COLOR-CRITICAL GRAPHS ON SURFACES

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COLOR-CRITICAL GRAPHS ON SURFACES

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DEDICATION

To my parents
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A graph is \((t + 1)\)-critical if it is not \(t\)-colorable, but every proper subgraph is. In this thesis, we study the structure of critical graphs on higher surfaces. One major result in this area is Carsten Thomassen’s proof that there are finitely many 6-critical graphs on a fixed surface. This proof involves a structural theorem about a precolored cycle \(C\) of length \(q\). In general terms, he proves that a coloring \(\phi\) of \(C\) can be extended inside the cycle, or there exists a subgraph \(H\) with at most \(5q^3\) vertices such that \(\phi\) can not be extended to a 5-coloring of \(H\). In Chapter 2, we provide an alternative proof that reduces the number of vertices in \(H\) to be cubic in \(q\). In Chapter 3, we find the nine 6-critical graphs among all graphs embeddable on the Klein bottle. Finally, in Chapter 4, we prove a result concerning critical graphs related to an analogue of Steinberg’s conjecture for higher surfaces. We show that if \(G\) is a 4-critical graph embedded on surface \(\Sigma\), with Euler genus \(g\) and has no cycles of length four through ten, then \(|V(G)| \leq 2442g + 37\).
CHAPTER I

INTRODUCTION

In this chapter, we will provide the graph theoretic context of the results to follow. In Section 1.1, we give descriptions of the basic terminology and structures used for our results. In Section 1.2, we explain how graphs can be drawn on surfaces other than the plane. In Section 1.3, we describe graph-theoretic problems related to finding color-critical graphs on fixed surfaces and state the contributions of this thesis to the research area. In Section 1.4, we describe a conjecture of Steinberg and our results related to this question.

1.1 Graph Theoretic Preliminaries

We follow the exposition of Bondy and Murty in [9]. A multigraph is an ordered triple \((V(G), E(G), \psi_G)\) consisting of a nonempty set \(V(G)\) of vertices, a set \(E(G)\), disjoint from \(V(G)\) of edges and an incidence function \(\psi_G\), that associates with each edge of \(G\) an unordered pair of (not necessarily distinct) vertices of \(G\). If \(e\) is an edge and \(u\) and \(v\) are vertices such that \(\psi_G(e) = uv\), then \(e\) joins \(u\) and \(v\), and we say that \(u\) and \(v\) are adjacent and that \(e\) is incident with \(u\) and \(v\). Define a loop as an edge whose endpoints are the same. All other edges are called links. A multigraph is a graph if it has no loops and no two of its links join the same pair of vertices. In this thesis, we will only consider graphs. One class of graphs are complete graphs which consist of graphs with vertex set \(V\) and an edge joining every pair of distinct vertices in \(V\). Graphs are usually represented in a pictorial manner with vertices appearing as points and edges represented by lines connecting the two vertices associated with the edge.

For a graph \(G = (V, E, \psi)\), if \(V' \subseteq V\), \(E' \subseteq E\), \(\psi'\) is the restriction of \(\psi\) to \(E'\), and
for every edge $e' \in E'$ both endpoints are in $V'$, then $G' = (V', E', \psi')$ is a subgraph of $G$. Given a graph $G = (V, E, \psi)$, if $X$ is a subset of vertices, we denote by $G[X]$ the subgraph with vertex set $X$ and edge set containing every edge of $G$ with both endpoints contained in $X$. Then $G[X]$ is the graph induced by $X$. A graph $H$ is a subdivision or topological minor of $G$ if $H = G$ or if $H$ is constructed by replacing some edges of $G$ with internally disjoint paths such that each of these paths has only its endpoints in common with $G$. Let $u$ and $v$ be vertices in graph $G$. If we identify $u$ and $v$, we delete $u$ and $v$ and create a new vertex $w$ that is adjacent to all vertices that were adjacent to $u$ or $v$. However, if vertex $x$ is adjacent to both $u$ and $v$, we only draw one edge between $w$ and $x$.

Two graphs $G$ and $H$ are isomorphic if there exists a bijection $f$ between $V(G)$ and $V(H)$ with the property that any two vertices $u$ and $v$ in $G$ are adjacent if and only if $f(u)$ and $f(v)$ are adjacent in $H$. In general, the problem of determining whether two graphs are isomorphic has undetermined complexity, and the problem of showing whether one graph has a subgraph isomorphic to another graph is $NP$-Complete. Some of our results involve lists characterizing graphs by whether or not they have a subgraph isomorphic to a list of graphs.

### 1.2 Graphs on Surfaces

We define a surface as a connected, compact, 2-dimensional manifold with empty boundary. We follow the exposition of Mohar and Thomassen [26] to describe how we view graphs on surfaces and ask the reader to refer to this text for further details. Two surfaces are homeomorphic if there exists a bijective continuous mapping between them.

Let $X$ be a topological space. An arc in $X$ is the image of a continuous one-to-one function $f : [0, 1] \rightarrow X$. We say a graph $G$ is embedded in a topological space $X$ if the vertices of $G$ are distinct elements of $X$ and every edge of $G$ is an arc
connecting in $X$ the two vertices it joins in $G$, such that its interior is disjoint from other edges and vertices. An embedding of a graph $G$ in topological space $X$ is an isomorphism of $G$ with a graph $G'$ embedded in $X$. A planar embedding of a graph is an embedding of $G$ in the plane. A topological space $X$ is arcwise connected if any two elements of $X$ are connected by an arc in $X$. The existence of an arc between two points of $X$ determines an equivalence relation whose equivalence classes are called the arcwise connected components, or the regions of $X$. A face of $C \subseteq X$ is an arcwise connected component of $X \setminus C$. A 2-cell embedding is an embedding where every face is homeomorphic to an open disk [26].

Graphs on different surfaces have different properties. For instance, it is not possible to construct a planar embedding of $K_5$ on the plane. However, on the torus, there exist planar embeddings for $K_5, K_6$ and $K_7$ (but no larger complete graphs). In order to describe graph-theoretic properties on surfaces, we first must be able to describe the structure of all possible surfaces. We can do this via the classification theorem of surfaces. Before stating this theorem, we must define a few terms.

One way to distinguish between surfaces is by classifying the surface as orientable or nonorientable. In a nonorientable surface, there exist curves where right and left interchange along the curve. A straightforward example of this is to consider a curve drawn parallel to the boundary on a Möbius strip. This allows for one-sided curves, curves which have no inside or outside. This is not the case in an orientable surface, where all closed curves have an inside and an outside.

One way to construct a surface is by taking a collection of pairwise disjoint polygons in the plane such that the sum of the number of edges in the collection of polygons is even. We also require all the polygons to be regular and have the same edge length. So, for instance, a triangle is simply a three-sided polygon whose edge lengths are all the same. Each side of each polygon will be oriented one of two ways. We can do this by associating an initial point and an ending point to each edge and
pictorially denote this by an arrow on the edge that points to the ending point. Then we can identify pairs of edges so that initial points agree. So all points in the union of these polygons still have open neighborhoods homeomorphic to the plane. The sides of these polygons and their endpoints determine a connected multigraph $G$ embedded on surface $S$. If all the polygons are triangles, then $G$ triangulates surface $S$. It can be shown [26] that every surface is homeomorphic to a triangulated surface.

The most basic surface that is considered in the classification theorem of surfaces is the sphere, denoted $S_0$. It can be constructed by taking four equilateral triangles and connecting them to create a regular tetrahedron. Given a surface $S$, we can perform additional operations to make a more complicated surface. In particular, we can add a handle, add a twisted handle or add a crosscap. First we will describe how to add a handle. Let $T_1$ and $T_2$ be two disjoint triangles in $S$ all of whose side lengths are the same. If surface $S$ is orientable, then configure the edges of $T_1$ and $T_2$ so that the directions of $T_1$’s edges are the opposite of $T_2$’s when each is viewed in a clockwise direction. Then if we remove the interiors of $T_1$ and $T_2$ and identify the edges of $T_1$ to the edges of $T_2$, this creates a new surface $S'$. We say that $S'$ is obtained by adding a handle to $S$. Notice that we can only add a handle to an orientable surface. Suppose that $S$ is orientable and the clockwise orientations of $T_1$ and $T_2$ are the same. If we remove the interior of $T_1$ and $T_2$ and we identify the edges of $T_1$ to the edges of $T_2$ then the resulting surface, call it $S''$, is the result of adding a twisted handle to $S$. If $S$ is non-orientable, then any handle added is a twisted handle. Finally, suppose that we have a simple closed disk, call it $T$. Suppose that we delete the interior of $T$ from $S$ and identify diametrically opposite points of $T$. This adds a crosscap to $S$. It can also be shown that adding two crosscaps is equivalent to adding a twisted handle.

We can now state the classification theorem of surfaces. It states that every surface is homeomorphic to either $S_g$, the surface obtained from the sphere by adding $g$ handles, or $N_k$, the surface obtained from the sphere by adding $k$ cross-caps. Using
this terminology, \( S_0 = \mathcal{N}_0 \) is the sphere, \( S_1 \) is the torus, \( \mathcal{N}_1 \) is the projective plane and \( \mathcal{N}_2 \) is the Klein bottle. One way that we can distinguish between these surfaces is that graphs embedded on these surfaces have different properties. In particular we have the following:

**Theorem 1.2.1.** \([26]\) Let \( \Sigma \) be a surface, and \( G \) be a graph that is 2-cell embedded in \( \Sigma \), with \( n \) vertices, \( q \) edges and \( f \) faces. Then \( \Sigma \) is homeomorphic to either \( S_h \) or \( \mathcal{N}_k \) where \( h \) and \( k \) are defined by the equations

\[
   n - q + f = 2 - 2h = 2 - k.
\]

With this theorem in mind, define the *Euler characteristic* of a surface \( \Sigma \) to be \( \chi(\Sigma) = 2 - 2h \) if \( \Sigma = S_h \) and \( \chi(\Sigma) = 2 - k \) if \( \Sigma = \mathcal{N}_k \). Also, define the *Euler genus* of surface \( \Sigma \), \( g(\Sigma) \) to be \( g(\Sigma) = 2 - \chi(\Sigma) \). We can now state Euler’s formula for surfaces.

