**GEORGIA INSTITUTE OF TECHNOLOGY**  
**OFFICE OF CONTRACT ADMINISTRATION**  
**PROJECT ADMINISTRATION DATA SHEET**

| **Project No.** | E-25-653 |  
| **Project Director:** | Jerry H. Ginsberg |  
| **Sponsor:** | Office of Naval Research 800 North Quincy Street |  

**Type Agreement:** Contract No. N00014-84-K-0713  
**Award Period:** From 9/1/84 to 10/31/87  
**Sponsor Amount:**  
- Estimated: $256,820  
- Funded: $79,700  
- Total to Date: $256,820  
- Cost Sharing Amount: $79,700  

**Title:** Improved Algorithms for Transient and Steady-State Fluid-Structural Interaction  

**Administrative Data:**  
1) Sponsor Technical Contact:  
Program Manager, Solid Mechanics  
Office of Naval Research  
800 North Quincy Street  
Arlington, VA 22217  
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Georgia Institute of Technology  
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Atlanta, GA 30332  
(404) 881-4374  
3) Sponsor Security Contact:  
Military Security Classification:  
4) Sponsor Proprietary Information:  
Company/Industrial Proprietary:  

**Restrictions:**  
- Attached Gov't Supplemental Information Sheet for Additional Requirements.  
- Foreign travel must have prior approval — Contact OCA in each case. Domestic travel requires sponsor approval where total will exceed greater of $500 or 125% of approved proposal budget category.  
- Equipment: Title vests with Acquisition less than $1,000 GTRC. Acquisition cost $1,000 or more determination is deferred until acquisition pursuant to DFARS 52.235-14  

**Comments:**  
- Funded through 8/31/85
### NOTICE OF PROJECT CLOSEOUT

**Closeout Notice Date:** 04/02/91

**Project No.:** E-25-653  
**Center No.:** R5831-0A0  
**Project Director:** GINSBERG J H  
**School/Lab:** MECH ENGR  
**Sponsor:** NAVY/OFN OF NAVAL RESEARCH  
**Contract/Grant No.:** N00014-84-K-0713  
**Contract Entity:** GTRC  
**Effective Completion Date:** 901231 (Performance) 910228 (Reports)

### Closeout Actions Required:

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**Comments:**

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**Project Under Main Project No.:**

**Subprojects Project No.:**

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: Final Patent Questionnaire sent to PDPI.
END-OF-THE-FISCAL-YEAR LETTER
CONTRACT NO. N00014-84-K-0713, CODE 432-F

IMPROVED ALGORITHMS FOR TRANSIENT AND STEADY-STATE
FLUID-STRUCTURE INTERACTION

DESCRIPTION OF RESEARCH

Simulation of the interaction between submerged structures and the
surrounding fluid currently places a very large demand on computer
resources. Analytical work is hampered by complications in the system
configuration and excitation. Discretized formulations that simultaneously
model the structure and fluid responses require an excessive number of
degrees of freedom. For these reasons, approximate methods to decouple the
fluid and structural responses have been developed. Particularly noteworthy
in this regard are the doubly asymptotic approximations, which match various
limits of the laws of fluid mechanics. However, the accuracy of all decou-
pling procedures depends on the time scales associated with the excitation,
the structural response, and the period of observation.

The present project is developing a direct representation of the fluid-
structure interaction that does not rely on approximate principles. The
foundation for the concept is the Kirchhoff-Helmholtz integral theorem that
governs, without approximation, the radiation and scattering of sound waves
from an arbitrary curved surface. This theorem has been employed directly
by other researchers to form a boundary element analysis. Our modeling
approach is based on a variational principle that is derived from the K-H
theorem, analogous to the derivation of Hamilton's principle from Newton's
laws. The virtue of the variational principle is that it ensures that
predictions of the response will be the best possible result within the
confines of the functional form that was initially assumed. Furthermore,
our formulation avoids a nonintegrable singularity that is often encountered
in direct applications of the K-H theorem.

The overall goal of the project is to develop procedures for implement-
ing this variational description of fluid-structure interaction in cases of
mono-frequency and transient excitation in typical structural
configurations. The three types of formulations that have been identified
are modal series expansions, finite elements, and finite differences. The
project will assess the computational merits of each through studies of
representative systems.

SIGNIFICANT RESULTS IN THE PAST YEAR

We selected the problem of radiation and scattering from a thin,
unbaffled disk for the first implementation of the variational principle.
This system displays all features of interaction problems, while still being
geofometrically uncomplicated. It has been investigated by other methods, so
it provides a strong basis for validation.

The primary feature of a thin disk suspended in an acoustic medium is
the sharpness of its edges, which leads to a strong diffraction effect. In
order to judge the degree to which the variational method can handle dif-
fraction, we began with the case of a disk vibrating at a single frequency,
which we have shown is equivalent in the rigid case to normal incidence of a
plane wave.

The variational principle in the single frequency case requires that
the action integral $K$ be stationary to small increments in the acoustic
potential and particle velocities normal to the surface. The general
dependence of $K$ in the single frequency case is

$$K = \iint \left\{ V\Phi_s(r), V\Phi_s(r'), v_n, v_n \right\} \, dS' \, dS$$

where $V\Phi_s(r)$ is the gradient of the potential parallel to the surface.
Also, $r$ and $r'$ denote two arbitrary locations on the surface.

The first technique we implemented was the modal expansion version of
the variational principle. This involved expressing the potential function
in a series of assumed mode shapes $\psi_i$ whose modal amplitudes $a_i$ are harmonic
at the frequency of the excitation. The stationarity of the $K$ integral
leads to a set of simultaneous amplitude-frequency-phase angle equations for
the modal amplitudes. A computer program was developed to carry out the
integrals over the surface area that arise in the process of substituting
the modal series into $K$. A major portion of the effort was involved in
correctly accounting for the singularity in the integrand of $K$ that arises
when the two points $r$ and $r'$ approach each other.

The pressure distributions we have obtained for the modal series have
been remarkably good. In Figure 1, we show a comparison of our results with
a direct solution of the wave equation using special functions. (For
brevity, only the imaginary part is shown. The results are equally good for
the real portion.) The primary parameter for these curves is $ka = \text{frequency} \times \text{radius} / \text{speed of sound}$. We were able to obtain close agreement
with various families of mode functions.

The next stage of the project involved implementing discretized ver-
sions of the variational principle. Finite element and finite difference
techniques both lead to a representation of the $K$ integral in terms of the
response at mesh points on the surface. We instituted parallel efforts to
implement each. The finite element version has been quite successful,
primarily because of its similarity to the modal expansion development. We
used a linear interpolation element centered on one mesh point, and extend-
ing to the neighboring mesh point on either side. The evaluation of the
finite element coefficients was achieved by a computer program. In Figure
2, we show a comparison of the results obtained for different primary
parameters in the finite element discretization. The parameter NBASES
refers to the number of elements included in the model, whereas NDIVS refers
to the number of divisions used to evaluate the finite element coefficients.
Once again, there is excellent agreement with the theoretical result. Notice
that the agreement is quite good for the most coarse resolution, but
there is significant improvement obtained by increasing the number of
elements.

The finite difference approach has thus far not met with equal success.
The initial approach was a straightforward central difference representation
that seemed to encounter numerical instability difficulties. We recently
began to reformulate the problem using a different discretization scheme,
but numerical results have not been obtained thus far.
PLANS FOR NEXT YEAR'S RESEARCH

The project will extend the implementation of the variational principle in two directions: greater generality of the excitation and more complicated structural configurations. The cases of arbitrary vibration of a disk, and of oblique incidence of a plane wave on a disk, will be studied, in order to remove the axisymmetric spatial dependence of the surface pressure. This effort will then be extended to incorporate the analysis of deformation effects for the case of an incident signal, which will involve a combined formulation featuring Hamilton's principle for the structural motion.

The generalized structural configuration will be a finite cylinder with flat ends. This system has much similarity in regard to diffraction with the thin disk problem. However, the dual nature of the geometry, with a finite cylinder that joins planar ends, has thus far permitted only approximate or numerical treatment. One difficulty for the modal formulation of the variational principle is identifying an appropriate set of mode functions that satisfy the geometrical boundary conditions. Also, the treatment of the integrand in the case where the points \( r \) and \( r' \) are situated on different types of surfaces requires careful consideration. The initial efforts for the finite cylinder will treat axial vibration as a rigid body, because limiting solutions of the problem are known. When the modal formulation has been verified, the finite element version will be implemented and the surface motion will be extended to cases where there is transverse motion on the cylindrical surface.

LIST OF PRESENTATIONS


LIST OF PUBLICATIONS

LIST OF PARTICIPANTS

Dr. Jerry H. Ginsberg, Professor of Mechanical Engineering.
Prof Allan D. Pierce, Regent's Professor of Mechanical Engineering.
Mr. Xiao-Feng Wu - graduate research assistant - Ph.D. candidate.
Mr. John DiMarco - graduate research assistant - M.S. candidate.

OTHER SPONSORED RESEARCH

J. H. Ginsberg
2. O.N.R. - code 425UA, "Theoretical and experimental investigations of nonlinearity in high-intensity sound beams," $171,500, 9/1/85 to 8/31/86.

A. D. Pierce

--- Modal analysis
□ Leitner's results

Figure 1: Analytical & modal series predictions for the pressure distribution on a disk (imaginary part).

Figure 2: Finite element & modal series predictions for the pressure distribution on a disk (imaginary part).
October 24, 1988

Dr. R. E. Whitehead,
director, Mechanics Division
Office of the Chief of Naval Research
800 N. Quincy Street
Arlington, VA 22217-5000

Dear Dr. Whitehead:

Enclosed you will find the "End-of the Fiscal Year Letter" and other documentation summarizing the research activity in contract number 87-K-0713, for which I am principal investigator. I hope my failure to meet the requested Sept. 30 date for submission of this material is not a serious inconvenience. Please call me if there are any questions regarding the information reported here.

Sincerely yours,

Jerry H. Ginsberg
Professor of Mechanical Engineering

cc: Dr. Albert J. Tucker, code 1132-SM
GT Office of Contract Administration
DESCRIPTION OF RESEARCH GOALS

The interaction between a vibrating submerged structure and the surrounding fluid, which features coupling between the surface pressure distribution and the structural displacement, is an inherent feature for sound radiation and target strength analyses. A variety of approaches have been implemented in the past, but each suffers from serious limitations. Formal mathematical analysis is suitable only for the simplest structural models, and full finite element descriptions of realistic structures and the surrounding medium lead to computer excessively large simulations. The doubly asymptotic expansion replaces the actual laws for the surface pressure with an approximate impedance-type boundary condition of uncertain accuracy. Boundary element formulations rationally represent the interaction phenomena without explicitly solving field equations for the fluid, at the expense of an enormous increase in computational effort due to the need to cover the surface with a reasonably fine mesh.

The variational principle we developed earlier in the project has the same foundation as boundary elements, in that it is derived from the Kirchhoff-Helmholtz integral theorem, but it provides the ability for one to optimize the errors associated with an assumed form of the pressure distribution. That is, if one begins with a representation of the pressure in terms of a sequence of functions having first order errors in comparison with the actual result, their combination derived from the variational principle will be second order. The functions implemented in this context may be a small number of analytical functions covering the entire or a large number of discrete elements covering segments of the wetted surface, depending on the insight and experience of the modeler. In previous years we demonstrated the validity of the principle for axisymmetric bodies undergoing known vibration, specifically, cylinders in axial and breathing motion and transversely oscillating disks.

SIGNIFICANT RESULTS IN THE PAST YEAR

Our efforts for the past year focussed on three aspects of the variational principle for fluid-structure interaction. Each of these efforts considered acoustic radiation phenomena, but we have already shown that the treatment of scattering of an ambient acoustic wave may be obtained as a straightforward extension. The problem of a disk in a finite baffle provided the framework for assessing the relative merits of various types of approximations for the pressure distribution and vibratory displacement. A substantial effort currently underway will generate an analytical/numerical model for radiation from general shell structures. The third effort has begun to address the basic steps required to incorporate asymmetrical excitations and inhomogeneous structural features, such as rib stiffeners.

Conventional applications of variational principles, such as the Rayleigh-Ritz method, employ analytical functions to represent the response variables in the entire domain, or at least in some sub-space characterized
by a common geometry. On the other hand, finite element descriptions can be derived from the same principles. Last year we extended the variational formulation to include the effect of elasticity for a centrally excited plate in a finite baffle. This year we developed a "discrete element" description of the interaction phenomena by using a sequence of functions covering segments of the wetted surface. These discrete elements may be employed in conjunction with either a modal or finite element formulation of the structural response. We found that the penalty, as measured by the number of unknowns, involved in using discrete elements rather than analytical functions, decreases as one increases either the frequency of the excitation or the relative width of the baffle, particularly if one does not have reasonable insight into the appropriate choice of analytical functions. In conjunction with these developments, we have assessed the convergence properties and accuracy of the various formulations. An interesting result is that a solution of the same problem based on the doubly asymptotic approximation was shown to give reasonably accurate results for the structural displacement, but poor results for the surface pressure distribution, which means that such an approach may not be useful for studies of acoustic radiation and scattering.

We are currently extending the treatment of structural elasticity to arbitrary shells of revolution, whose shape is described by specifying the outer cross-sectional radius as a parametric function of the arclength along the generating curve. This has led to alteration in our insight into the way in which surface displacement and pressure couple in the variational principle. In the flat geometry of a disk the manner in which the pressure amplitudes couple to the elasticity equations through a set of coefficients \([\xi]\) that are symmetric (except for a factor) to the coupling coefficients \([r]\) between displacement and pressure amplitudes in the acoustic equations. However, in the case of shells additional terms arise in \([\xi]\) whenever the surface has compound curvature. The general form of the coupled equations is

\[
\begin{bmatrix}
[K_s] - \Omega^2 [m_s] & [K_{sw}] & [0]
\end{bmatrix}
\begin{bmatrix}
(s)
\end{bmatrix}
- \begin{bmatrix}
(Q_s)
\end{bmatrix}
\begin{bmatrix}
[K_{sw}] & [K_w] - \Omega^2 [m_w] & [\xi]
\end{bmatrix}
\begin{bmatrix}
(w)
\end{bmatrix}
- \begin{bmatrix}
(Q_w)
\end{bmatrix}
\begin{bmatrix}
[0]
\end{bmatrix}
\begin{bmatrix}
(\xi)
\end{bmatrix}
\begin{bmatrix}
[a]
\end{bmatrix}
\begin{bmatrix}
(p)
\end{bmatrix}
\begin{bmatrix}
(0)
\end{bmatrix}
\]

where \((s)\) and \((w)\) represent amplitudes of the assumed mode functions for tangential and normal displacements of the structure, \((p)\) contains the amplitudes of the functions representing the surface pressure, and \((Q_s)\) and \((Q_w)\) represent the corresponding generalized forces. The stiffness and inertia coefficients appearing above are derived from Hamilton's principle for the structural motion.

The associated numerical modelling has been implemented for the case of an arbitrary body of revolution excited by a transverse excitation at its tip. The predictions are currently being checked for convergence and accuracy against the known analytical solution for spherical shells. The simplicity of that case is actually a very strong check on the results derived from the variational principle, since the analytical solution is expressed in terms of orthogonal Legendre functions, which must be verified by the numerical analysis using nonorthogonal functions. An interesting aspect is that the bulk of the computations seem to lie in the formulation
of the set of coefficients \([a]\) appearing above, which corresponds to the solution of the homogeneous problem. This part of the formulation is not affected by the structural modes, provided that one fixes the frequency and surface geometry, and changes neither the number nor type of functions representing the surface pressure. This suggests that ultimately a powerful design tool might emerge, in which alternative structural configurations may be assessed in a process resembling component mode synthesis for structural dynamics.

In regard to rib stiffeners, we ultimately will use the foregoing model of an arbitrary shell of revolution as the fundamental structure. However, in order to not delay exploration of the role of such structural elements, we are currently considering a flat elastic plate containing concentric stiffening rings. Two techniques for describing the structural dynamic aspects are being considered. In an integrated approach, a common set of modes satisfying all displacement and compatibility conditions between the plate and the stiffeners are employed. Alternatively, independent modes may be selected for the plate and the stiffeners, with compatibility enforced by means of internal reaction forces. Both methods obviously should ultimately lead to the same result. However, the internal forces may lead to spurious sources of acoustic radiation. The ability of the variational principle to eliminate such sources, and the corresponding rate of convergence, need to be assessed, with particular attention to the behavior as a function of the wavelength of the shell motion in comparison to the dimensions of the stiffeners.

PLANS FOR NEXT YEAR’S RESEARCH

The primary emphasis for the coming year is to begin to examine scale effects arising in the interaction of stiffeners and shells, particularly as to the relative significance of the physical dimensions of the shell and stiffener in comparison to the wavelengths of the structural response. We will follow a multi-pronged attack on this question using the flat disk and axisymmetric shell models. Attempts will be made to use overlapping mode functions having small and large scale resolutions, as well as travelling wave modes that can better simulate the small scale impedance effects of the stiffeners. Of course, a prerequisite to usage of the shell model is its validation, which requires examination of accuracy for more general configurations than a spherical shell, specifically, cylindrical shells with flat and hemispherical ends, and prolate spheroids. In each case, the evaluation will assess the validity and efficiency of the variational algorithms as a function of the frequency of excitation.

Our effort to generalize the variational principle to include asymmetrical effects is currently in the theoretical stage. We have shown that an arbitrary body of revolution which is subjected to a nonsymmetric excitation may be treated by a Fourier series expansion in the circumferential direction, with each circumferential number reducing to an equivalent axisymmetric problem. However, in this formulation, one must be careful to recognize asymmetries in the coefficients associated with non-self-adjointness of the basic acoustical equations. We will begin to quantify these concepts with the basic model of a flat disk.
LIST OF PUBLICATIONS/REPORTS/PRESENTATIONS

I.a. PAPERS IN REFEREED JOURNALS

I.b. CONFERENCE PROCEEDINGS

II. TECHNICAL REPORTS

III.a. INVITED PRESENTATIONS

III.b. CONTRIBUTED PRESENTATIONS
IV. BOOKS


LIST OF AWARDS


LIST OF PARTICIPANTS

Dr. Jerry H. Ginsberg, Professor of Mechanical Engineering.
Dr. Allan D. Pierce, Regent's Professor of Mechanical Engineering.
Dr. Xiao-Feng Wu - graduate research assistant - Ph.D. awarded June 1987.
Mr. Pei-Tai Chen - graduate research assistant - Ph.D. candidate.
Mr. J. Gregory McDaniel - graduate research assistant - M.S. candidate.

OTHER SPONSORED RESEARCH

J. H. Ginsberg


September 26, 1989

Dr. M. M. Reischman  
Director (Acting), Mechanics Division  
Office of the Chief of Naval Research  
800 N. Quincy Street  
Arlington, VA 22217-5000

Dear Dr. Reischman:

Enclosed you will find two copies of the "End-of the Fiscal Year Letter" summarizing the research activity in contract number N00014-84-K-0713, for which I am principal investigator. Please call me if there are any questions regarding the information reported here.

Sincerely yours,

Jerry H. Ginsberg  
The George W. Woodruff Chair in Mechanical Systems

cc: Dr. Phillip B. Abraham, code 1132-SM  
GT Office of Contract Administration
END-OF-THE-FISCAL-YEAR LETTER
CONTRACT NO. N00014-84-K-0713, CODE 1132SM

IMPROVED ALGORITHMS FOR TRANSIENT AND STEADY-STATE
FLUID-STRUCTURE INTERACTION

DESCRIPTION OF SCIENTIFIC RESEARCH GOALS

The interaction between a vibrating submerged structure and the surrounding fluid, which features coupling between the surface pressure distribution and the structural displacement, is an inherent feature for sound radiation and target strength analyses. A variety of approaches have been implemented in the past, but each suffers from serious limitations. Formal mathematical analysis using separation of variables or integral transform techniques is suitable only for the simplest structural models, while full finite element descriptions of realistic structures and the surrounding medium lead to excessively large computer simulations. The doubly asymptotic expansion and other approximate representations of the influence of the exterior fluid domain replace the actual laws for the surface pressure with an approximate impedance-type boundary condition of uncertain accuracy. Boundary element formulations rationally represent the interaction phenomena without explicitly solving field equations for the fluid, at the expense of an enormous increase in computational effort due to the need to cover the surface with a reasonably fine mesh.

The limitation of classical analytical techniques has led to a widely held belief that analytical solutions, in which the response is expressed in a functional form, can only be obtained in idealized systems whose exterior shape coincides with a constant coordinate surface of one of a set of orthogonal curvilinear coordinates (e.g. an infinite cylinder for a set of cylindrical coordinates). The variational acoustics principle (VAP) relating surface pressure and surface particle velocity, which we derived earlier, partially invalidates the foregoing statement. VAP assumes that the particle velocity on the surface is temporally harmonic with a known spatial distribution. For radiation problems this velocity is the structural velocity normal to the surface, while for scattering the velocity is the difference between the structural and incident normal velocities. VAP represents the surface response as a series expansion in a set of assumed basis functions. Since the solution derived from VAP is the coefficients of this series, the primary difference from the results of a classical analysis lies in the fact that VAP determines these coefficients by numerical techniques. However, if one considers the complicated form of most analytical solutions, whose coefficients can only be fully understood by numerical evaluations for specified system parameters, it is apparent that the difference in the manner in which the series coefficients are determined is not so significant.

Because VAP represents the surface pressure in a functional series form, it involves a substantially reduced set of unknowns in comparison to boundary element formulations, which are based on point-wise discretizations. In addition, VAP shares several other virtues with other variational principles. It is essentially an optimization process which selects the coefficients of the modal series so as to minimize the deviation of the derived solution relative to the true one. Indeed, convergence to the exact
solution is guaranteed if one progressively increases the number of independent basis functions, subject to computer limitations. The optimization abilities of other weighted residual methods are less powerful. Another feature of VAP is that it enables one to make use of prior experience with the problem, either from experiments or analysis by simpler techniques. Tailoring the basis functions to match this "expert" knowledge enables one to substantially reduce the number of unknown series coefficients.

Prior phases of the project were devoted to the derivation of VAP and to validating it through comparison with solutions derived by other analytical procedures. Now the primary objective of the project is to develop the capability of using VAP to derive a complete formulation of radiation and scattering problems in elastic structures, in which case both the surface pressure and surface displacement are unknown. The structural models ultimately will include realistic effects, such as stiffeners and bulkheads. An important aspect of the research effort is the identification of optimal methods for evaluating the various parts of the VAP formulation, in order to minimize numerical errors and maximize computer efficiency.

SIGNIFICANT RESULTS OF THE PAST YEAR

1. We completed analysis of the surface pressure and vibratory response of a circular elastic disk (membrane or plate) that is excited axisymmetrically, for the case where the supporting baffle is a finite annulus. This development involved simultaneous application of VAP and Hamilton's principle for the structural motion. The study was an extension of our earlier treatment of a disk in an infinite baffle. It allowed us to examine a variety of effects influencing the rate of convergence and accuracy. Extensive evaluations of the response in the frequency domain for the case of a very large baffle showed that the minor discrepancies from an earlier analysis of the infinite baffle problem were attributable to differences in plate theory. A comprehensive paper describing the merger of VAP and Hamilton's principle, as well as the convergence and accuracy issues, was submitted and recently accepted for publication in JASA.

2. We began an analysis of the vibration and acoustic radiation of submerged shells subjected to axisymmetric excitation. A complete solution in this case requires evaluation of two surface displacements components and the surface pressure distribution. The analytical solution of this problem reveals that the surface pressure and displacement are representable as series of Legendre functions, with no coupling between different orders of the functions. We first confirmed that when VAP and Hamilton's principle are implemented with Legendre functions, the equations for different orders do uncouple, and that the results match those obtained analytically. As a further test, we employed VAP and Hamilton's principle to solve the spherical shell problem with a straightforward Fourier series expansion. We found that the solution consisted of a linear combination of basis functions that precisely matched the Legendre functions in the analytical solution.

3. We generalized the treatment of spherical shells, such that we could address acoustic radiation from an arbitrary shell of revolution. The first test was the case of a prolate spheroidal shell. We showed that our solution using variational principles is very close to an earlier solution,
provided that the eccentricity is small. In the case of higher eccentricity, we found that disagreements between our solution and earlier papers seemed to be attributable to the fact that no prior investigation of slender prolate spheroids had succeeded in evaluating the in-vacuo modal properties associated with the upper vibrational branch, in which in-plane displacement is dominant. We are in the midst of developing a procedure to address this problem in an expedient manner. The concept is to introduce an artificial energy function, whose purpose is to alter the properties of the lower branch at high wave numbers, in order to assist convergence to the upper branch at low wave numbers. This effort is well underway and we expect to report on it at the next ASA meeting.

4. In order to develop the capability of treating non-ideal structures, we are currently extending the earlier solution for an elastic circular plate to include the effects of concentric stiffeners at arbitrary positions. We are examining the merits of two alternative formulations. In the first, the dynamic elastic effects of the structure are described in terms of a set of modes derived from an in-vacuo analysis of the entire structure. In contrast, the second formulation uses concepts of component mode synthesis, in which the individual modal properties of the plate and isolated stiffeners are tied together by constraint equations and Lagrangian multipliers. The virtue of the latter is that the modes are easier to obtain, but at a penalty of increasing the number of unknowns. A key question we will examine is the degree to which discrepancies between the responses from each approach are influential for far-field radiation patterns. To support that study, we used the Kirchhoff-Helmholtz integral theorem to develop a computer program for acoustic radiation from an axisymmetric body executing a known surface vibration. Results from this study will be reported at the next ASA meeting.

5. We recently began to generalize the implementation of the variational principle to treat nonaxisymmetric excitations of axisymmetric structures. The general approach here is to exploit the circumferential periodicity of all surface response variables by using Fourier series expansions in that direction. The variational principle then leads to uncoupled problems for each harmonic, each of which is governed by equations that are analogous to those for the axisymmetric problem. At the present time, we have completed the formal derivation of the equations governing the coefficients of the assumed mode series for the case of a flat elastic disk, either baffled or unbaffled.

PLANS FOR NEXT YEAR’S RESEARCH

1. A primary objective is to complete the analysis of the in-vacuo modes of a highly eccentric spheroidal shell. We will first endeavor to use the artificial energy concept discussed in the previous section. The virtue of this procedure is that it involves relatively small changes to the equations that are derived from the true kinetic and potential energies of the shell. Should that method fail, we will modify the analysis by using the Timoshenko formulation of transverse shear, in which the rotation of a transverse fiber of the shell is modeled independently of the mid-surface displacement. This approach should suffice because it addresses the fact that high wave numbers
lead to shear deformation. The sole reason we did not implement such a formulation initially is that it introduces many more unknowns, since each shear angle must be described by another series of basis functions.

2. We will continue the development of the general procedure for coupling VAP and Hamilton’s principle. A primary task in this effort is to improve the evaluation of the coefficients associated with VAP. A large part of this task is associated with the need to integrate the free space Green’s function over the surface in order to derive the system of coefficients associated with VAP. We will attempt to expedite the numerical algorithm by identifying ways in which the contribution of the singularities of the Green’s function may be treated efficiently and accurately through asymptotic representation of the singular part. Once an efficient algorithm is developed, we will validate the procedure by comparing our predictions for a cylindrical shell capped by hemispherical shells to those obtained from other formulations, most likely the CHIEF program.

3. We will continue to work on modeling stiffeners. Specifically, we will complete the analysis of stiffeners on an elastic plate, and then use the knowledge gained from that simple study to introduce stiffeners into the model of an arbitrary shell of revolution that is axisymmetrically excited.

4. We will complete the analysis of non-axisymmetric excitations of a disk, which involves development of algorithms suitable for evaluating the VAP coefficients. This result will be verified by determining the VAP-Hamilton’s principle solution for scattering of a plane wave obliquely incident on a plate, for which analytical solutions are available.

5. We will generalize the VAP description to address non-axisymmetric excitations of shells. This task will require Fourier series expansion of surface pressure and both displacement components. Particular emphasis will be placed in this effort on using recursion relations to expedite the description of each circumferential harmonic.
LIST OF PUBLICATIONS/REPORTS/PRESENTATIONS

I. PAPERS IN REFEREED JOURNALS


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<td>J. H. Ginsberg</td>
<td>Georgia Inst. of Technology</td>
<td>Fellow of the Society</td>
<td>Amer. Soc. of Mechanical Eng.</td>
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(Number Only)

Papers Submitted to Refereed Journals (and not yet published): 2
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Technical Reports Published: 0
Books Submitted for Publication: 0
Books Published: 0
Patents Filed: 0
Patents Granted: 0
Invited Presentations at Topical or Scientific/Technical Society Conferences: 3
Contributed Presentations at Topical or Scientific/Technical Society Conferences: 4
Honors/Awards/Prizes: 2
Number of Graduate Students: 4
Number of Post-Docs: 0
LIST OF PARTICIPANTS AND STATUS

1. Jerry H. Ginsberg, Principal Investigator, George W. Woodruff Chair in Mechanical Systems.

2. Pei-Tai Chen, graduate research assistant, Ph.D. candidate, citizen of Taiwan, completed all requirements except for thesis, expected completion date is June 1990.

3. Hyun-Gwon Kil, graduate research assistant, Ph.D. candidate, citizen of Taiwan, completed all requirements except for research proposal and thesis, expected completion date is Aug. 1991.


OTHER SPONSORED RESEARCH

J. H. Ginsberg

October 1, 1990

Dr. Spiro Lekoudis
Acting Director, Mechanics Division
Office of the Chief of Naval Research
800 N. Quincy Street
Arlington, VA 22217-5000

Dear Dr. Lekoudis:

Enclosed you will find two copies of the "End-of the Fiscal Year Letter" summarizing the research activity in contract number N00014-84-K-0713, for which I am principal investigator. Please call me if there are any questions regarding the information reported here.

Sincerely yours,

Jerry H. Ginsberg
The George W. Woodruff Chair
in Mechanical Systems

cc: Dr. Phillip B. Abraham, code 1132-SM
    GT Office of Contract Administration
IMPROVED ALGORITHMS FOR TRANSIENT AND STEADY-STATE FLUID-STRUCTURE INTERACTION

DESCRIPTION OF SCIENTIFIC RESEARCH GOALS

The interaction between a vibrating submerged structure and the surrounding fluid, which features coupling between the surface pressure distribution and the structural displacement, is an inherent feature for sound radiation and target strength analyses. A variety of approaches have been implemented in the past, but each suffers from serious limitations. Formal mathematical analysis using separation of variables or integral transform techniques is suitable only for the simplest structural models, while full finite element descriptions of realistic structures and the surrounding medium lead to excessively large computer simulations. One approach uses approximate impedance-type boundary condition of uncertain accuracy to model the fluid response. Boundary element formulations rationally represent the interaction phenomena without explicitly solving field equations for the fluid, at the expense of an enormous increase in computational effort due to the need to cover the surface with a reasonably fine mesh.

The limitation of classical analytical techniques has led to a widely held belief that analytical-type solutions can only be obtained in idealized systems. The surface variational acoustics principle (SVP) relating surface pressure and surface particle velocity, which we derived earlier, partially invalidates the foregoing statement. SVP, which is valid for temporally harmonic excitations, represents the spatial dependence of the surface response as a series expansion in a set of assumed basis functions. Since the solution derived from SVP is the coefficients of this series, the primary difference from the results of a classical analysis lies in the fact that SVP determines these coefficients by numerical techniques.

Because SVP represents the surface pressure in a functional series form, it involves a substantially reduced set of unknowns in comparison to boundary element formulations, which are based on point-wise discretizations. In addition, SVP is essentially an optimization process that selects the coefficients of the modal series so as to minimize the deviation of the derived solution relative to the true one. The optimization abilities of other weighted residual methods are less powerful. Another feature of SVP is that it enables one to make use of prior experience with the problem, either from experiments or analysis by simpler techniques. Tailoring the basis functions to match this "expert" knowledge enables one to substantially reduce the number of unknown series coefficients. Prior phases of the project were devoted to the derivation of SVP and to validating it through comparison with solutions for rigid body motion derived by other analytical procedures. Now the primary objective of the project is to develop the capability of using SVP to derive a complete formulation of radiation and scattering problems in elastic structures.
SIGNIFICANT RESULTS OF THE PAST YEAR

1. SVP for vibration of submerged shells

   The main effort last year was devoted to development of a general implementa-
   tion of SVP for axisymmetric vibration and radiation from submerged 
   shells of revolution. In this effort, the shape and thickness of the shell 
   were allowed to be arbitrary, and the mechanical excitation was limited only 
   by the restriction to axisymmetry. Our previous work regarding fluid-
   structure coupling involving elastic structures had considered only flat 
   plates in a baffle and spherical shells. Although SVP was found in both 
   cases to be extremely accurate relative to analytical solutions, we did not 
   deem such comparisons to be fair measures of the merits of SVP because of 
   the simplicity of both geometries. We therefore decided to apply the 
   general model to the case of a prolate ellipsoidal shell.

   A few prior investigations had considered the case where the inner and 
   outer surfaces of the shell are confocal ellipsoids, which corresponds to a 
   variable skin thickness. Our initial implementation of SVP used the ap-
   proach widely employed in structural acoustics of first analyzing the free 
   in-vacuo vibrational properties, in order to perform a modal truncation 
   before the fluid-structure interaction phenomena are addressed. In our 
   general procedure, the free vibration natural frequencies and modes of a 
   shell are obtained from Hamilton's principle in conjunction with the method 
   of assumed modes. When we attempted to compare our results with prior work, 
   we found that results for slender ellipsoids was very sparse. Further in-
   vestigation disclosed that the natural frequencies in certain ranges of 
   slenderness ratio were very close, in which case mode localization phenomena 
   became prominent. In order to explore this question we spun off a separate 
   general investigation of mode localization, which is discussed in the next 
   section.

   Mode localization has been suggested at ONR-sponsored workshops as 
   being a potential source of modeling error for numerical simulations. We 
   used the prolate spheroidal shell as a prototypical system to investigate 
   with the aid of SVP the validity of such concerns. We compared the response 
   of the submerged system when all in-vacuo structural modes are retained, 
   regardless of the frequency range in which they occur, to the results ob-
   tained from various modal truncations both below and above the frequency at 
   which mode localization occurs. We found that when the slenderness ratio is 
   such that in-vacuo modes localize, accurate results for surface vibration 
   and pressure can only be obtained by retaining all modes, whereas the mode 
   truncation gives good results when the slenderness ratio is such that 
   localization does not occur. Another result of note was that in neither 
   case does the submerged response display localization. At this juncture, 
   the results have been presented as an invited paper at the most recent ASA 
   meeting, and several papers describing the results for shells are being 
   prepared for journals.

2. The relationship between mode localization and eigenvalue veering

   Our initial observation of mode localization was obtained as a corol-
   lary of a parametric investigation of the dependence of the various in-vacuo 
   natural frequencies of a spheroidal shell as a function of the slenderness 
   ratio. We were concerned about the computational ability to obtain or-
   thogonal eigenfunctions when natural frequencies are close. Prior work had
showed in a general manner that the natural frequency branches often veer, rather than intersecting, and that such veering phenomena are associated with mode localization. Such was the case for the spheroidal shell, but all previous investigations were invalid for the behavior in the veering zone. We successfully performed an analysis of the behavior in the veering zone. It showed that the eigenfunctions when the natural frequencies are very close are linear combinations of the eigenfunctions outside the zone, and these combinations lead to the localization phenomena. This work will be presented at the AIAA/ASME/ASCE Structural Dynamics meeting in April, 1991, and it has been submitted for publication.

3. Modeling of stiffeners in conjunction with SVP

We addressed the effect of stiffeners in elastic plates in order to introduce greater realism into the structural dynamics models used for SVP. Since the effects of deviations from asymmetry were under separate investigation (see the next section), we considered only concentric ring stiffeners on an elastic plate. We successfully demonstrated two alternative formulations. In the first, the dynamic elastic effects of the structure are described in terms of a set of modes derived from an in-vacuo analysis of the entire structure. In this case the portions of the SVP equations associated with the elastic and inertial effects of the structure are altered with each change in stiffener properties. In contrast, the second formulation uses concepts of component mode synthesis, in which the individual modal properties of the plate and isolated stiffeners are tied together by constraint equations and Lagrangian multipliers. The virtue of the latter is that the modes are easier to obtain, and changes due to changes in the stiffener configuration leads to relatively small and easily determined alterations in the SVP, at a penalty of increasing the number of unknowns. We demonstrated the equivalence of both methods, from both analytical and computational viewpoints. We used the component mode synthesis approach to identify optimal stiffener for maximizing or minimizing surface vibration or far field pressure. We determined the latter by developing a computer program based on the Kirchhoff-Helmholtz integral theorem. Results from this study were reported at an ASA meeting and have been accepted for publication.

4. Nonaxisymmetric problems using SVP

Our initial generalization of SVP to treat nonaxisymmetric excitations of axisymmetric structures used Fourier series to represent the azimuthal dependence of all surface response variables, thereby exploiting their circumferential periodicity. In the first study it was assumed that the excitation is such that only a single azimuthal mode is excited. The variational principle then leads to equations for that harmonic that are analogous in form and computational effort to those for the axisymmetric case. This work, which used the flat elastic plate as a prototype, was presented as an M.S. thesis. Current work is extending this to the case where a full azimuthal series is required. The goal here is to develop optimal SVP algorithms for the successive evaluations of the azimuthal harmonics. The formal analysis has been completed and a computer program that will set up and solve the problem of a eccentrically loaded elastic plate is under development.
PLANS FOR NEXT YEAR'S RESEARCH

1. Extend the variational principle to treat nonaxisymmetric excitations of submerged elastic shells. The analytical steps required to address such situations for flat plates were described in the previous section. Extension of those concepts to curved shells involves substantial work because of the geometric complications inherent to an arbitrary surface of compound curvature. An alternative solution technique using a conventional boundary element code, e.g. CHIEF, will be implemented, in order to validate the SVP results.

2. Implement a constrained version of the variational principle, using Lagrangian multipliers, in order to treat the spurious resonance phenomenon. This occurs at frequencies that match the solution of the eigenvalue problem for the interior cavity. Several concepts, all based on the fact that the interior field calculated by SVP should vanish, are currently being assessed for simple geometries. We showed at an ASA meeting that Lagrangian multipliers are effective in conjunction with SVP as a means for enforcing side constraints. Hence, the main task is to identify which constraints most robustly address the loss of uniqueness. It should be noted that success in this task will also lead to substantial improvements in the ability of several conventional boundary element codes, including CHIEF and NASHUA, to handle these problematical frequencies.

3. Explore vibrational localization phenomena arising from structural irregularities, such as small deviations in the stiffness and spacing of stiffeners. This phenomenon, which is sometimes referred to as Andersen localization, was first encountered in solid-state physics. Numerous investigators believe it to be a potentially serious source of error for predictions of radiation and scattering from structures. The insights already obtained from our previous study of this problem will be extended to treat the stiffener question.

4. Several concepts using asymptotic mathematical analysis have used to describe the singular terms occurring in SVP, due to the presence of the free space Green's function. This study in the case of a plate decreased by an order of magnitude the numerical operations required to implement SVP. A preliminary test for a sphere has indicated that the improvement can result in a reduction of computer time by a factor of 10 with no loss in accuracy. Work needs to be done to implement these concepts for arbitrary fluid-loaded shapes.

5. Extend SVP to evaluate scattering of arbitrary incident acoustic waves. It is not difficult to extend the basic variational principle to include scattering, but most problems of interest (monostatic and bistatic scattering) involve nonaxisymmetric response. The virtue of the variational principle is that its' high accuracy is obtained from relatively small system models. This should make it possible to expedite substantially the frequency sweeps and angle of incidence scans associated with the general objectives of scattering studies. Furthermore, the high accuracy of the prediction of surface response will enable thorough investigation of the relationship between surface features and far field phenomena.
OFFICE OF NAVAL RESEARCH
PUBLICATIONS/PATENTS/PRESENTATIONS/HONORS REPORT
1 October 1989 through 30 September 1990

R&T Number: 4/324730

Contract/Grant Title: IMPROVED ALGORITHMS FOR TRANSIENT AND STEADY-STATE FLUID-STRUCTURE INTERACTION

Scientific Officer: Dr. Philip Abraham

Principal Investigator: Dr. Jerry H. Ginsberg

Mailing Address: Georgia Institute of Technology, School of M.E., Atlanta, GA 30332-0405

Phone Number: (404) 894-3265

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a. Number of Papers Submitted to Referred Journal but not yet published:
   1

b. Number of Papers Published in Referred Journals:
   (List Attached):
   2

c. Number of Books or Chapters Submitted but not yet Published:
   0

d. Number of Books or Chapters Published (List Attached):
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f. Number of Patents Filed:
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g. Number of Patents Granted (List Attached):
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h. Number of Invited Presentations at Workshops or Professional Society Meeting (List Attached):
   4

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   4

j. Honors/Awards/Prizes for Contract/Grant Employees:
   (List Attached, may include Society Awards/Offices, Promotions, Faculty Awards/Offices, etc.)
   3

k. Number of Graduate Students and Post-Docs Supported at least 25% this year on Contract/Grant:

   Grad Students:
   TOTAL 4
   Female 1
   Minority* 0

   Post Doc:
   TOTAL 0
   Female
   Minority

l. Degrees Granted (List Attached):

   2

* Minorities include Blacks, Aleuts, Amindians; Hispanics, etc.

NB: Asians are not considered an under-represented or minority group in science and engineering.
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2. BOOKS PUBLISHED: none

3. TECHNICAL REPORTS - none

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5. CONTRIBUTED PRESENTATIONS


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<td>Georgia Inst. of Technology</td>
<td>Invited to give keynote talk on &quot;Hot Topics in Structural Acoustics at next meeting.&quot;</td>
<td>Acoustical Society of America</td>
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<td>Organized GT Structural Mechanics seminar series, Fall 1989. Theme was Structural Acoustics</td>
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Papers Published in Refereed Journals: 2

Papers Published in Non-Refereed Journals: 0

Technical Reports Published: 0

Books Submitted for Publication: 0

Books Published: 0

Patents Filed: 0

Patents Granted: 0

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Contributed Presentations at Topical or Scientific/Technical Society Conferences: 4

Honors/Awards/Prizes: 3

Number of Graduate Students: 4 (1 female, 0 minority)

Number of Post-Docs: 0

Degrees Granted: 2
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1. Jerry H. Ginsberg, Principal Investigator, George W. Woodruff Chair in Mechanical Systems.

2. Pei-Tai Chen, graduate research assistant, Ph.D. candidate, citizen of Taiwan, completed all requirements except for thesis, expected completion date is June 1990.


4. Pearl Chu, graduate research assistant, M.S.M.E. (March 1990), citizen of U.S.A. Currently Staff Research Engineer, General Dynamics, Inc. Houston, TX.

5. Hyun-Gwon Kil, graduate research assistant, citizen of Taiwan. Now working on another ONR project at Georgia Tech.

OTHER SPONSORED RESEARCH

J. H. Ginsberg
VARIATIONAL PRINCIPLES FOR ACOUSTIC RADIATION AND DIFFRACTION FROM UNDERWATER STRUCTURES

Xiao-Feng Wu

School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

24 November 1987

Technical Report

Approved for public release; distribution unlimited.

Prepared for:

OFFICE OF NAVAL RESEARCH
DEPARTMENT OF THE NAVY
ARLINGTON, VA 22217

Acoustics & Dynamics Research Laboratory
Variational principles derived from the Kirchhoff-Helmholtz integral theorem applied to various acoustic radiation and diffraction problems. Specific examples include sound adiation from transversely vibrating thin disks and finite cylinders in axisymmetric scillations. The formulation has the acoustic pressure on the vibrating surface as the unknown variable, with the normal velocity of the surface taken as given. The general implementation of the variational formulation is the Rayleigh-Ritz technique in which the unknown surface pressure is represented by an expansion of preselected basis functions. The unknown coefficients corresponding to the basis functions are determined by a system of simultaneous equations. It is demonstrated that substantially less computational time in any cases can be achieved if one conscientiously uses physical insight and good common sense in selecting the basis functions for the expansion of the unknown surface pressure. Numerical results agree well with the previous ones obtained by using other numerical methods. The uniqueness of the variational formulation is also discussed. It is shown that solutions...
A variationally formulated acoustic radiation problems are (1) unique for the surface pressure distributions on infinitesimally thin disks and plate-like bodies for which each surface point is oscillating either in phase or 180° degree out of phase; (2) nonunique for the acoustic pressure on surfaces of bodies with finite volumes; and (3) unique for the total radiated acoustic power.
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B. Variational Method for Computing Surface Acoustic Pressure on Vibrating Bodies, Applied to Transversely Oscillating Disks — by X.-F. Wu, A. D. Pierce, and J. H. Ginsberg
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CHAPTER I

INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

In the practice of engineering acoustics, one is often concerned with the problem of predicting the sound radiation from an arbitrary object which is submerged in an infinite fluid medium and undergoing a harmonic oscillation. Analytical solutions to such problems are generally limited to cases for which the surface of the object conforms to one of eleven coordinate systems [1 - 3] so that the reduced wave equation (Helmholtz equation) can be separated. For objects with nonstandard shapes, numerical solutions have to be sought. Since the environment of the vibrating body is most appropriately idealized as being unbounded, those numerical techniques such as finite differences and finite element methods are often found impracticable to apply from the computational viewpoint. Consequently, an approach commonly adopted is to reformulate the problem as a boundary integral relation, known as the Kirchhoff-Helmholtz integral theorem [4], which expresses the acoustic field at any external point as a definite integral over the vibrating surface. (This theorem is stated further below in section 2.1 of this thesis.) Terms in the integrands involve both the acoustic pressure and normal component of velocity on the vibrating surface. These two surface quantities, however, are not independent of each other and cannot be specified simultaneously in the boundary conditions. In most acoustic radiation problems, the surface velocity is taken as given. Therefore, one has to solve for the surface acoustic pressure as the first step.
Many numerical methods have been developed in that regard during the past few decades. The use of the Kirchhoff-Helmholtz integral relation for determination of the external sound field is valid no matter how the surface pressures are obtained. In the limiting cases of low frequencies or high frequencies, the surface pressure can be estimated by using incompressible flow approximations [5] or plane wave approximations [6, 7]. If the frequencies span the whole domain, but the surface of the vibrating body differs only slightly from a prolate spheroid (or from other surfaces for which exact solutions are known), then the surface pressure can be calculated by means of the exact theory using spheroidal wavefunctions on the equivalent spheroid [8]. This procedure is particularly simple if the velocity distribution on the surface (times a scale factor) resembles a spheroidal-surface harmonic function [9, 10]. In the most general cases where both surfaces and frequencies are arbitrary, the surface pressure can be expressed as a Fredholm-type integral equation [8, 11–13]. Once the surface pressure is obtained, the acoustic field anywhere else is completely determined.

It can be shown that solutions to the scalar Helmholtz equation with an appropriate boundary condition for the exterior acoustic field are uniquely determined [14]. Solutions to integral equation formulations for the surface pressure, however, are sometimes nonexistent or nonunique when the driving frequency of the vibrating surface coincides with one of the characteristic eigenfrequencies for a related boundary value problem in the interior region. It should be pointed out here that such a nonexistence or nonuniqueness problem is an artifact of the mathematical formulation. It is the result of the assumption made on the existence of the integral equation in a particular form and this assumption simply fails at these characteristic frequencies.
1.2 Literature Review

1.2.1 Numerical Implementations Of Boundary Integral Methods

One of the earliest applications of boundary integral methods to the acoustic radiation problems was the *simple source method* proposed by Kupradze [15]. In this method, the acoustic field was assumed to be generated by a layer of point sources with unknown density distributed over the surface. Given the surface velocity as a boundary condition, one was able to formulate an integral equation for the unknown source density function. After solving for the unknown source density function, one could compute the surface pressure and then compute the external acoustic field.

Chen and Schweikert [11] numerically implemented this *simple source method*. They calculated the sound radiation from a circular piston set in a rigid sphere and from a stiffened cylinder in water. This *simple source method* was also numerically implemented by many other researchers. For example, Baron and his colleagues [16 — 18] used this method to solve the acoustic radiation from an elastic circular cylinder and a finite circular cylindrical shell. Hess [19] and Brundrit [20] employed this method for solutions of a plane wave scattered from rigid prolate and oblate spheroids.

In 1964, Chertock [8] proposed an alternate form of the integral equation known as the *surface Helmholtz integral equation*. [The equation appears in this thesis as Eq.(2.1 — 18).] This method had been previously put forth by Kupradze in 1952 [21]; it was later summarized in 1965 [22]. There, the acoustic pressure at a field point was expressed in the form of an integral in terms of surface pressure and normal velocity. Given the velocity boundary condition, an integral equation for the unknown surface pressure was formed by letting the field point approach the surface. In doing so, some care had to be taken in the passage to the limit because some terms in the integrands could become singular as one surface point approached the other.
Careful mathematical analysis resulted in a formulation where all integrands were nominally integrable, even though some had singularities. This method was used by Chertock to solve the acoustic radiation from axisymmetric radiators. Smith, Hunt, and Barach [23] also applied it to sonar transducers where they considered the sound field generated by a radiator consisting of a coaxial array of two piezoelectric transducers, each in the form of a circular ring.

Copley introduced what he called the *interior Helmholtz integral equation* method [12, 13] to solve the acoustic radiation from axisymmetric radiators. [The equation appears in this thesis as Eq. (5.1-1).] In this method, the surface pressure was determined by the vanishing of the integral obtained by restricting the field point to lie within the interior region. The advantage of this *interior Helmholtz integral equation* method was that all integrals involved in the formulation were regular, hence numerical computations were simplified. Copley applied this method to spheroidal and finite cylindrical surfaces by choosing the interior field points on the axis of symmetry.

In 1967, Schenck [24] pointed out some inherent mathematical difficulties in the boundary integral methods. In particular, he showed that at the characteristic frequencies corresponding to the interior (homogeneous) Dirichlet problem for the region enclosed by the surface of the radiator, the *simple source method* failed to yield a solution while the *surface Helmholtz integral equation* failed to yield a unique solution. Although the *interior Helmholtz integral equation* had unique solutions, the numerical implementation of such a formulation would encounter difficulties when the interior points coincided with the nodes of the standing waves associated with the interior homogeneous Dirichlet problem.

To circumvent the deficiencies of these formulations, Schenck presented the *combined Helmholtz integral equation formulation* (CHIEF). Basically, this improved method used the *surface Helmholtz integral equation* to obtain a system
of algebraic equations for the surface pressure at discrete points and then overde-
determined the solutions with additional compatible equations based on the interior Helmholtz integral equation with the interior points. Since such a system had more equations than unknowns, the least-square procedure was used to obtain the surface pressure. Schenck demonstrated that although the solutions based on the surface Helmholtz integral equation were nonunique at the characteristic frequencies, only one of them also satisfied the additional interior Helmholtz integral equation. Hence the solutions thus obtained were unique for all wavenumbers. However, choosing the appropriate interior points still remained a problem just as in the interior Helmholtz integral equation method. The CHIEF method, nevertheless, was successfully applied by Schenck to sound radiation from vibrating spheres, finite cylinders, and rectangular parallelepipeds.

Analogous to the CHIEF method, Piaszczyk and Klosner [25] in 1984 proposed an iterative overdetermination scheme which combined the surface Helmholtz integral equation with an integral relation, namely, the field pressure was expressed as a definite integral in terms of surface pressure and normal component of surface velocity. The solution was carried out by first choosing relatively simple impedance functions, for example, the plane wave impedance function on the vibrating surface. Given the velocity boundary condition, the approximate values of the pressure at appropriately chosen field points were calculated based on this impedance function using the integral relation. These field pressures were then utilized to form an overdetermined system of equations resulting from the surface Helmholtz integral equation. The overdetermined system of equations were solved with the least square procedure for the surface pressure. The surface pressures thus obtained were then used to recalculate the pressures at the exterior field points. Such a procedure continued itself until the convergence criterion was satisfied. Piaszczyk and
Klosner applied this method to the cylindrical transducer problems and obtained satisfactory results at characteristic frequencies.

Burton and Miller [26] developed another approach to circumvent the problem at the characteristic frequencies. They formed a particular linear combination of the surface Helmholtz integral representation and the differentiated Helmholtz integral representation which was derived by taking a normal derivative at a field point and subsequently letting the fielding point approach the surface. Particular care were taken in their formulation such that the resulting integrals were well behaved in the passage to the limit. Burton and Miller demonstrated that although the surface Helmholtz integral equation and its differentiated counterpart failed to yield unique solutions at the interior Dirichlet and Neumann eigenfrequencies, respectively, the combination of the two had unique solutions for all wavenumbers because these two equations had only one solution in common.

Burton and Miller's method was applied by Meyer et al [27, 28] to the acoustic radiation from an oscillating piston set in a rigid sphere. Since one of the integrands in this method was highly singular as two surface points approached one another, the second normal derivative of the free-space Green's function in one of the integrands was recast into a form involving the tangential derivative of the surface pressure. In the problem Meyer et al considered, the mixed boundary conditions were used, i.e. the velocity was specified on one part of the surface while the acoustic impedance was prescribed on the rest part of the sphere. Since the symmetry of the sphere was not taken advantage of in these computations, the numerical schemes employed by Meyer et al were applicable to general three dimensional sound radiation problems.

There are other approaches which guarantee unique solutions at all frequencies. For example, Ursell [29] pointed out that the uniqueness problem could be
avoided by use of a specially constructed Green's function instead of the free-space Green's function. However, this function was difficult to implement numerically as it contained computations of an infinite series. A modification to Ursell's method was made by Jones [30] in which the infinite series was replaced by a truncated one. The truncation itself, on the other hand, introduced additional bounds on the wavenumbers for which this method was applicable. Hence, the unique solution was limited in certain frequency ranges depending on the geometry.

1.2.2 Early Versions Of Variational Principles For Wave Propagation

The variational principles, which have found the greatest usefulness in solid mechanics, elasticity, and many other areas of physics [31, 32], have not drawn as much attention in applications to acoustic radiation and diffraction problems. Although the formulation of an integral equation as a variational principle dated back as early as in 1884 [33], it was scarcely used for practical calculations in wave diffraction and scattering until the late 1940's.

One of the first demonstrations of uses of variational principles seems to be given by Levine and Schwinger [34] in the diffraction of a scalar plane wave by an aperture in an infinite plane screen. In that problem, the wave function describing the total field (incident plus diffracted) at an arbitrary point in the space was expressed in terms of the aperture fields, with a boundary condition of vanishing of the wave function on the screen. The aperture fields were determined by an integral equation derived by requiring the continuity of the normal derivative of the wave function on traversing the plane of the aperture. Based on the reciprocity principle, i.e. the amplitudes of the plane wave were invariant with respect to reversal in the sense of excitation and observation along a pair of directions in the space, the diffracted amplitudes at large distances from the aperture were expressed in a form
which was stationary to small variations of the aperture fields arising from the incident waves. This expression was independent of the scale of the aperture fields and in turn led to what Levine and Schwinger termed as a first variational principle.

