s} \]

Therefore, the particular solutions of these equations are

\[ \pi_{ae} = d \xi_c J_z(k \sqrt{t}, \rho_s)H_1^{(1)}(k \sqrt{t}, \rho_s) \]

\[ \pi_{um} = 0 \]

5. Scalarization

In the regions of space where the source currents are zero, the only non-zero components of the electric and magnetic Hertz vectors are the axial components. Therefore, for the source distribution of an point Hertzian dipole, the problem is scalarized with the axial components of the homogeneous solutions only, superimposed with the particular solutions. Therefore,

\[ x = d x c J_z(k \sqrt{t}, \rho_s)H_1^{(1)}(k \sqrt{t}, \rho_s) \]

\[ y = d y c J_z(k \sqrt{t}, \rho_s)H_1^{(1)}(k \sqrt{t}, \rho_s) \]

\[ z = d z c J_z(k \sqrt{t}, \rho_s)H_1^{(1)}(k \sqrt{t}, \rho_s) + p_z \zeta_\ell(\pm) + q_z \zeta_\ell(\pm) \]

and

\[ x_{\ell m} = 0 \]

\[ y_{\ell m} = 0 \]

\[ z_{\ell m} = 0 \]

where \( p_\ell^+ \) and \( q_\ell^+ \) are arbitrary constants.
The cylindrical components of the Hertz vectors are

\[
\begin{align*}
\eta^\rho_{\text{le}} &= \pi^\rho_{\text{le}} \cos \varphi + \pi^\nu_{\text{le}} \sin \varphi = \mathcal{D}^{\rho}_{\text{le}} \mathcal{J}_{\text{le}} (k_o \nu k_p) \eta^{(1)}_{\text{le}} (k_o k_p), \\
\eta^\varphi_{\text{le}} &= -\pi^\nu_{\text{le}} \sin \varphi + \pi^\nu_{\text{le}} \cos \varphi = \mathcal{D}^{\nu}_{\text{le}} \mathcal{J}_{\text{le}} (k_o \nu k_p) \eta^{(1)}_{\text{le}} (k_o k_p), \\
\pi^z_{\text{le}} &= \pi^z_{\text{le}} = \mathcal{D}^{z}_{\text{le}} \mathcal{J}_{\text{le}} (k_o \nu k_p) \eta^{(1)}_{\text{le}} (k_o k_p) + \mathcal{P}^{(1)}_{\text{le}} \mathcal{Z}^{(1)}_{\text{le}} (k_o \nu k_p)
\end{align*}
\]

and

\[
\begin{align*}
\eta^\rho_{\text{lm}} &= 0 = 0, \\
\eta^\nu_{\text{lm}} &= 0 = 0, \\
\pi^z_{\text{lm}} &= \pi^z_{\text{lm}} = 0 + q^{(+)}_{\text{le}} \mathcal{Z}^{(+)}_{\text{le}} (k_o \nu k_p)
\end{align*}
\]

where

\[
\begin{align*}
d^\rho &= d^x \cos \varphi + d^y \sin \varphi, \\
d^\varphi &= -d^x \sin \varphi + d^y \cos \varphi, \\
d^z &= d^z
\end{align*}
\]

In terms of the transformed variables, the cylindrical components of the field vectors are

\[
\begin{align*}
\eta^\rho_{\text{le}} &= \left( \frac{2}{\rho^2} + k_o \nu k_p \right) \pi^\rho_{\text{le}} + \frac{iv}{\rho} \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \pi^\varphi_{\text{le}} + \frac{i}{\rho} k_o \nu \frac{\partial}{\partial \rho} \pi^z_{\text{le}} \\
\eta^\varphi_{\text{le}} &= \frac{iv}{\rho} \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \pi^\rho_{\text{le}} + \frac{1}{\rho^2} \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \frac{\partial^2}{\partial \rho^2} + k_o \nu k_p \pi^\varphi_{\text{le}} + \frac{i}{\rho} k_o \nu \frac{\partial}{\partial \rho} \pi^z_{\text{le}} \\
\pi^z_{\text{le}} &= \frac{i}{\rho} k_o \nu \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \pi^\rho_{\text{le}} + \frac{i}{\rho} k_o \nu \frac{\partial}{\partial \rho} \pi^\varphi_{\text{le}} + \frac{2}{\rho} \frac{t^2}{2} z \pi^z_{\text{le}} \\
\eta^\rho_{\text{lm}} &= \frac{i}{\rho} \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \pi^\rho_{\text{lm}} + \frac{i}{\rho} \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \pi^\varphi_{\text{lm}} + \frac{i}{\rho} k_o \nu \pi^z_{\text{lm}} \\
\eta^\nu_{\text{lm}} &= \frac{i}{\rho} \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \pi^\rho_{\text{lm}} + \frac{i}{\rho} \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \pi^\varphi_{\text{lm}} + \frac{i}{\rho} k_o \nu \pi^z_{\text{lm}} \\
\eta^\rho_{\text{le}} &= \left( \frac{2}{\rho^2} + k_o \nu k_p \right) \pi^\rho_{\text{le}} + \frac{i}{\rho} \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \pi^\varphi_{\text{le}} + \frac{i}{\rho} k_o \nu \frac{\partial}{\partial \rho} \pi^z_{\text{le}} \\
\eta^\nu_{\text{le}} &= \left( \frac{2}{\rho^2} + k_o \nu k_p \right) \pi^\nu_{\text{le}} + \frac{i}{\rho} \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \pi^\varphi_{\text{le}} + \frac{i}{\rho} k_o \nu \frac{\partial}{\partial \rho} \pi^z_{\text{le}} \\
\pi^z_{\text{le}} &= \left( \frac{2}{\rho^2} + k_o \nu k_p \right) \pi^z_{\text{le}} + \frac{i}{\rho} \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \pi^\varphi_{\text{le}} + \frac{i}{\rho} k_o \nu \frac{\partial}{\partial \rho} \pi^z_{\text{le}}
\end{align*}
\]
\[
\begin{align*}
\mathbf{h}_z &= \frac{it}{\rho} \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \eta_{z\rho} + \left( \frac{1}{\rho^2} - \frac{1}{\rho \partial \rho} - \frac{\partial^2}{\partial \rho^2} + k_o^2 \nu^2 \right) \eta_{\rho m} + ik_o \nu^2 \frac{1}{\rho} \eta_{z\rho}^m \\
&\quad + i\omega \epsilon_c \frac{\partial}{\partial \rho} \eta_{\rho m}^z - i\omega \epsilon_c \frac{ik_o^2 \nu^2}{\rho} \eta_{\rho m}^z \\
\eta_{z\rho}^m &= ik_o \nu^2 \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \eta_{\rho m}^z + ik_o \nu^2 \frac{1}{\rho} \eta_{\rho m}^\rho + k_o^2 \nu^2 \eta_{z\rho}^z \\
&\quad + i\omega \epsilon_c \frac{it}{\rho} \eta_{\rho m}^z - i\omega \epsilon_c \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \eta_{\rho m}^\rho
\end{align*}
\]

Therefore, in the wave number domain, the cylindrical components of the field vectors are

\[
\begin{align*}
e_{z\rho} &= ik_o \nu^2 \nu_p \eta_{z\rho}^z + i\omega \epsilon_c \eta_{\rho m}^z \left( \nu_p \eta_{z\rho}^z \right) + \eta_{z\rho}^\rho \\
&\quad + i\omega \epsilon_c \frac{it}{\rho} \eta_{\rho m}^z - i\omega \epsilon_c \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \eta_{\rho m}^\rho \\
e_{z\rho}^\rho &= ik_o \nu^2 \nu_p \eta_{z\rho}^z + i\omega \epsilon_c \eta_{\rho m}^z \left( \nu_p \eta_{z\rho}^z \right) + \eta_{z\rho}^\rho \\
e_{z\rho}^z &= k_o \nu^2 \nu_p \eta_{z\rho}^z + \eta_{z\rho}^z \\
\text{and}
\end{align*}
\]

\[
\begin{align*}
\eta_{z\rho} &= ik_o \nu^2 \nu_p \eta_{z\rho}^z + i\omega \epsilon_c \eta_{\rho m}^z \left( \nu_p \eta_{z\rho}^z \right) + \eta_{z\rho}^\rho \\
&\quad + i\omega \epsilon_c \frac{it}{\rho} \eta_{\rho m}^z - i\omega \epsilon_c \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \eta_{\rho m}^\rho \\
\eta_{z\rho}^\rho &= ik_o \nu^2 \nu_p \eta_{z\rho}^z + i\omega \epsilon_c \eta_{\rho m}^z \left( \nu_p \eta_{z\rho}^z \right) + \eta_{z\rho}^\rho \\
&\quad + i\omega \epsilon_c \frac{it}{\rho} \eta_{\rho m}^z - i\omega \epsilon_c \left( \frac{1}{\rho} + \frac{\partial}{\partial \rho} \right) \eta_{\rho m}^\rho \\
\eta_{z\rho}^z &= k_o \nu^2 \nu_p \eta_{z\rho}^z + \eta_{z\rho}^z
\end{align*}
\]

where

\[
\begin{align*}
e_{z\rho} &= \epsilon_{z\rho} \left\{ \left[ \frac{\nu^2}{\rho^2} + k_o^2 \nu^2 \right] d^\rho + \frac{i\omega}{\rho^2} d^\nu \right\} \left[ J_{1\nu} \left( k_o \nu \rho \right) \right] \\
&\quad + \left[ \frac{i\omega}{\rho} d^\nu + ik_o \nu^2 d^\nu \right] \frac{\partial}{\partial \rho} \left[ J_{1\nu} \left( k_o \nu \rho \right) \right]
\end{align*}
\]
\[ e^\varphi \equiv c_{zs} \left\{ \left[ -\frac{i z}{\rho^2} d^\rho + \frac{1}{\rho^2} + k_0^2 \varphi^2 \right] J_z(k_0 \varphi \rho) \right\} H_z^{(1)}(k_0 \varphi \rho) \\
+ \left[ -\frac{i z}{\rho} d^\rho - \frac{1}{\rho} d^\varphi \right] \frac{\partial}{\partial \rho} \left[ J_z(k_0 \varphi \rho) \right] H_z^{(1)}(k_0 \varphi \rho) \\
+ \left[ - d^\rho \right] \frac{\partial^2}{\partial \rho^2} \left[ J_z(k_0 \varphi \rho) \right] H_z^{(1)}(k_0 \varphi \rho) \right\} \]

\[ e^z \equiv c_{zs} \left\{ \left[ i k_0 \varphi^2 \frac{1}{\rho} d^\rho + i k_0 \varphi \frac{i z}{\rho} d^\varphi + k_0^2 \varphi^2 \right] J_z(k_0 \varphi \rho \rho) \right\} H_z^{(1)}(k_0 \varphi \rho) \\
+ \left[ i k_0 \varphi^2 \frac{1}{\rho} d^\rho \right] \frac{\partial}{\partial \rho} \left[ J_z(k_0 \varphi \rho) \right] H_z^{(1)}(k_0 \varphi \rho) \right\} \}

and

\[ \eta^\rho \equiv i \omega e_0 \zeta c_{zs} (1k_0 \varphi^2 d^\rho - \frac{i z}{\rho} d^z) J_z(k_0 \varphi \rho) H_z^{(1)}(k_0 \varphi \rho) \]

\[ \eta^z \equiv i \omega e_0 \zeta c_{zs} \left\{ \left[ -i k_0 \varphi^2 d^\rho \right] J_z(k_0 \varphi \rho) H_z^{(1)}(k_0 \varphi \rho) \\
+ \left[ d^z \right] \frac{\partial}{\partial \rho} \left[ J_z(k_0 \varphi \rho \rho) \right] H_z^{(1)}(k_0 \varphi \rho) \right\} \}

\[ \eta^z \equiv i \omega e_0 \zeta c_{zs} \left\{ \left[ \frac{i z}{\rho} d^\rho - \frac{1}{\rho} d^\varphi \right] J_z(k_0 \varphi \rho \rho) H_z^{(1)}(k_0 \varphi \rho \rho) \\
\left[ -d^\varphi \right] \frac{\partial}{\partial \rho} \left[ J_z(k_0 \varphi \rho \rho) \right] H_z^{(1)}(k_0 \varphi \rho \rho) \right\} \} \]
C. Plasma (Convection Currents)

A suitable model of a stationary plasma, consistent with the objectives of this study, is that of a certain number density \( N \), of electrons per unit volume, each with a charge \( q \), and a mass \( m \), free to move with a velocity \( \mathbf{V} \) under the influence of applied electromagnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) (part of which is a static biasing field \( \mathbf{B}_0 \)), and a spherical gravitational field \( \mathbf{G} \); but, subject to a damping force due to elastic collisions characterized by a collision frequency \( f_c \), and a shear or pressure/tension force due to the fluid motions of the electrons characterized by a stress dyad \( S \). It is assumed that all inelastic collisions, viz. ionization, excitation, attachment, and recombination, are negligible. It is also assumed that the plasma is in thermal equilibrium at a temperature \( T \).

The interactions of the electromagnetic waves with the other species of the plasma, viz. the ions and the neutrals, are neglected, since the ions and the neutrals are much heavier than the light electrons, and are nearly immobile at the microwave frequencies of interest in this study. Therefore, the convection current density in the plasma is determined by examining the motion of the free electrons only.

1. Governing Equations

In the frequency domain, the acceleration of the free electrons, as given by Newton's equation of motion, is

\[
\dot{\mathbf{V}} = \frac{\mathbf{F}_e}{m} + \frac{\mathbf{F}_i}{m} + \frac{\mathbf{F}_{\text{gravity}}}{m} + \frac{\mathbf{F}_{\text{fluid}}}{m} + \frac{\mathbf{F}_{\text{loss}}}{m}
\]

where

\[
\dot{\mathbf{V}} = -\mathrm{i}\omega \mathbf{V} + \nabla \times \mathbf{V} \quad \text{(convective time derivative)}
\]
and
\[ \frac{F_{e/m}}{m} = q(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \]
\[ F_{\text{gravity}} = m\mathbf{g} \]
\[ F_{\text{fluid}} = \frac{1}{N} \mathbf{q} \cdot \nabla \rho \]
\[ F_{\text{loss}} = -\omega_c (m\mathbf{V}) \quad (\omega_c = 2\pi f_c) \]

It is assumed that the plasma is present only at such altitudes for which the gravitational field is negligible in comparison to the electromagnetic fields present. Also, it is assumed that the velocity distribution of the free electrons is isotropic so that
\[ \mathbf{S} = \mathbf{i} \rho \]

where
\[ \rho = \text{scalar pressure} \]

then,
\[ \nabla \cdot \mathbf{S} = \mathbf{v} \rho \]

Explicitly, then, the hydrodynamic equation of motion is
\[ -i\omega \mathbf{V} + \nabla \cdot \mathbf{v} \mathbf{V} = \frac{q}{m} (\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \frac{1}{mN} \mathbf{v} \rho - \omega_c \mathbf{V} \]

The motion of the free electrons is also governed by the equation of continuity
\[ \nabla \cdot (\mathbf{v} \mathbf{N}) = -\mathbf{N} \]

where
\[ \mathbf{N} = -i\omega \mathbf{N} + \nabla \cdot \mathbf{v} \mathbf{N} \quad \text{(convective time derivative)} \]

and the equations of state
\[ P = N \kappa_0 T \quad \text{(ideal gas)} \]
\[ \rho = Nm \]

and
\[ a = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{\kappa_0 T}{m}} \quad \text{(acoustical speed)} \]
\[ \kappa_0 = 1.3804 \times 10^{-23} \quad \text{(Boltmann's constant)} \]
2. **Linearization**

Let
\[
\begin{align*}
V &= V_o + v \\
N &= N_o + n \\
T &= T_o + t \\
P &= P_o + p \\
\rho &= \rho_o + \rho
\end{align*}
\]
and
\[
\begin{align*}
E &= E_o + e \\
B &= B_o + b
\end{align*}
\]
where the first term in each equation denotes the average or bulk value of that property and the second term in each equation denotes the deviation of that property from the average value.

To obtain a linear system of governing equations, the separated terms in the above equations are substituted into the governing equations. When all nonlinear terms are neglected,
\[
\begin{align*}
-i \omega V + \frac{V_o \cdot v}{m} v &= \frac{q}{m} \left[ (E_o + e) + (V_o + v) \Lambda (\theta_o + b) \right] - \frac{\omega (V_o + v)}{c} - \frac{1}{mN_o} \nabla p \\
N_o \nabla \cdot V + \frac{V_o \cdot \nabla}{m} n &= +i \omega n - \frac{V_o \cdot \nabla a} \\
p &= \kappa_o (N_o t + nT_o) \\
\rho &= \rho_m
\end{align*}
\]
It is assumed that the plasma is at rest, so that
\[
V_o = 0
\]
and that the plasma has an isothermal temperature distribution, so that
\[
t = 0
\]
Also, it is assumed that the electrostatic bias in the plasma is zero; but, that the magnetostatic bias is non-zero and large with respect to the magnetic field, so that the magnetic field is negligible. The direction of the magnetostatic bias in the plasma is described by the direction angles $\beta^x, \beta^y, \beta^z$ with respect to the $x, y, z$ axes, respectively. Therefore,

$$E = e$$
$$B = bB_o$$

where

$$B_o \equiv \text{scalar magnetostatic bias}$$

$$b = \hat{x}b^x + \hat{y}b^y + \hat{z}b^z$$

and

$$b^i = \cos \beta^i \quad i = (x, y, z)$$

where the direction angles $\beta^i$ satisfy

$$\sum_{i=1}^{3} \cos^2 \beta^i = 1$$

With these assumptions, the linear system of governing equations reduces to

$$-i_0 \nabla \cdot \mathbf{v} = \frac{q}{m} \left( e + v \wedge \hat{b} \right) - \frac{1}{mN_0} \nabla p$$

$$N_0 \nabla \cdot \mathbf{v} = i_0 n$$

$$p = \rho \omega n$$

$$\rho = \rho_0$$

The linearized hydrostatic equation of motion is rewritten as

$$0 \cdot \mathbf{v} = \frac{q}{m} e - \frac{1}{mN_0} \nabla p$$

where

$$0 \equiv -i_0 \nabla \cdot \mathbf{v} + \omega_0 \mathbf{e}$$

and

$$\epsilon = -\nabla \hat{b} = (\hat{\gamma} - \hat{\gamma}z) \csn \beta^x + (\hat{\gamma}z - \hat{\gamma}x) \csn \beta^y + (\hat{\gamma}y - \hat{\gamma}y) \csn \beta^z$$

$$\hat{e} = \hat{xx} + \hat{yy} + \hat{zz}$$

*see Appendix I
where the collision terms are defined by

\[ \omega_c = 2nf_c \]
\[ r_c = \frac{\omega_c}{\omega} \]
\[ \Omega = 1 + ir_c \]

and the gyrotropic terms are defined by

\[ n_g = \frac{q B_o}{m} \]
\[ r_g = \frac{\omega_g}{\omega} \]

Then, the velocity of the free electrons is explicitly given by

\[ v = \Omega^{-1} \left( \frac{q}{m} e - \frac{1}{mN_o} \nabla \mathbf{p} \right) \]

where \( \Omega^{-1} \) is the inverse dyad of \( \Omega \), i.e.