**Theorem 1.2.2.** \([26]\) Let \( G \) be a multigraph which is 2-cell embedded in a surface \( \Sigma \). If \( G \) has \( n \) vertices, \( q \) edges and \( f \) faces in \( \Sigma \), then

\[
   n - q + f = \chi(\Sigma).
\]

With this theorem in mind we can see that graphs embedded on different surfaces have different structures. Notice that the torus and the Klein bottle have the same Euler characteristic but are somehow different. One way to distinguish between surfaces is by classifying the surface as *orientable* or *nonorientable*. Another way to distinguish between these surfaces is to study the type of cycles that can be drawn on each surface. On the Klein bottle, a nonorientable surface, it is possible to draw a *one-sided cycle*, which is a cycle whose boundary only has one side. It is impossible to do this on the orientable torus, as all cycles on the torus are *two-sided*, that is the boundary of these cycles have two sides. Note that two-sided cycles can also be drawn on a Klein bottle. One reason why the proof of a result we will describe in Section 1.3...
for the Klein bottle is more complicated than Thomassen’s analogous result for the torus is that the Klein bottle is nonorientable and leads to additional complications that will be described in later chapters.

Another important property of graphs embedded on surfaces is that cycles drawn on the surface may have different properties. A homotopy between two functions $f$ and $g$ from a space $X$ to a space $Y$ is a continuous map $G$ from $X \times [0,1] \to Y$ such that $G(x,0) = f(x)$ and $G(x,1) = g(x)$. Two spaces are homotopic if there is a homotopy between them. A contractible cycle is a cycle $C$ such that some parametrization of $C$ is homotopic to a constant map. We call it contractible because it can be contracted to a point. However, on some surfaces there also exist cycles which are noncontractible. The torus is one surface that contains non-contractible cycles. One metric useful in the study of embedded graphs is the property of face-width, also called representativity. The face-width of a graph $G$ embedded on surface $\Sigma$ is the smallest number $k$ such that $\Sigma$ contains a noncontractible closed curve that intersects the graph in $k$ points.

Finally, when studying graphs on surfaces, it is often useful to construct a two-dimensional representation of a surface that is easier work with and visualize than something in a higher dimension. To do this we again use polygons having pairs of edges with a starting and ending point that are identified with each other. As motivating examples we show depictions of the Klein bottle and the torus in two dimensions and an embedding of $K_6$ on the Klein bottle. Here opposite sides of the quadrilateral are identified.

### 1.3 Graph Coloring on Surfaces

One of the most well-studied areas of graph theory relates to graph coloring. Initially, mathematicians were interested in the coloring of maps. They wanted to determine the fewest number of colors necessary so that regions or countries that touched each
other on the map had different colors. This can be modeled using a graph where each region on a map corresponds to a vertex of a graph and if two regions share a nonzero length boundary segment, there is an edge between them in the graph. A *proper coloring* is a coloring of the vertices such that the vertices of each edge are colored differently. After a few failed attempts, the Four-Color Theorem was proved [3, 4, 32] and we now know that any such map (and therefore planar graph) can be properly colored using at most four colors.

There have been many other variants and generalizations to this graph coloring problem. In the *list coloring* problem, every vertex is given a list of colors that can be used, but unlike the traditional coloring problem, the lists may be different. If the lists of each vertex are at least size $k$ and the graph can be properly colored, we say

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{torus_klein_bottle.png}
\caption{A two-dimensional representation of a torus and Klein bottle. The torus is on the left and the Klein bottle is on the right.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{k6_klein_bottle.png}
\caption{A planar embedding of $K_6$ on the Klein bottle.}
\end{figure}
$G$ is $k$-choosable. In the edge coloring edges are given colors. There are other coloring problems of interest such as total coloring and circular coloring. Graph theorists are also interested in the colorability of particular classes of graphs. One famous result is that of Grötzsch [18], which states that every triangle free planar graph is 3-colorable.

We are also interested in coloring problems for graphs embedded on surfaces. A fundamental question in topological graph theory is as follows: Given a surface $\Sigma$ and an integer $t \geq 0$, which graphs drawn in $\Sigma$ are $t$-colorable? Heawood proved that if $\Sigma$ is not the sphere, then every graph in $\Sigma$ is $t$-colorable as long as $t \geq H(\Sigma) := \left\lfloor \frac{(7 + \sqrt{24\gamma + 1})}{2} \right\rfloor$, where $\gamma$ is the Euler genus of $\Sigma$, defined as twice the genus if $\Sigma$ is orientable and the cross-cap number otherwise. Ringel and Youngs [29] proved that the bound is best possible for all surfaces except the Klein bottle. Every graph embeddable in the Klein bottle requires only six colors, but Heawood’s bound says this bound is seven. This was shown by Franklin [15] in 1934. Dirac [11] and Albertson and Hutchinson [1] improved Heawood’s result by showing that every graph in $\Sigma$ is actually $(H(\Sigma) - 1)$-colorable, unless it has a subgraph isomorphic to the complete graph on $H(\Sigma)$ vertices.

There have been a number of results which, aside from being interesting by themselves have produced useful structural lemmas helpful for other problems in coloring and list coloring. Recall that a graph that is $k$-colorable is not necessarily $k$-choosable. For instance, every planar graph is 4-colorable by the Four Color Theorem, but not every planar graph is 4-choosable. The first example of such a graph was given by Voigt [44]. The following is a result of Carsten Thomassen in [37]:

**Theorem 1.3.1.** Every planar graph is 5-choosable.

A common technique when working on coloring problems is to consider a precolored cycle and see whether all precolorings extend. This is precisely the technique that Thomassen used when proving his theorem. In particular, he proves the following lemma:
Lemma 1.3.2. Suppose that every inner face of graph $G$ is bounded by a triangle and its outer face by a cycle $C = v_1 \cdots v_k v_1$. Suppose further that $v_1$ has already been colored with color 1 and $v_2$ has been colored with color 2. Suppose finally that with every other vertex of $C$ a list of at least three colors is associated, and with every vertex of $G - C$ a list of at least five colors. Then the coloring of $v_1$ and $v_2$ can be extended to a coloring of $G$ from the given lists.

Notice that the lemma almost immediately implies the theorem of Thomassen. First, suppose we are given a planar graph where every vertex is given a list of five colors. We will add edges to this graph until it is a maximally planar, that is, until no more edges can be drawn without forcing edges to cross. It is well-known that maximally planar graphs are triangulations, so the outer face of this graph has size three. Precolor two of the vertices of this triangle with colors on their respective lists such that these colors are different. The remaining graph satisfies the hypotheses of Thomassen’s lemma, and hence is 5-choosable.

Another characterization fundamental in the theory of graph coloring is that of critical graphs. We say that a graph is $(t + 1)$-critical if it is not $t$-colorable, but every proper subgraph is. We follow the notation as described in [26]. Dirac [12] proved that for every $t \geq 8$ and every surface $\Sigma$ there are only finitely many $t$-critical graphs on $\Sigma$. By a result of Gallai [16] this can be extended to $t = 7$. Finally, Thomassen [36] proved a deeper result.

Theorem 1.3.3. For every surface $\Sigma$, there are finitely many 6-critical graphs that embed in $\Sigma$.

In addition, he does not give an explicit upper bound for the number of vertices of 6-critical graphs on a fixed surface, but an analysis of his proof shows this bound is at least doubly exponential. This automatically gives a linear time algorithm to decide 5-coloring, because we can test whether a graph in a fixed surface has a particular
subgraph of bounded size in linear time [14]. However, the proof of Thomassen’s theorem [36] is rather long. In this paper, Thomassen proves a theorem that bounds the size of a minimal subgraph $H$ such that a precoloring of a cycle of length $q$ does not extend to $H$. In particular, he proves the following theorem:

**Theorem 1.3.4.** Let $G$ be a 2-connected plane graph with no separating triangle and with outer cycle $C$ of length $q$. Let $\phi$ be a 5-coloring of the vertices of $C$. Then $G$ contains a connected subgraph $H$ with at most $5q^3$ vertices such that either (i) or (ii) holds:

(i) $\phi$ cannot be extended to a 5-coloring of $H$,

(ii) $\phi$ can be extended to a 5-coloring of $H$ such that each vertex of $G - H$ which is joined to more than two colors of $H$ either has degree at most 4 or has degree 5 and is joined to two distinct vertices of $H$ of the same color. The coloring of $H$ in (ii) can be extended to a 5-coloring in $G$.

In Chapter 2, we improve this theorem by reducing the constant $5q^3$ to a bound that is cubic in $q$ by proving an analogous theorem with a slightly different outcome (ii) which is based upon our analysis. This is work that grew out of ongoing research with Ken-ichi Kawarabayashi which attempts to give a shorter proof of Thomassen’s theorem that there are finitely many 6-critical graphs on a fixed surface. We prove the following theorem that grew out of the methods we have developed so far:

**Theorem 1.3.5.** Let $G$ be a 2-connected plane graph with no separating triangle and with outer cycle $C$ of length $q$. Let $\phi$ be a 5-coloring of the vertices of $C$. Then $G$ contains a connected subgraph $H$, which includes the vertices of $C$, with at most $7(q - 2)^3$ vertices such that either (i) or (ii) holds:

(i) $\phi$ cannot be extended to a 5-coloring of $H$, or

(ii) there exists a 5-coloring $\phi'$ of the subgraph of $G$ induced by $H$ that extends $\phi$ such that for all $v \in V(G) - V(H)$, the following holds: either $v$ sees at most two colors...
of \( \varphi' \), or the number of colors in \( \varphi' \) seen by \( v \) plus the number of neighbors of \( v \) in \( G \setminus V(H) \) is at most four. In this case the 5-coloring \( \varphi' \) extends to a 5-coloring of \( G \).

Aside from Thomassen’s theorem that says there are finitely many 6-critical graphs on a fixed surface, for some small surfaces, we know the precise list of the 6-critical graphs. For instance, there are no 5-critical graphs on the plane by the Four-Color Theorem \([3, 4, 32]\). Hence there are also no 6-critical graphs on the plane. Albertson and Hutchinson \([1]\) proved that a graph in the projective plane is 5-colorable if and only if it has no subgraph isomorphic to \( K_6 \), the complete graph on six vertices.