An explicit application of this variational principle was to the case of normal incidence of a plane wave on a circular aperture mounted on an infinite plane screen. In particular, the transmission coefficient of the aperture, which was related to the diffracted amplitudes observed in the direction of incidence, was calculated in terms of expansions of the characteristic parameter $ka$, where $k$ was the wavenumber and $a$ was the radius of the aperture. Different trial functions for the wave functions were tested and numerical results were compared with the exact solutions obtained by Bouwkamp [35] using the oblate spheroidal wave functions. Since an infinite series expansion was involved in the solution of the first variational principle just as in the exact solution, numerical computations became progressively more difficult at higher $ka$ values ($ka > 10$).

To improve the accuracy in the high frequency range, Levine and Schwinger presented what they called a second variational principle [36] by considering the screen as an obstacle to the propagation of the incident wave through the free space. With the same boundary condition as previously, the wave function at an arbitrary point in the space was expressed in terms of the discontinuity in its normal derivative at the screen. (Such fields were analogous to the surface currents excited in a perfectly conducting screen by incident electromagnetic waves.) The discontinuity in the normal derivative of the aperture field was also governed by an integral equation. In addition to the reciprocity principle, however, introduced in this case were the residual functions which characterized the discontinuity in the velocity potential at the screen. This procedure led directly to a form, which was stationary to small changes of residual functions arising from a pair of incident waves, for the
desired spherical wave amplitudes at large distances from the aperture. The second variational principle was applied to the same circular aperture problem. The plane wave transmission cross section of the aperture was calculated as a power series in terms of $ka$ and results were compared with the exact solutions. It was shown that the two variational principles were individually adapted to opposite extremes of $ka$ values.

Levine and Schwinger's method constituted the basis on which the variational principles started to develop into a powerful tool for a quantitative analysis of wave propagation and diffraction phenomena. Such variational principles were adopted by Miles [37–39], Kato [40, 41], Blatt and Jackson [42], Papas [43], Levine and Papas [44], and many others [45–47] in the electromagnetic wave diffraction and neutron-proton scattering problems. With suitable changes in the formulation, the Schwinger-type variational principle was also applied to the problem of acoustic diffraction from a rigid plane disk [48], with the boundary condition of vanishing of the normal derivative of the scalar wave function on the disk surface.

Indeed, the Schwinger-type variational principle had drawn an immense attention in the 1950's. Copson [49] criticized this variational principle for many of the integrals involved in the analysis appeared to be divergent. However, in deriving what he called the "Levine-Schwinger's variational principle" in a mathematically sound way, Copson confined himself to the second kind of boundary value problem in which the normal derivative of the wave function vanished on the screen, while his criticism concerned the first kind of boundary value problem in which the wave function itself vanished on the screen. The divergent problem raised in his original criticism was later solved by Bouwkamp [50] using a similar method discussed previously by Maue [51] and Mitzner [52].
Bouwkamp [53] further exhibited that the variational principle could be used to obtain a rigorous solution of the diffraction problem by choosing correct basis functions (Sommerfeld-type [54]) for the aperture field. The coefficients corresponding to the basis functions were determined by an infinite system of linear equations resulting from the variational principle. Magnus [55, 56] thoroughly studied this system of equations and showed that for sufficiently small values of $ka$, solutions of the coefficients were unique and could be expanded in a convergent power series in $ka$. Bouwkamp [53] also suggested an alternative residual function for the second variational principle derived by Levine and Schwinger to solve the problem of wave diffraction from the circular aperture. Since the leading term of the transmission cross section given by the second variational principle departed from the correct behavior at low frequencies ($ka < 5$), the original residual function used by Levine and Schwinger was no good at all in that frequency range.

Based on the mathematical theory of diffraction, Erdélyi [57] in 1953 pointed out that the variational principles introduced by Levine and Schwinger were the natural extensions to higher frequencies of the Rayleigh's approximation for the diffraction of an incident wave on a finite plane screen. Moreover, their methods were often presented in such a manner that they were applicable to the analysis of the field far from the obstacle only. Erdélyi extended the analysis of the scattered field due to an incident wave to any point in the space.

Jones [58] demonstrated that the Schwinger-type variational principle was exactly equivalent to Galerkin's method of solving the integral equation. Furthermore, he showed that the trial functions which Levine and Schwinger used to solve the problem of diffraction by a circular aperture in an infinite screen happened to be the only choice that would lead to accurate results (in certain senses) at the low frequencies. In general, the determination of the special trial functions was
equivalent to solving the problem using a power series expansion in terms of the characteristic parameter $ka$.

Needless to say, the Schwinger-type variational principle was limited to wave diffractions from an aperture in an infinite plane screen or from a rigid disk in a free space. The diffraction by an aperture in a perfectly rigid (soft) screen is essentially identically to the diffraction by the complementary perfectly soft (rigid) disk.

In 1953, Morse and Fechbach [59] derived a variational principle for problems of the acoustic radiation from arbitrary objects based on the Kirchhoff-Helmholtz integral theorem. However, this variational formulation had found little usefulness in the practice. The reason was because one of integrands involved in the formulation contained such a high order singularity that numerical implementations of such an integral equation became impossible.

At roughly the same time, Gerjuoy and Saxon [60] also presented variational principles for the scattering amplitude in the general acoustic scattering problems and variational principles for the phase shifts of spherically symmetric scatters. They felt that the previously derived variational principles for wave scattering problems were confined to the cases in which both the wave function and its normal derivative were continuous across a surface of discontinuity. Consequently, these formulations were not applicable in acoustic scattering problems where the potential function was discontinuous across a surface bounding two media of different densities. The derivations of Gerjuoy and Saxon's variational principles were non-trivial, and no applications were made because of complications involved in the numerical implementations of these variational formulations.

In 1960, starting from the integral equation given by Baker and Copson [48] for the velocity potential at any field point, Sleator [61] derived a variational formulation for the problem of scalar scattering from a prolate spheroid. The stationary
expression was obtained by taking a normal derivative of the integral equation for
the velocity potential and subsequently letting the field point approach the spheroid,
with the boundary condition of vanishing of the velocity potential on the spheroidal
surface. The prolate spheroidal coordinate system was introduced and the trial
functions for the velocity potential were expanded in terms of the Legendre polyno-
mials which formed a complete and orthogonal set over the integration domain. The
corresponding coefficients were determined by an infinite set of simultaneous equa-
tions resulting from the variational principle. However, even if the infinite system
was replaced by a finite one, the computations still seemed formidable. Since the
triple integral expression for the Green's function was used, a seven-fold integration
resulted. Although six of the integrations were carried out, the seventh integration,
as Sleator admitted, appeared to be too complicated to deal with by any analytical
techniques, and the actual evaluation had to be done graphically. Application of
Sleator's method was to the nose-on back-scattering cross section of a particular
spheroid at a single wavenumber. The results, of course, were unsatisfactory be-
cause of the complications involved in the computations which prohibited the use
of more terms in the expansion of the basis functions.

1.2.3 Development Of Variational Principles

As the availability of more and more powerful computing machines, the vari-
atational principles are widely used to study sound radiation from flexible mechanical
systems. The basic approach is to combine variational principles in the fields of
acoustics and elastodynamics. Such a model will provide approximate equations of
motion, boundary conditions and compatibility conditions to describe the complete
system. The responses of the system are expressed in terms of surface displacements
or pressures and mechanical and acoustical damping coefficients. Examples are the
applications of variational principles to acousto-structural problems [62 – 65]. This is, however, beyond the scope of the present thesis.

On the other hand, little progress has been made in the development of a variational principle as a possible means for solving pure wave propagation and diffraction problems since 1960. This is probably due to the fact that the variational principles commonly used then contain integrations with such complexities that numerical implementations of these formulations seem virtually impossible. Another reason presumably is caused by the widespread belief that the acoustic radiation and diffraction problems can equally be solved by Galerkin's method [58] without resorting to the variational theorems at all.

A variational principle does not emerge naturally from the conventional Kirchhoff-Helmholtz integral theorem. The variational formulation derived by simply allowing the field point in the Kirchhoff-Helmholtz integral relation to approach the surface automatically leads to a nonself-adjoint integration kernel, which in turn results in a nonsymmetric matrix in the numerical computation. To obtain self-adjoint integration kernels, one may take a normal derivative of the Kirchhoff-Helmholtz integral relation with respect to a field point and subsequently let the field point approach the surface. These procedures lead to the Morse and Feshbach's variational principle [59] which, unfortunately, contains a non-integrable kernel. The order of singularity of one of the integrands in that formulation is so high that the integrand cannot be implemented using any kind of numerical techniques.

It seems to the present author that no further progress in developing a useful variational principle has been made during the subsequent two decades. The variational principles that are free from singularity problems and yet facilitate numerical implementations have not appeared until recently; they are independently derived by Hamdi [66] and Pierce [67]. [Pierce's equation appears in this thesis as
The vigor and vitality of such formulations are immediately seen in the applications to various acoustic radiation and diffraction problems [68–70]. The formulation has also been extended to take into account the elasticity in the radiation from an un baffled circular disk [71].

Actually, the singularity difficulties in the variational formulation [59] are surmountable using the techniques previously employed by Maue [51] and Stallybrass [72] in integral equations formulated for similar problems. The techniques constitute a series of mathematics manipulations, which include uses of a vector identity, properties of the free-space Green's function, integration by parts, and Stokes' theorem, to recast the second normal derivative with respect to the Green's function into a form involving a dot product of two tangential derivatives of some functions defined on the surface. The integrands thus obtained are well behaved or at most of Cauchy-type [73] principal values.

Based on the general techniques for constructing variational principles [74], Pierce [75] in 1986 derived another variational formulation for estimating stationary quantities of practical interest, for example, the total radiated acoustic power. [The equation appears in this thesis as Eq. (5.3–1).] This stationary expression is related to the previously derived variational principle for the surface acoustic pressure. In general the accuracy of the estimation of the surface pressure will be "OK" if one incorporates physical insight into the selection of basis functions for the unknown surface pressure distribution. The accuracy of the estimation of the total radiated power, however, will be excellent if the "OK" estimation for the surface pressure is used. The reason for that is due to the fact that the error term in the stationary expression for the power is at least one order higher than that for the surface pressure.
1.2.4 Uniqueness Problems

The variational principles developed so far for the acoustic radiation and diffraction problems are all based on the Kirchhoff-Helmholtz integral theorem. Naturally, a question arises as to whether solutions to these formulations are unique. To answer this question, it is necessary to examine the Euler-Lagrange equation corresponding to the variational formulation [67] in the related interior space. However this Euler-Lagrange equation does not explicitly match the form of the second kind of Fredholm integral equation for which the eigensolutions have been proved to exist [24]; and further it is not simply an integral equation, but an integrodifferential equation.

In a recent paper, Wu and Pierce [76] demonstrate that this integrodifferential equation is actually a reformed Fredholm integral equation of the second kind, which has nonunique solutions at particular wavenumbers. Nonetheless, for disks or plate-like bodies with infinitesimal thickness and for which each surface point is vibrating either in phase or 180° degree out of phase, the solution is unique. The reason for that is because the mode for the interior space is either symmetric or antisymmetric about the center plane of the disks or plate-like bodies. In the limit of zero thickness, the lowest eigenfrequency for the antisymmetric mode approaches the infinity, i.e. a rigid body motion. Consequently, the nonuniqueness problem will not occur in the practical frequency ranges. It is further shown [76] that solutions of the variationally formulated total radiated acoustic power are always unique for all wavenumbers even if the surface pressure is not unique.
1.3 Scope Of Present Thesis

The variationally formulated acoustic radiation and diffraction problems are presented in Chapter II. Since the formulations involve double surface integrals (quadruple integrations), numerical computations are nontrivial. The computations get further complicated as most integrands become singular when two surface points approach one another. These singularities, however, are shown to be of the Cauchy type. Furthermore, the contributions of singularities are all finite. Chapter II also explores the fact that numerical computations can be significantly simplified if the vibrating surface is of axisymmetry.

Chapter III and IV illustrate several numerical examples which include acoustic radiation from (1) a transversely oscillating unbaflled circular thin disk, (2) a finite circular cylinder pulsating in the radial direction with two end caps held fixed, and (3) a finite circular cylinder oscillating in the axial direction like a rigid body. Numerical solutions compare favorably in each case with previously obtained results. The far-field radiation patterns are also calculated and compared with those based on surface pressure distributions obtained by using other methods.

Chapter V discusses the uniqueness of solutions to the variationally formulated acoustic radiation problems. It is shown that the variational formulation for estimations of the surface pressure distribution fails to yield a unique solution at the interior Neumann eigenfrequencies. However, for disks or plate-like bodies, solutions based on the variational formulation are always unique. Furthermore, it is shown that the variational expression for the total radiated acoustic power always yields unique solutions for all wavenumbers even when the surface pressure may not be. The conclusions and recommendations for further research work are given in Chapter VI.
CHAPTER II

VARIATIONALLY FORMULATED ACOUSTIC RADIATION PROBLEMS

2.1 Derivations Of Formulations

The derivation of the variational principle used in this thesis for acoustic radiation problems has been demonstrated in detail in a report by Ginsberg, Pierce, and Wu [77] and in the author's MS thesis [78]. For completeness, we will give a brief review of the derivation in the following.

Consider an arbitrary object immersed in an infinite fluid medium of ambient density $\rho_o$ and sound speed $c$. The Kirchhoff-Helmholtz integral theorem [4] gives the complex acoustic pressure amplitude $p(x)$ at an external field point $x$ due to monochromatic excitation on the surface $S'$ of the object as a definite integral over the surface.

\[
p(x) = \frac{1}{4\pi} \int_{S'} \left\{ p(x_{S'}) [n(x_{S'}) \cdot \nabla' G(x \mid x_{S'})] - [n(x_{S'}) \cdot \nabla' p(x_{S'})] G(x \mid x_{S'}) \right\} dS'
\]

(2.1-1)

where $n(x_{S'})$ depicts the unit normal vector at a surface point $x_{S'}$ (with a prime $'$ denoting a coordinate system defined on the surface) pointing toward the external region; and the quantity $G(x \mid x_{S'})$ is the free-space Green's function defined by

\[
G(x \mid x_{S'}) = \frac{e^{ikR}}{R}
\]

(2.1-2)
where the wavenumber $k$ is $\omega/c$ with $\omega$ denoting the angular frequency of the excitation on the surface and $R$ is the distance between the field point and source point

$$R = |x - x_{S'}|$$ \hfill (2.1–3)

In what follows, our interest will be focused on radiation problems so that the surface velocity is presumed a given quantity, while the acoustic pressure is to be calculated. Euler’s equation of motion for a fluid particle combined with the continuity of the normal component of particle velocity on the surface allows one to replace the normal derivative of surface pressure in (2.1–1) by

$$-n(x_{S'}) \cdot \nabla' p(x_{S'}) = -i \omega \rho_o v_n(x_{S'}) = f_n(x_{S'})$$ \hfill (2.1–4)

where $v_n(x_{S'})$ is the outward normal component of the surface velocity, while the function $f_n$ is defined for the convenience sake only.

Equations (2.1–1) and (2.1–4) show that if the acoustic pressure and normal velocity on the surface $S'$ are known, then the acoustic pressure at any external point $x$ can be obtained from the definite integral (2.1–1). Our concern in this thesis is the evaluation of the surface acoustic pressure given the velocity distribution on $S'$. The equation governing these two surface quantities is formally derived from Eq.(2.1–1) by letting the external point $x$ approach the surface $S'$. When $x$ is brought to the surface $S$, both integrands in (2.1–1) become singular because $R = 0$ at $x_S = x_{S'}$. The singularity in $G(x|x_{S'})$ is integrable, but evaluation of the integral containing $n(x_{S'}) \cdot \nabla' G(x|x_{S'})$ requires careful consideration. In particular, one finds that the integral may have different values...
depending on whether one regards $x$ as having approached the surface from the exterior or from the interior.

The analysis here of the effect of the singularity is similar to that of Kellogg [79], in which the external point $x$ is brought to a location $x_s$ on the surface in a limiting operation. As shown in Fig. 1, the point $x_s$ is defined to be concurrent with the normal that intersects $x$, so

$$x = x_s + \epsilon n(x_s)$$  \hspace{1cm} (2.1 - 5)

where $\epsilon$ is the small perpendicular distance of $x$ from the surface. The region $\Delta S$ is a circular segment of $S'$, centered at $x_s$, with small radius $\delta$. The remainder of the surface is denoted as $S$. The required integral is evaluated by taking the limit as $\epsilon \to 0$ with $\delta$ fixed, and then the limit as $\delta \to 0$. The order in which the two limits are taken is important.

Given that $\epsilon$ and $\delta$ are both small compared with any characteristic dimensions of the surface $S'$ and given that $S'$ is smooth near the point $x_s$, it is appropriate to regard $\Delta S$ as having the shape of an elliptical bowl. The principal curvature radii, $R_I$ and $R_{II}$, of this bowl are those of the surface at location $x_s$. Since $\Delta S$ is small, the $z$ component of a generic point $x_{S'}$ on $\Delta S$ can be expanded into a power series in terms of $z$ and $y$ ($x/R_I \ll 1$ and $y/R_{II} \ll 1$), so one has

$$z \simeq -\frac{x^2}{2R_I} - \frac{y^2}{2R_{II}}$$  \hspace{1cm} (2.1 - 6)

(A convex surface would correspond to negative radii of curvature.) The unit outward normal vector $n(x_{S'})$ at the point $x_{S'}$ is approximated by
Figure 1. Integration when the field point \( \mathbf{x} \) approaches the surface.
\[ n(x_{S'}) \simeq e_z + \left( \frac{x}{R_I} \right) e_x + \left( \frac{y}{R_{II}} \right) e_y \] (2.1 - 7)

Since in this local coordinate system, the external point \( x \) is at \( e e_z \), while the \( z \)-coordinate of \( x_{S'} \) is as described by (2.1 - 6) above. Consequently, the vector \( x - x_{S'} \) is given by

\[ x - x_{S'} \simeq \left[ \epsilon + \left( \frac{x^2}{2R_I} \right) + \left( \frac{y^2}{2R_{II}} \right) \right] e_z - xe_x - ye_y \] (2.1 - 8)

and one derives

\[ R \left[ n(x_{S'}) \bullet \nabla' R \right] = n(x_{S'}) \bullet (x_{S'} - x) \simeq -\epsilon + \frac{x^2}{2R_I} + \frac{y^2}{2R_{II}} \] (2.1 - 9)

\[ n(x_{S'}) \bullet \nabla' G(x | x_{S'}) \simeq \left( \frac{1}{R^2} + \frac{k^2}{R} \right) \left( \epsilon - \frac{x^2}{2R_I} - \frac{y^2}{2R_{II}} \right) \] (2.1 - 10)

where in (2.1 - 10) the free-space Green's function is approximated by a truncated power series in terms of the small value \( R \) and the higher order terms are neglected. Moreover, on the right side of (2.1 - 10), it is consistent to replace the distance \( R \), wherever it appears, by

\[ R \simeq \sqrt{\epsilon^2 + x^2 + y^2} \] (2.1 - 11)

and, for the differential of area, to set

\[ dS' \simeq r dr d\gamma; \quad x = r \cos \gamma; \quad y = r \sin \gamma \] (2.1 - 12)

where the \( r \)-integration extends from 0 to \( \delta \), and the \( \gamma \)-integration extends from 0 to \( 2\pi \).
Since all points on $S$ are at a finite distance away from the external point $x$, the factor $n(x_{S'}) \cdot \nabla' G(x | x_{S'})$ has no singularities on $S$ when $\epsilon \to 0$. Furthermore, if one lets $\epsilon \to 0$ first and considers $r$ to be small but nonzero, then

$$\left[ n(x_{S'}) \cdot \nabla' G(x | x_{S'}) \right] dS' \to -\frac{1}{2} \left( \frac{\cos^2 \gamma}{R_I} + \frac{\sin^2 \gamma}{R_{II}} \right) \frac{1}{r} r dr d\gamma \quad (2.1 - 13)$$

The $1/r$ singularity here is cancelled by the factor in the area differential factor $r dr$. The integral in this limit is exactly the same as if one set $x$ to $x_S$ at the outset, and then did the integral over the surface $S'$. Hence, it is mathematically meaningful to integrate over the surface $S$ by taking the double limit such that first $\epsilon \to 0$ and then $\delta \to 0$. We therefore turn our attention to the contribution of $\Delta S$.

Let $F(x_{S'})$ be any continuous scalar function on $S'$. Then Eqs. (2.1 - 10) and (2.1 - 12) lead to

$$\iint_{\Delta S} F(x_{S'}) \left[ n(x_{S'}) \cdot \nabla' G(x | x_{S'}) \right] dS' \simeq \iint_{\Delta S} F(x_S) \left( \frac{\epsilon}{R^3} \right) r dr d\gamma$$

$$+ \iint_{\Delta S} F(x_S) \left[ A(\epsilon, r, \gamma) - \left( \frac{r}{R} \right)^2 B(\gamma) \right] \frac{1}{R} r dr d\gamma \quad (2.1 - 14)$$

where in the second term on the right side of (2.1 - 14) we abbreviate

$$A(\epsilon, r, \gamma) \simeq k^2 \epsilon - \frac{1}{2} k^2 r^2 \left( \frac{\cos^2 \gamma}{R_I} + \frac{\sin^2 \gamma}{R_{II}} \right) \quad (2.1 - 15a)$$

$$B(\gamma) \simeq \frac{1}{2} \left( \frac{\cos^2 \gamma}{R_I} + \frac{\sin^2 \gamma}{R_{II}} \right) \quad (2.1 - 15b)$$

Functions $A$ and $B$ are both finite; $A$ actually vanishes in the limit as both $\epsilon$ and $\delta$ go to zero. Also, we note that the quantity $r/R$ is always less than 1, regardless of
the value of $\epsilon$. Consequently, the second term on the right side of (2.1 - 14) is of
the order of magnitude of $\delta$; it therefore vanishes in the limit as $\delta \to 0$.

We now proceed to evaluate the first term on the right side of (2.1 - 14). With $\epsilon$ small, but positive, an appropriate change of integration variable is to the
polar angle $\theta$, defined such that

$$r = \epsilon \tan \theta; \quad R = \epsilon \sec \theta; \quad dr = \epsilon \sec^2 \theta \, d\theta; \quad \left(\frac{\epsilon}{R^3}\right)^r \, dr = \sin \theta \, d\theta \quad (2.1 - 16)$$

The integration limits on $\theta$ are 0 and $\tan^{-1}(\delta/\epsilon)$. In the limit as $\epsilon \to 0$ with $\delta$ fixed,
the upper limit approaches $\pi/2$. Substitution of (2.1 - 16) into (2.1 - 14) makes the integration trivial, the overall result being simply $2\pi F(x_S)$. Thus, we find that, as regards the integral over the entire surface, when $\epsilon$ goes from a finite positive value to 0,

$$\lim_{\epsilon \to 0} \int_{S'} F(x_{S'}) \left[ n(x_{S'}) \cdot \nabla' G(x_S | x_{S'}) \right] dS'$$

$$= 2\pi F(x_S) + \int_{S'} F(x_{S'}) \left[ n(x_{S'}) \cdot \nabla' G(x_S | x_{S'}) \right] dS' \quad (2.1 - 17)$$

The result of applying (2.1 - 17) to (2.1 - 1), with the pressure boundary condition (2.1 - 4), is therefore

$$p(x_S) = \frac{1}{2\pi} \int_{S'} \frac{p(x_{S'})}{n(x_{S'})} \left[ n(x_{S'}) \cdot \nabla' G(x_S | x_{S'}) \right] dS'$$

$$+ \frac{1}{2\pi} \int_{S'} f_n(x_{S'}) G(x_S | x_{S'}) dS' \quad (2.1 - 18)$$
The above integral equation has been used for numerical analysis of surface pressure [8]. One of the intrinsic sources of difficulty [80] is that, for certain discrete characteristic frequencies, the solution is not unique, because the homogeneous part of the integral equation (2.1 – 18) has eigensolutions (the multiplicative constant being arbitrary) at these frequencies. Another complication is that the integration kernel, \( n(x_{S'}) \bullet \nabla' G(x_S \mid x_{S'}) \), although integrable, is singular, so one is confronted with a singular integral equation. Also, the kernel is not symmetric in the interchange of \( x_S \) and \( x_{S'} \), so the linear operator associated with this integral equation is not self-adjoint. Hence the integral equation (2.1 – 18) does not lead in a natural manner to a variational principle.

The variational principle derived by Morse and Feshbach [59] is obtained by simply taking a normal derivative of Eq.(2.1 – 1) at an external point and then letting the external point approach the surface. The resulting principle features an integration kernel that is self-adjoint. However, one of the integrands becomes highly singular as two surface points approach one another, and the integral is ambiguous without careful definition of how the singular integrand is to be evaluated. It is not apparent how their principle could be numerically implemented for cases of interest.

In order to circumvent this singularity problem, we will use the method previously employed by Maue [51] and Stallybrass [72]. Let us begin by taking a normal derivative of Eq.(2.1 – 1) with respect to the acoustic pressure \( p \) at an extrior point, \( x = x_S + \epsilon n(x_S) \), with \( \epsilon \) small and positive,

\[
f_n(x_S) = - \frac{1}{4\pi} \int_{S'} p(x_{S'}) [n(x_S) \bullet \nabla] [n(x_{S'}) \bullet \nabla'] G(x \mid x_{S'}) dS' \\
- \frac{1}{4\pi} \int_{S'} f_n(x_{S'}) [n(x_S) \bullet \nabla G(x \mid x_{S'})] dS' \quad (2.1 - 19)
\]
Note that the integrand in the first term on the right side of (2.1 - 19) is highly singular as $x \to x_S$, so care must be taken in the numerical evaluation of this term in the limit as $\epsilon \to 0$. To this purpose, we follow an approach similar to that previously developed by Maue [51] and by Stallybrass [72].

First, we introduce a vector identity [81]

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \quad (2.1 - 20)$$

In order for such a vector identity to match the second normal derivative involved in the integrand of the first integral on the right side of Eq.(2.1 - 19), we rewrite (2.1 - 20) as

$$(\mathbf{c} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{d}) = (\mathbf{c} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{d}) - (\mathbf{b} \times \mathbf{a}) \cdot (\mathbf{c} \times \mathbf{d}) \quad (2.1 - 21)$$

because

$$\mathbf{V} \cdot \mathbf{V'} \equiv \mathbf{V}' \cdot \mathbf{V} \quad \text{and} \quad \mathbf{V} \times \mathbf{V'} \equiv -\mathbf{V'} \times \mathbf{V} \quad (2.1 - 22)$$

for any cartesian coordinate system. If we set

$$\mathbf{a} = \nabla; \quad \mathbf{d} = \nabla'; \quad \mathbf{c} = n(x_S); \quad \text{and} \quad \mathbf{b} = n(x_{S'}) \quad (2.1 - 23)$$

in Eq.(2.1 - 21), then we obtain

$$[n(x_S) \cdot \nabla][n(x_{S'}) \cdot \nabla'] = [n(x_S) \cdot n(x_{S'})](\nabla \cdot \nabla') - [n(x_{S'}) \times \nabla] \cdot [n(x_S) \times \nabla'] \quad (2.1 - 24)$$
Since $G(x \mid x')$ is symmetric with respect to interchange of $x$ and $x'$ and is a function only of $R$, one has, in particular,

$$\nabla G(x \mid x') = -\nabla' G(x \mid x')$$  \hspace{1cm} (2.1 - 25)

where the gradient $\nabla'$ is understood to be taken with respect to the primed coordinate system. Equation (2.1 - 25) allows one to replace the gradient of the Green's function with respect to the unprimed coordinate by the negative of the gradient with respect to the primed coordinate. Hence Eq. (2.1 - 24) when applied to the Green's function gives

$$[\mathbf{n}(x) \cdot \nabla][\mathbf{n}(x') \cdot \nabla']G(x \mid x') = [\mathbf{n}(x) \cdot \mathbf{n}(x')][\nabla \cdot \nabla' G(x \mid x')$$

$$- [\mathbf{n}(x) \times \nabla] \cdot [\mathbf{n}(x') \times \nabla']G(x \mid x')$$  \hspace{1cm} (2.1 - 26)

Also, because $G(x \mid x')$ satisfies the scalar Helmholtz equation for $x \neq x'$, we can make the substitution

$$\nabla \cdot \nabla' G(x \mid x') = -\nabla^2 G(x \mid x') = k^2 G(x \mid x')$$  \hspace{1cm} (2.1 - 27)

Eqs. (2.1 - 26) and (2.1 - 27) enable one to recast Eq. (2.1 - 19) into the form

$$f_n(x) = -\frac{1}{4\pi} \int_{S'} f_n(x') \left[ \mathbf{n}(x) \cdot \nabla G(x \mid x') \right] dS'$$

$$- \frac{k^2}{4\pi} \mathbf{n}(x) \cdot \int_{S'} \mathbf{n}(x') p(x') G(x \mid x') dS'$$

$$+ \frac{1}{4\pi} [\mathbf{n}(x) \times \nabla] \cdot \int_{S'} p(x') [\mathbf{n}(x') \times \nabla'] G(x \mid x') dS'$$  \hspace{1cm} (2.1 - 28)
where the operator $n(x_S) \times \nabla$ contained in the last integral on the right side of (2.1 - 28) is a derivative tangential to the surface. Note that the second tangential derivative of the Green's function in that term still presents a problem in taking the limit as $\epsilon \to 0$, so we integrate the last integral on the right side of (2.1 - 28) by parts

\[ \int \int_{S'} p(x_{S'}) [n(x_{S'}) \times \nabla'] G(x | x_{S'}) \, dS' = \int \int_{S'} [n(x_{S'}) \times \nabla'] \left\{ p(x_{S'}) G(x | x_{S'}) \right\} \, dS' \]

The first (surface) integral on the right side of (2.1 - 29) can be transformed into a line integral by the Stokes' theorem [82]. The integrand of the line integral (contained within the braces) is continuous (since $x$ is off the surface) and differentiable with respect to tangential coordinates on the surface. In the present situation, the surface $S'$ is a closed one. Accordingly, the length of the curve along which the line integral is carried out shrinks to zero. Consequently, the first term on the right side of (2.1 - 29) vanishes identically, and one obtains

\[ f_n(x_S) = -\frac{1}{4\pi} \int \int_{S'} f_n(x_{S'}) \left[ n(x_S) \cdot \nabla G(x | x_{S'}) \right] \, dS' \]

\[ - \frac{k^2}{4\pi} n(x_S) \cdot \int \int_{S'} n(x_{S'}) p(x_{S'}) G(x | x_{S'}) \, dS' \]

\[ - \frac{1}{4\pi} [n(x_S) \times \nabla] \cdot \int \int_{S'} [n(x_{S'}) \times \nabla'] G(x | x_{S'}) \, dS' \]

(2.1 - 30)

where $x$ is kept a slight distance away from the surface in the above so that all integrands are continuous. The desired Euler-Lagrange equation (an integrodifferential
equation) now emerges upon taking the limit as $\epsilon \to 0$, namely, letting the field point $x$ approach the surface

$$U_n(x_S) = -\frac{k^2}{4\pi} n(x_S) \cdot \iint_{S'} n(x_{S'}) p(x_{S'}) G(x_S \mid x_{S'}) dS'$$

$$- \frac{1}{4\pi} [n(x_S) \times \nabla] \cdot \iint_{S'} [n(x_{S'}) \times \nabla' p(x_{S'})] G(x_S \mid x_{S'}) dS'$$

(2.1 - 31)

where the function $U_n(x_S)$ is defined by

$$U_n(x_S) = f_n(x_S) + \lim_{\epsilon \to 0} \left\{ \frac{1}{4\pi} \iint_{S'} f_n(x_{S'}) \left[ n(x_S) \cdot \nabla G(x \mid x_{S'}) \right] dS' \right\}$$

$$= \frac{1}{2} f_n(x_S) + (PR) \frac{1}{4\pi} \iint_{S'} f_n(x_{S'}) \left[ n(x_S) \cdot \frac{x_S - x_{S'}}{R} \right] \frac{d}{dR} \left( \frac{e^{ikR}}{R} \right) dS'$$

(2.1 - 32)

The symbol (PR) in (2.1 - 32) stands for the principal value and $R$ is the distance between two surface points $| x_S - x_{S'} |$. The second version of Eq.(2.1 - 32) is derived by using the technique similar to that for which Eq.(2.1 - 17) is obtained. This latter version, however, should not be used in case of an unbaflled plate (flat or curved) with an infinitesimal thickness as will be discussed in the next Chapter.

We now derive the variational principle from Eq.(2.1 - 31) by multiplying each term in that equation by a virtual increment $\delta p(x)$ and integrate over the surface once again. To avoid possible confusions concerning the treatment of singularities in the derivation, we will temporarily keep $x$ a slight distance off the surface, $x = x_S + \epsilon n(x_S)$, with $\epsilon$ small and positive.
Integrate the last term on the right side of (2.1 - 33) by parts, use the Stokes' theorem, with the inner integral enclosed in large braces regarded as the function $F(x_S)$, to transfer, in effect, the tangential derivative operator $n(x_S) \times \nabla$ to $p(x)$, and then take the limit as $\epsilon \to 0$. Doing so yields

\[
\iint_{S} \delta p(x) U_n(x_S) dS = -\frac{k^2}{4\pi} \iint_{S} \iint_{S'} \delta p(x) p(x_S') [n(x_S) \cdot n(x_S')] G(x_S | x_S') dS' dS \\
- \frac{1}{4\pi} \iint_{S} \delta p(x) [n(x_S) \times \nabla] \cdot \left\{ \iint_{S'} [n(x_S') \times \nabla' p(x_S')] G(x_S | x_S') dS' \right\} dS
\]

(2.1 - 33)

Since we are seeking the surface pressure amplitude $p(x_S)$ given the normal surface velocity, the function $U_n(x_S)$ remains constant when $p(x_S)$ is varied. Symmetry of the free-space Green's function and the rules in the calculus of variations make it possible to replace, within the integrands, the variational factor of the generic form $F(x_{S'}) \delta F(x_S)$ by an alternative variational factor $\frac{1}{2} \delta [F(x_{S'}) F(x_S)]$. Also, since the integration limits do not change during the variation, the integral of the variation is the variation of the integral. Similarly, the sum of variations can be replaced by the variation of the sum. All this allows one to recast Eq.(2.1 - 34) as the variational principle

\[
\iint_{S} \delta p(x_S) U_n(x_S) dS = -\frac{k^2}{4\pi} \iint_{S} \iint_{S'} \delta p(x_S) p(x_S') [n(x_S) \cdot n(x_S')] G(x_S | x_S') dS' dS \\
+ \frac{1}{4\pi} \iint_{S} \iint_{S'} [n(x_S) \times \nabla \delta p(x_S)] \cdot [n(x_S') \times \nabla' \delta p(x_S')] G(x_S | x_S') dS' dS
\]

(2.1 - 34)
\[ \delta J[p] = 0 \quad (2.1 - 35) \]

where the functional \( J[p] \) is identified as

\[
J[p] = 4\pi \int_S p(x_S)U_n(x_S) dS + \frac{k^2}{2} \int_S \int_{S'} [n(x_S) \cdot n(x_{S'})] p(x_S) p(x_{S'}) G(x_S | x_{S'}) dS' dS - \frac{1}{2} \int_S \int_{S'} [n(x_S) \times \nabla p(x_S)] \cdot [n(x_{S'}) \times \nabla' p(x_{S'})] G(x_S | x_{S'}) dS' dS
\quad (2.1 - 36)
\]

where the function \( U_n(x_S) \) is given by (2.1 - 32). The functional \( J[p] \) is stationary to small changes in \( p(x_S) \). One cannot, however, in general state that \( J[p] \) has an extreme value (maximum or minimum) when the trial function \( p(x_S) \) is equal to the true complex pressure distribution on the surface. Such a maximum or minimum principle probably does not exist in wave propagation problems, except in the zero frequency limit. If the trial function is taken to be the actual function plus \( \epsilon F(x_S) \), where \( \epsilon \) is small and \( F(x_S) \) is a fixed admissible trial function, then \( J[p] \) can be regarded as a function of \( \epsilon \), and \( \frac{dJ}{d\epsilon} \) must be zero at \( \epsilon = 0 \). The sign of either the real or imaginary part of the second derivative \( \frac{d^2J}{d\epsilon^2} \), however, depends on the choice of this fixed admissible trial function \( F(x_S) \). In general, the class of admissible trial functions is restricted to function that are continuous over the surface and piece-wise differentiable with respect to displacement on the surface so that \( n(x_S) \times \nabla p(x_S) \) exists almost everywhere and is square integrable.

In the typical Rayleigh-Ritz procedure, the unknown surface pressure is replaced by an expansion of preselected basis functions. Let us assume that
\[ p(x_S) = \sum_{n=1}^{N} C_n P_n(x_S) \]  \hspace{1cm} (2.1 - 37)

where \( P_n(x_S) \) is the \( n \)th chosen basis function and \( C_n \) is the corresponding coefficient to be determined. Substituting (2.1 - 37) into (2.1 - 36) gives the functional \( J[p] \) as

\[ J[p] = \sum_{n=1}^{N} B_n C_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} A_{nm} C_n C_m \]  \hspace{1cm} (2.1 - 38)

The variation of \( J[p] \) is obtained from virtual increments in \( C_n \). Since the functional \( J[p] \) is stationary and the increments \( \delta C_n \) are arbitrary, we have

\[ \frac{\partial J}{\partial C_n} = 0; \quad n = 1, 2, \ldots, N \]  \hspace{1cm} (2.1 - 39)

which results in a system of \( N \) simultaneous equations for the unknown coefficients \( C_n \)

\[ [A]\{C\} = \{B\} \]  \hspace{1cm} (2.1 - 40)

where elements of the column vector \( \{B\} \) and the square matrix \( [A] \) are given by

\[ B_n = -4\pi \int_S \int_{S'} U_n(x_S) P_n(x_S) dS \]  \hspace{1cm} (2.1 - 41)

\[ A_{nm} = k^2 \int_S \int_{S'} \left[ n(x_S) \cdot n(x_{S'}) \right] P_n(x_S) P_m(x_{S'}) G(x_S \mid x_{S'}) dS' dS \]

\[ - \int_S \int_{S'} \left[ n(x_S) \times \nabla P_n(x_S) \right] \cdot \left[ n(x_{S'}) \times \nabla' P_m(x_{S'}) \right] G(x_S \mid x_{S'}) dS' dS \]  \hspace{1cm} (2.1 - 42)
Suppose that the matrix \([A]\) is non-singular, then coefficients \(C_n\) can be obtained by solving (2.1 - 40) using, for example, the Gauss-Jordan elimination method.

\[
\{C\} = [A]^{-1}\{B\} \tag{2.1 - 43}
\]

Substituting \(C_n\) into Eq.(2.1 - 37) gives the approximate solution of the surface pressure \(p(x_s)\). The acoustic pressure at an external point \(x\) can now be determined by Eq.(2.1 - 1) and the acoustic radiation problem is considered solved.

2.2 Formulations For Axisymmetric Surfaces

If the body of interest is of axisymmetry, formulations for elements of the column vector \(\{B\}\) and the square matrix \([A]\) given by (2.1 - 41) and (2.1 - 42) can be greatly simplified. In these circumstances, the integrands of the double surface integrations depend only on the difference of the two angular coordinates. A simple transformation of angular variables enables one to replace the double integrations over angles by \(2\pi\) times an integration over the difference in angular coordinates. Consequently, the initially four-fold integrations inherent to the variational formulation are reduced to three-fold integrations, one integration over the difference of the angles and two integrations over coordinates corresponding to linear distance along the generator of the axisymmetric surface. Furthermore, since the angular integration is independent of the acoustic pressure and velocity profile distributions on the surface, this integral can be carried out at the outset, with results stored and used over and over again in a series of computations, as if it were a completely tabulated standard function. The subsequent calculations only require a two-fold integration, rather than a four-fold integration. A significant simplification in numerical computations is thus achieved.
In the following, our interest will be confined to axially symmetric cases. Equations (2.1 - 41) and (2.1 - 42) will accordingly be reexpressed in the forms outlined in the above observations.

Suppose that the object under consideration is an arbitrary body of revolution formed by rotating the area generator around the central axis. The position of a generic point on the axisymmetric surface can be described, for example, by the cylindrical coordinates \((r, z, \Theta)\) (See Fig. 2). An alternative description which is more efficient from the computational viewpoint, however, will be to use one parameter instead of two, in addition to the angular variable \(\Theta\). To this end, we shall express the radial and longitudinal variables \(r\) and \(z\), respectively, in terms of the position index \(s\) which is defined along the generator of the surface of revolution.

The integrations over two angular variables \(\Theta\) and \(\Theta'\) are both carried out from 0 to \(2\pi\). Because of the axisymmetry, the integrations are coupled only through the difference of the two angular coordinates \((\Theta - \Theta')\). A simple transformation of angular variables thus enables one to replace this difference \((\Theta - \Theta')\) in the first angular integration with respect to, say \(\Theta'\) by \(\theta\). Such a angular coordinate transformation removes the dependence of the integrands upon the second angular variable \(\Theta\). Therefore the integration over \(\Theta\) from 0 to \(2\pi\) yields simply a factor \(2\pi\). The dimensionality of the integration is thus reduced by one. Since this \(2\pi\) factor appears in all double surface integrations, we devide through Eq.(2.1 - 38) by this factor so that it drops out in the later expressions. Also, because of the axisymmetry, the angular integration is independent of the acoustic pressure and velocity distribution on the surface, hence this integral can be carried out at the outset with the results stored in the form of some tabulated coefficient arrays.
Figure 2. Nomenclature of an axisymmetric surface. The surface point \( x_s \) can be described by the cylindrical coordinates \((r, z, \Theta)\) or, alternatively, by the position index \( s \) defined along the generator of the axisymmetric surface.
The subsequent calculations involved in the elements $B_n$ and $A_{nm}$ only require a two-fold integration over coordinates corresponding to the linear distance along the generator of the axisymmetric surface.

\[
B_n = -4\pi \int_{-S_{\text{max}}}^{+S_{\text{max}}} U_n(s) p_n(s) r(s) \, ds 
\]  
(2.2 - 1)

\[
A_{nm} = \int_{-S_{\text{max}}}^{+S_{\text{max}}} \int_{-S_{\text{max}}}^{+S_{\text{max}}} F_2(s, s') \, ds' \, ds - \int_{-S_{\text{max}}}^{+S_{\text{max}}} \int_{-S_{\text{max}}}^{+S_{\text{max}}} F_3(s, s') \, ds' \, ds 
\]  
(2.2 - 2)

where $+S_{\text{max}}$ and $-S_{\text{max}}$ are two extremities of the generator of the body of revolution. (Note that in Fig. 2 we choose the point where $s = 0$ to be the center of the generator of the axisymmetric surface so that $+S_{\text{max}} = -S_{\text{max}}$. In a more general case, one can set different distances measured from the center to the extremeties of the generator.)

The function $U_n(s)$ involved in the integrand in Eq.(2.2 - 1) is accordingly given by

\[
U_n(s) = -i \frac{1}{2} \rho_o c k v_n(s) - \frac{1}{4\pi} \int_{-S_{\text{max}}}^{+S_{\text{max}}} F_1(s, s') \, ds' 
\]  
(2.2 - 3)

where we abbreviate the functions $F_1$ to $F_3$ which appear in the integrands in Eqs.(2.2 - 2) and (2.2 - 3) to be

\[
F_1(s, s') = i \rho_o c k v_n(s') \mathcal{H}_1(s, s') 
\]  
(2.2 - 4)

\[
F_2(s, s') = k^2 p_n(s) p_m(s') \mathcal{H}_2(s, s') 
\]  
(2.2 - 5)

\[
F_3(s, s') = \frac{d p_n(s)}{d s} \frac{d p_m(s')}{d s'} \mathcal{H}_3(s, s') 
\]  
(2.2 - 6)
with coefficients $\mathcal{H}_1$ to $\mathcal{H}_3$ being integrations of the free-space Green's function over the angular variable $\theta$

$$\mathcal{H}_1(s, s') = \int_0^{2\pi} \left[ n(s) \cdot \frac{x(s) - x(s')}{R} \right] \frac{d}{dR} \left( \frac{e^{ikR}}{R} \right) \tau(s') \, d\theta$$  \hspace{1cm} (2.2 - 7)

$$\mathcal{H}_2(s, s') = \int_0^{2\pi} [n(s) \cdot n(s')] \frac{e^{ikR}}{R} \tau(s) \tau(s') \, d\theta$$  \hspace{1cm} (2.2 - 8)

$$\mathcal{H}_3(s, s') = \int_0^{2\pi} \frac{e^{ikR}}{R} \cos \theta \tau(s) \tau(s') \, d\theta$$  \hspace{1cm} (2.2 - 9)

where $n(s)$ is the unit normal vector at the position $s$.

Clearly, the major computations lie in Eqs.\((2.2 - 7)\) to \((2.2 - 9)\) because all integrands become singular when $R = 0$ at $s = s'$ and $\theta = 0$. In the preceding section, we have shown that the singularities contained in these integrands are at most Cauchy-type. In the next section, we will demonstrate that contributions from these singularities are all finite.

To get an overall picture of the nature of singularities involved in these integrations, let us examine $\mathcal{H}_1$ which is of a higher order singularity than both $\mathcal{H}_2$ and $\mathcal{H}_3$. Writing out the normal derivative of the Green's function in \((2.2 - 7)\), we obtain

$$\left[ n(s) \cdot \frac{x(s) - x(s')}{R} \right] \frac{d}{dR} \left( \frac{e^{ikR}}{R} \right)$$

$$= \left\{ \tau(s) - \tau(s') \cos \theta \frac{dz(s)}{ds} - [z(s) - z(s')] \frac{dr(s)}{ds} \right\} \left( ik - \frac{1}{R} \right) \frac{e^{ikR}}{R^2}$$  \hspace{1cm} (2.2 - 10)

where
\[ R = \sqrt{r^2(s) + r^2(s') - 2r(s)r(s') \cos \theta + [z(s) - z(s')]^2} \]  
(2.2 - 11)

Let \( \theta = \pi - 2\alpha \), substitute (2.2 - 10) and (2.2 - 11) into (2.2 - 7), and then group terms according to the order of singularities. Doing so, we obtain

\[ \mathcal{H}(s, s') = i k \eta \int_0^{\frac{\pi}{2}} e^{i\Phi} \, d\alpha - \frac{\eta}{M} \int_0^{\frac{\pi}{2}} \frac{e^{i\Phi}}{D} \, d\alpha \\
+ i k (m\zeta - \eta) \int_0^{\frac{\pi}{2}} \frac{e^{i\Phi}}{D^2} \, d\alpha - \frac{(m\zeta - \eta)}{M} \int_0^{\frac{\pi}{2}} \frac{e^{i\Phi}}{D^3} \, d\alpha \]  
(2.2 - 12)

where we define

\[
\begin{align*}
\Phi &= kMD \\
D &= \sqrt{1 - m \sin^2 \alpha} \\
m &= \frac{4r(s)r(s')}{M^2} \\
M &= \sqrt{[r(s) + r(s')]^2 + [z(s) - z(s')]^2} \\
\zeta &= [r(s) + r(s')] \frac{dz(s)}{ds} - [z(s) - z(s')] \frac{dr(s)}{ds} \\
\eta &= 2r(s') \frac{dz(s)}{ds} 
\end{align*}
\]

(2.2 - 13)

Obviously, the last three integrands on the right side of (2.2 - 12) are singular when \( D = 0 \), i.e. when \( s = s' \) and \( \alpha = \frac{\pi}{2} \). To make Eq.(2.2 - 12) more amenable to numerical computations, we split the singular parts out from the regular parts using the identity

\[ e^{i\Phi} = 1 + i 2 \sin(\Phi/2) e^{i(\Phi/2)} \]  
(2.2 - 14)
This identity allows us to rewrite (2.2 - 12) essentially as follows

\[ \mathcal{H}_1(s, s') = \text{(coefficient } I) \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m \sin^2 \alpha}} \, d\alpha \]
\[ + \text{(coefficient } II) \int_0^{\frac{\pi}{2}} \sqrt{1 - m \sin^2 \alpha} \, d\alpha \]
\[ + \text{(regular parts) } \] (2.2 - 15)

One recognizes immediately that the first two terms on the right side in the above are the first and second kinds of complete elliptic integrals, respectively, and can be approximated by series expansions with an error of order \(|e(m)| \leq 2 \times 10^{-8}\) [83]. The coefficients \(I\), \(II\), and (regular parts) are found to be

(coefficient \(I\)) \[ = -\frac{k^2 M^2 (m \zeta - \eta) + 2\eta}{2M} \] (2.2 - 16)

(coefficient \(II\)) \[ = -\frac{m \zeta - \eta}{M(1 - m)} \] (2.2 - 17)

(regular parts) \[ = \int_0^{\frac{\pi}{2}} \text{(regular integrands) } d\alpha \] (2.2 - 18)

where

(regular integrands) \[ = ik \eta \left[ e^{i\Phi} - \frac{\sin(\Phi/2)}{\Phi/2} e^{i(\Phi/2)} \right] \]
\[ + \frac{k^3 M^2 (m \zeta - \eta)}{2} \left[ \frac{(\Phi/2) - \sin(\Phi/2)}{\Phi/2} \right] e^{i(\Phi/2)} \]
\[ - \frac{\left[ \frac{(\Phi/4) - \sin(\Phi/4)}{\Phi/4} \right] e^{i(\Phi/4)}}{4} + i \left\{ \frac{\left[ \frac{(\Phi/2) - \sin(\Phi/2)}{(\Phi/2)^3} \right] e^{i(\Phi/2)}}{2} \right. \]
\[ - \frac{\sin(\Phi/4)}{(\Phi/4)} e^{i(\Phi/4)} + \frac{\sin(\Phi/8)}{(\Phi/8)} \frac{e^{i(\Phi/8)}}{4} \} \] (2.2 - 19)
where \( m, M, \zeta, \eta, \), and \( \Phi \) are given in (2.2 - 13). One sees that (2.2 - 19) is indeed finite as \( \Phi \to 0 \) or \( D \to 0 \), because the generic functions \( \sin x/x \), \( (x - \sin x)/x^2 \), and \( (x - \sin x)/x^3 \) are all finite as \( x \to 0 \).

In a similar manner, by separating singularities out from the integrands of (2.2 - 8) and (2.2 - 9) with use of the identity (2.2 - 14), one is able to rewrite coefficients \( \mathcal{H}_2(s, s') \) and \( \mathcal{H}_3(s, s') \) as sums of the first and second kinds of the complete elliptic integrals plus some regular terms.

\[
\mathcal{H}_2(s, s') = \begin{cases} 
  mM \Psi_1 & \text{if} & [n(s) \cdot n(s')] = +1 \\
  0 & \text{if} & [n(s) \cdot n(s')] = 0 \\
  -mM \Psi_1 & \text{if} & [n(s) \cdot n(s')] = -1 \\
  M \Psi_2 & \text{if} & [n(s) \cdot n(s')] = \cos \theta 
\end{cases} 
\tag{2.2 - 20}
\]

\[
\mathcal{H}_3(s, s') = M \Psi_2 
\tag{2.2 - 21}
\]

where we abbreviate functions \( \Psi_1 \) and \( \Psi_2 \) as

\[
\Psi_1 = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m \sin^2 \alpha}} d\alpha + i k M \int_0^{\frac{\pi}{2}} \sin(\Phi/2) e^{i(\Phi/2)} \frac{\sin(\Phi/2)}{(\Phi/2)} \sin(\Phi/2) e^{i(\Phi/2)} \sqrt{1 - m \sin^2 \alpha} d\alpha 
\tag{2.2 - 22}
\]

\[
\Psi_2 = (2 - m) \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m \sin^2 \alpha}} d\alpha - 2 \int_0^{\frac{\pi}{2}} \sqrt{1 - m \sin^2 \alpha} d\alpha
\]

\[
+ i \int_0^{\frac{\pi}{2}} [k M (2 - m) \frac{\sin(\Phi/2)}{(\Phi/2)} e^{i(\Phi/2)} - 4 \sin(\Phi/2) e^{i(\Phi/2)} \sqrt{1 - m \sin^2 \alpha}] d\alpha 
\tag{2.2 - 23}
\]

Although formulations developed in the above are limited to axisymmetric surfaces under axisymmetric excitations, one should note that the procedures described here are easily extended to cases where the surface velocity \( v_n \) also depends
on the angular coordinate Θ. In such cases, the quantity $f_n$ defined in Eq.(2.1 - 4) can be expanded in a Fourier series, such as

$$f_n(s, Θ) = \sum_{j=-\infty}^{\infty} f_{nj}(s)e^{i(jΘ)}$$

(2.2 - 24)

where

$$f_{nj}(s) = -i\omega\rho_0 v_{nj}(s)$$

(2.2 - 25)

The surface pressure $p$ can also be expanded in such a series

$$p(s, Θ) = \sum_{j=-\infty}^{\infty} p_{j}(s)e^{i(jΘ)}$$

(2.2 - 26)

The important simplification which results because the body is axisymmetric is that $p_j(s)$ depends only on $f_{nj}(s)$ for the same index $j$. This means that one can develop a variational principle for any particular $p_j(s)$ which is independent of those that correspond to other values of $j$. Such variational principles for $j \neq 0$ are not markedly different from what given previously; the chief distinction is that an extra factor $e^{i(2jΘ)}$ enters into the integrands of coefficients $\mathcal{H}_{1j}(s, s')$, $\mathcal{H}_{2j}(s, s')$, and $\mathcal{H}_{3j}(s, s')$,

$$\mathcal{H}_{1j}(s, s') = \int_{0}^{2π} \int_{0}^{2π} \left[ n(s) \cdot \frac{\mathbf{x}(s) - \mathbf{x}(s')}{R} \right] d\frac{d}{R} \left( \frac{e^{ikR}}{R} \right) e^{i(2jΘ)} r(s') dΘdΘ'$$

(2.2 - 27)

$$\mathcal{H}_{2j}(s, s') = \int_{0}^{2π} \int_{0}^{2π} \left[ n(s) \cdot n(s') \right] \frac{e^{ikR}}{R} e^{i(2jΘ)} r(s) r(s') dΘdΘ'$$

(2.2 - 28)

$$\mathcal{H}_{3j}(s, s') = \int_{0}^{2π} \int_{0}^{2π} \frac{e^{ikR}}{R} \cos Θ e^{i(2jΘ)} r(s) r(s') dΘdΘ'$$

(2.2 - 29)

where

$$R = \sqrt{r^2(s) + r^2(s') - 2r(s)r(s') \cos(Θ - Θ') + [z(s) - z(s')]^2}$$

(2.2 - 30)
Accordingly, the functions $F_{1j}(s, s')$, $F_{2j}(s, s')$, and $F_{3j}(s, s')$ are found to be

\[
F_{1j}(s, s') = i\rho_0 c k v_{nj}(s')\mathcal{H}_{1j}(s, s')
\]

\[
F_{2j}(s, s') = k^2 P_{nj}(s) P_{mj}(s')\mathcal{H}_{2j}(s, s')
\]

\[
F_{3j}(s, s') = \frac{dP_{nj}(s)}{ds} \frac{dP_{mj}(s')}{ds'} \mathcal{H}_{3j}(s, s')
\]

The elements of the column vector \( \{B\} \) and square matrix \([A]\) are then given by

\[
B_{nj} = -4\pi \int_{-s_{\text{max}}}^{+s_{\text{max}}} U_{nj}(s) P_{nj}(s) r(s) ds
\]

\[
A_{nmj} = \int_{-s_{\text{max}}}^{+s_{\text{max}}} \int_{-s_{\text{max}}}^{+s_{\text{max}}} F_{2j}(s, s') ds' ds - \int_{-s_{\text{max}}}^{+s_{\text{max}}} \int_{-s_{\text{max}}}^{+s_{\text{max}}} F_{3j}(s, s') ds' ds
\]

where the function $U_{nj}(s)$ in the integrand on the right side of Eq.(2.2 - 34) reads

\[
U_{nj}(s) = -\frac{1}{2} \omega_0 \rho_0 v_{nj}(s') \int_0^{2\pi} e^{i(2j\Theta)} d\Theta - \frac{1}{4\pi} \int_{-s_{\text{max}}}^{+s_{\text{max}}} F_{1j}(s, s') ds'
\]

The first integral on the right side of Eq.(2.2 - 36) is identically zero for $j \neq 0$. The case in which $j = 0$ corresponds to the axisymmetric excitation. In such a circumstance, the formulations just described in the above reduce to those which have been given previously.
2.3 Integrations With Weak Singularities

In this section, we shall examine the singularities separated out from the integrands of the coefficients $\mathcal{H}_1(s, s')$ to $\mathcal{H}_3(s, s')$. Let us start from $\mathcal{H}_1(s, s')$ [See Eq. (2.2 - 12)] since it has the highest order of singularity and consider the case in which the basis functions are assumed to be continuous and have piece-wise tangential derivatives on the surface.

