Rotation Algorithm Artifacts in Stereoscopic Images

by

Larry F. Hodges & David F. McAllister

GIT-GVU-91-02
July 1991

Graphics, Visualization & Usability Center

Georgia Institute of Technology
Atlanta GA 30332-0280
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Larry F. Hodges, MEMBER SPIE
Georgia Institute of Technology
College of Computing
Atlanta, Georgia 30332

David F. McAllister
North Carolina State University
Department of Computer Science
Raleigh, North Carolina 27695-8206

Abstract. We examine the effect of using rotations for generation of the left- and right-eye perspective views of a stereoscopic image. We show that this approach to stereoscopic display of perspective views results in vertical parallax between the left- and right-eye views and we present an analytic expression that characterizes this parallax in terms of the center of rotation, the location of the image plane, and the angle of rotation. We also derive an analytic expression that shows that the rotation of the perspective views results in a semicylindrical stereo window with center at approximately \((0, 0, R/2)\) and radius \(R/2\), where \(R\) is the distance from the center of rotation to the center of projection. When this semicylindrical window is mapped to a flat display surface, relative depth relationships can be distorted.

Subject terms: stereoscopic displays; parallax; rotations; perspective views; stereo windows.

Optical Engineering 29(8), 973-976 (August 1990).

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1. Introduction
2. Equations for left- and right-eye views
3. Vertical parallax
4. Shape of the stereo window
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1. INTRODUCTION

The past five years have seen significant advances in the computer generation and display of stereoscopic images. The introduction of liquid crystal shutters, circular polarization, and 120 Hz refresh rates have resulted in bright, flickerless, true three-dimensional display stations that can be viewed by an observer wearing passively polarized glasses.\(^{1-4}\) Scientists and engineers in many application areas have recognized that the depth perception provided by a true three-dimensional display system is an important part of understanding spatial relationships between objects in a scene. Such areas include medical imaging, crystallography, cartography, remote positioning, meteorology, CAD/CAE, and visualization of scientific and engineering data.

Several methods have appeared in the literature for computing the left- and right-eye views of the stereo pair. Grotch,\(^5\) Roese and Mc Cleary,\(^6\) and Wixon\(^7\) describe stereoscopic computer graphic images based on rotations. Lipscomb approximates rotations with shears.\(^8\) Baker produces the left- and right-eye views with a lateral shift along the \(x\)-axis, along with a lateral shift of the resulting images relative to each other.\(^1\) Penna and Patterson suggest choosing two different eye positions and a single image plane.\(^9\) The generation of images using some of these methods introduces unintended artifacts that affect the user’s perception of the information in the scene or his ability to fuse the views into a single stereoscopic image.

In this paper we examine the effect of using rotations for generation of the left- and right-eye perspective views of a stereoscopic image that is to be displayed on a planar display medium. Such a medium could be a time-multiplexed CRT, a nonpolarizing projection screen, anaglyphs, or left- and right-eye views mounted side-by-side. In the following discussion we will refer to any display medium as a display screen. We will show that rotations introduce vertical parallax between the left- and right-eye views. This effect has been pointed out by several others, including Baker\(^1\) and Butts and McAllister.\(^10\) We derive an exact expression describing the amount of vertical parallax in terms of the center of rotation, location of the image plane, and the amount of rotation. This expression allows us to compute the maximum vertical parallax to the maximum allowable parallax, based on the results of Fender and Julesz.\(^11\) We will also show that rotations produce a nonplanar stereo window that can distort spatial relationships in the image when displayed on a flat screen. This distortion has been shown by numerical and graphical examples by Saunders.\(^12\) We derive an analytic expression that fully characterizes this distortion.