\[ \Omega \Omega^{-1} = \mathbb{I} = \Omega^{-1} \Omega \]

The inverse of \( \Omega \) is determined by examining its individual components in matrix form. This process yields\(^*\)

\[ \Omega^{-1} = \frac{i \Omega}{\Omega^2 - r_g} \left[ \mathbb{I} - i \left( \frac{r_g}{\Omega} \right) c + \left( \frac{r_g}{\Omega} \right)^2 b \right] \]

where

\[ b = b = \hat{b} = xx \text{csn} s^2 g^x + xy \text{csn} s^x \text{csn} s^y + xz \text{csn} s^x \text{csn} s^z + \hat{y} \text{csn} s^y \text{csn} s^x + \hat{y} \text{csn} s^2 g^y + \hat{z} \text{csn} s^y \text{csn} s^z + \hat{z} \text{csn} s^z \text{csn} s^x + \hat{z} \text{csn} s^x \text{csn} s^y + \hat{z} \text{csn} s^2 g^z \]

Therefore

\[ v = \frac{i \Omega}{\Omega^2 - r_g} \left[ \mathbb{I} - i \left( \frac{r_g}{\Omega} \right) c + \left( \frac{r_g}{\Omega} \right)^2 b \right] \cdot \left( \frac{q}{m} e - \frac{1}{mN_o} \nabla \mathbf{p} \right) \]

The convection current density in the plasma is defined by

\[ j_{\text{conv}} = N_o q v \]

\(^*\) see Appendix II
Therefore,

\[ J_{\text{conv}} = \mathbf{\mathcal{\Sigma}} \cdot \mathbf{E} + \mathbf{\chi} \cdot \nabla \mathbf{p} \]

where

\[ \mathbf{\mathcal{\Sigma}} = -i \omega \varepsilon_0 \mathbf{r}_p \frac{\mathbf{r}}{\Omega - r_g} \left[ \frac{1}{\mathbf{\mathcal{\Omega}}} - i \left( \frac{\mathbf{r}}{\Omega} \right) \right] - \left( \frac{\mathbf{r}}{\Omega} \right) \]

\[ \mathbf{\chi} = -i \omega \varepsilon_0 \mathbf{r}_p \frac{\mathbf{r}}{N_0 q} \frac{\Omega^2}{\Omega^2 - r_g^2} \left[ \frac{1}{\mathbf{\mathcal{\Omega}}} - i \left( \frac{\mathbf{r}}{\Omega} \right) \right] - \left( \frac{\mathbf{r}}{\Omega} \right) \]

where the plasma terms are defined by

\[ \mathbf{\varepsilon}_p = \sqrt{\frac{N_0 q^2}{m_\varepsilon_0}} \]

\[ \mathbf{\mu}_p = \frac{m_\varepsilon_0}{m_\varepsilon_0} \]

\[ \mathbf{\rho}_p = \frac{m_\varepsilon_0}{m_\varepsilon_0} \]

Also,

\[ \mathbf{\mathcal{\Sigma}} = \mathbf{\mathcal{\Omega}} \]

\[ \mathbf{\mu}_p = \frac{m_\varepsilon_0}{m_\varepsilon_0} \]

3. Maxwell's Equations

Substituting the constitutive parameters into the independent Maxwell curl equations gives

\[ -\nabla \times \mathbf{E} = -i \omega \mu_0 \mathbf{H} \]

\[ +\nabla \times \mathbf{H} = -i \omega \varepsilon_0 \mathbf{E} + \mathbf{\mathcal{\Sigma}} \cdot \mathbf{E} + \mathbf{\chi} \cdot \nabla \mathbf{p} + \mathbf{J}_d \]

To put the above equations into a symmetric form, a complex relative permittivity dyad \( \varepsilon_c \), a complex relative permeability dyad \( \mu_c \), and a complex compressivity dyad \( \mathcal{C}_c \) are defined by
\[ \zeta_c = \begin{cases} \zeta_{c+} + \zeta_{c_+} + \zeta_{c_+} & \text{(if } b = z) \\ \zeta_{c_+} + \zeta_{c_{+}} + \zeta_{c_{+}} & \text{(if } b = z) \end{cases} \]

where

\[ \zeta_{c_+} = 1 - \frac{r^2}{\Omega} \frac{\Omega}{\Omega^2 - r^2} \]

and

\[ \xi_{c_+} = \frac{r}{N_q} \frac{\Omega}{\Omega^2 - r^2} \]

\[ \xi_{c} = -\frac{r}{N_q} \frac{\Omega}{\Omega^2 - r^2} \]

\[ \xi_{c_{+}} = -\frac{r}{N_q} \frac{\Omega}{\Omega^2 - r^2} \]

\[ \xi_{c_{+}} = \frac{1}{N_q} \frac{r^2}{\Omega} \]

\[ \zeta'' \text{=} \zeta'' \text{=} \zeta'' \text{=} \zeta'' \text{=} \zeta'' \text{=} \zeta'' \text{=} \zeta'' \text{=} \zeta'' \text{=} \zeta'' \]
and the dyad $\mathbf{g}$ is the transverse part of the unit dyad $\mathbf{i}$. Then, in the frequency domain

$$-\nabla \mathbf{E} = -i\omega \mathbf{c} \cdot \mathbf{H}$$
$$+\nabla \mathbf{H} = -i\omega \mathbf{c} \cdot \mathbf{E} + \nabla \mathbf{\varphi} + \mathbf{J}_a$$

The dependent Maxwell divergence equations are determined by taking the divergence of both sides of the independent Maxwell curl equations. To simplify the resulting operations, the curl equations are written in the unsymmetric form

$$-\nabla \mathbf{E} = -i\omega \mathbf{c} \cdot \mathbf{H}$$
$$+\nabla \mathbf{H} = -i\omega \mathbf{c} \cdot \mathbf{E} + N_0 \mathbf{\varphi} + \mathbf{J}_a$$

Then, when the equation of continuity and the equation of state are used,

$$\nabla \cdot \mathbf{H} = 0$$
$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \frac{q}{\kappa_0 T_0} \mathbf{p} + \frac{1}{i\omega \sigma_0} \nabla \cdot \mathbf{J}_a$$

A further divergence relationship is obtained by taking the divergence of both sides of the symmetric form of the Maxwell curl equations.

$$\nabla \cdot (\mathbf{c} \cdot \mathbf{E} + \mathbf{\varphi}) = \frac{1}{i\omega \sigma_0} \nabla \cdot \mathbf{J}_a$$

In a source free region, the reduced Maxwell equations are

$$-\nabla \mathbf{E} = -i\omega \mathbf{c} \cdot \mathbf{H}$$
$$+\nabla \mathbf{H} = -i\omega \mathbf{c} \cdot \mathbf{E} + \nabla \mathbf{\varphi}$$

and

$$\nabla \cdot \mathbf{H} = 0$$
$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \frac{q}{\kappa_0 T_0} \mathbf{p}$$
$$\nabla \cdot (\mathbf{c} \cdot \mathbf{E} + \mathbf{\varphi}) = 0$$

4. Helmholtz' Equations

The field forms of the Helmholtz wave equations are determined by taking the curl of both sides of the independent Maxwell curl equations. When the resulting curl-curl operation is replaced with the Laplacian operator and
when the divergence equations are substituted into the resulting equations, the field forms of the Helmholtz wave equations become

\begin{align*}
(\nabla^2 + k_{o \omega}^2) \cdot E + (\frac{1}{\varepsilon_o} \cdot \frac{\mu_o}{\varepsilon_o} \cdot \frac{\alpha}{\varepsilon_o} \cdot i + k_{o \omega c}^2) \cdot \nabla p &= -i \omega \mu_o (i + \frac{1}{k_o^2} \cdot \nabla \cdot J)

\nabla^2 i \cdot H - i \omega \varepsilon_o \nabla (\zeta_c \cdot E + \xi_c \cdot \nabla p) &= -\nabla \cdot J
\end{align*}

In a source free region, the reduced field forms of the Helmholtz wave equations are

\begin{align*}
(\nabla^2 + k_{o \omega}^2) \cdot E + (\frac{1}{\varepsilon_o} \cdot \frac{\mu_o}{\varepsilon_o} \cdot \frac{\alpha}{\varepsilon_o} \cdot i + k_{o \omega c}^2) \cdot \nabla p &= 0

\nabla^2 i \cdot H - i \omega \varepsilon_o \nabla (\zeta_c \cdot E + \xi_c \cdot \nabla p) &= 0
\end{align*}

5. Cylindrical Systems (Axial/Transverse)

For the cylindrical system of this study, in which an axial direction \( z \) and a transverse plane \( r^t = r^t(\rho, \phi) \) can be identified, it is desirable to separate the field/potential relationships into an axial and a transverse part.

a. Axial/Transverse Relations (Ohms' Law)

In the wave number domain, the axial and the transverse parts of the Maxwell equations are

\begin{align*}
\nabla \cdot e^t &= -i \omega \mu_o h^z \\
\nabla \cdot h^t &= -i \omega \varepsilon_o (\zeta^t e^z + i k_o \zeta^z p)
\end{align*}

\text{axial}

and

\begin{align*}
\nabla e^z &= -i k_o \zeta^z e^t \\
\nabla h^z &= -i \omega \varepsilon_o (\zeta e^z + i k_o \zeta^z p)
\end{align*}

\text{transverse}
also
\[ \nabla \cdot h = -ik_0 \mu_0 \varepsilon_0 \nabla z \]
\[ \nabla \cdot t = -ik_0 \mu_0 \varepsilon_0 z + \frac{1}{\varepsilon_0} \frac{q}{\varepsilon_0 \varepsilon_0} p \]

where
\[ \zeta^2 = \zeta^2 \]
\[ \zeta_c^t = \zeta_{\perp c}^t + i\zeta_{\perp c} \]

and
\[ \zeta^2 = \zeta^2 \]
\[ \zeta_c^t = \zeta_{\perp c}^t + i\zeta_{\perp c} \]

if
\[ \hat{b} = \hat{z} \]

The transverse parts of the Maxwell equations are regrouped in the form
\[ +ik_0 \mu_0 \varepsilon_0 \nabla \cdot t + i\omega \varepsilon_0 \mu_0 \nabla \cdot h = +t \cdot \nabla z \]
\[ -i\omega \varepsilon_0 \mu_0 \nabla \cdot c \cdot t + ik_0 \mu_0 \varepsilon_0 \nabla \cdot h = +t \cdot \nabla z + i\omega \varepsilon_0 \mu_0 \nabla \cdot c \cdot t \]

These equations are solved simultaneously for the transverse components of the field vectors explicitly in terms of the axial components of the field vectors and the pressure term. Therefore,
\[ \frac{t}{t} = \frac{1}{k_0^\Delta} (\zeta^* \nabla z) \cdot (\zeta_c - \kappa \nabla z) \cdot t \nabla z - k_0^\Delta \zeta_c \cdot \nabla \cdot t \]
\[ \frac{h}{h} = \frac{1}{k_0^\Delta} (\zeta^* \nabla z) \cdot t \nabla z + i\omega \varepsilon_0 \mu_0 \nabla \cdot c \cdot t \cdot z - \omega \varepsilon_0 k_0 \nabla z \cdot c \cdot \nabla \cdot t \]

where
\[ \zeta^*_{\perp c} = \zeta_{\perp c} - i\zeta_{\perp c} \]

and
\[ \Delta = (\zeta^2_{\perp} - \kappa z^2)^2 - \zeta^2_{\perp} = (\zeta^2_{\perp} - \zeta^2_{\perp}) - 2\zeta_{\perp} \kappa z^2 + \kappa z^4 \]
These expressions represent hybrid mode propagation in the plasma.

Therefore, when the dyadic operators are expanded, the transverse cylindrical components of the field vectors become

\[ e^\varphi = +i k_0 z \left( \frac{\partial}{\partial \rho} + \frac{i \zeta}{\rho} \frac{\partial}{\partial \varphi} \right) e^{-i \omega_o \left( \frac{\partial}{\partial \rho} - \frac{\eta}{\rho} \frac{\partial}{\partial \varphi} \right)} h - k_0^2 \left( \xi_1 \frac{\partial}{\partial \rho} + \frac{\xi_2}{\rho} \frac{\partial}{\partial \varphi} \right) p \]

\[ e^\theta = +i k_0 z \left( \frac{\partial}{\partial \rho} + \frac{i \zeta}{\rho} \frac{\partial}{\partial \varphi} \right) e^{-i \omega_o \left( \frac{\partial}{\partial \rho} + \frac{\eta}{\rho} \frac{\partial}{\partial \varphi} \right)} h - k_0^2 \left( \xi_2 \frac{\partial}{\partial \rho} + \frac{\xi_1}{\rho} \frac{\partial}{\partial \varphi} \right) p \]

and

\[ h^0 = +i k_0 z \left( \frac{\partial}{\partial \rho} + \frac{i \zeta}{\rho} \frac{\partial}{\partial \varphi} \right) e^{-i \omega_o \left( \frac{\partial}{\partial \rho} + \frac{\eta}{\rho} \frac{\partial}{\partial \varphi} \right)} h - k_0^2 \left( \xi_1 \frac{\partial}{\partial \rho} + \frac{\xi_2}{\rho} \frac{\partial}{\partial \varphi} \right) p \]

\[ h^\varphi = +i k_0 z \left( \frac{\partial}{\partial \rho} + \frac{i \zeta}{\rho} \frac{\partial}{\partial \varphi} \right) e^{-i \omega_o \left( \frac{\partial}{\partial \rho} + \frac{\eta}{\rho} \frac{\partial}{\partial \varphi} \right)} h - k_0^2 \left( \xi_2 \frac{\partial}{\partial \rho} + \frac{\xi_1}{\rho} \frac{\partial}{\partial \varphi} \right) p \]

where

\[ \zeta \equiv \frac{\zeta_+}{k_0 \Delta} \]

\[ \eta \equiv \frac{\zeta_+ - \zeta^-}{k_0 \Delta} \]

\[ \xi \equiv \zeta_+ \eta - \zeta^- \zeta \]

and

\[ \xi_1 \equiv \xi_+ \eta - \xi^- \zeta \]

\[ \xi_2 \equiv \xi_+ \zeta - \xi^- \eta \]

b. Wave Equations

Similarly, the axial and transverse parts of the Helmholtz wave equations are

\[ [\left( \nabla^2 - k_0^2 \zeta \right) \frac{\xi}{\rho} + k_0^2 \frac{\xi}{\rho} \frac{\partial}{\partial \varphi}] \cdot \frac{\xi}{t} + \left( - \frac{1}{\varepsilon_0} \frac{\varepsilon}{\mu} \frac{\partial}{\partial t} \frac{\xi}{t} + k_0^2 \frac{\xi}{\rho} \frac{\partial}{\partial \varphi} \right) \cdot \nabla \xi = 0 \]
\[
\left[ \nabla^2 + k_o^2 \left( \zeta - \zeta^2 \right) \right] e^z + \left( - \frac{1}{\varepsilon_o} - \frac{q}{\kappa_o T_o} + k_o^2 \zeta^2 \right) ik_o \kappa p = 0
\]
and
\[
\left[ (\nabla^2 - k_o^2 \zeta^2) \right] \xi \cdot h + i \omega e \zeta = \left[ \nabla^2 \left( \zeta e^z + ik_o \kappa \xi p \right) - ik_o \kappa \left( \frac{t \cdot e}{\xi_{\parallel}} + \frac{t \cdot \xi}{e} \right) \right] = 0
\]
\[
\left[ \nabla^2 - k_o^2 \zeta^2 \right] h + i \omega e \zeta \cdot c \cdot \left( \frac{t \cdot e}{\xi_{\parallel}} + \frac{t \cdot \xi}{e} \right) = 0
\]
also
\[
\nabla \cdot \left( \zeta e^z + ik_o \kappa \xi p \right) = 0
\]

Therefore, when the dyadic operators are expanded and when the axial parts of the Maxwell equations are substituted into the resulting equations, the axial parts of the Helmholtz wave equations become

\[
\nabla^2 e^z + k_o^2 (\zeta - \zeta^2) e^z + 0 \cdot h + ik_o \kappa \left( - \frac{1}{\varepsilon_o} - \frac{q}{\kappa_o T_o} + k_o^2 \zeta^2 \right) p = 0
\]

\[
\nabla^2 h^z + i \omega e \zeta k_o \kappa \left[ \frac{\xi_{\perp}^z}{\xi_{\perp}} (\zeta_{\perp} - \zeta_{\parallel}) - \zeta_{\parallel} \right] e^z + k_o (\zeta_{\perp} - \zeta_{\parallel}) \frac{2 \zeta_{\parallel}}{\xi_{\perp}} h^z
\]

\[
- \left[ \omega \left( \frac{\xi_{\perp}^z}{\xi_{\perp}} - \zeta_{\parallel} \right) \frac{q}{\kappa_o T_o} - \omega e k_o \kappa \zeta_{\parallel} \frac{\xi_{\perp}^z}{\xi_{\perp}} \right] p = 0
\]

\[
\nabla^2 p - ik_o \kappa \frac{\zeta_{\perp} - \zeta_{\parallel}^z}{\xi_{\perp}} e^z + i \omega e \zeta_{\parallel} h^z + \left( \frac{1}{\varepsilon_o} \frac{\xi_{\parallel}}{\xi_{\perp}} \frac{q}{\kappa_o T_o} - k_o^2 \zeta_{\parallel} \frac{\xi_{\parallel}}{\xi_{\perp}} \right) p = 0
\]

6. Diagonalization

To simplify the notation in what follows, let the vector field \( \mathbf{f} \) be such that its components are

\[
(f_i) = \begin{pmatrix}
    e^z \\
    h^z \\
    p
\end{pmatrix}
\]
and let the dyadic wave number $k$ be such that its components are

$$
\begin{bmatrix}
k^2(o \zeta - o z^2) & 0 & i k o \zeta (k o^2 - \frac{1}{e o} o T o)
\end{bmatrix}
$$

$$
\begin{bmatrix}
i o k o \zeta [\frac{e_1}{o} (o \zeta - o z^2) - o z] & k^2(o \zeta - o z^2 \frac{e_1}{o} \zeta) & -o k o \zeta \frac{e_1}{o} \zeta + \frac{1}{e o} o \zeta \zeta o T o
\end{bmatrix}
$$

Then

$$(\nabla^2 \frac{1}{o} + k) \cdot \Omega = 0$$

Let

$$\Omega = O \cdot \Pi$$

where $O$ is the dyadic operator which will diagonalize the dyadic wave number $k$, i.e.

$$(\nabla^2 \frac{1}{o} + d) \cdot \Pi = 0$$

where

$$d = O^{-1} \cdot k \cdot O$$

and $O^{-1}$ is the inverse dyad of $O$, i.e.