In section 2.1, it has been shown that the Cauchy principal value of the integral involving the normal derivative with respect to the free-space Green's function exists. In particular, the integral on the right side of (2.2 - 3) is interpreted as the result of integrating over the area $S$ in a limiting process by taking first $\epsilon \to 0$ ($\epsilon > 0$) and then $\delta \to 0$. The limit of this integral is exactly the same as if one set the field point on the surface at the outset and then carried out the integral over the surface. Hence in evaluating the integral on the right side of (2.2 - 3), the position indices $s$ and $s'$ are understood to represent two points on the surface $S'$.

As these two surface points approach one another, we have $s \to s'$ and $\alpha \to \frac{\pi}{2}$. So the limiting value of $m \to 1$ and the denominator $D$ in integrands of (2.2 - 12) approaches zero by definition of (2.2 - 13). The exponential function is nearly unity because the exponent $\Phi$ becomes infinitely small in this limit. The quantities $\zeta$ and $\eta$, however, are finite because the direction cosines $\frac{dr(s)}{ds}$ and $\frac{d\nu(s)}{ds}$ always remain finite. Consequently, the coefficient $\mathcal{H}_1(s, s')$ given by (2.2 - 12) to the leading order in small value $D$ is

$$\mathcal{H}_1(s, s') \sim \int_0^{\frac{\pi}{2}} \frac{1}{D^3} d\alpha$$

(2.3 - 1)

where $D = \sqrt{1 - m \sin^2 \alpha}$, as defined in (2.2 - 13). The integrand in (2.3 - 1) is highly singular as $D \to 0$, i.e. as $m \to 1$ and $\alpha \to \frac{\pi}{2}$. However, it can be shown
that the integral of \((2.3 - 1)\) is actually one of special cases of the third kind of the complete elliptic integral \([84]\) defined by

\[
\Pi(n; \varphi \setminus \beta) = \int_0^\varphi (1 - n \sin^2 \theta)^{-1} (1 - \sin^2 \beta \sin^2 \theta)^{-1} \, d\theta \tag{2.3 - 2}
\]

Let the upper limit \(\varphi\) be \(\pi\) and the parameter \(n\) be \(\sin^2 \beta\). With these substitutions, the integral \((2.3 - 2)\) reduces to

\[
\Pi(\sin^2 \beta; \frac{\pi}{2} \setminus \beta) = \sec^2 \beta \, E(\frac{\pi}{2} \setminus \beta) - \frac{\tan^2 \beta \sin \pi}{2 \Delta(\frac{\pi}{2})} \tag{2.3 - 3}
\]

where the symbol \(\Delta\) is the delta amplitude

\[
\Delta(\varphi) = \sqrt{1 - m \sin^2 \varphi} \tag{2.3 - 4}
\]

For \(0 \leq m < 1\), \(\Delta(\frac{\pi}{2})\) is nonzero, so the second term on the right side of \((2.3 - 3)\) vanishes identically (since \(\sin \pi \equiv 0\)). The quantity \(\beta\) in \((2.3 - 3)\) is the modular angle defined in the canonical forms of complete elliptic integrals

\[
\beta = \arcsin(m) \tag{2.3 - 5}
\]

where \(m\) is the parameter given in \((2.2 - 13)\).

Substituting \((2.3 - 4)\) into \((2.3 - 3)\) then yields

\[
\Pi(m; \frac{\pi}{2} \setminus \beta) = \frac{1}{1 - m} \, E(\frac{\pi}{2} \setminus \beta) \tag{2.3 - 6}
\]

which corresponds to the second term on the right side of \((2.2 - 15)\) with the symbol \(E(\frac{\pi}{2} \setminus \beta)\) indicating the complete elliptic integral of the second kind.
\[
E\left(\frac{\pi}{2} \mid \beta\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - m \sin^2 \alpha} \, d\alpha \quad (2.3 - 7)
\]

The polynomial approximation of \(E\left(\frac{\pi}{2} \mid \beta\right)\) is given by

\[
E(m) = (1 + a_1 m_1 + \cdots + a_4 m_1^4)
- (b_1 + b_2 m_1 + \cdots + b_4 m_1^3) m_1 \ln(m_1) + \epsilon(m) \quad (2.3 - 8)
\]

with an error of order \(|\epsilon(m)| < 2 \times 10^{-8}\). The coefficients of \(a_1\) to \(a_4\) and \(b_1\) to \(b_4\) in \((2.3 - 8)\) are as follows

\[
\begin{align*}
a_1 &= 0.44325141463 & b_1 &= 0.24998368310 \\
b_2 &= 0.0626061220 & b_3 &= 0.09200180037 \\
a_3 &= 0.04757383546 & b_4 &= 0.04069697526 \\
a_4 &= 0.01736506451 & b_4 &= 0.00526449639
\end{align*}
\quad (2.3 - 9)
\]

The parameter \(m_1\) in \((2.3 - 8)\) is complementary to \(m\)

\[
m_1 = 1 - m \quad (2.3 - 10)
\]

Obviously, as \(m \to 1, m_1 \to 0\). Since the limit of \(m_1 \ln(m_1)\) is zero as \(m_1\) goes to zero, we have \(E(m) \to 1\). So the second kind of the complete elliptic integral is finite. The value of the integral \((2.3 - 6)\), however, becomes infinite as \(m \to 1\). Although the derivation of the variational formulation indicates that the requisite integrals must exist and singularities in the integrands must be integrable, it is instructive to demonstrate this afresh taking into account the explicit knowledge of
singularities involved in integrations (2.2 - 7) to (2.2 - 9). In the following, we will show that the contribution of (2.3 - 6) to the integral (2.2 - 1) is finite.

Consider the case in which two surface points are close to one another, namely, \( m \to 1 \) and \( a \to \frac{\pi}{2} \), so \( D \to 0 \). From the previous analyses, one sees that the coefficient \( \mathcal{H}_1(s, s') \) to leading order in \( D \), with substitution of (2.3 - 6), is given by

\[
\mathcal{H}_1(s, s') \simeq -\frac{(mc - \eta)}{M(1 - m)} E(m) + \text{(higher order terms)}
\]  \hspace{1cm} (2.3 - 11)

Substituting (2.3 - 11) into (2.2 - 4), one finds that the contribution of the second integral on the right side of (2.2 - 3) to leading order in \( D \) is

\[
\frac{1}{4\pi} \int_{-S_{max}}^{+S_{max}} F_1(s, s') ds' \simeq \text{(constant)} \int_{-1}^{1} \frac{(mc_o - \eta_o)}{M(1 - m)} E(m)v_n(s'_o) ds'_o
\]

\[
+ \text{(higher order terms)}
\]  \hspace{1cm} (2.3 - 12)

Note that the integral on the right side in the above has been nondimensionalized.

As \( x_S \) approaches \( x_{S'} \), we have \( r_o(s_o) \to r_o(s'_o) \) and \( z_o(s_o) \to z_o(s'_o) \). Hence \( M \to [r_o(s_o) + r_o(s'_o)] \) and \( (1 - m) \to \frac{[r_o(s_o) - r_o(s'_o)]^2}{[r_o(s_o) + r_o(s'_o)]^2} \). Also, since the direction cosines \( \frac{dz_o(s_o)}{ds_o} \) and \( \frac{dr_o(s_o)}{ds_o} \) are always finite, we have \( \zeta_o \to [r_o(s_o) + r_o(s'_o)] \frac{dz_o(s_o)}{ds_o} \) and \( \eta_o \to 2r_o(s'_o) \frac{dz_o(s_o)}{ds_o} \). Therefore \( (mc_o - \eta_o) \to [r_o(s_o) - r_o(s'_o)] \frac{dz_o(s_o)}{ds_o} \). Moreover, since the differential of length \( ds'_o = \sqrt{1 + \left[ \frac{dz_o(s'_o)}{dr_o(s'_o)} \right]^2} \ dr_o(s'_o) \), the integral over \( ds'_o \) in (2.3 - 12) can be transformed into the one over \( dr_o(s'_o) \). The second kind of the complete elliptic integral \( E(m) \) is nearly unity in this limit. For simplicity, the velocity distribution \( v_n(s'_o) \) is set to constant over the entire surface. The subscript
will be suppressed in what follows for brevity and all quantities will be understood
dimensionless unless otherwise stated.

With these substitutions, the integral of (2.3 - 12), to leading order in regards
to small distance between two surface points, becomes

\[ \int_0^1 \frac{r + r'}{|r - r'|} \sqrt{1 + \left( \frac{dz'}{dr'} \right)^2} \, dr' \]

where the position index \( s' \) is omitted for simplicity. In the above integration, the
quantity \( \sqrt{1 + \left( \frac{dz'}{dr'} \right)^2} \, dr' \) is always finite even though \( \frac{dz'}{dr'} \) may be infinite at some
particular location. Without loss of generality, let us consider the integral with
\( \sqrt{1 + \left( \frac{dz'}{dr'} \right)^2} \, dr' \) approximated by \( dr' \), for example, at a fixed \( z' \) plane, so one has

\[ \int_0^1 \frac{r + r'}{|r - r'|} \sqrt{1 + \left( \frac{dz'}{dr'} \right)^2} \, dr' \simeq \int_0^1 \frac{r}{|r - r'|} \, dr' + \int_0^1 \frac{r'}{|r - r'|} \, dr' \quad (2.3 - 13) \]

The integrands in (2.3 - 13) have a singularity at \( r = r' \). In general, one cannot
simply cross these singular points in carrying out the integrations. Instead, one
should draw a small circle around, say, the variable \( r \), with radius \( \delta \), and take the
limit as \( \delta \to 0 \) after the integration. In this situation, integrations are carried
along the generator, so the line is broken into three segments, namely, from 0 to
\( (r - \delta) \), then from \( (r - \delta) \) to \( (r + \delta) \), and finally from \( (r + \delta) \) to 1. Accordingly, the
integrations in (2.3 - 13) are found to be
It is understood that the \( r \)-coordinate remains constant in integrating with respect to the \( r' \)-coordinate in the above. It turns out that all terms involving \( \delta \) offset each other before taking the limit as \( \delta \to 0 \). Hence one can actually carry out integrations in (2.3 - 13) with respect to \( r' \) directly from 0 to 1 without regard to the singularity at \( r = r' \). Since the basis functions \( P_n(s) \) are assumed continuous on the surface and since the limiting value of \( \epsilon \ln(\epsilon) \) for any small \( \epsilon \) (\( \epsilon > 0 \)) is finite as \( \epsilon \to 0 \), the values of the integrations in (2.3 - 14) with respect to the \( r \)-coordinate are all bounded. Therefore, the contribution of Eq.(2.3 - 6) or equivalently, the second term on the right side of (2.2 - 15) to the integral in (2.2 - 1) is finite.

It should be emphasized here that a severer numerical difficulty would occur in Eq.(2.3 - 14) at \( r = 1 \) if there were no further integrations with respect to the \( r \)-coordinate, because the integrands increase without bound at \( r = 1 \). Such is the case when one solves for the surface acoustic pressure based on the integral equation
given by (2.1 – 18), since the integrands with the same order of singularities are integrated over the surface only once. Although the principal value of the integral still exists, the integrations will be less amenable for numerical computations than those discussed here.

We continue our examination of the coefficient $\mathcal{H}_1(s,s')$. At first glance, the next trouble spot will be the integrand involving the factor $\frac{1}{D^2}$ on the right side of Eq.(2.2 – 12). Fortunately, this term disappears later in the expansion in terms of small value $D$, which is demonstrated as follows.

Making use of the identity (2.2 – 14), we can rewrite the last two integrals on the right side of (2.2 – 12) as

\[
\begin{align*}
\int_0^{\frac{\pi}{2}} e^{i\Phi} d\alpha &= \int_0^{\frac{\pi}{2}} \frac{1}{D^2} d\alpha - 2k(m\zeta - \eta) \int_0^{\frac{\pi}{2}} \frac{\sin(\Phi/2)e^{i(\Phi/2)}}{D^2} d\alpha \\
- \frac{(m\zeta - \eta)}{M} \int_0^{\frac{\pi}{2}} \frac{e^{i\Phi}}{D^3} d\alpha &= - \frac{(m\zeta - \eta)}{M} \int_0^{\frac{\pi}{2}} \frac{1}{D^3} d\alpha - i \frac{2(m\zeta - \eta)}{M} \int_0^{\frac{\pi}{2}} \frac{\sin(\Phi/2)e^{i(\Phi/2)}}{D^3} d\alpha
\end{align*}
\]

The first term on the right side of (2.3 – 15) has just been examined in the above. So we turn our attention to the second integral on the right side of (2.3 – 16). This integral is rewritten in the following

\[
\begin{align*}
- \frac{2(m\zeta - \eta)}{M} \int_0^{\frac{\pi}{2}} \left[ \frac{\sin(\Phi/2) - (\Phi/2) + (\Phi/2)}{D^3} \right] e^{i(\Phi/2)} d\alpha \\
= - \frac{2(m\zeta - \eta)}{M} \int_0^{\frac{\pi}{2}} \left[ \frac{\sin(\Phi/2) - (\Phi/2)}{D^3} \right] e^{i(\Phi/2)} d\alpha \\
- \frac{i(m\zeta - \eta)}{M} \int_0^{\frac{\pi}{2}} \frac{\Phi}{D^3} d\alpha + \frac{2(m\zeta - \eta)}{M} \int_0^{\frac{\pi}{2}} \frac{\Phi \sin(\Phi/4)e^{i(\Phi/4)}}{D^3} d\alpha
\end{align*}
\]

(2.3 – 17)
Since \( \Phi = kMD \), as given in (2.2 - 13), one finds that the second term on the right side of (2.3 - 17) is identical to the first term on the right side of (2.3 - 15) except with a minus sign, so these two integrals cancel each other.

We now come down to the integral involving the factor \( \frac{1}{D} \) in the coefficient \( \mathcal{H}_1(s, s') \) [See Eq.(2.2 - 12)]. Once again, utilizing the identity (2.2 - 14), one is able to separate out the singular part as

\[
\int_0^{\pi} \frac{1}{D} \, d\alpha = \int_0^{\pi} \frac{1}{\sqrt{1 - m \sin^2 \alpha}} \, d\alpha
\]

which corresponds to the first integral on the right side of (2.2 - 15). One identifies immediately that Eq.(2.3 - 18) is just the first kind of the complete elliptic integral \( K(m) \) whose standard polynomial approximation is given by

\[
K(m) = [c_0 + c_1 m_1 + \cdots + c_4 m_1^4] \\
- [d_0 + d_1 m_1 + \cdots + d_4 m_1^4] \ln(m_1) + \epsilon(m)
\]

where \( m \) and \( m_1 \) are given in (2.2 - 13) and (2.3 - 10), respectively. The error of this approximation is of order \( |\epsilon(m)| \leq 2 \times 10^{-8} \). The coefficients \( c_0 \) to \( c_4 \) and \( d_0 \) to \( d_4 \) in (2.3 - 19) are

\[
\begin{align*}
    c_0 &= 1.38629436112 & d_0 &= 0.5 \\
    c_1 &= 0.09666344259 & d_1 &= 0.12498593597 \\
    c_2 &= 0.03590092383 & d_2 &= 0.06880248576 \\
    c_3 &= 0.03742563713 & d_3 &= 0.03328355346 \\
    c_4 &= 0.01451196212 & d_4 &= 0.00441787012
\end{align*}
\]
With substitution of the polynomial approximation for the first kind of the complete elliptic integral in the first term on the right side of Eq. (2.2 - 15), one finds the corresponding contribution to the integral in Eq. (2.2 - 3) to be

\[ I = \frac{1}{4\pi} \int_{-S_{\text{max}}}^{+S_{\text{max}}} i \rho_c k v_{n}(s') \frac{K(m)}{2M} \left( \frac{1}{2} M^2 (m \zeta - \eta) + 2\eta \right) ds' \]  

(2.3 - 21)

The integrand in the above has a logarithmic singularity when two surface points approach one another. As before, we nondimensionalize the integral in (2.3 - 21) and consider the case in which \( x_s \) is brought close to \( x_{s'} \). As \( r \rightarrow r' \) and \( z \rightarrow z' \), we have \( M \rightarrow (r + r') \), \( m \rightarrow 1 \), \( \zeta \rightarrow (r + r') \frac{dz}{ds} \), \( \eta \rightarrow 2r' \frac{dz}{ds} \), and \( K(m) \rightarrow \frac{1}{2} \ln \left| \frac{r-r'}{r-r''} \right| \). The velocity profile \( v_{n}(s') \) is again assumed constant over the surface for simplicity. Neglecting the higher order terms and examining the integral of (2.3 - 21) at a fixed \( z' \) plane so that \( ds' \rightarrow dr' \), we obtain

\[ I \rightarrow \int_{0}^{1} \left( |r-r'| \ln |r-r'| - |r-r'| \ln |r+r'| + 2r' \ln |r-r'| \right) dr' 
- 2r' \ln |r+r'| + \frac{1}{|r+r'|} \ln |r-r'| - \frac{1}{|r+r'|} \ln |r+r'| \right) dr' 
\]  

(2.3 - 22)

From the previous analyses, it is not difficult to see that all integrands in the above are integrable and the integrations can be carried out directly from 0 to 1. The value of \( I \) is bounded and therefore the contribution corresponding to the singular integrand (2.3 - 18) is finite.

We have just demonstrated that the contributions of singularities contained in the coefficient \( H_1(s,s') \) are all finite. Singularities involved in the coefficients \( H_2(s,s') \) and \( H_3(s,s') \) are at most of the order of \( \frac{1}{D} (D \rightarrow 0) \). The value of the
integral involving this singularity has been shown to be bounded. Therefore the contributions of singularities involved in $\mathcal{H}_2(s, s')$ and $\mathcal{H}_3(s, s')$ are also finite.

Consequently, one may conclude from the above that the integrations of the free-space Green's function (2.2 - 7) to (2.2 - 9) can be evaluated numerically without much effort, even though they have weak singularities.

2.4 Numerical Schemes For Double Integrations

Having done the angular integrations (2.2 - 7) to (2.2 - 9), we now focus on the remaining double integrals given by Eqs.(2.2 - 1) and (2.2 - 2). The integrands of these integrals are relatively simple as one can see from Eqs.(2.2 - 3) to (2.2 - 6); they are merely algebraic multiplications among surface velocity, basis functions, and coefficients $\mathcal{H}_1(s, s')$ to $\mathcal{H}_3(s, s')$ which have been tabulated at the outset.

For the single integration contained in Eq.(2.2 - 1), the Simpson's 1/3 rule will be adequate to yield satisfactory results. On the other hand, however, one must be careful in choosing numerical schemes for the remaining two-dimensional integrations because most of such schemes assume the integrands to be finite along the diagonal line $s = s'$. As shown in section 2.3, most integrands have singularities at $s = s'$. Although these singularities are integrable, one cannot explicitly ask for values of integrands at those singular points. To circumvent this difficulty, we use a special numerical integration scheme (See Figs. 3 to 5) which evaluates the integrands only at interior points.
Figure 3. Numerical integration scheme using interior points to avoid singularities along the diagonal line.
Figure 4. Locations of interior points in an internal square.
Figure 5. Location of interior points in a triangle along the edges of the square.
Figure 3 shows that the square $2S_{\text{max}} \times 2S_{\text{max}}$ over which integrations are carried out is segmented into $2W(W - 1)$ slanted squares in the interior and $4W$ triangles along the edges of the square. For each of the interior squares, a nine-point integration rule is used so that the integral over each of the interior squares is approximated by the area of the square times the weighted values of the integrands evaluated at nine points. The locations of these points and corresponding weighting factors are so selected that the integration will be exact if the integrand is a polynomial up to the fifth degree [85].

Because of the inclination of interior squares, the locations of nine points have to be recalculated. Let the perimeters of the outer square be equally divided into $W$ subdivisions so that each interior square has a length of $\sqrt{2}h$ ($h = \frac{S_{\text{max}}}{W}$) on a side, a diagonal of $2h$, and an area of $2h^2$. The coordinates of nine points relative to the center of the interior square are then given by

\[
\begin{align*}
sq(1) &= 0 & sq'(1) &= 0 \\
 sq(2) &= -\sqrt{\frac{3}{20}} h & sq'(2) &= +\sqrt{\frac{3}{20}} h \\
 sq(3) &= +\sqrt{\frac{3}{20}} h & sq'(3) &= +\sqrt{\frac{3}{20}} h \\
 sq(4) &= +\sqrt{\frac{3}{20}} h & sq'(4) &= -\sqrt{\frac{3}{20}} h \\
 sq(5) &= -\sqrt{\frac{3}{20}} h & sq'(5) &= -\sqrt{\frac{3}{20}} h \\
 sq(6) &= -\sqrt{\frac{3}{5}} h & sq'(6) &= 0 \\
 sq(7) &= 0 & sq'(7) &= +\sqrt{\frac{3}{5}} h \\
 sq(8) &= +\sqrt{\frac{3}{5}} h & sq'(8) &= 0 \\
 sq(9) &= 0 & sq'(9) &= -\sqrt{\frac{3}{5}} h
\end{align*}
\]

(2.4 - 1)
where it is understood that the symbol \( t \) denotes the primed coordinate in Fig. 3 and the corresponding weights are

\[
\begin{align*}
W_{sq}(1) &= \frac{32}{81} h^2 \\
W_{sq}(2) &= W_{sq}(3) = W_{sq}(4) = W_{sq}(5) = \frac{20}{81} h^2 \\
W_{sq}(6) &= W_{sq}(7) = W_{sq}(8) = W_{9}(9) = \frac{50}{324} h^2
\end{align*}
\]

where the subscript \( sq \) stands for squares.

To achieve the same order of accuracy, we choose a seven-interior-point scheme for triangles along perimeters of the outer square. Each triangle will then have a height \( h \) and an area \( h^2 \). The coordinates of seven points relative to the geometrical center of the triangle are obtained by mapping the \( 45^\circ-45^\circ-90^\circ \) triangle to an equilateral one. Thus for triangles on the line \( s = -S_{max} \), we have

\[
\begin{align*}
tr_b(1) &= 0 \\
tr_b(2) &= +\left(\frac{\sqrt{15} - 1}{7}\right) h \\
tr_b(3) &= 0 \\
tr_b(4) &= -\left(\frac{\sqrt{15} - 1}{7}\right) h \\
tr_b(5) &= 0 \\
tr_b(6) &= +\left(\frac{\sqrt{15} + 1}{7}\right) h \\
tr_b(7) &= -\left(\frac{\sqrt{15} + 1}{7}\right) h
\end{align*}
\]

\[
\begin{align*}
tr'_b(1) &= \frac{1}{3} h \\
tr'_b(2) &= +\left(\frac{6 + \sqrt{15}}{21}\right) h \\
tr'_b(3) &= +\left(\frac{9 - 2\sqrt{15}}{21}\right) h \\
tr'_b(4) &= +\left(\frac{6 + \sqrt{15}}{21}\right) h \\
tr'_b(5) &= +\left(\frac{9 + 2\sqrt{15}}{21}\right) h \\
tr'_b(6) &= +\left(\frac{6 - \sqrt{15}}{21}\right) h \\
tr'_b(7) &= +\left(\frac{6 - \sqrt{15}}{21}\right) h
\end{align*}
\]

where the subscript \( b \) denotes triangles along the bottom perimeter of the outer square.
The locations of seven points for triangles along other three perimeters of the outer square can be deduced from (2.4 - 3).

\[
\begin{aligned}
tr_l(j) &= +tr_r^l(j) \\
tr_r(j) &= +tr_b(j) \\
tr_r(j) &= +tr_b^r(j) \\
tr_u(j) &= -tr_b(j) \\
tr_u(j) &= -tr_b^u(j) \\
\end{aligned}
\]

\[j = 1, 2, \ldots, 7\] \hspace{1cm} (2.4 - 4a)

\[
\begin{aligned}
tr_r(j) &= -tr_b^l(j) \\
tr_r(j) &= -tr_b(j) \\
\end{aligned}
\]

\[j = 1, 2, \ldots, 7\] \hspace{1cm} (2.4 - 4b)

where the subscripts \(l\), \(r\), and \(u\) represent left, right, and upper perimeters of the outer square, respectively. The corresponding weighting coefficients are

\[
\begin{aligned}
W_{tr}(1) &= \frac{270}{1200} h^2 \\
W_{tr}(2) &= W_{tr}(3) = W_{tr}(4) = \frac{(155 + \sqrt{15})}{1200} h^2 \\
W_{tr}(5) &= W_{tr}(6) = W_{tr}(7) = \frac{(155 - \sqrt{15})}{1200} h^2 \\
\end{aligned}
\]

\[\] \hspace{1cm} (2.4 - 5)

Therefore, the total number of operation steps required in evaluating a single two-dimensional integration will be \([9 \times 2W(W - 1)] + (7 \times 4W)\). Such a series of computations repeats itself for every element of \(\{B\}\) and \([A]\). Since one only computes coefficients \(H_1(s, s')\) to \(H_3(s, s')\) once, an increase in the number of basis functions, or in other words, the sizes of the square matrix \([A]\) and the column vector \(\{B\}\) will not appreciably retard the overall numerical computations for a fixed number of integration subdivisions.
CHAPTER III

ACOUSTIC RADIATION FROM A CIRCULAR THIN DISK
IN RIGID TRANSVERSE OSCILLATIONS

3.1 Improper Integral For A Thin Disk

Before solving the problem of sound radiation from a circular thin disk in rigid transverse oscillations, we shall state a few guidelines concerning the improper integral (2.1 - 29). We rewrite this integral as follows

\[ U_n(x_S) - f_n(x_S) = \lim_{\epsilon \to 0} \left\{ \frac{1}{4\pi} \iint_{S'} f_n(x_{S'}) \left[ n(x_S) \cdot \nabla G(x \mid x_{S'}) \right] dS' \right\} \quad (3.1 - 1a) \]

\[ = -\frac{1}{2} f_n(x_S) + (PR) \frac{1}{4\pi} \iint_{S'} f_n(x_{S'}) \left[ n(x_S) \cdot \frac{x_S - x_{S'}}{R} \right] \frac{d}{dR} \left( \frac{e^{i k R}}{R} \right) dS' \quad (3.1 - 1b) \]

For an unbaffled thin disk, the surface \( S' \) consists of two contiguous sheets with an infinitesimal separation between them. The unit normal vector at any point on the surface is either in the positive or negative \( z \) direction because the side area of a thin disk is negligible (See Fig. 6).

If we stipulate that the subscripts + and − denote the front and back sides of the disk, then \( R_{++} \) and \( R_{--} \) indicate distances between two points on the same side while \( R_{+-} \) and \( R_{-+} \) imply distances between two points on the opposite sides of the disk, respectively. Since the separation of two contiguous sheets of the disk is much less than \( \epsilon \) while the limit is being taken, the distances \( R_{++} \) and \( R_{+-} \) will always be in the directions perpendicular to both \( n(x_{S+}) \) and \( n(x_{S-}) \).
Figure 6. Transversely oscillating unbaffled circular rigid thin disk with radius $a$. 
Consequently, the limiting value of the integral on the right side of (3.1 - 1a) vanishes identically and one obtains the simple result

\[ U_n(x_S) = f_n(x_S) \]  

(3.1 - 2)

for thin unbaffled plate-like bodies (alternatively referred to as laminas). This result, however, will not emerge easily from (3.1 - 1b) because one has to distinguish the order of integration and then passing to the limit in carrying out the Cauchy principal value.

Such a difficulty can be overcome by a fail-safe method developed from the identity

\[ \lim_{\epsilon \to 0} \iint_{S'} n(x_{S'}) \cdot \nabla \left( \frac{1}{R} \right) dS' = 0 \]  

(3.1 - 3)

The validity of this identity is verified below. First, we notice from (2.1 - 3) that \( R \) is of reciprocity in regards to interchange of \( x \) and \( x_{S'} \), hence

\[ \nabla \left( \frac{1}{R} \right) = -\nabla' \left( \frac{1}{R} \right) \]  

(3.1 - 4)

Substitute (3.1 - 4) into (3.1 - 3) and use the Gauss theorem to transform the surface integral (3.1 - 3) into a volume integral

\[ -\lim_{\epsilon \to 0} \iint_{S'} n(x_{S'}) \cdot \nabla' \left( \frac{1}{R} \right) dS' = -\lim_{\epsilon \to 0} \iiint_{V'} \nabla' \cdot \nabla' \left( \frac{1}{R} \right) dV' \]

\[ = -\lim_{\epsilon \to 0} \iiint_{V'} \nabla'^2 \left( \frac{1}{R} \right) dV' \]  

(3.1 - 5)
For $\varepsilon \neq 0$, the function $\frac{1}{R}$ satisfies the Laplace's equation. Therefore the integrand on the right side of (3.1 - 5) is identically zero. Equation (3.1 - 1) then follows.

Making use of the identity (3.1 - 3), we can rewrite Eq.(3.1 - 1a) as

$$U_n(x_s) - f_n(x_s) = \lim_{\varepsilon \to 0} \left\{ \frac{1}{4\pi} \int_{S'} f_n(x_{s'}) \left[ n(x_s) \cdot \nabla G(x | x_{s'}) \right] \right\} dS'$$

$$= \frac{1}{4\pi} \int_{S'} \left\{ \left[ f_n(x_{s'}) n(x_s) - f_n(x_s) n(x_{s'}) \right] \cdot \frac{x_s - x_{s'}}{R} \right\} \frac{d}{dR} \left( \frac{e^{ikR}}{R} \right) dS'$$

$$+ \frac{1}{4\pi} \int_{S'} f_n(x_s) \left[ n(x_{s'}) \cdot \frac{x_s - x_{s'}}{R} \right] \frac{d}{dR} \left( \frac{e^{ikR}}{R} - \frac{1}{R} \right) dS' \quad (3.1 - 6)$$

What we have done here is that we replace $f_n(x_{s'}) n(x_s)$ in (3.1 - 1a) by the difference of itself evaluated at two different points $[f_n(x_{s'}) n(x_s) - f_n(x_s) n(x_{s'})]$. To compensate for this discrepancy, we add the second integral on the right side of (3.1 - 6) and tactically insert an additional term $-\frac{1}{R}$ in the derivative with respect to $R$. The surface integration of this added factor $n(x_{s'}) \cdot \nabla' \left( \frac{1}{R} \right)$ has been shown to be zero [See Eq.(3.1 - 3)], hence it in effect changes nothing. The order of singularity in the reformed version (3.1 - 6), however, has been reduced to $\frac{1}{R}$, which is integrable as $R \to 0$. So one need not be concerned with explicitly taking the Cauchy principal values in Eq.(3.1 - 6).

Such a representation (3.1 - 6) will be computationally useful in case of a finite length circular cylinder in rigid transverse vibrations. As the length $L$ of the cylinder goes to zero, the area integrations in (3.1 - 6) are over only the top ($z = L/2$) and bottom ($z = -L/2$) surfaces of the cylinder. If $x_s$ is a point on the top surface, then $n(x_s)$ is in the positive $z$ direction, while $n(x_{s'})$ may be in either positive or negative $z$ direction. With $L \to 0$, one has in any case that $(x_s - x_{s'})$ is perpendicular to both $n(x_s)$ and $n(x_{s'})$, so that both the first and second integrals
on the right side of (3.1 - 6) vanish trivially, and one retrieves the thin disk result, i.e. $U_n(x_S) = f_n(x_S)$.

One cannot draw such a conclusion from the integral in Eq.(3.1 - 1b), because the principal value applies only to the integral over the top surface (given that $x_S$ is on the top surface). For the integral over the bottom surface, one must in general evaluate the integral for a finite $L$ and then take the limit as $L \to 0$. One would get a different (and incorrect) answer if one jumped inside the integral and took the limit as $L \to 0$ before evaluating the integration. Such a problem does not arise, however, for integrals in (3.1 - 6) because integrands are sufficiently well-behaved and non-singular that the order of taking the limit and carrying out the integrations can be freely interchanged.

3.2 Variational Formulation For A Thin Disk

As demonstrated in the preceding section, the integral involving the normal derivative with respect to the free-space Green's function entirely disappears and $U_n(x_S)$ is simply equal to $f_n(x_S)$ for a thin disk. Hence the variational formulation is greatly simplified. The singularities involved in other integrands are at most of order $\frac{1}{R}$ ($R \to 0$), which is well behaved and integrable.

Since the disk is assumed to be vibrating as a rigid body normal to its faces, the velocity on the front side $v_n(x_{S+})$ is 180° out of phase with that on the back side $v_n(x_{S-})$, namely, as one surface moves inwards, the other moves outwards simultaneously, and vice versa. It follows that the actual acoustic pressure must have comparable behavior, i.e. it must be antisymmetric with respect to reflection through the plane on which the disk nominally lies. We choose to restrict our class of admissible trial functions for $p$ to be those that have this property only. Thus we obtain
\[ v_n(x_{s+}) = -v_n(x_{s-}) \quad (3.2 - 1a) \]
\[ p(x_{s+}) = -p(x_{s-}) \quad (3.2 - 1b) \]
\[ v_n(x_{s+}) p(x_{s+}) = v_n(x_{s-}) p(x_{s-}) \quad (3.2 - 1c) \]

Also, since the unit normal vectors on opposite sides of the disk are oppositely directed, we have

\[ n(x_{s+}) p(x_{s+}) = n(x_{s-}) p(x_{s-}) \quad (3.2 - 2a) \]
\[ n(x_{s+}) \times \nabla p(x_{s+}) = n(x_{s-}) \times \nabla p(x_{s-}) \quad (3.2 - 2b) \]

These relations also hold if the dummy integration variable \( x_s \) is replaced by the dummy integration variable \( x_{s'} \).

Equations (3.2 - 1) and (3.2 - 2) substantially simplify the functional \( J[p] \) given by (2.1 - 36). When each of the integrals in (2.1 - 36) is decomposed into the contributions of the front and back surfaces, we find that \( J[p] \) may be expressed in terms of integrals extended over the front side of the disk only; specifically,

\[
J[p] = -i8\pi \rho_o ck \int_{S^+} v_n(x_S) p(x_S) dS_+ \\
+ 2k^2 \int_{S^+} \int_{S^+} p(x_S) p(x_S') G(x_S | x_{s'}) dS_+ dS_+ \\
- 2 \int_{S^+} \int_{S^+} [e_z \times \nabla p(x_S)] \cdot [e_z \times \nabla' p(x_{s'})] G(x_S | x_{s'}) dS'_+ dS_+ \\
(3.2 - 3)
\]
where $x_s$ and $x_{s'}$, are implicitly situated on the front side of the disk $S_+$. For convenience sake, we shall nondimensionalize the functional $J[p]$ in the above. To this end, we scale the distance by the radius of the disk $a$ and the acoustic pressure by $\rho_0 c v_o$, where $\rho_0$ and $c$ are the ambient density and the sound speed, respectively, and $v_o$ is the amplitude of the surface velocity such that on the front side of the disk, the complex amplitude of the normal velocity is

$$v_n(x_s) = v_o$$  \hspace{1cm} (3.2 - 4)

Let a hat $\hat{}$ denote the nondimensional quantity. Equation (3.2 - 3) then becomes

$$\hat{J}[\hat{p}] = \frac{J[p]}{4\rho_0^2 c^2 v_o^2 a} = -i 2\pi (ka) \int_{\hat{S}_+} \hat{p}(\hat{x}_s) d\hat{S}_+$$

$$+ \frac{(ka)^2}{2} \int_{\hat{S}_+} \int_{\hat{S}_+'} \hat{p}(\hat{x}_s) \hat{p}(\hat{x}_{s'}) \hat{G}(\hat{x}_s \mid \hat{x}_{s'}) d\hat{S}_+ d\hat{S}_+$$

$$- \frac{1}{2} \int_{\hat{S}_+} \int_{\hat{S}_+'} [e_z \times \hat{\nabla} \hat{p}(\hat{x}_s)] \cdot [e_z \times \hat{\nabla}' \hat{p}(\hat{x}_{s'})] \hat{G}(\hat{x}_s \mid \hat{x}_{s'}) d\hat{S}_+ d\hat{S}_+$$  \hspace{1cm} (3.2 - 5)

where

$$\hat{p} = \frac{p}{\rho_0 c v_o}; \hspace{0.5cm} \hat{G} = aG; \hspace{0.5cm} \hat{\nabla} = a\nabla; \hspace{0.5cm} \text{and} \hspace{0.5cm} d\hat{S}_+ = \frac{1}{a^2} dS_+.$$  \hspace{1cm} (3.2 - 6)

For brevity, the hats will be omitted in the following and all quantities will be understood dimensionless unless otherwise stated.

Because of the axisymmetry of the surface velocity, the surface pressure amplitude must only be a function of the radial distance from the center of the disk. Using polar coordinates with $r$ denoting the dimensionless distance in the radial direction, we obtain
\[ p(x_s) = p(r) \]  
\[ \left[ e_z \times \nabla p(x_s) \right] \cdot \left[ e_z \times \nabla' p(x_{s'}) \right] = \frac{dp(r)}{dr} \frac{dp(r')}{dr'} \cos \Theta \]  
\[ G(x_s | x_{s'}) = \frac{e^{ikaR}}{R} \]  
\[ R = \sqrt{r^2 + r'^2 - 2rr' \cos(\Theta - \Theta')} \]  
\[ dS_+ = r \, d\Theta \, dr \]  

Since the integrand for the double integrations over area depends only on the relative angle, say \( \theta = \Theta - \Theta' \), a simple transformation of angular coordinates to \( \theta \) enables one of angular integrations to be done trivially, yielding a factor of \( 2\pi \). This leads to the following simple form for the scaled functional that appears in (3.2 - 5), with the final form divided through by the factor \( 2\pi \),

\[
\frac{J}{2\pi} = -i2\pi(ka) \int_0^1 p(r) r \, dr + \frac{(ka)^2}{2} \int_0^1 \int_0^1 p(r) p(r') G_1(r, r') \, dr' \, dr - \frac{1}{2} \int_0^1 \int_0^1 \frac{dp(r)}{dr} \frac{dp(r')}{dr'} G_2(r, r') \, dr' \, dr
\]  

where the angular integrations are abbreviated by

\[
G_1(r, r') = \int_0^{2\pi} \frac{e^{ikaR}}{R} \, r r' \, d\theta
\]  
\[
G_2(r, r') = \int_0^{2\pi} \frac{e^{ikaR}}{R} \cos \theta \, r r' \, d\theta
\]
The differential factor \( G_1(r, r') dr'dr \) and \( G_2(r, r') dr'dr \) in Eq.(3.2 - 8) can be interpreted as describing the effect of an annulus of radius \( r \) and width \( dr \) on another annulus of radius \( r' \) and width \( dr' \). We shall refer to these functions \( G_1(r, r') \) and \( G_2(r, r') \) as the integrated Green's functions. These functions, in fact, can be obtained directly from coefficients \( H_2(s, s') \) and \( H_3(s, s') \) given by Eqs.(2.2 - 20) and (2.2 - 21), with a proper change of dummy integration variables, \( s \rightarrow r \) and \( s' \rightarrow r' \).

[Since there is no \( z \)-component, the quantities \( \zeta \) and \( \eta \) in (2.2 - 13) are identically zero. Hence the coefficient \( H_1(s, s') \) which corresponds to the improper integration vanishes trivially as expected.] For clearness, however, we shall rederive expressions for angular integrations (3.2 - 9) and (3.2 - 10).

Let \( \theta = \pi - 2\alpha \), then the trigonometric half-angle formula gives

\[
\cos \theta = -\cos(2\alpha) = -(1 - 2\sin^2\alpha) = \frac{2 - m}{m} - \frac{2}{m}D^2
\]

where \( D = \sqrt{1 - m\sin^2\alpha} \) and the parameter \( m \) is the same as previously defined in (2.2 - 13) with \( z \)-component set to zero

\[
m = \frac{4rr'}{(r + r')^2}
\]

Similarly, the distance between two surface points is \( R = MD \), where the quantity \( M \) becomes simply

\[
M = (r + r')
\]

With these substitutions, the angular integration (3.2 - 9) reduces to
where $\Phi = kMD$, as defined in (2.2 - 13). Replacing the exponential function on the right side of (3.2 - 14) by the identity (2.2 - 14), we obtain

$$ \int_0^{2\pi} \frac{e^{iR}}{R} \cos \theta \, d\theta = mM \int_0^{\frac{\pi}{2}} \frac{e^{i\Phi}}{D} \, d\alpha \quad (3.2 - 14) $$

The first term on the right side of (3.2 - 15) is the first kind of the complete elliptic integral, which has a logarithmic singularity at $m = 1$, and the second term is finite for all values of $\Phi$.

In a similar manner, we utilize Eq.(3.2 - 11) to rewrite Eq.(3.2 - 10) as

$$ \int_0^{2\pi} \frac{e^{iR}}{R} \cos \theta \, d\theta = mM \int_0^{\frac{\pi}{2}} \frac{e^{i\Phi}}{D} \left( \frac{2 - m}{m} - \frac{2}{M} D^2 \right) \, d\alpha \quad (3.2 - 16) $$

Using the technique for which $G_1(\sigma, r')$ is derived in the above, one can replace the factor $\frac{e^{i\Phi}}{D}$ on the right side of (3.2 - 16) by

$$ G_2(\sigma, r') = mM \int_0^{\frac{\pi}{2}} \left[ 1 + i \frac{2 \sin(\phi/2)}{D} e^{i(\Phi/2)} \right] \left( \frac{2 - m}{m} - \frac{2}{M} D^2 \right) \, d\alpha $$

$$ = M \left[ (2 - m)K(m) - 2E(m) \right] $$

$$ + iM \int_0^{\frac{\pi}{2}} \left[ kM(2 - m) \frac{\sin(\phi/2)}{(\Phi/2)} e^{i(\Phi/2)} - 4D \sin(\Phi/2) e^{i(\Phi/2)} \right] \, d\alpha \quad (3.2 - 17) $$
Figure 7. The acoustic pressure distributions on the surface of the transversely oscillating unbaffled circular rigid thin disk at $ka = 0$. Solid line: Analytic solution (3.4 - 1). Dashed line: Approximate solution based on 5 half-range cosine basis functions (3.4 - 2).
The integrated Green's functions $G_1(r,r')$ and $G_2(r,r')$ are exactly the same as $mM\Psi_1$ and $M\Psi_2$ given by Eqs.(2.2 - 21) and (2.2 - 22), respectively.

The basis functions selected to represent the pressure distribution on the surface of the disk must satisfy all requirements that are imposed by the manner of derivation of the variational principle on admissible trial functions. In particular, these must be continuous and differentiable, so that derivatives such as $\frac{dp(r)}{dr}$ exist. The derivation exploited the fact that the pressure at corresponding locations on the front and back sides of the disk differ only in signs. However, because the thickness of the disk is infinitesimal, the pressure on two surfaces at the edge must be equal. Both conditions can only be satisfied if the pressure vanishes at the edge, so

$$p(r) = 0; \quad \text{at } r = 1 \quad (3.2 - 18)$$

Since the admissible trial functions must be continuous over the disk surface, each of the basis functions must satisfy the above condition. Thus the general trial functions formed as a linear combination of $N$ basis functions can be written as

$$p(r) = \sum_{n=0}^{N} C_n P_n(r); \quad P_n(1) = 0 \quad (3.2 - 19)$$

where $C_n$ are unknown coefficients corresponding to the preselected basis functions $P_n(r)$. Note that the condition $\frac{dp(r)}{dr} = 0$ at $r = 0$, which results from axisymmetry, is a natural boundary condition that will emerge from the variational principle for the exact solution, but not a requirement that must be imposed at the outset on trial functions and on variations. It is not necessary that the basis functions $P_n(r)$ satisfy this condition.

Substituting Eq.(3.2 - 19) into (3.2 - 8) then yields the functional $J[p]$
\[
\frac{J[p]}{2\pi} = \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} A_{nm} C_n C_m - \sum_{n=1}^{N} B_n C_n \tag{3.2-20}
\]

where the elements of \( B_n \) and \( A_{nm} \) are given by

\[
B_n = i2\pi(ka) \int_{0}^{1} P_n(r) r \, dr \tag{3.2-21}
\]
\[
A_{nm} = (ka)^2 \int_{0}^{1} \int_{0}^{1} G_1(r,r')P_n(r)P_m(r') \, dr' \, dr - \int_{0}^{1} \int_{0}^{1} G_2(r,r') \frac{dP_n(r)}{dr} \frac{dP_m(r')}{dr'} \, dr' \, dr \tag{3.2-22}
\]

The selection of basis functions \( P_n(r) \) will be addressed in the next section. The variation of \( J[p] \) is obtained from virtual increments in each of the pressure coefficients \( C_n \). Those increments are arbitrary, so \( \delta J = 0 \) for all admissible variations in the basis functions \( P_n(r) \) requires that

\[
\frac{\partial J}{\partial C_n} = 0; \quad n = 1, 2, \ldots, N \tag{3.2-23}
\]

This requirement results in a system of \( N \) simultaneous equations for the unknown coefficients \( C_n \)

\[
[A] \{C\} = \{B\} \tag{3.2-24}
\]

where the elements of the column vector \( \{B\} \) and the square matrix \( [A] \) are given by (3.2-21) and (3.2-22), respectively. After the set of linear equations (3.2-24) is solved for \( C_n \), it is a simple matter to recreate the spatial distribution of surface pressure \( p(r) \) from Eq.(3.2-19).
3.3 Selection Of Basis Functions

The basis functions selected to represent the pressure distributions on surfaces of the thin disk must satisfy all requirements imposed by the variational principle on admissible trial functions. In particular, they must be continuous and have piece-wise tangential derivatives on the surface. Because there is a wide latitude in the selection of basis functions based on the requirement (3.2 - 18) for the unknown surface pressure distribution, whatever insight, intuition, or experience one has regarding the problem at hand can be used to shorten the overall computational effort. Such an example is demonstrated below.

Near the edge of the disk the exact solution for the acoustic pressure on the surface must behave to leading order in the distance from the edge in the same manner as would a solution of Laplace’s equation near a knife edge. Thus if $\epsilon$ is the radial distance from the edge and $\theta$ is the angle about the edge, such that $\theta$ is 0 on the front and $2\pi$ on the back sides of the disk, respectively, then one has

\[
\frac{\partial^2 p}{\partial \epsilon^2} + \frac{1}{\epsilon} \frac{\partial p}{\partial \epsilon} + \frac{1}{\epsilon^2} \frac{\partial^2 p}{\partial \theta^2} = 0
\]  
(3.3 - 1)

for sufficiently small $\epsilon$. The pressure $p$ must be finite near the edge and satisfy the rigid wall boundary condition

\[
\frac{\partial p}{\partial \epsilon} = 0; \quad \text{at } \theta = 0 \text{ and } 2\pi
\]  
(3.3 - 2)

This boundary value problem can be solved by using the technique of separation of variables, with the result to the leading order in $\epsilon$,

\[
p \simeq C + D\sqrt{\epsilon} \cos(\theta/2)
\]  
(3.3 - 3)
where $C$ and $D$ are constants. For the particular case of sound radiation from a thin disk in rigid transverse vibrations, symmetry and continuity require that $p$ be zero at the edge. Hence the constant $C$ is identically zero and on the front side, where $\theta = 0$, the implication of the above equation is

$$p \sim \sqrt{\epsilon} \quad (3.3 - 4)$$

where $\epsilon$ denotes a small distance from the edge of the disk.

The above deduced behavior must be exhibited by the exact solution for $p$ of either the integrodifferential equation or the variational principle. It is not a requirement that must be imposed at the outset on the trial functions or the basis functions. The approximate solution based on any admissible trial functions, so long as the number of basis functions is sufficiently large, is expected, in aggregate, to approach the behavior of Eq. (3.3 - 4).

However, since we know the result (3.3 - 4) in advance of a detailed numerical solution, a strong argument can be made that it should be incorporated into the trial functions at the outset, for then a faster convergence toward the exact solution might be achieved. One possibility for doing this is to include in each of basis functions the factor $\sqrt{1 - r^2}$, which turns out to be the exact result [86] for the pressure distribution on the disk in the limit as $ka \to 0$. Doing so, however, introduces another singularity into one of the integrands of the integrals that define the matrix elements in (3.2 - 22), because then $\frac{dP_n(r)}{dr}$ will be infinite at $r = 1$. The singularity is integrable, but integration over singular integrands is an inherent source of numerical difficulties. Such difficulties are often surmountable with a change of integration variable that transforms the original integrand into one in which the singularity does not appear.
With the purpose just described in mind, we denote the desired transformation as

\[ r = g(x) \]  

(3.3 - 5)

The factor \( \frac{dP_n(r)}{dr} dr \) correspondingly becomes

\[
\frac{dP_n(r)}{dr} dr = \frac{dP_n[g(x)]}{dx} dx
\]

(3.3 - 6)

If \( g(x) \) is chosen appropriately, then \( \frac{dP_n[g(x)]}{dx} \) will be finite at the value of \( x \) where \( g(x) = 1 \), even if \( \frac{dP_n}{dx} \) is not. It is convenient to have the limits of the domain of \( x \) matches those of \( r \), so we introduce the requirement that

\[ g(0) = 0 \quad \text{and} \quad g(1) = 1 \]  

(3.3 - 7)

The choice for the integration variable transformation function \( g \) could be different for different basis functions, but it is advantageous for one to require that it be the same for all elements within the same set, so that no symmetry properties are lost; this is assumed to have been done in the numerical computations discussed in the next section.

For the particular basis function \( P_1(r) = \sqrt{1 - r^2} \), we set

\[ g(x) = \sin\left(\frac{\pi x}{2}\right) \]  

(3.3 - 8)

which satisfies Eq.(3.3 - 7) and the derivatives in (3.3 - 6) are transformed to

\[
\frac{dP_1[g(x)]}{dx} = \frac{d}{dx}\left[\sqrt{1 - \sin^2\left(\frac{\pi x}{2}\right)}\right] = -\frac{\pi}{2} \sin\left(\frac{\pi x}{2}\right)
\]

(3.3 - 9)
which is well-behaved in the integration domain \( x \in [0, 1] \).

One could construct various sets of basis functions with the features just described. The set of basis functions adopted here is obtained by multiplying each term of a power series expansion \( r^{2(n-1)} \) by the factor \( \sqrt{1-r^2} \), so

\[
P_n(r) = r^{2(n-1)} \sqrt{1-r^2}; \quad n = 1, 2, \ldots, N \tag{3.3-10}
\]

The coordinate transformation (3.3-8) then gives

\[
P_n \left[ \sin \left( \frac{\pi x}{2} \right) \right] = \left[ \sin \left( \frac{\pi x}{2} \right) \right]^{2(n-1)} \cos \left( \frac{\pi x}{2} \right); \quad n = 1, 2, \ldots, N \tag{3.3-11}
\]

Accordingly, the derivatives with respect to \( x \) are

\[
\frac{dP_n \left[ \sin \left( \frac{\pi x}{2} \right) \right]}{dx} = \pi(n-1) \left[ \sin \left( \frac{\pi x}{2} \right) \right]^{(2n-3)} \cos^2 \left( \frac{\pi x}{2} \right) \\
- \frac{\pi}{2} \left[ \sin \left( \frac{\pi x}{2} \right) \right]^{(2n-1)}; \quad n = 1, 2, \ldots, N \tag{3.3-12}
\]

which is finite for all \( x \in [0, 1] \).

Consequently, elements \( B_n \) and \( A_{nm} \) take the forms

\[
B_n = i2\pi(ka) \int_0^1 \Upsilon_n(x) \, dx \quad n = 1, 2, \ldots, N \tag{3.3-13}
\]

\[
A_{nm} = (ka)^2 \int_0^1 \int_0^1 G_1 \left[ \sin \left( \frac{\pi x}{2} \right), \sin \left( \frac{\pi y}{2} \right) \right] \Upsilon_n(x) \Upsilon_m(y) \, dx \, dy \\
- \int_0^1 \int_0^1 G_2 \left[ \sin \left( \frac{\pi x}{2} \right), \sin \left( \frac{\pi y}{2} \right) \right] \frac{d\Upsilon_n(x)}{dx} \frac{d\Upsilon_m(y)}{dy} \, dx \, dy \\
\quad n, m = 1, 2, \ldots, N \tag{3.3-14}
\]
where

\[
\begin{align*}
\Upsilon_n(x) &= \frac{\pi}{2} \left[ \sin \left( \frac{\pi x}{2} \right) \right]^{(2n-1)} \cos^2 \left( \frac{\pi x}{2} \right) \quad n = 1, 2, \ldots, N \\
\frac{d\Upsilon_n(x)}{dx} &= \pi(n - 1) \left[ \sin \left( \frac{\pi x}{2} \right) \right]^{(2n-3)} \cos^2 \left( \frac{\pi x}{2} \right) - \frac{\pi}{2} \left[ \sin \left( \frac{\pi x}{2} \right) \right]^{(2n-1)} 
\end{align*}
\]

(3.3-15) (3.3-16)

These relations, of course, hold for the dummy integration variable \( y \). Because these two variables, \( x \) and \( y \), undergo the same coordinate transformations, the singularities contained in the integrands of the integrated Green's functions continue to lie on the diagonal, which is now the line \( x = y \).