2. EQUATIONS FOR LEFT- AND RIGHT-EYE VIEWS

We will model the rotation approach as follows. Assume a right-handed coordinate system with eye point at \((0, 0, 0)\) and an image plane parallel to the \(x-y\) plane and located a distance \(d\) along the \(z\)-axis from the eye point. We assume device-independent screen coordinates in the range \(-1 \leq x \leq 1\) and \(-1 \leq y \leq 1\). We do a \(y\)-rotation about a point \((0,0,R)\) through an angle \(-\phi/2\) to produce the right-eye view and a \(y\)-rotation about the point \((0,0,0)\) through an angle \(+\phi/2\) to produce the left-eye view [Fig. (1)].

We begin by considering a point \(P_0 = (x_0, y_0, z_0)\). The rotated points become

\[
P_{0 \text{ left}} = \left( x_0\cos(\phi/2) + (z_0 - R)\sin(\phi/2), y_0, (z_0 - R)\cos(\phi/2) - x_0\sin(\phi/2) + R \right)
\]

\[
P_{0 \text{ right}} = \left( x_0\cos(\phi/2) - (z_0 - R)\sin(\phi/2), y_0, (z_0 - R)\cos(\phi/2) + x_0\sin(\phi/2) + R \right)
\]

After projection onto the image plane we have

\[
P_{0 \text{ left}}' = \left( \frac{x_{0 \text{ left}}}{w_{0 \text{ left}}}, \frac{y_{0 \text{ left}}}{w_{0 \text{ left}}}, \frac{z_{0 \text{ left}}}{w_{0 \text{ left}}} \right)
\]

\[
P_{0 \text{ right}}' = \left( \frac{x_{0 \text{ right}}}{w_{0 \text{ right}}}, \frac{y_{0 \text{ right}}}{w_{0 \text{ right}}}, \frac{z_{0 \text{ right}}}{w_{0 \text{ right}}} \right)
\]

Paper 2730 received May 4, 1989; revised manuscript received April 2, 1990; accepted for publication April 26, 1990.
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+ x-axis

\[ x_i = \frac{d \cdot x_0 \cos(\phi/2) + (z_0 - R) \sin(\phi/2))}{(z_0 - R) \cos(\phi/2) - x_0 \sin(\phi/2) + R} \]  

- x-axis

\[ y_i = \frac{d y_0}{(z_0 - R) \cos(\phi/2) - x_0 \sin(\phi/2) + R} \]  

3. VERTICAL PARALLAX

From Eqs. (4) and (6) the vertical parallax \( V \), equal to \( y_i - y_r \), is given by

\[ V = \frac{d y_0}{(z_0 - R) \cos(\phi/2) - x_0 \sin(\phi/2) + R} - \frac{d y_0}{(z_0 - R) \cos(\phi/2) + x_0 \sin(\phi/2) + R} \]  

\[ = \frac{2d x_0 \sin(\phi/2)}{[(z_0 - R) \cos(\phi/2) + R]^2 - x_0^2 \sin^2(\phi/2)} \]  

From the numerator of this expression we note that the vertical parallax is zero when either \( x_0 \) or \( y_0 \) is zero. Furthermore, \( V \) is an increasing function of \( x_0 y_0 \) and a decreasing function of \( z_0 \). Since the parallax increases as the denominator tends toward zero, it follows from Eq. (8) that large vertical parallax occurs when \( x_0 \) is approximately equal to

\[ x_0 = \frac{R(1 - \cos(\phi/2))}{\sin(\phi/2)} \]  

**TABLE I.** Maximum amount of absolute vertical parallax with a rotational angle of \( 6^\circ \) and \( z = R \), measured as a percentage relative to device-independent screen coordinates for values of \( R \) and \( d \) between 1 and 10. (\( R = \) the distance from the center of rotation to the center of projection; \( d = \) distance from center of projection to the projection plane.)

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**TABLE II.** Vertical parallax values in which fusion can be maintained and for fusion after breakdown. The observer is assumed to be a distance \( R \) (scaled to display device coordinates) from the display screen. Values are shown as a percentage relative to device-independent coordinates for \( R \) between 1 and 10. (\( R = \) the distance from the center of rotation to the center of projection.)