$$O^{-1} \cdot O^{-1} = i = O^{-1} \cdot O$$

a. **Eigenvalues**

The eigenvalues of this equation are determined by the cubic equation

$$|k - \lambda |^3 = \lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0 = 0$$

where

$$c_0 = -k_{11} k_{22} k_{33} + k_{11} k_{23} k_{32} + k_{12} k_{21} k_{33} + k_{13} k_{22} k_{31} - k_{12} k_{31} k_{23} - k_{13} k_{21} k_{32}$$

$$c_1 = k_{11} (k_{22} + k_{33}) + k_{22} k_{33} - k_{12} k_{21} - k_{13} k_{31} - k_{23} k_{32}$$

$$c_2 = - (k_{11} + k_{22} + k_{33})$$
The three roots of this equation are

\[ \lambda_i = \bar{x}_i - \frac{c^2}{3} \quad i = (1,2,3) \]

where

\[ \bar{x}_i = \begin{cases} \frac{s}{2} + \frac{d}{2} \sqrt{\frac{3}{2}} & i = (1,2,3) \\ \frac{s}{2} - \frac{d}{2} \sqrt{\frac{3}{2}} & \end{cases} \]

where

\[ s = c_+ + c_- \]
\[ d = c_+ - c_- \]

and

\[ c_+ = \sqrt{\frac{k_0}{2} + q} \]
\[ c_- = \sqrt{\frac{k_0}{2} - q} \]

and

\[ q = \sqrt{\frac{k^2}{4} - \frac{k^3}{27}} \]

and

\[ k_0 = \frac{1}{27}(2c_2^3 - 9c_2c_1^2 + 27c_0) \]
\[ k_1 = \frac{1}{3}(3c_1 - c_2^2) \]

ii. Eigenfunctions

The eigenfunctions of this equation are determined by the equation

\[ (k - \lambda_i) \cdot T_i = 0 \quad i = (1,2,3) \]

where the components of the transformation vector \( T \) are

\[ (T_i^1) = \begin{pmatrix} t_1^1 \\ t_2^1 \\ t_3^1 \end{pmatrix} \quad i = (1,2,3) \]
Therefore,
\[
(k_{11} - \lambda_i) t_i^1 + k_{12} t_i^2 + k_{13} t_i^3 = 0
\]
\[
k_{21} t_i^1 + (k_{22} - \lambda_i) t_i^2 + k_{23} t_i^3 = 0 \quad i = (1,2,3)
\]
\[
k_{31} t_i^1 + k_{32} t_i^2 + (k_{33} - \lambda_i) t_i^3 = 0
\]

This system of dependent equations is solved simultaneously for the components of the transformation vector $T$ and gives

\[
t_i^1 = \frac{k_{13}(\lambda_i - k_{22}) + k_{12}k_{23}}{(\lambda_i - k_{11})(\lambda_i - k_{22}) - k_{12}k_{21}} t_i^3
\]

\[
t_i^2 = \frac{k_{23}(\lambda_i - k_{11}) + k_{21}k_{13}}{(\lambda_i - k_{11})(\lambda_i - k_{22}) - k_{12}k_{21}} t_i^3
\]

\[
t_i^3 = \frac{k_{31}(\lambda_i - k_{22}) + k_{32}k_{21}}{(\lambda_i - k_{11})(\lambda_i - k_{22}) - k_{12}k_{21}} t_i^3
\]

or

\[
t_i^2 = \frac{k_{21}(\lambda_i k_{33}) + k_{23}k_{31}}{(\lambda_i - k_{22})(\lambda_i - k_{33}) - k_{23}k_{32}} t_i^1
\]

\[
t_i^3 = \frac{k_{31}(\lambda_i - k_{22}) + k_{32}k_{21}}{(\lambda_i - k_{22})(\lambda_i - k_{33}) - k_{23}k_{32}} t_i^1
\]

or

\[
t_i^1 = \frac{k_{12}(\lambda_i - k_{33}) + k_{13}k_{32}}{(\lambda_i - k_{11})(\lambda_i - k_{33}) - k_{13}k_{31}} t_i^2
\]

\[
t_i^3 = \frac{k_{32}(\lambda_i - k_{11}) + k_{31}k_{12}}{(\lambda_i - k_{11})(\lambda_i - k_{33}) - k_{13}k_{31}} t_i^2
\]

These relationships are normalized by letting

\[
t_i^1 = 1
\]
\[
t_i^2 = 1
\]
\[
t_i^3 = 1
\]
The components of the dyadic operator $\Omega$ are simply

$$
\begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} \\
\Omega_{21} & \Omega_{22} & \Omega_{23} \\
\Omega_{31} & \Omega_{32} & \Omega_{33}
\end{bmatrix} = \begin{bmatrix}
1 & \frac{k_{12}(\lambda_2 - k_{33}) + k_{13}k_{32}}{(\lambda_2 - k_{11})(\lambda_2 - k_{33}) - k_{13}k_{31}} & \frac{k_{13}(\lambda_3 - k_{22}) + k_{12}k_{23}}{(\lambda_3 - k_{11})(\lambda_3 - k_{22}) - k_{12}k_{21}} \\
\frac{k_{21}(\lambda_1 - k_{33}) + k_{23}k_{31}}{(\lambda_1 - k_{22})(\lambda_1 - k_{33}) - k_{23}k_{32}} & 1 & \frac{k_{23}(\lambda_3 - k_{11}) + k_{21}k_{13}}{(\lambda_3 - k_{11})(\lambda_3 - k_{22}) - k_{12}k_{21}} \\
\frac{k_{31}(\lambda_1 - k_{22}) + k_{32}k_{21}}{(\lambda_1 - k_{22})(\lambda_1 - k_{33}) - k_{23}k_{32}} & \frac{k_{32}(\lambda_2 - k_{11}) + k_{31}k_{12}}{(\lambda_2 - k_{11})(\lambda_2 - k_{33}) - k_{13}k_{31}} & 1
\end{bmatrix}
$$

and the components of the diagonal dyad $d$ are simply

$$
[d_{jk}] = [\lambda_j \delta_{jk}] = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \\
0 & \lambda_3
\end{bmatrix}
$$

where the Kronecker delta function is defined by

$$
\delta_{jk} = \begin{cases} 
0 & \text{if } j \neq k \\
1 & \text{if } j = k
\end{cases}
$$

If the components of the transformed vector $\Pi$ are denoted by

$$
(\pi_\perp) = \begin{bmatrix}
\pi_e \\
\pi_m \\
\pi_p
\end{bmatrix}
$$

then, the uncoupled Helmholtz wave equations are

\[ e^z = 0_{11} \pi_e + 0_{12} \pi_m + 0_{13} \pi_p \]
\[ h^z = 0_{21} \pi_e + 0_{22} \pi_m + 0_{23} \pi_p \]
\[ p = 0_{31} \pi_e + 0_{32} \pi_m + 0_{33} \pi_p \]

and

$$
(\nabla^2 + \lambda_i) \pi_\perp = 0 \quad i = (1, 2, 3)
$$
where
\[ \lambda_i = k_o^2 \gamma_i \quad i = (1,2,3) \]

7. Solutions

In a cylindrical coordinate system
\[ \nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\rho^2} \frac{\partial^2}{\partial z^2} \]

Since the solution is periodic in \( \varphi \) with a period of \( 2\pi \), it is desirable to expand the solution in the form
\[ s(\rho, \varphi, k_z, \omega) = \sum_{n=-\infty}^{\infty} s_n(\rho, k_z, \omega) e^{in\varphi} \]
\[ s_n(\rho, k_z, \omega) = \frac{1}{2\pi\omega} \int_{0}^{2\pi} d\varphi \ s(\rho, \varphi, k_z, \omega) e^{-in\varphi} \]

which is just a Fourier series expansion in \( \varphi \). The term \( n \) is assumed to be an integer. In what follows, let any transformed scalar \( s_n \) represent a function of \( \rho, k_z \), and \( \omega \), i.e. \( s_n = s_n(\rho, k_z, \omega) \).

Therefore, the uncoupled Helmholtz wave equations in terms of the transformed variables are

\[ \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \left( k_o \gamma_i \rho \right)^2 - \ell^2 \right] \pi_{li} = 0 \quad i = (1,2,3) \]

The complementary solutions of these equations are
\[ \pi_{li} = c_{li}^{(\pm)} z_{li}^{(\pm)} (k_o \gamma_i \rho) \quad i = (1,2,3) \]

where
\[ c_{li}^{(\pm)} z_{li}^{(\pm)} (k_o \gamma_i \rho) \quad i = (1,2,3) \]

represents any linear combination of the Bessel functions of the complex argument \( k_o \gamma_i \rho \) and the integer order \( \ell \). This is valid for each index \( i \).
Therefore,

\[ e^z_z = 0_{11}e^{(+)}_z Z^{(+)}_z (k_o \gamma_1 \rho) + 0_{12}^1u^{(+)}_z Z^{(+)}_z (k_o \gamma_2 \rho) + 0_{13}^1\rho^{(+)}_z Z^{(+)}_z (k_o \gamma_3 \rho) \]

\[ h^z_z = 0_{21}e^{(+)}_z Z^{(+)}_z (k_o \gamma_1 \rho) + 0_{22}^2u^{(+)}_z Z^{(+)}_z (k_o \gamma_2 \rho) + 0_{23}^2\rho^{(+)}_z Z^{(+)}_z (k_o \gamma_3 \rho) \]

\[ p^z_z = 0_{31}e^{(+)}_z Z^{(+)}_z (k_o \gamma_1 \rho) + 0_{32}^3u^{(+)}_z Z^{(+)}_z (k_o \gamma_2 \rho) + 0_{33}^3\rho^{(+)}_z Z^{(+)}_z (k_o \gamma_3 \rho) \]

where \( \varepsilon^\pm_{z}, u^\pm_{z}, \rho^\pm_{z} \) denote the various arbitrary constants. Also

\[ e^\varphi_z = \left( + i k_o \gamma_1^2 \varepsilon^2_{z} - i w_{o I}^{} \gamma_2^{} - k_o \varepsilon_{z}^2 \rho \right) \]

\[ h^\varphi_z = \left( + i k_o \gamma_1^2 \varepsilon^2_{z} - i w_{o I}^{} \gamma_2^{} - k_o \varepsilon_{z}^2 \rho \right) \]

\[ p^\varphi_z = \left( + i k_o \gamma_1^2 \varepsilon^2_{z} - i w_{o I}^{} \gamma_2^{} - k_o \varepsilon_{z}^2 \rho \right) \]

where

\[ e^{i z}_{z} = \frac{\partial e^{z}_{z}}{\partial \rho} = 0_{11}k_o \gamma_1^2 \varepsilon^{z}_{z} (k_o \gamma_1 \rho) + 0_{12}k_o \gamma_2^2 \varepsilon^{z}_{z} (k_o \gamma_2 \rho) + 0_{13}k_o \gamma_3^2 \varepsilon^{z}_{z} (k_o \gamma_3 \rho) \]

\[ h^{i z}_{z} = \frac{\partial h^{z}_{z}}{\partial \rho} = 0_{21}k_o \gamma_1^2 \varepsilon^{z}_{z} (k_o \gamma_1 \rho) + 0_{22}k_o \gamma_2^2 \varepsilon^{z}_{z} (k_o \gamma_2 \rho) + 0_{23}k_o \gamma_3^2 \varepsilon^{z}_{z} (k_o \gamma_3 \rho) \]

\[ p^{i z}_{z} = \frac{\partial p^{z}_{z}}{\partial \rho} = 0_{31}k_o \gamma_1^2 \varepsilon^{z}_{z} (k_o \gamma_1 \rho) + 0_{32}k_o \gamma_2^2 \varepsilon^{z}_{z} (k_o \gamma_2 \rho) + 0_{33}k_o \gamma_3^2 \varepsilon^{z}_{z} (k_o \gamma_3 \rho) \]
Explicitly,

\[ e^p = e^p_e \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + \mu^p \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^p \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

\[ + e^p \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + \mu^p \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^p \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

\[ e^\varphi = e^\varphi_e \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + \mu^\varphi \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^\varphi \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

\[ + e^\varphi \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + \mu^\varphi \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^\varphi \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

\[ e^z = e^z_e \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + \mu^z \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^z \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

\[ + e^z \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + \mu^z \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^z \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

and

\[ h^p = h^p_e \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + h^p \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^p \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

\[ + h^p \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + \mu^p \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^p \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

\[ h^\varphi = h^\varphi_e \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + h^\varphi \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^\varphi \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

\[ + h^\varphi \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + \mu^\varphi \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^\varphi \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

\[ h^z = h^z_e \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + h^z \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^z \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

\[ + h^z \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + \mu^z \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho^z \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

and

\[ p = p_e \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + p \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho p \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]

\[ + p \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_1 \rho \right) + \mu p \mu \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_2 \rho \right) + \rho p \rho \left( \pm \right) \left( \pm \right) \left( k_0 \gamma_3 \rho \right) \]
where

\[ \varepsilon_{\rho} = \frac{iZ}{\rho} (ik_0 \nabla \cdot \mathbf{\eta}_{11} + i\omega_0 \mathbf{\eta}_{21} - k_0^2 \mathbf{\gamma}_{31} ) \]

\[ \varepsilon_{\rho}' = k_0 \gamma_1 (ik_0 \nabla \cdot \mathbf{\eta}_{11} - i\omega_0 \mathbf{\eta}_{21} - k_0^2 \mathbf{\gamma}_{31} ) \]

\[ \mu_{\rho} = \frac{iZ}{\rho} (ik_0 \nabla \cdot \mathbf{\eta}_{12} + i\omega_0 \mathbf{\eta}_{22} - k_0^2 \mathbf{\gamma}_{32} ) \]

\[ \mu_{\rho}' = k_0 \gamma_2 (ik_0 \nabla \cdot \mathbf{\eta}_{12} - i\omega_0 \mathbf{\eta}_{22} - k_0^2 \mathbf{\gamma}_{32} ) \]

\[ \rho_{\rho} = \frac{iZ}{\rho} (ik_0 \nabla \cdot \mathbf{\eta}_{13} + i\omega_0 \mathbf{\eta}_{23} - k_0^2 \mathbf{\gamma}_{33} ) \]

\[ \rho_{\rho}' = k_0 \gamma_3 (ik_0 \nabla \cdot \mathbf{\eta}_{13} - i\omega_0 \mathbf{\eta}_{23} - k_0^2 \mathbf{\gamma}_{33} ) \]

and

\[ \varepsilon_{\omega} = \frac{iZ}{\rho} (ik_0 \nabla \cdot \mathbf{\eta}_{11} + i\omega_0 \mathbf{\eta}_{21} - k_0^2 \mathbf{\gamma}_{31} ) \]

\[ \varepsilon_{\omega}' = k_0 \gamma_1 (-ik_0 \nabla \cdot \mathbf{\eta}_{11} - i\omega_0 \mathbf{\eta}_{21} + k_0^2 \mathbf{\gamma}_{31} ) \]

\[ \mu_{\omega} = \frac{iZ}{\rho} (ik_0 \nabla \cdot \mathbf{\eta}_{12} - i\omega_0 \mathbf{\eta}_{22} - k_0^2 \mathbf{\gamma}_{32} ) \]

\[ \mu_{\omega}' = k_0 \gamma_2 (-ik_0 \nabla \cdot \mathbf{\eta}_{12} - i\omega_0 \mathbf{\eta}_{22} + k_0^2 \mathbf{\gamma}_{32} ) \]

\[ \rho_{\omega} = \frac{iZ}{\rho} (ik_0 \nabla \cdot \mathbf{\eta}_{13} - i\omega_0 \mathbf{\eta}_{23} - k_0^2 \mathbf{\gamma}_{33} ) \]

\[ \rho_{\omega}' = k_0 \gamma_3 (-ik_0 \nabla \cdot \mathbf{\eta}_{13} - i\omega_0 \mathbf{\eta}_{23} + k_0^2 \mathbf{\gamma}_{33} ) \]

and

\[ \epsilon_{z} = 0_{11} \]

\[ \epsilon_{z}' = 0 \]

\[ \mu_{z} = 0_{12} \]

\[ \mu_{z}' = 0 \]

\[ \rho_{z} = 0_{13} \]

\[ \rho_{z}' = 0 \]
and

\[ \epsilon^h_z = i \frac{e}{\rho (-i \omega_o \xi_0_{11} + i k o \mu \xi_{1021} + \omega_o k o \mu \xi_{1031})} \]

\[ \epsilon^h'_z = k o \gamma_1 (i \omega_o \xi_{011} + i k o \mu \eta_{021} - \omega_o k o \mu \xi_{1031}) \]

\[ \mu^h_z = i \frac{e}{\rho (-i \omega_o \xi_0_{12} + i k o \mu \xi_{1022} + \omega_o k o \mu \xi_{1032})} \]

\[ \mu^h'_z = k o \gamma_2 (i \omega_o \xi_{012} + i k o \mu \eta_{022} - \omega_o k o \mu \xi_{1032}) \]

\[ \rho^h_z = i \frac{e}{\rho (-i \omega_o \xi_0_{13} + i k o \mu \xi_{1023} + \omega_o k o \mu \xi_{1033})} \]

\[ \rho^h'_z = k o \gamma_3 (i \omega_o \xi_{013} + i k o \mu \eta_{023} - \omega_o k o \mu \xi_{1033}) \]

and

\[ \epsilon^h_{\bar{z}} = \frac{e}{\rho (-i \omega_o \xi_{011} + i k o \mu \eta_{021} - \omega_o k o \mu \xi_{1031})} \]

\[ \epsilon^h'_{\bar{z}} = k o \gamma_1 (i \omega_o \xi_{011} - i k o \mu \xi_{1021} - \omega_o k o \mu \xi_{1031}) \]

\[ \mu^h_{\bar{z}} = i \frac{e}{\rho (i \omega_o \xi_{012} + i k o \mu \eta_{022} - \omega_o k o \mu \xi_{1032})} \]

\[ \mu^h'_{\bar{z}} = k o \gamma_2 (i \omega_o \xi_{012} - i k o \mu \xi_{1022} - \omega_o k o \mu \xi_{1032}) \]

\[ \rho^h_{\bar{z}} = i \frac{e}{\rho (i \omega_o \xi_{013} + i k o \mu \eta_{023} - \omega_o k o \mu \xi_{1033})} \]

\[ \rho^h'_{\bar{z}} = k o \gamma_3 (i \omega_o \xi_{013} - i k o \mu \xi_{1023} - \omega_o k o \mu \xi_{1033}) \]

and

\[ \epsilon^h_z = 0_{21} \]

\[ \epsilon^h_z' = 0 \]

\[ \mu^h_z = 0_{22} \]

\[ \mu^h_z' = 0 \]

\[ \rho^h_z = 0_{23} \]

\[ \rho^h_z' = 0 \]
\[ e^p_z \equiv 0_{31} \]
\[ e^{p'}_z \equiv 0 \]
\[ \mu^p_z \equiv 0_{32} \]
\[ \mu^{p'}_z \equiv 0 \]
\[ \rho^p_z \equiv 0_{33} \]
\[ \rho^{p'}_z \equiv 0 \]
VI. Moving Media

The problem of determining how the field vectors behave in the presence of a moving medium is solved by studying certain aspects of the Special Theory of Relativity. Since the experimental basis and the development of this theory are described in detail in many places, only a brief summary of the key points needed in this study is presented.

A. Flat Spacetime

The following notation is used throughout the remainder of this section of the study. The time coordinate \( t \) is denoted by \( x^0 \) or \( x_0 \) and the space coordinate \( r \) is separated into the rectangular components \( x^1, x^2, \) and \( x^3 \) or \( x_1, x_2, \) and \( x_3. \) If the time coordinate is measured in the same unit as each of the space coordinates, then the mathematical expressions presented in the remainder of this section of the study are more symmetrical in form. This is accomplished by arbitrarily setting the speed of light \( c \) in vacuum equal to unity. This set of geometrized units is also used throughout the remainder of this section of the study.