Thomassen characterized \([39]\) the 6-critical graphs on the torus and found there are four. We now give a description of this result. First, if \( K, L \) are graphs, then define \( K + L \) as the graph obtained from the union of a copy of \( K \) with a disjoint copy of \( L \) by adding all edges between \( K \) and \( L \). The graph \( H_7 \) is obtained by taking two copies of \( K_4 \), deleting edge \( xy \) in one of the copies and deleting edge \( uv \) in the other copy, adding edge \( yv \) and identifying vertices \( x \) and \( u \). The graph \( T_{11} \) is obtained from a cycle of length 11 by adding edges joining all pairs of vertices at distance two or three.

**Theorem 1.3.6.** A graph in the torus is 5-colorable if and only if it has no subgraph isomorphic to \( K_6 \), \( C_3 + C_5 \), \( K_2 + H_7 \) or \( T_{11} \).

In addition, one structural result useful in Thomassen’s paper is another statement about precolored outer cycles.

**Lemma 1.3.7.** Let \( G \) be a planar graph with outer cycle \( S : x_1x_2 \cdots x_kx_1, k \leq 6 \). Let \( c \) be a 5-coloring of \( G[S] \). Then \( c \) can be extended to a 5-coloring of \( G \) if and only if none of (i), (ii), (iii) below hold:

(i) \( S \) has five colors and \( G - S \) has a vertex joined to all five colors of \( S \),

(ii) \( k = 6 \), and \( S \) has precisely four colors. The graph \( G - S \) contains two adjacent vertices each joined to all four colors of \( S \), and
In this thesis, we find the nine 6-critical graphs on the Klein bottle. This was joint work with Chenette, Postle, Streib and Thomas. It should be noted that another group [19] found these nine 6-critical graphs on the Klein bottle, but their proof relies upon a computer search. Our result is entirely self-contained and does not require the use of a computer.

**Theorem 1.3.8.** A graph in the Klein bottle is 5-colorable if and only if it has no subgraph isomorphic to $K_6$, $C_3 + C_5$, $K_2 + H_7$ or any of the graphs $L_1, L_2, \ldots, L_6$.

The graphs $L_1, L_2, \ldots, L_6$ are defined in Figure 3. It should be noted that three of the graphs on this list are also 6-critical graphs on the torus. The reason that $T_{11}$ is not on this list is that it is not embeddable on the Klein bottle. In fact,
there is a result of Lawrencenko and Negami [22] which says there are no 6-regular graphs embeddable on both the torus and the Klein bottle. Part of our result involves extending the preceding lemma of Thomassen to the case of precolored 7-cycles.

Our result settles Problem 3 of Thomassen in [36]. This result also implies that in order to test 5-colorability of a graph $G$ drawn in the Klein bottle, all we must check is whether the graph has a subgraph isomorphic to one of these nine graphs. By the algorithms of [14] and [24], we obtain the following.

**Corollary 1.3.9.** There exists an explicit linear-time algorithm to decide whether an input graph embeddable in the Klein bottle is 5-colorable.

Another corollary of our theorem involves a complicated result concerning Eulerian triangulations of Kral, Mohar, Nakamoto, Pangrac and Suzuki [21]. An *Eulerian* triangulation is a triangulation where all its vertices have even degree. Here is the corollary.

**Corollary 1.3.10.** An Eulerian triangulation of the Klein bottle is 5-colorable if and only if it has no subgraph isomorphic to $K_6$.

By inspection, each of the graphs from our main theorem has a subgraph isomorphic to a subdivision of $K_6$. So we can deduce the following corollary.

**Corollary 1.3.11.** If a graph in the Klein bottle is not 5-colorable, then it has a subgraph isomorphic to a subdivision of $K_6$.

Our result is also related to Hajos’ conjecture, which states that for every integer $k \geq 1$, if a graph $G$ is not $k$-colorable, then it has a subgraph isomorphic to a subdivision $K_{k+1}$. Hajos’ conjecture is known to be true for $k = 1, 2, 3$ but false for all $k \geq 6$. The cases $k = 4$ and $k = 5$ remain open. Further, in [41] (Conjecture 6.3), Thomassen conjectured that Hajos’ conjecture holds for every graph in the projective plane or the torus. His results imply that it suffices to prove this conjecture for $k = 4$,
but this is still open. One may also try to extend Thomassen’s conjecture to graphs in the Klein bottle. The previous corollary then implies it would suffice to prove the conjecture for $k = 4$.

Another related conjecture of Thomassen [41] (Conjecture 6.2) states that every graph which triangulates some surface satisfies Hajos’ conjecture. He also noted that this holds for $k \leq 4$ for every surface by a deep theorem of Mader [23] and that it holds for the projective plane and the torus by [39]. So the previous corollary implies that Thomassen’s second conjecture holds for graphs in the Klein bottle. This conjecture was disproved for general surfaces by Mohar [25] and qualitatively stronger counterexamples were found by Rodl and Zich [31].

### 1.4 Steinberg’s conjecture and higher surfaces

A famous result in graph coloring is that of Grötzsch’s theorem [18], which states that every triangle-free planar graph is 3-colorable. A shorter and more general proof of this statement was shown by Thomassen, who also proved an analogous statement for list coloring. The girth of a graph is the length of its smallest cycle. Thomassen showed [42] that every planar graph of girth at least five is also 3-list-colorable, meaning that it is possible to properly color the vertices of graph $G$ where $|L(v)| = 3$ for each $v \in G$.

In addition, there are results related to Grötzsch’s theorem for higher surfaces. In [40] Thomassen showed that every graph on the torus with girth at least five is 3-colorable. In addition, he showed that a graph embeddable on the projective plane with no contractible 3-cycle nor 4-cycle is 3-colorable. This work was extended by Gimbel and Thomassen [17] who proved that a triangle-free graph in the projective plane is 3-colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane. A quadrangulation is an embedded graph where every face is of length four. In addition, they show that every graph of girth
at least six and genus two is 3-colorable. The \textit{girth} of a graph is the length of its smallest cycle.

A famous related conjecture is due to Steinberg [35], which states that any planar graph without 4-cycles or 5-cycles is 3-colorable. One paradigm suggested by Erdős for Steinberg’s conjecture is to find the minimum $k$ such that a planar graph that excludes cycles from length 4 to length $k$ can be 3-colored. Steinberg’s conjecture is that $k = 5$. There have been a series of papers that have tightened the value of $k$. A number of these papers are based on \textit{discharging} arguments. In a discharging argument particular structures of a graph (such as faces, edges or vertices) are assigned an initial charge. Based on discharging rules germane to the problem, charge is transferred between graph structures, but the total charge is preserved. This often leads to a contradiction after the sum of the charges is counted in a different way. One basic result that can be shown using discharging is that if the minimum degree of a planar graph is five, then either it has an edge with endpoints both of degree five or one with endpoints of degrees five and six. For a more detailed explanation of discharging see [9].

In 1995, Sanders and Zhao [34] showed that $k \leq 9$. Borodin [5] also proves this same result in a separate paper. Both employ similar discharging arguments. Later, Salavatipour [33] showed that $k \leq 8$, and the most current result is of Borodin et al [6] who showed that $k \leq 7$. In this paper, the authors are able to show the stronger theorem that every proper 3-coloring of a cycle containing eight to eleven vertices can be extended to a 3-coloring of the entire graph.

Aside from the Erdős-type results, there have been other papers that describe more structural properties. One such structural characteristic that has been investigated is that of adjacent (also called intersecting) triangles. By only including triangles spaced far apart, we can reduce the cycle restrictions dramatically. In fact, for plane graphs, if we only exclude 5-cycles and triangles, Borodin and Raspaud [8], with a correction
from Xu [45] prove that if the triangles are at least distance four from each other, then the graph is 3-colorable. Distance between triangles is counted as the minimum distance between vertices of different triangles. Borodin and Raspaud also conjecture that the distance restriction could be strengthened to no intersecting 3-cycles. This conjecture is best possible because there exist planar graphs without 5-cycles that require four colors, as well as planar 4-chromatic graphs without intersecting triangles.

![Image](image1.png)

**Figure 4:** A planar 4-chromatic graph that has no 4-cycles.

![Image](image2.png)

**Figure 5:** A planar 4-chromatic graph that has no 5-cycles.

These graphs are relatively small (compared to similar counterexamples we will describe for similar choosability problems) as they contain 16 and 10 vertices, respectively. Baogang Xu has strengthened this result so that the distance between
intersecting triangles can be reduced to three [46]. In addition, Borodin [7] et al showed that planar graphs without 5- and 7-cycles without adjacent triangles are 3-colorable.

This condition given by Erdős actually also holds for arbitrary surfaces. Zhao [49] showed that for every surface $\Sigma$, there exists some $k$ so that if $G$ is a graph embedded on a surface $\Sigma$, and there are no cycles of length 4 up to length $k$, then $G$ is 3-colorable. However, this bound is far from tight for graphs of higher genus. In particular, the bound is $11 - 12\chi(\Sigma)$. So for a double torus, say, this requires the exclusion of cycles up to length 23. Zhao also shows that by using a method similar to his $k \leq 9$ paper, for surfaces with characteristic zero and above, we also have that $k \leq 9$. This bound could be far from tight, and it is still open whether $k$ could be reduced further or if there are any counterexamples to Steinberg’s conjecture for graphs with Euler characteristic at least zero.

For finitely many exceptions, we obtain a stronger bound than Zhao using as our guide the stronger statement proved in [6]. In particular, we prove the following theorem.

**Theorem 1.4.1.** Let $\Sigma$ be a surface of Euler genus $g$. If $G$ is 4-critical and has no cycles of length four through ten, then $|V(G)| \leq 2442g + 37$.

The proof of this theorem involves an inductive argument coupled with a discharging process. This is the main objective of Chapter 4.
CHAPTER II

SIX-CRITICAL GRAPHS ON A PRECOLORED CYCLE

2.1 Introduction

As stated in the introduction, Carsten Thomassen, in [36] shows that there are finitely many 6-critical graphs on a fixed surface. One step of his proof involves a structural theorem about a precolored cycle $C$ of length $q$. In general terms, he proves that the coloring $\phi$ of $C$ can be extended inside the cycle, or there exists a subgraph $H$ with at most $5^q \times 3$ vertices such that $\phi$ cannot be extended to a 5-coloring of $H$. In this chapter, we provide an alternative proof that reduces the number of vertices in $H$ to be cubic in $q$.