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The term \( \phi \) is always small, so \( \sin(\phi/2) \) is small, which means that \( x_0 \) must be large compared to \( z_0 \) for the denominator of Eq. (8) to be close to zero. Since with device-independent screen coordinates, viewable projected points have \( x \) and \( y \) coordinates \((x, y)\) such that \(|x| \leq 1 \) and \(|y| \leq 1 \) and the projection is of the form \( x_0 = xR/d \) and \( y_0 = yR/d \), this case may be ignored when \( d \gg 1 \). The maximum amount of vertical parallax for a point near the stereo window (\( z = R \)) measured as a percentage of the height of the display screen is given in Table I for various values of \( R \) and \( d \). The table assumes that \( \phi = 6^\circ \). The amount of vertical parallax is greatest at the corners of the screen and diminishes toward the center. Smaller values of \( \phi \) decrease the maximum vertical parallax, but they also decrease the horizontal parallax that produces the stereoscopic depth effect.

In their experiments with random-dot stereograms, Fender and Julesz found that a breakdown of fusion of the left- and right-eye images occurs with vertical parallax of about 20 arcmin. Assuming that the observer is located a distance \( R \) (scaled to display device coordinates) from the display screen then this value, as a percentage of screen height, may be calculated as \( R(\tan(1/3))/2 \). Once breakdown occurs, however, they also found that the views do not re-fuse unless they are brought much closer together, to approximately 6 arcmin \( R(\tan(1/10))/2 \). These values are shown in Table II as a percentage of screen.
height for values of $R$ between 1 and 10. As an example, assume
device-independent coordinates in the range $-1 \leq x \leq 1$ and
$-1 \leq y \leq 1$ scaled to a 30 cm vertical height CRT screen and
d = 5 units. Therefore an observer would have to sit a minimum of
approximately 75 cm from the CRT according to the maximum
breakdown limit ($R = 5$) and 120 cm from the CRT by the
maximum re- fusion limit ($R = 8$).

4. SHAPE OF THE STEREO WINDOW

In a stereoscopic image, the plane of the display screen is usually
referred to as the stereo window. Objects in the scene with
positive horizontal parallax between the left- and right-eye views
appear to lie behind the stereo window, objects with no hori-
zontal parallax appear to lie in the plane of the stereo window,
and objects with negative horizontal parallax appear to lie in
front of the stereo window. The horizontal parallax $H$ is equal
to $x_l - x_r$. Therefore, from Eqs. (3) and (5)

$$H = \frac{d[x_0 \cos(\phi/2) + (z_0 - R)\sin(\phi/2)]}{(z_0 - R)\cos(\phi/2) - x_0 \sin(\phi/2) + R}$$

$$- \frac{d[x_0 \cos(\phi/2) - (z_0 - R)\sin(\phi/2)]}{(z_0 - R)\cos(\phi/2) + x_0 \sin(\phi/2) + R}.$$  

(10)

This expression reduces to

$$H = \frac{d[x_0^2 + (z_0 - R)^2 \sin(\phi/2) + 2R(z_0 - R)\sin(\phi/2)]}{[(z_0 - R)\cos(\phi/2) + R]^2 - [x_0 \sin(\phi/2)]^2}.$$  

(11)

At the stereo window, horizontal parallax $H = x_{lw} - x_{rw} = 0$, so we have

$$\frac{d[x_0^2 + (z_0 - R)^2 \sin(\phi/2) + 2R(z_0 - R)\sin(\phi/2)]}{[(z_0 - R)\cos(\phi/2) + R]^2 - [x_0 \sin(\phi/2)]^2} = 0.$$  

(12)

Setting the numerator to zero, dividing through by $d(\sin(\phi))$, and
then completing the square results in

$$x_0^2 + \left[ z_0 - R + \frac{R\sin(\phi/2)}{\sin(\phi)} \right]^2 = \left[ \frac{R\sin(\phi/2)}{\sin(\phi)} \right]^2,$$  

(13)

which is the equation for a circle in the $x-z$ plane with center
$(0,0,R[1 - \sin(\phi/2)/\sin(\phi)])$ and radius $\frac{R\sin(\phi/2)}{\sin(\phi)}$. Since
the values of $\phi$ are small, usually $\leq 10^\circ$, $2\sin(\phi/2) = \sin(\phi)$
to three decimal places. Using this simplification we have a circle
with center at $(0,0,R/2)$ and with radius $R/2$. Since part of the
circle is outside the view volume, the result is a semicylindrical
stereo window. This is shown in Fig. 2.