3.4 Numerical Results

As a check of validity of the variational formulation, we first calculate the acoustic pressure on the surface of the thin disk at low frequency limit (\( ka \to 0 \)) so that we can compare numerical results with the analytical solution [86]. Suppose that the disk is oscillating at a constant frequency and that the velocity on the front side is \( v_0 e^{-i\omega t} \). Then at \( ka \ll 1 \), the analytical solution for the surface pressure distribution is found to be

\[
- \frac{p}{\rho_o c v_o} = -i \frac{2}{\pi} (ka) \sqrt{1 - r^2}
\]

(3.4-1)

where \( 0 \leq r \leq 1 \). Note that the first basis function in (3.3-10) is exactly the same as the shape function given in the above. Therefore, use of these preselected basis functions (3.3-10) enables one to test two things: (1) when \( N = 1 \), the variational formulation should yield \( C_1 \approx -i \frac{2}{\pi} \) and (2) when \( N > 1 \), contributions from other basis functions should be much smaller than the first.
Table 1 lists the amplitudes as a function of the number of basis functions $N$ used in expansion of (3.3 – 10). In the numerical computations, the coordinate transformation (3.3 – 8) is employed and the number of integration subdivisions is set at $W = 10$. All calculations in this thesis are done on the CDC CYBER 855-B. The accuracy of results obtained by the variational principle is remarkable. It is shown that the amplitude of the first basis function is indeed very close to the exact value given by (3.4 – 1) at $ka \ll 1$, while the amplitudes corresponding to other basis functions are at least five orders of magnitudes smaller.

One may argue that the basis functions used in this case happen to match the true shape of the pressure distribution on the surface of the disk. A further test of the ability of the variational principle to predict the correct pressure distribution is then to use trial functions which merely satisfy the geometric boundary condition, i.e. $p = 0$, at the edge. One of many such choices for admissible trial functions is the half-range cosine series

$$P_n(r) = \cos \left[ \frac{(2n - 1)\pi}{2} r \right]; \quad n = 1, 2, \ldots, N \quad (3.4 - 2)$$

The results of using the above basis functions with $N = 5$ are depicted in Figure 7 and compare favorably with the exact solution (3.4 – 1). Actually, the prediction is identical to a Fourier series expansion of Eq.(3.4 – 1) using Eq.(3.4 – 2).

The stationary value for the transversely oscillating unbaffled circular rigid thin disk at $ka \leq 0$ is calculated using basis functions (3.3 – 10) and (3.4 – 2), respectively. The results are tabulated as a function of the number of basis functions $N$ in Table 2. Because of the stationary property of the variational principle, this stationary value using $p$ correct to the first order is correct to the second order.
Table 1. Rayleigh-Ritz approximations for the surface acoustic pressure of the transversely oscillating unbaflled circular rigid thin disk.

Coefficients $C_n$ corresponding to basis functions

$$r^{2(n-1)}\sqrt{1-r^2} \quad n = 1, 2, \ldots, N.$$  

Analytical solution at $ka \to 0 : C_1 = -0.63662$

<table>
<thead>
<tr>
<th>N</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.63875$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$-0.63875$</td>
<td>$0.27369 \times 10^{-6}$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$-0.63875$</td>
<td>$-0.21376 \times 10^{-5}$</td>
<td>$0.27127 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
Table 2. The stationary value for the transversely oscillating unbaffled circular rigid thin disk at $ka = 0$.

a) Stationary values computed with basis functions $r^{2(n-1)\sqrt{1-r^2}}$.

<table>
<thead>
<tr>
<th>N</th>
<th>Stationary value</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.66695</td>
<td>0.05%</td>
</tr>
<tr>
<td>2</td>
<td>0.66694</td>
<td>0.04%</td>
</tr>
</tbody>
</table>

b) Stationary values computed with basis functions $\cos\left(\frac{2(n-1)\pi}{2}r\right)$.

<table>
<thead>
<tr>
<th>N</th>
<th>Stationary value</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.55280</td>
<td>17.1%</td>
</tr>
<tr>
<td>2</td>
<td>0.60604</td>
<td>4.2%</td>
</tr>
</tbody>
</table>
Table 2 shows that when the trial function takes the true shape of the surface pressure distribution, the accuracy of the numerical result is remarkable. While for an arbitrary but admissible trial function, the accuracy of the approximate solution improves as the number of basis functions increases.

Next, we shall examine the cases at moderate \( ka \) regimes. In the previous low frequency limit \((ka \to 0)\), the pressure amplitude is a negative imaginary quantity. In other words, the pressure is in phase with the acceleration because the disk is assumed to oscillate at a constant frequency \( e^{-i \omega t} \). As \( ka > 0 \), the pressure has both real and imaginary parts.

Figures 8 to 12 demonstrate numerical results obtained by using the variational principle with basis functions (3.3 - 10) and (3.4 - 2), respectively, and with 10 integration subdivisions at \( ka = 1, 2, 3, 4, \) and 5. The results are compared with those obtained by Leitner [82] using expansions of oblate spheroidal wave functions for the complementary problem of sound scattering from a motionless circular disk. Excellent agreements of numerical results are achieved. Furthermore, the slope discontinuity at the edge of the disk seems to be modeled better by the variational principle with basis functions (3.3 - 10). Another highlight of Figs. 8 and 9 is that the imaginary parts of Leitner's results for \( ka = 1 \) and 2 are interchanged.

Having solved the acoustic pressure on the surface of the disk, we are able to predict the far-field radiation patterns using the Kirchhoff-Helmholtz integral relation (2.1 - 1). Here we calculate the far-field acoustic pressure fields from the same thin disk in rigid transverse oscillations at \( ka = 1, 2, 3, 4, \) and 5. The values of the complex amplitudes of the far-field acoustic pressures given by Leitner are plotted against the angle \( \alpha \) which is measured from the central axis of the disk, such that \( \alpha = 0^\circ \) corresponding to the axis on the front side, \( \alpha = 90^\circ \) to the edge, while \( \alpha = 180^\circ \) to the back side of the disk, etc. In order to depict the acoustic radiation
beam patterns, we here plot all values of the complex amplitudes of the far-field acoustic pressures in polar coordinates and normalize the results with respect to the values evaluated at $\alpha = 0^\circ$.

To be consistent with Leitner's results, we calculate the far-field acoustic pressures at the distance $\lambda$ defined by Leitner [87]

$$\lambda = \frac{2\pi a}{ka} \quad (3.4-3)$$

which is measured from the center of the disk surface. Hence for the radius of the disk $a = 1$ and the dimensionless wavenumber $ka = 1$ the acoustic pressure is evaluated at a point which is $2\pi$ distance away from the surface of the disk.

The far-field radiation beam patterns based on the surface pressure distributions calculated by using the variational formulation and by using Leitner's method are depicted in Figs. 13 to 17, respectively. The agreements are remarkable in each case, namely, $ka = 1, 2, 3, 4, \text{ and } 5$, even though there are some discrepancies in the surface pressure distributions around the edge of the disk.
Figure 8. The acoustic pressure distributions on the surface of the transversely oscillating unbaflled circular rigid thin disk at $ka = 1$. 

*Solid lines:* Approximate solution using basis functions $(3.3 - 10)$ with $N = 2$ and with integration subdivisions $W = 10$. *Chain-dotted lines:* Approximate solution using basis functions $(3.4 - 2)$ with $N = 5$ and $W = 10$. *Dashed lines:* Leitner's results.
Figure 9. The acoustic pressure distributions on the surface of the transversely oscillating unbaffled circular rigid thin disk at \( ka = 2 \).

Solid lines: Approximate solution using basis functions (3.3 - 10) with \( N = 2 \) and with integration subdivisions \( W = 10 \). Chain-dotted lines: Approximate solution using basis functions (3.4 - 2) with \( N = 5 \) and \( W = 10 \). Dashed lines: Leitner's results.
Figure 10. The acoustic pressure distributions on the surface of the transversely oscillating unbaffled circular rigid thin disk at $ka = 3$.

**Solid lines:** Approximate solution using basis functions (3.3 - 10) with $N = 5$ and with integration subdivisions $W = 10$. **Chain-dotted lines:** Approximate solution using basis functions (3.4 - 2) with $N = 5$ and $W = 10$. **Dashed lines:** Leitner’s results.
Figure 11. The acoustic pressure distributions on the surface of the transversely oscillating unbaflled circular rigid thin disk at $ka = 4$. 
**Solid lines:** Approximate solution using basis functions (3.3 - 10) with $N = 5$ and with integration subdivisions $W = 10$. **Chain-dotted lines:** Approximate solution results using basis functions (3.4 - 2) with $N = 5$ and $W = 10$. **Dashed lines:** Leitner's results.
Figure 12. The acoustic pressure distributions on the surface of the transversely oscillating unbaffled circular rigid thin disk at $ka = 5$. 

Solid lines: Approximate solution using basis functions (3.3 - 10) with $N = 5$ and with integration subdivisions $W = 10$. Chain-dotted lines: Approximate solution using basis functions (3.4 - 2) with $N = 5$ and $W = 10$. Dashed lines: Leitner's results.
Figure 13. Far-field radiation beam pattern from the transversely oscillating unabaffled circular rigid thin disk at $ka = 1$. **Solid line:** Approximate solution using basis functions (3.3 - 10). **Dashed line:** Leitner's result.
Figure 14. Far-field radiation beam pattern from the transversely oscillating unbaffled circular rigid thin disk at $ka = 2$. Solid line: Approximate solution using basis functions (3.3 - 10). Dashed line: Leitner's result.
Figure 15. Far-field radiation beam pattern from the transversely oscillating unbaffled circular rigid thin disk at $ka = 3$. Solid line: Approximate solution using basis functions (3.3 — 10). Dashed line: Leitner’s result.
Figure 16. Far-field radiation beam pattern from the transversely oscillating unbaffled circular rigid thin disk at $ka = 4$. Solid line: Approximate solution using basis functions (3.3 - 10). Dashed line: Leitner's result.
Figure 17. Far-field radiation beam pattern from the transversely oscillating unbaflled circular rigid thin disk at $ka = 5$. Solid line: Approximate solution using basis functions (3.3–10). Dashed line: Leitner's result.
CHAPTER IV

ACOUSTIC RADIATION FROM FINITE CYLINDERS
IN AXISYMMETRIC VIBRATIONS

4.1 Radially Oscillating Finite Cylinder

This Chapter studies sound radiation from a less idealized object — finite length circular cylinder (See Fig. 18). One case is when the cylindrical surface is pulsating in the radial direction with two end caps held fixed; the other is when the ends move back and forth in the axial direction like a rigid body with the normal component of the surface velocity being zero on the cylindrical side. Since the finite cylinder simultaneously possesses three basic geometrical features that can occur in practice: a flat surface, a curved surface, and a sharp edge, its study plays an important role in many engineering applications.

In this section we shall focus on the radially oscillating finite cylinder problem. There exist no applicable analytical solutions because the finite cylinder is not a natural surface of a coordinate system for which the Helmholtz equation can be separated. Hence we use formulations developed in section 2.2 for axisymmetric surfaces to calculate approximate solutions of the surface acoustic pressure distribution. Because of a finite length $L$, the contribution from the cylindrical side is no longer negligible; all integrated Green's functions $\mathcal{H}_1(s,s')$ to $\mathcal{H}_3(s,s')$ are to be evaluated. Therefore numerical computations in this case will be quite involved as compared with those in the idealized thin disk case demonstrated in the preceding Chapter.

Following the steps outlined in section 2.2, we first reexpress the cylindrical coordinates $r$ and $z$ in terms of the position index $s$ defined along the generator of the surface of revolution.
Figure 18. Nomenclature of the finite cylinder with radius $a$ and half-length $L$. 
where $S_{\text{max}}$ is measured from the center to the extremity of the generator while $S_{\text{edge}}$, from the center to the edge that separates the cylindrical wall and the end cap. In the present case, we have $S_{\text{max}} = a + L$ and $S_{\text{edge}} = L$, where $a$ is the radius and $L$ is the half-length of the finite cylinder (See Fig. 18). The derivatives of $r$ and $z$ with respect to $s$ are given, respectively, by

$$
\frac{dr(s)}{ds} = \begin{cases} 
+1 & \text{if } -S_{\text{max}} \leq s < -S_{\text{edge}} \\
0 & \text{if } -S_{\text{edge}} \leq s < +S_{\text{edge}} \\
-1 & \text{if } +S_{\text{edge}} < s \leq +S_{\text{max}} 
\end{cases} \quad (4.1 - 3)
$$

$$
\frac{dz(s)}{ds} = \begin{cases} 
0 & \text{if } -S_{\text{max}} \leq s < -S_{\text{edge}} \\
+1 & \text{if } -S_{\text{edge}} < s < +S_{\text{edge}} \\
0 & \text{if } +S_{\text{edge}} < s \leq +S_{\text{max}} 
\end{cases} \quad (4.1 - 4)
$$

Note that the above derivatives are discontinuous at $S = \pm S_{\text{edge}}$

$$
\left. \frac{dr(s)}{ds} \right|_{s=-S_{\text{edge}}^{-}} = +1 \quad \text{and} \quad \left. \frac{dr(s)}{ds} \right|_{s=+S_{\text{edge}}^{-}} = 0 \quad \text{and} \quad \left. \frac{dr(s)}{ds} \right|_{s=+S_{\text{edge}}^{+}} = -1 \quad (4.1 - 5)
$$
With these substitutions, we can calculate quantities defined in (2.2 - 13) and then calculate the integrated Green's functions given by Eqs.(2.2 - 15), (2.2 - 20), and (2.2 - 21), respectively. In carrying out those integrations, the numerical integration scheme depicted in Figure 3 is used. The integrands are evaluated at each of the interior points with results stored in coefficient arrays $H_1(s, s')$ to $H_3(s, s')$. These coefficient arrays together with the preselected basis functions are then substituted into Eqs. (2.2 - 3) to (2.2 - 6) for evaluations of the elements $B_n$ and $A_{nm}$ given by Eqs.(2.2 - 1) and (2.2 - 2). The unknown coefficients $C_n$ corresponding to the preselected basis functions are determined by solving the system of $N$ simultaneous equations (2.1 - 40).

To compare with the previously published numerical results [12, 88], we set the velocity profile on the cylindrical surface to be such that it is unity between $0 < |s| < 0.8 S_{\text{edge}}$, then tapers off linearly to zero at edges and keeps zero on the end caps.

$$v_n(s) = \begin{cases} 1 & \text{if } 0 \leq |s| \leq 0.8 S_{\text{edge}} \\ 5 \left(1 - \frac{s}{S_{\text{edge}}} \right) & \text{if } 0.8 S_{\text{edge}} \leq |s| \leq S_{\text{edge}} \\ 0 & \text{if } S_{\text{edge}} \leq |s| \leq S_{\max} \end{cases}$$

Note that $v_n(s)$ is symmetric about the plane which intersects the generator line at $s = 0$. Accordingly, the surface acoustic pressure $p(s)$ must behave comparably. This symmetry requires that the derivative of $p(s)$ with respect to $s$ vanish at $s = 0$. 

\[ \frac{dz(s)}{ds} \bigg|_{s = -S_{\text{edge}}} = 0 \quad \text{and} \quad \frac{dz(s)}{ds} \bigg|_{s = +S_{\text{edge}}} = 0 \] (4.1 - 6)
Also, since \( p(s) \) is axially symmetric, we have \( \frac{dp(s)}{ds} = 0 \) at \( s = \pm S_{\text{max}} \). However, this latter requirement is a natural boundary condition, hence it is not necessary for \( p(s) \) to satisfy this condition. From the potential flow theorem [89], one is able to predict that the derivative of the surface pressure is singular at edges of the finite cylinder as \( \epsilon^{-\frac{1}{2}} (\epsilon \to 0) \), where \( \epsilon \) is a small distance measured from the edge of the finite cylinder. Since such a singularity is well behaved, it will not cause any problem in the numerical computations.

Without loss of generality, we choose the trigonometric functions as the set of admissible trial functions for the unknown surface pressure

\[
p(s) = \sum_{n=1}^{N} C_n P_n(s) \tag{4.1 - 8}
\]

where

\[
P_n(s) = \cos \left[ \frac{(n - 1)\pi}{S_{\text{max}}} s \right] \tag{4.1 - 9}
\]

Note that the above basis functions satisfy both the geometric and natural boundary conditions, namely, \( \frac{dp_n(s)}{ds} = 0 \) at \( s = 0 \) and \( s = \pm S_{\text{max}} \).

As illustrative examples, we here calculate the surface pressure \( p(s) \) under different driving frequencies and different characteristic dimensions: (1) \( ka = 1 \) and \( L/a = 2 \); (2) \( ka = 2 \) and \( L/a = 1 \), and (3) \( ka = 2 \) and \( L/a = 2 \) three cases.

First, we determine an appropriate grid size of the numerical integration scheme and the number of basis functions that are required to obtain stationary numerical results. Plotted in Figs. 19 to 21 are the real and imaginary parts of \( p(s) \) with different integration subdivisions, but with the number of basis functions
fixed at $N = 5$. It is shown in each case that the numerical results reach a reason-
ably stabilized state at 12 integration subdivisions. This is because the Gaussian
quadrature formulation used here has a relatively high accuracy [with an error of
order $O(h^6)$]. Figures 22 to 24 display the real and imaginary parts of $p(s)$ at fixed
integration subdivisions, but with a different number of basis functions. Curves in
Figs. 22 to 24 show that for the cases under consideration, 5 basis functions will
be sufficient to yield stationary results. These experiments suggest that one can
obtain reasonably good numerical results with 12 integration subdivisions and 5
trigonometric basis functions (4.1–9). The approximate solutions based on these
5 trigonometric basis functions and 12 integration subdivisions are compared with
the previously published numerical results calculated by Copley [12] and by Fenlon
[88] using the general collocation method and the Galerkin method. Figures 25
to 27 demonstrate that at $ka = 1$ and $L/a = 2$, the pressure distributions on the
surface of the finite cylinder predicted by the trigonometric basis functions agree
quite well with those predicted by other methods. However, the discrepancies grow
in the surface pressure distributions predicted by the trigonometric basis functions
and by other methods as the driving frequencies and the characteristic dimensions
increase, which suggest that a better trial function for the unknown surface pressure
distribution should be used. The trigonometric functions are most convenient and
therefore most frequently adopted in many engineering applications, for example,
the Fourier series expansions. They are of course not the best trial functions. A
better trial function should be such that it incorporates one's physical insight and
prior knowledge of the problem at hand so that it is able to yield a good approxima-
tion yet with relatively few computations. Such an example will be demonstrated
in the section that follows.
Figure 19. Approximate solutions of the real and imaginary parts of the acoustic pressure distributions on the surface of the radially oscillating finite cylinder at \( ka = 1 \) and \( L/a = 2 \). The results are based on 5 trigonometric basis functions (4.1 - 9) but with different integration subdivisions \( W \). Chain-dotted line: \( W = 6 \); Dashed line: \( W = 12 \); and Solid line: \( W = 24 \).
Figure 20. Approximate solutions of the real and imaginary parts of the acoustic pressure distributions on the surface of the radially oscillating finite cylinder at $ka = 2$ and $L/a = 1$. The results are based on 5 trigonometric basis functions (4.1 - 9) but with different integration subdivisions $W$. Chain-dotted line: $W = 6$; Dashed line: $W = 12$; and Solid line: $W = 24$. 
Figure 21. Approximate solutions of the real and imaginary parts of the acoustic pressure distributions on the surface of the radially oscillating finite cylinder at $ka = 2$ and $L/a = 2$. The results are based on 5 trigonometric basis functions (4.1 - 9) but with different integration subdivisions $W$. Chain-dotted line: $W = 6$; Dashed line: $W = 12$; and Solid line: $W = 24$. 

\[
\frac{p}{\rho_0 c v_n}
\]
Figure 22. Approximate solutions of the real and imaginary parts of the acoustic pressure distributions on the surface of the radially oscillating finite cylinder at $ka = 1$ and $L/a = 2$. The results are based on 24 integration subdivisions but with a different number $N$ of basis functions ($4.1 - 9$). Chain-dotted line: $N = 5$; Dashed line: $N = 8$; and Solid line: $N = 10$. 
Figure 23. Approximate solutions of the real and imaginary parts of the acoustic pressure distributions on the surface of the radially oscillating finite cylinder at $ka = 2$ and $L/a = 1$. The results are based on 24 integration subdivisions but with a different number $N$ of basis functions (4.1 - 9). Chain-dotted line: $N = 5$; Dashed line: $N = 8$; and Solid line: $N = 10$. 
Figure 24. Approximate solutions of the real and imaginary parts of the acoustic pressure distributions on the surface of the radially oscillating finite cylinder at $ka = 2$ and $L/a = 2$. The results are based on 24 integration subdivisions but with a different number $N$ of basis functions (4.1 – 9). Chain-dotted line: $N = 5$; Dashed line: $N = 8$; and Solid line: $N = 10$. 
Figure 25. Real and imaginary parts of the acoustic pressure distributions on the surface of the radially oscillating finite cylinder at $ka = 1$ and $L/a = 2$. Solid line: Variational principle’s results; Dashed line: Fenlon’s results; and $\bullet$: Copley’s results.
Figure 26. Real and imaginary parts of the acoustic pressure distributions on the surface of the radially oscillating finite cylinder at $ka = 2$ and $L/a = 1$. 

*Solid line:* Variational principle's results; 
*Dashed line:* General collocation method’s results; and *Chain-dotted line:* Galerkin method’s results.
Figure 27. Real and imaginary parts of the acoustic pressure distributions on the surface of the radially oscillating finite cylinder at $ka = 2$ and $L/a = 2$. *Solid line:* Variational principle’s results; *Dashed line:* General collocation method’s results; and *Chain-dotted line:* Galerkin method’s results.
4.2 Axially Oscillating Finite Cylinder

This section concerns the sound radiation from a finite cylinder oscillating back and forth in the axial direction as a rigid body. (The previously solved thin disk problem is actually the limiting case of the present one in which the finite length \( L \to 0 \).) In comparison with the radially oscillating cylinder problem discussed in the preceding section, the only difference in the present problem is the change of the prescribed velocity distribution on the surface of the finite cylinder. Under the assumed rigid transverse oscillation, the normal components of the surface velocity on two end caps are 180° out of phase and zero on the cylindrical wall

\[
v_n(s) = \begin{cases} 
-v_o & \text{if } -S_{\text{max}} \leq s \leq -S_{\text{edge}} \\
0 & \text{if } -S_{\text{edge}} < s < +S_{\text{edge}} \\
+v_o & \text{if } +S_{\text{edge}} \leq s \leq +S_{\text{max}} 
\end{cases} \tag{4.2 - 1}
\]

Clearly, \( v_n(s) \) is an odd function of \( s \). Correspondingly, the surface pressure \( p(s) \) must also be odd about \( s = 0 \) plane. This and the continuity condition together require that \( p(s) \) be zero at \( s = 0 \), a necessary condition for all admissible trial functions to satisfy.

\[
p(s) = 0 \quad \text{at } s = 0 \tag{4.2 - 2}
\]

One could construct as many basis functions satisfying the above condition as one wishes. Once again, however, we use the trigonometric functions as the set of admissible trial functions to describe the unknown surface pressure distribution. The purpose of doing so is to demonstrate the fact that with the variational principle, one can use any admissible trial function to obtain an approximate solution for the problem at hand. Needless to say, the numerical solution based on an arbitrary
admissible trial function will not be the best. Here one may set, for example, the trial functions to be of the form

\[ P_n(s) = \sin \left[ \frac{(2n - 1)\pi}{2S_{\text{max}}} s \right] \quad n = 1, 2, \ldots, N \]  

(4.2 - 3)

which satisfies the geometric boundary condition (4.2 - 2). Note that in each of the basis functions (4.2 - 3), we have embedded the feature resulting from the axisymmetry of the pressure distribution on the surface of the finite cylinder, i.e.

\[ \frac{dP_n(s)}{ds} = 0 \quad \text{at} \quad s = \pm S_{\text{max}} \]  

(4.2 - 4)

Again, this requirement (4.2 - 4) is not a necessary condition, but a natural boundary condition which will emerge from the variational principle for the exact solution. However, imposing of such a condition at the outset on basis functions \( P_n(s) \) will sometimes speed up the convergence of numerical results toward the correct answers.

The numerical algorithm used in the thesis to solve the system of \( N \) simultaneous equations for the unknown coefficients \( C_n \) corresponding to the preselected basis functions for the acoustic pressure distribution on an axisymmetrically vibrating surface is written in such a way that the velocity profile and preselected basis functions are specified in two separate subroutines. Consequently, substituting the changes in the velocity profile and basis functions, one is able to calculate the surface pressure distribution corresponding to the axially vibrating cylinder problem.

Here we choose to calculate the surface pressure at (1) \( ka = 1 \) and \( L/a = 0.5 \), (2) \( ka = 2 \) and \( L/a = 0.5 \), and (3) \( ka = 1 \) and \( L/a = 1 \) three different cases. Figures 28 to 30 show the real and imaginary parts of the surface pressure at fixed integration subdivisions, but with a different number of basis functions (4.2 - 3). The curves
indicate that 5 basis functions and 10 integration subdivisions seem sufficient for yielding reasonably stationary numerical results.

Since there are no previously published numerical results available for the axially vibrating finite cylinder, the variational principle's results are compared with those computed using the SHIP program [90]. Figures 31 to 33 depict the numerical results obtained by using 5 basis functions (4.2 - 3) and 10 integration subdivisions and by using the SHIP program. In each case, we also compare the numerical results using the same basis functions (4.2 - 3) but with a coordinate transformation (3.3 - 8) which has previously been used in the thin disk problem. It is clearly demonstrated that the numerical results with the coordinate transformation give better agreements with those obtained by the SHIP program. The reason for that is because the integration scheme described in Fig. 3 uses a uniform mesh size in $s$ and $s'$ plane, while the transformation (3.3 - 8) maps the points into a nonuniform mesh in $z$ and $y$ plane, so that the density of integration points increases monotonically as $z \to 1$ and $y \to 1$. Thus, the transformation gives a better description of, and greater emphasis on, the behavior near the edges that separate the cylindrical wall and end caps, where diffraction effects are most significant.

Next, we demonstrate that a better approximation of the surface pressure distribution can be obtained when one incorporates the physical insight and prior knowledge into the selection of basis functions for the unknown surface pressure.

As mentioned earlier that there exist no analytical solutions for the finite cylinder problem except at some limiting cases. For instance, as the length $L \to 0$, the finite cylinder reduces to the thin disk. If at the same time $ka \to 0$, then the potential flow near the edge of the disk can be solved exactly (See section 3.3). It is therefore advantageous for one to take into account of this knowledge in the construction of basis functions for the unknown surface pressure $p(s)$. One way of
doing so is to simply take the exact shape of the potential function on the surface of the disk at the low frequency limit as one of the basis functions, since $p(s)$ is proportional to the potential function. However, for a finite cylinder with nonzero thickness, $p(s)$ will not vanish at the edge as would for the thin disk. To compensate for this discrepancy, we add a constant to it so that the trial function for the pressure distribution on two end caps take the form

$$p_{\text{end}}(r) = \pm [C_1 P_{\text{end},1}(r) + C_2 P_{\text{end},2}(r)] \quad (4.2-5)$$

where the basis functions

$$P_{\text{end},1}(r) = \left\{ \begin{array}{l}
1 \\
0 \leq r \leq 1
\end{array} \right. \quad (4.2-6)$$

$$P_{\text{end},2}(r) = \sqrt{1 - r^2} \quad 0 \leq r \leq 1$$

The plus and minus signs in the above correspond to the upper and lower ends (See Fig. 18), because the surface pressure is odd about the mid-plane of the finite cylinder. The dummy variable $r$ in (4.2-6) is dimensionless and measured from the center to the edge of the end cap, and coefficients $C_1$ and $C_2$ are to be determined by the variational principle. Note that the above basis functions also satisfy the axisymmetry requirement, $\frac{dP_{\text{end}}(r)}{dr} = 0$, at the center of the end cap $r = 0$.

The basis functions for the pressure distribution on the cylindrical wall must match those for the end caps. The simplest choice for such basis functions is found to be

$$p_{\text{cy}}(z) = C_1 P_{\text{cy},1}(z) \quad (4.2-7)$$

where $P_{\text{cy},1}(z)$ is simply
\[ P_{cy,1}(z) = z \quad -1 \leq z \leq +1 \quad (4.2 - 8) \]

where \( z \) is dimensionless and measured from the mid-plane of the cylinder such that \(|z| = 1\) coincides with the edges that separate the cylindrical wall and the end caps. Clearly, \( p_{cy}(z) \) is an odd function of \( z \) and \( p_{cy}(z) = 0 \) at \( z = 0 \) while at the edges, i.e. at \(|z| = 1\), \( p_{cy}(z) \) matches \( p_{end}(r) \) at \( r = 1 \). Consequently, Eqs.\((4.2 - 5)\) and \((4.2 - 7)\) are compatible and satisfy the required boundary condition. Since the coefficients \( C_1 \) and \( C_2 \) are the same, they must be varied simultaneously.

It is noticed that the basis functions now take different forms for different areas of the finite cylinder, which differs from previous cases where the same basis functions cover the whole vibrating surfaces. Because of this, we devise a more specialized computer program which makes use of different coordinate systems to evaluate contributions from different areas, one from the cylindrical wall and the other from the end caps.

Since both \( v_n \) and \( p \) are perfectly antisymmetric about \( z = 0 \) plane, one does not need to integrate over the entire surface of the finite cylinder. Instead, one can simplify the computations by decomposing integrals over the whole surface into those over either the cylindrical surface or the ends, and subsequently transferring the resulting individual integrals to two integrals, one integrating over the upper half of the cylindrical surface and the other integrating over the upper end cap only. The derivation of such a decomposition process is exactly the same as those for which Eq.\((3.2 - 8)\) is derived, but of course much lengthier than the former [91].

Consider, for example, the first integral on the right side of Eq.\((2.1 - 36)\) with the function \( U_n \) replaced by Eq.\((3.1 - 6)\) so that one need not be concerned with explicitly taking the Cauchy principal values in carrying out the integrations.
\[ 4\pi \int_S p(x_S) U_n(x_S) \, dS = 4\pi \int_S p(x_S) f_n(x_S) \, dS + \int_S \int_S \left\{ p(x_S) [f_n(x_{S'}) n(x_S) - f_n(x_S) n(x_{S'})] \cdot \frac{x_S - x_{S'}}{R} \right\} \frac{d}{dR} \left( \frac{e^{ikR}}{R} \right) \, dS' \, dS \]

\[ + \int_S \int_S p(x_S) f_n(x_S) \left[ n(x_S) \cdot \frac{x_S - x_{S'}}{R} \right] \frac{d}{dR} \left( \frac{e^{ikR} - 1}{R} \right) \, dS' \, dS \quad (4.2 - 9) \]

The decomposition of the first term on the right side in the above is relatively simple

\[ 4\pi \int_S p(x_S) f_n(x_S) \, dS = \]

\[ 4\pi \left[ \int_u p_u(r) f_u(r) \, dS + \int_l p_l(z) f_u(z) \, dS + \int_w p_w(z) f_l(z) \, dS \right] \quad (4.2 - 10) \]

where subscripts \(u\), \(l\), and \(w\) represent the upper end, lower end, and cylindrical wall, respectively. The function \(f\) is the same as previously defined in (2.1 - 4).

From Eq.(4.2 - 1), we know that

\[ f_u(z) = 0 \quad \text{and} \quad f_u(r) = -f_l(r) \quad (4.2 - 11) \]

Also, we have

\[ p_u(r) = -p_l(r) \quad (4.2 - 12) \]

because the surface pressures on two ends are 180° out of phase. Consequently, Eq.(4.2 - 10) reduces to

\[ 4\pi \int_S p(x_S) f_n(x_S) \, dS = 8\pi \int_u p_u(r) f_u(r) \, dS \quad (4.2 - 13) \]
The integral initially integrating over the whole surface of the finite cylinder is now transformed to the one integrating over the upper end only. The double surface integrations, however, will not be as simple. In general, one has

\[
\iint_S \iint_{S'} = \iint_u \iint_{u'} + \iint_u \iint_{v'} + \iint_u \iint_{w'} + \iint_u \iint_{w'} + \iint_u \iint_{l'} + \iint_l \iint_{u'} + \iint_l \iint_{v'} + \iint_l \iint_{w'} + \iint_l \iint_{l'}
\]  

(4.2 - 14)

The position vectors and corresponding unit normal vectors for points on each of the upper end, lower end, and side wall can be described in terms of the cylindrical coordinates

\[
\begin{align*}
x_u &= +Le_z + re_r \\
x_l &= -Le_z + re_r \\
x_w &= ze_z + ae_r
\end{align*}
\]

\[
\begin{align*}
n_u &= +e_z \\
n_l &= -e_z \\
n_w &= e_r
\end{align*}
\]  

(4.2 - 15)

The above equation also holds for the primed coordinates. From Eq.(4.2 - 15) one finds that

\[
\begin{align*}
n_u \cdot \frac{x_u - x_{u'}}{R_{uu'}} &= n_{u'} \cdot \frac{x_u - x_{u'}}{R_{uu'}} = n_l \cdot \frac{x_l - x_{l'}}{R_{ll'}} = n_{l'} \cdot \frac{x_l - x_{l'}}{R_{ll'}} \\
&= n_w \cdot \frac{x_w - x_{w'}}{R_{ww'}} = n_{w'} \cdot \frac{x_w - x_{w'}}{R_{ww'}} \equiv 0
\end{align*}
\]  

(4.2 - 16)

where \(R_{ij}\) is the distance between two surface points located on surfaces \(i\) and \(j\), respectively. Substituting (4.2 - 15) and (4.2 - 16) along with properties (4.2 - 11) and (4.2 - 12) into the last two integrals on the right side of (4.2 - 9) gives
\[
\begin{align*}
\int_{S} \int_{S'} \left\{ p(x_S) \left[ f_n(x_{S'}) n(x_S) - f_n(x_S) n(x_{S'}) \right] \cdot \frac{x_S - x_{S'}}{R} \right\} \frac{d}{dR} \left( \frac{e^{ikR}}{R} \right) dS' dS \\
+ \int_{S} \int_{S'} \left[ p(x_S) f_n(x_S) \left[ n(x_{S'}) \cdot \frac{x_S - x_{S'}}{R} \right] \right. & \left. \frac{d}{dR} \left( \frac{e^{ikR} - 1}{R} \right) \right] dS' dS \\
= \int_{S} \int_{S'} p_u(z) f_u(r')(a - r') \left[ \frac{(ikR_{u'w} - 1)e^{ikR_{u'w}}}{R_{u'w}^3} - \frac{(ikR_{l'w} - 1)e^{ikR_{l'w}}}{R_{l'w}^3} \right] dS' dS \\
- \int_{S} \int_{S'} p_u(r) f_u(r)(a - r')(\frac{1}{R_{u'w}^3} + \frac{1}{R_{l'w}^3}) dS' dS \\
- \int_{S} \int_{S'} p_u(r) f_u(r) \frac{4L}{R_{u'w}^3} \left[ (ikR_{u'w} - 1)e^{ikR_{u'w}} - 1 \right] dS' dS \\
\end{align*}
\]

(4.2 - 17)

where the integrations in the above are integrated over either the upper end or the cylindrical wall.

In a similar manner, one can decompose the second term on the right side of (2.1 - 36) into the form of Eq.(4.2 - 14). Notice that the unit normals on the side wall and on two end caps are always perpendicular to each other, hence all integrands involving the dot products of either \( n_w \) and \( n_u \) or \( n_u \) and \( n_l \) vanish identically. Therefore one obtains

\[
\begin{align*}
\frac{k^2}{2} \int_{S} \int_{S'} [n(x_S) \cdot n(x_{S'})] p(x_S) p(x_{S'}) G(x_S \mid x_{S'}) dS' dS \\
= \frac{k^2}{2} \int_{S} \int_{S'} p_u(r) p_u'(r') \left( \frac{e^{ikR_{uu'}}}{R_{uu'}} + \frac{e^{ikR_{ul'}}}{R_{ul'}} \right) dS' dS \\
+ \frac{k^2}{2} \int_{S} \int_{S'} p_u(z) p_u'(z') \frac{e^{ikR_{wu'}}}{R_{wu'}} \cos \theta dS' dS \\
\end{align*}
\]

(4.2 - 18)

Accordingly, the last integral on the right side of (2.1 - 36) is found to be
\[-\frac{1}{2} \iiint_S \left[ n(x_S) \times \nabla p(x_S) \right] \bullet \left[ n(x_{S'}) \times \nabla' p(x_{S'}) \right] G(x_S | x_{S'}) \, dS' \, dS \]

\[
= - \iiint_u \iiint_{u'} \frac{dp_u(r)}{dr} \frac{dp_{w'}(r')}{dr'} \left( \frac{e^{ikR_{uu'}}}{R_{uu'}} + \frac{e^{ikR_{uw'}}}{R_{uw'}} \right) \cos \theta \, dS' \, dS \\
+ \iiint_u \iiint_{w'} \frac{dp_u(r)}{dr} \frac{dp_{w'}(z')}{dz'} \left( \frac{e^{ikR_{uw'}}}{R_{uw'}} + \frac{e^{ikR_{iw'}}}{R_{iw'}} \right) \cos \theta \, dS' \, dS \\
- \frac{1}{2} \iiint_w \iiint_{w'} \frac{dp_w(z)}{dz} \frac{dp_{w'}(z')}{dz'} \frac{e^{ikR_{ww'}}}{R_{ww'}} \cos \theta \, dS' \, dS \tag{4.2 - 19}
\]

Combining Eqs. (4.2 - 10), (4.2 - 17) to (4.2 - 19) then gives the functional

\[ J[p] = 8\pi \iiint_u p_n(r) f_n(r) \, dS \]

\[ + \iiint_w \iiint_{w'} p_u(r) f_{u'}(r')(a - r') \left[ \frac{(ikR_{uu'}w - 1)e^{ikR_{uw'}w}}{R_{uu'}^3} - \frac{(ikR_{iw'}w - 1)e^{ikR_{iw'}w}}{R_{iw'}^3} \right] \, dS' \, dS \\
- \iiint_u \iiint_{w'} p_u(r) f_u(r)(a - r) \left( \frac{1}{R_{uu'}} + \frac{1}{R_{iw'}} \right) \, dS' \, dS \\
- \iiint_u \iiint_{w'} p_u(r) f_u(r) \frac{4L}{R_{uu'}} \left[ (ikR_{uu'} - 1)e^{ikR_{uw'}w} - 1 \right] \, dS' \, dS \\
+ k^2 \iiint_u \iiint_{w'} p_u(r) f_{u'}(r') \left( \frac{e^{ikR_{uw'}w}}{R_{uu'}} + \frac{e^{ikR_{uw'}w}}{R_{uw'}} \right) \, dS' \, dS \\
+ \frac{k^2}{2} \iiint_w \iiint_{w'} p_u(z) f_{w'}(z') \frac{e^{ikR_{ww'}w}}{R_{ww'}} \cos \theta \, dS' \, dS \\
- \iiint_u \iiint_{u'} \frac{dp_u(r)}{dr} \frac{dp_{u'}(r')}{dr'} \left( \frac{e^{ikR_{uu'}}}{R_{uu'}} + \frac{e^{ikR_{uw'}}}{R_{uw'}} \right) \cos \theta \, dS' \, dS \\
+ \iiint_u \iiint_{w'} \frac{dp_u(r)}{dr} \frac{dp_{w'}(z')}{dz'} \left( \frac{e^{ikR_{uw'}}}{R_{uw'}} + \frac{e^{ikR_{iw'}}}{R_{iw'}} \right) \cos \theta \, dS' \, dS \\
- \frac{1}{2} \iiint_w \iiint_{w'} \frac{dp_w(z)}{dz} \frac{dp_{w'}(z')}{dz'} \frac{e^{ikR_{ww'}}}{R_{ww'}} \cos \theta \, dS' \, dS \tag{4.2 - 20}
\]

Since \( p_w(z) \) is an odd function of \( z \), one can further decompose the integrals over the cylindrical surface in the above into contributions from the upper half of the cylindrical side only. It is always convenient for one to deal with dimensionless
quantities in the numerical computations. So we nondimensionalize the above functional by setting \( \hat{p} = \frac{p}{\rho_0 c_{v_n}} \), \( \hat{r} = \frac{r}{a} \), and \( \hat{z} = \frac{z}{L} \). Doing so, we obtain an expression for the functional given in Eq.(4.2 - 20), with the surface pressure replaced by the basis functions (4.2 - 6) and (4.2 - 8)

\[
\frac{J}{2\pi \rho_0 c_{v_n}} = \frac{1}{2} \sum_{n=1}^{2} \sum_{m=1}^{2} A_{nm} C_n C_m - \sum_{n=1}^{2} B_n C_n \quad (4.2 - 21)
\]

Since the functional \( J \) is stationary, the variational principle requires that \( \frac{\partial J}{\partial C_n} = 0 \), which yields the system of equations (2.1 - 40) for the unknown coefficients \( C_n \).

The corresponding elements of \( B_n \) and \( A_{nm} \) are given by

\[
B_n = i 4\pi (ka) \int_0^1 P_{end,n}(x) x \, dx
\]

\[
- i (kL) \int_0^1 \int_0^1 \left[ P_{end,n}(x) CB - P_{cy,n}(y) CA \right] dx \, dy \quad (4.2 - 22)
\]

\[
A_{nm} = \int_0^1 \int_0^1 \left\{ (ka)^2 P_{end,n}(x) P_{end,m}(y) CC - \frac{dP_{end,n}(x)}{dx} \frac{dP_{cy,m}(y)}{dy} CD \\
+ (kL)^2 P_{cy,n}(x) P_{cy,m}(y) CE - \frac{dP_{cy,n}(x)}{dx} \frac{dP_{cy,m}(y)}{dy} CF \\
+ \left[ \frac{dP_{end,n}(x)}{dx} \frac{dP_{cy,m}(y)}{dy} + \frac{dP_{end,m}(y)}{dy} \frac{dP_{cy,n}(x)}{dx} \right] CG \right\} dx \, dy
\]

\[
(4.2 - 23)
\]

where coefficients \( CA \) to \( CG \) in the above are defined as follows
\[ CA = G_A(m_5) - G_A(m_6) \]
\[ CB = G_B \]
\[ CC = m_1 M_1 G_1(m_1) + m_2 M_2 G_1(m_2) \]
\[ CD = M_1 [(2 - m_1) G_1(m_1) - 2G_2(m_1)] + M_2 [(2 - m_2) G_1(m_2) - 2G_2(m_2)] \]
\[ CE = M_3 [(2 - m_3) G_1(m_3) - 2G_2(m_3)] - M_4 [(2 - m_4) G_1(m_4) - 2G_2(m_4)] \]
\[ CF = M_3 [(2 - m_3) G_1(m_3) - 2G_2(m_3)] + M_4 [(2 - m_4) G_1(m_4) - 2G_2(m_4)] \]
\[ CG = M_5 [(2 - m_5) G_1(m_5) - 2G_2(m_5)] + M_6 [(2 - m_6) G_1(m_6) - 2G_2(m_6)] \]

where the parameters \( m_\ell \) and \( M_\ell, \ell = 1, 2, \ldots, 6 \) are given, respectively, by

\[
\begin{align*}
m_1 &= \frac{4xy}{M_1^2} \\
m_2 &= \frac{4xy}{M_2^2} \\
m_3 &= \frac{4}{M_3^2} \\
m_4 &= \frac{4}{M_4^2} \\
m_5 &= \frac{4x}{M_5^2} \\
m_6 &= \frac{4x}{M_6^2}
\end{align*}
\]
\[
\begin{align*}
M_1 &= \sqrt{(x + y)^2 + \left(\frac{2L}{a}\right)^2} \\
M_2 &= (x + y) \\
M_3 &= 2\sqrt{1 + \left(\frac{L}{2a}\right)^2 (x - y)^2} \\
M_4 &= 2\sqrt{1 + \left(\frac{L}{2a}\right)^2 (x + y)^2} \\
M_5 &= \sqrt{(1 + x)^2 + \left(\frac{L}{a}\right)^2 (1 - y)^2} \\
M_6 &= \sqrt{(1 + x)^2 + \left(\frac{L}{a}\right)^2 (1 + y)^2}
\end{align*}
\]

The functions \( G_A(m_5), G_A(m_6), G_B, G_1(m_\ell), \) and \( G_2(m_\ell) \) in Eq. (4.2 - 24) are

\[
G_1(m_\ell) = K(m_\ell) + i(ka)M_\ell \int_0^\frac{\pi}{2} \frac{\sin(\Phi_\ell/2)}{(\Phi_\ell/2)} e^{i(\Phi_\ell/2)} d\alpha \quad (4.2 - 26)
\]
\[
G_2(m_\ell) = E(m_\ell) + i2 \int_0^\frac{\pi}{2} D_\ell \sin(\Phi_\ell/2)e^{i(\Phi_\ell/2)} d\alpha \quad (4.2 - 27)
\]
with the function $\mathcal{G}_\ell$ appearing on the right sides of Eqs. (4.2—28) and (4.2—29) given by

$$
\mathcal{G}_\ell = \int_0^{\frac{\pi}{2}} \left[ \left( \frac{\Phi_\ell/4 - \sin(\Phi_\ell/4)}{\Phi_\ell/2} \right)^2 e^{i(\Phi_\ell/4)} \right] e^{i\Phi_\ell/2} \\
- \frac{\sin(\Phi_\ell/4) e^{i(\Phi_\ell/4)}}{\Phi_\ell/4} - \frac{\sin(\Phi_\ell/8) e^{i(\Phi_\ell/8)}}{\Phi_\ell/8} \\
- \frac{1}{4} \left\{ \frac{\left( \frac{\Phi_\ell/2}{4} - \sin(\Phi_\ell/2) \right)}{\Phi_\ell/2} \right\} e^{i\Phi_\ell/2} \\
+ \frac{\left( \frac{\Phi_\ell/2}{4} - \sin(\Phi_\ell/2) \right)}{\Phi_\ell/2} e^{i\Phi_\ell/2} \right) \tag{4.2—30}
$$

where $\Phi_\ell = kM_\ell D_\ell$, $D_\ell = \sqrt{1 - m_\ell \sin^2 \alpha}$, and $\ell$ can be 1, 2, ..., 6 in the above. The symbols $K(m_\ell)$ and $E(m_\ell)$ are the complete elliptic integrals of the first and second kinds, respectively. (The hats in the above have been omitted for brevity and all variables are understood to be dimensionless.)

Figures 34 to 36 depict the approximate solutions of the real and imaginary parts of the surface pressure distributions based on basis functions (4.2—6) and (4.2—8) with 10 integration subdivisions. The numerical results compare favorably with the SHIP program's results in each case. To test the ability of the variational principle to predict the surface pressure distributions, we also calculate and plot numerical results based on extensions of basis functions (4.2—5) and (4.2—7).
\[ p_{\text{end}}(\tau) = \pm \left[ C_1 P_{\text{end},1}(\tau) + C_2 P_{\text{end},2}(\tau) + C_n P_{\text{end},n}(\tau) \right] \quad (4.2 - 31) \]

\[ p_{\text{cy}}(z) = C_1 P_{\text{cy},1}(z) + C_n P_{\text{cy},n}(z) \quad (4.2 - 32) \]

where

\[ P_{\text{end},n}(\tau) = r^{2(n-2)} \sqrt{1 - \tau^2} \quad n \geq 3 \quad (4.2 - 33) \]

\[ P_{\text{cy},n}(z) = z^{(2n-5)}(1 - z^2) \quad n \geq 3 \quad (4.2 - 34) \]

Note that the extended basis functions (4.2 - 31) and (4.2 - 32) for the surface pressure distributions on the end caps and on the cylindrical side match at the edges and satisfy the required boundary condition (4.2 - 2).

We emphasize here that there can be more than one kind of admissible trial functions even though they are based on one's best knowledge of the problem and good common sense. For example, in the transversely oscillating finite cylinder problem, one can construct the basis functions for the unknown surface pressure distribution based on the solutions of the potential flow around a right angle (0 ≤ θ ≤ \( \frac{3\pi}{2} \)) rather than a knife edge (0 ≤ θ ≤ 2π). Whatever basis functions one use, however, the variational principle will always pick among various possible coefficients the ones that are optimal to the chosen basis functions. The closer the trial functions to the true shape of the surface pressure distributions, the better the approximations are.

Figures 34 to 36 depict numerical results based on 5 extended basis functions (4.2 - 31) and (4.2 - 33) and 10 integration subdivisions. The curves show no appreciable improvements comparing with the results based on 2 basis functions.
(4.2 - 6) and (4.2 - 8). However, numerical results depicted in Figs. 34 to 36 do improve upon the ones based on 5 trigonometric basis functions plotted in Figs. 31 to 33.

A further test of the formulations and numerical algorithms developed here is to see if the numerical solutions of the transversely oscillating finite cylinder reduce to those of previously solved thin disk problem when the length of the finite cylinder becomes infinitely small. Figures 37 and 38 demonstrate the numerical results based on 5 trigonometric basis functions at $ka = 0$ and $ka = 1$, respectively, with the length $L = 0.5, 0.25, 0.005, 0.0005, 0.00005$, and 0. Figures 39 and 40 show the same trends using 2 basis functions (4.2 - 6) and (4.2 - 8) at $ka = 0$ and $ka = 1$ with $L$ varying monotonically from 0.5 to 0. The curves in Figs. 37 to 40 show that as $L \to 0$, the numerical results of the transversely oscillating finite cylinder indeed reduce to those of the thin disk problem (See Figs. 7 and 8). Since the improper integral involving the normal derivative of the free-space Green's function is evaluated in the form of (3.1 - 6), no numerical difficulties are encountered in taking the limit as $L \to 0$. An ill condition does occur, however, in numerical computations based on direct numerical implementations of the Helmholtz integral equation, for example the SHIP program, in the limit as the length $L$ approaches zero. This is because the integrands become highly singular as $L \to 0$. 
Figure 28. Approximate solutions of the real and imaginary parts of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 1$ and $L/a = 0.5$. The results are based on 10 integration subdivisions but with a different number $N$ of basis functions (4.2 - 3). *Chain-dotted line: $W = 2$; Dashed line: $W = 4$; and Solid line: $W = 5$.*
Figure 29. Approximate solutions of the real and imaginary parts of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 2$ and $L/a = 0.5$. The results are based on 10 integration subdivisions but with a different number $N$ of basis functions (4.2 - 3). Chain-dotted line: $W = 2$; Dashed line: $W = 4$; and Solid line: $W = 5$. 
Figure 30. Approximate solutions of the real and imaginary parts of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 1$ and $L/a = 1$. The results are based on 10 integration subdivisions but with a different number $N$ of basis functions (4.2 - 3). Chain-dotted line: $W = 2$; Dashed line: $W = 4$; and Solid line: $W = 5$. 
**Figure 31.** Real and imaginary parts of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 1$ and $L/a = 0.5$. *Chain-dotted line:* Variational principle's results without coordinate transformation; *Dashed line:* Variational principle's results with coordinate transformations $(3.3 - 8)$; and *Solid line:* SHIP program's results.
Figure 32. Real and imaginary parts of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 2$ and $L/a = 0.5$. Chain-dotted line: Variational principle’s results without coordinate transformation; Dashed line: Variational principle’s results with coordinate transformations (3.3 – 8); and Solid line: SHIP program’s results.
Figure 33. Real and imaginary parts of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 1$ and $L/a = 1$. Chain-dotted line: Variational principle's results without coordinate transformation; Dashed line: Variational principle's results with coordinate transformations (3.3 - 8); and Solid line: SHIP program's results.
Figure 34. Real and imaginary parts of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 1$ and $L/a = 0.5$. Chain-dotted line: Variational principle's results based on basis functions (4.2 - 6) and (4.2 - 8); Dashed line: Variational principle's results based on basis functions (4.2 - 31) and (4.2 - 33); and Solid line: SHIP program's results.
Figure 35. Real and imaginary parts of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 2$ and $L/a = 0.5$. Chain-dotted line: Variational principle's results based on basis functions (4.2 - 6) and (4.2 - 8); Dashed line: Variational principle's results based on basis functions (4.2 - 31) and (4.2 - 33); and Solid line: SHIP program's results.
Figure 36. Real and imaginary parts of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 1$ and $L/a = 1$. **Chain-dotted line:** Variational principle’s results based on basis functions (4.2 – 6) and (4.2 – 8); **Dashed line:** Variational principle’s results based on basis functions (4.2 – 31) and (4.2 – 33); and **Solid line:** SHIP program’s results.
Figure 37. Approximate solutions of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at \(ka = 0\) with the finite length \(L = 0.5, 0.25, 0.05, 0.005, 0.0005, 0.00005, \) and 0. The results are based on 5 trigonometric basis functions (4.2 – 3).
Figure 38. Approximate solutions of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 1$ with the finite length $L = 0.5, 0.25, 0.05, 0.005, 0.0005, 0.00005$, and $0$. The results are based on 5 trigonometric basis functions (4.2 - 3).
Figure 39. Approximate solutions of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 0$ with the finite length $L = 0.5, 0.25, 0.05, 0.005, 0.0005, 0.00005$, and $0$. The results are based on 2 basis functions $(4.2 - 6)$ and $(4.2 - 8)$. 
Figure 40. Approximate solutions of the acoustic pressure distributions on the surface of the axially oscillating finite cylinder at $ka = 1$ with the finite length $L = 0.5, 0.25, 0.05, 0.005, 0.0005, 0.00005,$ and $0$. The results are based on 2 basis functions (4.2 - 6) and (4.2 - 8).
4.3 Far-Field Radiation Patterns

Once we obtain the acoustic pressure distributions on the surface of the finite cylinder in axisymmetric vibrations, it is just a simple matter to use the Kirchhoff-Helmholtz integral relation (2.1 - 1) for calculating the far-field radiation patterns. As before, we calculate the acoustic pressure at a distance \( \lambda = \frac{2\pi a}{ka} \), which is measured from the center of the finite cylinder, then normalize the results and use the polar coordinate system to plot out the radiation beam patterns.