2.2 The Proof

We begin this section with definitions related to list coloring that will help us prove the main result of this chapter.

Definition 2.2.1. Let $G$ be a graph. A list-assignment is a function $L$ which assigns to every vertex $v \in V(G)$ a set $L(v)$ of natural numbers, which are called admissible colors for that vertex. An $L$-coloring of $G$ is an assignment of admissible colors to all vertices of $G$, i.e., a function $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for every $v \in V(G)$, such that for every edge $uv$ we have $c(u) \neq c(v)$. If $k$ is an integer and $|L(v)| \geq k$ for every $v \in V(G)$, then $L$ is a $k$-list-assignment. The graph is $k$-list-colorable (or $k$-choosable) if it admits an $L$-coloring for every $k$-list-assignment $L$. If $L(v) = \{1, 2, \ldots, k\}$ for every $v$, then every $L$-coloring is referred to as a $k$-coloring of $G$. If $G$ admits an $L$-coloring ($k$-coloring), then we say that $G$ is $L$-colorable ($k$-colorable).
We say that a vertex sees a color if it is adjacent to at least one precolored vertex of that color. Further, a coloring \( \phi \) of a subset \( H \) of \( G \) extends to \( G \) if there exists a proper coloring of \( G \) after all the vertices in \( H \) are precolored by \( \phi \). Given a cycle \( C \), and a vertex \( v \) adjacent to three consecutive vertices of \( C \), called, in order, \( x_1, x_2, x_3 \), define the operation \( * \) such that \( C * v \) defines a new cycle \( C' \) where the path \( x_1x_2x_3 \) is substituted by \( x_1vx_3 \). In this case, we say that \( C' \) is a rerouting of \( C \) via \( v \).

We now state the following lemma, which will be useful in the proof of Theorem 2.2.4. Notice that in the statement of the lemma and its proof, coloring refers to 5-coloring.

**Lemma 2.2.2.** Let \( G \) be a plane graph with outer cycle \( C_1 \) that has no separating triangles. Let \( L \) be a set of vertices in \( C_1 \). Let \( \phi \) be a coloring of the subgraph of \( C_1 \) induced by \( L \). Then there exists a cycle \( C'_1 \) obtained from \( C_1 \) via a series of rerouting operations via only vertices distance at most two from \( C_1 \) such that either:

(i) \( C'_1 \) has a chord, or some vertex \( v \) in the open disk bounded by \( C'_1 \) has two neighbors on \( C'_1 \) that are at least distance three apart on \( C'_1 \), or

(ii) there exists a set \( L' \) of \( G \) that includes \( L \), and a coloring \( \phi' \) of the subgraph of \( G \) induced by \( L' \) that extends \( \phi \) such that for all \( v \in V(G) - L' \), the following holds: If \( v \) does not lie in the closed disk bounded by \( C'_1 \), then \( v \) is a vertex of \( G \setminus (V(C'_1) \cup L') \) and the number of colors in \( \phi' \) seen by \( v \) plus the number of neighbors of \( v \) in \( G \setminus V(L') \) is at most four. Further, either \( |V(C_1) \cap L'| < |L| \) or every \( v \) not in \( L' \) in the closed disk bounded by \( C'_1 \) sees at most two colors of \( \phi' \). We also require that the subgraph induced by \( L' \) is connected.

**Proof.** If there are no vertices in \( G - C_1 \) that are adjacent to three or more vertices of \( L \), then we are finished because \( C_1 \) and \( L \) satisfy the conclusion of (ii) as \( C'_1 \) and \( L' \), respectively.

Before continuing, we introduce some notation. Let \( D \) be a cycle of vertices that are all distance at most two from \( C_1 \). Define a **tripod** on \( D \) to be a vertex, \( y \), that
is adjacent to at least three vertices on \( D \). For the purposes of this lemma, we may assume that \( y \) must be adjacent to exactly three consecutive vertices on \( D \), else condition (i) of the lemma is satisfied. Let \( y \) be a tripod that is adjacent to \( x_1, x_2, x_3 \), in that order on some cycle \( D \). We say that \( y \) is an initial tripod if vertex \( x_2 \) is a member of \( C_1 \) and if any of the \( x_i \)'s belong to \( C_1 \), then they are members of \( L \). Vertices \( x_1 \) and \( x_3 \) need not be members of \( C_1 \).

We now describe how to construct \( C'_1 \). Start with \( C_1 \), and then if there exists an initial tripod \( v \), construct \( C_2 \) by rerouting \( C_1 \) via \( v \). Continue this process if there are additional initial tripods, creating cycles \( C_3, \ldots \) until there are no more initial tripods. Call this graph \( C'' \). Notice that by construction, every vertex in \( C' \) is either on \( C_1 \) or distance one from \( C_1 \). Define a candidate vertex to be a vertex of \( C' \) not on \( C_1 \). The reason we call this vertex a candidate vertex is because it is a candidate for being a member of \( L' \). Suppose that \( E \) is the set of candidate vertices associated with the cycle \( C'' \) in graph \( G \). Notice there are at most \( q \) candidate vertices and hence at most \( q \) initial tripod reroutings. If there are no vertices (that are neither candidate vertices nor in \( C_1 \)) that are adjacent to three candidate vertices or vertices in \( L \), then let \( C'_1 \) be \( C'' \) and let \( L' \) be equal to \( L \cup E \). Color the vertices of \( E \) properly to obtain \( \phi' \) in this case. Then \( L' \) and \( \phi' \) satisfy proposition (ii). Further, suppose there are no vertices \( w \) that satisfy condition (i) where \( C'_1 \) is \( C' \).

Before we continue with the proof, we make some general observations about the structure of candidate vertices which will be useful later. If a vertex is a candidate vertex, then it is on a path of candidate vertices such that one of these vertices is adjacent to three vertices of \( C_1 \), which we label \( v_1, v_2, v_3 \) in that order. To see this, consider how initial tripods are generated. The first initial tripod, with candidate vertex \( x_1 \), must be adjacent to three vertices of \( C_1 \). The next initial tripod has a candidate vertex, \( x_2 \) which either is also adjacent to three vertices of \( C_1 \) or adjacent to \( x_1 \) and two vertices of \( C_1 \). Notice that if \( x_2 \) was adjacent to exactly two vertices
of $C_1$, then these two vertices must be either $v_3$ and $v_4$ or $v_0$ and $v_1$, where $v_0$ is a member of $C_1$ adjacent to $v_1$ and $v_4$ is a member of $C_1$ adjacent to $v_3$. Otherwise the rerouting $C'_1 = C_1 \ast x_1 \ast x_2$ satisfies condition (ii) of this lemma after properly coloring $x_1, x_2$ and adding them to $L'$. Subsequent candidate vertices, say $x_3, \ldots$ must also be adjacent to at least two vertices of $C_1$ unless they are adjacent to two separate paths of candidate vertices (each of which must contain a candidate vertex adjacent to three vertices of $C_1$) or both ends of a single path of candidate vertices. The only way this latter case could occur is then if the single path of candidate vertices traverses around the entire cycle. Even in these special cases, these candidate vertices are still adjacent to one vertex of $C_1$.

We may assume there exists a vertex $v$, not on $C_1$ adjacent to three consecutive vertices of $C'$, call them $x_1, x_2, x_3$, such that if any of these vertices are on $C_1$ then these vertices are in $L$. We first consider the situation when at least one of $x_1, x_2, x_3$ is a member of $C_1$, and hence a member of $L$.

Suppose that $x_1$ and $x_3$ and both are on $C_1$, but $x_2$ is not (else $v$ would be an initial tripod). Then the rerouting of $C_1$, $C'_1 = C_1 \ast v$ satisfies the condition of proposition (ii). Notice that $v$ does not receive a color, so the condition $|C_1 \cap L'| < |L|$ is satisfied. This completes the case when both $x_1, x_3 \in C_1$.

Suppose that $x_1$ is a member of $C_1$. Then $x_2$ is not a member of $C_1$ as then $v$ would have been an initial tripod. Suppose that $x_3$ is not on $C_1$. Without loss of generality, let $x_1 = v_1$. Suppose that $x_2$ is adjacent to exactly two vertices on $C_1$, call them $v_1, v_2$. Also suppose that $x_3$ is adjacent to exactly three vertices of $C_1$. Then $x_2$ is an vertex of degree four. We can construct $C'_1$ by $C'_1 = C_1 \ast x_3 \ast x_2 \ast v$. Also, color vertex $x_3$ properly and include it in $L'$. Then $C'_1$ and $L'$ satisfy condition (ii).

Suppose instead that $x_3$ is adjacent to exactly two vertices of $C_1$, call them $v_2$ and $v_3$. In this case, $x_3$ must be adjacent to another candidate vertex, which is part of a path of candidate vertices $x_4, \ldots$ that includes a vertex, call it $x_k$ that is adjacent to
three vertices of $C_1$. In this case, let $C'_1$ be the path $x_1vx_3x_4\cdots x_k$, edge $x_kv_l$, where $v_l$ is the vertex furthest from $x_1$ on $C_1$ in the direction of $v_3$ that traverses $v_1,v_2,v_3$ in that order, and the path $v_lv_{l+1}\cdots x_1$. Observe that $x_4 \neq x_2$ since $x_2$ has two neighbors of $C_1$ and it is not adjacent to any other candidate vertices. In addition, properly color vertices $x_3,x_4,\cdots,x_k$ and make them part of $L'$. Again condition (ii) is satisfied as $|L' \cup C_1| < |L|$. Now, suppose that $x_3$ is adjacent to one vertex of $C_1$. This is a contradiction to our assumption that $x_2$ is adjacent to two vertices of $C_1$.

Observe that $x_2$ must have been an initial tripod before $x_3$, but $x_2$ is not adjacent to any other candidate vertices. If $x_2$ were adjacent to other candidate vertices, then $v$ would satisfy condition (i) as it would be adjacent to at least four vertices on a cycle, say $C_1$, constructed in the process of creating $C'$. So $x_2$ must be adjacent to three vertices on $C_1$. This contradicts our assumption that $x_2$ is adjacent to exactly two vertices of $C_1$. This concludes the case when $x_1 \in C_1$, $x_3 \notin C_1$ and $x_2$ is adjacent to exactly two vertices of $C_1$.