Since points on the stereo window seem to lie in the plane
of the display screen, a curved stereo window distorts relative
depth relationships in the image. We demonstrate this distortion
with the random-dot stereogram shown in Fig. 3. Random-dot
stereograms have been used extensively by Julesz and others to
study stereopsis by removing monocular cues and providing only
parallax differences between left- and right-eye views. To create
Fig. 3, we began with a rectangular polygon located in the
$z = R/2$ plane. Both the polygon and the background are textured
using a random-dot pattern. The polygon was then projected
onto the background using the rotated perspective projection for
generating left- and right-eye views. The stereo image should
appear to be a rectangle sitting in front of the display screen

[Fig. 4(a)]. Instead we get quite a different image than the one
that our data describes. When the curved stereo window [Fig.
4(b)] is mapped to a flat display surface, the rectangle is similarly
deformed [Fig. 4(c)]. Since this image was created on a CRT
with integer pixel locations, roundoff of dot locations to the
nearest pixel causes the polygon to break into sections that appear
to lie in different depth planes [Fig. 4(d)].

5. SUMMARY

This approach to stereoscopic display of perspective views results
in vertical parallax between the left- and right-eye per-
spective views. We have presented an analytic expression that
characterizes this parallax in terms of the center of rotation, the
location of the image plane, and the angle of rotation. Using
this expression we show that the maximum parallax occurs at
the corners of the display screen and is minimized along the $x$
and $y$ axes of the screen. This expression allows us to choose a
center of rotation and image plane distance that puts the maxi-
mum vertical parallax within acceptable limits based on the
results of Fender and Julesz.

We have also derived an analytic expression that shows that
the rotation of perspective views results in a semicylindrical
Fig. 4. Schematic illustration of distortion in Fig. 3.

The stereo window with center approximately \((0, 0, R/2)\) and radius \(R/2\). Since this semicylindrical window is mapped to a planar display medium, relative depth relationships can be distorted.

We would also like to note that, in general, generation and display of stereoscopic computer graphics images is a young field with much that is still unknown or vaguely defined. Our analytic results are based on relatively straightforward criteria that are familiar to anyone who works in 3-D display. There are many other critical engineering issues that cannot be addressed until further research is done to resolve human factor and visual perception issues.

6. REFERENCES

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Several methods have appeared in the literature for computing the left- and right-eye views of the stereo pair. Grotch, Roese and McCleary, and Wixson describe stereoscopic computer graphic images based on rotations. Lipscomb approximates rotations with shears. Baker produces the left- and right-eye views with a lateral shift along the x-axis, along with a lateral shift of the resulting images relative to each other. Penna and Patterson suggest choosing two different eye positions and a single image plane. The generation of images using some of these methods introduces unintended artifacts that affect the user's perception of the information in the scene or his ability to fuse the views into a single stereoscopic image.

In this paper we examine the effect of using rotations for generation of the left- and right-eye perspective views of a stereoscopic image that is to be displayed on a planar display medium. Such a medium could be a time-multiplexed CRT, a nonpolarizing projection screen, anaglyphs, or left- and right-eye views mounted side-by-side. In the following discussion we will refer to any display medium as a display screen. We will show that rotations introduce vertical parallax between the left- and right-eye views. This effect has been pointed out by several others, including Baker and Butts and McAllister. We derive an exact expression describing the amount of vertical parallax in terms of the center of rotation, location of the image plane, and the amount of rotation. This expression allows us to compare the maximum vertical parallax to the maximum allowable parallax, based on the results of Fender and Julesz. We will also show that rotations produce a nonplanar stereo window that can distort spatial relationships in the image when displayed on a flat screen. This distortion has been shown by numerical and graphical examples by Saunders. We derive an analytic expression that fully characterizes this distortion.