An event is defined as a point in spacetime and is denoted by its contravariant components \( \tilde{x} = (x^0, x^1, x^2, x^3) \), or \( \tilde{x}; \) or by its covariant components \( \tilde{x} = (x^0, x_1, x_2, x_3) \), or \( \tilde{x} \). The contravariant and covariant components of an event in spacetime are related by the metric dyad, i.e.

\[
x^\mu = m_{\mu\nu} x^\nu
\]

or

\[
x_\mu = m_{\mu\nu} x^\nu.
\]

The metric dyad \( m \) of the flat spacetime of special relativity theory
is such that its components are

\[
\begin{bmatrix}
[\mathbf{w}^\mu_{\nu}] = [\mathbf{w}_{\mu\nu}] = \\
-1 & 0 \\
1 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

B. Lorentz Transformation

The Lorentz transformation relates the coordinates \( \mathbf{x}^\ell \) in the inertial reference frame S to the coordinates \( \mathbf{x'}^\ell \) in the inertial reference frame \( S' \). The Lorentz transformation is a consequence of the postulate that the speed of light in vacuum has the same value in all inertial reference frames. To derive the Lorentz transformation it is only necessary to assume that the transformation is linear. With this assumption, the Lorentz transformation of the coordinates between the system S and the system \( S' \) becomes

\[
\mathbf{x}^\mu = \Lambda^\mu_{\nu} \mathbf{x'}^\nu
\]

or

\[
\mathbf{x'}^\mu = \Lambda'^{\mu}_{\nu} \mathbf{x}^\nu
\]

where the transformation dyad \( \Lambda \) is such that its components are

\[
\begin{bmatrix}
[\Lambda^\mu_{\nu}] = \\
\frac{\gamma}{\beta \gamma} & \frac{\beta \gamma}{\beta \gamma} \\
\beta \gamma & \frac{1}{1 + \frac{\gamma}{\beta \gamma} (\gamma - 1)} \\
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
[\Lambda'^{\mu}_{\nu}] = \\
\frac{\gamma}{\beta \gamma} & \frac{-\beta \gamma}{\beta \gamma} \\
-\beta \gamma & \frac{1}{1 + \frac{\gamma}{\beta \gamma} (\gamma - 1)} \\
\end{bmatrix}
\]

where

\[
\beta = \frac{\gamma}{c} \leq 1
\]

\[
\gamma = \frac{1}{\sqrt{1 - \beta^2}} \geq 1
\]

and

\[
\gamma = \hat{\gamma}_{\hat{v}}
\]

where

\[
\hat{v} = \frac{\gamma}{\hat{v}}
\]

and

\[
\mathbf{v} = \hat{\mathbf{v}} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}
\]
v represents the velocity of the system S' relative to the system S.

In the previous notation
\[
t = \gamma (t' + \beta \gamma \cdot \mathbf{r}')
\]
\[
\mathbf{r} = \left[ 1 + (\gamma - 1)v^2 \right] \cdot \mathbf{r}' + \beta \gamma vt'
\]

If the coordinates undergo a proper Lorentz transformation, then a 4-vector is defined as a set of four quantities \( v^\mu \) that obey the transformation law
\[
v'^\nu = \Lambda^\mu_\nu v^\mu
\]
or
\[
v'^\nu = \Lambda^\mu_\nu v^\mu
\]

and a 4-dyad is defined as a set of \( 4^2 \) quantities \( d^{\mu\nu} \) that obey the transformation law
\[
d'^{\mu\nu} = \Lambda^\mu_\xi \Lambda^\nu_\eta d^{\xi\eta}
\]
or
\[
d'^{\mu\nu} = \Lambda^\mu_\xi \Lambda^\nu_\eta d^{\xi\eta}
\]

C. Maxwell's Equations

The electric field intensity \( \mathbf{E} \) and the magnetic flux density \( \mathbf{B} \) are written as the elements of an antisymmetric field dyad \( \mathfrak{F} \) such that the two homogeneous Maxwell equations
\[
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0
\]
\[
\nabla \cdot \mathbf{B} = 0
\]
are written in the covariant form
\[
\nabla \times F^{\mu\nu} = 0
\]
where
\[
\nabla = \frac{\partial}{\partial x^\nu}
\]

and the dual operation \( * \) is defined by
\[
*d^{\mu\nu} = \frac{1}{2!} \varepsilon^{\mu\nu\zeta\eta} d^{\zeta\eta}
\]
for any antisymmetric dyad $\underline{d}$, where the Schwartz-Kristoffel symbol is defined by

$$
\varepsilon^{\alpha\beta\gamma\delta} = \begin{cases} 
+1 & \text{if } \alpha^\beta\gamma\delta \text{ form an even permutation of 0123} \\
-1 & \text{if } \alpha^\beta\gamma\delta \text{ form an odd permutation of 0123} \\
0 & \text{otherwise}
\end{cases}
$$

Explicitly, the field dyad is such that its components are defined by

$$
\begin{bmatrix}
\varepsilon^x & \varepsilon^y & \varepsilon^z \\
-\varepsilon^x & 0 & -\varepsilon^y \\
-\varepsilon^y & 0 & \varepsilon^x \\
-\varepsilon^z & -\varepsilon^y & 0 \\
\end{bmatrix}
$$

Similarly, the electric flux density $\underline{E}$ and the magnetic field vector intensity $\underline{H}$ are written as the elements of an antisymmetric field dyad $\underline{\gamma}$ such that the two inhomogeneous Maxwell equations

$$
\nabla \cdot \underline{E} = \rho_0 \\
\nabla \times \underline{H} - \frac{\partial \underline{E}}{\partial t} = \underline{j}_0
$$

are written in the covariant form

$$
\nabla \underline{j}^\mu = \rho_0
$$

where the source charge-density 4-vector $\underline{j}_0^\mu$ is defined as

$$
\underline{j}_0 = (\rho_0, \underline{\rho}_0)
$$

Explicitly, the field dyad is such that its components are defined by

$$
\begin{bmatrix}
\kappa^x & \kappa^y & \kappa^z \\
-\kappa^x & 0 & -\kappa^y \\
-\kappa^y & 0 & \kappa^x \\
-\kappa^z & -\kappa^y & 0 \\
\end{bmatrix}
$$
D. Transformations

Since the fields $\mathcal{E}$ and $\mathcal{B}$ are elements of the field dyad $\mathcal{F}^{\mu\nu}$, their transformation properties are determined by

$$\mathcal{F}^{\mu\nu} = \Lambda^{\mu^\prime}_{\mu} \Lambda^{\nu^\prime}_{\nu} \mathcal{F}^{\mu^\prime\nu^\prime}$$

With the transformation dyad $\Lambda$ from the system $S'$ to the system $S$, the above equation gives the transformed fields

$$\mathcal{E} = \{\gamma \mathcal{E} + (1 - \gamma) \mathcal{E}'\} \cdot \mathcal{B}^\prime - \beta \mathcal{E}^\prime \cdot \mathcal{E}^\prime$$

$$\mathcal{B} = \{\gamma \mathcal{B} + (1 - \gamma) \mathcal{B}'\} \cdot \mathcal{E}^\prime + \beta \mathcal{E}^\prime \cdot \mathcal{E}^\prime$$

Similarly, since the fields $\mathcal{A}$ and $\mathcal{H}$ are elements of the field dyad $\mathcal{F}^{\mu\nu}$, their transformation properties are determined by

$$\mathcal{F}^{\mu\nu} = \Lambda^{\mu^\prime}_{\mu} \Lambda^{\nu^\prime}_{\nu} \mathcal{F}^{\mu^\prime\nu^\prime}$$

With the transformation dyad $\Lambda$ from the system $S'$ to the system $S$, the above equation gives the transformed fields

$$\mathcal{A} = \{\gamma \mathcal{A} + (1 - \gamma) \mathcal{A}'\} \cdot \mathcal{H}^\prime - \beta \mathcal{A}^\prime \cdot \mathcal{H}^\prime$$

$$\mathcal{H} = \{\gamma \mathcal{H} + (1 - \gamma) \mathcal{H}'\} \cdot \mathcal{A}^\prime + \beta \mathcal{A}^\prime \cdot \mathcal{A}^\prime$$

The phase $\Phi$ of a wave in the system $S$ is defined by

$$\Phi \equiv \kappa \cdot \tau \equiv k_0 \cdot \kappa \cdot \tau - \omega t$$

Since the frequency $\omega$ and the wave 3-vector $k_0 \cdot \kappa$ are the elements of a wave 4-vector $k$, their properties are determined by

$$k^{\mu} = \Lambda^{\mu^\prime}_{\mu} k^{\nu^\prime}$$

With the transform dyad $\Lambda$ from the system $S'$ to the system $S$, the above equation gives the transformed variables

*The dyad $\mathcal{E}$ is now referenced to the direction of $\mathcal{B}$, i.e.

$$\mathcal{E} = -k^\mu \mathcal{B}$$
\( \omega = \gamma (\omega' + k'_{o} \cdot \hat{v}) \)

\( k'_{o} = \left[ i + \left( \frac{\omega}{c} - 1 \right) \frac{\delta}{\omega} \right] \cdot k'_{o} + \beta'_{v} \omega' \)

From the invariance of the phase of a wave from the system \( S' \) to the system \( S \),

\( k'_{o} \cdot \hat{r} - \omega t = \phi = k'_{o} \cdot \hat{r}' - \omega' t' \)

These results also hold in terms of the transformed variables \( \omega \) and \( k'_{o} \).

Unlike the wave frequency, the natural frequencies of the plasma, viz. the plasma frequency, the gyrotropic frequency, and the collision frequency, do not have a wave number associated with them; therefore, they do not have the property of phase invariance. The relationships between the various natural frequencies are

\( \omega'_{p} = \frac{1}{\gamma} \omega_{p} \) (plasma frequency)

\( \omega'_{c} = \gamma \omega_{c} \) (collision frequency)

and

\( \omega'_{g} = \left( \gamma i + (1-\gamma) \frac{\delta}{\omega} \right) \cdot \omega_{g} \) (gyrotropic frequency)

Also

\( a' = \gamma a \) (acoustical velocity)

and

\( n' = \frac{1}{\gamma} n \) (number density)

\( p' = p \) (pressure)

\( T' = \gamma T \) (temperature)

E. Axial Motion

For the special case of relative motion in the \( z \) direction, i.e.,

\( \hat{\beta} = \hat{z} \)

and

\( \beta'_{z} = \frac{\nu_{z}}{c} \)

\( \gamma'_{z} = \frac{1}{\sqrt{1 - \beta'_{z}}} \)
the transformed coordinates reduce to the expressions

\[ t = \gamma^2 (t' + \beta^2 z') \]
\[ z = \gamma^2 (z' + \beta^2 t') \]

and

\[ \tau = \tau' \]

The equations for the transformed fields reduce to the expressions

\[ \varepsilon_z = \varepsilon_z' \]
\[ \varepsilon_z = \gamma (\varepsilon_z' - \varepsilon c \cdot \varepsilon t') \]
\[ \beta_z = \beta_z' \]
\[ \beta_z = \gamma (\beta_z' - \varepsilon c \cdot \varepsilon t') \]
\[ \varrho_z = \varrho_z' \]
\[ \varrho_z = \gamma (\varrho_z' - \varepsilon c \cdot \varepsilon t') \]
\[ \mu_z = \mu_z' \]
\[ \mu_z = \gamma (\mu_z' - \varepsilon c \cdot \varepsilon t') \]

Similarly, the expressions for \( \omega \) and \( k_o \kappa \) reduce to

\[ \omega = \gamma (\omega' + \beta \kappa_z' \omega') \]
\[ k_o \kappa = \gamma (k_o \kappa' + \beta \omega') \]

and

\[ k_o \kappa = k_o \kappa' \]

also

\[ k_o \kappa_z - \omega t = 0 = k_o \kappa_z' - \omega' t' \]

F. Constitutive Relations

In the system \( S' \) the constitutive relations in an anistropic plasma in the frequency domain are

\[ \begin{align*}
D' &= \varepsilon_{\omega c} \mathbf{E}' = \varepsilon_0 \left[ (c_1^t + i c_2^t \varepsilon) \cdot \mathbf{E}' + \hat{\varepsilon} \mathbf{E}'^t \right] \\
B' &= \mu_{\omega c} \mathbf{H}' = \mu_0 H'
\end{align*} \]
When the primed field variables in the system $s'$ are expressed in terms of the unprimed field variables in the system $s$, the constitutive relations become

\[
\begin{align*}
\gamma_z(D^z + \beta_z c \cdot H^t) + \vec{z}D^z &= \varepsilon_o(c'_t + i\zeta'_t) \cdot (E^t + \beta_z c \cdot B^t) + \hat{z}\varepsilon_o c'_t E^z \\
\gamma_z(B^t - \beta_z c \cdot E^t) + \vec{z}B^z &= \mu_o \gamma_z(H^t - \beta_z c \cdot D^t) + \hat{z}\mu_o H^z
\end{align*}
\]

The axial and transverse components of these equations are separated as follows:

\[
\begin{align*}
D^z &= \frac{1}{\mu_o} B^z \\
H^t &= \frac{1}{\mu_o} B^z \\
\begin{align*}
D^t + \beta_z c \cdot H^t &= \gamma_o(c'_t + i\zeta'_t) \cdot (E^t + \beta_z c \cdot B^t) \\
B^t - \beta_z c \cdot E^t &= \mu_o(H^t - \beta_z c \cdot D^t)
\end{align*}
\]  

After some rearrangement the transverse equations become

\[
\begin{align*}
\varepsilon_o(c'_t + i\zeta'_t) \cdot (E^t + \beta_z c \cdot B^t) = -\beta_z c \cdot D^t + \varepsilon_o(\zeta'_t \cdot c - i\zeta'_t \cdot c - i\zeta'_t \cdot B^t)
\end{align*}
\]

It is possible to solve these equations simultaneously for the transverse field vectors $D^t$ and $H^t$ in terms of the transverse field vectors $E^t$ and $B^t$. The constitutive relations in the system $s$ then become

\[
\begin{align*}
D^t &= \gamma_o^2 B^t \left[ \left( \frac{1}{\beta_z} \varepsilon_o c'_t - \beta_z \frac{1}{\mu_o} \right) \varepsilon_o \varepsilon_o c'_t \right] \cdot E^t \\
+ \gamma_o^2 B^t \left[ \left( \frac{1}{\beta_z} \varepsilon_o c'_t - \frac{1}{\mu_o} \right) \varepsilon_o \varepsilon_o c'_t \right] \cdot B^t
\end{align*}
\]
\[
\begin{align*}
H^t &= \gamma^2 \beta \left[ \left( \frac{1}{\beta} \mu_0 \rho_0 \right) \mathbf{e}_z + \beta \rho_0 \mathbf{e}_\rho \right] t - \beta \mathbf{e}_z \mathbf{e}_z \mathbf{e}_z + \mathbf{E}^t \\
+ \gamma^2 \beta \left[ \left( \rho_0 \mathbf{e}_z - \frac{1}{\mu_0} \right) \mathbf{e}_\rho \mathbf{e}_\rho \mathbf{e}_\rho + \mathbf{E}^t \right]
\end{align*}
\]

Therefore, the cylindrical components of the field vectors in the wave number domain are

\[
\begin{align*}
\mathbf{e}_\rho &= \gamma^2 \left( \rho_0 \mathbf{e}_\rho + \mu_0 \beta \rho_0 \mathbf{h}^* \right) \\
\mathbf{e}_\varphi &= \gamma^2 \left( \mathbf{e}_\varphi - \mu_0 \beta \rho_0 \mathbf{h}^* \right) \\
\mathbf{e}_z &= \mathbf{e}_z^* \\
\text{and} \\
\mathbf{h}_\rho &= \gamma^2 \left[ \mathbf{h}_\rho^* - \rho_0 \beta \rho_0 \left( \mathbf{e}_\rho^* \mathbf{e}_\rho^* + \mathbf{i} \mathbf{e}_\rho^* \mathbf{e}_\rho^* \right) \right] \\
\mathbf{h}_\varphi &= \gamma^2 \left[ \mathbf{h}_\varphi^* + \rho_0 \beta \rho_0 \left( \mathbf{e}_\varphi^* - \mathbf{i} \mathbf{e}_\varphi^* \mathbf{e}_\varphi^* \right) \right] \\
\mathbf{h}_z &= \mathbf{h}_z^*
\end{align*}
\]

Similarly, in terms of the transformed variables,

\[
\begin{align*}
\mathbf{e}_\rho^* &= \gamma^2 \left( \mathbf{e}_\rho^* + \mu_0 \beta \rho_0 \mathbf{h}^* \right) \\
\mathbf{e}_\varphi^* &= \gamma^2 \left( \mathbf{e}_\varphi^* - \mu_0 \beta \rho_0 \mathbf{h}^* \right) \\
\mathbf{e}_z^* &= \mathbf{e}_z^* \\
\text{and} \\
\mathbf{h}_\rho^* &= \gamma^2 \left[ \mathbf{h}_\rho^* - \rho_0 \beta \rho_0 \left( \mathbf{e}_\rho^* \mathbf{e}_\rho^* + \mathbf{i} \mathbf{e}_\rho^* \mathbf{e}_\rho^* \right) \right] \\
\mathbf{h}_\varphi^* &= \gamma^2 \left[ \mathbf{h}_\varphi^* + \rho_0 \beta \rho_0 \left( \mathbf{e}_\varphi^* - \mathbf{i} \mathbf{e}_\varphi^* \mathbf{e}_\varphi^* \right) \right] \\
\mathbf{h}_z^* &= \mathbf{h}_z^*
\end{align*}
\]

also

\[p = p^*\]
VII. Rest Frame

The results derived in the previous section on the special theory of relativity are used to determine the fields in the presence of the moving plasma flow field.

The inertial rest frame $S$ is taken to be at rest with respect to the antenna. Therefore, the region 0, the recirculation region inside the plasma shell, has a zero velocity. Also, the region $\infty$, the undisturbed atmospheric region outside the plasma shell, is assumed to be so tenuous as to have no motional effects on the radiation; therefore, its velocity is effectively zero.

In the $i^{th}$ cylindrical layer of the plasma shell ($1 \leq i \leq n$), an inertial rest frame $S_i$ is taken to be at rest with respect to the plasma in that layer. Notice that the velocities of the adjacent layers need not be the same.

In what follows, let a subscript 0 or $\infty$ denote that the particular variable so subscripted is to be evaluated with the given parameters of the region 0 or $\infty$, respectively. Also, let a subscript $i$ and a superscript $'$ denote that the particular variable so indicated is to be evaluated with the given parameters in the rest frame $S_i$ of the $i^{th}$ layer of the plasma shell.

A. Region 0

Since the region 0 contains the origin, the linear combination

$$p_{z_0} (\pm) Z_{\pm} (k_o \chi_o^t o)$$

becomes

$$p_{z_0} J_{\pm} (k_o \chi_o^t o)$$

and the linear combination

$$q_{z_0} (\pm) Z_{\pm} (k_o \chi_o^t o)$$

becomes

$$q_{z_0} J_{\pm} (k_o \chi_o^t o)$$
where \( \rho_0 \) and \( q_0 \) are arbitrary constants to be determined later in the study.