Now suppose that $x_1 \in C_1$, $x_3 \notin C_1$ and $x_2$ is adjacent to exactly three vertices of $C_1$. Then we must give $x_3$ an explicit color so that $x_2$ can satisfy the conditions for an isolated vertex stated in condition (ii) of the lemma. Now, without loss of generality, say that the neighbors of $x_2$ on $C_1$, labeled $x_1 = v_1,v_2,v_3$ in that order are colored $\alpha,1,2$, respectively. Notice that $x_3$ is also adjacent to $v_3$. If $\alpha \notin \{1,2\}$ then let $\phi'(x_3)$ be 1 or $\alpha$. If $\alpha \in \{1,2\}$, then let $\phi'(x_3)$ be an arbitrary color that keeps the coloring proper at this point. Set $x_3 \in L'$. If $x_3$ is adjacent to three vertices of $C_1$, then at the time of either $x_2$ or $x_3$’s addition as an initial tripod, one of $x_2$ and $x_3$ was adjacent to four vertices of the cycle (depending upon whether $x_2$ or $x_3$ was added first). Then condition (i) of the lemma is obtained. Now suppose that $x_3$ is adjacent to exactly two vertices of $C_1$, called $v_3$ and $v_4$. Let $C'_1 = C_1 \ast x_2 \ast x_3 \ast v$. Let $L' = L \cup \{x_3\}$. If $x_3$ is adjacent to only one vertex of $C_1$, then it must be adjacent to a candidate vertex other than $x_2$, which is part of a path of candidate vertices $x_4,\ldots$
that includes a vertex, call it $x_k$ that is adjacent to three vertices of $C_1$. In this case, let $C'_1$ be the path $x_1v_3x_4 \cdots x_k$, edge $x_kv_l$, where $v_l$ is the vertex furthest from $x_1$ on $C_1$ in the direction of $v_3$, that traverses vertices $v_1,v_2,v_3$ in that order, and the path $v_lv_{l+1} \cdots x_1$. In addition, properly color vertices $x_4, \ldots, x_k$ and make them part of $L'$. Again condition (ii) is satisfied as $|L' \cup C_1| < |L|$.

Finally, suppose that $x_1 \in C_1$, $x_3 \not\in C_1$ and $x_2$ is adjacent to exactly one vertex of $C_1$ and this vertex must be $x_1$. Notice that $x_1,x_2,x_3$ are consecutive vertices of $C'$. Also notice that $x_3$ must be adjacent to at least two vertices of $C'$, as it is a candidate vertex adjacent to a candidate vertex with only one neighbor on $C'$. Now $x_2$ must be adjacent to another candidate vertex, call it $x_0$. This vertex must also be adjacent to at least two vertices of $C'$. But then vertex $v$, adjacent to $x_1,x_2,x_3$ can not be drawn.

An analogous set of arguments hold by symmetry if $x_3$ was the member of $C_1$ initially identified instead of $x_1$.

From the results of the previous paragraphs, we may assume that vertex $v$ is adjacent to three candidate vertices. We now describe how to color candidate vertices and reroute $C'$ to construct the cycle $C'_1$, the set $L'$ and associated coloring $\phi'$ that we desire. For each of these situations, here is the general set-up. Let $v$ be a vertex adjacent to three candidate vertices of $C'$, which we call, in order, $x_1,x_2,x_3$. For the purposes of this proof, let $(a,b,c)$ be the number of vertices $x_1,x_2,x_3$, respectively are adjacent to in $C_1$.

Case 1: (2,3,2) Suppose that $x_2$’s neighbors on $C_1$, denoted $v_1,v_2,v_3$ are colored $\alpha$, 1 and 2, respectively. First suppose that $\alpha \not= \{1,2\}$. Then let $\phi'(x_1)$ equal 1 or 2 and let $\phi'(x_3)$ 1 or $\alpha$. Similarly, if $\alpha = 2$, then let $\phi'(x_1)$ and $\phi'(x_3)$ be colored with 1 or 2, with the additional flexibility that an additional color (namely, 3, 4, or 5) can be used amongst $\phi'(x_1)$ and $\phi'(x_3)$ as the argument still applies if $x_2$ sees three colors of $\phi'$. Let the two neighbors of $x_1$ on $C_1$ be denoted $v_0,v_1$, and the two neighbors of $x_3$ on $C_1$ be denoted $v_3,v_4$. Notice that vertex $v$ and $x_2$ are not members of $L'$,
but $x_1$ and $x_3$ are now members of $L'$. Let $C'_1 = C_1 \ast x_2 \ast x_3 \ast x_1 \ast v$. Observe that condition (ii) is satisfied as $|L' \cup C_1| < |L|$. 

Case 2: (3,2,2) Observe that the (2,2,3) case is identical. Suppose that vertex $x_1$ is adjacent to $v_1, v_2, v_3$ in $C_1$. Again, we want to color via $\phi'$ vertices $x_1$ and $x_3$ such that, under $\phi'$, $x_2$ sees at most three colors. So suppose that without loss of generality $v_3$ and $v_4$ are colored 1 and 2, respectively. Now, let $\phi'(x_3)$ be 1 or a color in $\{3,4,5\}$ that is not the color of $v_1$ nor the color of $v_2$. We now define $\phi'(x_1)$. If $\phi'(x_3) = 1$, then color $x_1$ with a proper color. If $\phi'(x_3) \neq 1$, then let $\phi'(x_1) = \phi'(x_3)$. Notice that vertex $v$ and $x_2$ are not members of $L'$. So $C'_1 = C_1 \ast x_1 \ast x_2 \ast x_3 \ast v$. Observe that condition (ii) is satisfied as $|L' \cup C_1| < |L|$. 

Case 3: (2,2,2) Again, we want to color via $\phi'$ $x_1$ and $x_3$ such that, under $\phi'$, $x_2$ sees at most three colors. So suppose that without loss of generality $x_2$ is adjacent to $v_1, v_2, v_3$ on $C_1$ and $v_1, v_2$ are colored 1 and 2 respectively. Suppose that $x_1$ is adjacent to $v_0, v_1$ and $x_3$ is adjacent to $v_2, v_3$. Now, vertex $x_1$ can be colored 2 or 3 (as it is adjacent to a vertex on $C'_1$ colored 1, and vertex $x_3$ can be colored 1 or 3, as it is adjacent to the vertex on $C_1$ colored 2. Suppose, without loss of generality that there exists a vertex adjacent to three vertices of $C_1$ along the path of candidate vertices $x_3, x_4, \ldots$. Suppose this vertex is $x_k$, which is adjacent to vertices $v_{l-2}, v_{l-1}, v_l$ on $C_1$. Here $v_l$ is furthest from $v_1$ on $C_1$ in the direction of $C_1$ that traverses $v_1, v_2, v_3$ in that order. Notice that $v$ and $x_2$ are not members of $L'$. In addition, properly color vertices $x_4, \ldots, x_k$ and make them part of $L'$. So $C'_1$ is the cycle $v_0 x_1 v x_3 x_4 \cdots x_k v_l v_{l+1} \cdots v_0$. Observe that condition (ii) is satisfied as $|L' \cup C_1| < |L|$. 

Observe that cases involving two threes in $(a, b, c)$ except (3,1,3) are impossible because the process of generating $C'$ would have constructed a vertex $v$ adjacent to more than three vertices on $C_k$, for some $k$. So the remaining cases include at least one vertex that has a 1 in $(a, b, c)$. Also let $d/e$ denote the situation where either that vertex is adjacent to either $d$ or $e$ vertices on $C'_1$. 

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Case 4: (2/3,1,2/3) Observe that vertex $x_2$ has degree four, so we may color $x_1, x_3$ arbitrarily but properly. In (3,1,3), let $C'_1 = C_1 \ast x_1 \ast x_3 \ast x_2 \ast v$ and add $x_1, x_3$ to $L'$. This satisfies condition (ii). For cases when $a = 2, c = 2$ or when $a, c = 2$, we will have to travel along a path of candidate vertices moving away from $x_2$ starting with $x_1$ (when $a = 2$) or $x_3$ (when $c = 2$), or both when $a, c = 2$ before $C'_1$ returns to $C_1$.

Let $P_1$ be the path obtained when $a = 2$, starting at $x_1$ in the direction away from $x_2$ and continuing until a vertex of degree three is reached along this path of candidate vertices. Similarly define $P_2$ to be the path obtained when $c = 2$, starting at $x_3$ in the direction away from $x_2$ and continuing until a vertex of degree three is reached along this path of candidate vertices. Color the vertices of $P_1$ or $P_2$ or both if they are constructed and include these vertices in $L'$. Again, condition (ii) is satisfied.

Case 5: (1,2,2/3) or (2/3,2,1) Without loss of generality, suppose we are in the case (1,2,2/3). The arguments for Cases 2 and 3 hold for this case, except that vertex $x_1$ has one less restriction in its set of possible colors. When $C'_1$ is constructed, instead of directly traveling to $C_1$ from $x_1$, construct a path of candidate vertices $P_1$ as described in Case 4 so that $C'_1$ returns to $C_1$ once a vertex on $P_1$ is reached that is adjacent to three vertices of $C'_1$. Properly color the vertices of $P_1$ and include them in $L'$. Further, if $c = 2$ we will create an analogous path of candidate vertices $P_2$ starting at $x_3$ and ending at a candidate vertex adjacent to three vertices of $C_1$. Now $C'_1$ travels along $P_2$ before returning to $C'_1$. Properly color the vertices of $P_2$ and include them in $L'$. Again, condition (ii) is satisfied.

Case 6: (1,3,1/2) or (1/2,3,1) Without loss of generality, suppose we are in the case (1,3,1/2). The arguments for Case 1 hold in for this case in the coloring of $x_1$ and $x_3$, except that vertex $x_1$ or (both $x_1$ and $x_3$) has one less restriction in its set of possible colors. When $C'_1$ is constructed, instead of directly traveling to $C_1$ from $x_1$ or $x_3$, if $x_1$ or $x_3$ (or both) is a vertex adjacent to one vertex of $C'_1$, then construct paths $P_1$ and $P_2$, for $x_1, x_3$ respectively as in Cases 4 and 5. If $P_1$ and $P_2$ are used, properly
color these paths of candidate vertices and include them in $L'$. Again, condition (ii) is satisfied.