As illustrative examples, we calculate the complex amplitudes of the far-field acoustic pressures from the radially oscillating finite cylinder at \( ka = 1, L/a = 2 \) and \( ka = 2, L/a = 1 \). Figures 41 and 42 show the agreements of radiation beam patterns based on surface pressure distributions obtained by using the variational principle and by using the general collocation method. We also calculate the complex amplitudes of the far-field acoustic pressures from the the axially oscillating finite cylinder at \( ka = 1, L/a = 0.5 \) and \( ka = 2, L/a = 0.5 \). The beam patterns based on surface pressure distributions obtained by using the variational formulation with basis functions (4.2 - 6) and (4.2 - 8) and by using the SHIP program are plotted in Figs. 43 and 44. In each case, namely, the radially oscillating and axially oscillating, the calculated complex amplitudes of the far-field acoustic pressures are normalized with respect to the values evaluated at the angle \( \theta = 0 \). Figures 41 to 44 demonstrate that the far-field radiation beam patterns are relatively insensitive to changes in the surface pressure distributions.
Figure 41. Far-field radiation beam patterns based on the radially oscillating finite cylinder at $ka = 1$ and $L/a = 2$. Solid line: Based on Fenlon's results; Dashed line: Based on variational principle's results.
Figure 42. Far-field radiation beam patterns based on the radially oscillating finite cylinder at $ka = 2$ and $L/a = 1$. Solid line: Based on Fenlon's results; Dashed line: Based on variational principle's results.
Figure 43. Far-field radiation beam patterns based on the axially oscillating finite cylinder at $ka = 1$ and $L/a = 0.5$. Solid line: Based on basis functions (4.2 - 6) and (4.2 - 8); Chain-dotted line: Based on trigonometric basis functions (4.2 - 3); Dashed line: Based on the SHIP program's results.
Figure 44. Far-field radiation beam patterns based on the axially oscillating finite cylinder at $ka = 2$ and $L/a = 0.5$. Solid line: Based on basis functions (4.2 - 6) and (4.2 - 8); Dashed line: Based on the SHIP program's results.
CHAPTER V

UNIQUENESS OF SOLUTIONS TO
ACOUSTIC RADIATION PROBLEMS

5.1 Kirchhoff-Helmholtz Integral Equation

The Kirchhoff-Helmholtz integral theorem is powerful in that it transfers a three dimensional partial differential equation to a two dimensional surface integral equation. Hence one is released from considering the infinite domains which are usually associated with the acoustic radiation problems. The computer storage requirements for numerical solutions are accordingly significantly reduced. Also, since this integral equation can handle arbitrary geometries and boundary conditions, the great efforts involved in extensively modifying computer programs for changes in geometries and boundary conditions are eliminated. Consequently, use of such a formulation is advantageous from the computational viewpoint. On the other hand, solutions of the integral equations obtained are nonunique when the driving frequency corresponds to one of the eigenfrequencies of a related interior boundary value problem.

To display these eigenfrequencies, let us examine the Kirchhoff-Helmholtz integral theorem for interior and exterior regions. The integral equation formulations for the exterior region have been given by (2.1 - 1). For the interior region, the same formulations also hold except the plus sign on the left side of (2.1 - 1) should be replaced by the minus sign. Consequently, one has
\[
\frac{1}{4\pi} \iint_{S'} \left\{ p(x_{S'}) [n(x_{S'}) \cdot \nabla' G(x \mid x_{S'})] + f_n(x_{S'}) G(x \mid x_{S'}) \right\} dS'
\]

\[
= \begin{cases} 
0 & x \in E \\
-\frac{1}{2} p(x) & x \in \partial D \\
-\rho(x) & x \in D
\end{cases}
\]

(5.1 - 1a) \quad (5.1 - 1b) \quad (5.1 - 1c)

where \( E, \partial D, \) and \( D \) indicate, respectively, the exterior region, boundary, and interior region; all other quantities in the above are defined the same as before.

Taking the normal derivative of (5.1 - 1c) at an internal point \( x \) then gives

\[
f_n(x) = \frac{1}{4\pi} \iint_{S'} p(x_{S'}) [n(x_{S}) \cdot \nabla] [n(x_{S'}) \cdot \nabla'] G(x \mid x_{S'}) dS'
\]

+ \( \frac{1}{4\pi} \iint_{S'} f_n(x_{S'}) [n(x_S) \cdot \nabla G(x \mid x_{S'})] dS' \)

(5.1 - 2)

Note that the integrand in the first term on the right side of Eq.(5.1 - 2) is highly singular as the field point \( x \) approaches the surface. To continue our analysis here, we need first to reduce the order of singularities contained in that integrand. Section 2.1 has demonstrated that the second normal derivative of the free-space Green's function can be recast into a form involving derivatives tangential to the surface. The same procedure is repeated here, namely, we apply: (1) the vector identity (2.1 - 20), properties of the free-space Green's function (2.1 - 25) to (2.1 - 27), (3) integration by parts, and (4) the Stokes' theorem to the first integral on the right side of Eq.(5.1 - 2). Doing so yields

\[
f_n(x) = \frac{1}{4\pi} \iint_{S'} f_n(x_{S'}) [n(x_S) \cdot \nabla G(x \mid x_{S'})] dS'
\]

+ \( \frac{k^2}{4\pi} \iint_{S'} [n(x_S) \cdot n(x_{S'})] p(x_{S'}) G(x \mid x_{S'}) dS' \)

+ \( \frac{1}{4\pi} \iint_{S'} [n(x_S) \times \nabla] \cdot [n(x_{S'}) \times \nabla' p(x_{S'})] G(x \mid x_{S'}) dS' \)

(5.1 - 3)
The integrands in the above are now well behaved or at most of Cauchy type singularity which is integrable as the field point approaches the surface. The limiting version as \( x \rightarrow x_s \) from the interior region is accordingly given by

\[
\begin{align*}
    f_n(x_s) &= \frac{1}{2\pi} \int_{S'} f_n(x_s') \left[ n(x_s) \cdot \frac{x_s - x_s'}{R} \right] \frac{d}{dR} \left( \frac{e^{ikR}}{R} \right) dS' \\
    &\quad + \frac{k^2}{2\pi} \int_{S'} \left[ n(x_s) \cdot n(x_s') \right] p(x_s') G(x_s \mid x_s') dS' \\
    &\quad + \frac{1}{2\pi} \int_{S'} \left[ n(x_s) \times \nabla \right] \cdot \left[ n(x_s') \times \nabla' p(x_s') \right] G(x_s \mid x_s') dS' \\
\end{align*}
\]

Now we consider two common boundary value problems, namely,

(\textbf{Dirichlet problem}) \quad p(x_s) = \mathcal{P}(x_s) \quad x_s \in \partial D \quad (5.1 - 5)

and

(\textbf{Neumann problem}) \quad f_n(x_s) = \mathcal{V}(x_s) \quad x_s \in \partial D \quad (5.1 - 6)

where \( \mathcal{P}(x_s) \) and \( \mathcal{V}(x_s) \) are prescribed on the boundary.

Substituting, respectively, boundary conditions (5.1 - 5) and (5.1 - 6) into Eq. (5.1 - 4) and using integral operator notations yield two integral equations for the interior region

\[
\begin{align*}
    (I - \mathcal{M}_k) f_n &= \mathcal{N}_k(\mathcal{P}) \quad x_s \in \partial D \quad (5.1 - 7) \\
    \mathcal{N}_k(p) &= (I - \mathcal{M}_k^T) \mathcal{V} \quad x_s \in \partial D \quad (5.1 - 8)
\end{align*}
\]
for determining the surface velocity $v_n$ given the surface pressure distribution $P(x_S)$ [See Eq.(5.1-7)] or determining the surface pressure distribution $p$ given the surface velocity $V(x_S)$ [See Eq.(5.1-8)]. The function $f_n$ here is defined for convenience sake only [See Eq.(2.1-4)]. The symbol $I$ in Eqs.(5.1-7) and (5.1-8) stands for the identity operator while $M_k$ and $N_k$ are integral operators defined as follows,

\[
M_k(f_n) = \frac{1}{2\pi} \int_{S'} \int_{S'} f_n(x_S') [n(x_S) \cdot \frac{x_S - x_{S'}}{R}] \frac{d}{dR} \left( \frac{e^{ikR}}{R} \right) dS' \quad (5.1-9)
\]

\[
N_k(p) = \frac{k^2}{2\pi} \int_{S'} \int_{S'} [n(x_S) \cdot n(x_{S'})] p(x_{S'}) G(x_S | x_{S'}) dS' \\
+ \frac{1}{2\pi} \int_{S'} [n(x_S) \times \nabla] \cdot [n(x_{S'}) \times \nabla' p(x_{S'})] G(x_S | x_{S'}) dS' \quad (5.1-10)
\]

The transpose of $M_k$ is defined as the one obtained by interchanging $x_S$ and $x_{S'}$ in the integration kernel,

\[
M_k^T(p) = \frac{1}{2\pi} \int_{S'} \int_{S'} p(x_{S'}) [n(x_{S'}) \cdot \frac{x_{S'} - x_S}{R}] \frac{d}{dR} \left( \frac{e^{ikR}}{R} \right) dS' \quad (5.1-11)
\]

It is clear that

\[
M_k^T \neq M_k \quad (5.1-12)
\]

because of the normal derivative of the free-space Green's function involved in the integration kernel. Nevertheless we have

\[
(M_k^T)^T \equiv M_k \quad (5.1-13)
\]
On the other hand, the operator $N_k$ is symmetric in the sense that

$$N_k^T \equiv N_k \quad (5.1 - 14)$$

Similarly, with substitution of the boundary condition (5.1 - 6) into the Helmholtz integral equation (2.1 - 18) and its differentiated counterpart (2.1 - 31) for the exterior region, respectively, we obtain

$$(I - M_k^T) p = \mathcal{H}_k(V) \quad x_S \in \partial D \quad (5.1 - 15)$$

$$N_k(p) = -(I + M_k)V \quad x_S \in \partial D \quad (5.1 - 16)$$

where the integral operator $\mathcal{H}_k$ is defined as

$$\mathcal{H}_k(V) = \frac{1}{2\pi} \iint_{S'} V(x_s')G(x_S \mid x_s') dS' \quad (5.1 - 17)$$

It is observed that the homogeneous versions of (5.1 - 7) and (5.1 - 15) are adjoint to one another, while the homogeneous versions of (5.1 - 8) and (5.1 - 16) are a self-adjoint pair. From the Fredholm Alternative Theorem [92], we learn that solutions to Eqs.(5.1 - 15) and (5.1 - 16) for the external acoustic fields are uniquely determined unless the corresponding homogeneous equations

$$(I - M_k^T)p = 0 \quad (5.1 - 18)$$

$$N_k(p) = 0 \quad (5.1 - 19)$$
have non-trivial solutions. The same theorem also applies to the interior region, namely, solutions to Eqs. (5.1 - 7) and (5.1 - 8) are unique if and only if the corresponding homogeneous parts

\[(I - \mathcal{M}_k)f_n = 0 \quad (5.1 - 20)\]
\[\mathcal{N}_k(p) = 0 \quad (5.1 - 21)\]

have non-trivial solutions.

In what follows, we first demonstrate that Eqs. (5.1 - 20) and (5.1 - 21) indeed have nonzero solutions whenever \(k\) coincides, respectively, with the interior Dirichlet eigenfrequencies \(k_D\) and interior Neumann eigenfrequencies \(k_N\). We then prove that if Eqs. (5.1 - 20) and (5.1 - 21) have nonzero solutions at these eigenfrequencies, so do Eqs. (5.1 - 18) and (5.1 - 19), and vice versa.

To verify the existence of eigensolutions to Eqs. (5.1 - 20) and (5.1 - 21), let us consider the interior region and assume a representation of the eigenfunction \(p\) as a double-layer potential,

\[p(x) = \frac{1}{4\pi} \int_{S'} \sigma(x_{S'}) \left[ n(x_{S'}) \cdot \nabla' G(x \mid x_{S'}) \right] dS' \quad x \in D \quad (5.1 - 22)\]

where \(\sigma(x_{S'})\) is known as the layer density.

Take the limit as \(x \to \partial D\), i.e. let the point \(x\) in the interior region approach the surface from the inside. The continuity property of the potential function then gives
\[ p(x_S) = -\frac{1}{2} \sigma(x_S) + \frac{1}{4\pi} \int_{\Omega} \sigma(x_S') \left[ n(x_S') \cdot \frac{x_S - x_{S'}}{R} \right] \frac{d}{dR} \left( \frac{e^{ikR}}{R} \right) dS' \]

\[ x_S \in \partial D \quad (5.1 - 23) \]

Substituting the homogeneous Dirichlet boundary condition

\[ p(x_S) = 0 \quad x_S \in \partial D \quad (5.1 - 24) \]

into Eq.\((5.1 - 23)\) yields an integral equation for \(\sigma\)

\[ (I - M^T_k)\sigma = 0 \quad x_S \in \partial D \quad (5.1 - 25) \]

The adjoint version of \((5.1 - 25)\)

\[ (I - M_k)\xi = 0 \quad x_S \in \partial D \quad (5.1 - 26) \]

is identical to Eq.\((5.1 - 20)\). Hence Eq.\((5.1 - 20)\) does have non-trivial solutions at \(k_D\).

Similarly, if we take a normal derivative of \((5.1 - 22)\) at an interior point \(x\) and use the same technique for which Eq.\((5.1 - 3)\) is derived to obtain

\[ \frac{\partial p(x)}{\partial n} = \frac{k^2}{4\pi} \int_{\Omega} [n(x_S) \cdot n(x_{S'})] \sigma(x_{S'}) G(x | x_{S'}) dS' \]

\[ + \frac{1}{4\pi} \int_{\Omega} [n(x_S) \times \nabla] \cdot [n(x_{S'}) \times \nabla' \sigma(x_{S'})] G(x | x_{S'}) dS' \]

\[ (5.1 - 27) \]
Then let the field point \( x \) approach the surface from the inside and substitute the homogeneous Neumann boundary condition

\[
\frac{\partial p(x_S)}{\partial n} = 0 \quad x_S \in \partial D \tag{5.1 - 28}
\]

Doing so, we obtain

\[
\mathcal{N}_k(\sigma) = 0 \quad x_S \in \partial D \tag{5.1 - 29}
\]

which is exactly the same as Eq.(5.1 - 21), hence it confirms the existence of non-trivial solutions of (5.1 - 21) at \( k \in k_N \).

Now we display the relations of the eigenfrequencies between the interior and exterior regions.

**Theorem.** Let \( \mathcal{L} \) be a square integrable kernel over the two dimensional surface \( S \) [93]. If the homogeneous equation

\[
\mathcal{L}(\phi) = 0 \tag{5.1 - 30}
\]

has nonzero solutions, then

\[
\mathcal{L}^\dagger(\psi) = 0 \tag{5.1 - 31}
\]

also has nonzero solutions.

**Proof.** Let \( \{ \vartheta_1, \vartheta_2, \ldots \} \) be a complete set of basis functions for the domain \( S \) and the function \( \phi \) be expansible in terms of \( \vartheta_n \) in the following way
\[
\phi = \sum_{n} \alpha_n \vartheta_n \quad n = 1, 2, \ldots \quad (5.1-32)
\]

where at least one of the coefficients \(\alpha_n\) is nonzero. Substituting Eq. (5.1-32) into (5.1-30), premultiplying by \(\{\vartheta_1, \vartheta_2, \ldots\}^T\), and then integrating over \(S\) once again results in an infinite system of simultaneous equations

\[
\sum_{n} \iint_{S} \vartheta_m \mathcal{L}(\alpha_n \vartheta_n) \, dS = 0 \quad m, n = 1, 2, \ldots, \quad (5.1-33)
\]

or in the matrix form

\[
\begin{pmatrix}
L_{11} & L_{12} & \cdots \\
L_{21} & L_{22} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots
\end{pmatrix} = 0 \quad (5.1-34)
\]

where the elements of \([L]\) are defined by

\[
L_{nm} = \iint_{S} \vartheta_m \mathcal{L}(\vartheta_n) \, dS \quad n, m = 1, 2, \ldots \quad (5.1-35)
\]

It is "permissible" to truncate the infinite expansion to a finite one, hence we can assume that

\[
\phi \approx \sum_{n=1}^{N} \alpha_n \vartheta_n \quad (5.1-36)
\]

where \(N\) is a large number. Substituting (5.1-36) into (5.1-33) then gives

\[
[L]\{a\} = 0 \quad (5.1-37)
\]

where \([L]\) is an \(N \times N\) square matrix and \(\{a\}\) are the unknown coefficients.
Suppose that Eq.(5.1 - 37) has nonzero solutions. This is true if and only if the determinant of \([L]\) vanishes

\[
\begin{vmatrix}
L_{11} & L_{12} & \cdots & L_{1N} \\
L_{21} & L_{22} & \cdots & L_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
L_{N1} & L_{N2} & \cdots & L_{NN}
\end{vmatrix} = 0 \quad (5.1 - 38)
\]

This has been proven in many textbooks [94]. The theorem of interest here is whether there is a set of nonzero \(\beta_n\) such that

\[
\psi = \sum_{n=1}^{N} \beta_n \varphi_n \quad (5.1 - 39)
\]

is a solution of

\[
\mathcal{L}^\dagger(\psi) = 0 \quad (5.1 - 40)
\]

The preceding analysis indicates that such a case holds if

\[
[L^\dagger] \{ \beta \} = 0 \quad (5.1 - 41)
\]

has nonzero solutions or if

\[
\begin{vmatrix}
L_{11}^\dagger & L_{12}^\dagger & \cdots & L_{1N}^\dagger \\
L_{21}^\dagger & L_{22}^\dagger & \cdots & L_{2N}^\dagger \\
\vdots & \vdots & \ddots & \vdots \\
L_{N1}^\dagger & L_{N2}^\dagger & \cdots & L_{NN}^\dagger
\end{vmatrix} = 0 \quad (5.1 - 42)
\]

where

\[
L_{mn}^\dagger = \iint_S \varphi_n \mathcal{L}^\dagger(\varphi_m) \, dS \quad (5.1 - 43)
\]
From the definition of an adjoint operator [95], we have

$$\iint_{S} \varphi_{n} L^{\dagger}(\varphi_{m}) \, dS = \iint_{S} L(\varphi_{n}) \varphi_{m} \, dS = \iint_{S} \varphi_{m} L(\varphi_{n}) \, dS$$  (5.1 - 44)

Hence $[L^{\dagger}]$ is the transpose of $[L]$ which is formed by interchanging rows and columns — reflecting about the diagonal. Since the determinant of a matrix is invariant with respect to such interchanges of rows and columns, we have

$$\det [L^{\dagger}] = \det [L]$$  (5.1 - 45)

Consequently, if there is a nonzero solution for (5.1 - 37), then there must be a nonzero solution for (5.1 - 41). Equivalently, if Eq.(5.1 - 30) has a non-trivial solution, so does Eq.(5.1 - 31). A rigorous proof for integral operators can be found in reference [96]. The restriction here is that the integration kernel must be square integrable, which may not be true for cases in which we are interested. However, Kellogg [97] has established that singular kernels such as those appearing in our problems are integrable for the case where $k = 0$, and his arguments are readily extensible to the present cases.

We now show that even at $k \in k_{D}$, the inhomogeneous equation (5.1 - 15) is still soluble. To this end, let us examine Eq.(5.1 - 4) with $p(x_{S})$ replaced by $\mu(x_{S})$ for the interior region $D$. The function $\mu$ is assumed to satisfy the homogeneous Dirichlet boundary condition

$$\mu(x_{S}) = 0 \quad x_{S} \in \partial D$$  (5.1 - 46)

Substituting the above boundary condition into (5.1 - 4) gives
\((I - \mathcal{M}_k) \frac{\partial \mu}{\partial n} = 0\) \quad x_S \in \partial D \quad (5.1 - 47)

The transpose of (5.1 - 47) is exactly the same as (5.1 - 25) with \(\sigma\) replaced by \(\frac{\partial \mu}{\partial n}\). Hence from the preceding analysis, we know that Eq.(5.1 - 47) has eigensolutions at \(k \in k_D\). Applying the Green's second identity to \(\mu\) and the function \(\mathcal{H}_k(\mathcal{V})\) on the right side of (5.1 - 15) for the interior region \(D\) then leads to

\[
0 = \iiint_S \left[ \mathcal{H}_k(\mathcal{V}) \frac{\partial \mu(x_S)}{\partial n} - \mu(x_S) \frac{\partial \mathcal{H}_k(\mathcal{V})}{\partial n} \right] dS \quad x_S \in \partial D \quad (5.1 - 48)
\]

Since \(\frac{\partial \mu}{\partial n}\) is proportional to \(\sigma\) and since \(\mu = 0\) on the boundary, Eq.(5.1 - 48) reduces to

\[
0 = \iiint_S \mathcal{H}_k(\mathcal{V}) \sigma(x_S) dS \quad x_S \in \partial D \quad (5.1 - 49)
\]

which is just the required compatibility condition. Consequently, Eq.(5.1 - 15) is soluble for all values of \(k\), but at \(k \in k_D\) solutions to (5.1 - 15) will not be unique because any combination of eigenfunctions may constitute a solution.

The particular solution of (5.1 - 16) may or may not exist at \(k \in k_N\). If the right side of (5.1 - 16) is orthogonal to all eigenfunctions of the adjoint homogeneous version of (5.1 - 16) at \(k \in k_N\), then solutions to the exterior region exist but will be nonunique. For example, in the case of scattering from a rigid surface, the normal derivative of the total wave function is zero on the boundary

\[
\frac{\partial p_{\text{total}}(x_S)}{\partial n} = 0 \quad x_S \in \partial D \quad (5.1 - 50)
\]
Substitution of (5.1 - 50) into (2.1 - 18) for the exterior region with \( p(x_S) \) replaced by \( p_{\text{total}}(x_S) \) yields an integral equation in terms of the given incident wave function \( p_{\text{inc}}(x_S) \)

\[
N_k(p_{\text{total}}) = -\frac{\partial p_{\text{inc}}(x_S)}{\partial n}, \quad x_S \in \partial D \tag{5.1-51}
\]

The homogeneous part of (5.1 - 51) has been shown to have nonzero solutions at \( k \in k_N \). To demonstrate that \( \frac{\partial p_{\text{inc}}(x_S)}{\partial n} \) on the right side of (5.1 - 51) satisfies the compatibility condition, let us consider the interior region again and replace \( p(x_S) \) in (5.1 - 4) by a function \( \nu(x_S) \) which satisfies the homogeneous Neumann boundary condition

\[
\frac{\partial \nu(x_S)}{\partial n} = 0, \quad x_S \in \partial D \tag{5.1-52}
\]

With this substitution, Eq.(5.1 - 4) becomes

\[
N_k(\nu) = 0, \quad x_S \in \partial D \tag{5.1-53}
\]

which is identical to Eq.(5.1 - 29), hence Eq.(5.1 - 53) has eigensolutions at \( k \in k_N \). Now apply the Green’s second identity to \( \nu(x_S) \) and \( p_{\text{inc}}(x_S) \).

\[
0 = \int_S \left[ \nu(x_S) \frac{\partial p_{\text{inc}}(x_S)}{\partial n} - p_{\text{inc}}(x_S) \frac{\partial \nu(x_S)}{\partial n} \right] dS
\]

\[
= \int_S \nu(x_S) \frac{\partial p_{\text{inc}}(x_S)}{\partial n} dS, \quad x_S \in \partial D \tag{5.1-54}
\]

because \( \frac{\partial \nu(x_S)}{\partial n} = 0 \) from (5.1 - 52). The compatibility condition (5.1 - 54) thus holds and solutions to (5.1 - 51) at \( k \in k_N \) exist but, again, are nonunique.
However, in the radiation cases solutions to (5.1 - 16) at eigenfrequencies $k \in k_N$ will not exist in general, because the source term on the right side of that equation may be arbitrary such that the compatibility condition is violated. In those circumstances, the eigenfunctions need to be represented as combinations of single and double layer potential functions and the source term needs to be redefined so as to satisfy the compatibility condition [98].

In summary, we have two integral equations (5.1 - 15) and (5.1 - 16) for the external acoustic field. Both of them fail by nonuniqueness at some particular frequencies, the first (5.1 - 15) at the interior Dirichlet eigenfrequencies $k \in k_D$, while the second (5.1 - 16) at the interior Neumann eigenfrequencies $k \in k_N$.

5.2 Variationally Formulated Surface Acoustic Pressure

The variational formulation for evaluation of the surface acoustic pressure with the normal component of the surface velocity taken as given has been derived in section 2.1. The corresponding Euler-Lagrange equation (2.1 - 31) is rewritten here in the integral operator notation

$$\mathcal{N}_k(p) = -(I + \mathcal{M}_k)f_n \quad \mathbf{x}_S \in \partial D$$

(5.2 - 1)

Replacing the function $f_n$ on the right side in the above by the boundary condition (5.1 - 6) gives exactly the same integral equation (5.1 - 16). Consequently, from the preceding section we conclude that the integral equation (5.2 - 1) has nonunique solutions at the interior Neumann eigenfrequencies $k_N$.

The example shown below is a finite circular cylinder oscillating back and forth along the axial direction as a rigid body. The surface pressure distribution has been obtained in section 4.2 using a variational principle. Here we want to show
that solutions to such a formulation are nonunique when the driving frequencies coincide with the interior Neumann eigenfrequencies $k_N$.

The corresponding interior problem in this case resembles a one-dimensional wave propagating in the longitudinal direction of a duct with two rigid ends.

$$\frac{d^2 p(x)}{dx^2} + k^2 p(x) = 0 \quad 0 \leq x \leq L$$

Solving the above differential equation and substituting the rigid-wall boundary condition at both ends,

$$\frac{dp(x)}{dx} = 0 \quad x = 0 \text{ and } L$$

one obtains eigenfrequencies $k_N$

$$k_N = \frac{(2n - 1)\pi}{L} \quad n = 1, 2, \cdots$$

where $L$ is the length of the finite cylinder.

Figure 45 plots the real and imaginary parts of the acoustic pressure evaluated at the center of the end cap of the finite cylinder in the vicinity of the first eigenfrequency $k_1 = \frac{\pi}{L}$. In computing these curves, the trigonometrical basis functions (4.2 - 3) are used. The ill-conditioning in the numerical computations is clearly depicted around the first eigenfrequency $k_1 = \frac{\pi}{L}$ in both real and imaginary parts.

This ill-conditioning, however, does not occur in cases of disks or plate-like bodies for which each surface point is vibrating either in phase or 180° out of phase. To verify this statement, let us consider a finite sized thin rectangular plate with dimensions $L_x \times L_y \times L_z$, where $L_z$ is the thickness of the plate (See Fig. 46). Using
the technique of separation of variables, one readily finds the Helmholtz equations for the interior region in $x$, $y$, and $z$ components

\[
\begin{align*}
X'' + k_z^2 X &= 0 \\
Y'' + k_y^2 Y &= 0 \\
Z'' + k_z^2 Z &= 0
\end{align*}
\]

(5.2 — 5)

where the separation constants are related to each other through

\[
k_z^2 + k_y^2 + k_z^2 = k^2
\]

(5.2 — 6)

Substituting the rigid-wall boundary conditions at $x = L_x$, $y = L_y$, and $z = L_z$, respectively, one obtains

\[
k = \pi \sqrt{\left( \frac{n_x}{L_x} \right)^2 + \left( \frac{n_y}{L_y} \right)^2 + \left( \frac{n_z}{L_z} \right)^2}
\]

(5.2 — 7)

For finite values of $L_x$ and $L_y$, the lowest eigenfrequency $k$ approaches infinity as the thickness of the plate $L_z$ becomes infinitely small.

\[
k \to \infty \quad \text{as} \quad L_z \to 0
\]

(5.2 — 8)

Consequently, one will not encounter the ill-conditioning problem in numerical computations in practical ranges of wavenumber $k$ and the external acoustic field is uniquely determined.
Figure 45. Dimensionless total radiated acoustic power $\frac{1}{2kL} Im(\mathcal{J})$ and surface acoustic pressure $\frac{p}{\rho_e c v_o}$ evaluated at the center of the end cap of the finite cylinder as a function of $kL$. - - - - : $Re(\frac{p}{\rho_e c v_o})$; $\cdots \cdots$ : $Im(\frac{p}{\rho_e c v_o})$; and $\cdots$ : $\frac{1}{2kL} Im(\mathcal{J})$. 

$kL = 3.1416$
Figure 46. A transversely vibrating thin rectangular plate with dimensions $L_x \times L_y \times L_z$, where $L_z$ is the thickness of the plate.
5.3 Variationally Formulated Total Radiated Power

In many practical applications, our interests are often concerned with accurate estimations of some stationary physical quantities such as the total radiated acoustic power. A variational principle is thus advantageous in that it enables one to obtain an estimate of a required quantity with an error of second order by making a mediocre approximation of some other quantity with an error of the first order. Such a variational formulation can be derived by use of the Gerjuoy-Rao-Spruch technique [74] as the one obtained by Pierce [75]. In the integral operator notations adopted in the present paper, we have

\begin{align}
\mathcal{J} = & \iint_S \bar{p} f_n^* dS \\
+ & \iint_S \bar{\lambda}_1 [(I - M_k^T) \bar{p} - \mathcal{H}_k(f_n)] dS \\
+ & \iint_S \bar{\lambda}_2 [N_k(\bar{p}) + (I + M_k)f_n] dS \\
\end{align}

where \( f_n^* \) is the complex conjugate of \( f_n \) which is a prescribed function on the boundary, \( \bar{\lambda}_1 \) and \( \bar{\lambda}_2 \) are trial functions for the arbitrary Lagrange multipliers \( \lambda_1 \) and \( \lambda_2 \), and the integral operators \( M_k, N_k, \) and \( \mathcal{H}_k \) are defined in Eqs.(5.1 – 9), (5.1 – 10), and (5.1 – 17), respectively. The variational principle (5.3 – 1) states that if errors in estimates of \( \bar{p}, \bar{\lambda}_1, \) and \( \bar{\lambda}_2 \) are of the first order, then the error in the estimation of the stationary quantity \( \mathcal{J} \) is of the second order. In particular, if the trial function \( \bar{p} \) happens to be the true function \( p \), then the last two terms on the right side of (5.3 – 1) vanish identically and the expression for the total radiated acoustic power reduces to

\[ P_{\text{power}} = \frac{1}{2 \omega \rho} \text{Im}(\mathcal{J}) \]
Equation (5.3–1) requires that the quantity \( J \) be stationary to small changes in \( \bar{p}, \bar{\lambda}_1, \) and \( \bar{\lambda}_2 \). This requirement results in three simultaneous equations known as the Euler-Lagrange equations

\[(I - M_k^T)\ddot{p} = \mathcal{H}_k(f_n) \tag{5.3–3}\]

\[N_k(\ddot{p}) = -(I + M_k)f_n \tag{5.3–4}\]

\[N_k(\bar{\lambda}_2) = -(I - M_k^T)\bar{\lambda}_1 - f_n^* \tag{5.3–5}\]

The first two equations (5.3–3) and (5.3–4) in the above are quite familiar to us [compare with Eqs.(5.1–15) and (5.1–16)]. Both of them suffer from the nonuniqueness difficulties, the former at \( k \in k_D \) while the latter at \( k \in k_N \). The third equation (5.3–5) looks new to us. However, this last equation (5.3–5) is proportional to the adjoint of (5.3–4). This will become obvious if one sets, since \( \bar{\lambda}_1 \) and \( \bar{\lambda}_2 \) are arbitrary Lagrange multipliers,

\[\bar{\lambda}_1 = -\frac{1}{2} f_n^* \tag{5.3–6}\]

\[\bar{\lambda}_2 = \frac{1}{2} p_{aux} \tag{5.3–7}\]

where \( p_{aux} \) is the solution of an auxiliary exterior problem with the surface value \( f \) specified as

\[f_{aux,n} = f_n^* \tag{5.3–8}\]

Substituting (5.3–6) and (5.3–7) into (5.3–5) then gives
\[ N_k(p_{aux}) = -(I + M_k^T)f_{aux,n} \]  

(5.3 - 9)

which is the transpose of Eq.(5.3 - 4). From the Theorem established in section 5.1, we conclude that the homogeneous part of Eq.(5.3 - 9) also has non-trivial solutions at \( k \in k_N \). Nevertheless, Eqs.(5.3 - 3) and (5.3 - 4) together always define a unique solution for all wavenumbers \( k \) because Eqs.(5.3 - 3) and (5.3 - 4), have only one solution in common which is the required boundary value \( p \ [98] \).

We continue our examination of Eq.(5.3 - 1). By replacing the Lagrange multipliers \( \lambda_1 \) and \( \lambda_2 \) by Eqs.(5.3 - 6) and (5.3 - 7), we find after some manipulations that the expression for the stationary quantity \( J \) reduces to

\[
\begin{align*}
J &= \frac{1}{2} \iint_S (\tilde{p} f_{aux,n} + \tilde{p}_{aux} f_n)\,dS \\
&+ \frac{1}{2} \iint_S [f_{aux,n} M_k(\tilde{p}) + \tilde{p}_{aux} M_k(f_n)]\,dS \\
&+ \frac{1}{2} \iint_S \tilde{p}_{aux} N_k(\tilde{p})\,dS \\
&+ \frac{1}{2} \iint_S f_{aux,n} H_k(f_n)\,dS 
\end{align*}
\]  

(5.3 - 10)

Note that the last term on the right side in the above is merely a constant because both \( f_{aux,n} \) and \( f_n \) are prescribed on the boundary. Since the functions \( \tilde{p} \) and \( \tilde{p}_{aux} \) are to be varied independently, the variational principle \( \delta J = 0 \) yields Eqs.(5.3 - 4) and (5.3 - 9) as the corresponding Euler-Lagrange equations. Equation (5.3 - 3), however, is not retrieved in this case even though it is used during the construction of the stationary expression.

The stationary formulation (5.3 - 10) can be greatly simplified if each surface point is vibrating either in phase or 180° out of phase with other points such as
in the example of an axially oscillating finite cylinder. The complex conjugate of
the function \( f_n^* \) is then \( \mathcal{K} f_n \), where \( \mathcal{K} \) is a complex constant of magnitude unity.
Accordingly, from the linearity of Eq.(5.3-9), one has \( \tilde{p}_{aux} = \mathcal{K}\tilde{p} \). Hence the
functions \( \tilde{p} \) and \( \tilde{p}_{aux} \) need not be varied separately in the variational principle.
Instead, one can substitute \( \tilde{p}_{aux} \) with \( \mathcal{K}\tilde{p} \) in Eq.(5.3-10) at the outset. Since one
can always choose the time origin such that \( \mathcal{K} \) is unity, no generality is lost if one
simply sets \( \tilde{p}_{aux} \) to \( \tilde{p} \) and \( f_{aux,n} \) to \( f_n \), with the stipulation that \( f_n \) is real (positive
or negative) at each point on the surface. Doing such yields

\[
J = \iint_S p(x_S) f_n(x_S) \, dS \\
+ \frac{1}{2\pi} \iiint_{S'} p(x_S) f_n(x_{S'}) \left[ n(x_S) \cdot \frac{x_{S} - x_{S'}}{R} \right] \frac{d}{dR} \left( \frac{e^{ikR}}{R} \right) \, dS' \, dS \\
+ \frac{k^2}{4\pi} \iiint_{S'} [n(x_S) \cdot n(x_{S'})] p(x_S)p(x_{S'}) G(x_S \mid x_{S'}) \, dS' \, dS \\
- \frac{1}{4\pi} \iiint_{S'} [n(x_S) \times \nabla p(x_S)] \cdot [n(x_{S'}) \times \nabla' p(x_{S'})] G(x_S \mid x_{S'}) \, dS' \, dS \\
+ \frac{1}{4\pi} \iiint_{S'} f_n(x_S)f_n(x_{S'}) G(x_S \mid x_{S'}) \, dS' \, dS
\]  \quad (5.3-11)

Note that the tildes in the above expression have been suppressed for brevity.
The corresponding Euler-Lagrange equation derived from \( \delta J = 0 \) becomes simply Eq.(5.3-4). The stationary expression given by Eq.(5.3-11) will suffice for
estimations of the acoustic pressure on the vibrating surface. The introduction of
the acoustic pressure \( p_{aux} \) for the auxiliary exterior problem into the expression for
\( J \) given by Eq.(5.3-10) is to guarantee that the estimated radiated acoustic power
be accurate to the second order.

In what follows, we calculate the quantity related to the total radiated acous-
tic power for the axially vibrating finite cylinder. As before we first nondimen-
sonalize Eq.(5.3-11) by scaling the surface pressure \( p \) by \( \rho_o c \nu_o \) and the distance
by the length of the finite cylinder $L$, then replace $p$ by preselected basis functions given by Eq.(4.2 - 3), with the resultant equation written as

$$\mathcal{J} = \sum_{n=1}^{N} B_n C_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} A_{nm} C_n C_m + (constant)$$  \hspace{1cm} (5.3 - 12)

where the (constant) corresponds to the last integral on the right side of (5.3 - 11) and elements $B_n$ and $A_{nm}$ are defined by Eqs.(2.1 - 41) and (2.1 - 42), respectively. All quantities in the above are understood to be dimensionless. Substituting $\mathcal{J}$ into Eq.(5.3 - 2) then gives the total radiated acoustic power

$$P_{\text{power}} = \frac{1}{2kL} \text{Im}(\mathcal{J})$$  \hspace{1cm} (5.3 - 13)

In the preceding section, we have computed the coefficients $C_n$ for the axially oscillating finite cylinder at the vicinity of the first eigenvalue $kL = \pi$. So here we substitute these values of $C_n$ into Eq.(5.3 - 12) for the stationary quantity $\mathcal{J}$ and then substitute $\mathcal{J}$ into Eq.(5.3 - 13) to calculate the quantity related to the total radiated acoustic power. The results (See Fig. 45) show that the calculated total radiated power is indeed stationary for all wavenumbers even when the surface pressure is not.

In the last part of this Chapter, we illustrate a numerical example to show that the stationary expression can yield accurate estimates with relatively little effort. Considered here is the total radiated acoustic power from the transversely oscillating unbaffled circular rigid thin disk. Since the disk is infinitesimally thin and since the surface pressure $p$ and normal component of the surface velocity $v_n$ are perfectly antisymmetric in regard to the front and back sides of the disk, the
surface integrals in the stationary expression (5.3 - 1) for the radiated power can be extended over only the front side of the disk.

\[ J = 8\pi \int_0^1 p(r) f_n(r) r \, dr 
+ 2(k\alpha)^2 \int_0^1 \int_0^{2\pi} p(r)p(r') \, G \, rrr' \, dr \, dr' 
- 2 \int_0^1 \int_0^{2\pi} \frac{dp(r)}{dr} \frac{dp(r')}{dr'} \, G \cos \theta \, rrr' \, dr \, dr' \]  
(5.3 - 14)

Under the assumed transverse oscillations, the velocity amplitude \( v_n \) is unity for the entire front surface of the disk. The surface acoustic pressure \( p \) in the above is expanded in terms of the preselected basis functions

\[ p(r) = \sum_{n=1}^{N} C_n r^{2(n-1)} \sqrt{1 - r^2} \]  
(5.3 - 15)

Since the coefficients \( C_n \) have been solved in Chapter III, the function \( p(r) \) is known. Substituting \( p(r) \) into Eq.(5.3 - 14) then gives the estimate of the total radiated acoustic power.

Computations for the power and the surface pressure evaluated at the center of the disk \( r = 0 \) are carried out for \( ka = 3 \) and \( ka = 5 \), respectively, and tabulated as a function of the number of basis functions \( N \). The results are normalized to the values at \( N = 6 \). Such a normalization is appropriate since the convergences shown in Table 3 and Table 4 are actually complete at \( N = 6 \). Table 3 and 4 depict that for \( N = 1 \), errors in the estimate of the radiated power are about 5%, while errors in the estimate of the surface pressure at the center of the disk are off by more than 25% and a factor of 2, respectively. At \( N = 3 \), errors in power is as small
as $3 \times 10^{-6}$ and $16 \times 10^{-4}$, while errors in pressure is approximately $3 \times 10^{-4}$ and $14 \times 10^{-2}$, respectively, the latter being nearly 100 times as large as the former.

Another interesting feature exhibited in Table 3 and 4 is that both the total radiated acoustic power and the surface acoustic pressure oscillate as they converge toward the limiting values. This feature confirms the fact that the variational principles for the total radiated acoustic power (5.3 - 1) and for the surface acoustic pressure (2.1 - 36) are neither maximum nor minimum, but just stationary principles.
Table 3. Total radiated acoustic power and surface acoustic pressure evaluated at the center of the transversely oscillating unbaffled circular rigid thin disk at $ka = 3$.

The surface pressure $p$ is based on basis functions

$$C_n r^{2(n-1)} \sqrt{1-r^2} \quad n = 1, 2, \ldots, N.$$

The total radiated power and surface pressure are normalized to the values of $N = 6$.

| $N$ | $|p(0)|$ | $P_{\text{power}}$ |
|-----|--------|-------------------|
| 1   | 0.782099 | 1.055820          |
| 2   | 0.975671 | 1.001071          |
| 3   | 0.999271 | 0.999993          |
| 4   | 1.000222 | 1.000000          |
| 5   | 1.000130 | 1.000000          |
| 6   | 1.000000 | 1.000000          |
Table 4. Total radiated acoustic power and surface acoustic pressure evaluated at the center of the transversely oscillating unbaflled circular rigid thin disk at $ka = 5$.

The surface pressure $p$ is based on basis functions

$$C_n r^{2(n-1)} \sqrt{1 - r^2} \quad n = 1, 2, \ldots, N.$$  

The total radiated power and surface pressure are normalized to the values of $N = 6$.

| $N$ | $|p(0)|$ | $P_{\text{power}}$ |
|-----|---------|-----------------|
| 1   | 2.306093 | 1.051240        |
| 2   | 1.948102 | 0.951187        |
| 3   | 1.137665 | 0.998393        |
| 4   | 1.009615 | 0.999957        |
| 5   | 0.999587 | 1.000000        |
| 6   | 1.000000 | 1.000000        |
CHAPTER VI

CONCLUSIONS AND DISCUSSIONS

The research work reported here is the continuation of the present author's MS Thesis. The variational principles previously derived from the Kirchhoff-Helmholtz integral theorem are extended to calculate the acoustic pressure on the surface of an axisymmetric body in a general state of axisymmetric vibrations. The computed surface pressure and the given normal surface velocity are then taken as inputs into the Kirchhoff-Helmholtz integral relation for determination of the far-field radiation patterns. The formulations developed here also apply, with a suitable reinterpretation of quantities, to the problem of scattering of sound from an axisymmetric rigid body when the incident wave propagates parallel to the axis or emanates from a point source on the axis.

Although no assumptions on the shape of the vibrating body are made in the derivations of the variational formulations, numerical examples studied here are all confined to bodies of axisymmetry. Such a confinement of one's attention to axisymmetric cases in the initial exploratory implementations is desirable, because then one can develop insight and relevant experience on the uses and limitations of variational techniques with modest and more accessible computational sources. Due to the axisymmetry, one of the angular integrations involved in the double surface integrals can be done trivially. Accordingly, the initially four-fold integrations inherent to the variational formulation are reduced to three-fold integrations, one integration over the angle and two integrations over coordinates corresponding to linear distance along the generator of the axisymmetric surface. Furthermore, since
the angular integration is independent of the acoustic pressure and velocity profile distributions on the surface, this integral can be carried out at the outset, with the results stored and used over and over again in a series of calculations, as if it were a completely tabulated standard function. Hence, the subsequent applications only require a two-fold integration, rather than a four-fold integration. The numerical computations are thus significantly simplified.

When the Rayleigh-Ritz technique is incorporated in the variational principle, the unknown surface pressure is expressed as an expansion in terms of basis functions. The coefficients of the basis functions are determined by a system of simultaneous equations resulting from the variational principle. In general one needs a complete set of basis functions and an infinite number of terms to guarantee an exact answer. However, if an approximate answer is desired, it may be possible to use a relatively small number of basis functions and yet achieve the desired accuracy. Such instances are given in this thesis.

Specific examples studied in this thesis include the transversely oscillating unbaflled circular rigid thin disk and finite circular cylinder in axisymmetric vibrations. These examples demonstrate a number of features which support the contention that a variational principle may greatly improve our capabilities for numerical solutions of acoustic radiation and scattering problems. Because there is a wide latitude in the selection of basis functions, one can use whatever insight, intuition, or experience one has regarding the problem at hand to shorten the overall computational effort.

Here, for example, we know at the outset the form of the pressure distribution near the edge of the disk, and we take the advantage of this knowledge. We also know the exact solution in one limiting case, that of $ka \to 0$, so we take our first basis function to be of the correct form in this limit. (It should be pointed out here that
such basis functions are not appropriate for high $ka$ values, for example, $ka > 10$.)
The effect of this judicious selection of basis functions for the unknown surface pressure is that we can improve upon results previously computed by considerably more arduous methods with 5 or fewer basis functions. The same is true in the case of an axially vibrating finite cylinder for which an excellent agreement with the results computed by using the SHIP program is achieved with 2 basis functions. For the general trigonometrical functions as admissible trial functions, the computed surface acoustic pressure distributions compare reasonably well with those obtained by other numerical methods.

While numerical examples demonstrated in this thesis are restricted to axisymmetric cases and full advantages is taken of such symmetry (as should any well-thought-out implementations of alternative formulations), the necessity of doing higher order integrations (four-fold integrations) for non-axisymmetric cases should not discourage use of the variational formulation. Gaussian quadrature techniques exist for multi-dimensional integrals and the approximate results, regardless of number of dimensions, are just judiciously weighted sums of values of the integrands at a finite number of judiciously selected points. Contrary to what might be one’s initial naive thinking (based on, say, a casual generalization of the trapezoidal rule) on the subject, the number of required terms in the sum to achieve a given accuracy does not increase exponentially with the number of dimensions. One may speculate at the outset (although a proof is beyond the scope of the present thesis) that the number of arithmetic operations required to achieve a given overall accuracy of the surface pressure is roughly the same for all competently devised implementations of various competing formulations of the problem. The intrinsic advantage of the variational formulation is that one can do much better than a priori expected, providing that the right basis functions are selected. Insofar as one
can codify the thought processes that an expert would make in selecting such basis functions, then the variational principle formulations should result in faster calculations. The tradeoff is that the algorithms which embody these thought processes may result in lengthier programs. The author's viewpoint is that overall trends in computational hardware make the development of lengthier but faster running programs worthwhile.

Discussed in the last part of the thesis is the uniqueness of solutions to the variationally formulated acoustic radiation problems. Because the variational principles developed here take the conventional Kirchhoff-Helmholtz integral theorem as their bases, estimations of surface pressure distributions based on the variational formulations are shown to be nonunique when the wavenumber $k$ corresponds to the interior Neumann eigenfrequencies $k_N$. An illustrative numerical example is the finite circular cylinder oscillating back and forth in the axial direction. However, such a nonuniqueness does not occur in cases of disks or plate-like bodies because the lowest eigenvalue approaches infinity when the thickness of the disk or plate becomes infinitely small. It is also demonstrated that the variationally formulated total radiated acoustic power is always unique for all frequencies, even though the surface pressure is not unique. Estimations of the total radiated power are also shown to be much more accurate than those of the surface pressure distributions, because the error term in the former is at least one order higher than that in the latter.

The variational formulations derived in this thesis can be extended easily to estimate the acoustic pressure distribution on other surfaces, for example the oblate spheroids and prolate spheroids. For a smooth surface with continuous or piece-wise continuous tangential derivatives, one in general does not need a large number of basis functions (which corresponds to a big system of simultaneous equations) to
approximate the unknown surface pressure distribution. Also, since the Gaussian quadrature formulation used in the numerical integration is of a high accuracy, one can obtain an accurate numerical integration with relatively fewer integration subdivisions. For example, the numerical examples (See Chapters III and IV) show that 5 trigonometric functions (selected as examples of arbitrary admissible trial functions) and 10 integration subdivisions will enable one to obtain reasonably stationary numerical results, and a further increase in the number of basis functions does not appreciably improve the numerical results. On the other hand, good approximations are achieved at the first place with fewer basis functions which incorporate one's physical insight and prior knowledge. Hence the selection of the basis functions is crucial for one to obtain a good approximation of the surface pressure distribution with relatively less computational time and effort.

In so far as the acoustic radiation problems are concerned, the variational formulations take the surface velocity profile as given, while the acoustic pressure distribution on the vibrating surface is the only unknown variable. Once this has been done, one may extend the present variational formulations to couple the variational principles to describe the structural vibrations for acoustic radiation problems in which both the surface velocity and pressure distributions are to be found. The combined variational formulations will then be of generality and greater importance because they take into account of the elasticity of the vibrating object. The results will be a highly accurate, yet computationally simple, description of fluid-structure interaction.
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Stationary Variational Expressions for Radiated and Scattered Acoustic Power and Related Quantities

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Stationary Variational Expressions for Radiated and Scattered Acoustic Power and Related Quantities

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Abstract—The construction of stationary expressions for quantities of physical interest such as radiated power and target strength is discussed broadly for acoustic problems involving radiation or scattering from finite objects of arbitrary shape. The Kirchhoff–Helmholtz integral corollaries of the wave equation, which express acoustic pressure at either interior or exterior points in terms of pressure and its normal derivative over any closed surface, yield for both interior and exterior problems two mathematically dissimilar but related functional relations between surface field quantities. One of these is the better known surface Helmholtz integral equation; the other is a differential-integral relation which involves the tangential derivatives of pressure on the surface. The four linear operators involved in these functional relations are studied and it is found that two are self-adjoint, while the other two are an adjoint pair. A general technique for constructing variational expressions recently developed by Gerjuoy et al. [28] is adapted to acoustic radiation and scattering problems with the functional relations taken as the primary governing relations. Included examples are stationary expressions for the radiated power when either the normal velocity or the pressure are specified on the surface (the other quantity being unknown) and the target strength for scattering from a rigid object. The adjoint relations allow simple physical interpretations for the Lagrange multipliers that arise in the theory, such that the guesses for good trial functions can take advantage of existing physical insight. It is demonstrated with a specific example (transversely vibrating disk) that the resulting estimate for radiated power is substantially more accurate than that of the trial function for surface pressure which was inserted into the stationary expression.

NOTE

Some of the symbols in this paper denote vectors. The context should indicate whether or not a vector is designated; in particular, \( \xi, \eta, x, \pi, \) and \( v \) represent vectors unless otherwise stated.

I. INTRODUCTION

VARIATIONAL formulations of physical problems are often convenient departure points for introducing approximations; they help to maintain a degree of self-consistency that might otherwise be absent. Also, they offer some assurance that, when one seeks to approximate a solution by a function of restricted (but initially incompletely specified) type with, for example, various adjustable parameters, the variational formulation will automatically select, from among all such functions of such a type, one which in some sense is optimal.

A disconcerting feature of many variational formulations, however, is that one often does not necessarily know at the outset the actual sense in which such an approximate solution is optimal. Nevertheless, it is a strong philosophical tenet among many theoretical physicists, engineers, and applied mathematicians that, all other things being seemingly equal and provided the variational formulation was constructed in a natural manner, without any artificial mathematical contrivances, the approximation yielded by the variational formulation is vastly to be preferred.

There is a class of variational principles, however, whose appeal does not rest on aesthetic considerations alone. These are those of the generic form \( \delta J = 0 \) where the stationary quantity \( J \) is a measurable physical quantity \( J \). Rayleigh's principle [1] for vibrations, in which \( J \) is the square of the natural frequency, is perhaps the best known of such variational principles. Others are Fermat's principle of stationary travel time for optical [2] and acoustical [3] ray paths and the principle of minimum potential energy [4] in statics. Ideally, one would prefer principles that enable one to "keep score" on the goodness of alternative approximations. For example, when applying Rayleigh's principle to determine the lowest natural frequency and the fundamental mode's shape, one takes the "best" choice as being that approximate trial function which, when inserted into the variational expression, yields the lowest estimate for \( \omega^2 \), and the latter is indisputably an upper bound to the actual lowest \( \omega^2 \). True minimum or maximum principles are relatively scarce, and are probably nonexistent for wave propagation problems, except in the limit of extremely low frequencies. Nevertheless, stationarity principles do retain some features that could be of considerable interest to one seeking improved algorithms for physical problems.

One might, for example, place high priority on the accurate estimate of the physical quantity \( J \) which corresponds to the variational expression \( J \). The latter could be a functional of some other physical quantity \( \psi(\eta) \) that depends on some position parameter \( \eta \). One might have some insight into the nature of the unknown function \( \psi(\eta) \) and use this insight and the variational principle to refine the quantitative estimate of the function. The resulting approximation to the actual \( \psi(\eta) \) might be characterized as being "good but not great," or simply as being "mediocre." However, this mediocre estimate for \( \psi(\eta) \), when inserted into the variational expression, should result in a highly accurate estimate of the physical quantity \( J \). The reason for this is that the error in \( J \) should be of second order when the error in \( \psi(\eta) \) is taken as first order.

Variational principles with the feature just described were developed for a number of wave propagation applications during the 1940's and 1950's by Schwinger, Levine, Liepmann, and others [5]-[9]; a good summary of such can be found in the second volume of Morse and Feshbach's 1953 treatise [10]; a number of papers [11]-[16] appeared that were in this general spirit and specifically directed toward acoustical applications. However, the enthusiasm within the acoustical
community for such methods waned considerably during the subsequent two decades [17], presumably because of the computational efforts involved and the formidable sophistication of the analytical techniques that had been used in prior implementations of such variational formulations. Another factor was undoubtedly the widespread belief that anything accomplishable using a variational formulation could be equally well accomplished using Galerkin's method [18]. (Debating this point would unduly lengthen the present paper.)

In regard to acoustic radiation and scattering by bodies of nonstandard shape, the variational principles commonly presented in the literature up until relatively recently had integrals over singular integrands, and the procedure for direct numerical evaluation seemed undefined [19]. In retrospect, however, this was easily circumventable [20] using techniques applied by Maue [21] and Stallybrass [22] in earlier formulations of integral equations for similar problems; variational principles not involving singularity difficulties were independently developed by Hamdi [23] and by the present author [24], and were used in a number of applications by Ginsberg, Wu, and others [25]-[27]. However, these were not associated at the outset with the stationarity of a desirable physical quantity, so the question of the sense in which the variational solution is optimal remained unanswered.

Relatively recently, W. Möhring called the author's attention to a 1983 paper (which is actually the culmination of a long series of papers) by Gerjuoy et al. [28] that provides the key to the answer to the question just posed and which, moreover, gives a general technique for constructing variational principles that allow accurate estimation of any physical quantity of interest. The present paper shows how this technique can be applied to the formulation of novel and practical variational principles for acoustic radiation and scattering.