In the rest frame of the antenna in the region 0, the cylindrical components of the field vectors in the wave number domain are

\[
\begin{align*}
e^p_{\ell_0} &= ik_0^2 \frac{z}{\rho} \rho \phi J^t(k_0^t \rho) + i \omega_0 \eta_{o} \frac{z}{\rho} \rho \phi J^t(k_0^t \rho) + e^p_{\ell_0} \\
e^\varphi_{\ell_0} &= ik_0^2 \frac{z}{\rho} \rho \phi J^t(k_0^t \rho) - i \omega_0 \eta_{o} k_0^t \rho \phi J^t(k_0^t \rho) + e^\varphi_{\ell_0} \\
e^z_{\ell_0} &= k_0^2 t^2 \rho \phi J^t(k_0^t \rho) + e^z_{\ell_0}
\end{align*}
\]

and

\[
\begin{align*}
h^p_{\ell_0} &= ik_0^2 \frac{z}{\rho} \rho \phi J^t(k_0^t \rho) - i \omega_0 \eta_{o} \frac{z}{\rho} \rho \phi J^t(k_0^t \rho) + \eta^p_{\ell_0} \\
h^\varphi_{\ell_0} &= ik_0^2 \frac{z}{\rho} \rho \phi J^t(k_0^t \rho) + i \omega_0 \eta_{o} k_0^t \rho \phi J^t(k_0^t \rho) + \eta^\varphi_{\ell_0} \\
h^z_{\ell_0} &= k_0^2 t^2 \rho \phi J^t(k_0^t \rho) + \eta^z_{\ell_0}
\end{align*}
\]

where

\[
\begin{align*}
e^p_{\ell_0} &= c \left\{ \left[ \left( \frac{2}{\rho^2} + k_0^2 z^2 \right) \rho^2 + \frac{1}{\rho} \rho^2 \right] J^t(k_0^t \rho) \right. \\
& \quad \left. - \rho \right\} \left. \frac{\partial}{\partial \rho} \right\} \left[ J^t(k_0^t \rho) \right] \\
& \quad + \left( \frac{1}{\rho} \rho^2 + 1 \right) \frac{\partial}{\partial \rho} \left[ J^t(k_0^t \rho) \right] \\
& + \left. \frac{1}{\rho} \rho^2 \frac{\partial}{\partial \rho} \right\} \left[ J^t(k_0^t \rho) \right]
\end{align*}
\]

\[
\begin{align*}
e^\varphi_{\ell_0} &= c \left\{ \left[ - \frac{1}{\rho} \rho^2 + \left( \frac{1}{\rho} \rho^2 + k_0^2 z^2 \right) \rho^2 + \frac{1}{\rho} \rho^2 \right] \rho^2 \right. \\
& \quad \left. - \rho \right\} \left. \frac{\partial}{\partial \rho} \right\} \left[ J^t(k_0^t \rho) \right] \\
& + \left( \frac{1}{\rho} \rho^2 + 1 \right) \frac{\partial}{\partial \rho} \left[ J^t(k_0^t \rho) \right] \\
& + \left. \frac{1}{\rho} \rho^2 \frac{\partial}{\partial \rho} \right\} \left[ J^t(k_0^t \rho) \right]
\end{align*}
\]

\[
\begin{align*}
e^z_{\ell_0} &= c \left\{ \left[ ik_0^2 \frac{1}{\rho} \rho^2 + ik_0^2 \frac{1}{\rho} \rho^2 \right] \rho^2 + k_0^2 t^2 \rho^2 \right. \\
& \quad \left. - \rho \right\} \left. \frac{\partial}{\partial \rho} \right\} \left[ J^t(k_0^t \rho) \right] \\
& + \left( \frac{1}{\rho} \rho^2 + 1 \right) \frac{\partial}{\partial \rho} \left[ J^t(k_0^t \rho) \right]
\end{align*}
\]
and
\[
\eta_{lo} = i \omega e_{o} \epsilon_{o} c_{l} \zeta_{l} \left( ik_{o}^{t} z d \varphi - \frac{1}{\varphi} d^{2} \right) J_{l} \left( k_{o}^{t} \varphi \rho_{o} \right) H_{l}^{(1)} \left( k_{o}^{t} \rho_{o} \right)
\]
\[
\eta_{lo}^{p} = i \omega e_{o} \epsilon_{o} c_{l} \zeta_{l} \left( -ik_{o}^{t} z d \varphi \right) J_{l} \left( k_{o}^{t} \varphi \rho_{o} \right) H_{l}^{(1)} \left( k_{o}^{t} \rho_{o} \right)
\]
\[
+ \left( d^{2} \right) \frac{\partial}{\partial \rho} \left[ J_{l} \left( k_{o}^{t} \varphi \rho_{o} \right) H_{l}^{(1)} \left( k_{o}^{t} \rho_{o} \right) \right]
\]
\[
\eta_{lo}^{z} = i \omega e_{o} \epsilon_{o} c_{l} \zeta_{l} \left[ \left( \frac{1}{\rho} \rho d \varphi - \frac{1}{\varphi} d^{2} \right) J_{l} \left( k_{o}^{t} \varphi \rho_{o} \right) H_{l}^{(1)} \left( k_{o}^{t} \rho_{o} \right) \right]
\]
\[
\left[ \frac{-d^{2}}{\rho} \right] \frac{\partial}{\partial \varphi} \left[ J_{l} \left( k_{o}^{t} \varphi \rho_{o} \right) H_{l}^{(1)} \left( k_{o}^{t} \rho_{o} \right) \right]
\]
also
\[
c_{l} = -i \frac{1}{4i \sqrt{2 \pi}} \frac{i s s}{\omega e_{o} \epsilon_{o}} e^{i \varphi - \lambda_{o}^{t} z_{o} e^{i \omega s}}
\]

B. Region i

In the rest frame of the plasma in the region i (1 ≤ i ≤ n), the linear combinations
\[
e_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + e_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-} + \epsilon_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + \epsilon_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-}
\]
\[
\mu_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + \mu_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-} + \mu_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + \mu_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-}
\]
\[
\rho_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + \rho_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-} + \rho_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + \rho_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-}
\]
are explicitly chosen to be
\[
e_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + e_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-} + e_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + e_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-}
\]
\[
\mu_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + \mu_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-} + \mu_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + \mu_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-}
\]
\[
\rho_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + \rho_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-} + \rho_{l}^{+} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{+} + \rho_{l}^{-} \left( k_{o}^{t} \varphi \rho_{o} \right)_{l}^{-}
\]
where \( \epsilon_{l}^{+}, \mu_{l}^{+}, \rho_{l}^{+} \) are arbitrary constants to be determined later in the study.

In the rest frame of the antenna in the region i (1 ≤ i ≤ n), the cylindrical components of the field vectors in the wave number domain are
\[ e_{\xi} = e_{\xi} (J) e_{\xi} + e_{\xi} (N) e_{\xi} + \tilde{e}_{\xi} (J) e_{\xi} + \tilde{e}_{\xi} (N) e_{\xi} + \tilde{\mu}_{\xi} (J) \mu_{\xi} + \tilde{\mu}_{\xi} (N) \mu_{\xi} + \tilde{\rho}_{\xi} (J) \rho_{\xi} + \tilde{\rho}_{\xi} (N) \rho_{\xi} \]

\[ e_{\varphi} = e_{\varphi} (J) e_{\varphi} + e_{\varphi} (N) e_{\varphi} + \tilde{e}_{\varphi} (J) e_{\varphi} + \tilde{e}_{\varphi} (N) e_{\varphi} + \tilde{\mu}_{\varphi} (J) \mu_{\varphi} + \tilde{\mu}_{\varphi} (N) \mu_{\varphi} + \tilde{\rho}_{\varphi} (J) \rho_{\varphi} + \tilde{\rho}_{\varphi} (N) \rho_{\varphi} \]

\[ e_{\pi} = e_{\pi} (J) e_{\pi} + e_{\pi} (N) e_{\pi} + \tilde{e}_{\pi} (J) e_{\pi} + \tilde{e}_{\pi} (N) e_{\pi} + \tilde{\mu}_{\pi} (J) \mu_{\pi} + \tilde{\mu}_{\pi} (N) \mu_{\pi} + \tilde{\rho}_{\pi} (J) \rho_{\pi} + \tilde{\rho}_{\pi} (N) \rho_{\pi} \]

\[ e_{\pi} (J) e_{\pi} + e_{\pi} (N) e_{\pi} + \tilde{e}_{\pi} (J) e_{\pi} + \tilde{e}_{\pi} (N) e_{\pi} + \tilde{\mu}_{\pi} (J) \mu_{\pi} + \tilde{\mu}_{\pi} (N) \mu_{\pi} + \tilde{\rho}_{\pi} (J) \rho_{\pi} + \tilde{\rho}_{\pi} (N) \rho_{\pi} \]

\[ + e_{\pi} (J) e_{\pi} + e_{\pi} (N) e_{\pi} + \tilde{e}_{\pi} (J) e_{\pi} + \tilde{e}_{\pi} (N) e_{\pi} + \tilde{\mu}_{\pi} (J) \mu_{\pi} + \tilde{\mu}_{\pi} (N) \mu_{\pi} + \tilde{\rho}_{\pi} (J) \rho_{\pi} + \tilde{\rho}_{\pi} (N) \rho_{\pi} \]

and

\[ h_{\xi} = h_{\xi} (J) h_{\xi} + h_{\xi} (N) h_{\xi} + \tilde{h}_{\xi} (J) h_{\xi} + \tilde{h}_{\xi} (N) h_{\xi} + \tilde{\mu}_{\xi} (J) \mu_{\xi} + \tilde{\mu}_{\xi} (N) \mu_{\xi} + \tilde{\rho}_{\xi} (J) \rho_{\xi} + \tilde{\rho}_{\xi} (N) \rho_{\xi} \]

\[ h_{\varphi} = h_{\varphi} (J) h_{\varphi} + h_{\varphi} (N) h_{\varphi} + \tilde{h}_{\varphi} (J) h_{\varphi} + \tilde{h}_{\varphi} (N) h_{\varphi} + \tilde{\mu}_{\varphi} (J) \mu_{\varphi} + \tilde{\mu}_{\varphi} (N) \mu_{\varphi} + \tilde{\rho}_{\varphi} (J) \rho_{\varphi} + \tilde{\rho}_{\varphi} (N) \rho_{\varphi} \]

\[ h_{\pi} = h_{\pi} (J) h_{\pi} + h_{\pi} (N) h_{\pi} + \tilde{h}_{\pi} (J) h_{\pi} + \tilde{h}_{\pi} (N) h_{\pi} + \tilde{\mu}_{\pi} (J) \mu_{\pi} + \tilde{\mu}_{\pi} (N) \mu_{\pi} + \tilde{\rho}_{\pi} (J) \rho_{\pi} + \tilde{\rho}_{\pi} (N) \rho_{\pi} \]

and

\[ \rho_{\xi} = \rho_{\xi} (J) \rho_{\xi} + \rho_{\xi} (N) \rho_{\xi} + \tilde{\rho}_{\xi} (J) \rho_{\xi} + \tilde{\rho}_{\xi} (N) \rho_{\xi} + \tilde{\mu}_{\xi} (J) \mu_{\xi} + \tilde{\mu}_{\xi} (N) \mu_{\xi} + \tilde{\rho}_{\xi} (J) \rho_{\xi} + \tilde{\rho}_{\xi} (N) \rho_{\xi} \]

where

\[ \tilde{\varepsilon}_{\xi} (Z) = \varepsilon_{\xi} \tilde{Z} (k_{o} y_{1i} \rho) + \varepsilon_{\xi} \tilde{Z} (k_{o} y_{1i} \rho) \]

\[ \tilde{\mu}_{\xi} (Z) = \mu_{\xi} \tilde{Z} (k_{o} y_{1i} \rho) + \mu_{\xi} \tilde{Z} (k_{o} y_{1i} \rho) \]

\[ \tilde{\rho}_{\xi} (Z) = \rho_{\xi} \tilde{Z} (k_{o} y_{1i} \rho) + \rho_{\xi} \tilde{Z} (k_{o} y_{1i} \rho) \]

\[ \tilde{\varepsilon}_{\varphi} (Z) = \varepsilon_{\varphi} \tilde{Z} (k_{o} y_{1i} \rho) + \varepsilon_{\varphi} \tilde{Z} (k_{o} y_{1i} \rho) \]

\[ \tilde{\mu}_{\varphi} (Z) = \mu_{\varphi} \tilde{Z} (k_{o} y_{1i} \rho) + \mu_{\varphi} \tilde{Z} (k_{o} y_{1i} \rho) \]

\[ \tilde{\rho}_{\varphi} (Z) = \rho_{\varphi} \tilde{Z} (k_{o} y_{1i} \rho) + \rho_{\varphi} \tilde{Z} (k_{o} y_{1i} \rho) \]

\[ \tilde{\varepsilon}_{\pi} (Z) = \varepsilon_{\pi} \tilde{Z} (k_{o} y_{1i} \rho) + \varepsilon_{\pi} \tilde{Z} (k_{o} y_{1i} \rho) \]

\[ \tilde{\mu}_{\pi} (Z) = \mu_{\pi} \tilde{Z} (k_{o} y_{1i} \rho) + \mu_{\pi} \tilde{Z} (k_{o} y_{1i} \rho) \]

\[ \tilde{\rho}_{\pi} (Z) = \rho_{\pi} \tilde{Z} (k_{o} y_{1i} \rho) + \rho_{\pi} \tilde{Z} (k_{o} y_{1i} \rho) \]
\[ e_{t1}^z(z) = e_{t1}^z(k_o \gamma_{t1}^p) + e_{t1}^z(k_o \gamma_{t1}^p) \]
\[ \mu_{t1}^z(z) = \mu_{t1}^z(k_o \gamma_{t1}^p) + \mu_{t1}^z(k_o \gamma_{t1}^p) \]
\[ \rho_{t1}^z(z) = \rho_{t1}^z(k_o \gamma_{t1}^p) + \rho_{t1}^z(k_o \gamma_{t1}^p) \]

and
\[ h^0_{t1}(z) = h^0_{t1}(k_o \gamma_{t1}^p) + h^0_{t1}(k_o \gamma_{t1}^p) \]
\[ \mu_{t1}^{h0}(z) = \mu_{t1}^{h0}(k_o \gamma_{t1}^p) + \mu_{t1}^{h0}(k_o \gamma_{t1}^p) \]
\[ \rho_{t1}^{h0}(z) = \rho_{t1}^{h0}(k_o \gamma_{t1}^p) + \rho_{t1}^{h0}(k_o \gamma_{t1}^p) \]
\[ h^0_{t1}(z) = h^0_{t1}(k_o \gamma_{t1}^p) + h^0_{t1}(k_o \gamma_{t1}^p) \]
\[ \mu_{t1}^{h0}(z) = \mu_{t1}^{h0}(k_o \gamma_{t1}^p) + \mu_{t1}^{h0}(k_o \gamma_{t1}^p) \]
\[ \rho_{t1}^{h0}(z) = \rho_{t1}^{h0}(k_o \gamma_{t1}^p) + \rho_{t1}^{h0}(k_o \gamma_{t1}^p) \]
\[ h^0_{t1}(z) = h^0_{t1}(k_o \gamma_{t1}^p) + h^0_{t1}(k_o \gamma_{t1}^p) \]
\[ \mu_{t1}^{h0}(z) = \mu_{t1}^{h0}(k_o \gamma_{t1}^p) + \mu_{t1}^{h0}(k_o \gamma_{t1}^p) \]
\[ \rho_{t1}^{h0}(z) = \rho_{t1}^{h0}(k_o \gamma_{t1}^p) + \rho_{t1}^{h0}(k_o \gamma_{t1}^p) \]

and
\[ p_{t1}^z(z) = p_{t1}^z(k_o \gamma_{t1}^p) + p_{t1}^z(k_o \gamma_{t1}^p) \]
\[ p_{t1}^{h0}(z) = p_{t1}^{h0}(k_o \gamma_{t1}^p) + p_{t1}^{h0}(k_o \gamma_{t1}^p) \]
\[ p_{t1}^{h0}(z) = p_{t1}^{h0}(k_o \gamma_{t1}^p) + p_{t1}^{h0}(k_o \gamma_{t1}^p) \]
\[ p_{t1}^{h0}(z) = p_{t1}^{h0}(k_o \gamma_{t1}^p) + p_{t1}^{h0}(k_o \gamma_{t1}^p) \]

where
\[ Z = \sum_{j=0}^{N} z_j \]
Also,
\[ \varepsilon_{e,\beta}^{\mu} = \gamma_i (\varepsilon_{e,\beta} + \mu_{e,\beta}^{\mu} h_{\beta}^{\mu}) \]
\[ \varepsilon_{e,\beta}^{\mu'} = \gamma_i (\varepsilon_{e,\beta}^{\mu'} + \mu_{e,\beta}^{\mu'} h_{\beta}^{\mu'}) \]
\[ \mu_{e,\beta}^{\mu} = \gamma_i (\mu_{e,\beta}^{\mu} + \mu_{e,\beta}^{\mu} h_{\beta}^{\mu}) \]
\[ \mu_{e,\beta}^{\mu'} = \gamma_i (\mu_{e,\beta}^{\mu'} + \mu_{e,\beta}^{\mu'} h_{\beta}^{\mu'}) \]

and
\[ \varepsilon_{e,\beta}^{\rho} = \gamma_i (\varepsilon_{e,\beta}^{\rho} - \mu_{e,\beta}^{\rho} h_{\beta}^{\rho}) \]
\[ \varepsilon_{e,\beta}^{\rho'} = \gamma_i (\varepsilon_{e,\beta}^{\rho'} - \mu_{e,\beta}^{\rho'} h_{\beta}^{\rho'}) \]
\[ \mu_{e,\beta}^{\rho} = \gamma_i (\mu_{e,\beta}^{\rho} - \mu_{e,\beta}^{\rho} h_{\beta}^{\rho}) \]
\[ \mu_{e,\beta}^{\rho'} = \gamma_i (\mu_{e,\beta}^{\rho'} - \mu_{e,\beta}^{\rho'} h_{\beta}^{\rho'}) \]

and
\[ \varepsilon_{e,\beta}^{z} = \gamma_i (\varepsilon_{e,\beta}^{z} + \mu_{e,\beta}^{z} h_{\beta}^{z}) \]
\[ \varepsilon_{e,\beta}^{z'} = \gamma_i (\varepsilon_{e,\beta}^{z'} + \mu_{e,\beta}^{z'} h_{\beta}^{z'}) \]
\[ \mu_{e,\beta}^{z} = \gamma_i (\mu_{e,\beta}^{z} + \mu_{e,\beta}^{z} h_{\beta}^{z}) \]
\[ \mu_{e,\beta}^{z'} = \gamma_i (\mu_{e,\beta}^{z'} + \mu_{e,\beta}^{z'} h_{\beta}^{z'}) \]
Also,

\[ -h_{\xi}^{\phi} = \gamma_{1}[e_{\xi}^{h} - e_0^{\beta}(\zeta_{\xi}^{\phi} - i\zeta_{\xi}^{\phi})] \]

\[ -h_{\xi}^{\rho} = \gamma_{1}[e_{\xi}^{h} - e_0^{\beta}(\zeta_{\xi}^{\rho} - i\zeta_{\xi}^{\rho})] \]