2. EQUATIONS FOR LEFT- AND RIGHT-EYE VIEWS
We will model the rotation approach as follows. Assume a right-handed coordinate system with eye point at (0,0,0) and an image plane parallel to the x-y plane and located a distance d along the z-axis from the eye point. We assume device-independent screen coordinates in the range $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. We do a z-rotation about a point (0,0,R) through an angle $-\phi/2$ to produce the right-eye view and a z-rotation about the point (0,0,R) through an angle $+\phi/2$ to produce the left-eye view [Fig. (1)].

We begin by considering a point $P_0 = (x_0, y_0, z_0)$. The rotated points become

$$P_{0\text{ left}} = (x_0 \cos(\phi/2) + (z_0 - R) \sin(\phi/2), y_0, (z_0 - R) \cos(\phi/2) - x_0 \sin(\phi/2) + R)$$

(1)

$$P_{0\text{ right}} = (x_0 \cos(\phi/2) - (z_0 - R) \sin(\phi/2), y_0, (z_0 - R) \cos(\phi/2) + x_0 \sin(\phi/2) + R)$$

(2)

After projection onto the image plane we have

Paper 2730 received May 4, 1989; revised manuscript received April 2, 1990; accepted for publication April 26, 1990.
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Fig. 1. Geometry for rotation.

\[ x_t = \frac{d(x_0 \cos(\phi/2) + (z_0 - R) \sin(\phi/2))}{(z_0 - R) \cos(\phi/2) - x_0 \sin(\phi/2) + R} \]  

(3)

\[ y_t = \frac{dy_0}{(z_0 - R) \cos(\phi/2) - x_0 \sin(\phi/2) + R} \]  

(4)

\[ z_t = \frac{d(x_0 \cos(\phi/2) - (z_0 - R) \sin(\phi/2))}{(z_0 - R) \cos(\phi/2) + x_0 \sin(\phi/2) + R} \]  

(5)

\[ y_r = \frac{dy_0}{(z_0 - R) \cos(\phi/2) + x_0 \sin(\phi/2) + R} \]  

(6)

3. VERTICAL PARALLAX

From Eqs. (4) and (6) the vertical parallax \( V \), equal to \( y_t - y_r \), is given by

\[ V = \frac{dy_0}{(z_0 - R) \cos(\phi/2) - x_0 \sin(\phi/2) + R} \]  

(7)

\[ = \frac{2dx_0 \sin(\phi/2)}{(z_0 - R) \cos(\phi/2) + R} \]  

(8)

From the numerator of this expression we note that the vertical parallax is zero when either \( x_0 \) or \( y_0 \) is zero. Furthermore, \( V \) is an increasing function of \( x_0 \) and a decreasing function of \( z_0 \). Since the parallax increases as the denominator tends toward zero, it follows from Eq. (8) that large vertical parallax occurs when \( z_0 \) is approximately equal to

\[ z_0 + R[1 - \cos(\phi/2)] \]  

(9)

\[ \sin(\phi/2) \]

TABLE I. Maximum amount of absolute vertical parallax with a rotational angle of 6° and \( z = R \), measured as a percentage relative to device-independent screen coordinates for values of \( R \) and \( d \) between 1 and 10. (\( R \) = the distance from the center of rotation to the center of projection, \( d \) = distance from center of projection to the projection plane.)

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TABLE II. Vertical parallax values in which fusion can be maintained and for fusion after breakdown. The observer is assumed to be a distance \( R \) (scaled to display device coordinates) from the display screen. Values are shown as a percentage relative to device-independent coordinates for \( R \) between 1 and 10. (\( R \) = the distance from the center of rotation to the center of projection.)