(Variational principles are used in a variety of contexts in the physical sciences; that to which many readers are primarily exposed during their professional education is that of giving a succinct—and possibly more fundamental—statement of the mathematical relations governing some phenomenon. For example, Hamilton's principle in classical mechanics, via the Euler theorem, yields the familiar Lagrange's equations. As succinct—and possibly more the physical sciences; that to which many readers are primarily scattering.

With any given closed surface $S$ one may in general associate a number of different types of acoustic radiation and scattering problems. Although one may be interested in solving only one such problem, some knowledge of related problems involving the same surface may be helpful in producing a good approximation for the problem of interest. In general, one distinguishes interior and exterior problems [29]. For the former, the complex amplitude of the acoustic pressure $p_{\text{int}}$ satisfies the scalar Helmholtz equation throughout the volume $V$ enclosed by $S$. For exterior problems, the complex amplitude $p_{\text{ext}}$ satisfies the Helmholtz equation throughout the region external to $S$. In many cases such exterior problems can be idealized such that the external region is unbounded in all directions; even when there are external boundaries, the solution for an unbounded external environment may be a valuable building block in the construction (using ray concepts or the method of images) of the solution for the bounded external region. The present discussion assumes, for simplicity, that the external region is unbounded.

Exterior problems can be classified as radiation or scattering problems. For radiation problems, the complex pressure amplitude $p_{\text{ext}}$ satisfies the Sommerfeld radiation condition at great distances from the source. For scattering problems, $p_{\text{ext}}$ is regarded as a sum of an incident part $p_{\text{ext, inc}}$ and a scattered part $p_{\text{ext, sc}}$, where the former is taken as given throughout the external region, while the latter satisfies the Sommerfeld radiation condition. Consequently, general statements that one might make for radiated fields apply equally well to scattered fields. For brevity, the expression $p_{\text{ext}}$ is understood throughout the present section to refer to either the radiated field for a radiation problem or the scattered field for a scattering problem.

Following well-known procedures [30], one can derive Kirchhoff-Helmholtz integral corollaries to the wave equation for both the interior problem and the exterior problem. These can be written in such a form that they relate the appropriate acoustic pressure $p$ on the surface to a quantity $f$ which is the negative of the outward normal component of the gradient of $p$ at the surface. Alternately, because of Euler’s equation of motion and the implicit use of $e^{-iat}$ time dependence, one has

$$-i\omega u_\xi(\xi) = -\nabla_\xi p(\xi)$$

and

$$f = -i\omega v_\xi$$

where $u_\xi(\xi)$ is the normal component $n(\xi) \cdot u(\xi)$ of the complex fluid velocity vector amplitude $u(\xi)$ at the surface. One can regard $f$ as a convenient grouping of symbols, either as a constant times the normal velocity, or as a constant times the normal acceleration, or as the normal component of the apparent body force, per unit volume, exerted on the fluid at the surface. Because of the latter identification, the use of the symbol \[ f \] seems appropriate.

For the interior problem one has

$$\text{Re} \{ x, p_{\text{int}} f_{\text{int}} \} = \begin{cases} -p_{\text{int}}(x) & \text{if } x \text{ inside } V \\ 0 & \text{if } x \text{ outside } V \end{cases}$$

1 Note, for example, the small number of papers in the Special Session on variational Techniques in Acoustics at the Spring 1970 meeting of the Acoustical Society of America.
while for the exterior problem one has
\[ \mathcal{M} \{ x, p_{\text{ex}}, f_{\text{ex}} \} = \begin{cases} p_{\text{ex}}(x) & \text{if } x \text{ outside } V \\ 0 & \text{if } x \text{ inside } V \end{cases} \] (3)

Both (2) and (3) use the abbreviation
\[ \mathcal{M} \{ x, p, f \} = \frac{1}{4\pi} \int \left[ f(\xi) G(x|\xi) \right] \cdot \nabla G(x|\xi) \, dS_{\xi} \] (4)

where
\[ G(x|\xi) = \frac{e^{ikR}}{R} \] (5)

is the so-called free-space Green's function with
\[ R = |x - \xi| \] (6)
denoting the distance between "source" and "receiver" points. In the integrand of (4), the point \( \xi \) (after evaluation of any requisite normal derivatives) is understood to range over the surface \( S \), with the point \( x \) held fixed during the integration. The unit outward normal vector \( \eta(\xi) \) points out of the enclosed volume \( V \) at the surface point \( \xi \).

In (4) one should note that the integral \( \mathcal{M} \) is a function of the point \( x \), but a functional (function of a function) of the function arguments \( p \) and \( f \).

From either (2) or (3), one can derive two types of relations (distinguished by subscripts I and II) between the surface values of \( p \) and \( f \). One such type of relation results when the off-surface point \( x \) is allowed to approach an arbitrary but fixed surface point \( \eta \). For one of the terms in the integral defining the quantity \( \mathcal{M} \), the limit as \( x \) approaches \( \eta \) of the integral is not the same as the integral over the limit of the integrand as \( x \) approaches \( \eta \). Instead one has
\[ \lim_{x \to \eta} \int f(\xi) n(\xi) \cdot \nabla G(x|\xi) \, dS_{\xi} = \pm 2\pi p(\eta) + \int f(\xi) n(\xi) \cdot \nabla G(\eta|\xi) \, dS_{\xi} \] (7)
where
\[ x_{\xi} = \eta + \epsilon n(\eta). \] (8)
The plus sign in (7) applies if the limit is taken with \( \epsilon \) kept positive during the process, while the minus sign applies if it is kept negative during the process. With this subtlety taken into account, one obtains
\[ p_{\text{in}}(\eta) + \mathcal{L}_{\text{I}} \{ \eta, p_{\text{in}} \} = -3\mathcal{C}_{\text{I}} \{ \eta, f_{\text{ex}} \} \] (9)
\[ p_{\text{ex}}(\eta) - \mathcal{L}_{\text{I}} \{ \eta, p_{\text{ex}} \} = 3\mathcal{C}_{\text{I}} \{ \eta, f_{\text{ex}} \} \] (10)

and
\[ \mathcal{L}_{\text{II}} \{ \eta, p \} = \frac{1}{2\pi} \int f(\xi) n(\xi) \cdot \nabla G(\eta|\xi) \, dS_{\xi} \] (11)
\[ 3\mathcal{C}_{\text{II}} \{ \eta, f \} = \frac{1}{2\pi} \int f(\xi) G(\eta|\xi) \, dS_{\xi} \] (12)
such that the symbols \( \mathcal{L}_{\text{I}} \) and \( \mathcal{C}_{\text{I}} \) can be regarded as linear operators which operate on the surface values of \( p \) and \( f \), respectively, with the resultant in each case being a function of the position of the surface point \( \eta \).

The second type of surface relationship is obtained by taking the gradient of both sides of (2) or (3), subsequently setting \( x \) to \( \eta + \epsilon n(\eta) \), where \( \eta \) is an arbitrary point on the surface, taking the dot product with \( n(\eta) \), then taking the limit as \( \epsilon \) goes to zero. To express the limit of one of the integrals as the integral of a limit, it is appropriate to make use of a relation previously derived by Maue [21] and by Stallybrass [22], holding for \( \epsilon \neq 0 \), to the effect that
\[ (n(\eta) \cdot \nabla_{\eta}) (n(\xi) \cdot \nabla_{\xi}) G(x|\xi) = k^2 n(\eta) \cdot n(\xi) G(x|\xi) \] (13)

(The derivation makes use of the symmetry of the Green's function and of the fact that the Green's function satisfies the Helmholtz equation for \( x \neq \xi \).) Moreover, a version of Stokes' theorem allows a type of integration by parts, by which the operator \( n(\xi) \times \nabla_{\xi} \) is transferred, with the usual accompanying change in sign, from \( G(x|\xi) \) to \( p(\xi) \). A relation analogous to (7) is also used to transform one of the terms. In such a manner, one obtains the relations
\[ \mathcal{L}_{\text{II}} \{ \eta, p_{\text{in}} \} = f_{\text{in}}(\eta) - 3\mathcal{C}_{\text{II}} \{ \eta, f_{\text{ex}} \} \] (14)
\[ -\mathcal{L}_{\text{II}} \{ \eta, p_{\text{ex}} \} = f_{\text{ex}}(\eta) + 3\mathcal{C}_{\text{II}} \{ \eta, f_{\text{ex}} \} \] (15)

where
\[ \mathcal{L}_{\text{II}} \{ \eta, p \} = (n(\eta) \times \nabla_{\eta}) \cdot \frac{1}{2\pi} \int (n(\xi) \times \nabla_{\xi} p(\xi)) G(\eta|\xi) \, dS_{\xi} \] (16)
\[ + \frac{k^2}{2\pi} \int n(\eta) \cdot n(\xi) p(\xi) G(\eta|\xi) \, dS_{\xi} \]
\[ 3\mathcal{C}_{\text{II}} \{ \eta, f \} = \frac{1}{2\pi} \int f(\xi) n(\eta) \cdot \nabla_{\xi} G(\eta|\xi) \, dS_{\xi}. \] (17)

In regard to (16), one should note that the operator \( n(\eta) \times \nabla_{\eta} \) involves only derivatives tangential to the surface, so that the integral on which it acts needs only be evaluated at surface points \( \eta \).

III. ADJOINT OPERATORS

Having now identified characteristic surface operators \( \mathcal{L}_{\text{I}}, \mathcal{L}_{\text{II}}, \mathcal{C}_{\text{I}}, \mathcal{C}_{\text{II}} \), it is appropriate for the derivations given further below involving variational expressions to identify their corresponding adjoint operators. The notation of the present paper distinguishes such by the superscript \( \dagger \); their definitions are such that if \( \Phi(\eta) \) and \( \Psi(\eta) \) are functions defined over the surface \( S \), then, for example
\[ \int \Psi(\eta) \mathcal{L}_{\text{I}} \{ \eta, \Phi \} \, dS_{\eta} = \int \Phi(\eta) \mathcal{L}_{\text{I}}^{\dagger} \{ \eta, \Psi \} \, dS_{\eta}. \] (18)

Using such a definition of an adjoint operator, one identifies
What is desired is that, when slightly erroneous functions are inserted into (24) to calculate \( j \), the error in the consequently calculated power via (25) be at most only of second order in the variations of \( p e \). The slightly erroneous functions are those which are here designated as \( P e \). The error in the power will be of second order if the expression on the right side of (24) is stationary under independent variations of these three quantities. This requirement, with application of the techniques of the calculus of variations, yields the three equations

\[
\begin{align*}
L_1^t \{ \eta, p \} &= \frac{1}{2\pi} \int \int \rho(\xi) n(\eta) \cdot \nabla_e G(\xi | \eta) \, dS_t = 3C_1^t \{ \eta, p \} \\
3C_1^t \{ \eta, f \} &= \frac{1}{2\pi} \int \int f(\xi) G(\xi | \eta) \, dS_t = 3C_1 \{ \eta, f \} \\
L_1^t \{ \eta, p \} &= (n(\eta) \times \nabla_e) \cdot \frac{1}{2\pi} \int \int (n(\xi) \times \nabla_e p(\xi)) G(\xi | \eta) \, dS_t \\
&\quad + \frac{k^2}{2\pi} \int \int n(\eta) \cdot n(\xi) p(\xi) G(\xi | \eta) \, dS_t \\
&= L_1^t \{ \eta, f \}. 
\end{align*}
\] (19)

Thus the operators \( C_1^t \) and \( L_1^t \) are self-adjoint, while \( C_1^t \) and \( L_1^t \) are adjoints of each other (an adjoint pair).

IV. STATIONARY EXPRESSION FOR RADIATED POWER

A quantity often of interest is the net radiated power for a radiation problem or the net scattered power for a scattering problem. With the nomenclature and conventions introduced in the previous sections, this can be expressed as one-half of the real part of the surface integral of \( p_{ext}(n)u_{ext}(n) \) or as

\[
\text{Power} = \frac{1}{4\omega^2} \int \int \left[ p_{ext} f_{ext} - p_{ext}^* f_{ext} \right] \, dS_t. 
\] (23)

What follows, the method of Gerjuoy et al. [28] is adopted to derive variational expressions that will yield accurate estimates of this quantity when one or the other of the quantities \( f_{ext} \) or \( p_{ext} \) is specified on the surface \( S \). For scattering problems, \textit{a priori} knowledge of \( f_{ext} \) on the surface would be appropriate for the problem of scattering by a perfectly rigid body, while \textit{a priori} knowledge of \( p_{ext} \) on the surface would be appropriate for scattering by a perfectly soft (where sum of incident and scattered pressure vanishes on \( S \)).

To construct an appropriate stationary expression which elds the radiated power when \( f_{ext}(n) \) is known at the outset, one sets

\[
\begin{align*}
\mathcal{J} &= \int \int \rho_{ext}(n) f^*(n) \, dS_t \\
&\quad + \int \int \Phi_t(n) \left[ \rho_{ext}(n) - L_1^t \{ \eta, \rho_{ext} \} - 3C_1^t \{ \eta, f \} \right] \, dS_t \\
&\quad + \int \int \Phi_t(n) \left[ L_1^t \{ \eta, \rho_{ext} \} + f(n) + 3C_1^t \{ \eta, f \} \right] \, dS_t. 
\end{align*}
\] (24)

where \( \rho_{ext}(n) \) is a trial function for \( p_{ext}(n) \), while \( \Phi_t \) and \( \Phi_t \) are regarded as trial functions for variables \( \Phi_t(n) \) and \( \Phi_t(n) \). The untilded \( \Phi \)'s will eventually be defined as the exact (although \textit{a priori} unknown) solutions of related problems. Note that the subscript "ext" has been dropped from \( f_{ext} \); since \( f(\eta) \) is here a prescribed boundary value, it is not necessarily associated with the exterior region.

If \( \rho_{ext} \) is identically \( p_{ext} \), then the second and third terms on the right side of (24) vanish because of (10) and (15), and the radiated power is

\[
\text{Power} = \frac{1}{2\omega^2} \text{Im} \{ \mathcal{J} \}. 
\] (25)

What is desired is that, when slightly erroneous functions are inserted into (24) to calculate \( \mathcal{J} \), the error in the consequently calculated power via (25) be at most only of second order in the variations of \( p_{ext}, \Phi_t, \) and \( \Phi_t \). The slightly erroneous functions are those which are here designated as \( P_{ext} \), \( \Phi_t \), and \( \Phi_t \). The error in the power will be of second order if the expression on the right side of (24) is stationary under independent variations of these three quantities. This requirement, with application of the techniques of the calculus of variations, yields the three equations

\[
\begin{align*}
\rho_{ext}(n) - L_1^t \{ \eta, \rho_{ext} \} - 3C_1^t \{ \eta, f \} &= 0 \\
L_1^t \{ \eta, \rho_{ext} \} + f(n) + 3C_1^t \{ \eta, f \} &= 0 \\
f^*(n) + \Phi_t(n) - L_1^t \{ \eta, \Phi_t \} + L_1^t \{ \eta, \Phi_t \} &= 0. 
\end{align*}
\] (26a) (26b) (26c)

The first two of the above three equations are the same as (10) and (15); the development of Section II shows that they are mathematically consistent (not contradictory). This implies that one is at liberty to choose either \( \Phi_t \) or \( \Phi_t \) to be zero, or to impose any independent relation between these two quantities.

Ideally, one would want the unknown functions that are used in the estimate of the radiated power to have tangible physical interpretations, for such would facilitate their estimation. To this purpose, one rewrites (26c) using the adjoint relations (19) and (20) as

\[
\begin{align*}
f^*(n) + \Phi_t(n) - 3C_1^t \{ \eta, \Phi_t \} + L_1^t \{ \eta, \Phi_t \} &= 0. 
\end{align*}
\] (27)

Then, after a comparison with (14) and (15), a simple choice becomes apparent:

\[
\begin{align*}
\Phi_t(n) &= -\frac{1}{2} f^*(n) \\
\Phi_t(n) &= \frac{1}{2} \rho_{ext, aux}(n) 
\end{align*}
\] (28a) (28b)

where \( \rho_{ext, aux}(x) \) is the solution of an auxiliary exterior problem with the surface value of \( f \) specified as

\[
\rho_{ext}(n) = f^*(n). 
\] (28c)

Given the choice described by (28a)-(28c), the expression for the stationary quantity \( \mathcal{J} \) reduces (omitting the tildes and
the subscript "ext" for brevity) after some manipulations to

\[ \delta \mathcal{J} = \frac{1}{2} \int \{ P(\eta)f_{\text{aux}}(\eta) + p_{\text{aux}}(\eta)f(\eta) \} \, dS_{\eta} \]

\[ + \frac{1}{4\pi} \int \int \int \{ P(\eta)f_{\text{aux}}(\xi) + p_{\text{aux}}(\eta)f(\xi) \} n(\eta) \]

\[ \cdot \nabla_{\eta} G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} - \frac{1}{4\pi} \int \int \int (n(\eta) \times \nabla_{\eta} p_{\text{aux}}(\eta)) \]

\[ \cdot (n(\xi) \times \nabla_{\xi} p(\xi))G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} \]

\[ + \frac{k^2}{4\pi} \int \int \int n(\eta) \cdot n(\xi) p_{\text{aux}}(\eta)p(\xi)G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} \]

\[ + \frac{1}{4\pi} \int \int \int f_{\text{aux}}(\eta)f(\xi)G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} \]

(29)

In the variation principle \( \delta \mathcal{J} = 0 \) that corresponds to (29), the functions \( p \) and \( p_{\text{aux}} \) are to be varied independently, so the corresponding Euler-Lagrange equations are

\[ f_{\text{aux}}(\eta) + 3C_{\text{II}} \{ \eta, f_{\text{aux}} \} + \mathcal{L}_{\text{II}} \{ \eta, p_{\text{aux}} \} = 0 \]

(30a)

\[ f(\eta) + 3C_{\text{II}} \{ \eta, f \} + \mathcal{L}_{\text{II}} \{ \eta, p \} = 0 \]

(30b)

both of which are of the form of (15). Note that one does not recover (10), even though it was used during the construction.

If the acoustic pressure \( p_{\text{aux}}(\eta) \) (rather than \( f_{\text{aux}}(\eta) \)) is specified on the surface \( S \) at the outset and \( f_{\text{ext}}(\eta) \) is unknown, an analogous stationary expression for the radiated power can be similarly constructed. Such a development yields

\[ \text{Power} = \frac{1}{2\omega p} \text{Im} \{ \mathcal{J} \} \]

(31)

\[ \mathcal{J} = - \frac{1}{2} \int \{ P(\eta)f_{\text{aux}}(\eta) + p_{\text{aux}}(\eta)f(\eta) \} \, dS_{\eta} \]

\[ + \frac{1}{4\pi} \int \int \int \{ P(\eta)f_{\text{aux}}(\xi) + p_{\text{aux}}(\eta)f(\xi) \} n(\eta) \]

\[ \cdot \nabla_{\eta} G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} - \frac{1}{4\pi} \int \int \int (n(\eta) \times \nabla_{\eta} p_{\text{aux}}(\eta)) \]

\[ \cdot (n(\xi) \times \nabla_{\xi} p(\xi))G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} \]

\[ + \frac{k^2}{4\pi} \int \int \int n(\eta) \cdot n(\xi) p_{\text{aux}}(\eta)p(\xi)G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} \]

\[ + \frac{1}{4\pi} \int \int \int f_{\text{aux}}(\eta)f(\xi)G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} \]

(32)

where

\[ p_{\text{aux}}(\eta) = P^*(\eta) \]

(33)

and \( \mathcal{J} \) is stationary to first order when either \( f(\eta) \) or \( f_{\text{aux}} \) are varied.

Note that the sole difference between the form of (33) and that of (29) is the sign of the first term. The Euler-Lagrange equations ensuing from \( \delta \mathcal{J} = 0 \) are considerably different, however, these being

\[ p_{\text{aux}}(\eta) - \mathcal{L}_{\text{I}} \{ \eta, p_{\text{aux}} \} - 3C_{\text{I}} \{ \eta, f_{\text{aux}} \} = 0 \]

(34a)

\[ p(\eta) - \mathcal{L}_{\text{I}} \{ \eta, p \} - 3C_{\text{I}} \{ \eta, f \} = 0 \]

(34b)

and are of the form of (10) rather than of (15).

Returning to (29), which yields a variational expression for the acoustic pressure \( p(\eta) \), one may note that a considerable simplification results if each point on the surface is vibrating either exactly in phase or exactly 180° out of phase with other points. Then \( f^*(\eta) = Kf(\eta) \), where \( K \) is a complex constant of magnitude unity. It follows from the linearity of (30) that \( p_{\text{aux}} = Kp \), so one need not vary \( p \) and \( p_{\text{aux}} \) separately when using the variational principle. Instead, one can insert \( Kp \) for \( p_{\text{aux}} \) at the outset into (29). Since one can always choose the time origin such that \( K \) is unity, no generality is lost if one simply sets \( D = \text{aux} \) to \( p \) and \( f_{\text{aux}} \) to \( f \), with the stipulation that \( f(\eta) \) is real (positive or negative) at each point on the surface. Doing such yields

\[ \delta \mathcal{J} = \int \{ P(\eta)f(\eta) \} \, dS_{\eta} + \frac{1}{2\pi} \int \int \int p(\eta)f(\xi)n(\eta) \]

\[ \cdot \nabla_{\eta} G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} - \frac{1}{4\pi} \int \int \int (n(\eta) \times \nabla_{\eta} p(\eta)) \]

\[ \cdot (n(\xi) \times \nabla_{\xi} p(\xi))G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} \]

\[ + \frac{k^2}{4\pi} \int \int \int n(\eta) \cdot n(\xi) p(\eta)p(\xi)G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} \]

\[ + \frac{1}{4\pi} \int \int \int f(\eta)f(\xi)G(\eta|\xi) \, dS_{\xi} \, dS_{\eta} \]

(35)

The corresponding Euler-Lagrange equation derived from \( \delta \mathcal{J} = 0 \) is (15).

Insofar as one desires a variational principle (without any guarantee that the estimated radiated power be accurate to second order) for the acoustic pressure (for the exterior problem) on the surface, then that corresponding to (35) will suffice, regardless of whether or not \( f(\eta) \) is real; the introduction of the acoustic pressure \( p_{\text{aux}} \) for the auxiliary exterior problem into the expression for \( \mathcal{J} \) is necessary only if one desires that the radiated power estimate be of second order in the estimate of surface acoustic pressure. A simplified variational principle, equivalent to that of (35), has been previously derived by other methods independently by Hamdi [23] and the present author [24].

V. STATIONARY EXPRESSION FOR TARGET STRENGTH

Another example of interest to underwater acoustics is the calculation of target strength \( (TS) \), which is a type of logarithmic measure of the apparent area of a body in backscatter, the definition [30] being

\[ TS = 10 \ln \left( \frac{\sigma_{\text{BS}}}{\pi R^2} \right) \]

(36)
where $\sigma_{bs}$ is the backscattering cross section and $R_0$ is a reference length normally taken as 1 m. The backscattering cross section is $4\pi$ times the differential cross section in the backward direction, the latter being the limiting value of square of distance times intensity back toward the source, divided by the incident intensity at the scattering object. A value for the far-field backscattered intensity can be formally divided by the incident intensity at the scattering object. A value for the far-field backscattered intensity can be formally divided by the incident intensity at the scattering object. A value for the far-field backscattered intensity can be formally divided by the incident intensity at the scattering object.

In what follows, we abbreviate this quantity simply as $\sigma_{bs}$.

For simplicity, attention is restricted here to when the scattering object is perfectly rigid, so that the normal component of the scattered part of the fluid velocity at the surface $S$ is opposite to that of the incident wave. The incident wave is a plane wave, so its particle velocity is $k/\omega \rho$ times the acoustic pressure. Consequently, one has

$$ f(n) = ik \cdot \eta e^{ik \cdot r}. \tag{39} $$

In what follows, we abbreviate this quantity simply as $f(\eta)$ and $p(n)$ simply as $p(\eta)$.

It is evident that a stationary expression for target strength will result if one has a stationary expression for $\alpha$, so one can follow the procedure outlined in Section IV and set

$$ \alpha = \iint e^{ik \cdot \eta} [f(\eta) + ik \cdot n(\eta) p(\eta)] \, ds_1 $$

and

$$ \phi(\eta) = \iint [\Psi_1(\eta) - \xi_1(\eta, \eta_1)] \, ds_1 $$

$$ \Psi_2(\eta) = \iint [\Psi_1(\eta) - \xi_2(\eta, \eta_1)] \, ds_1. \tag{40} $$

Varying $\Psi_1$ and $\Psi_2$ once again yields (26a) and (26b), while varying $p$ yields

$$ ik \cdot n(\eta) e^{ik \cdot \eta} + \Psi_1(\eta) - \xi_1(\eta, \eta_1) + \xi_2(\eta, \eta_1) = 0 \tag{41} $$

or

$$ f(\eta) + \Psi_1(\eta) - \xi_1(\eta, \eta_1) + \xi_2(\eta, \eta_1) = 0. \tag{42} $$

Comparison of the latter with (15) leads to the choice of

$$ \phi(\eta) = -\frac{1}{2} f(\eta) \tag{43} $$

such that $\Psi_{\Pi}$ can consequently be identified as

$$ \Psi_{\Pi}(\eta) = \frac{1}{2} p(\eta). \tag{44} $$

Subsequent substitutions of (43) and (44) into (40) reduce the stationary expression for $\alpha$ to

$$ \alpha = \iint [f(\eta) + ik \cdot n(\eta) p(\eta)] \, ds_1 + \frac{1}{2\pi} \iint f(\eta) \cdot n(\eta) \, ds_1 $$

$$ \cdot \nabla \xi G(\eta) \, ds_1 \, ds_1 $$

$$ + \frac{k^2}{4\pi} \iint n(\eta) \cdot n(\xi) p(\eta) p(\xi) G(\eta, \xi) \, ds_1 \, ds_1 $$

$$ + \frac{1}{4\pi} \iint f(\eta) \cdot n(\eta) \cdot \nabla \xi G(\eta) \, ds_1 \, ds_1. \tag{45} $$

Here the quantity $f(\eta)$ is understood to be given by (39), while $p(\eta)$ is the scattered part of the acoustic pressure at the rigid body’s surface, when the incident acoustic pressure wave has unit amplitude. The above expression is stationary to small variations in the dependent variable $p$.

Comparison of (45) with (35) indicates that the variational principles, $\delta \phi = 0$ and $\delta \alpha = 0$, are equivalent if the $f$’s appearing in the two expressions are the same.

**VI. Numerical Example**

To demonstrate that stationary expressions constructed along the lines described in this paper can yield accurate estimates with relatively minimal effort, the power radiated by an un baffled rigid disk in transverse sinusoidal motion is considered here. The disk is taken as infinitesimally thin, of radius $a$, and oriented such that its faces are parallel to the $x$-$y$ plane with its center nominally at the origin. The disk is moving backward and forward in the $z$ direction with velocity $v_c \sin (\omega t)$, where $v_c$ is a constant.

For this example, one can use (35) for the calculation of the radiated power because the velocity on the surface is everywhere in phase. The quantity $f$ is $\omega v_c$ on the $+z$ side and $\omega v_c$ on the $-z$ side. This antisymmetry of $f$ in $z$ requires that $p$ also be antisymmetric; so considerable simplification results in (35). With appropriate multiplications of each integral by a constant (0, 2, or 4), one need only carry out integrations over the front side of the disk. Some subtlety is involved, however, with the term

$$ \frac{1}{2\pi} \iint f(\eta) \cdot n(\eta) \cdot \nabla \xi G(\eta) \, ds_1 \, ds_1. $$

In the limit as the disk thickness goes to zero, there is no contribution when $\eta$ and $\xi$ are on the same side of the disk, but when they are on opposite sides, considerations such as those that enter into the derivation of (7) apply, with the net result that the above term reduces to

$$ \iint f(\eta) \, ds_1. $$
which effectively doubles the first term on the right side of (35). Taking full advantage of the symmetry subsequently yields

\[ J = 4 \int \int p(\eta) f(\eta) \, dA_\eta - \frac{1}{\pi} \int \int (e_3 \times \nabla_p p(\eta)) \cdot (e_3 \times \nabla_p p(\eta)) \, dA_\eta \]

\[ + \frac{k^2}{\pi} \int \int p(\eta) p(\xi) G(\eta|\xi) \, dA_\eta \, dA_\xi \]

(46)

where here the area integrals (no longer surface integrals) extend over only the front surface of the disk.

At the present author's request, calculations based on this expression were carried out by X.-F. Wu for several values of \( ka \) with trial functions consisting of a linear combination of \( N \) basis functions, the sum having the generic form

\[ p(r) = \sum_{n=1}^{N} C_n [1 - (r/a)^2]^{1/2}(r/a)^{2(n-1)}. \]

(47)

These particular basis functions were selected because: a) in the limit of \( N \rightarrow \infty \), they become a complete set for radially symmetric functions that are analytic at the disk's center (hence only even powers of \( r \) in any expansion about \( r = 0 \)); b) the \( n = 1 \) basis function corresponds, apart from a multiplicative constant, to the exact solution in the limit as \( ka \rightarrow 0 \); c) because it is \( a \) priori known that, regardless of the value of \( ka \), the exact solution for \( p(r) \) must go to zero when \( r \rightarrow a \) as \( (a - r)^{1/2} \). (One has considerable latitude in choosing such basis functions, but any valid prior insight exercised in their selection should help to achieve better results with a fixed number of functions. A systematization of the art of selecting basis functions is a topic for further study.)

Calculations were carried out for \( N = 1, 2, 3, 4, 5, \) and 6. For each such \( N \) the coefficients \( C_n \) were determined by a Rayleigh–Ritz procedure: (47) was inserted into (46), the requisite integrals were evaluated, and the coefficients \( C_n \) were determined from the system of linear algebraic equations resulting from setting each of the derivatives \( \partial J/\partial C_n \) to zero. These so-derived coefficients were then substituted back into the expression for \( J \) to derive the corresponding \( N \)-term approximation for radiated power via (25). These coefficients were also inserted back into (47) with \( r \) set to 0 to determine the \( N \)-term approximation for the acoustic pressure amplitude \( |p(0)| \) at the center of the disk.

The contention of the present paper's theoretical development is that the calculated values for the radiated power should converge much more rapidly with increasing \( N \) than do the values for acoustic pressure at the center of the disk. This of course was the case. The following numbers, normalized to the values for \( N = 6 \), are representative.

Calculations for \( ka = 5 \):

\[ \begin{array}{lll}
N = 1: & |p(0)| = 2.3061 & \text{Power} = 1.0512 \\
N = 2: & |p(0)| = 1.9481 & \text{Power} = 0.9512 \\
N = 3: & |p(0)| = 1.1377 & \text{Power} = 0.9984 \\
N = 4: & |p(0)| = 1.0096 & \text{Power} = 1.0000 \\
N = 5: & |p(0)| = 0.9996 & \text{Power} = 1.0000.
\end{array} \]

Thus, for \( N = 1 \), the radiated power estimate is only 5 percent in error, even though the pressure at the center of the disk is off by more than a factor of 2. At \( N = 3 \) the error in power is approximately \( 16 \times 10^{-4} \), while the error in acoustic pressure at disk center is approximately \( 14 \times 10^{-4} \), the latter being nearly 100 times as large as the former. (The normalization to the \( N = 6 \) result rather than to the exact result should be appropriate here, because the convergence appears to be virtually complete at \( N = 6 \). The problem is solvable in closed form in the limit of \( ka = 0 \), but the "exact" solution for arbitrary \( ka \) has the form of an infinite series involving spheroidal wave functions, and it is highly likely that this series is slowly convergent for moderate values of \( ka \). Development of a computer algorithm using spheroidal wave functions that would give the exact answer to a specified accuracy is a nontrivial task that did not seem worthwhile within the context of illustrating the principal point of the present paper.)

**VII. CONCLUDING REMARKS**

The variational principles exhibited in the present paper were selected primarily for convenience of presentation. Variational principles can also be developed that take the internal structure (e.g., its elastic properties) of the body into account. This could be done, for example, by including additional terms to incorporate the equations governing the internal dynamics, with Lagrange-multiplier-function coefficients identified during the process of constructing the variational principle via the Gerjuoy, Rau, and Spruch technique. One could expand the internal displacement fields in terms of a finite number of internal basis functions (modal shape functions) with undetermined coefficients and then express the normal acceleration field at the body's surface in terms of these coefficients. The pressure field on the surface would be expanded in terms of surface basis functions with a second set of undetermined coefficients. Whatever variational principle is developed would yield as a by-product a complete set of linear algebraic equations whose solution would produce values for the two sets of coefficients. (A calculation of such a nature has recently been reported by Ginsberg [27] for the example of sound radiation from a vibrating point-driven circular elastic plate, although the governing equations were not derived with explicit use of the Gerjuoy, Rau, and Spruch technique.)

The extent to which computational algorithms based on variational formulations can compete with other techniques for acoustic radiation and scattering calculations remains a subject for further investigation. One intriguing question is, what are the optimum accuracy requirements that should be imposed on the evaluation of the multiple integrals that arise in Rayleigh–Ritz implementations of variational techniques? In the general case, one must contend with four-fold integration, and this would seem to intrinsically require many arithmetic operations, but such is not necessarily so if one uses an algorithm that approximates the integral by the integration "volume" times a weighted average of the integrand at a finite number of judiciously selected points in the integration parameter space. A relatively small number of points might suffice if the point
therefore not be regarded as necessarily intimidating. Also, there are difficulties because the integrands of interest will often have singularities, but these will always be integrable singularities and one might be able to separate each integrand into a sum of a simple singular term and a term which is nonsingular, the integration over the first term being performed analytically. The higher order integrations should therefore not be regarded as necessarily intimidating. Also, the consistent and intelligent use of a variational principle should allow one to achieve comparable accuracy with a smaller number of basis functions than would be required when one uses a generic numerical approach not specifically tailored to the problem of interest. Given such considerations, the author is optimistic that, with additional research on the selection and weighting coefficient selection procedures took advantage of the known analytical properties of the integrand. There are difficulties because the integrands of interest will often have singularities, but these will always be integrable singularities and one might be able to separate each integrand into a sum of a simple singular term and a term which is nonsingular, the integration over the first term being performed analytically. The higher order integrations should therefore not be regarded as necessarily intimidating. Also, the consistent and intelligent use of a variational principle should allow one to achieve comparable accuracy with a smaller number of basis functions than would be required when one uses a generic numerical approach not specifically tailored to the problem of interest. Given such considerations, the author is optimistic that, with additional research on the details of their numerical implementation, variational techniques may fare well when pragmatically compared with other computational techniques.

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Variational Method for Computing Surface Acoustic Pressure on Vibrating Bodies, Applied to Transversely Oscillating Disks

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Variational Method for Computing Surface Acoustic Pressure on Vibrating Bodies, Applied to Transversely Oscillating Disks

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Abstract—A variational principle derived from the Kirchhoff-Helmholtz integral relation can be applied to acoustic radiation and diffraction problems. An illustrative example discussed here is that of sound radiation from a flat rigid circular disk in transverse oscillation. The variational formulation has the surface pressure as the unknown variable, with the normal velocity of the surface taken as given. The Rayleigh-Ritz method used in determining approximate solutions in terms of truncated expansions of basis functions encounters some numerical problems in the evaluation of integrals with singular integrands. The integrands are nevertheless integrable and techniques are described for handling the singularities. Another potential source of difficulty is that the tangential derivative of the surface pressure for the exact solution must be infinite at the edge of the disk. One makes use of prior knowledge of such a fact by using basis functions with the correct dependence on radial distance near the disk edge. Because basis functions in the Rayleigh-Ritz procedure have been selected with the aid of prior insight into the nature of the true solution, accurate results are obtained with a relatively small number of basis functions. The numerical solutions agree well with results calculated by Leitner in 1949.

I. INTRODUCTION

The prediction of acoustic radiation fields from vibrating bodies is a recurrent problem of practical importance. Analytical solutions for such problems are generally limited to cases for which the surface of the object conforms to a suitable coordinate system so that the wave equation can be separated. An approach commonly used for objects with nonstandard shapes is to reformulate the problem as one or more integral equations involving quantities defined on the surface. Solution of such equations enables one to express [1] the acoustic field at any point exterior to the object as a definite integral over the surface of the object. Terms in the integrand involve both the surface pressure and the normal acceleration of the surface. Then, however, one of the integrands becomes so highly singular that the integral is not well defined. It is not easily determined how their principle would be numerically implemented for cases of interest. Another potential source of difficulties for the construction of a useful variational principle is that the commonly cited integral relation derived from the Kirchhoff-Helmholtz corollary has an integration kernel that is not self-adjoint. A self-adjoint kernel can be obtained in principle as was done by Morse and Feshbach by taking a normal derivative of the Kirchhoff-Helmholtz integral relation for acoustic radiation problems. In their formulation, however, one of the integrands is so highly singular that the integral is not well defined. It is not easily determined how their principle would be numerically implemented for cases of interest. Another potential source of difficulties for the construction of a useful variational principle is that the commonly cited integral relation derived from the Kirchhoff-Helmholtz corollary has an integration kernel that is not self-adjoint. A self-adjoint kernel can be obtained in principle as was done by Morse and Feshbach by taking a normal derivative of the Kirchhoff-Helmholtz integral relation for acoustic radiation problems. In their formulation, however, one of the integrands is so highly singular that the integral is not well defined. It is not easily determined how their principle would be numerically implemented for cases of interest. Another potential source of difficulties for the construction of a useful variational principle is that the commonly cited integral relation derived from the Kirchhoff-Helmholtz corollary has an integration kernel that is not self-adjoint. A self-adjoint kernel can be obtained in principle as was done by Morse and Feshbach by taking a normal derivative of the Kirchhoff-Helmholtz integral relation for acoustic radiation problems. In their formulation, however, one of the integrands is so highly singular that the integral is not well defined. It is not easily determined how their principle would be numerically implemented for cases of interest.

II. VARIATIONAL FORMULATION

The variational principle used in this paper results from the standard Kirchhoff-Helmholtz integral relation for acoustic pressure $p(x)$ at an external point:

$$p(x) = \frac{1}{4\pi} \int \int f_0(x') G(x|x') dA'$$

$$+ \frac{1}{4\pi} \int p(x') [n(x') \cdot \nabla' G(x|x')]_{x' \to x} dA'$$

(1)

with the integration extending over the surface $S$ of the vibrating body. Here all field quantities are assumed to oscillate at a constant frequency, with the characteristic time dependence $\exp(-i\omega t)$. The primes distinguish the dummy variables of integration, so $p(x)$ and $p(x')$ are acoustic...
pressures at an external point \( x \) and at a surface point \( x' \), respectively. The symbol \( G \) stands for the free-space Green’s function

\[
G(x|x') = \frac{e^{i\omega R}}{R}
\]  

(2)

where \( R = |x - x'| \) is the distance between the source point \( x' \) and the field point \( x \). The quantity \( f_n(x') \) is defined as

\[
f_n(x'_i) = -i\omega \rho_0 v_n(x'_i)
\]  

(3a)

and, because this quantity is the product of the complex amplitude of the normal surface acceleration and the density \( \rho_0 \) of the fluid, it can be regarded (from Newton’s second law) as the normal component of the apparent force per unit volume experienced by the fluid immediately adjacent to the surface. Here \( v_n \) is the outward normal component of velocity on the surface. The above also applies for diffraction of sound by a rigid body if \( p(x'_i) \) is interpreted as the scattered part of the acoustic pressure and \( f_n \) is taken as

\[
f_n(x'_i) = \left[ \frac{\partial p_{inc}(x'_i)}{\partial n} \right]_{x'=x'_i}
\]  

(3b)

where \( p_{inc}(x'_i) \) is the pressure of the incident wave. The fact that the pressure \( p(x'_i) \) and the apparent surface value \( f_n(x'_i) \) of the force density cannot be independently specified becomes evident when one lets \( x \) approach a surface point \( x_i \) in (1). With careful attention to the change in values of integrals when the order of integration and passing to limit are interchanged, one finds the integral relation (1) to be

\[
p(x_i) = \frac{1}{2\pi} \iint f_n(x'_i) G(x_i|x'_i) \, dA'
\]  

\[
+ \frac{1}{2\pi} \int p(x'_i)[\n(x_i) \cdot \nabla G(x_i|x'_i)]_{x'=x'_i} \, dA'
\]  

(4)

where the second integral can be regarded as a principal value [19], with \( x_i \) being flush on the surface. Because this relates the surface values of \( f_n(x_i) \) and \( p(x_i) \), the two are not independent. Here, and in what follows, we assume that \( f_n(x_i) \) is given and that the unknown variable is the surface pressure \( n(x_i) \).

Equation (4) is not self-adjoint for \( p(x_i) \), so it is not ideally suitable for the derivation of a variational principle. A more suitable expression results from taking the normal derivative of (1) at a point \( x = x_i + \epsilon n(x_i) \) with \( \epsilon \) regarded as small and positive; doing so yields

\[
f_n(x_i) = -\frac{1}{4\pi} \iint f_n(x'_i)n(x_i) \cdot \nabla G(x|x'_i) \, dA'
\]  

\[
- \frac{1}{4\pi} \int p(x'_i)[\n(x_i) \cdot \nabla] \times [n(x'_i) \cdot \nabla] G(x|x'_i)]_{x'=x'_i} \, dA'
\]  

(5)

where \( x \) is understood to be \( x_i + \epsilon n(x_i) \) and \( x_i \) is an arbitrary point on the surface \( S \).

Although the first integrand on the right side of (5) is singular in the limit \( \epsilon \to 0 \), it is integrable [20]. The second integrand, however, has a nonintegrable singularity in the limit of \( \epsilon \to 0 \). Hence we postpone taking the limit as \( \epsilon \to 0 \) in order to carry out the derivation with all integrands continuous on the surface.

To reduce the complications associated with the singularity in the second integrand on the right side of (5), we use the following vector identity:

\[
(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c). \tag{6}
\]

It follows from this and the properties of the free-space Green’s function \( G(x|x') \) that

\[
\[n(x') \cdot \nabla'](n(x_i) \cdot \nabla] G = [n(x') \cdot n(x_i)](\nabla \cdot \nabla') G
\]

\[
- [n(x') \times \nabla'] \cdot [n(x_i) \times \nabla] G. \tag{7}
\]

Note that \( n(x) \) is constant in regard to differentiation with respect to the primed coordinate system. The symmetry of the Green’s function gives \( G(x|x') = G(x'|x) \) and \( \nabla G(x|x') = -\nabla' G(x|x') \). Also \( G(x|x') \) satisfies the Helmholtz equation \( \nabla^2 G + k^2 G = 0 \), for \( \epsilon \neq 0 \). The above relations allow one to rewrite the second integral on the right side of (5) as

\[
- \frac{1}{4\pi} \iint \int p(x'_i)[\n(x_i) \cdot \nabla'] \times [n(x') \cdot \nabla'] G(x|x')]_{x'=x'_i} \, dA'
\]

\[
= -\frac{k^2}{4\pi} \iint \int p(x'_i)[n(x_i) \cdot n(x'_i)] G(x|x'_i) \, dA'
\]

\[
+ \frac{1}{4\pi} \int [p(x'_i)] [n(x'_i) \cdot \nabla'] \times [n(x'_i) \times \nabla] G(x|x'_i)]_{x'=x'_i} \, dA'. \tag{8}
\]

The operator \( \mathbf{n}(x_i) \times \nabla \) involves the derivative tangential to the surface only. Hence it is meaningful when applied to a function defined on the surface.

Substituting (8) into (5), integrating the last term on the right side of (8) by parts, and then using the Stokes’ theorem, one finds

\[
f_n(x_i) = -\frac{1}{4\pi} \iint f_n(x'_i)n(x_i) \cdot \nabla G(x|x'_i) \, dA'
\]

\[
- \frac{k^2}{4\pi} \iint p(x'_i)[n(x'_i) \cdot n(x_i)] G(x|x'_i) \, dA'
\]

\[
- \frac{1}{4\pi} [n(x_i) \times \nabla]
\]

\[
\iint \{G(x|x') [n(x') \times \nabla'] p(x')\} \, dA'. \tag{9}
\]
in such terms at the outset. Also, since \( n(x_i) \times \nabla \) is a tangential derivative, one can set \( x \) to be \( x \) in the argument of the Green's function at the outset. Interchanging the order of passing to limit and integration does change the value of the first term on the right side of (9), so one should be careful to specify the intended order. However, since this term does not involve the dependent variable \( p(x'_s) \), the entire term can be regarded simply as a known source term in what follows.

To derive the variational expression from (9), we multiply each term by \( \delta p(x_s) \) and integrate once again over the surface with \( \epsilon \) kept small and positive. Next, repeat as before the procedure of integrating the last term on the right side of (9) by passing to limit and integration does change the value of the subsequent terms and use the Stokes' theorem, then let \( \epsilon \to 0 \). Doing so, one obtains

\[
\delta \left\{ 2\pi \int p(x_s) f_n(x_s) \, dA + \int p(x) \right\} \times \lim_{\epsilon \to 0} \int f_n(x'_s) n(x) \cdot \nabla G(x | x'_s) \, dA' \}
\]

\[
+ \frac{k^2}{2} \int \int \int p(x'_s) p(x'_s) \] 

\[
	imes [n(x_s) \cdot n(x'_s)] G(x | x'_s) \, dA' \] 

\[
- \frac{1}{2} \int \int \int [n(x_s) \times \nabla p(x)] \] 

\[
\cdot [n(x'_s) \times \nabla p(x'_s)] G(x | x'_s) \, dA' \, dA \}
\]

\[= 0. \quad (10)\]

Note that all terms are now well defined in the limit \( x_s \to x'_s \) and the integration kernels in the last two terms on the right side of (10) are symmetric under the exchange of \( x_s \) and \( x'_s \). Equivalently, (10) can be written as \( \delta J[p] = 0 \), where

\[
J[p] = 2\pi \int p(x_s) U_n(x_s) \, dA \]

\[
+ \frac{k^2}{2} \int \int \int p(x'_s) p(x'_s) [n(x_s) \cdot n(x'_s)] \]

\[
	imes G(x | x'_s) \, dA' \] 

\[
- \frac{1}{2} \int \int \int [n(x_s) \times \nabla p(x)] \] 

\[
\cdot [n(x'_s) \times \nabla p(x'_s)] G(x | x'_s) \, dA' \, dA \]

\[= 0. \quad (12)\]

The expression \( J[p] \) has accordingly been shown to be stationary to variations in \( p(x_s) \). Once \( p(x_s) \) is found, the external acoustic field is determined by (1).

The second term in the expression for \( U_n(x_s) \) can be expressed as \( - (1/2) f_n(x_s) \) plus the Cauchy principal value of an integral using a technique similar to those by which (4) is derived from (1). However, for the disk, \( U_n(x_s) \) is just \( f_n(x_s) \), and the second term vanishes identically.

(12)
for sufficiently small $\epsilon$, with the rigid wall boundary condition
\[
\frac{\partial p}{\partial \alpha} = 0, \quad \text{at } \alpha = 0 \text{ and } \alpha = 2\pi
\]

plus the requirement that $p$ be bounded or zero near the edge. This "boundary-value problem" can be solved by the method of separation of variables, with the result that, to leading order in $\epsilon$
\[
p = A + B\sqrt{\epsilon} \cos \left(\frac{\alpha}{2}\right)
\]

where $A$ and $B$ are "constants." For the particular case of sound generated by a disk in rigid-body transverse vibration, symmetry and continuity require that $p$ be zero at the edge, so the constant $A$ is identically zero. Thus on the front surface, where $\alpha = 0$, the implication of the above equation is
\[
p = \text{distance from edge} \frac{\epsilon}{2}
\]

near the edge of the disk.

Since the exact behavior at the edge of the thin disk is known, it is advantageous to incorporate this known feature into the basis functions at the outset. Then a faster convergence toward the exact solution and an accurate numerical result with a relatively small number of basis functions can be achieved. Hence we set
\[
p(x) = \sum_{n=1}^{N} C_n P_n(x)
\]

where
\[
P_n(x) = r^{2(n-1)} \sqrt{a^2 - r^2}.
\]

The first basis function, i.e., $n = 1$, turns out to be (apart from a multiplication constant) the exact solution of the surface pressure distribution as $ka \to 0$.

Note that the basis functions $P_n$ depend on $r$ only and that each goes to zero at the edge of the disk. The lack of angular dependence is due to the axisymmetry of the problem, and the vanishing of $P_n$ at $r = a$ is mandatory if the trial functions are to be admissible.

The stationary expression (18) leads to the system of equations (14) for the unknown coefficients $C_n$. The elements of the matrix $[A]$ and the column vector $[B]$, after an appropriate nondimensionalization process, are given by
\[
A_{nm} = \int_0^r \int_0^r (ka)^3 P_n P_m
\]

\[
\times \left[ \int_0^{2\pi} \frac{e^{ikaR_0}}{R_0 r_0' \cos \Phi} d\Phi \right] dr_0 \, dr_0'
\]

\[
- \int_0^r \int_0^r \frac{dP_n}{dr_0} \frac{dP_m}{dr_0'} dr_0 \, dr_0'
\]

\[
\times \left[ \int_0^{2\pi} \frac{e^{ikaR_0}}{R_0 r_0' \cos \Phi} d\Phi \right] dr_0 \, dr_0'
\]

\[
B_n = i2\pi (ka) v_n \int_0^r P_n r_0 \, dr_0
\]
where $P_n$ and $P_m$ are the preselected basis functions, $r_0$ and $r_0'$ are the dimensionless radial variables

\[
\begin{align*}
    r_0 &= \frac{r}{a} \\
    r_0' &= \frac{r'}{a}
\end{align*}
\] (27)

and $R_0$ is the dimensionless distance between $r_0$ and $r_0'$

\[
R_0 = \sqrt{r_0^2 + r_0'^2 - 2r_0r_0'\cos \Phi},
\] (28)

The subscript zero is omitted for brevity in what follows and all variables are understood to be dimensionless.

where

\[
\begin{align*}
    r' &= 1, \quad \text{if} \quad r = 1, \quad \text{and} \quad \Phi = -x
\end{align*}
\]

When $r = r'$, this can be demonstrated if one rewrites the Green's function factor as

\[
\frac{e^{ikar}}{R} = \frac{1}{R} + \frac{(e^{ikar} - 1)}{R}.
\] (29)

and considers the $1/R$ term only. One notes, for example, for one of the integrals in (25) that

\[
\int_0^{2\pi} \frac{r'}{r} d\Phi = (r + r')mK(m)
\] (30)

where

\[
m = \frac{4rr'}{(r + r')^2}, \quad 0 < m < 1
\] (31)

and $K(m)$ is the complete elliptic integral [22] of the first kind. When $r \rightarrow r'$ or $m \rightarrow 1$, one finds

\[
K(m) \rightarrow 1 + \ln \left( \frac{16}{1 - m} \right).
\] (32)

Consequently, $K(m)$ grows logarithmically without bound as $m \rightarrow 1$ and the integrand for the double integration over $r$ and $r'$ has a singularity of the form $\ln |r - r'|$.

To circumvent problems arising from this singularity in the numerical computation, an integration algorithm is designed (Fig. 2) such that one does not evaluate the integrands at points where $r = r'$.

Another potential source of difficulties in the numerical computation is that the derivative $dp_\alpha/dr$ of the exact surface pressure is infinite at the edge of the disk. This is obvious from the expression (24). Since the trial functions have been selected to also have this property, there will be singularities in the integrand of the second term in (25) at $r = 1$ and $r' = 1$. One has, in particular,

\[
\frac{dP_\alpha}{dr} = 2(n - 1)r^{2(n - 3)} - \frac{r^{2(n - 1)}}{\sqrt{1 - r^2}}.
\] (33)

The first term is bounded at $r = 1$, but the second is not. To remove this singularity at $r = 1$, one makes use of a coordinate transformation

\[
r = \sin \left( \frac{\pi}{2} x \right)
\] (34)

such that

\[
\frac{dP_\alpha}{dx} = \pi(n - 1) \left[ \sin \left( \frac{\pi}{2} x \right) \right]^{(2n - 3)}
\]

is finite at $x = 1$, i.e., $r = 1$.

Plotted in Fig. 3 are the numerical results of the dimensionless surface pressure $p/p_{\infty}$ versus the dimensionless radial distance $r/a$ for various $ka$ values. With the complex amplitude of the disk velocity $u_0$, taken as being real, one distinguishes real and imaginary parts of $p$ as being in phase and $90^\circ$ out of phase with $u_0$. The solid curves are calculated with $N \leq 5$ by the method discussed in the present paper, while the dashed lines are taken from Leitner [23, figs. 1 and 2]. Here we have corrected an apparent mislabeling of $ka = 1$ and $ka = 2$ curves for the imaginary parts in [23, fig. 2]. Because of our choice of basis functions, all of our computed curves have infinite slopes at $r/a = 1$, as must be the case for the exact solution. The accuracy of Leitner's results, on the other hand, is not sufficient to exhibit this feature.

IV. CONCLUDING REMARKS

The variational principle exhibited here can be applied to a large variety of problems and is readily extended to circumstances when the elastic properties of the vibrating body are to be taken into account or to scattering by elastic bodies.

The example of radiation from the vibrating disk treated here demonstrates a number of features which support the contention that variational principles may significantly improve our capabilities for numerical solution of acoustic radiation and scattering problems. Because there is a wide latitude in the selection of basis functions, whatever insight, intuition, or experience one has regarding the problem at hand can be used to shorten the overall computational effort. Here, for example, we knew the form of the pressure near the edge of the disk at the outset, and we took advantage of this knowledge. We also knew the exact solution in one limiting case, that of $ka \rightarrow 0$, so we took our first function to be of the
in conjunction with the possibility of using a very small number of basis functions, makes the variational method a viable alternative to direct numerical solutions of integral equations.

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and began working toward the Ph.D. degree in the School of Mechanical Engineering at the Georgia Institute of Technology. During this latter period, he was supported by NASA- and ONR-funded research as a Research Assistant.

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Allan D. Pierce (M'63), for photograph and biography please see this issue, p. 411.

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A variational principle governing the acoustic pressure on the exterior of an arbitrary body is derived from the Kirchhoff-Helmholtz integral relation. The principle is valid for acoustic radiation and diffraction problems. The general principle is specialized to the case of a thin body and then illustrated by an example of sound radiation from a flat, rigid, circular disk in transverse oscillation. The variational principle has the surface pressure as the unknown variable, with the velocity normal to the surface taken as given. The Rayleigh-Ritz method is used to determine a solution in terms of truncated expansions of basis functions. The basis functions employed are polynomial and trigonometric functions, and piece-wise linear functions leading to a finite element description. The results compare very well with previous analytical estimations.
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1. INTRODUCTION

The prediction of sound radiation from vibrating bodies is a recurrent problem of practical importance in engineering science. Analytical solutions for such problems are generally limited to cases for which the surface of the object conforms to a suitable coordinate system such that the wave equation can be separated. For objects with nonstandard shapes, an approach commonly used is to reformulate the problem as a boundary integral relation, known as the Kirchhoff-Helmholtz integral theorem [Pierce, 1981], which enables one to express the acoustic field at any point exterior to the object as a definite integral over the surface of the object. Terms in the integrands involve both the pressure and the normal acceleration of the surface. These two surface quantities, however, are not independent and cannot be prescribed simultaneously. In most acoustic problems, one of the two is specified and it is necessary to solve the integral equation for the other unknown surface quantity.