\[ -h_{\mu}^{\phi} = \gamma_{1}[e_{\mu}^{h} - e_0^{\beta}(\zeta_{\mu}^{\phi} - i\zeta_{\mu}^{\phi})] \]

\[ -h_{\mu}^{\rho} = \gamma_{1}[e_{\mu}^{h} - e_0^{\beta}(\zeta_{\mu}^{\rho} - i\zeta_{\mu}^{\rho})] \]

\[ -h_{\rho}^{\phi} = \gamma_{1}[e_{\rho}^{h} - e_0^{\beta}(\zeta_{\rho}^{\phi} - i\zeta_{\rho}^{\phi})] \]

\[ -h_{\rho}^{\rho} = \gamma_{1}[e_{\rho}^{h} - e_0^{\beta}(\zeta_{\rho}^{\rho} - i\zeta_{\rho}^{\rho})] \]

and

\[ -h_{\xi}^{\phi} = \gamma_{1}[e_{\xi}^{h} - e_0^{\beta}(\zeta_{\xi}^{\phi} + i\zeta_{\xi}^{\phi})] \]

\[ -h_{\xi}^{\rho} = \gamma_{1}[e_{\xi}^{h} - e_0^{\beta}(\zeta_{\xi}^{\rho} + i\zeta_{\xi}^{\rho})] \]

\[ -h_{\mu}^{\phi} = \gamma_{1}[e_{\mu}^{h} - e_0^{\beta}(\zeta_{\mu}^{\phi} + i\zeta_{\mu}^{\phi})] \]

\[ -h_{\mu}^{\rho} = \gamma_{1}[e_{\mu}^{h} - e_0^{\beta}(\zeta_{\mu}^{\rho} + i\zeta_{\mu}^{\rho})] \]

\[ -h_{\rho}^{\phi} = \gamma_{1}[e_{\rho}^{h} - e_0^{\beta}(\zeta_{\rho}^{\phi} + i\zeta_{\rho}^{\phi})] \]

\[ -h_{\rho}^{\rho} = \gamma_{1}[e_{\rho}^{h} - e_0^{\beta}(\zeta_{\rho}^{\rho} + i\zeta_{\rho}^{\rho})] \]

and

\[ -h_{\xi}^{z} = e_{\xi}^{z} \]

\[ -h_{\xi}^{z'} = e_{\xi}^{z'} \]

\[ -h_{\mu}^{z} = e_{\mu}^{z} \]

\[ -h_{\mu}^{z'} = e_{\mu}^{z'} \]

\[ -h_{\rho}^{z} = e_{\rho}^{z} \]

\[ -h_{\rho}^{z'} = e_{\rho}^{z'} \]
where

\[ e_{\xi i} = \frac{iL}{\rho} (ik'z'\xi_{i1} + iw'\mu_o \eta_{1i} - k_o^2 \xi_{i1}) \]
\[ e_{\xi i} = k'v_{i1} (ik'z'\xi_{i1} - iw'\mu_o \eta_{1i} - k_o^2 \xi_{i1}) \]
\[ \mu_{\xi i} = \frac{iL}{\rho} (ik'z'\xi_{i1} + iw'\mu_o \eta_{1i} - k_o^2 \xi_{i1}) \]
\[ \mu_{\xi i} = k'v_{i1} (ik'z'\xi_{i1} - iw'\mu_o \eta_{1i} - k_o^2 \xi_{i1}) \]
\[ \rho_{\xi i} = \frac{iL}{\rho} (ik'z'\xi_{i1} + iw'\mu_o \eta_{1i} - k_o^2 \xi_{i1}) \]
\[ \rho_{\xi i} = k'v_{i1} (ik'z'\xi_{i1} - iw'\mu_o \eta_{1i} - k_o^2 \xi_{i1}) \]

and

\[ e_{\xi i} = \frac{iL}{\rho} (ik'z'\xi_{i1} + iw'\mu_o \eta_{1i} - k_o^2 \xi_{i1}) \]
\[ e_{\xi i} = k'v_{i1} (-ik'z'\xi_{i1} - iw'\mu_o \eta_{1i} + k_o^2 \xi_{i1}) \]
\[ \mu_{\xi i} = \frac{iL}{\rho} (ik'z'\xi_{i1} + iw'\mu_o \eta_{1i} - k_o^2 \xi_{i1}) \]
\[ \mu_{\xi i} = k'v_{i1} (-ik'z'\xi_{i1} - iw'\mu_o \eta_{1i} + k_o^2 \xi_{i1}) \]
\[ \rho_{\xi i} = \frac{iL}{\rho} (ik'z'\xi_{i1} + iw'\mu_o \eta_{1i} - k_o^2 \xi_{i1}) \]
\[ \rho_{\xi i} = k'v_{i1} (-ik'z'\xi_{i1} - iw'\mu_o \eta_{1i} + k_o^2 \xi_{i1}) \]

and

\[ e_{\xi i} = 0_{11} \]
\[ e_{\xi i} = 0 \]
\[ \mu_{\xi i} = 0_{12} \]
\[ \mu_{\xi i} = 0 \]
\[ \rho_{\xi i} = 0_{13} \]
\[ \rho_{\xi i} = 0 \]
and

\begin{align*}
\epsilon_{zi}^h &= \frac{i\omega}{\rho}(i\omega\epsilon_i\xi_i'\delta_{i1}1 + k'\eta\xi_i'\delta_{i1}21 + \omega\epsilon_o k'\eta\xi_i'\delta_{i1}31) \\
\epsilon_{zi}^h' &= k'\eta\xi_i'\delta_{i1}1 + k'\eta\xi_i'\delta_{i1}21 - \omega\epsilon_o k'\eta\xi_i'\delta_{i1}31 \\
\mu_{zi}^h &= \frac{i\omega}{\rho}(i\omega\epsilon_i\xi_i'\delta_{i1}12 + k'\eta\xi_i'\delta_{i1}22 + \omega\epsilon_o k'\eta\xi_i'\delta_{i1}32) \\
\mu_{zi}^h' &= k'\eta\xi_i'\delta_{i1}12 + k'\eta\xi_i'\delta_{i1}22 - \omega\epsilon_o k'\eta\xi_i'\delta_{i1}32 \\
\rho_{zi}^h &= \frac{i\omega}{\rho}(i\omega\epsilon_i\xi_i'\delta_{i1}13 + k'\eta\xi_i'\delta_{i1}23 + \omega\epsilon_o k'\eta\xi_i'\delta_{i1}33) \\
\rho_{zi}^h' &= k'\eta\xi_i'\delta_{i1}13 + k'\eta\xi_i'\delta_{i1}23 - \omega\epsilon_o k'\eta\xi_i'\delta_{i1}33
\end{align*}

and

\begin{align*}
\epsilon_{zi}^h &= \frac{i\omega}{\rho}(i\omega\epsilon_i\xi_i'\delta_{i1}1 + k'\eta\xi_i'\delta_{i1}21 - \omega\epsilon_o k'\eta\xi_i'\delta_{i1}31) \\
\epsilon_{zi}^h' &= k'\eta\xi_i'\delta_{i1}1 - k'\eta\xi_i'\delta_{i1}21 + \omega\epsilon_o k'\eta\xi_i'\delta_{i1}31 \\
\mu_{zi}^h &= \frac{i\omega}{\rho}(i\omega\epsilon_i\xi_i'\delta_{i1}12 + k'\eta\xi_i'\delta_{i1}22 - \omega\epsilon_o k'\eta\xi_i'\delta_{i1}32) \\
\mu_{zi}^h' &= k'\eta\xi_i'\delta_{i1}12 + k'\eta\xi_i'\delta_{i1}22 + \omega\epsilon_o k'\eta\xi_i'\delta_{i1}32 \\
\rho_{zi}^h &= \frac{i\omega}{\rho}(i\omega\epsilon_i\xi_i'\delta_{i1}13 + k'\eta\xi_i'\delta_{i1}23 - \omega\epsilon_o k'\eta\xi_i'\delta_{i1}33) \\
\rho_{zi}^h' &= k'\eta\xi_i'\delta_{i1}13 + k'\eta\xi_i'\delta_{i1}23 + \omega\epsilon_o k'\eta\xi_i'\delta_{i1}33
\end{align*}

and

\begin{align*}
\epsilon_{zi}^h &= 0'_{21} \\
\epsilon_{zi}' &= 0 \\
\mu_{zi}^h &= 0'_{22} \\
\mu_{zi}' &= 0 \\
\rho_{zi}^h &= 0'_{23} \\
\rho_{zi}' &= 0 \end{align*}
and

\[ \varepsilon_{z_1}^p = 0 \]

\[ \varepsilon_{z_1}^{p'} = 0 \]

\[ \mu_{z_1}^p = 0 \]

\[ \mu_{z_1}^{p'} = 0 \]

\[ \rho_{z_1}^p = 0 \]

\[ \rho_{z_1}^{p'} = 0 \]
\[\varepsilon_{\rho o}^e(J) = \frac{2}{3} k o_{\nu o} t J^t(k o_{\nu o})\]
\[\mu_{\rho o}^e(J) = i \omega o_{\nu o} \eta o_{\rho o} J(k o_{\nu o})\]
\[\varepsilon_{\rho o}^\phi(J) = \frac{2}{3} k o_{\nu o} \eta o_{\rho o} J(k o_{\nu o})\]
\[\mu_{\rho o}^\phi(J) = -i \omega o_{\nu o} k o_{\nu o} t J^t(k o_{\nu o})\]
\[\varepsilon_{\rho o}^z(J) = k o_{\nu o} t 2 J(k o_{\nu o})\]

and

\[\varepsilon_{\rho o}^e(J) = \frac{2}{3} k o_{\nu o} t J^t(k o_{\nu o})\]
\[\mu_{\rho o}^e(J) = i \omega o_{\nu o} \eta o_{\rho o} J(k o_{\nu o})\]
\[\varepsilon_{\rho o}^\phi(J) = \frac{2}{3} k o_{\nu o} \eta o_{\rho o} J(k o_{\nu o})\]
\[\mu_{\rho o}^\phi(J) = -i \omega o_{\nu o} k o_{\nu o} t J^t(k o_{\nu o})\]
\[\varepsilon_{\rho o}^z(J) = k o_{\nu o} t 2 J(k o_{\nu o})\]

and

\[\varepsilon_{\rho o}^e(H) = \frac{2}{3} k o_{\nu o} t H^t(1)(k o_{\nu o})\]
\[\mu_{\rho o}^e(H) = i \omega o_{\nu o} \eta o_{\rho o} H^t(1)(k o_{\nu o})\]
\[\varepsilon_{\rho o}^\phi(H) = \frac{2}{3} k o_{\nu o} \eta o_{\rho o} H^t(1)(k o_{\nu o})\]
\[\mu_{\rho o}^\phi(H) = -i \omega o_{\nu o} k o_{\nu o} t H^t(1)(k o_{\nu o})\]
\[\varepsilon_{\rho o}^z(H) = k o_{\nu o} t 2 H^t(1)(k o_{\nu o})\]

and

\[\varepsilon_{\rho o}^e(H) = \frac{2}{3} k o_{\nu o} t H^t(1)(k o_{\nu o})\]
\[\mu_{\rho o}^e(H) = i \omega o_{\nu o} \eta o_{\rho o} H^t(1)(k o_{\nu o})\]
\[\varepsilon_{\rho o}^\phi(H) = \frac{2}{3} k o_{\nu o} \eta o_{\rho o} H^t(1)(k o_{\nu o})\]
\[\mu_{\rho o}^\phi(H) = -i \omega o_{\nu o} k o_{\nu o} t H^t(1)(k o_{\nu o})\]
\[\varepsilon_{\rho o}^z(H) = k o_{\nu o} t 2 H^t(1)(k o_{\nu o})\]
Since the region $\omega$ contains only outwardly traveling waves, the linear combination $p_{t,\omega}(+)^{Z(t)}(k_{o,\omega}^t,\rho)$ becomes
\[ p_{t,\omega} H_l'(k_{o,\omega}^t,\rho) \]
and the linear combination $q_{t,\omega}(+)^{Z(t)}(k_{o,\omega}^t,\rho)$ becomes
\[ q_{t,\omega} H_l'(k_{o,\omega}^t,\rho) \]
where $p_{t,\omega}$ and $q_{t,\omega}$ are arbitrary constants to be determined later in the study.

In the rest frame of the antenna in the region $\omega$, the cylindrical components of the field vectors in the wavenumber domain are

\[ e_{t,\omega} = ik_o^2 \frac{z t}{\rho} H_l'(k_{o,\omega}^t,\rho)_{t,\omega} + i\omega_o^\tau \eta_o \frac{z t}{\rho} H_l'(k_{o,\omega}^t,\rho)_{t,\omega} \]
\[ e_{t,\omega} = ik_o^2 \frac{z t}{\rho} H_l'(k_{o,\omega}^t,\rho)_{t,\omega} - i\omega_o^\tau \eta_o \frac{z t}{\rho} H_l'(k_{o,\omega}^t,\rho)_{t,\omega} \]
\[ e_{t,\omega} = k_{o,\omega}^2 \frac{z t}{\rho} H_l'(k_{o,\omega}^t,\rho)_{t,\omega} \]

and

\[ h_{t,\omega} = ik_o^2 \frac{z t}{\rho} H_l'(k_{o,\omega}^t,\rho)_{t,\omega} - i\omega_o^\tau \eta_o \frac{z t}{\rho} H_l'(k_{o,\omega}^t,\rho)_{t,\omega} \]
\[ h_{t,\omega} = ik_o^2 \frac{z t}{\rho} H_l'(k_{o,\omega}^t,\rho)_{t,\omega} + i\omega_o^\tau \eta_o \frac{z t}{\rho} H_l'(k_{o,\omega}^t,\rho)_{t,\omega} \]
\[ h_{t,\omega} = k_{o,\omega}^2 \frac{z t}{\rho} H_l'(k_{o,\omega}^t,\rho)_{t,\omega} \]
VIII. Boundary Values

The boundary conditions on the field vectors in the rest frame of the antenna are simply that the tangential components of the electric and the magnetic field vectors are continuous across each layer of the plasma shell. Also, the pressure deviation is continuous across each layer of the plasma shell; however, the normal derivative of the pressure deviation is discontinuous across each layer of the plasma shell by an amount equal to the difference between the product of the number density and the normal component of the Lorentz force on each side of the discontinuity.

To simplify the solution of the boundary value problem, the unknown constants in the $i^{th}$ layer of the plasma shell ($1 \leq i \leq n$) are combined as the elements of a single vector $c_{zi}$ such that its components are given by

$$
(c_{zi}) =
\begin{pmatrix}
    e_{zi}^+ \\
    e_{zi}^- \\
    \phi_{zi}^+ \\
    \phi_{zi}^- \\
    \rho_{zi}^+ \\
    \rho_{zi}^-
\end{pmatrix}
\quad i = (1, \ldots, n)
$$

As special cases, in the regions 0 and $\infty$, let

$$
(c_{z0}) =
\begin{pmatrix}
    p_{z0} \\
    0 \\
    q_{z0} \\
    0 \\
    0 \\
    0
\end{pmatrix}
\quad (c_{z\infty}) =
\begin{pmatrix}
    p_{z\infty} \\
    0 \\
    q_{z\infty} \\
    0 \\
    0 \\
    0
\end{pmatrix}
$$
\[ e^0_{\infty} (J) = \frac{i k_o}{2} z t J'(k_o t) \]
\[ e^0_{\infty} (J) = i \omega \eta \phi \]
\[ e^0_{\infty} (J) = k o z t J'(k_o t) \]
\[ e^0_{\infty} (J) = -i \omega \eta \phi \]
\[ e^0_{\infty} (J) = k o z t J'(k_o t) \]
\[ e^0_{\infty} (J) = k o z t J'(k_o t) \]
\[ e^0_{\infty} (J) = k o z t J'(k_o t) \]
\[ e^0_{\infty} (J) = k o z t J'(k_o t) \]

and

\[ e^0_{\infty} (H) = \frac{i k_o}{2} z t H(1) \]
\[ e^0_{\infty} (H) = i \omega \eta \phi \]
\[ e^0_{\infty} (H) = k o z t H(1) \]
\[ e^0_{\infty} (H) = -i \omega \eta \phi \]
\[ e^0_{\infty} (H) = k o z t H(1) \]

and

\[ e^0_{\infty} (H) = \frac{i k_o}{2} z t H(1) \]
\[ e^0_{\infty} (H) = i \omega \eta \phi \]
\[ e^0_{\infty} (H) = k o z t H(1) \]
\[ e^0_{\infty} (H) = -i \omega \eta \phi \]
\[ e^0_{\infty} (H) = k o z t H(1) \]

and

\[ e^0_{\infty} (H) = \frac{i k_o}{2} z t H(1) \]
\[ e^0_{\infty} (H) = i \omega \eta \phi \]
\[ e^0_{\infty} (H) = k o z t H(1) \]
\[ e^0_{\infty} (H) = -i \omega \eta \phi \]
\[ e^0_{\infty} (H) = k o z t H(1) \]
Then, in the region 0,
\[ \varepsilon_n^{e_{zo}} = \varepsilon_n^{e_{zo}} \cdot c_{zo} + \varepsilon_n^{n_{zo}} \]
\[ \varepsilon_t^{e_{zo}} = \varepsilon_t^{e_{zo}} \cdot c_{zo} + \varepsilon_t^{n_{zo}} \]
\[ h_n^{e_{zo}} = h_n^{n_{zo}} \cdot c_{zo} + h_n^{n_{zo}} \]
\[ h_t^{e_{zo}} = h_t^{n_{zo}} \cdot c_{zo} + h_t^{n_{zo}} \]

where
\[
\begin{bmatrix}
\varepsilon_n^{e_{zo}} \\
\varepsilon_t^{e_{zo}} \\
h_n^{e_{zo}} \\
h_t^{e_{zo}}
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_{e_{zo}}(J) & \varepsilon_{e_{zo}}(N) & \mu_{e_{zo}}(J) & \mu_{e_{zo}}(N) & \rho_{e_{zo}}(J) & \rho_{e_{zo}}(N) \\
\varepsilon_{e_{zo}}(J) & \varepsilon_{e_{zo}}(N) & \mu_{e_{zo}}(J) & \mu_{e_{zo}}(N) & \rho_{e_{zo}}(J) & \rho_{e_{zo}}(N) \\
\varepsilon_{e_{zo}}(J) & \varepsilon_{e_{zo}}(N) & \mu_{e_{zo}}(J) & \mu_{e_{zo}}(N) & \rho_{e_{zo}}(J) & \rho_{e_{zo}}(N) \\
\varepsilon_{e_{zo}}(J) & \varepsilon_{e_{zo}}(N) & \mu_{e_{zo}}(J) & \mu_{e_{zo}}(N) & \rho_{e_{zo}}(J) & \rho_{e_{zo}}(N)
\end{bmatrix}
\]

and
\[
\begin{bmatrix}
\varepsilon_n \\
\varepsilon_t \\
h_n \\
h_t
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_{e_{zo}}(J) & \varepsilon_{e_{zo}}(N) & \mu_{e_{zo}}(J) & \mu_{e_{zo}}(N) & \rho_{e_{zo}}(J) & \rho_{e_{zo}}(N) \\
\varepsilon_{e_{zo}}(J) & \varepsilon_{e_{zo}}(N) & \mu_{e_{zo}}(J) & \mu_{e_{zo}}(N) & \rho_{e_{zo}}(J) & \rho_{e_{zo}}(N) \\
\varepsilon_{e_{zo}}(J) & \varepsilon_{e_{zo}}(N) & \mu_{e_{zo}}(J) & \mu_{e_{zo}}(N) & \rho_{e_{zo}}(J) & \rho_{e_{zo}}(N) \\
\varepsilon_{e_{zo}}(J) & \varepsilon_{e_{zo}}(N) & \mu_{e_{zo}}(J) & \mu_{e_{zo}}(N) & \rho_{e_{zo}}(J) & \rho_{e_{zo}}(N)
\end{bmatrix}
\]
In the region $i$,

\[
\begin{align*}
\varepsilon^R_{zi} &= \varepsilon^R_{zi} \cdot \varepsilon_{zi} \\
\varepsilon^T_{zi} &= \varepsilon^T_{zi} \cdot \varepsilon_{zi} \\
\varepsilon^H_{zi} &= \varepsilon^H_{zi} \cdot \varepsilon_{zi} \\
\varepsilon^P_{zi} &= \varepsilon^P_{zi} \cdot \varepsilon_{zi}
\end{align*}
\]

where

\[
\begin{bmatrix}
\varepsilon^R_{zi} \\
\varepsilon^T_{zi} \\
\varepsilon^H_{zi} \\
\varepsilon^P_{zi}
\end{bmatrix} =
\begin{bmatrix}
\varepsilon^R_{zi}(J) & \varepsilon^R_{zi}(N) & \varepsilon^R_{zi}(J) & \varepsilon^R_{zi}(N) \\
\varepsilon^T_{zi}(J) & \varepsilon^T_{zi}(N) & \varepsilon^T_{zi}(J) & \varepsilon^T_{zi}(N) \\
\varepsilon^H_{zi}(J) & \varepsilon^H_{zi}(N) & \varepsilon^H_{zi}(J) & \varepsilon^H_{zi}(N) \\
\varepsilon^P_{zi}(J) & \varepsilon^P_{zi}(N) & \varepsilon^P_{zi}(J) & \varepsilon^P_{zi}(N)
\end{bmatrix}
\]