<table>
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The term \( \phi \) is always small, so \( \sin(\phi/2) \) is small, which means that \( x_0 \) must be large compared to \( z_0 \) for the denominator of Eq. (8) to be close to zero. Since with device-independent screen coordinates, viewable projected points have \( x \) and \( y \) coordinates \((x_0, y_0)\) such that \( |x_0| \leq 1 \) and \( |y_0| \leq 1 \) and the projection is of the form \( x_t = x d/z \) and \( y_t = y d/z \), this case may be ignored when \( d \gg 1 \). The maximum amount of vertical parallax for a point near the stereo window (\( z = R \)) measured as a percentage of the height of the display screen is given in Table I for various values of \( R \) and \( d \). The table assumes that \( \phi = 6° \). The amount of vertical parallax is greatest at the corners of the screen and diminishes toward the center. Smaller values of \( \phi \) decrease the maximum vertical parallax, but they also decrease the horizontal parallax that produces the stereoscopic depth effect.

In their experiments with random-dot stereograms, Fender and Julesz found that a breakdown of fusion of the left- and right-eye images occurs with vertical parallax of about 20 arcmin. Assuming that the observer is located a distance \( R \) (scaled to display device coordinates) from the display screen, then this value, as a percentage of screen height, may be calculated as \( R \tan(1/3) / 2 \). Once breakdown occurs, however, they also found that the views do not re-fuse unless they are brought much closer together, to approximately 6 arcmin \( R \tan(1/10) / 2 \). These values are shown in Table II as a percentage of screen
height for values of $R$ between 1 and 10. As an example, assume
device-independent coordinates in the range $-1 \leq x, y \leq 1$ and
$-1 \leq z \leq 1$ scaled to a 30 cm vertical height CRT screen and
d = 5 units. Therefore an observer would have to sit a minimum
of approximately 75 cm from the CRT according to the maxi-
mum breakdown limit ($R = 5$) and 120 cm from the CRT by
the maximum re-fusion limit ($R = 8$).

4. SHAPE OF THE STEREO WINDOW

In a stereoscopic image, the plane of the display screen is usually
referred to as the stereo window. Objects in the scene with
positive horizontal parallax between the left- and right-eye views
appear to lie behind the stereo window, objects with no hori-
izontal parallax appear to lie in the plane of the stereo window,
and objects with negative horizontal parallax appear to lie in
front of the stereo window. The horizontal parallax $H$ is equal
to $x_i - x$. Therefore, from Eqs. (3) and (5)

$$H = \frac{d(x_0 \cos(\phi/2) + (z_0 - R) \sin(\phi/2))}{(z_0 - R) \cos(\phi/2) - x_0 \sin(\phi/2) + R} - \frac{d(x_0 \cos(\phi/2) - (z_0 - R) \sin(\phi/2))}{(z_0 - R) \cos(\phi/2) + x_0 \sin(\phi/2) + R}.$$  

(10)

This expression reduces to

$$H = \frac{d[(x_0^2 + (z_0 - R)^2) \sin(\phi) + 2R(z_0 - R) \sin(\phi/2)]}{[(z_0 - R) \cos(\phi/2) + R]^2 - [x_0 \sin(\phi/2)]^2}.$$  

(11)

At the stereo window, horizontal parallax $H = x_{lw} - x_{rw} = 0$, 
so we have

$$\frac{d[(x_0^2 + (z_0 - R)^2) \sin(\phi) + 2R(z_0 - R) \sin(\phi/2)]}{[(z_0 - R) \cos(\phi/2) + R]^2 - [x_0 \sin(\phi/2)]^2} = 0.$$  

(12)

Setting the numerator to zero, dividing through by $d(\sin(\phi))$, and
then completing the square results in

$$x_0^2 + (z_0 - R + \frac{R \sin(\phi/2)}{\sin(\phi)})^2 = \left(\frac{R \sin(\phi/2)}{\sin(\phi)}\right)^2.$$  

(13)

which is the equation for a circle in the $x$-$z$ plane with center
$(0, 0, R(1 - \sin(\phi/2)/\sin(\phi)))$ and radius $= (R \sin(\phi/2))/\sin(\phi)$. Since
the values of $\phi$ are small, usually $\approx 10^\circ$, $2 \sin(\phi/2) = \sin(\phi)$
to three decimal places. Using this simplification we have a circle
with center at $(0, 0, R/2)$ and with radius $R/2$. Since part of the
circle is outside the view volume, the result is a semicylindrical
stereo window. This is shown in Fig. 2.