Case 7: (1, 1/2, 1) This case is impossible because (as described in the observations about candidate vertices) in every path of candidate vertices containing a pair of candidate vertices adjacent to one vertex of $C_1$, say $x_i$ and $x_j$, there must be a candidate vertex adjacent to three vertices of $C_1$ between $x_i$ and $x_j$ along this path of candidate vertices.

Finally, notice that $L'$ is connected because all vertices that are added to $L'$ are adjacent to a vertex in $L$. Further notice that $L'$ does not include any vertices inside the open disk bounded by $C'_1$. \hfill \Box

**Lemma 2.2.3.** Let $G$ be a plane graph with outer cycle $C_1$ of length $q$ that has no separating triangles. Let $\phi$ be a coloring of $C_1$. Then there exists a cycle $C'_1$ obtained from $C_1$ via a series of rerouting operations, where there are at most $3q^2$ vertices in $G$ outside of the open disk bounded by $C'_1$, such that either:

(i) $C'_1$ has a chord, or some vertex $v$ in the open disk bounded by $C'_1$ has two neighbors on $C'_1$ that are at least distance three apart on $C'_1$, or

(ii) there exists a set $L'$ of $G$, consisting only of vertices outside the open disk bounded by $C'_1$ that includes $C_1$ and a coloring $\phi'$ of the subgraph of $G$ induced by $L'$ that extends $\phi$ such that for all $v \in V(G) - L'$ the following holds: If $v$ lies in the closed disk bounded by $C'_1$, then $v$ sees at most two colors of $\phi'$. Otherwise, the number of colors in $\phi'$ seen by $v$ plus the number of neighbors of $v$ in $G \setminus V(L')$ is at most four.

**Proof.** Suppose $G$, $C_1$, $\phi$ are described as in the statement of the lemma. By Lemma 2.2.2 there exists a cycle $C'_1$ that either satisfies condition (i) or condition (ii) of Lemma 2.2.2.

First suppose that $C'_1$ satisfies condition (i) of Lemma 2.2.2. Notice that by Lemma 2.2.2, $C'_1$ uses vertices distance at most two from $C_1$. At most $q$ vertices are distance one from $C_1$ are generated and at most $q$ vertices distance two from $C_1$ are
generated from the process in Lemma 2.2.2. So this satisfies the condition that the graph inside $C'_1$ excludes at most $3q^2$ vertices from $G$. Thus $C'_1$ satisfies condition (i) of this lemma.

Suppose instead that $C'_1$ satisfies condition (ii) of Lemma 2.2.2. If every $v$ in the closed disk bounded by $C'_1$ sees at most two colors of $\phi'$, then condition (ii) of this lemma is satisfied. Call this condition (ii)(a) of Lemma 2.2.2. The lemma is satisfied because all the vertices of $G$ not inside $C'_1$ are either in the $L'$ given in Lemma 2.2.2 or are vertices that satisfy the conditions in condition (ii) for vertices in $G \setminus (V(C'_1) \cup L')$. Note that $L'$ is connected. Observe that in this iteration of Lemma 2.2.2, the graph inside $C'_1$ excludes at most $2q$ vertices from $G$. If the other conclusion of condition (ii) from Lemma 2.2.2, $|C_1 \cap L'| < |L|$, holds (call this condition (ii)(b)), then we will again apply Lemma 2.2.2 but now $C_1$ is the $C'_1$ generated by the lemma. Notice that this time $|L|$ is at most $q - 1$. Also observe that there are at most $2q$ vertices of $G$ outside of this new $C_1$.

So we will apply Lemma 2.2.2 repeatedly until either condition (i) of Lemma 2.2.2 holds or condition (ii)(a) of Lemma 2.2.2 holds. Notice that condition (ii)(b) may only be satisfied $q$ times in a row because in the first iteration of Lemma 2.2.2, $|L| = q$ and the number of precolored vertices decreases at least by one after each iteration of Lemma 2.2.2. Notice after $q$ iterations of Lemma 2.2.2 at most $2q(q)$ vertices of $G$ have been excluded from the given $C'_1$ and its interior. Thus, after $q + 1$ iterations of Lemma 2.2.2, one of the two conclusions that satisfies this lemma is obtained and at most $2q(q + 1) \leq 3q^2$ vertices are excluded from $G$ in the provided $C'_1$. This concludes the proof of the lemma.

\[\square\]

**Theorem 2.2.4.** Let $G$ be a 2-connected plane graph with no separating triangle and with outer cycle $C$ of length $q$. Let $\phi$ be a 5-coloring of the vertices of $C$. Then $G$ contains a connected subgraph $H$, which includes the vertices of $C$, with at most $7(q - 2)^3$ vertices such that either (i) or (ii) holds:
(i) \( \phi \) can not be extended to a 5-coloring of \( H \), or

(ii) there exists a 5-coloring \( \phi' \) of the subgraph of \( G \) induced by \( H \) that extends \( \phi \) such that for all \( v \in V(G) - V(H) \), the following holds: either \( v \) sees at most two colors of \( \phi' \), or the number of colors in \( \phi' \) seen by \( v \) plus the number of neighbors of \( v \) in \( G \setminus V(H) \) is at most four. In this case the 5-coloring \( \phi' \) extends to a 5-coloring of \( G \).

Proof. This proof is by induction on the size of \( q \). Let \( f_q \) be the maximum size of a graph \( H \), described in the theorem when cycle \( C \) is length \( q \). In either case assume that cycle \( C \) is length \( q \). First, suppose that \( \phi \) can be extended to a 5-coloring of \( G \). We will now apply Lemma 2.2.3 to this situation. If condition (ii) of Lemma 2.2.3 is obtained, then the subgraph induced by \( L' \) given in Lemma 2.2.3 is an \( H \), with associated coloring \( \phi' \) that satisfies conclusion (ii) of this theorem. Observe that by Lemma 2.2.3, the size of the subgraph induced by \( L' \) is at most \( 3q^2 \), which is less than \( 7(q - 2)^3 \). Further, we show that the 5-coloring \( \phi' \) extends to a 5-coloring of \( G \). To do this, observe that by Lemma 2.2.3, we obtain a cycle \( C_1' \) and coloring \( \phi' \) of a set of at most \( 3q^2 \) vertices such that every vertex contained in \( C_1' \) sees at most two colors of \( \phi' \). Notice that there exists some cycle \( C_1'' \) that contains all vertices inside \( C_1' \) but not necessarily vertices of \( C_1' \) such that each of these vertices has at least three colors available. Let \( M \) be the graph contained in the closed disk bounded by \( C_1'' \). Triangulate each component of \( M \). We can then apply Lemma 1.3.2 to show that there exists a 5-coloring that extends inside \( C_1'' \). By construction of \( C_1' \), the vertices on \( C_1' \) and outside of \( C_1' \), but not in \( C_1'' \) may now be properly colored. Thus the coloring of \( \phi' \) extends to \( G \).

Suppose instead case (i) of Lemma 2.2.3 occurs. This condition gives a cycle \( C_1' \) in \( G \). Let \( K \) be the set of vertices of \( G \) outside or on \( C_1' \). Notice that by Lemma 2.2.3 that the number of vertices of \( K \) is at most \( 3q^2 \). The lemma gives \( \phi' \), a coloring of \( L' \), a subset of \( K \). Further, the lemma gives that there exists either a chord or a path of length two that separates the graph inside \( C_1' \) into two graphs \( G_1, G_2 \), with outer
cycles $D_1, D_2$, respectively whose lengths are each smaller than $q$. Since $\phi$ extends to $G$, there exists a coloring $\phi_0$ that extends $\phi$ to a 5-coloring of $G$. Let coloring $\phi_1$ be coloring $\phi_0$ restricted to $G \setminus G_1$, and let $\phi_2$ be the coloring $\phi_0$ restricted to $G \setminus G_2$. We may then apply induction on $G_1$ and $G_2$, using colorings $\phi_1$ and $\phi_2$ respectively. Let the length of $D_1$ be $l_1$ and the length of $D_2$ be $l_2$. If a chord was used to partition $G_1, G_2$, then it follows that $f_q$ satisfies the recurrence relation $f_q \leq f_{l_1+1} + f_{l_2+1} + 3q^2$, $q \geq 4$. Similarly, if a path of length two was used to partition $G_1, G_2$, then $f_q$ satisfies the recurrence relation $f_q \leq f_{l_1+2} + f_{l_2+2} + 3q^2$, $q \geq 6$. We will obtain a bound for $f_q$ after we consider the case when $\phi$ does not extend to a 5-coloring of $G$. This completes the case when $\phi$ extends to a 5-coloring of $G$.

Now suppose that $\phi$ does not extend to a 5-coloring of $G$. Let $H$ be a minimal subgraph that does not extend $\phi$. So we may assume that $M = C \cup H$ is a $C$-critical graph. Then when we apply Lemma 2.2.3, condition (ii) can not occur, as this would provide for a 5-coloring of $G$. This is because, after triangulating the graph inside the open disk bounded by $C'_1$, we can use Lemma 1.3.2. Thus, we may assume that condition (i) of Lemma 2.2.3 occurs. This condition gives a cycle $C'_1$ in $G$. Let $K$ be the set of vertices of $G$ outside or on $C'_1$. Notice that by Lemma 2.2.3 the number of vertices of $K$ is at most $3q^2$. The lemma gives $\phi'$, a coloring of $L'$, a subset of $K$. Further, the lemma gives that there exists either a chord or a path of length two that separates the graph inside $C'_1$ into two graphs $G_1, G_2$, with outer cycles $D_1, D_2$, respectively whose lengths, denoted $l_1, l_2$, respectively, are each smaller than $q$. Notice that $\phi'$ can be extended to $K$ because every vertex not in $L'$ in $K$ satisfies condition (ii) of Lemma 2.2.3. Call this coloring $\phi''$.