Many numerical methods have been developed in that regard, for example, by Chen and Schweikert [1963], Banaugh and Goldsmith [1963], Chertock [1964], Copley [1967, 1968], Schenck [1968], Burton and Miller [1971], Bell et al [1978, 1979] for acoustic radiation and scattering problems. The present report describes a variational formulation of the fluid-structure interaction, which is derived from the Kirchhoff-Helmholtz integral theorem. The variational principle is attractive in that it always selects, from among various adjustable parameters of a chosen set of approximating functions, the ones that are optimal. It retains the accuracy of the boundary integral approaches, while offering the computational efficiency of approximate decoupling techniques, such as doubly asymptotic approximation developed by Geers [1971].

Although the formulation of an integral equation as a variational principle dated back as early as in 1884 [Volterra], it was not used for practical calculations in wave diffraction or scattering until the late 1940's. Levine and Schwinger [1948, 1949] employed a variational principle to study the diffraction of a scalar plane wave by an aperture in an infinite plane screen. Levine [1950], Bouwkamp [1954], and Sleator [1960] further discussed variational principles for acoustic diffraction and scattering problems.
In general, derivation of a useful variational principle from an integral equation requires that the integration kernel be symmetric, and therefore self-adjoint. The variational principle derived by Morse and Feshbach [1953] was obtained by simply taking a normal derivative of the Kirchhoff-Helmholtz integral relation for the pressure at an exterior point, and then letting the exterior point approach the surface. The resulting principle features an integration kernel that is self-adjoint. However, one of the resulting integrands becomes highly singular, and the integral is ambiguous without careful definition of how the singular integral is to be evaluated. It is not apparent how their principle could be numerically implemented for cases of interest.

In order to circumvent this singularity problem, the present report utilizes a method previously employed by Maue [1949] and Stallybrass [1967]. By using some mathematical identities associated with the free-space Green's functions, the integrand with non-integrable singularity is recast into a form involving the tangential derivative of the surface pressure. The singularities contained in the resultant integrands are at most of Cauchy type [Churchill, 1960], and therefore well behaved in the limit as the external point approaches the surface.

After we derive the present variational principle for an arbitrary body, we shall specialize it to situations where the body is slender, such that wetted surfaces are proximate. The application of the present variational principle to a circular disk with infinitesimal thickness in rigid transverse vibration is demonstrated. The numerical results agree remarkably well with those obtained by Leitner [1949].

2. KIRCHHOFF-HELMHOLTZ INTEGRAL THEOREM

The variational principle discussed in the present report is based on the standard Kirchhoff-Helmholtz integral theorem [Pierce, 1981] for the complex pressure amplitude $p$ at an external field point $\mathbf{x}$ due to monochromatic excitation on a closed surface $S$. It is an exact corollary of the Helmholtz wave equation and the Sommerfeld radiation condition that, for $\mathbf{x}$ external to $S$, one has
\[ p(\hat{x}) = -\frac{1}{4\pi} \iint_S \hat{n}(\xi) \cdot \nabla G(\hat{x} | \xi) \, dA_{\xi} + \frac{1}{4\pi} \iint_S p(\xi) \hat{n}(\xi) \cdot \nabla G(\hat{x} | \xi) \, dA_{\xi} \]  
where $\xi$ is a source point on $S$, the operator $\nabla_{\xi}$ denotes the gradient at that location, and $dA_{\xi}$ is an element of area surrounding the source point. The quantity $G(\hat{x} | \xi)$ is the free space Green's function for the pressure at the field point $\hat{x}$ associated with a point source at $\xi$ oscillating as $e^{-i\omega t}$; that is,

\[ G(\hat{x} | \xi) = \frac{1}{R} e^{ikR} \]  
where $k = \omega/c$ (with $c$ denoting the ambient speed of sound), and $R$ is the distance between the source and field points,

\[ R = |\hat{x} - \xi| \]  

Our interest here shall be focused on radiation problems, such that the surface velocity is presumed a given quantity, while the acoustic pressure is to be calculated. Euler's equation of motion for a fluid (the fluid dynamic counterpart of Newton's second law), combined with continuity of the normal component of particle velocity on the surface leads to

\[ \hat{n}(\xi) \cdot \nabla p(\xi) = i\omega \nu_n(\xi) \]  
where $\nu_n(\xi)$ is the complex amplitude of the (outward) normal component of the surface velocity.

Equations (1) and (4) show that, if the pressure and normal velocity on the surface $S$ are known, then the pressure at any external point not on $S$ may
be obtained from a definite integral. Our concern in this study is the
evaluation of the pressure on $S$ corresponding to the velocity distribution
$v_n(\xi)$. Although we shall derive a different interrelationship between these
two physical quantities on the surface, it is instructive to first review the
derivation of a better known relationship [Schenck, 1968] that has often been
used in earlier numerical studies. This relation is formally obtained by
letting $\xi$ approach the surface $S$.

3. FIRST INTEGRAL EQUATION FOR SURFACE PRESSURE

In the situation where $\xi$ lies on $S$, such that $\xi = \xi$, both integrals in
Eq. (1) have a singularity at $\xi = \xi$, because $R = 0$. The singularity in $G(\xi; \xi)$
is integrable, but evaluation of the integral containing $V(\xi)G(\xi; \xi)$ requires
careful consideration. In particular, one finds that the value of the
integral may have different values depending on whether one regards $\xi$ as
having approached the surface from the exterior or from the interior.

The analysis here of the effect of the singularity is similar to that of
Kellogg [1953], in which the external point $\xi$ is brought to a location $\xi$ on
the surface in a limiting operation. As shown in Fig. 1, point $\xi$ is defined
to be concurrent with the normal that intersects $\xi$, so

$$\hat{\xi} = \hat{\xi} + \epsilon \hat{n}(\xi)$$

(5)

where $\epsilon$ is the small perpendicular distance of $\xi$ from the surface. The region
$S'$ is a circular segment of $S$, centered at $\xi$, with small radius $\sigma$. The
remainder of the surface is denoted as $S''$. The required integral is evaluated
by taking the limit as $\epsilon \rightarrow 0$ with $\sigma$ fixed, and then taking the limit as $\sigma \rightarrow 0$.
The order in which the two limits are taken is important.

Given that $\epsilon$ and $\sigma$ are both small compared with any characteristic dimensions of the surface $S$ and given that $S$ is smooth near the point $\xi$, it is appropriate to regard $S'$ as having the shape of an elliptical bowl. The
Figure 1. Integration when the field point approaches the surface.
principal curvature radii, $R_I$ and $R_{II}$, of this bowl are those of the surface at location $\xi$. In a local coordinate system with origin at $\xi$, the surface $S'$ is locally described by

$$z = -\frac{x^2}{2R_I} - \frac{y^2}{2R_{II}}$$  \hspace{2cm} (6)

(A convex surface would correspond to negative radii of curvature.) The unit outward normal vector $\hat{n}(\xi)$ at a generic point $\xi$ on $S'$ is given approximately by

$$\hat{n}(\xi) \approx \hat{e}_z + \left(\frac{x}{R_I}\right)\hat{e}_x + \left(\frac{y}{R_{II}}\right)\hat{e}_y$$  \hspace{2cm} (7)

In this local coordinate system, the external point $\bar{x}$ is at $\epsilon\hat{e}_z$, while the $z$-coordinate of $\xi$ is as described by Eq. (6) above. Consequently, the vector $\bar{x} - \xi$ is given by

$$R = \bar{x} - \xi \approx [\epsilon + (x^2/2R_I) + (y^2/2R_{II})]\hat{e}_z - x\hat{e}_x - y\hat{e}_y$$  \hspace{2cm} (8)

and one derives

$$R \cdot \hat{n}(\xi) \cdot \psi R \approx -\epsilon + (x^2/2R_I) + (y^2/2R_{II})$$  \hspace{2cm} (9)

$$\hat{n}(\xi) \cdot \psi R \hat{G}(\xi) \hat{x} \approx (1/R^3 + k^2/2R) [\epsilon - (x^2/2R_I) - y^2/2R_{II}]$$  \hspace{2cm} (10)

where, in the latter expression, the neglected terms are of the order of unity or smaller when $\epsilon$ and $\sigma$ are both much smaller than $R_I$ and $R_{II}$. Moreover, on the right side of Eq. (10), it is consistent to replace the distance $R$, wherever it appears, by

$$R \approx (\epsilon^2 + r^2)^{1/2}; \hspace{1cm} r^2 = x^2 + y^2$$  \hspace{2cm} (11)
and, for the differential of area, to set
\[ dA_\xi = rdr \, d\phi; \quad x = r \cos \phi; \quad y = r \sin \phi \quad (12) \]

where, for the surface \( S' \), the \( r \)-integration extends from 0 to \( \sigma \), and the \( \phi \)-integration extends from 0 to \( 2\pi \).

Since all points on \( S'' \) are at a finite distance from \( \xi \), the factor \( \hat{n}(\xi) \cdot \nabla G(\xi \xi) \) has no singularities on \( S'' \) when \( \epsilon \to 0 \). Moreover, if one lets \( \epsilon \to 0 \) first and considers \( r \) to be small but nonzero, then

\[ \hat{n}(\xi) \cdot \nabla G(\xi \xi) \, dA_\xi = -(1/2) (R^{-1} \cos^2 \phi + R^{-1} \sin^2 \phi) (1/r) \, rdr \, d\phi \quad (13) \]

The \( 1/r \) singularity here is cancelled by the factor in the area differential factor \( rdr \). The integral in this limit is exactly the same as if one set \( \xi \) to \( \xi \) at the outset, and then did the integral over the surface \( S \). Hence, it is mathematically meaningful to integrate over the surface \( S'' \) by taking the double limit such that first \( \epsilon \to 0 \), then \( \sigma \to 0 \). We therefore turn our attention to the contribution of \( S' \).

Let \( F(\xi) \) be any continuous scalar function on \( S \). Then Eqs. (10) and (12) lead to

\[ \iint_{S'} F(\xi) \hat{n}(\xi) \cdot \nabla G(\xi \xi) \, dA_\xi \]

\[ = \iint_{S'} F(\xi) \left( \epsilon/R^3 \right) \, rdr \, d\phi \]

\[ + \iint_{S'} F(\xi) \left[ A(\epsilon, r, \phi) - (r/R)^2 B(\phi) \right] R^{-1} \, rdr \, d\phi \quad (14) \]
where in the latter term we abbreviate

\[ A(e, r, \phi) = \frac{1}{2} k^2 e - \frac{1}{4} k^2 r^2 (R_{I}^{-1} \cos^2 \phi + R_{II}^{-1} \sin^2 \phi) \]  

(15a)

\[ B(\phi) = \frac{1}{2} (R_{I}^{-1} \cos^2 \phi + R_{II}^{-1} \sin^2 \phi) \]  

(15b)

Functions A and B are both finite; A actually vanishes in the limit as both \( e \) and \( \sigma \) go to zero. Also, we note that the quantity \( r/R \) is always less than 1, regardless of the value of \( e \). Consequently, the second term in Eq. (14) is of the order of magnitude of \( \sigma \); it therefore vanishes in the limit \( \sigma \to 0 \).

We now proceed to evaluate the first term in Eq. (14). With \( e \) small, but positive, an appropriate change of integration variable is to the polar angle \( \theta \), defined such that

\[ r = e \tan \theta; \quad R = e \sec \theta; \quad dr = e \sec^2 \theta d\theta; \quad \frac{e}{R^3} r \, dr = \sin \theta \, d\theta \]  

(16)

The integration limits on \( \theta \) are 0 and \( \tan^{-1}(\sigma/e) \). In the limit as \( e \to 0 \) with \( \sigma \) fixed, this upper limit approaches \( \pi/2 \). The substitution of Eq. (16) into Eq. (14) makes the integration trivial, the overall result being simply \( 2\pi F(\xi) \). Thus, we find that, as regards the integral over the entire surface, when \( e \) goes from a finite positive value to 0,

\[
\lim_{e \to 0} \int_{S} F(\xi) \hat{n}(\xi) \cdot \nabla_{\zeta} G(\xi, \xi) \, dA_{\xi} \\
= 2\pi F(\xi) + \int_{S} F(\xi) \hat{n}(\xi) \cdot \nabla_{\zeta} G(\xi, \xi) \, dA_{\xi}
\]  

(17)

The result of applying Eq. (17) to the Kirchhoff-Helmholtz integral equation (1), with the pressure boundary condition given by Eq. (4), is
\[
\mathbf{p}(\xi) = -\frac{1}{2\pi} \iint_{S} i\omega \mathbf{V}_{n}(\xi') \mathbf{G}(\xi,\xi') \, dA_{\xi'} \\
+ \frac{1}{2\pi} \iint_{S} \mathbf{p}(\xi) \cdot \mathbf{v}_{\xi} \mathbf{G}(\xi,\xi') \, dA_{\xi'}
\]  
(18)

The above integral equation has been used often for numerical analysis of surface pressure, but the numerical implementations have not led to especially accurate results for many cases of practical interest. One of the intrinsic sources of difficulty [Rogers, 1973] is that, for certain discrete characteristic frequencies, the solution is not unique, because the homogeneous integral equation has eigensolutions (the multiplicative constant being arbitrary) at these frequencies. Another complication is that the integration kernel, \( \mathbf{v}_{\xi} \mathbf{G}(\xi,\xi') \), although integrable, is singular, so one is confronted with a singular integral equation. Also, the kernel is not symmetric in the interchange of \( \xi \) and \( \xi' \), so the linear operator associated with this integral equation is not self-adjoint.

4. SECOND INTEGRAL EQUATION FOR SURFACE PRESSURE

A strong case can be made that a variational principle is often a useful intermediate step for developing robust numerical methods for solving an integral equation. However, the integral equation (18) does not lead in a natural manner to a variational principle, because the integration kernel is not self-adjoint. Consequently, we have made use of a second integral equation (strictly speaking, a differential-integral equation) that applies to the same problem [Meyer et. al., 1978]. Although this equation is less well known, it nevertheless has the standard Kirchhoff-Helmholtz integral theorem as its foundation. We begin by using Eq.(1) to form the normal derivative of the pressure on the surface. Because of the singularity of the integral terms for a field point at the surface, we shall temporarily keep \( \xi \) off \( S \) by making use of Eq. (5). Thus, we form
\[ \hat{n}(\xi) \cdot \nabla_x p(\xi) \]
\[ = -\frac{1}{4\pi} \int_S \omega \phi n(\xi) \hat{n}(\xi) \cdot \nabla_x G(x|\xi) \, dA \]
\[ + \frac{1}{4\pi} \int_S p(\xi) \left[ \hat{n}(\xi) \cdot \nabla_x \right] \left[ \hat{n}(\xi) \cdot \nabla_x \right] G(x|\xi) \, dA \]  
(19)

where \( \nabla_x \) denotes the gradient at \( x \).

Note that the integrand in the second term of Eq. (19), in the limit as \( \xi \to x \), is highly singular, so the appropriate expression for numerical evaluation of this term in the limit of \( \varepsilon \to 0 \) has to be developed with some care. To this purpose, we follow an approach similar to that previously developed by Maue [1949] and Stallybrass [1967]. We first introduce the vector identity

\[ (\hat{a} \times \hat{b}) \cdot (\hat{c} \times \hat{d}) = (\hat{a} \cdot \hat{c})(\hat{b} \cdot \hat{d}) - (\hat{a} \cdot \hat{d})(\hat{b} \cdot \hat{c}) \]  
(20)

so that

\[ (\hat{e}_i \times \nabla_x \times \nabla_x) \cdot (\hat{e}_j \times \nabla_x \times \nabla_x) G = (\hat{e}_i \cdot \hat{e}_j)(\nabla_x \cdot \nabla_x) G - (\hat{e}_j \cdot \hat{e}_i)(\nabla_x \cdot \nabla_x) G \]  
(21)

where \( \hat{e}_i \) and \( \hat{e}_j \) are unit vectors (possibly the same) in any cartesian coordinate system and \( G \) is the free-space Green's function (2). Since \( G(x|\xi) \) is a function only of \( R = |x-\xi| \), \( \nabla_x G(x|\xi) = -\nabla_x G(x|\xi) \). This allows us to replace the third term on the right side of the above equation by an analogous expression, in which the subscripts \( i \) and \( j \) are interchanged. After making such a change, we multiply each term by \( n_i(\xi) n_j(\xi) \), and then sum over \( i \) and \( j \) from 1 to 3. A rearrangement of terms leads to
\[
\left(\mathbf{n}(x) \cdot \nabla \right) G = \left(\mathbf{n}(x) \cdot \mathbf{v} \right) G - \left(\mathbf{n}(x) \times \mathbf{v} \right) G 
\]

(22)

Also, because \( G(x,x) \) has the property that \( \nabla \times G(x,x) \) is the negative of \( \nabla x G(x,x) \), and because it satisfies the scalar Helmholtz equation, we can make the substitution

\[
\nabla x \cdot \nabla G(x,x) = - \nabla^2 G(x,x) = k^2 G(x,x) 
\]

(23)

These two latter relations allow Eq. (19) to be recast in the form

\[
\begin{align*}
\mathbf{n}(x) \cdot \nabla p(x) &= - \frac{1}{4\pi} \iint_S \mathbf{n}(x) \cdot \nabla G(x,x) \cdot \mathbf{n}(x) \times G(x,x) \, dA \\
+ & \frac{k^2}{4\pi} \nabla \cdot \iint_S p(x) G(x,x) \times \nabla G(x,x) \, dA \\
- & \frac{1}{4\pi} \left[ \mathbf{n}(x) \times \nabla x \right] \cdot \iint_S p(x) [\mathbf{n}(x) \times \nabla] G(x,x) \, dA
\end{align*}
\]

(24)

The second derivative of the Green's function will present a problem when \( \epsilon \to 0 \), so we integrate the last term by parts, such that
The first integral on the right side vanishes. In order to demonstrate this, we recall a version of Stokes' theorem [Less, 1950]. Let \( F \) be any scalar function that is continuous and that is differentiable with respect to tangential coordinates in \( S'' \), and let \( C \) denote the curve bounding \( S'' \). Then

\[
\iint_S \left[ \mathbf{n}(\hat{t}) \times \nabla F \right] dA \xi = \oint_C F d\mathbf{\hat{t}} \xi \tag{26}
\]

where the line element \( d\mathbf{\hat{t}} \) progresses along \( C \) in the counter-clockwise sense when the unit normal \( \mathbf{n} \) points toward the observer. In the present situation, \( F \) represents the factor contained within braces in the first integral in Eq. (25). Since \( e \neq 0 \), this factor \( F \) is finite. Also, since \( S \) is a closed surface, one can regard the open surface \( S'' \) as being all of \( S \) except for a small patch surrounding any arbitrarily chosen point on \( S \); in the limit as the dimensions of this patch shrink to zero, the curve \( C \) shrinks to zero length. It follows that the right side of Eq. (26) vanishes when \( S'' \) becomes \( S \). As a result, Eq. (24) becomes
\[ \hat{n}(\xi) \cdotp \nabla_x \varphi(x) = -\frac{1}{4\pi} i \omega \int_S \hat{n}(\xi) \cdotp \nabla_x G(x, \xi) \, dA_x \]
\[ + \frac{k^2}{4\pi} \hat{n}(\xi) \cdotp \int_S \hat{n}(\xi) \cdotp G(x, \xi) \, p(\xi) \, dA_x \]
\[ + \frac{1}{4\pi} [ \hat{n}(\xi) \times \nabla_x ] \cdotp \int_S G(x, \xi) \, \hat{n}(\xi) \times \nabla_x p(\xi) \, dA_x \]  

Equation (27)

The desired differential-integral equation emerges after one takes the limit of Eq. (27) as \( \epsilon \to 0 \); doing so yields

\[ i \omega \rho U_n(\xi) = \frac{k^2}{4\pi} \hat{n}(\xi) \cdotp \int_S \hat{n}(\xi) \cdotp G(x, \xi) \, p(\xi) \, dA_x \]
\[ + \frac{1}{4\pi} [ \hat{n}(\xi) \times \nabla_x ] \cdotp \int_S G(x, \xi) \, \hat{n}(\xi) \times \nabla_x p(\xi) \, dA_x \]  

Equation (28)

where we abbreviate

\[ U_n(\xi) = \varphi_n(\xi) + \lim_{\epsilon \to 0} \left[ \frac{1}{4\pi} \int_S \varphi_n(\xi) \cdotp \nabla_x G(x, \xi) \, dA_x \right] \]  

Equation (29a)

\[ = \frac{1}{2} \varphi_n(\xi) + \frac{1}{4\pi} \text{PR} \int_S \varphi_n(\xi) \cdotp \hat{n}(\xi) \cdotp \frac{z-\xi}{R} \, dR \left[ \frac{e^{iR}}{R} \right] dA_x \]  

Equation (29b)

In the latter version, the symbol \( \text{PR} \) implies principal value and \( R \) is understood to be \( |z-\xi| \). The equivalence of these two versions of Eq. (29) can be demonstrated in a manner similar to that used in deriving Eq. (17). The
second version, however, has limited use; it should not be used, for example, if the vibrating body is an unbaftled plate (curved or not curved) with thickness idealized as being infinitesimal.

5. DERIVATION OF THE VARIATIONAL PRINCIPLE

We shall derive the variational principle from Eq. (28). The first step is to multiply each term of that equation by a virtual increment \( \delta p(\xi) \) in the actual pressure amplitude at the surface point \( \xi \). Each term is then integrated over the surface \( S \), with the parameter \( \xi \) distinguishing the points on the surface during this integration. The result is then that

\[
\int_0^L \int_0^L \int_0^L \left[ \left( \frac{v}{\xi} \right) G(\xi) \right] \, dA \,
\]

An analysis (involving Stokes' theorem) similar to that described in Eqs. (24) and (25) applies to the last term in Eq. (30) with the inner integral (enclosed in large braces) regarded as a function \( F(\xi) \). Such a procedure enables one, in effect, to integrate by parts, transferring (with a change in sign) the operator \( \hat{\mathbf{n}}(\xi) \times \nabla_s \) to the factor \( \delta p(\xi) \). The result is that

\[
\frac{1}{\pi} \int_0^L \int_0^L \int_0^L \left[ \left( \frac{v}{\xi} \right) G(\xi) \right] \, dA \,
\]
We are seeking the pressure amplitude $p$ at points on the surface $S$, given the normal velocity amplitude distribution $v_n(t)$. This means that $v_n(t)$ should be held constant when $p$ is varied. Symmetry of the Green's function and the usual rules of the calculus of variables makes it possible to replace, within the integrands, variational factors of the generic form $F(t)\delta F(t)$ by alternative variational factors $(1/2)\delta[F(t)F(t)]$. Also, because the integration limits do not change during the variation, the integral of the variation is the variation of the integral. Similarly, a sum of variations can be replaced by the variation of the sum. All this allows us to recast Eq. (31) as the variational principle

$$\delta J[p] = 0$$

where the functional $J[p]$ can be identified as
\[
J[p] = - i \omega \iint_S p(\xi) \ 4\pi U_n(\xi) \ dA \ 
\]

\[
+ \frac{k^2}{2} \iint_S \iint_S \left( \hat{n}(\xi) \cdot \hat{n}(\xi') \right) p(\xi) \ p(\xi') \ G(\xi \xi') \ dA \ dA \ 
\]

\[
- \frac{1}{2} \iint_S \iint_S \left[ \hat{n}(\xi) \times \nabla_p(\xi') \right] \left[ \hat{n}(\xi) \times \nabla_p(\xi') \right] G(\xi \xi') \ dA \ dA \ 
\] (33)

The implication of Eq. (32) is that the functional \( J \) is stationary to small changes \( \delta p(\xi) \) in the pressure distribution \( p(\xi) \). One cannot, however, in general state that \( J \) has an extreme value (maximum or minimum) when the trial function \( p(\xi) \) is equal to the true complex pressure distribution. If the trial function is taken to be the actual function plus \( \varepsilon f(\xi) \), where \( \varepsilon \) is small and \( f(\xi) \) is a fixed admissible function, then \( J[p] \) can be regarded as a function of \( \varepsilon \), and \( \frac{dJ}{d\varepsilon} \) must be zero at \( \varepsilon = 0 \). The sign of either the real or imaginary part of the second derivative \( \frac{d^2J}{d\varepsilon^2} \), however, depends on the choice of this fixed admissible function \( f(\xi) \). In general, the class of admissible trial functions is restricted to functions that are continuous over the surface and piece-wise differentiable with respect to displacements on the surface, such that \( \hat{n}(\xi) \times \nabla_p(\xi) \) exists almost everywhere and is square integrable.

One possible trial function is simply a position-independent factor times the actual function. The stationarity condition is satisfied when this factor is unity, but differentiation with respect to this factor introduces additional factors of 1 and 2, respectively, for those terms in Eq. (33) which are linear and bilinear in the surface pressure. Consequently, after differentiating the functional with respect to the factor and then setting the factor equal to 1, one derives
\[
\begin{align*}
&i \omega \rho \iint_{S} p(\zeta) 4\pi U_{n}(\zeta) \, dA_{S} = \\
&+ \kappa^{2} \iint_{S} \left[ \hat{\mathbf{n}}(\zeta) \cdot \hat{\mathbf{n}}(\xi) \right] p(\zeta) p(\xi) G(\zeta, \xi) \, dA_{\zeta} dA_{\xi} \\
&- \iint_{S} \left[ \hat{\mathbf{n}}(\zeta) \times \nabla_{S} p(\zeta) \right] \cdot \left[ \hat{\mathbf{n}}(\xi) \times \nabla_{S} p(\xi) \right] G(\zeta, \xi) \, dA_{\zeta} dA_{\xi}
\end{align*}
\]

(34)

where the inserted function \( p(\xi) \) is the actual complex pressure on the surface. (Note that the foregoing is merely the integral over \( dA_{S} \) of Eq. (28).) Substitution of Eq. (34) into Eq. (33) reveals that the stationary value of the functional \( J \) is numerically equal to

\[
J[p_{\text{true}}] = - \frac{1}{2} i \omega \rho \iint_{S} p_{\text{true}}(\zeta) 4\pi U_{n}(\zeta) \, dA_{S}
\]

(35)

Consequently, if a trial function for \( p(\zeta) \), that is correct to first order, is inserted into the functional \( J[p] \) stated in Eq. (33), one obtains an estimate, accurate to second order, of the surface integral of \( p_{\text{true}}(\zeta) U_{n}(\zeta) \). In some cases, the latter may be of principal interest and may have an important physical identification; an accurate method for estimating its value would then be of great use.

6. INTEGRALS INVOLVING SURFACE VELOCITY

A few guidelines may be stated at the outset concerning the improper integral that causes the distinction between \( U_{n} \) and \( v_{n} \) appearing in Eqs. (29). We rewrite those relations as
For thin platelike structures that are unbaffled, the surface $S$ consists of two contiguous sheets with infinitesimal separation between them. If these sheets are labelled by the subscripts $I$ and $II$, then adjacent points on the front and back of the surface can be labelled $\xi_I$ and $\xi_{II}$. The geometry requires

$$\hat{n}(\xi_I) = -\hat{n}(\xi_{II})$$

so, if the surfaces are vibrating together as a unit, one must have

$$v_n(\xi_I) = -v_n(\xi_{II})$$

Insofar as the integral on the right side of Eq. (36a) is concerned, the separation between $\xi_I$ and $\xi_{II}$ is much less than $\varepsilon$ while the limit is being taken. Consequently $R_I$ and $R_{II}$ are the same for all finite $\varepsilon$; the opposite signs in Eq. (38) therefore require the integrations over the two sides of $S$ to be equal but opposite for all finite $\varepsilon$; these integrals cancel and one is left with the simple result

$$U_n = v_n$$

for thin unbaffled plate-like bodies (alternately referred to as laminas).
For less specialized circumstances, a fail-safe method for evaluating the integral in Eq. (36a), without the complications of taking a limit or of separately distinguishing circumstances when portions of the body are laminae, can be developed from the identity

$$\lim_{\epsilon \to 0} \int_S \hat{n}(\xi) \cdot \nabla_x (R^{-1}) \, dA_{\xi} = 0 \quad (40)$$

which follows from (i) the fact that $\nabla_x$ applied to $1/R$ is equivalent to $-\nabla_{\xi}$ applied to $1/R$, (ii) Gauss's theorem, and (iii) the fact that $1/R$ satisfies Laplace's equation. Consequently, one can write

$$4\pi \left[ U_n(\xi) - V_n(\xi) \right] = \lim_{\epsilon \to 0} \int_S \left[ V_n(\xi) \hat{n}(\xi) \cdot \nabla_x G(\xi) \right] \, dA_{\xi}$$

$$= \int_S \left[ V_n(\xi) \hat{n}(\xi) \cdot \nabla_x G(\xi) \right] \cdot \frac{\xi - \hat{\xi}}{R} \frac{d}{dR} \left[ \frac{e^{ikR}}{R} \right] \, dA_{\xi}$$

$$+ V_n(\xi) \int_S \hat{n}(\xi) \cdot \frac{\xi - \hat{\xi}}{R} \frac{d}{dR} \left[ \frac{e^{ikR}}{R} - \frac{i}{R} \right] \, dA_{\xi} \quad (41)$$

Here the singularities in the integrands of the integrals in the second version are at most of order $1/R$ (which is integrable), so one need not be concerned with explicitly taking the limit as $\epsilon \to 0$.

An example for which the above representation might be computationally useful is that of a finite length circular cylinder vibrating as a rigid body parallel to its axis. If the length $L$ of the cylinder goes to zero, then one has a circular disk (discussed in greater depth in the next section) in
transverse vibration, and Eq. (39) should apply, \( U_n = v_n \). This limit, however, does not emerge very easily from Eq. (36b), but it does pop out of Eq. (41). In the limit of \( L \to 0 \), the area integrals in (41) are over only the top \( (z = L/2) \) and bottom surfaces \( (z = -L/2) \) of the cylinder. If \( \xi \) is a point on the top surface, then \( \hat{N}(\xi) \) is in the direction of the unit vector \( \hat{e}_z \), while \( \hat{N}(\xi) \) may be in either the \( \hat{e}_z \) or \(-\hat{e}_z\) directions. With \( L \to 0 \), one has in any case that \( \xi-\xi \) is perpendicular to both \( \hat{N}(\xi) \) and \( \hat{N}(\xi) \), so both the first and second terms in Eq. (41) vanish trivially in this limit. One cannot draw such a conclusion for the integral in Eq. (36b), because the principal value applies only for the integral over the top surface (given that \( \xi \) is on the top surface). For the integral over the bottom end, one must in general evaluate that integral for finite \( L \), then take the limit as \( L \to 0 \). One would get a different (and incorrect) answer if one jumped inside the integral and took the limit as \( L \to 0 \) before evaluating the limit. Such problems do not arise, however, for the integrals in the latter version of Eq. (41), because the integrands are sufficiently well-behaved and non-singular that the order of taking the limit and carrying out the integrations can be freely interchanged.

7. SPECIAL CASE OF A THIN DISK

Because there are no gradients of the Green's function within the integrands of the second and third terms of the functional \( J[p] \) in Eq. (33), one need not characterize those integrals as being Cauchy principal values. Consequently, the formalism is easily adapted to a slender body, for which one region of the surface \( S \) is infinitesimally close to a different region of \( S \).

For example, consider the disk in Fig. 2, which is shown on edge. Let us denote variables on the upper and lower surfaces by a subscript plus or minus, respectively. The disk is assumed to be vibrating as a rigid body normal to its faces, so the analysis given in the preceding section applies and one has \( U_n = v_n \). Also, as stated in Eq. (38), the surface velocities \( v_n(\xi_+) \) and \( v_n(\xi-) \) are 180° out-of-phase, that is, when the upper surface moves inward, the lower surface moves outward, and vice versa. It follows that the actual
Figure 2. Field and surface points for a thin disk.
acoustic pressure must have comparable behavior, and must be antisymmetric with respect to reflection through the plane on which the disk nominally lies. We choose to restrict our class of admissible trial functions for \( p \) to include only functions that have this property. Thus we have

\[
v_n(\xi_-) = -v_n(\xi_+), \quad p(\xi_-) = -p(\xi_+) \tag{42}
\]

\[
v_n(\xi_-) p_n(\xi_-) = v_n(\xi_+) p(\xi_+) \tag{43}
\]

Also, since the unit normal vectors on opposite sides of the disk are oppositely directed, we have

\[
\mathbf{n}(\xi_-) p(\xi_-) = \mathbf{n}(\xi_+) p(\xi_+); \quad \mathbf{n}(\xi_-) \times \mathbf{\nabla} p(\xi_-) = \mathbf{n}(\xi_+) \times \mathbf{\nabla} p(\xi_+) \tag{44}
\]

These relations, of course, also hold if the dummy integration variable \( \xi \) is replaced by the dummy integration variable \( \xi' \).

Equations (42-44) substantially simplify the functional \( J[p] \). When each of the integrals in Eq. (33) is decomposed into the contributions of the upper and lower surfaces, we find that \( J[p] \) may be expressed in terms of integrals extending over the upper surface only; specifically,

\[
J[p] = -8\pi i \omega \int_{S_+} v_n(\xi) p(\xi) \, dA_\xi + 2k^2 \int_{S_+} \int_{S_+} p(\xi) p(\xi') G(\xi' \xi) \, dA_\xi \, dA_{\xi'}
\]

\[
-2 \int_{S_+} \int_{S_+} [\mathbf{e}_z \times \mathbf{\nabla} p(\xi)] [\mathbf{e}_z \times \mathbf{\nabla} p(\xi')] G(\xi' \xi) \, dA_\xi \, dA_{\xi'} \tag{45}
\]

where \( \xi \) and \( \xi' \) are dimensionless variables and can...
8. ANALYSIS OF A RIGID DISK WITH DISTRIBUTED BASIS FUNCTIONS

In this section we shall represent the pressure distribution on the surface of an unbaffled rigid disk in a series of assumed basis functions, which are each in general nonzero at every point on the disk. It is convenient to begin by nondimensionalizing the functional $J$ in Eq. (45). For this purpose we scale distance by the radius $a$ of the disk, and scale pressure by $\rho c v_0$, where $\rho$ and $c$ are the ambient density and sound speed, and $v_0$ is the amplitude of the surface velocity, such that on the plus side of the disk, the normal velocity's complex amplitude is given by

$$v_n(\xi) = v_0$$  \hspace{1cm} \text{(46)}

Let an overcarat denote a nondimensional quantity. We then find that Eq. (45) becomes

$$J[\hat{p}] = \frac{J}{4\rho^2 c^2 v_0^2 a^4} =$$

$$- 2\pi ka \int_{S_+} \hat{p}(\xi) \, dA_{\xi} + \frac{(ka)^2}{4} \int_{S_+} \int_{S_+} \hat{p}(\xi) \hat{p}(\xi) \hat{G}(\xi) \, dA_{\xi} dA_{\zeta}$$

$$- \frac{1}{2} \int_{S_+} \int_{S_+} [\hat{\varepsilon}_2 \times \hat{\psi}_2 \hat{p}(\zeta)] \cdot [\hat{\varepsilon}_2 \times \hat{\psi}_2 \hat{p}(\xi)] \hat{G}(\xi) \, dA_{\xi} dA_{\zeta}$$  \hspace{1cm} \text{(46)}

where

$$\hat{p} = \frac{p}{\rho c v_0}; \quad \hat{\nabla} = a \nabla; \quad \hat{G} = a \hat{G} = \frac{e^{ikaR}}{R}; \quad dA = \frac{1}{a^2} \, dA$$  \hspace{1cm} \text{(47)}
Because of the axial symmetry of the surface velocity, the pressure amplitude on the disk must only be a function of the radial distance from the center of the disk. Polar coordinates for the disk are depicted in Fig. 3, where \( r_a \) and \( s_a \) are defined as the radial distances to the points \( \zeta \) and \( \xi \), respectively. Thus \( r \) and \( s \) are dimensionless variables that may be regarded as the possible arguments (which replace \( \zeta \) and \( \xi \)) of the scaled dimensionless pressure. Correspondingly, we find that

\[
\hat{p}(\zeta) = \hat{p}(r); \quad \hat{p}(\xi) = \hat{p}(s);
\]

\[
\hat{\nabla}_\zeta \hat{p}(\zeta) = \hat{p}'(r) \hat{e}_r; \quad \hat{\nabla}_\xi \hat{p}(\xi) = \hat{p}'(s) \hat{e}_s;
\]

\[
\hat{G}(\zeta, \xi) = \frac{e^{ika\hat{R}}}{\hat{R}}
\]

where a prime indicates a derivative with respect to the argument, and \( \hat{e}_r \) and \( \hat{e}_s \) are the radial unit vectors shown in Fig. 3. The nondimensional distance \( \hat{R} \) is found from the law of cosines to depend on the relative polar angle \( \theta \) (equal to \( \theta_r - \theta_s \)), according to the well known relation

\[
\hat{R} = (r^2 + s^2 - 2rs \cos \theta)^{1/2}
\]

The scaled elements of area in this axisymmetric situation may be taken as

\[
d\hat{A}_\zeta = r d\theta_r dr, \quad d\hat{A}_\xi = s d\theta_s ds
\]

Since the integrand for the double integrations over area depends only on the relative angle \( \theta \), a simple transformation to \( \theta \) as a variable of integration, replacing, say, \( \theta_r \), enables one of the angular integrations to be done trivially, yielding a factor of \( 2\pi \). This leads to the following simple form for the scaled functional that appears in Eq. (46)
Figure 3. Position coordinates for dual integration.
\[
\frac{j}{2\pi} = -2\pi i \kappa \int_0^1 \hat{p}(r) r \, dr + \frac{(\kappa a)^2}{2} \int_0^1 \int_0^1 \hat{p}(r) \hat{p}(s) C_1(r|s) \, dr \, ds
\]

\[
- \frac{1}{2} \int_0^1 \int_0^1 \hat{p}'(r) \hat{p}'(s) C_2(r|s) \, dr \, ds
\]

(51)

where

\[
C_1(r|s) = rs \int_0^{2\pi} \frac{e^{i\kappa a \hat{R}}}{\hat{R}} \, d\theta
\]

(51a)

\[
C_2(r|s) = rs \int_0^{2\pi} \frac{e^{i\kappa a \hat{R}}}{\hat{R}} \cos \theta \, d\theta
\]

(52b)

Note that the factor $2\pi$ occurs in Eq. (51) because the integrands in Eq. (46) are dependent on the polar angles $\theta_s$ and $\theta_r$ only through their difference, which is here denoted as $\theta$. The differential factors $C_1(r|s) \, ds \, dr$ and $C_2(r|s) \, ds \, dr$ can be interpreted as describing the effect of an annulus of radius $s$ and width $ds$ on another annulus of radius $r$ and width $dr$. We shall refer to these functions $C_1$ and $C_2$ as the integrated Green's functions.

The basis functions selected to represent the pressure distribution on the surface of the disk must satisfy all requirements that are imposed by the manner of derivation of the variational principle on admissible trial functions. In particular, these must be continuous and differentiable, so that derivatives such as $\hat{p}'(r)$ exist. The derivation exploited the fact that the pressure at corresponding locations on the upper and lower surface differ only in sign. However, because the thickness of the disk is infinitesimal, the pressure on the two surfaces at the edge must be equal. Both conditions can
only be satisfied if the pressure vanishes at the edge, so

\[ \hat{p}(r) = 0 \quad \text{at } r = 1 \]  

(53)

Since the admissible trial functions must be continuous over the disk's surface, each of the basis functions must satisfy the above condition. Thus the general trial function formed as a linear combination of N basis functions can be written

\[ \hat{p}(r) = \sum_{n=1}^{N} P_n \Psi_n(r); \quad \Psi_n(1) = 0 \]  

(54)

Note that the condition \( p' = 0 \) at \( r = 0 \), which results from axisymmetry, is a natural boundary condition that will emerge from the variational principle for the exact solution, but it is not a requirement that must be imposed at the outset on trial functions and on variations. It is not necessary that the basis functions \( \Psi_n(r) \) satisfy this condition.

When we substitute Eq. (54) and a similar expression for \( p(s) \) into Eq. (51), we find that \( \hat{J} \) is a quadratic polynomial in the complex coefficients \( P_n \):
\[
\hat{J} = \frac{1}{2} \sum_{j=1}^{N} \sum_{n=1}^{N} a_{jn} \rho_j \rho_n - \sum_{n=1}^{N} b_n \rho_n
\]  
\[a_{jn} = a_{nj} = (ka)^2 \int_{0}^{1} \int_{0}^{1} \psi_j(r) \psi_n(s) c_1(r,s) \, dr \, ds\]  
\[b_n = 2\pi k a \int_{0}^{1} r \psi_n(r) \, dr\]  

The task of evaluating the coefficients \(a_{jn}\) and \(b_n\) shall be addressed in the next section. Once they are known, the variation of \(J[p]\) is obtained from virtual increments in each of the pressure coefficients. Those increments are arbitrary, so \(\delta\hat{J} = 0\) for all admissible variations in the trial function (54) requires that

\[
\frac{\delta \hat{J}}{\delta \rho_n} = 0 \quad \text{for } n = 1, 2, \ldots, N
\]  

In view of the form of \(\hat{J}\) indicated by Eq. (55), we conclude that the coefficients are governed by

\[\{a\} \{P\} = \{b\}\]  

where the elements of the square array \(\{a\}\), and of the vectors \(\{b\}\) and \(\{P\}\) are the \(a_{mn}\), \(b_n\), and \(P_n\), respectively. After the set of linear equations represented by Eq. (59) has been solved for the coefficients \(P_n\), it is a simple matter to recreate the spatial distribution \(\hat{\rho}(r)\) from Eq. (54).
9. EVALUATION OF THE INTEGRATED GREEN'S FUNCTIONS

A singularity in each of the integrated Green's functions, given by Eqs. (52), arises when the source radial coordinate s approaches the field radial coordinate r. In such a circumstance, the separation distance \( R \), which appears in the denominator of the respective integrands, vanishes for \( \theta = 0 \). Our approach to evaluating the \( C_j(rls) \) will separate out their singular parts as elliptic integrals.

The analysis and the writing of the arguments of the elliptic integrals are facilitated if one introduces the abbreviations

\[
m = \frac{4rs}{(r+s)^2}; \quad \alpha = ka(r+s)
\]

so that Eqs. (52) become

\[
C_1(rls) = \frac{1}{2} m(r+s) \int_0^\pi \frac{e^{i\alpha D}}{D} \, d\theta \\
C_2(rls) = \frac{1}{2} m(r+s) \int_0^\pi \frac{e^{i\alpha D}}{D} \cos \theta \, d\theta
\]

where

\[
D = \frac{\hat{R}}{r+s} = \left[1 - m \cos^2(\theta/2)\right]^{1/2}
\]

The factor \( \cos \theta \) in the integrand of the expression (61a) for \( C_2(rls) \) is eliminated with the aid of a trigonometric identity.
\[
\cos \theta = 2 \cos^2(\theta/2) - 1 = \frac{2-m}{m} - \frac{2}{m} D^2
\]  

As a result, the integrated Green's functions may be obtained from two basic integrals, \(G_1\) and \(G_2\), according to

\[
C_1(r_is) = m(r+s)G_1(a,m) \quad (64a)
\]
\[
C_2(r_is) = (r+s)[(2-m)G_1(a,m) - 2G_2(a,m)] \quad (64b)
\]

where

\[
G_1(a,m) = \frac{1}{2} \int_0^\pi \frac{e^{iaD}}{D} \, d\theta 
\]  

(65a)

\[
G_2(a,m) = \frac{1}{2} \int_0^\pi D \, e^{iaD} \, d\theta
\]  

(65b)

Both of the above expressions reduce to elliptic integrals when \(a\) is set to zero. To highlight the resemblance of the functions \(G_j(a,m)\) to elliptic integrals, Euler's identity and the trigonometric half-angle formulas are combined. The exponential factor then becomes

\[
e^{iaD} = \left[ 1 - 2 \sin^2 \left( \frac{aD}{2} \right) \right] + 2i \sin \left( \frac{aD}{2} \right) \cos \left( \frac{aD}{2} \right)
\]  

(66a)

\[
= 1 + 2i \sin \left( \frac{aD}{2} \right) \exp \left( \frac{iaD}{2} \right)
\]  

(66b)

We now introduce a change of variable for the polar angle \(\theta\),
\[ \phi = \frac{\pi}{2} - \frac{\theta}{2} \]  \hspace{1cm} (67)

so that the distance parameter \( D \) becomes

\[ D = \left(1 - m \sin^2 \phi\right)^{1/2} \]  \hspace{1cm} (68)

Substitution of Eqs. (66)-(68) converts Eqs. (65) to

\[ G_1(\alpha, m) = K(m) + 2i \int_0^{\pi/2} \frac{1}{D} \sin \left( \frac{\phi D}{2} \right) \exp \left( \frac{i \phi D}{2} \right) d\phi \]  \hspace{1cm} (69a)

\[ G_2(\alpha, m) = E(m) + 2i \int_0^{\pi/2} D \sin \left( \frac{\phi D}{2} \right) \exp \left( \frac{i \phi D}{2} \right) d\phi \]  \hspace{1cm} (69b)

where \( K(m) \) and \( E(m) \) are the complete elliptic integrals [Milne-Thomson, 1972] of the first and second kind, respectively,

\[ K(m) = \int_0^{\pi/2} \frac{d\phi}{(1-m \sin^2 \phi)^{1/2}} \]  \hspace{1cm} (70a)

\[ E(m) = \int_0^{\pi/2} (1-m \sin^2 \phi)^{1/2} d\phi \]  \hspace{1cm} (70b)

The singularity associated with the field point approaching the source point corresponds now to \( m = 1 \) and \( \phi = \pi/2 \), in which case \( D = 0 \). Both integrands in Eqs. (69) have a finite limit as \( D \to 0 \), so their evaluation
involves a straightforward application of conventional numerical methods. We use Simpson's rule, based on subdividing the interval $0 \leq \phi \leq \pi/2$ sufficiently finely to resolve the oscillations in the sinusoidal functions forming the integrand. Corresponding values for the elliptic integrals were obtained from polynomial approximations [Milne-Thomson, 1972]. Such are conveniently expressed in terms of the complementary parameter

$$m_1 = 1 - m$$

(71)

Then, to an absolute error that does not exceed $2 \times 10^{-8}$, we have

$$K(m) = \sum_{j=0}^{4} (a_j - b_j \ln m_1)m_1^j$$

(72a)

$$E(m) = 1 + \sum_{j=1}^{4} (c_j - d_j \ln m_1)m_1^j$$

(72b)

where the numerical coefficients are
\begin{align*}
a_0 &= (1/2) \ln(16) \quad b_0 = 0.5 \\
&= 1.38629436112 \\
a_1 &= 0.09666344259 \quad b_1 = 0.12498593597 \\
a_2 &= 0.03590092383 \quad b_2 = 0.06880248576 \\
a_3 &= 0.03742563713 \quad b_3 = 0.03328355346 \\
a_4 &= 0.01451196212 \quad b_4 = 0.00441787012 \\
c_1 &= 0.44325141463 \quad d_1 = 0.24998368310 \\
c_2 &= 0.06260601220 \quad d_2 = 0.09200180037 \\
c_3 &= 0.04757383546 \quad d_3 = 0.04069697526 \\
c_4 &= 0.01736506451 \quad d_4 = 0.00526449639 \\
\end{align*}

The logarithmic singularity in \( K(m) \) as \( m \to 1 \) causes both of the integrated Green's functions \( C_j(r|\mathbf{s}) \) to become singular as \( s \to r \). Although the manner of derivation of Eq. (56), which defines the matrix coefficient \( a_{ij} \), indicates that the requisite integrals must exist, and that the singularities in the integrands must be integrable, it is instructive to demonstrate this afresh taking into account the explicit knowledge of the logarithmic singularities. We accordingly digress to give a brief proof that the contribution of the singularity of \( K(m) \) to these integrals is finite.

When \( m \) is very close to unity, \( m \approx 1 \), the elliptic integral \( E(m) \) is also nearly unity (and therefore finite), but the elliptic integral of the first kind has the behavior

\[ K(m) = \frac{1}{2} \ln \left( \frac{16}{1 - m} \right) \]  

(74)

For the integrations involved in the expression (56) for the matrix coefficient \( a_{ij} \), the region in the integration plane surrounding the singularity can be taken to be the strip along the diagonal described by \( r - \Delta \leq s \leq r + \Delta \), where \( \Delta \) is very small. In this region, the dominant contributions to the \( C_j(r|\mathbf{s}) \)
factors in the integrand are identified from Eqs. (64) and (69) to be

\[ C_1(r|s) = 2rmK(m); \quad C_2(r|s) = 4rK(m) \]

If one assumes that the derivatives \( \phi'(r) \) of the basis functions are finite, then the corresponding contributions from such singular terms to the integrals in Eq. (56), which define the \( a_{jn} \), can be generically written

\[
\{\text{Contribution to } a_{jn}\} \approx \int_0^{r+\Delta} \int_{r-\Delta}^{r+\Delta} F(r,s) r K(m) \, ds \, dr + O(\Delta) \tag{75}
\]

where, with the assumption previously stated, \( F(r,s) \) denotes a factor that is finite in the domain of integration.

Because of the smallness of \( \Delta \), the mean value theorem of the integral calculus allows us to neglect fluctuations in \( F(r,s) \) resulting from changing \( s \), so we replace \( F(r,s) \) by \( F(r,r) \). Then, in order to perform the integral over the domain of \( s \), we change the variable of integration to \( \xi \), such that

\[
s = r(1+\xi), \quad 1-m = \left( \frac{s-r}{s+r} \right)^2 \approx \frac{1}{4} \xi^2; \quad K(m) \approx \frac{1}{2} \ln \left[ \frac{64}{\xi^2} \right] \tag{76}
\]

Such substitutions transform Eq. (75) to
\{\text{Contribution to } a_{jn}\} \approx \int_0^{\Delta/r} F(r,r) (2r^2) \ln(8r/\xi) \, dr + O(\Delta) \quad (77a)

\int_0^1 F(r,r) (2r\Delta) \ln(8r/\Delta) \, dr + O(\Delta) \quad (77b)

The integrand factor \((2r\Delta)\ln(8r/\Delta)\) is bounded in absolute value for all combinations of \(r\) and \(\Delta\), given that both are between 0 and 1. This is clear because \(x[\ln(1/x)]\) is less than \(e^{-1} \approx 0.37\). Thus, providing \(F(r,r)\) is finite for all \(r\) between 0 and 1, the integral in Eq. (77b) is bounded, and it goes to zero when \(\Delta\) goes to zero.

As is explained further below, it is advantageous to use basis functions \(\Psi(r)\) that go to zero like \((1-r)^{1/2}\) when \(r \to 1\). Such would require \(\Psi'\) to have an integrable singularity at \(r=1\). However, the above demonstration of boundedness would no longer be valid because \(F(r,s)\) would be singular when either \(r\) or \(s\) are unity. Thus, there would be a confluence of two or three singularities at the point \(r=s=1\). To check whether this precludes the guaranteed existence of finite matrix coefficients, it is sufficient to examine the existence of the integral

\begin{equation}
\{\text{Contribution to } a_{jn}\} \approx M \int_1^{1-\varepsilon} \frac{1}{1-\varepsilon} \frac{1}{(1-r)^{1/2}(1-s)^{1/2}} \frac{\ln[64/(r-s)^2]}{ds \, dr} \quad (78)
\end{equation}

where \(M\) is finite and \(\varepsilon\) is small compared to unity. The above is an integral over a small square of side length \(\varepsilon\) with a corner at the point \(r=s=1\). Alternatively, it is sufficient to consider integration over the quadrant of a circle centered at \(r=s=1\) and having radius \(\varepsilon\). With the latter understanding, we let \(1-r\) be \(a\varepsilon \cos\phi\) and \(1-s\) be \(a\varepsilon \sin\phi\), such that \(ds \, dr\) is replaced by
\[ \varepsilon^2 \alpha \, d\alpha \, d\phi \] and \( \alpha \) ranges from 0 to 1, while \( \phi \) ranges from 0 to \( \pi/2 \). The quadrant integral corresponding to the examination exercise posed by Eq. (78) above then becomes

\[
\{\text{Contribution to} \ a_{jn}\} = M_\varepsilon \int_{\alpha=0}^{\alpha=1} \int_{\phi=0}^{\phi=\pi/2} \ln\left\{64/[\varepsilon^2 \alpha^2 (\cos\phi - \sin\phi)^2]\right\} \frac{d\alpha}{(\sin\phi)^{1/2} (\cos\phi)^{1/2}}
\]

\[
= 4M_\varepsilon \ln\{8\varepsilon/e\} \int_{\phi=0}^{\phi=\pi/4} \frac{d\phi}{(\sin\phi)^{1/2} (\cos\phi)^{1/2}}
\]

\[
- 4M_\varepsilon^2 \int_{\phi=0}^{\phi=\pi/4} \frac{\ln(\cos\phi - \sin\phi) \, d\phi}{(\sin\phi)^{1/2} (\cos\phi)^{1/2}} \quad (79)
\]

Both of the indicated integrations over \( \phi \) in the latter expression exist and have values of the order of unity, even though the integrands have one and two singularities, respectively. Both terms are finite and, moreover, they go to zero when \( \varepsilon \) goes to zero.

One may conclude from the analysis of the present section that the integrated Green's functions may be evaluated numerically without much effort, even though the original integrals defining these quantities had integrands with singularities. Both of these integrated Green's functions have a logarithmic singularity at \( r=s \), so when these functions appear as factors in an integrand, as in Eq. (56), the integration scheme must not explicitly ask for values at such arguments. However, the presence of these singularities does not cause any of the anticipated integrands to be nonintegrable.
10. EVALUATION OF THE MATRIX ELEMENTS

Near the edge of the disk the exact solution for the acoustic pressure on the surface must behave to leading order in the distance from the edge in the same manner as would a solution of Laplace's equation near a knife edge. Thus if $a$ is radial distance from the edge and if $\phi$ is angle about the edge, such that $\phi$ is 0 on the front side and $2\pi$ on the back side, then

$$\frac{\partial^2 p}{\partial a^2} + a^{-1} \frac{\partial p}{\partial a} + a^{-2} \frac{\partial^2 p}{\partial \phi^2} = 0$$

(80)

for sufficiently small $a$. The pressure $p$ must be finite near the edge, and it must satisfy the rigid wall boundary conditions

$$\frac{\partial p}{\partial a} = 0 \text{ at } \phi = 0 \text{ and at } \phi = 2\pi$$

(81)

This "boundary value problem" can be solved by the method of separation of variables, with the result that, to leading order in $a$,

$$p = A + B a^{1/2} \cos(\phi/2)$$

(82)

where $A$ and $B$ are "constants". For the particular case of sound generated by a disk in rigid body transverse vibration, symmetry and continuity require that $p$ be zero at the edge, so the "constant" $A$ is identically zero. Hence, on the front surface, where $\phi = 0$, the implication of the above equation is

$$p \propto (\text{distance from edge})^{1/2}$$

(83)

near the edge of the disk.