In the region $\infty$

\[
\begin{align*}
\varepsilon^R_{zi,\infty} &= \varepsilon^R_{zi,\infty} \cdot \varepsilon_{zi,\infty} \\
\varepsilon^T_{zi,\infty} &= \varepsilon^T_{zi,\infty} \cdot \varepsilon_{zi,\infty} \\
\varepsilon^H_{zi,\infty} &= \varepsilon^H_{zi,\infty} \cdot \varepsilon_{zi,\infty} \\
\varepsilon^P_{zi,\infty} &= \varepsilon^P_{zi,\infty} \cdot \varepsilon_{zi,\infty}
\end{align*}
\]
where
\[
\left[ e_{\perp \infty} \right] = \begin{bmatrix}
\hat{e}_{\perp \infty}^{\rho} (J) & \hat{e}_{\perp \infty}^{\rho} (N) & \hat{e}_{\perp \infty}^{\rho} (J) & \hat{e}_{\perp \infty}^{\rho} (N) & \hat{e}_{\perp \infty}^{\rho} (J) & \hat{e}_{\perp \infty}^{\rho} (N) \\
\hat{\mu}_{\perp \infty}^{\varphi} (J) & \hat{\mu}_{\perp \infty}^{\varphi} (N) & \hat{\mu}_{\perp \infty}^{\varphi} (J) & \hat{\mu}_{\perp \infty}^{\varphi} (N) & \hat{\mu}_{\perp \infty}^{\varphi} (J) & \hat{\mu}_{\perp \infty}^{\varphi} (N) \\
\hat{\mu}_{\perp \infty}^{\zeta} (J) & \hat{\mu}_{\perp \infty}^{\zeta} (N) & \hat{\mu}_{\perp \infty}^{\zeta} (J) & \hat{\mu}_{\perp \infty}^{\zeta} (N) & \hat{\mu}_{\perp \infty}^{\zeta} (J) & \hat{\mu}_{\perp \infty}^{\zeta} (N)
\end{bmatrix}
\]

\[
\left[ e_{t \infty} \right] = \begin{bmatrix}
\hat{e}_{t \infty}^{\rho} (J) & \hat{e}_{t \infty}^{\rho} (N) & \hat{e}_{t \infty}^{\rho} (J) & \hat{e}_{t \infty}^{\rho} (N) & \hat{e}_{t \infty}^{\rho} (J) & \hat{e}_{t \infty}^{\rho} (N) \\
\hat{\mu}_{t \infty}^{\varphi} (J) & \hat{\mu}_{t \infty}^{\varphi} (N) & \hat{\mu}_{t \infty}^{\varphi} (J) & \hat{\mu}_{t \infty}^{\varphi} (N) & \hat{\mu}_{t \infty}^{\varphi} (J) & \hat{\mu}_{t \infty}^{\varphi} (N) \\
\hat{\mu}_{t \infty}^{\zeta} (J) & \hat{\mu}_{t \infty}^{\zeta} (N) & \hat{\mu}_{t \infty}^{\zeta} (J) & \hat{\mu}_{t \infty}^{\zeta} (N) & \hat{\mu}_{t \infty}^{\zeta} (J) & \hat{\mu}_{t \infty}^{\zeta} (N)
\end{bmatrix}
\]

\[
\left[ \eta_{\perp \infty} \right] = \begin{bmatrix}
\hat{\eta}_{\perp \infty}^{\rho} (J) & \hat{\eta}_{\perp \infty}^{\rho} (N) & \hat{\eta}_{\perp \infty}^{\rho} (J) & \hat{\eta}_{\perp \infty}^{\rho} (N) & \hat{\eta}_{\perp \infty}^{\rho} (J) & \hat{\eta}_{\perp \infty}^{\rho} (N) \\
\hat{\mu}_{\perp \infty}^{\varphi} (J) & \hat{\mu}_{\perp \infty}^{\varphi} (N) & \hat{\mu}_{\perp \infty}^{\varphi} (J) & \hat{\mu}_{\perp \infty}^{\varphi} (N) & \hat{\mu}_{\perp \infty}^{\varphi} (J) & \hat{\mu}_{\perp \infty}^{\varphi} (N) \\
\hat{\mu}_{\perp \infty}^{\zeta} (J) & \hat{\mu}_{\perp \infty}^{\zeta} (N) & \hat{\mu}_{\perp \infty}^{\zeta} (J) & \hat{\mu}_{\perp \infty}^{\zeta} (N) & \hat{\mu}_{\perp \infty}^{\zeta} (J) & \hat{\mu}_{\perp \infty}^{\zeta} (N)
\end{bmatrix}
\]

\[
\left[ \eta_{t} \right] = \begin{bmatrix}
\hat{\eta}_{t}^{\rho} (J) & \hat{\eta}_{t}^{\rho} (N) & \hat{\eta}_{t}^{\rho} (J) & \hat{\eta}_{t}^{\rho} (N) & \hat{\eta}_{t}^{\rho} (J) & \hat{\eta}_{t}^{\rho} (N) \\
\hat{\mu}_{t}^{\varphi} (J) & \hat{\mu}_{t}^{\varphi} (N) & \hat{\mu}_{t}^{\varphi} (J) & \hat{\mu}_{t}^{\varphi} (N) & \hat{\mu}_{t}^{\varphi} (J) & \hat{\mu}_{t}^{\varphi} (N) \\
\hat{\mu}_{t}^{\zeta} (J) & \hat{\mu}_{t}^{\zeta} (N) & \hat{\mu}_{t}^{\zeta} (J) & \hat{\mu}_{t}^{\zeta} (N) & \hat{\mu}_{t}^{\zeta} (J) & \hat{\mu}_{t}^{\zeta} (N)
\end{bmatrix}
\]

In the above formulas, a superscript \( t \) denotes the transverse components of the field vectors with respect to the interface, i.e., the \( \varphi, \zeta \) components, and a superscript \( n \) denotes the normal components of the field vectors with respect to the interface, i.e., the \( \rho \) components.

Therefore, the boundary conditions at the interface \( \rho = \rho_o \) are

\[
e_{\perp \infty} |_{\rho=\rho_o} = e_{t \perp} |_{\rho=\rho_o}
\]

\[
h_{\perp \infty} |_{\rho=\rho_o} = h_{t \perp} |_{\rho=\rho_o}
\]

\[
0 = p_{\perp 1} |_{\rho=\rho_o}
\]

\[
0 = \frac{d p_{\perp 1}}{d \rho} |_{\rho=\rho_o} - q N_1 e_{n \perp \perp} |_{\rho=\rho_o}
\]

or

\[
e_{\perp \infty} |_{\rho=\rho_o} \cdot c_{\perp \infty} + e_{t \perp} |_{\rho=\rho_o} = e_{t \perp} |_{\rho=\rho_o} \cdot c_{\perp \infty}
\]

\[
h_{\perp \infty} |_{\rho=\rho_o} \cdot c_{\perp \infty} + h_{t \perp} |_{\rho=\rho_o} = h_{t \perp} |_{\rho=\rho_o} \cdot c_{\perp \infty}
\]

\[
0 = p_{\perp 1} |_{\rho=\rho_o} \cdot c_{\perp \infty}
\]

\[
0 = \left( \frac{d p_{\perp 1}}{d \rho} |_{\rho=\rho_o} - q N_1 e_{n \perp \perp} |_{\rho=\rho_o} \right) \cdot c_{\perp \infty}
\]

at the interface \( \rho = \rho_i \) \( (1 \leq i \leq n - 1) \)

\[
e_{t \perp} |_{\rho=\rho_i} = e_{t \perp, i+1} |_{\rho=\rho_i}
\]
\[ \frac{d N_i}{d \tau} + \sum_{j \neq i} N_j \frac{d \theta_i}{d \tau} = -N_i \frac{d \theta_i}{d \tau} \]

or

\[ \frac{d \theta_i}{d \tau} = 0 \]
These boundary conditions are summarized as

\[
\begin{align*}
\mathbf{m}_{\mathbf{z}_0} | \rho = \rho_0 \cdot \mathbf{c}_{\mathbf{z}_0} + \frac{k_0}{\rho_0} | \rho = \rho_1 &= \mathbf{m}_{\mathbf{z}_1} | \rho = \rho_0 \cdot \mathbf{c}_{\mathbf{z}_1} \\
\mathbf{m}_{\mathbf{z}_i} | \rho = \rho_i \cdot \mathbf{c}_{\mathbf{z}_i} &= \mathbf{m}_{\mathbf{z}_{i+1}} | \rho = \rho_i \cdot \mathbf{c}_{\mathbf{z}_{i+1}} \quad (1 \leq i \leq n - 1) \\
\mathbf{m}_{\mathbf{z}_n} | \rho = \rho_n \cdot \mathbf{c}_{\mathbf{z}_n} &= \mathbf{m}_{\mathbf{z}_\infty} | \rho = \rho_n \cdot \mathbf{c}_{\mathbf{z}_\infty}
\end{align*}
\]

where the dyads \( \mathbf{m}_{\mathbf{z}_0}, \mathbf{m}_{\mathbf{z}_i}, \mathbf{m}_{\mathbf{z}_\infty} \) are such that their components are given by

\[
\begin{align*}
\mathbf{m}_{\mathbf{z}_0} &= \begin{bmatrix} \mathbf{t} & \mathbf{e}_{\mathbf{z}_0} \\ \mathbf{e}_{\mathbf{z}_0}^T & \mathbf{n}_{\mathbf{z}_0} \end{bmatrix} \\
\mathbf{m}_{\mathbf{z}_i} &= \begin{bmatrix} \mathbf{t} & \mathbf{e}_{\mathbf{z}_i} \\ \mathbf{e}_{\mathbf{z}_i}^T & \mathbf{n}_{\mathbf{z}_i} \end{bmatrix} \\
\mathbf{m}_{\mathbf{z}_\infty} &= \begin{bmatrix} \mathbf{t} & \mathbf{e}_{\mathbf{z}_\infty} \\ \mathbf{e}_{\mathbf{z}_\infty}^T & \mathbf{n}_{\mathbf{z}_\infty} \end{bmatrix}
\end{align*}
\]

and the vector \( \mathbf{k}_0 \) is such that its components are given by

\[
(\mathbf{k}_0) = \begin{bmatrix} \mathbf{t} \\ \mathbf{e}_{\mathbf{z}_0} \\ \mathbf{n}_{\mathbf{z}_0} \\ 0 \\ 0 \end{bmatrix}
\]
These equations are combined to yield

\[ \begin{align*}
&\begin{bmatrix}
  m_{\infty} & \rho_{i} \\
  m_{o} & \rho_{n}
\end{bmatrix}
  \begin{bmatrix}
  c_{\infty} & c_{n}
\end{bmatrix}
  =
  \begin{bmatrix}
  k & 0
\end{bmatrix}
\end{align*} \]

where

\[ m_{n} = \begin{bmatrix}
  m_{i} & \rho_{i}
\end{bmatrix}
\]

This system of equations is solved for the remaining undetermined constants \( p_{\infty}, q_{\infty} \) by matrix theory.
IX. Integral Expansions

In the region $\omega$, the integral expressions for the cylindrical components of the field vectors in the frequency domain are

$$E^\varphi = \sum_{\nu=\pm \infty} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \left[ i \kappa z \frac{d}{\rho} H^{(1)}_\kappa (k_\infty \rho \omega) \right] e^{i\varphi}$$

$$E^z = \sum_{\nu=\pm \infty} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \left[ i \kappa z \frac{d}{\rho} H^{(1)}_\kappa (k_\infty \rho \omega) \right] e^{i\varphi}$$

and

$$H^\varphi = \sum_{\nu=\pm \infty} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \left[ i \kappa z \frac{d}{\rho} H^{(1)}_\kappa (k_\infty \rho \omega) \right] e^{i\varphi}$$

$$H^z = \sum_{\nu=\pm \infty} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \left[ i \kappa z \frac{d}{\rho} H^{(1)}_\kappa (k_\infty \rho \omega) \right] e^{i\varphi}$$

The constants $p_{\omega}$ and $q_{\omega}$ are known from the work performed in the previous section of this study.

A. Integration Path

If the region $\omega$ is assumed to be nonconducting,

$$\sigma_\infty = 0$$

Then $\kappa_\infty$ reduces to the real number

$$\kappa_\infty = \sqrt{\epsilon_\infty \mu_\infty} = \sqrt{\mu_\infty \epsilon_\infty}$$
The integration in the complex $k^z$ plane is along the real axis of $k^z$ from $-\infty$ to $+\infty$ with an indentation below the branch point at $k^z = +k_0 K_\infty$ and above the branch point at $k^z = -k_0 K_\infty$, as shown in Figure 2. No indentations are required if $K_\infty$ is allowed to have a vanishingly small but finite positive imaginary part corresponding to the presence of some conductivity in the region $\infty$.

B. Radiation Fields

In the present case, it is assumed that $k_o K_\infty > 1$. This corresponds to evaluating the fields in the radiation zone of the antenna, since the distance from the point of observation to the outer radius of the cylinder is large compared with the wavelength. With this assumption, the radiation fields, correct to order $1/\rho$, reduce to the expressions

\[
E^{\rho} = \sum_{v=-\infty}^{\infty} \int_{v=-\infty}^{+\infty} \frac{d k^z}{2\pi i} \left( k_0^0 K_\infty^0 \right) (k_0^0 K_\infty^0 e^{i t}) H^1(1) e^{ik_0 K_\infty^0 z}
\]

\[
E^{\varphi} = \sum_{v=-\infty}^{\infty} \int_{v=-\infty}^{+\infty} \frac{d k^z}{2\pi i} i w \left( k_0^0 K_\infty^0 \right) (k_0^0 K_\infty^0 e^{i t}) H^1(1) e^{ik_0 K_\infty^0 z}
\]

\[
E^{z} = \sum_{v=-\infty}^{\infty} \int_{v=-\infty}^{+\infty} \frac{d k^z}{2\pi} k_0^0 K_\infty^0 (k_0^0 K_\infty^0 e^{i t}) H^1(1) e^{ik_0 K_\infty^0 z}
\]

and

\[
H^{\rho} = \sum_{v=-\infty}^{\infty} \int_{v=-\infty}^{+\infty} \frac{d k^z}{2\pi i} \left( k_0^0 K_\infty^0 \right) H^1(1) e^{i t} e^{ik_0 K_\infty^0 z}
\]

\[
H^{\varphi} = \sum_{v=-\infty}^{\infty} \int_{v=-\infty}^{+\infty} \frac{d k^z}{2\pi i} i w \left( k_0^0 K_\infty^0 \right) H^1(1) e^{i t} e^{ik_0 K_\infty^0 z}
\]

\[
H^{z} = \sum_{v=-\infty}^{\infty} \int_{v=-\infty}^{+\infty} \frac{d k^z}{2\pi} k_0^0 K_\infty^0 H^1(1) e^{i t} e^{ik_0 K_\infty^0 z}
\]
Fig. 2. Integration Paths
C. Spherical System

If the spherical transformation

\[ \rho = r \sin \vartheta \]
\[ \varphi = \varphi \]
\[ z = r \cos \vartheta \]

is performed, the resulting theta and phi components of the radiation fields are given by

\[ E^\vartheta = \sum_{l=0}^{\infty} \left[ \frac{d}{d\kappa_o^2} \sqrt{2\pi} \right] H^{(1)}(k_o \vartheta) \cos \vartheta + \frac{k_o \vartheta z}{2} \]
\[ E^\varphi = \sum_{l=0}^{\infty} \left[ \frac{d}{d\kappa_o^2} \sqrt{2\pi} \right] i l \varphi \sin \vartheta - \frac{k_o \vartheta z}{2} \]

and

\[ H^\vartheta = -\frac{k_o \vartheta}{u_o \gamma_o} E^\vartheta \]
\[ H^\varphi = +\frac{k_o \vartheta}{u_o \gamma_o} E^\varphi \]

D. Asymptotic Expansion

Also, under the assumption that \( k_o \vartheta >> 1 \), it is permissible to replace the Hankel functions and their derivatives by the first terms of their asymptotic expansions

\[ H^{(1)}(k_o \vartheta \sin \vartheta) \approx \frac{2}{\pi k_o \vartheta \sin \vartheta} \left[ ik_o \vartheta \sin \vartheta - i \frac{\pi}{2} - i \frac{\pi}{4} \right] \]
\[ H^{(1)'}(k_o \vartheta \sin \vartheta) \approx i \frac{2}{\pi k_o \vartheta \sin \vartheta} \left[ ik_o \vartheta \sin \vartheta - i \frac{\pi}{2} - i \frac{\pi}{4} \right] \]

When these expansions are used, the theta and the phi components of the electric field vectors become
E\phi = - \sum_{\nu=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\nu z}{\sqrt{2\pi}} k_0^2 (\mu_\infty \cos \vartheta + \mu_\infty \sin \vartheta) p_{t2} \exp \left\{ \begin{array}{l} \frac{2}{\pi k_0^2 \mu_\infty r \sin \vartheta} \\ -i k_0^2 r \sin \vartheta - i \frac{\pi}{2} - i \frac{\pi}{4} i \varphi \\ \mu_\infty z \\ \end{array} \right\}

E\phi = \sum_{\nu=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\nu z}{\sqrt{2\pi}} \omega_0 \gamma k_0^2 q \exp \left\{ \begin{array}{l} \frac{2}{\pi k_0^2 \mu_\infty r \sin \vartheta} \\ -i k_0^2 r \sin \vartheta - i \frac{\pi}{2} - i \frac{\pi}{4} i \varphi \\ \mu_\infty z \\ \end{array} \right\}

E. Saddle Point Integration

The resulting integrals are now in the form

\[ \sum_{\nu=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\nu z}{\sqrt{2\pi}} f(k_0^2 \mu_\infty^2 \cos \vartheta + \mu_\infty \sin \vartheta) p_{t2} \exp \left\{ \begin{array}{l} \frac{2}{\pi k_0^2 \mu_\infty r \sin \vartheta} \\ -i k_0^2 r \sin \vartheta - i \frac{\pi}{2} - i \frac{\pi}{4} i \varphi \\ \mu_\infty z \\ \end{array} \right\} \]

where

\[ f(k_0^2 \mu_\infty^2) = \begin{cases} -k_0^2 (\mu_\infty \cos \vartheta + \mu_\infty \sin \vartheta) p_{t2} e^{i \frac{\pi}{4} \varphi} \\ -i k_0^2 r \sin \vartheta - i \frac{\pi}{4} i \varphi \\ \omega_0 \gamma k_0^2 q \exp \left\{ \begin{array}{l} \frac{2}{\pi k_0^2 \mu_\infty r \sin \vartheta} \\ -i k_0^2 r \sin \vartheta - i \frac{\pi}{2} - i \frac{\pi}{4} i \varphi \\ \mu_\infty z \\ \end{array} \right\} \end{cases} \]

These integrals can be evaluated by the methods of saddle point integration if the integration is transformed to the complex \( \alpha \) plane by means of the substitution

\[ \mu_\infty^2 = \mu_\infty \cos \alpha \]

This leads to the form

\[ \sum_{\nu=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(k_0 \mu_\infty \cos \alpha) p_{t2} \exp \left\{ \begin{array}{l} \frac{2}{\pi k_0 \mu_\infty r \sin \vartheta} \\ \mu_\infty r \cos (\alpha - \vartheta) \\ \end{array} \right\} \]

where the path of integration in the complex \( \alpha \) plane is shown in Figure 2.