Since points on the stereo window seem to lie in the plane
of the display screen, a curved stereo window distorts relative
depth relationships in the image. We demonstrate this distortion
with the random-dot stereogram shown in Fig. 3. Random-dot
stereograms have been used extensively by Julesz and others
to study stereopsis by removing monocular cues and providing only
parallax differences between left- and right-eye views. To cre-
ate Fig. 3, we began with a rectangular polygon located in the
$z = R/2$ plane. Both the polygon and the background are textured
using a random-dot pattern. The polygon was then projected
onto the background using the rotated perspective projection for
generating left- and right-eye views. The stereo image should
appear to be a rectangle sitting in front of the display screen

[Fig. 4(a)]. Instead we get quite a different image than the one
that our data describes. When the curved stereo window [Fig.
4(b)] is mapped to a flat display surface, the rectangle is similarly
deformed [Fig. 4(c)]. Since this image was created on a CRT
with integer pixel locations, roundoff of dot locations to the
nearest pixel causes the polygon to break into sections that appear
to lie in different depth planes [Fig. 4(d)].

5. SUMMARY

This approach to stereoscopic display of perspective views
results in vertical parallax between the left- and right-eye per-
spective views. We have presented an analytic expression that
characterizes this parallax in terms of the center of rotation, the
location of the image plane, and the angle of rotation. Using
this expression we show that the maximum parallax occurs at
the corners of the display screen and is minimized along the $x$
and $y$ axes of the screen. This expression allows us to choose a
center of rotation and image plane distance that puts the max-
imum vertical parallax within acceptable limits based on the
results of Fender and Julesz.

We have also derived an analytic expression that shows that
the rotation of perspective views results in a semicylindrical

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Fig. 4. Schematic illustration of distortion in Fig. 3.

stereo window with center at approximately (0, 0, R/2) and radius R/2. Since this semicylindrical window is mapped to a planar display medium, relative depth relationships can be distorted.

We would also like to note that, in general, generation and display of stereoscopic computer graphic images is a young field with much that is still unknown or vaguely defined. Our analytic results are based on relatively straightforward criteria that are familiar to anyone who works in 3-D display. There are many other critical engineering issues that cannot be addressed until further research is done to resolve human factor and visual perception issues.

6. REFERENCES


Larry F. Hodges joined the faculty of Georgia Tech in the fall of 1988 after completing his Ph.D. in computer engineering at North Carolina State University. He also holds an MS in computer studies from NCSU and a BA with a double major in mathematics and physics from Elon College. Dr. Hodges has presented numerous technical courses and invited talks in raster graphics, true 3-D display technologies, and stereographics. He was course chair for "Steresographics" at SIGGRAPH '89 and was 1989–90 national co-chair for the Society for Information Display’s Special Technology Committee on 3-D Display. He is a member of the Association for Computing Machinery, IEEE Computer Society, Society for Information Display, and SPIE.

David F. McAllister is a professor in the Computer Science Department of North Carolina State University. He has developed and taught short courses in graphics for industry and at major computer science and display technology conferences. He is the author of numerous papers on curve and surface representation and three-dimensional display technology. His research interests include computer graphics and software reliability. Dr. McAllister received his MS and BS in mathematics from Purdue University and the University of North Carolina, Chapel Hill, respectively. He obtained his Ph.D. in computer science from the University of North Carolina, Chapel Hill in 1972. He is a member of the Association for Computing Machinery and IEEE.