Notice that graphs $G_1$ and $G_2$ are $D_1$-critical and $D_2$-critical, respectively and so we may apply induction to each of these graphs. If there is a chord that partitions $G_1$ and $G_2$, then $f_q \leq f_{l_1+1} + f_{l_2+1} + 3q^2$, for $q \geq 4$. If instead there is a path of length two that partitions $G_1$ and $G_2$, then $f_q$ satisfies the recurrence relation
We now must give a bound for \( f_q \). To do this we must ensure that \( f_q \) is larger than the inductive bound given from the recurrence relations described above and give appropriate base cases. First, observe that \( f_3 = 4 \). To see this, observe that the first vertex inside the triangle must be adjacent to all three vertices inside the triangle. Then the next vertex added to this graph will create a separating triangle, a contradiction. We now will estimate \( f_4 \).

If we begin with a 4-cycle, we may have to proceed through Lemma 2.2.2 adding, by Lemma 2.2.3, at most \( 3q^2 = 48 \) additional vertices to \( H \) before we find a chord that separates the graph into two 3-cycles. We know \( f_3 = 4 \), so \( f_4 \leq 56 \). We can make a similar argument to bound \( f_5 \). Lemma 2.2.3 implies that at most \( 3q^2 = 75 \) additional vertices are added to \( H \) before we find a chord that separates the graph into a 4-cycle and a 3-cycle. So then \( f_5 \leq 75 + f_4 + f_3 \leq 135 \).

The inductive argument gives that for every \( q \geq 4 \), either \( f_q \leq 3q^2 \), or there exist \( l_1, l_2 \) such that \( 2 \leq l_1, l_2 \leq q - 2 \), \( l_1 + l_2 = q \) and

\[
f_q \leq f_{l_1+1} + f_{l_2+1} + 3q^2,
\]

or there exist \( l_1, l_2 \) such that \( 2 \leq l_1, l_2 \leq q - 2 \), \( l_1 + l_2 = q \) and

\[
f_q \leq f_{l_1+2} + f_{l_2+2} + 3q^2.
\]

We will show that \( f_q \leq 7(q - 2)^3 \). For \( q \leq 6 \), we will prove this directly. Two paragraphs above, we have shown that \( f_3 = 4 \), \( f_4 \leq 56 \) and \( f_5 \leq 135 \). Notice that the claimed bound for \( f_q \) in these cases gives \( f_3 \leq 5 \), \( f_4 \leq 56 \), \( f_5 \leq 189 \). Suppose that \( q = 6 \), then \( f_6 \) must satisfy the inequalities \( f_6 \leq f_5 + f_5 + 3q^2 \), \( f_6 \leq f_5 + f_3 + 3q^2 \), and \( f_6 \leq f_4 + f_4 + 3q^2 \). Notice that the first of these three inequalities dominates the other two since \( f_5 \geq f_4 \geq f_3 \). So, \( f_6 \leq 135 + 135 + 3(6)^2 = 378 \). This satisifes \( f_6 \leq 7(4)^3 = 448 \). Thus, our bound holds for \( f_6 \). For \( q \geq 7 \) we have
\[ f_q \leq f_{l_1+2} + f_{l_2+2} + 3q^2 \leq 7l_1^3 + 7l_2^3 + 3q^2 \]
\[ \leq 7(q-3)^3 + 7(3)^3 + 3q^2 \leq 7(q-2)^3. \]

Notice that the last inequality holds for all \( q \geq 7 \) as the inequality \(-63q^2 + 189q + 3q^2 \leq -42q^2 + 84q - 56 \) holds. Thus, we have obtained a cubic bound for \( |V(H)| \).

In particular, we have shown that if \( \phi \) can not be extended to a 5-coloring of \( H \) then \( |V(H)| \leq 7(q-2)^3 \). \( \square \)

**Theorem 2.2.5.** Let \( G \) be a plane graph with outer cycle \( C \) of length \( q \). Let \( \phi \) be a 5-coloring of the vertices of \( C \). If the coloring of \( \phi \) does not extend to \( G \), then there exists a connected subgraph \( H \) of \( G \) that includes \( C \), of size at most \( 7(q-2)^3 \), for which \( \phi \) can not be extended.

**Proof.** By Theorem 2.2.4, we know that either \( \phi \) can not be extended to a 5-coloring of \( H \), where \( H \) has at most \( 7(q-2)^3 \) vertices and so the theorem holds, or condition (ii) of Theorem 2.2.4 is obtained. If the latter case holds, observe that by Theorem 2.2.4, the 5-coloring \( \phi' \) obtained from Theorem 2.2.4 can be extended to a 5-coloring of \( G \). \( \square \)
CHAPTER III

FIVE-COLORING GRAPHS ON THE KLEIN BOTTLE

3.1 Introduction

In this chapter we turn our attention to 5-coloring graphs on relatively simple surfaces. Even for graphs of low genus, the most interesting value of \( t \) for the \( t \)-colorability problem on a fixed surface seems to be \( t = 5 \). By the Four-Color Theorem every graph in the sphere is 4-colorable, but on every other surface there are graphs that cannot be 5-colored. Albertson and Hutchinson [1] proved that a graph in the projective plane is 5-colorable if and only if it has no subgraph isomorphic to \( K_6 \), the complete graph on six vertices. Thomassen [39] proved the analogous (and much harder) result for the torus, as follows. If \( K, L \) are graphs, then by \( K + L \) we denote the graph obtained from the union of a copy of \( K \) with a disjoint copy of \( L \) by adding all edges between \( K \) and \( L \). The graph \( H_7 \) is depicted in Figure 6 and the graph \( T_{11} \) is obtained from a cycle of length 11 by adding edges joining all pairs of vertices at distance two or three.

**Theorem 3.1.1.** A graph in the torus is 5-colorable if and only if it has no subgraph

\[ H_7 \]

Figure 6: The graph \( H_7 \)
Our objective in this chapter to prove the analogous result for the Klein bottle, the following. The graphs $L_1, L_2, \ldots, L_6$ are defined in Figure 7. Lemma 3.4.2 explains the relevance of most of these graphs.

**Theorem 3.1.2.** A graph in the Klein bottle is 5-colorable if and only if it has no subgraph isomorphic to $K_6$, $C_3 + C_5$, $K_2 + H_7$, or any of the graphs $L_1, L_2, \ldots, L_6$.

Theorem 3.1.2 settles a problem of Thomassen [36, Problem 3]. It also implies that in order to test 5-colorability of a graph $G$ drawn in the Klein bottle it suffices to test subgraph isomorphism to one of the graphs listed in Theorem 3.1.2. In order to describe aspects of a linear-time algorithm to test subgraph isomorphism, we must first define the notion of treewidth. To obtain the treewidth of a graph, we must first transform the graph into a tree-decomposition. A tree-decomposition of a graph $G$ is
a pair \((T, W)\), where \(T\) is a tree and \(W\) is a set of subsets \(W_t\) of \(V(G)\) (sometimes called "bags"), one for each vertex \(t\) of \(T\) such that:

- \(\cup_{t \in V(T)} W_t = V(G)\).

- For each edge \(e = uv\) of \(G\), there is a \(t \in V(T)\) such that \(u, v \in W_t\).

- For each vertex \(u \in V(G)\), the subtree of \(T\) induced by \(\{t|u \in W_t\}\) is connected.

In words, these conditions mean that every vertex is in at least one bag. They also imply that for each edge of \(G\), there exists some bag that contains both vertices of the edge. Further, for each particular vertex of \(G\), the subgraph consisting of the bags containing a particular vertex is connected.

For some tree-decomposition \((T, W)\), \(w(T, W)\) is the maximum of \(|W_t|\) over all vertices of \(T\). The treewidth of a graph \(G\) is defined to be the minimum value of \(w(T, W)\) over all tree-decompositions \((T, W)\) of \(G\). The standard definition of treewidth calls for the treewidth of a graph to actually be \((\max |W_t|) - 1\) so that trees have treewidth 1.

![Figure 8](image_url)

**Figure 8:** An example of a graph, a decomposition and a tree-decomposition derived from this decomposition.

A family \(\mathcal{F}\) of graphs has the diameter-treewidth property if there is some function \(f(D)\) such that every graph in \(\mathcal{F}\) with diameter at most \(D\) has treewidth \(f(D)\). An algorithm of Eppstein shows that
Theorem 3.1.3. [14] [13] There exists a linear-time algorithm for the subgraph isomorphism decision problem for all families of graphs that have the diameter-treewidth property.

Initially the result was proved only for planar graphs but the diameter-treewidth property result allows us to extend this algorithm to other families of graphs. The idea of the algorithm is that the graph is partitioned into pieces with small treewidth and then dynamic programming is applied to each piece.

A graph $H$ is a minor of $G$ if it can be obtained from $G$ via edge contractions and edge deletions. We say that a family is minor-closed if it is closed under the minor operations of edge deletion and edge contraction. An apex graph is a graph $G$ such that for some vertex $v$, after $v$ is deleted from $G$, the remaining graph $G - v$ is planar.

In [13] Eppstein proves the following:

**Theorem 3.1.4.** If $\mathcal{F}$ is a minor-closed family of graphs, then $\mathcal{F}$ has the diameter-treewidth property if and only if $\mathcal{F}$ does not contain all apex graphs.

Notice that the family of graphs embeddable on a surface $\Sigma$ does not contain all apex graphs since maximal apex graphs have $4n - 10$ edges and by Euler’s formula, graphs embeddable on $\Sigma$ with genus $g$ has at most $3n + O(g)$ edges.

Another useful algorithm is one of [24]. He shows that for an arbitrary fixed surface $\Sigma$, there is a linear-time algorithm that, for each graph $G$, either finds an embedding of $G$ in $\Sigma$ or finds a subgraph of $G$ that is homeomorphic to a minimal forbidden subgraph for embeddability in $\Sigma$. In the algorithm, the minimal forbidden subgraph is one of a bounded number of edges, so this gives a constructive proof of a result of Robertson and Seymour [30] that for each closed surface, there are only finitely many minimal forbidden subgraphs.

Using the algorithms of [14], [13] and [24] above we obtain the following corollary.

**Corollary 3.1.5.** There exists an explicit linear-time algorithm to decide whether an
input graph is embeddable in the Klein bottle, and if it is embeddable in the Klein bottle, whether it is 5-colorable.