The next step is to transform the contour to the path of steepest descent. The path is defined by

\[ \cos (\alpha - \vartheta) = 1 + i \chi^2 \]

where \( \chi \) is to range from \(-\infty\) to \(+\infty\).

The integral then becomes
\[
\sum_{\nu=-\infty}^{+\infty} \int_{\nu}^{+\infty} \frac{1}{1 + i\nu^2} f[k_0 \nu \cos \left( \cos^{-1}(1 + i\nu^2) + \theta \right)]
\]

\[
\frac{1}{\nu} \sqrt{k_0 \nu}(1 + i\nu^2) r
\]

The integral is now expanded in a power series in \(\nu^2\) and the integration is performed term by term. This leads to

\[
\sqrt{2} \frac{e^{ik_0 \nu \cos \theta}}{r} \sum_{\nu=-\infty}^{+\infty} f(k_0 \nu \cos \theta)
\]

where the remaining terms contain higher powers of \(1/r\).

By the use of the asymptotic expansion of the integrals, the theta and phi components of the field vectors become

\[
E^\theta = i \sqrt{\frac{2}{\pi}} k_0^2 \nu r \sin \theta \frac{e^{ik_0 \nu \cos \theta}}{r}
\]

\[
E^\phi = -i \sqrt{\frac{2}{\pi}} \frac{\nu \nu}{\nu_0} k_0 \nu \sin \theta \frac{e^{ik_0 \nu \cos \theta}}{r}
\]

and

\[
H^\theta = \frac{k_0 \nu}{\nu_0} E^\phi
\]

\[
H^\phi = -\frac{k_0 \nu}{\nu_0} E^\theta
\]

Note that \(p_{\nu_0}\) and \(q_{\nu_0}\) are evaluated at the value of \(k_0 \nu \cos \theta\).
X. Results

The point source character of the antenna becomes evident on considering the power radiated into the far field of the antenna. The Poynting vector is

\[ s = \frac{e^*}{Z_\infty} \]

where

\[ |e|^2 = |e|^2 + |e|^2 = e^*e^* + e^*e^* \]

In terms of the above analysis, the Poynting vector has the form

\[ s = \frac{\varphi}{r^2} \frac{f(\varphi, \psi)}{Z_\infty} \]

where

\[ f(\varphi, \psi) = \left( \sqrt{\frac{2}{\pi}} k_o \frac{2}{2} \sin \psi \right)^2 \left[ \left( \sum_{\varphi=-\infty}^{+\infty} \frac{P(t, \infty)}{k_o \frac{2}{2} \cos \theta} \right)^2 + \frac{2}{Z_{\infty}} \left( \sum_{\varphi=-\infty}^{+\infty} q(t, \infty) \right)^2 \right] \]

In discussing the power flow in the far field of the antenna, it is convenient to use instead of the Poynting vector the intensity of the radiation. The intensity is defined by

\[ I = r^2 |s| = \frac{f(\varphi, \psi)}{Z_\infty} \]

which is independent of the radial distance.
The intensity distribution in the far field of the antenna is conveniently specified in terms of the gain of the antenna with respect to an axial Hertzian dipole radiator. The gain is defined by

$$G = \frac{I}{<I_o>}$$

where

$$<I_o> = \frac{P_{rad}}{4\pi}$$

and

$$P_{rad} = \int_0^\pi \int_0^{2\pi} r^2 \sin \theta P_o d\psi d\phi = \int_0^\pi \int_0^{2\pi} I_o \sin \theta$$

where

$$I_o = \frac{r^2 P_o}{\frac{1}{2} Z} = \frac{i L_s}{4\pi} \left( k_o \kappa \sin \theta \right)^2$$

Therefore,

$$G = \frac{f(\theta, \phi)}{Z}$$

$$G = \frac{1}{3} \frac{Z}{Z} \left( \frac{i L_s}{4\pi} \right)^2 \left( k_o \kappa \right)^2$$
XI. Conclusions

The gain of a Hertzian dipole antenna in the presence of a cylindrically stratified, moving, linear, anisotropic, conducting, and compressible plasma shell has been calculated for arbitrary positions of the antenna with respect to the plasma shell. In particular, the antenna can be off-axis with respect to the axis of the plasma shell and can be oriented in any direction with respect to the axial direction and the transverse plane of the plasma shell. The gain of the antenna was referenced to a Hertzian dipole antenna in vacuum, located at the origin of the cylindrical coordinate system of the plasma shell and directed in the positive axial direction.

The resulting expression for the gain of the antenna was given as an explicit function of the given parameters of the plasma shell, viz. the number density, collision frequency, temperature, bias, and velocity of each layer of the plasma shell.

Due to the complexity of the results of this study, a parametric study of the gain of the antenna as a function of the parameters of the plasma shell was not attempted at this time. Only the theoretical aspects of this problem were presented here. A numerical study of the gain of the antenna will be undertaken and will be reported on in a future study.
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APPENDIX I

Projection Operators
In what follows, let the operator which projects any vector onto itself be denoted by \( i \), i.e.,

\[
v = i \cdot v
\]

Also, let the dual of any vector \( \mathbf{v} \) be denoted by \( *\mathbf{v} \), i.e.,

\[
(*\mathbf{v})_{jk} = \sum_{\ell} \varepsilon^{j\ell k} \mathbf{v}_{\ell}
\]

where the Kristoffel symbol \( \varepsilon^{j\ell k} \) is defined as

\[
\varepsilon^{j\ell k} = \begin{cases} 
+1 & \text{if } jk\ell \text{ forms an even permutation of } 1,2,3 \\
-1 & \text{if } jk\ell \text{ forms an odd permutation of } 1,2,3 \\
0 & \text{otherwise}
\end{cases}
\]

If the projection of the vector \( \mathbf{v} \) onto the plane normal to the axial direction \( \mathbf{a} \) is denoted by \( \mathbf{v}^t \), then

\[
\mathbf{v}^t = \frac{\mathbf{a}}{||\mathbf{a}||} \cdot \mathbf{v}
\]

where the projection operator \( \frac{\mathbf{a}}{||\mathbf{a}||} \) is defined as

\[
\frac{\mathbf{a}}{||\mathbf{a}||} = i - \frac{\mathbf{a}}{||\mathbf{a}||}
\]

If the projection of the vector \( \mathbf{v} \) onto the plane normal to the axial direction \( \mathbf{a} \), followed by a rotation through \( \theta \) radians about the axial direction \( \mathbf{a} \), is denoted by \( \mathbf{v}^s \), then

\[
\mathbf{v}^s = \mathbf{r}_{\theta} \cdot \mathbf{t} \cdot \mathbf{v}
\]

where the rotation operator \( \mathbf{r}_{\theta} \) is defined as

\[
\mathbf{r}_{\theta} = \cos\theta - *\mathbf{a} \sin\theta
\]

For the special case of \( \theta = \pi/2 \), let the projection operator \( \mathbf{s} \) be defined as

\[
\mathbf{s} = \mathbf{r}_{\pi/2} \cdot \mathbf{t}
\]

then

\[
\mathbf{v}^s = \mathbf{s} \cdot \mathbf{v}
\]
If the dyad $c$ is introduced by letting
\[ c = \frac{r}{11} \equiv -\hat{a} \]
then $v^s$ and $v^t$ are related by
\[ v^s = c \cdot v^t \]
or
\[ v^s = \hat{a} \wedge v^t \]
Also, since the inverse of $c$ is just $-c$,
\[ c \cdot c = -t \]
If
\[ \hat{a} = \hat{x}a^x + \hat{y}a^y + \hat{z}a^z \]
and
\[ a^i = \cos \alpha_i \quad i = (x,y,z) \]
where the direction angles $\alpha_i$ satisfy
\[ \sum_{i=1}^{3} \cos^2 \alpha_i = 1 \]
then
\[ \hat{a} = +\hat{x}a^z - \hat{z}a^x - \hat{y}a^y + \hat{y}a^z + \hat{z}a^x + \hat{y}a^x - \hat{y}a^x \]
or
\[ \hat{a} = +\hat{x} \cos \alpha^z - \hat{z} \cos \alpha^x - \hat{y} \cos \alpha^y + \hat{y} \cos \alpha^z + \hat{z} \cos \alpha^x + \hat{y} \cos \alpha^y \]
Since
\[ i = \hat{x} + \hat{y} + \hat{z} \]
then
\[ \hat{a} = \hat{x} \cos^2 \alpha^x + \hat{y} \cos \alpha^x \cos \alpha^y + \hat{z} \cos \alpha^x \cos \alpha^z + \hat{y} \cos \alpha^y \cos \alpha^z \]
\[ + \hat{y} \cos \alpha^z \cos \alpha^x + \hat{y} \cos \alpha^y \cos \alpha^z + \hat{z} \cos \alpha^y \cos \alpha^z \]
\[ \hat{c} = \hat{x} \sin \alpha^x - \hat{y} \cos \alpha^x \cos \alpha^y - \hat{z} \cos \alpha^x \cos \alpha^z \]
\[ - \hat{y} \cos \alpha^y \cos \alpha^z + \hat{y} \sin \alpha^y \cos \alpha^z - \hat{z} \cos \alpha^y \cos \alpha^z \]
\[ - \hat{z} \cos \alpha^z \cos \alpha^x - \hat{y} \cos \alpha^z \cos \alpha^y + \hat{z} \sin \alpha^z \cos \alpha^x \]
\[ - \hat{z} \cos \alpha^z \cos \alpha^x - \hat{y} \cos \alpha^z \cos \alpha^y + \hat{z} \sin \alpha^z \cos \alpha^x \]
and

\[ \psi \wedge \psi \wedge \psi = \psi^2 \psi^a - \psi^a \psi^t \psi^t \]

\[ + \psi^t \psi^t \psi^t \psi^t + \psi^a \psi^t \psi^a - \psi^a \psi^t \psi^t \]
\[
\mathbf{r} = 2\alpha \sin 2\psi \cos \phi - \frac{2\alpha}{\phi} \left( \cos \alpha \cos \beta \cos \psi + \cos \alpha \sin \beta \right) - 2\frac{\alpha}{\phi} \left( \cos \alpha \cos \beta \cos \alpha - \cos \alpha \sin \beta \right)
\]  

\[
- \frac{2\alpha}{\phi} \left( \cos \alpha \cos \beta \cos \psi + \cos \alpha \sin \beta \right) + \frac{\alpha}{\phi} \left( \cos \alpha \cos \beta \cos \psi + \cos \alpha \sin \beta \right) \]

\[
- \frac{2\alpha}{\phi} \left( \cos \alpha \cos \beta \cos \alpha - \cos \alpha \sin \beta \right) - \frac{\alpha}{\phi} \left( \cos \alpha \cos \beta \cos \alpha - \cos \alpha \sin \beta \right) + 2\alpha \sin \alpha \cos \beta
\]

and

\[
\mathbf{s} = \left( \frac{\alpha}{\phi} - \frac{\beta}{\phi} \right) \cos \beta \left( \frac{\alpha}{\phi} - \frac{\beta}{\phi} \right) + \left( \frac{\alpha}{\phi} - \frac{\beta}{\phi} \right) \cos \beta \left( \frac{\alpha}{\phi} - \frac{\beta}{\phi} \right) \cos \gamma
\]

\[
\mathbf{c} = \left( \frac{\alpha}{\phi} - \frac{\beta}{\phi} \right) \cos \beta \left( \frac{\alpha}{\phi} - \frac{\beta}{\phi} \right) + \left( \frac{\alpha}{\phi} - \frac{\beta}{\phi} \right) \cos \beta \left( \frac{\alpha}{\phi} - \frac{\beta}{\phi} \right) \cos \gamma
\]

The following identities among the projection operators are evident:

\[
(\mathbf{v}^{s})^{t} = -\mathbf{v}^{t}
\]

\[
\mathbf{v}^{t} \cdot \mathbf{v}^{s} = 0
\]

\[
\mathbf{u}^{t} \cdot \mathbf{v}^{s} = -\mathbf{v}^{t} \cdot \mathbf{u}^{s}
\]

\[
\mathbf{u}^{s} \cdot \mathbf{v}^{s} = \mathbf{u}^{t} \cdot \mathbf{v}^{t}
\]

\[
(\mathbf{u} \wedge \mathbf{v})^{a} = \mathbf{v}^{t} \cdot \mathbf{u}^{s} = -\mathbf{u}^{t} \cdot \mathbf{v}^{s}
\]

\[
(\mathbf{u} \wedge \mathbf{v})^{t} = \mathbf{a} \cdot \mathbf{s} - \mathbf{u}^{s} \cdot \mathbf{a}^{t}
\]

or

\[
\mathbf{c} \cdot \mathbf{c} \cdot \mathbf{v}^{t} = -\mathbf{v}^{t}
\]

\[
\mathbf{v}^{t} \cdot \mathbf{c} \cdot \mathbf{v}^{t} = 0
\]

\[
\mathbf{u}^{t} \cdot (\mathbf{c} \cdot \mathbf{v}^{t}) = -\mathbf{v}^{t} \cdot (\mathbf{c} \cdot \mathbf{u}^{t})
\]

\[
\mathbf{c} \cdot \mathbf{u}^{t} \cdot \mathbf{c} \cdot \mathbf{v}^{t} = \mathbf{u}^{t} \cdot \mathbf{v}^{t}
\]

\[
(\mathbf{u} \wedge \mathbf{v})^{a} = \mathbf{v}^{t} \cdot \mathbf{c} \cdot \mathbf{u}^{t} = -\mathbf{u}^{t} \cdot \mathbf{c} \cdot \mathbf{v}^{t}
\]

\[
(\mathbf{u} \wedge \mathbf{v})^{t} = \mathbf{a} \cdot \mathbf{c} \cdot \mathbf{v}^{t} - \mathbf{c} \cdot \mathbf{u}^{t} \mathbf{a} \cdot \mathbf{v} = \mathbf{c} \cdot (\mathbf{u} \cdot \mathbf{a} \cdot \mathbf{v}^{t} - \mathbf{u}^{t} \cdot \mathbf{a} \cdot \mathbf{v}) = -\mathbf{c} \cdot (\mathbf{u} \cdot \mathbf{v}^{t} - \mathbf{u}^{t} \cdot \mathbf{v})
\]

If a similar notation is used for the differential operator \(\nabla\), i.e.,

\[
\nabla^{t} = \mathbf{v}^{t} \cdot \nabla
\]

\[
\nabla^{s} = \mathbf{v}^{s} \cdot \nabla
\]

and

\[
\nabla^{s} = \mathbf{c} \cdot \nabla^{t}
\]

then

\[
\nabla \wedge \nabla = -\mathbf{a} \nabla^{t} \cdot \mathbf{c} \cdot \mathbf{v}^{t} - \mathbf{c} \cdot (\nabla^{t} \mathbf{a} \cdot \nabla - \mathbf{a} \cdot \nabla)
\]
APPENDIX II

Inverses
Let
\[ 0 = i\omega \Omega + \omega g \]
then
\[
[0] = \begin{bmatrix}
-i\omega \Omega & -\omega g \cos \alpha^z & +\omega g \cos \alpha^y \\
+\omega g \cos \alpha^z & -i\omega \Omega & -\omega g \cos \alpha^x \\
-\omega g \cos \alpha^y & +\omega g \cos \alpha^x & -i\omega \Omega
\end{bmatrix}
\]

The determinant \( \Delta \) of \( 0 \) is
\[ \Delta = i\omega \Omega (\omega \Omega^2 - \omega g^2) \]
and the inverse \( \left[ 0 \right]^{-1} \) of \( 0 \) is
\[
\left[ 0 \right]^{-1} = \frac{1}{i\omega \Omega (\omega \Omega^2 - \omega g^2)} \begin{bmatrix}
1 & 0 & 0 \\
-\omega \Omega & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & -\cos \alpha^x & +\cos \alpha^y \\
+\cos \alpha^x & 0 & -\cos \alpha^z \\
-\cos \alpha^y & +\cos \alpha^x & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cos \alpha^x & \cos \alpha^z & \cos \alpha^y \\
\cos \alpha^z & \cos \alpha^y & \cos \alpha^x \\
\cos \alpha^y & \cos \alpha^x & \cos \alpha^z
\end{bmatrix}
\]

\[
+ \frac{1}{i\omega \Omega (\omega \Omega^2 - \omega g^2)} \begin{bmatrix}
\omega \Omega^2 & 0 & 0 \\
0 & \omega \Omega^2 & 0 \\
0 & 0 & \omega \Omega^2
\end{bmatrix}
\]
Therefore,

\[ o^\pm_1 = \frac{-\omega^2 \Omega \pm 1 + i \omega \omega_0}{\omega^2 \Omega^2 \omega_0^2 + \omega^2} \cdot \frac{c + \omega^2 b}{i \omega \Omega (\omega^2 \Omega^2 - \omega^2)} \]

After normalization,

\[ o^\pm_1 = \frac{i \Omega}{\omega} \left[ \frac{1}{\Omega^2 - r^2_g} \left( \frac{r_g}{\Omega} \right) c - \left( \frac{r_g}{\Omega} \right)^2 b \right] \]