The above deduced behavior must be exhibited by the exact solution for $p$ of either the differential integral equation or of the variational principle. It is not, however, a requirement that must be imposed at the outset on the trial functions or the basis functions. The derived approximate solution, if the number of basis functions is sufficiently large, can be expected, in the aggregate, to approximate the behavior of Eq. (83).
Since we know the result (83) in advance of a detailed numerical solution, a strong argument can be made that it should be incorporated into the trial functions at the outset, for then a faster convergence toward the exact solution might be achieved. One possibility for doing this is to include a factor \((1-r^2)^{1/2}\) in each of the \(\Psi_n(r)\). One could take, for example, \(\Psi_1(r) = (1-r^2)^{1/2}\), which turns out to be the exact result [Pierce, 1981] for the pressure distribution on the disk in the limit \(ka \rightarrow 0\). Doing so, however, introduces another singularity into one of the integrands of the integrals that define the matrix elements \(a_{jm}\) in Eq. (56). Clearly, \(\Psi_1'(r)\) is infinite at \(r=1\). The singularity is integrable, but integration over singular integrands is an inherent source of numerical difficulties. Such difficulties are often surmountable with a change of integration variable that transforms the original integrand into one in which the singularity does not appear.

With the purpose just described in mind, we denote the desired transformation as

\[
    r = g(u); \quad s = g(w) \quad (84)
\]

Then

\[
    \Psi_m'(r) \ dr = \frac{d}{du} \Psi_m[g(u)] \ du \quad (85)
\]

If \(g(u)\) is chosen appropriately, then \((d/du)\Psi_m(g(u))\) will be finite at the value of \(u\) where \(g(u)=1\), even if \(\Psi_m(1)\) is not. It is convenient to have the limits of the domain of \(u\) and \(w\) match those of \(r\) and \(s\), so we introduce the requirement that

\[
    g(0) = 0; \quad g(1) = 1 \quad (86)
\]

The choice for the integration variable transformation function \(g\) could be different for different basis functions, but it is advantageous for us to require that it be the same for all elements within the same set, such that no symmetry properties are lost; this is assumed to have been done in the discussion that follows.
In the case of the basis function $\psi_1 = (1-r^2)^{1/2}$, we shall set

$$g(u) = \sin \left( \frac{\pi u}{2} \right)$$

This satisfies Eqs. (86), and the derivative in Eq. (85) transforms to

$$\frac{d}{du} \psi_1[g(u)] = \frac{d}{du} \left[ 1 - \sin^2 \left( \frac{\pi u}{2} \right) \right]^{1/2}$$

$$= - \frac{\pi}{2} \sin \left( \frac{\pi u}{2} \right)$$

which is well-behaved for all $u$.

To simplify the notation, we let $\Gamma_n(u)$ denote the $n$-th basis function when it is regarded as a function of the alternate integration variable $u$, so that

$$\Gamma_n(u) \equiv \psi_n[g(u)]$$

Then substitution of Eqs. (84), (85), and (89) into Eq. (56) yields

$$a_{jn} = a_{nj} = (ka)^2 \int_0^1 \int_0^1 g'(u)g'(w)\Gamma_j(u)\Gamma_n(w)C_1(g(u)g(w)) \, dw \, du$$

$$- \int_0^1 \int_0^1 \Gamma'_j(u)\Gamma'_n(w)C_1(g(u)g(w)) \, dw \, du$$

Because the source variable $r$ and field variable $s$ undergo the same transformation, the singularity contributed by the integrated Green's functions continues to lie on the diagonal, which is now the line $w=u$. This line can be taken to be a boundary line for the integration domain, by using the simple identity
\[
\int_0^1 \int_0^1 F(u,w) \, dw \, du = \int_0^1 \int_0^1 \left[ F(u,w) + F(w,u) \right] dw \, du \tag{91}
\]

where the function \( F(u,w) \) is arbitrary. With the recognition that the integrated Green's functions are unchanged by interchange of their arguments, the application of this identity to Eq. (90) yields

\[
a_j n = \left( \frac{ka}{2} \right) \int_0^1 \int_0^1 \left[ \Gamma_j(u) \Gamma_n(w) + \Gamma_j(w) \Gamma_n(u) \right] g'(u)g'(w) C_1(g(u)lg(w)) \, dw \, du
\]

\[
- \int_0^1 \int_0^1 \left[ \Gamma_j'(u) \Gamma_n(w) + \Gamma_j'(w) \Gamma_n'(u) \right] C_2(g(u)lg(w)) \, dw \, du \tag{92}
\]

In a similar manner, the "forcing vector" components \( b_n \), given previously by Eq. (57), when expressed in terms of the new variable of integration \( u \), become

\[
b_n = 2\pi ka \int_0^1 g(u)g'(u)\Gamma_n(u) \, du \tag{93}
\]

The integrations required to evaluate the components \( b_n \) may be effected by direct numerical quadrature; Simpson's 1/3 rule should be adequate. The interval \( 0 \leq u \leq 1 \) is regarded as composed of \( J \) intervals, where \( J \) is even; then

\[
b_n = 2\pi ka \sum_{j=0}^{J} \Gamma_n(u_j)g(u_j)g'(u_j)\nu_j \tag{94}
\]
where \( u_j = j/J \) is the endpoint of the j-th integration interval. The weighting coefficients \( \nu_j \) for Simpson's "1/3" rule are \( 1/(3J), 4/(3J), 2/(3J), 4/(3J), \ldots, 4/(3J), \) and \( 1/(3J) \) for \( j = 0, 1, 2, 3, \ldots, J-1, \) and \( J. \) The J-th term in the above sum is identically zero because, as remarked previously, one must have \( \Gamma_n(1) = 0. \) Recall that all admissible basis functions must vanish at the edge of the disk.

The double integrations required for the evaluation of the matrix coefficients \( a_{jn} \) in Eq. (90) cannot be evaluated by sequential application of a one-dimensional integration rule. Such an approach, which would discretize \( u \) and \( w \) in the same manner, would tacitly assume that the integrand be finite along the line \( w=u \) (where, in fact, the integrand is singular), since the integration algorithm would use values of the integrand evaluated at points along this line. We circumvent this difficulty by using the alternative expression, Eq. (92), and by employing an area integration rule based on interior points. As shown in Fig. 4, the triangular integration domain, which is bounded by the three lines \( w=0, u=1, \) and \( w=u, \) is segmented into \( K(K-1) \) slanted squares, and \( 2K \) triangles, based on \( K \) divisions along the lines \( w=0 \) and \( u=1. \) The value of \( K \) need not be the same as the number of intervals \( J \) that are used in the evaluation of the \( b_n. \) For integration over each of the interior squares, one uses a nine-point integration rule [Davis and Polonsky, 1972], such that the integral over a square is approximated by the area of the square times a weighted average of the values of the integrand at nine interior points. The locations of these points and the weighting factors are selected such that the integration will be exact if the integrand is a polynomial of the fifth degree. The relative location of these interior points within a single square is shown in Fig. 5.

Let \( 2\Delta = 1/K \) be the width of a subdivision interval along the line \( w=0 \) between \( u=0 \) and \( u=1. \) Each of the interior squares will then have length \( 2^{1/2} \Delta \) on a side, diagonals of length \( 2\Delta, \) and areas \( 2\Delta^2. \) The centerpoints of these interior squares are at
Figure 4. Discretization of integration domain into segments.
Figure 5. Numerical integration points for a square segment.
\[ u_{km,c} = 2k\Delta + (m-1)\Delta; \quad w_{km,c} = m\Delta \]
\[
\begin{align*}
\left\{
\begin{array}{l}
m = 1, \ldots, 2k-2k \\
k = 1, \ldots, K-1
\end{array}
\right.
\]

such that squares distinguished by the same value of the index \( k \) have centerpoints that lie along a common line parallel to the diagonal line \( w=u \); squares with the same \( k \) are numbered by the index \( m \) in the order in which they are encountered as one moves up this line toward increasing \( u \) and \( w \). The coordinates of the nine integrand sampling points within a given square and the corresponding weights (which sum to unity) are

\[ u_{km}(q) = u_{km,c} + a(q)\Delta; \quad w_{km}(q) = w_{km,c} + b(q)\Delta; \quad q = 1, 2, \ldots, 9 \]

\[
\begin{array}{ll}
a(1) = 0 & b(1) = 0 \\
a(2) = -(3/20)^{1/2} & b(2) = +(3/20)^{1/2} \\
a(3) = +(3/20)^{1/2} & b(3) = +(3/20)^{1/2} \\
a(4) = +(3/20)^{1/2} & b(4) = -(3/20)^{1/2} \\
a(5) = -(3/20)^{1/2} & b(5) = -(3/20)^{1/2} \\
a(6) = -(3/5)^{1/2} & b(6) = 0 \\
a(7) = 0 & b(7) = +(3/5)^{1/2} \\
a(8) = +(3/5)^{1/2} & b(8) = 0 \\
a(9) = 0 & b(9) = -(3/5)^{1/2}
\end{array}
\]

\[ \nu_{sq}(1) = 16/81; \quad \nu_{sq}(2) = \nu_{sq}(3) = \nu_{sq}(4) = \nu_{sq}(5) = 10/81 \]
\[ \nu_{sq}(6) = \nu_{sq}(7) = \nu_{sq}(8) = \nu_{sq}(9) = 25/324 \]  

(96)

The integrations over the \( K \) triangles that fill the gaps left by the squares along the line \( w=0 \) (lower edge, abbreviated le) can be similarly performed using a seven-point integration rule. The lower edges of these triangles have their centers at \( u = \Delta, 3\Delta, 5\Delta, \ldots, (2K-1)\Delta \); each has height \( \Delta \) and area \( \Delta^2 \). The coordinates of the integrand sampling points within the \( k \)-th triangle and the corresponding weights are derived by mapping the \( 45^\circ-45^\circ-90^\circ \)
triangle to an equilateral triangle and then using a standard integration scheme for the latter. [One of the expressions on page 483 in the Dover edition of Abramowitz and Stegun's Handbook of Mathematical Functions has an error. What appears there as -(15)\(1/2+1\) should be -\([15]1/2\+1\).] The result of such a derivation yields

\[
\begin{align*}
\nu_{\text{tri},1}(q) &= (2k-1)\Delta + a(q)\Delta; & \nu_{\text{tri},1}(q) &= \beta(q)\Delta; & q &= 1, 2, \ldots, 7 \\
a(1) &= 0 & \beta(1) &= 1/3 \\
a(2) &= 0 & \beta(2) &= [9 + 2(15)^{1/2}] / 21 \\
a(3) &= -[1 + (15)^{1/2}] / 7 & \beta(3) &= [6 - (15)^{1/2}] / 21 \\
a(4) &= +[1 + (15)^{1/2}] / 7 & \beta(4) &= [6 - (15)^{1/2}] / 21 \\
a(5) &= 0 & \beta(5) &= [9 - 2(15)^{1/2}] / 21 \\
a(6) &= -[(15)^{1/2}-1] / 7 & \beta(6) &= [6 + (15)^{1/2}] / 21 \\
a(7) &= +[(15)^{1/2}-1] / 7 & \beta(7) &= [6 + (15)^{1/2}] / 21 \\

\nu_{\text{tri},1}(1) &= 270 / 1200; & \nu_{\text{tri},1}(2) &= \nu_{\text{tri},1}(3) = \nu_{\text{tri},1}(4) &= 155 - (15)^{1/2} / 1200 \\
\nu_{\text{tri},1}(5) &= \nu_{\text{tri},1}(6) = \nu_{\text{tri},1}(7) &= 155 + (15)^{1/2} / 1200 \tag{97}
\end{align*}
\]

Similarly, for the \(K\) triangles with hypotenuses along the line \(u=1\) (right edge, abbreviated re), one has

\[
\begin{align*}
u_{\text{re},1}(q) &= 1 - \beta(q)\Delta; & w_{\text{re},1}(q) &= (2k-1)\Delta + a(q)\Delta; & q &= 1, 2, \ldots, 7 \tag{98}
\end{align*}
\]

with the \(a(q), \beta(q), \) and \(\nu_{\text{tri},1}(q)\) the same as in Eq. (97).

The integration scheme just described gives the following approximate representation for the double integral of any function \(F(u,w)\) over the unit square:

- 48 -
\[ \int_{0}^{1} \int_{0}^{1} F(u,w) \, dw \, du = \int_{0}^{1} \int_{0}^{1} \left[ F(u,w) + F(w,u) \right] \, dw \, du \]

\[ = \sum_{k=1}^{K-1} \sum_{m=1}^{2K-2k} \sum_{q=1}^{9} 2 \Delta^2 S(u_{km}(q), w_{km}(q)) \nu_{sq}(q) \]

\[ + \sum_{k=1}^{K} \sum_{q=1}^{7} \Delta^2 S(u_{le,k}(q), w_{le,k}(q)) \nu_{tr1}(q) \]

\[ + \sum_{k=1}^{K} \sum_{q=1}^{7} \Delta^2 S(u_{re,k}(q), w_{re,k}(q)) \nu_{tr1}(q) \]  \hspace{1cm} (99)

where

\[ \Delta = 1/(2K); \quad S(u,w) = F(u,w) + F(w,u) \]  \hspace{1cm} (100)

For the computation of a particular \( a_{jn} \), the function \( F(u,w) \) is identified as

\[ \text{Integrand } F(u,w) \text{ for } a_{jn} = (ka)^2 \Gamma_j(u) \Gamma_n(w) g'(u) g'(w) C_1(g(u) g(w)) \]

\[ - \Gamma_j'(u) \Gamma_n'(w) C_2(g(u) g(w)) \]  \hspace{1cm} (101)

Many numerical operations are required to evaluate the matrix elements \( a_{jn} \) using Eqs. (99-101) for all combinations of \( j \) and \( n \). One saving comes from the recognition that the sampled values of the integrated Green's functions appearing there are independent of the \( (j,n) \) pair under consideration. Hence, the values of \( g'(u) g'(w) C_1(g(u) g(w)) \) and \( C_2(g(u) g(w)) \) for all points \( (u,w) \) used in the sums in Eq. (99) may be stored in arrays, and then recalled whenever necessary to calculate the integrand (101) at the corresponding points.
11. FINITE ELEMENT BASIS FUNCTIONS

The implication thus far has been that the set of basis functions consists of a few analytical functions satisfying the boundary condition $\hat{p}(1)=0$. However, the generic trial functions will also be admissible if the basis functions are not smooth curves, but merely continuous, with a finite number of derivative discontinuities.

Suppose one seeks to represent the surface pressure distribution $\hat{p}(r)$ by $N$ discrete values at evenly spaced radial distances $r_n=(n-1)/N$. As shown in Fig. 6, a linear interpolation between such discrete values may be represented as a superposition of piecewise linear functions. Let $\Psi_n(r)$ denote a linear finite element basis function centered at $r_n$. We define this basis function to be unity at $r=r_n$, and to vanish beyond the adjacent points $r_{n-1}$ and $r_{n+1}$. A simple algorithm for evaluating these basis functions defines the local distance

\[ d = r - r_n = r - (n-1)\Delta; \quad \Delta = 1/N \]  

and then sets

\[ \Psi_n(r) = \begin{cases} 1-|d|/\Delta & \text{if } |d|<\Delta \\ 0 & \text{if } |d|>\Delta \end{cases} \]  
\[ \Phi_n(r) = \begin{cases} -1/\Delta & \text{if } 0\leq d<\Delta \\ 1/\Delta & \text{if } -\Delta<d<0 \\ 0 & \text{if } |d|>\Delta \end{cases} \]  

When we use these finite element basis functions to form a generic trial function as in Eq. (54), it is immediately apparent that the coefficients of the basis functions are identical to the discrete pressure values, that is,
Figure 6a. Linear interpolation of a discretized pressure.

Figure 6b. Finite element basis functions.
\[ P_n = \hat{p}(r_n) \quad \text{where} \quad r_n = (n-1)\Delta \] (104)

In principle, we could employ the finite element basis functions directly using the general formulation and numerical scheme developed in the preceding section. However, doing so would lose potential computational savings resulting from the fact that the finite element basis functions vanish over a large portion of the range \( 0 \leq r \leq 1 \). Such savings may be very significant, because many basis functions may be required for an accurate description of the pressure distribution.

For such a purpose, it is first convenient to split the overall sum of the type in Eq. (99) for the evaluation of a given \( a_{jn} \) using Eqs. (100) and (101) into two partial sums \( S_{jn} \) and \( S_{nj} \), such that

\[ a_{jn} = a_{nj} = S_{jn} + S_{nj} \] (105)

where

\[ S_{jn} = \sum_{k=1}^{K-1} \sum_{m=1}^{2K-2k} \sum_{q=1}^{9} 2\Delta^2 F_{jn}(u_{km}(q), w_{km}(q)) \nu_{sq}(q) \]

\[ + \sum_{k=1}^{K} \sum_{q=1}^{7} \Delta^2 F_{jn}(u_{le,k}(q), w_{le,k}(q)) \nu_{trj}(q) \]

\[ + \sum_{k=1}^{K} \sum_{q=1}^{7} \Delta^2 F_{jn}(u_{re,k}(q), w_{re,k}(q)) \nu_{trj}(q) \] (106)

\[ F_{jn}(u,w) = (ka)^2 \Gamma_j(u)\Gamma_n(w)g'(u)g'(w) C_1(g(u)g(w)) \]

\[ - \Gamma_j(u)\Gamma_n'(w) C_2(g(u)g(w)) \] (107)

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Since the sum in Eq. (106) uses only points for which the \( w \) coordinate is less than the \( u \) coordinate, the sum defining the \( S_{jn} \) will be identically zero if all of the \( u \)'s for which \( \Gamma_j(u) \) is nonzero are less than all of the \( u \)'s for which \( \Gamma_n(u) \) is nonzero.

By definition, \( \Gamma_j(u) = \Psi_j(g(u)) \). For the finite element basis functions defined by Eqs. (103), the products \( \Psi_j(r)\Psi_n(s) \) and \( \psi_j(r)\psi_n(s) \) are nonzero only in the domain \( R_{jn} \) defined by

\[
R_{jn}: \quad 0 \leq \frac{1-2}{N} \leq r \leq \frac{1}{N}, \quad 0 \leq \frac{n-2}{N} \leq s \leq \frac{n}{N}
\]  

(108)

This domain is depicted in Fig. 7a. Because the boundaries of \( R_{jn} \) correspond to constant values of \( r \) or \( s \), the transformation of \( r=g(u) \) and \( s=g(w) \) maps \( r_{jn} \) into the region \( U_{jn} \) in the \( u-w \) plane, as depicted in Fig. 7b. This region is defined by

\[
U_{jn}: \quad 0 \leq g^{-1}\left(\frac{1-2}{N}\right) \leq u \leq g^{-1}\left(\frac{1}{N}\right); \quad 0 \leq g^{-1}\left(\frac{n-2}{N}\right) \leq w \leq g^{-1}\left(\frac{n}{N}\right)
\]  

(109)

where \( g^{-1} \) denotes the inverse transformation.

Figure 7b also depicts the region covered by several integration area-elements (i.e., those squares and triangles used in the breaking up of the overall integration domain in the \( u-w \) plane). Any area-element that does not overlap the finite element domain \( U_{jn} \) may be skipped in the formation of the partial sum \( S_{jn} \). Furthermore, when an area element only partially overlaps \( U_{jn} \), the contributions from integrands evaluated at points \( (u,w) \) that do not satisfy Eq. (109) may also be skipped. This observation reduces the number of computations because it avoids operations on terms that would eventually be found to vanish.
Figure 7a. Domain of a finite element in physical variables.

Figure 7b. Domain of a finite element in integration variables.
Because Eqs. (105), (106), and (107) are equivalent to Eqs. (99), (100), and (101), we may also employ the partial sums to evaluate the coefficients \(a_{jn}\) corresponding to distributed basis functions. Hence, the only difference in the algorithms for distributed and finite element basis functions is that the latter should be checked for overlap according to Eq. (109). We also implemented an analogous overlap check in the computation of the summation for \(b_n\), Eq. (94), in the case of finite element basis elements.

12. NUMERICAL RESULTS

An analytical solution for the pressure distribution along the face of a thin disk is available for the limiting case of very low frequencies, \(ka \ll 0\). Thus, we chose this case for the first validation of the variational principle and its implementation. When the pressure on the positive \(z\) face is \(v_0 \exp(-i\omega t)\), with \(ka \ll 1\), then the pressure is

\[
p = -\frac{2}{\pi} \rho cv_0 ka(1 - r^2)^{1/2}
\]

(110)

In addition to providing a simple expression for comparison with numerical results, this form provides an indirect method of verification. If this mode is used as one of several modes, the amplitudes of the others should be very small when \(ka \ll 1\). We tested the accuracy of this hypothesis by redefining the modal amplitudes to be \(\Phi_j/ka\) and selecting the modes to be

\[
\Phi_j = r^2(j - 1)(1 - r^2)^{1/2}
\]

(111)

Table 1 gives the modal amplitudes as a function of the number of modes \(N\) in the series expansion. These results were developed by applying the sine transformation for the radial distance, Eq. (87), with the number of integration segments set at \(K = 10\). The accuracy of the variational principle is remarkable. Note that the first mode is very close to the value in Eq. (110), while the other modes are three orders of magnitude smaller.
Rayleigh-Ritz approximations of the acoustic pressure on the surface of a transversely vibrating rigid circular disk. The number of integration intervals is set at 20 in each case.

<table>
<thead>
<tr>
<th>Approximate solutions at ( \text{ka} \rightarrow 0 )</th>
<th>Coefficients ( C_n ) corresponding to the shape functions ( \frac{2(n-1)}{\sqrt{1-r^2}} ), ( n = 1, 2, \ldots N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 1 )</td>
<td>( C_1 )</td>
</tr>
<tr>
<td></td>
<td>( -0.63875 )</td>
</tr>
<tr>
<td>( N = 2 )</td>
<td>( -0.63875 )</td>
</tr>
<tr>
<td>( N = 3 )</td>
<td>( -0.63875 )</td>
</tr>
<tr>
<td>Analytical solution at ( \text{ka} \rightarrow 0 )</td>
<td>( -0.63662 )</td>
</tr>
</tbody>
</table>
We also checked the finite element results for \( ka = 0 \) against Eq. (110). For this, we must consider the general effect of inadequate choices for the number of elements \( N \) and the number of integration segments \( K \). The value of \( N \) must be sufficiently large to represent the pressure as a sequence of discrete values. We estimated that \( N \geq 5 \) was appropriate for the pressure given by Eq. (110). When the number of elements has been set, the value of \( K \) must be sufficient to place several integration points within the domain covered by a finite element. We found that sensible results are obtained if \( K \geq N \).

The transformation in Eq. (87) was used in the evaluation of the finite elements model, although it is not necessary to do so. We will discuss the reasons for this choice later. Comparison of finite element predictions for several values of \( K \) when \( N = 5 \) and 10 are shown in Figure 8 and 9, respectively. The agreement between the finite element formulation and the analytical solution is evident. Clearly, ten elements provide better resolution of the pressure distribution, but the results exhibit substantial numerical noise with increasing \( K \).

Such noise could have two causes. The computations were performed on a VAX 11/750, which due to hardware limitations in its current configuration slows drastically when performing double precision complex arithmetic in programs that access virtual memory. However, we checked the results with comparable calculations on a CDC CYBER 785, which doubles the arithmetic precision because the word size on the CYBER is twice as large as that on the VAX. Nevertheless, we found that the results changed very little. We have concluded that the primary source of error generation for finite elements is the usage of the piecewise linear modes in Eqs. (103). As the interval covered by each element decreases, continuous analytical derivatives are modeled in those modes as large values that change sign at the center point. When \( ka = 0 \), the integrand forming the coefficient array \([A]\) depends solely on these derivatives, so it is not surprising to see the predictions display numerical noise. We believe that these problems would be ameliorated through the application of smoother finite elements, such as polynomials.
Figure 8. Pressure distribution when $ka=0$ using five finite elements ($N=5$). ---: $K=5$; ·····: $K=10$; ---: $K=20$; — : Eq. (110).
Figure 9. Pressure distribution when $ka=0$ using ten finite elements ($N=10$).

- ---: $K=10$; --- ---: $K=20$; ---: Eq. (110).
The modes in Eq. (111) are such that one matches the analytical solution. A fairer test is the ability of the variational principle to predict the correct pressure distribution when the modes that are selected merely satisfy the boundary condition that \( p = 0 \) at the edge. Our choice for this evaluation was a half-range cosine series,

\[
\psi_j = \cos \left[ \left( 2j - 1 \right) \frac{\pi r}{2} \right]
\]

Figure 10 shows that an expansion using five of these modes compares quite favorably with Eq. (110). Indeed, the prediction is virtually identical to a Fourier series expansion of Eq. (110) using Eq. (112).

The results in Figure 10 were computed using the transformation in Eq. (87). Such a transformation is not required for these functions, nor for the finite element modes. Both Eq. (112) and Eq. (103) give finite values for \( \psi_j \) at \( r = 1 \). Hence it would be possible in either case to perform the numerical integrations leading to \([A]\) with an identity transformation of \( g(u) = 1 \).

When we employed the identity transformation, we found that favorable comparison with the analytical solution required more modes or finite elements. This observation has a ready explanation. The integration scheme described by Figures 4 and 5 uses a uniform mesh in the \( u-w \) plane. The transformation in Eq. (87) maps these points into a nonuniform mesh in the \( r-s \) plane, such that the density of integration points increases monotonically as \( r \to 1 \) and \( s \to 1 \). Thus, the transformation gives a better description of, and greater emphasis on, the behavior near the edge, where diffraction effects are most significant.

Based on these observations, we decided to perform all evaluations of the pressure distribution for non-zero \( ka \) using Eq. (87). Also, the nature of the slope singularity at \( r = 1 \) is not altered when \( ka \neq 0 \). For this reason, further evaluations using analytical modes shall only be based on Eq. (111).
Figure 10. Pressure distribution when \( ka=0 \) using sinusoidal nodes: 

\[ \text{--- : } N=5 \text{ & } K=10; \quad \text{--- : Eq. (110)}. \]
Data with which the results of the variational principle may be compared is somewhat sparse. Fortunately, the problem of diffraction of sound by a circular disk is analogous to the present radiation problem. Consider a plane wave at normal incidence to a thin circular disk that is fixed in space. Let $p_s$ denote the scattered acoustic wave, and let the $z$ axis coincide with the axis of symmetry. For a wave incident on the face $z=0^+$, corresponding to propagation in the direction of decreasing $z$, the total acoustic pressure is

\[ p = [p_s(r,z) + p_0 \exp(-ikaz)] \exp(-i\omega t) \]  

(113)

where $r$ and $z$ are nondimensional cylindrical coordinates and $p_0$ is the amplitude of the incident wave.

Since the disk is stationary, the acceleration of a fluid particle must vanish at $z=0$, which leads to

\[ \frac{dp_s}{dz} \bigg|_{z=0} = ika \ p_0 \]  

(114)

For comparison, we find in Eq. (4) that the boundary condition for acoustic radiation from a disk is

\[ \frac{\partial p}{\partial z} \bigg|_{z=0} = i\omega p v_0 \]  

(115)

when the disk velocity is $v_0 \exp(-i\omega t)$. Comparison of Eqs. (114) and (115) shows that results for scattering may be applied to the radiation problem by letting $p_0 = \rho c v_0$.

The diffraction problem was solved by Leitner [1949] using oblate spheroidal wave functions. Results from that analysis were obtained for $ka=1,2,3,4,$ and $5$, so those are the cases we shall consider here. When $ka>0$, the pressure amplitude is a negative imaginary quantity, corresponding to pressure that is in phase with the acceleration, in other words, a virtual mass impedance. When $ka > 0$, the pressure has both real and imaginary parts.
Figures 11 - 15 compare the results for the analytical modes in Eq. (111) with those obtained by Leitner. The analytical mode results were obtained for series truncation at $N=5$, while 10 finite elements were used.

The analytical modes give excellent predictions. Indeed, the slope discontinuity at the edge seems to be modeled better by the variational principle. Another highlight of Figures 13 and 14 is that they show that the earlier results for $ka = 1$ and 2 were interchanged.

The finite element model seems to work reasonably well. However the accuracy is not as good as that obtained from the analytical modes. One might think that the discrepancies are due to the relatively small number of elements and integration points. Figure 16 shows the pressure distribution for $ka=5$ using more elements and a finer integration mesh. Little improvement is obtained by increasing the number of integration points, whereas increasing the number of elements increases the numerical noise associated with the piecewise linear nature of the elements.

The numerous computer runs we made led us to inquire about the relationship between the execution time and the values of $K$ and $N$. We derived an estimate from the recognition that the bulk of the computations are associated with three distinct operations:

1. evaluation of the integrated Green's functions $C_j(g(\epsilon_k l)Ig(w_k l))$,
2. formation of the coefficients $a_{jn}$,
3. solution of the simultaneous equations to find the modal amplitudes.

It is clear from Figure 4 that the number of integration points is proportional to $K^2$, so the time required for phase 1 may be expected to have a comparable dependency. Note that the number of modes is irrelevant for this task. For the second phase, Eqs. (106) and (107) indicate that each coefficient $a_{jn}$ is obtained from a summary over $K^2$ terms. Since the number of these coefficients is proportioned to $N^2$, we expect that the time in the second phase will be proportional to $N^2K^2$. Finally, the time required to
Figure 11. Pressure distribution when $ka=1$.

- - - - : analytical modes ($N=5, K=10$);
- - - - - - : finite element modes ($N=10, K=20$);
- - - - - - - - : Leitner [1949].
Figure 12. Pressure distribution when ka=2.

——: analytical modes (N=5, K=10);
-- --: finite element modes (N=10, K=20);
•••••: Leitner [1949].
Figure 13. Pressure distribution when $ka=3$.

- analytical modes ($N=5$, $K=10$);
- finite element modes ($N=10$, $K=20$);

Leitner [1949].
Figure 14. Pressure distribution when ka=9.

— — —: analytical modes (N=5, K=10);
— — — —: finite element modes (N=10, K=20);
........: Leitner [1949].
Figure 15. Pressure distribution when $ka=5$.

- - -: analytical modes $N=5, K=10$;
---: finite element modes $(N=10, K=20$;
••••: Leitner [1949].
Figure 16. Influence of number of modes and integration points using finite elements, $ka=5$. $N=10 \& K=20$; $N=10 \& K=40$; $N=20 \& K=40$. 

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solve the system equations is independent of \( K \). The complex equation solver we employed (LEQ2C in the IMSL library) seems to require \( N^2 \) operation.

These heuristic evaluations led us to expect that the computation time would fit \( T = C_1 K^2 + C_2 N^2 K^2 + C_3 N^2 \). Reasonably close fits for the runs we made on a VAX 11/750 (with a floating point processor but no array processor) are

analytical modes:

\[
T(\text{sec}) \approx 0.75K^2 + 0.01K^2 N^2 + 0.14N^2
\]

finite elements

\[
T(\text{sec}) \approx 0.66 K^2 + 0.14 N^2
\] (93)

The differences between these estimates are attributable to the reduction in the number of operations introduced by the overlap check. For example, the time in phase (2) is not dependent on \( K^2 N^2 \) because the area covered by a finite element pair in the \((r,s)\) integration domain is inversely proportional to \( N^2 \).
8. CONCLUSIONS

We have derived a variational principle for the pressure on the exterior surface of an arbitrary body in harmonic motion. The principle which was obtained from the Kirchhoff-Helmholtz integral theorem, relates the surface pressure directly to the normal component of velocity. It avoids the need to solve the wave equation for the exterior domain subject to compatibility conditions at the surface. Current approximate methods for decoupling the surface pressure from the exterior domain, such as the doubly asymptotic approximation, rely on assumptions regarding the relationship between pressure and particle velocity. In contrast, the variational principle relies on no assumptions beyond those associated with linearized acoustic theory.

We specialized the variational principle to the case of a thin circular disk vibrating transversely. Comparable techniques may be employed to describe other thin bodies, such as curved panels and bars. We developed two techniques for applying the variational principle to the evaluation of the surface pressure. Both involved expansion of the pressure in a modal series, with the difference being whether the modes are analytical functions covering the entire surface, or finite elements covering only a segment of the surface.

The results obtained by using analytical modes were found to be remarkably accurate. Trial functions that match the singularities encountered in diffraction of an edge give the best results, but reasonable convergence was obtained for more general functions. The finite element formulation predicted the pressure less accurately. We concluded from computations for a variety of parameters that piecewise linear finite elements generate significant numerical noise when the number of elements is increased. We expect that the usage of polynomial elements would significantly improve the results.

We employed both formulations to obtain the pressure for nondimensional radii in the range $ka \leq 5$. We could have presented results for $ka > 5$, but have not done so because of a lack of data to use for a comparison. The trends suggested by the present results are that higher $ka$ would require more
analytical modes or elements in order to accommodate the more rapid spatial variation of pressure. Correspondingly, the mesh of integration points would have to be finer.

The variational principle may be extended to other geometrics. One of our current efforts is devoted to a finite length cylinder in axial vibration. Systems lacking axisymmetry, such as a transversely vibrating cylinder, may be treated by these techniques, except that more modes and more numerical integration points would be required to treat the higher dimensionality of such systems.

Another of our current efforts is devoted to coupling the variational principle to a modal description of structural vibration. The result will be a highly accurate, yet computationally simple, description of fluid-structure interaction. We have also shown that the variational principle may be readily modified to describe the scattering of acoustic waves.
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Dear Phil,

I regret the delay in forwarding the final report for contract no. N00014-84-K-0713. I am not sure now what the cause was, but I suspect that I slipped up because at first I thought that the new funding would be in the form of a continuation, rather than a new project. At any rate, you will find three copies of the report enclosed. In addition, I will send the report through the official channels, starting with the Georgia Tech Office of Contracts Administration. Needless to say, please call should you have any questions, or need additional information.

I am looking forward to our discussions at the ASA meeting. As I said in our phone conversation, I will be there from Monday evening until Thursday afternoon.

Best regards,

Jerry H. Ginsberg
The George W. Woodruff Chair
in Mechanical Systems
FINAL REPORT ON

IMPROVED ALGORITHMS FOR TRANSIENT AND
STEADY-STATE FLUID-STRUCTURE INTERACTION

SPONSORED BY

THE OFFICE OF NAVAL RESEARCH

CONTRACT NO. N00014-84-K-0713, CODE 1132SM

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I. OVERVIEW

The interaction between a vibrating submerged structure and the surrounding fluid, which features coupling between the surface pressure distribution and the structural displacement, is an inherent feature for sound radiation and target strength analyses. A variety of approaches have been implemented in the past, but each suffers from serious limitations. Formal mathematical analysis using separation of variables or integral transform techniques is suitable only for the simplest structural models, while full finite element descriptions of realistic structures and the surrounding medium lead to excessively large computer simulations. One approach uses approximate impedance-type boundary condition of uncertain accuracy to model the fluid response. Boundary element formulations rationally represent the interaction phenomena without explicitly solving field equations for the fluid, at the expense of an enormous increase in computational effort due to the need to cover the surface with a reasonably fine mesh.

The limitation of classical analytical techniques has led to a widely held belief that analytical-type solutions can only be obtained in idealized systems. The surface variational acoustics principle (SVP) relating surface pressure and surface particle velocity, which we derived earlier, partially invalidates the foregoing statement. SVP, which is valid for temporally harmonic excitations, represents the spatial dependence of the surface response as a series expansion in a set of assumed basis functions. Since the solution derived from SVP is the coefficients of this series, the primary difference from the results of a classical analysis lies in the fact that SVP determines these coefficients by numerical techniques.

Because SVP represents the surface pressure in a functional series form, it involves a substantially reduced set of unknowns in comparison to boundary element formulations, which are based on point-wise discretizations. In addition, SVP is essentially an optimization process that selects the coefficients of the modal series so as to minimize the deviation of the derived solution relative to the true one. Because the assumed basis functions for the series expansion span the entire wetted surface, the
optimization covers the entire domain simultaneously. In contrast, improve-
ment in conventional boundary element approaches only reduces the error over
the subdomain of each element. Another feature of SVP is that it enables
one to make use of prior experience with the problem, either from experi-
ments or analysis by simpler techniques. Tailoring the basis functions to
match this "expert" knowledge enables one to substantially reduce the number
of unknown series coefficients.

The initial effort within the project was devoted to deriving the
variational principle and validating it through the solution of canonical
problems whose solutions had already been obtained by other methods. This
work involved cylinder, spherical, and disk geometries. Because of the
presence of singularities in the formulation associated with the free space
Green's function, much effort was expended in the early phase of the project
to develop numerical techniques that were both efficient and robust. That
effort was a continuing theme of the research program, whenever the
generality of the principle was extended.

The next major research effort involved developing the capability to
use SVP to evaluate the vibration and acoustic radiation properties of elas-
tic structures that are subjected to a known mechanical excitation. This
work required coupling SVP with a suitable description of the dynamic
properties of elastic structures. Although an interface between SVP and any
technique for structural analysis could be developed, it was deemed ap-
propriate to the general thrust of SVP as an analytically based variational
formulation to use the method of assumed modes for structural vibrations,
which is sometimes (erroneously) called the Rayleigh-Ritz formulation. This
formulation was first applied to problems regarding an elastic disk in an
annular baffle of arbitrary size. It was then generalized to treat shells
in the shape of an arbitrary body of revolution and arbitrary thickness.
Until the last year of the project, all work had assumed that the problem
was axisymmetric. An initial extension of the overall formulation to situa-
tions where an elastic body of revolution is subjected to a nonsymmetric
load was completed in the closing stages of the project.
The versatility and generality of the SVP formulation led to efforts to apply it to problems of practical interest. One study was devoted to developing techniques whereby the effects of rib stiffeners could be included in the formulation in an efficient manner, in order to permit studies of the effect of stiffener placement and size on the acoustic radiation properties. This effort showed how component mode synthesis, which is a widely used tool for modeling the response of complicated structures in terms of smaller and simpler substructures, could be employed within the overall SVP formulation.

An interesting area of research was identified in the course of applying SVP to study the vibration and acoustic radiation of prolate spheroidal shells. It was found that there were substantial gaps in the available knowledge of the in-vacuo vibrational properties of such shells in the case of very slender geometries, where the aspect ratio is greater than four. These gaps were identified to correspond to cases of irregular behavior of the dry-structure eigensolutions that arise when natural frequencies associated with bending and extensional modes become close. A general theory for the parametric dependence of eigensystems having closely spaced eigenvalues was developed. At the closure of the project, knowledge of this general behavior was being applied to explain the anomalies in the vibrational properties of spheroidal shells. These anomalies correspond to mode localization, which had been identified as a potential source for enhanced acoustic radiation that is usually not well-described by conventional numerical modeling techniques. As a result, work underway at the close of the project was using SVP to explore the degree to which mode localization in the in-vacuo spheroidal shell influences the acoustic radiation of the shell when it is submerged.

II. RESEARCH TECHNIQUE

When the field point in the Kirchhoff-Helmholtz integral theorem is brought to the surface, one obtains an integral equation for the surface pressure resulting from a specified surface velocity. This equation, which is the foundation for many boundary element formulations, is not suitable
for forming a variational principle, because it is not symmetric between the field point and source points on the surface. An alternative equation governing the surface velocity is the normal derivative of the Kirchhoff-Helmholtz integral theorem, which is incorporated into some boundary element formulations in order to address the issue of spurious interior cavity resonances. SVP is derived from the latter equation. According to SVP, the true surface pressure distribution is that which extremizes the value of a functional $J[p(\xi)]$ with the normal component of surface velocity $v(\xi)$ held fixed, that is, $\delta J = 0$, where

\[
J[p] = \frac{1}{2} \iint_S \left[ k^2 n(x) \cdot n(\xi) P(x) P(\xi) - [n(x) \times \nabla P(x)] \cdot [n(\xi) \times \nabla P(\xi)] \right] G(\xi | x) \, ds_x \, ds_x - 4\pi \rho \omega \iint_S U_n(x) P(x) \, ds_x
\]

(1)

\[
U_n(x) = v_n(x) \nabla \cdot n(x) \cdot \nabla G(\xi | x) \, ds_x
\]

(2)

In broad terms, the assumed modes implementation of SVP consists of selecting a set of $N$ basis functions $\psi_j(\xi)$ to represent the surface pressure. The only conditions that these basis functions must satisfy are those necessary to ensure that the surface pressure is continuous. The corresponding series expansion is

\[
P(\xi) = \sum_{j=1}^{N} P_j \psi_j(\xi)
\]

(3)

The complex coefficients $P_j$ are determined by requiring that this series extremize the variational functional $J$. The normal direction and differential area element for the surface $S$ may be determined from the functional description of $S$. Thus, substitution of the series representation, followed by integrations over the surface, converts $J$ to a quadratic sum, given by
It is in the evaluation of the coefficients $A_{jn}$ and $B_{jm}$ that most of the computational effort resides, because the integrals defining these coefficients contains the Green's function between pairs of points on the surface, and the Green's function becomes singular when a pair of points are very close. The issue of integrability was a key part of the early project work, and the effort to enhance the efficiency of numerical integrations across the singularity continues to the present.

When $J$ has the form of eq. (4), the process of extremizing it becomes that of finding the local extrema of a function of several variables. The result is to require that $\frac{\partial J}{\partial P_j} = 0$, so that

$$[A] [P] = (B)$$

(5)

In cases where the surface velocity is specified, as it is when a rigid body executes a known vibrational motion, the coefficients $B_j$ may be computed directly, so that evaluation of the surface pressure reduces to solution of the eq. (5) for the pressure coefficients, followed by substitution into the series, eq. (3), in order to form the pressure distribution.

When the vibrating surface is elastic, it is necessary to couple the pressure equations to a structural dynamics description. All such techniques inherently represent the displacement in terms of a set of generalized coordinates such that

$$u(x) = \sum_{m=1}^{M} \phi_{m}(x) q_{m} \exp(-i\omega t)$$

(6)

In the case of finite elements, the basis functions $\phi_{m}(x)$ are the interpolating functions spanning each element, while in the conventional assumed modes formulations they span the entire domain of the structure. The corresponding surface velocity is
\[ v(x) = -i\omega \sum_{m=1}^{M} \Phi_n(x) q_n, \quad \Phi_n = \phi_n(\xi) \cdot n(\xi), \xi \text{ on } S \]  (7)

The equations of motion for the structure have the standard form,

\[
\begin{bmatrix}
[K] - i\omega [C] - \omega^2 [M]
\end{bmatrix} (q) = (Q^P) + (Q)  
\]  (8)

where \([K]\), \([C]\), and \([M]\) are the stiffness, damping, and inertia matrices, respectively. Also, \((Q^P)\) and \((Q)\) are the generalized forces due to the surface pressure and force excitation, respectively. When eqs. (3) and (6) are used to form the virtual work due to the surface pressure, the former is found to be expressible in terms of the pressure coefficients according to

\[ (Q^P) = - [A] (P)  \]  (9)

Furthermore, using eq. (7) to form the coefficients \(B_j\) in the pressure equations leads to

\[ (B) = [\Gamma] (q)  \]  (10)

The coefficient matrices \([A]\) and \([\Gamma]\) both represent fluid-structure coupling effects. The former represents the structural loading due to the surface pressure, while the latter represents the pressure excitation created by the structural motion. The combination of eqs. (5), (8), (9), and (10) is

\[
\begin{bmatrix}
[A] & -[\Gamma]
\end{bmatrix}
\begin{bmatrix}
[K - i\omega C - \omega^2 M]
\end{bmatrix}
\begin{bmatrix}
(P)
\end{bmatrix} = \begin{bmatrix}
(0)
\end{bmatrix} - \begin{bmatrix}
(Q)
\end{bmatrix}  
\]  (11)

The evaluation of the surface pressure and structural displacement in this situation involves solution of the simultaneous equations (11), followed by formation of the respective series, eqs. (3) and (6).

A general methodology for applying SVP through eq. (11) to bodies of revolution was in the process of development when the project ended. It is based on parametric representation of the functions defining the cylindrical
coordinates of points on the wetted surface. In the case of axisymmetric motion of shells, the pressure functions $\psi_n$ and surface displacement functions $\Phi_m$ are then taken as sets of complete functions (sinusoidal, Bessel, etc.) of the parameter used to represent the shape.

III. RESEARCH ACHIEVEMENTS

For the purposes of the present discussion, the research phases associated with the project are separated below. In actuality, some efforts were performed simultaneously, or later phases were begun before all the details of earlier work were finalized. In broad terms, the work may be categorized as to whether it was concerned with the basic development and implementation of eq. (5) in systems where the wetted surface was that of a vibrating rigid body, or basic developments pertaining to the generation of eq. (11) for elastic bodies subjected to harmonic excitation, or research questions arising from the application of eq. (11). Each category is discussed below.

A. Development of the Basic Principle and Application to Radiation from Vibrating Rigid Bodies

The first stage of the research program was devoted to the derivation of the variational principle. After the derivation was completed, it was necessary to establish techniques to integrate over the singularity of the Green's function. Rather than trying to do so in a general manner, several canonical acoustic radiation problems were selected. In addition to providing a specific geometrical configuration for performing the various operations, selection of these problems provided a high degree of confidence in the overall approach.

The first problem to be addressed was radiation from an thin disk executing transverse vibration as a rigid body. This problem is particularly nasty because the edge condition is associated with a large diffraction effect. We found that our results using any of several alternative basis functions were at least as accurate as early analyses using spheroidal wave
functions, and in the process corrected data derived from earlier works. We also showed that one could use as basis functions local interpolating functions spanning subdomains of the surface, in the manner of conventional boundary element formulations. Not surprisingly, doing so required substantially more functions than SVP formulation using analytical functions spanning the entire surface. Furthermore, implementing SVP efficiently when interpolating functions are used was found to involve more complicated algorithms, so further work has focussed on describing the surface pressure with analytical functions.

The other configuration to receive attention was a finite length rigid cylinder executing either axial translation or a uniform breathing mode. The results in this case were validated by comparison with the NRL SHIP program. A formalism was developed for numerically accounting for the singularities when the axial length to decrease without limit, so that the limiting case of a vibrating disk could be used as another verification of the SVP program. In this effort, we explored the question of spurious resonances that occur in boundary element formulations when the frequency is such that the interior cavity has an eigenstate. It was shown that the solution loses uniqueness at the eigenfrequencies of the Neumann problem for the interior, which was expected since SVP is equivalent to boundary element formulations using the normal derivative of the surface Helmholtz integral equation. However, even at those frequencies, the SVP prediction was proven to yield the correct value for radiated power. A later study proved that if interior cavity resonances are of concern, reformulating SVP as an isoperimetric problem associated with auxiliary constraint conditions on the acoustic field within the interior guarantees that SVP will yield unique results at all frequencies.

B. Variational Principles for Submerged Elastic Structures

The first study to couple SVP with structural dynamics equations for an elastic structure addressed an circular membrane or elastic plate an annular baffle of arbitrary outer radius. The excitation in this first study was taken to be axisymmetric, so there was no azimuthal variation of the surface pressure and transverse displacement. In the limit as the outer radius of
the baffle becomes very large, with fluid loading applied to only one side of the disk, one should recover from SVP the solution for a disk in an infinite baffle. The correctness of this result was confirmed by comparing it to an analytical solution using spheroidal wave functions. The main discrepancy between the SVP result and the earlier work was shown to be due to use in the earlier result of a correction for transverse shear and rotatory inertia, which becomes important for higher frequencies. The SVP result also showed that the frequency scan in the previous evaluation was not adequate to fully identify the peak response at the fundamental fluid-loaded resonance of the plate. The robustness of SVP was demonstrated by the fact that the results were independent of the type of assumed basis functions selected, and that convergence to the final solution could be obtained with few functions. An important feature of the SVP formalism is that with minor alteration, one can obtain the solution for the corresponding rigid body vibration. This feature is useful for extensions to problems in acoustic scattering, where one often wishes to identify the portion of a scattered signal that is due to specular reflection, as opposed to structural re-radiation.

The initial generalizations of SVP to treat nonaxisymmetric excitations of axisymmetric structures used Fourier series to represent the azimuthal dependence of all surface response variables, thereby exploiting their circumferential periodicity. In the first study it was assumed that the excitation is such that only a single azimuthal mode is excited. The variational principle then leads to equations for that harmonic that are analogous in form and computational effort to those for the axisymmetric case. This work, which used the flat elastic plate as a prototype, was presented as an M.S. thesis. Work nearing completion at the close of the project was extending this to the case where a full azimuthal series is required. The goal here was to develop optimal SVP algorithms for the successive evaluations of the azimuthal harmonics. The analysis was completed recently and used to solve the problem of an eccentrically loaded elastic plate is under development.

The main effort in the closing phase of the project was devoted to development of a general implementation of SVP for axisymmetric vibration
and radiation from submerged shells of revolution. In this effort, the shape and thickness of the shell were allowed to be arbitrary, and the mechanical excitation was limited only by the restriction to axisymmetry. Our previous work regarding fluid-structure coupling involving elastic structures had considered only flat plates in a baffle and spherical shells. Although SVP was found in both cases to be extremely accurate relative to analytical solutions, such comparisons were not deemed to be sufficient measures of the merits of SVP because of the simplicity of both geometries. It therefore was decided to apply the general model to the case of a prolate ellipsoidal shell.

A few prior investigations had considered the case where the inner and outer surfaces of the shell are confocal ellipsoids, which corresponds to a variable skin thickness. The initial SVP implementation used the approach widely employed in structural acoustics of first analyzing the free in-vacuo vibrational properties, in order to perform a modal truncation before the fluid-structure interaction phenomena are addressed. In the general procedure, the in-vacuo free vibration natural frequencies and modes of a shell are obtained from Hamilton's principle in conjunction with the method of assumed modes. An attempt to compare the results with prior work disclosed that results for slender ellipsoids was very sparse. Further investigation showed that the natural frequencies in certain ranges of slenderness ratio were very close, in which case mode localization phenomena became prominent. In order to explore this question a separate general investigation of mode localization, which is discussed in the next section, was pursued..

Mode localization has been suggested at ONR-sponsored workshops as being a potential source of modeling error for numerical simulations. The prolate spheroidal shell was used as a prototypical system to investigate with the aid of SVP the validity of such concerns. The response of the submerged system when all in-vacuo structural modes are retained, regardless of the frequency range in which they occur, was compared to the results obtained from various modal truncations both below and above the frequency at which mode localization occurs. It was shown that when the slenderness ratio is such that in-vacuo modes localize, accurate results for surface vibration and pressure can only be obtained by retaining all modes, whereas
the mode truncation gives good results when the slenderness ratio is such that localization does not occur. Another result of note was that in neither case does the submerged response display localization.

C. New Developments in Vibration Theory and Their Application to the Variational Formulation

A widely used tool for structural dynamics is component mode synthesis, in which the equations of motion for subregions of an elastic structure are formulated individually. The synthesis of the overall response of the entire system is achieved by coupling the subregions by accounting for the compatibility of their displacements through constraint conditions, and by treating their interaction forces either explicitly, or through Lagrangian multipliers. There are two primary advantages to this technique. Specifically, the modeling effort required to treat the separate regions is substantially less than that required to model the entire system simultaneously. In addition, should one wish to consider alterations in the configuration of one subregion, it is not necessary to reformulate the entire model.

Component mode synthesis has not been applied in the published literature to problems in acoustic radiation from vibrating elastic bodies. This matter was addressed in the program by modeling the effect of stiffeners in elastic plates. This problem was of interest because it introduced greater realism into the structural dynamics models used for SVP. As discussed earlier, the effects of deviations from asymmetry were under separate investigation, so the stiffeners were taken to be concentric elastic rings. Two alternative formulations were successfully demonstrated. In the first, the dynamic elastic effects of the structure are described in terms of a set of modes derived from an in-vacuo analysis of the entire structure. In this case the portions of the SVP equations associate with the elastic and inertial effects of the structure are altered with each change in stiffener properties. In contrast, the second formulation uses concepts of component mode synthesis, in which the individual modal properties of the plate and isolated stiffeners are tied together by constraint equations and Lagrangian
multipliers. The virtue of the latter is that the modes are easier to obtain, and changes due to changes in the stiffener configuration leads to relatively small and easily determined alterations in the SVP, at a penalty of increasing the number of unknowns. The equivalence of both methods was demonstrated, from both analytical and computational viewpoints. The component mode synthesis approach was used to identify the optimal stiffener for maximizing or minimizing surface vibration or far field pressure. The latter was determined by developing a computer program for the exterior pressure field based on the Kirchhoff-Helmholtz integral theorem.

Another area of significant effort was the phenomenon of mode localization, which was identified in the course of a parametric investigation of the dependence of the various in-vacuo natural frequencies of a spheroidal shell as a function of the slenderness ratio. Prior work had showed in a general manner that the natural frequency branches often veer, rather than intersecting, when they are plotted as a function of a system parameter, and that such veering phenomena are associated with mode localization. Such was the case for the spheroidal shell. However, all previous investigations were incapable of describing the actual within the veering zone. An asymptotic analysis of this behavior was successfully performed. The work proved that veering of the eigenvalue loci is accompanied by extreme sensitivity of the mode shapes on the system parameter. The research showed that the eigenfunctions when the natural frequencies are very close are linear combinations of the eigenfunctions outside the zone, and these combinations lead to the localization phenomena. Knowledge that response properties might change drastically when a system parameter is altered slightly causes great concern for the ability of conventional formulations, which do not account for these phenomena, to robustly predict the response of actual systems.
IV. CHRONOLOGICAL LISTING OF PROJECT PUBLICATIONS AND PRESENTATIONS


April 12, 1991

Dr. Phillip B. Abraham, code 1132-SM
Director, Mechanics Division
Office of the Chief of Naval Research
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Arlington, VA 22217-5000

Dear Phil,

I regret the delay in forwarding the final report for contract no. N00014-84-K-0713. I am not sure now what the cause was, but I suspect that I slipped up because at first I thought that the new funding would be in the form of a continuation, rather than a new project. At any rate, you will find three copies of the report enclosed. In addition, I will send the report through the official channels, starting with the Georgia Tech Office of Contracts Administration. Needless to say, please call should you have any questions, or need additional information.

I am looking forward to our discussions at the ASA meeting. As I said in our phone conversation, I will be there from Monday evening until Thursday afternoon.

Best regards,

Jerry H. Ginsberg
The George W. Woodruff Chair in Mechanical Systems