Proof. First, we use Mohar’s algorithm to determine whether a graph $G$ is embeddable on the Klein bottle. We then use the algorithm of Eppstein to see if $G$ contains a subgraph isomorphic to one of the nine subgraph-minimal 6-critical graphs on the Klein bottle. If all subgraphs of $G$ are not isomorphic to any of the nine graphs, then $G$ is 5-colorable. Otherwise, it is not 5-colorable. \hfill $\square$

We will show that with the sole exception of $K_6$, none of the graphs listed in Theorem 3.1.2 can be a subgraph of an Eulerian triangulation of the Klein bottle. Thus we deduce the following theorem of Král’, Mohar, Nakamoto, Panigrác and Suzuki [21].

**Corollary 3.1.6.** An Eulerian triangulation, $G$, of the Klein bottle is 5-colorable if and only if it has no subgraph isomorphic to $K_6$.

Before proving this corollary, we require an additional lemma about defective Eulerian triangulation. A defective triangulation is a triangulation where all the vertices have even degree except for exactly two adjacent vertices.

**Lemma 3.1.7.** There are no defective Eulerian triangulations of the plane.

Proof. Suppose for purposes of contradiction there was a defective Eulerian triangulation, $G$, with adjacent vertices $u$ and $v$, each of odd degree. Delete edge $uv$ from $G$, creating a unique face bounded by a 4-walk. Every vertex of this new graph, $G'$, has even degree and hence $G'$ is bipartite. Consider the geometric dual of $G'$, call it $G''$. In a geometric dual, faces of $G'$ become vertices of $G''$ and there is an edge between two vertices of $G''$ if they share an edge in $G'$. In our case, $G'$ is 3-regular except for a single vertex of degree four. This is a contradiction because $G''$ is also bipartite and the number of edges leaving each part of the bipartition is a multiple of three. \hfill $\square$
We can now prove the corollary.

Proof. Let $G$ be an Eulerian triangulation of the Klein bottle. If $\chi(G) \geq 7$, then $G$ contains some 7-critical subgraph $H$, which must have minimum degree six. But by Euler’s formula, any graph on the Klein bottle that has minimum degree six is 6-regular. But by a result of Nakamoto and Sasanuma [27], all 6-regular graphs on the Klein bottle are 5-colorable. Thus $\chi(G) \leq 6$.

We now will show that $\chi(G) = 6$ if and only if it has a $K_6$ subgraph. If $G$ contains $K_6$, then it follows that $\chi(G) = 6$ by the first paragraph.

Now, suppose that $\chi(G) = 6$ but $G$ does not contain a subgraph of $K_6$. By Theorem 3.1.2, $G$ must contain $M$, one of the other eight 6-critical graphs on the Klein bottle. Fix an embedding of $G$. Consider the embedding of $M$ inside the embedding of $G$. Let $T$ be any 3-face of $M$ with vertices $v_1, v_2, v_3$. Let $G_T$ be the subgraph of $G$ induced by $T$ and the vertices in the interior of $T$.

We now claim that the parity of $deg_M(v_i)$ is equal to $deg_{G_T}(v_i)$ for each of $v_1, v_2, v_3$. We first show that the number of $i$ such that $deg_M(v_i) \neq deg_{G_T}(v_i)$ modulo 2 must be even. Consider $G[G_T \setminus T]$, the graph of $G$ induced by $G_T \setminus T$. All of the vertices in this graph must have even degree. But all edges with one end in $T$ and the other in $G[G_T \setminus T]$ are only counted once toward the sum of degrees in $G[G_T \setminus T]$. So if there are an odd number of edges leaving $G[G_T \setminus T]$, the sum of the degrees in $V(G[G_T \setminus T])$ would be odd, a contradiction. This proves the statement after the claim. So the parity may change only one zero or two of the vertices of $T$. However, if the parity of two of the vertices change, then $G[G_T]$ is a defective Eulerian triangulation. This contradicts the previous lemma as $T$ is embeddable in the plane. Thus, the claim follows and so we may assume the parity of $deg_M(v_i)$ is equal to $deg_{G_T}(v_i)$ for each of $v_1, v_2, v_3$.

To finish the proof, we will investigate each of the remaining eight graphs individually. First suppose that $M$ is isomorphic to $L_3$ or $L_4$. Each are triangulations, so it
follows by the claim that $G = M$. However, both $L_3$ and $L_4$ contain vertices of odd degree, a contradiction.

Now suppose that $M$ is isomorphic to $C_3 + C_5$ or $K_2 + H_7$. Each of these graphs has a unique 4-cycle, and at least five vertices of odd degree, so there at least one vertex, $v$, of odd degree not on the unique 4-cycle. This contradicts the claim as the parity of $deg_M(v)$ is different from $deg_{G_T}(v)$.

Suppose that $M$ is isomorphic to $L_1$ or $L_2$. Both $L_1$ and $L_2$ contain all 3-faces except for one 5-face. Both graphs contain seven vertices of degree five. Again we obtain a contradiction to the claim.

Finally, suppose that $M$ is isomorphic to $L_5$ or $L_6$. This time, each graph contains all 3-faces except for two 5-faces. In addition, these 5-faces share two vertices. This means there are at most a total of eight distinct vertices on the two 5-faces. However, each graph has at least nine vertices of degree five. We again obtain a contradiction to the claim. This completes the proof of the corollary.

$$\square$$

It follows by inspection that each of the graphs from Theorem 3.1.2 has a subgraph isomorphic to a subdivision of $K_6$. Thus we deduce the following corollary.

**Corollary 3.1.8.** If a graph in the Klein bottle is not 5-colorable, then it has a subgraph isomorphic to a subdivision of $K_6$.

This is related to Hajós’ conjecture, which states that for every integer $k \geq 1$, if a graph $G$ is not $k$-colorable, then it has a subgraph isomorphic to a subdivision $K_{k+1}$. Hajós’ conjecture is known to be true for $k = 1, 2, 3$ and false for all $k \geq 6$. The cases $k = 4$ and $k = 5$ remain open. In [41, Conjecture 6.3] Thomassen conjectured that Hajós’ conjecture holds for every graph in the projective plane or the torus. His results [39] imply that it suffices to prove this conjecture for $k = 4$, but that is still open. Likewise, one might be tempted to extend Thomassen’s conjecture to graphs.
in the Klein bottle; Corollary 3.1.8 then implies that it would suffice to prove this extended conjecture for \( k = 4 \) as Corollary 3.1.8 handles the \( k = 5 \) case and the results for all other values of \( k \) are described above.

Thomassen proposed yet another related conjecture [41, Conjecture 6.2] stating that every graph which triangulates some surface satisfies Hajós’ conjecture. He also pointed out that this holds for \( k \leq 4 \) for every surface by a deep theorem of Mader [23], and that it holds for the projective plane and the torus by [39]. Thus Corollary 3.1.8 implies that Thomassen’s conjecture described at the beginning of this paragraph holds for graphs in the Klein bottle. For general surfaces the conjecture was disproved by Mohar [25] with a relatively small counterexample. Qualitatively stronger counterexamples were found by Rödl and Zich [31].

Our proof of Theorem 3.1.2 follows closely the argument of [39], and therefore we assume some familiarity with that paper, but include in this thesis the results of anything taken from [39]. We proceed as follows. First we show, using the description of all 6-regular graphs in the Klein bottle, that every 6-regular graph in the Klein bottle is 5-colorable. That allows us to select a minimal counterexample \( G_0 \) and a suitable vertex \( v_0 \in V(G_0) \) of degree five. If every two neighbors of \( v_0 \) are adjacent, then \( G_0 \) has a \( K_6 \) subgraph and the result holds. We may therefore select two non-adjacent neighbors \( x \) and \( y \) of \( v_0 \). Let \( G_{xy} \) be the graph obtained from \( G_0 \) by deleting \( v_0 \), identifying \( x \) and \( y \) and deleting all resulting parallel edges. If \( G_{xy} \) is 5-colorable, then so is \( G_0 \), as is easily seen. Thus we may assume that \( G_{xy} \) has a subgraph isomorphic to one of the nine graphs on our list, and it remains to show that either \( G_0 \) can be 5-colored, or it has a subgraph isomorphic to one of the nine graphs on the list. In Chapter 3.2, we prove a series of structural lemmas that will be useful to our proof. In Chapter 3.3, we show that \( G_{xy} \) must have a subgraph to \( K_6 \). In Chapter 3.4, we handle the case when \( G_{xy} = K_6 \).
3.2 Lemmas

Our first lemma is an extension of [39, Lemma 4.1], which proves the same result for cycles of length at most six. If \( C \) is a subgraph of a graph \( G \) and \( c \) is a coloring of \( C \), then we say that a vertex \( v \in V(G) - V(C) \) sees a color \( \alpha \) on \( C \) if \( v \) has a neighbor \( u \in V(C) \) such that \( c(u) = \alpha \).

**Lemma 3.2.1.** Let \( G \) be a plane graph with an outer cycle \( C \) of length \( k \leq 7 \), and let \( c \) be a 5-coloring of \( G[V(C)] \). Then \( c \) cannot be extended to a 5-coloring of \( G \) if and only if \( k \geq 5 \) and the vertices of \( C \) can be numbered \( x_1, x_2, \ldots, x_k \) in order such that one of the following conditions hold:

(i) some vertex of \( G - V(C) \) sees five distinct colors on \( C \),

(ii) \( G - V(C) \) has two adjacent vertices that both see the same four colors on \( C \),

(iii) \( G - V(C) \) has three pairwise adjacent vertices that each see the same three colors on \( C \),

(iv) \( G \) has a subgraph isomorphic to the first graph shown in Figure 9, and the only pairs of vertices of \( C \) colored the same are either \( \{x_5, x_2\} \) or \( \{x_5, x_3\} \), and either \( \{x_4, x_6\} \) or \( \{x_4, x_7\} \),

(v) \( G \) has a subgraph isomorphic to the second graph shown in Figure 9, and the only pairs of vertices of \( C \) colored the same are exactly \( \{x_2, x_6\} \) and \( \{x_3, x_7\} \),

(vi) \( G \) has a subgraph isomorphic to the third graph shown in Figure 9, and the only pairs of vertices of \( C \) colored the same are exactly \( \{x_2, x_6\} \) and \( \{x_3, x_7\} \).

**Proof.** Clearly, if one of (i)–(vi) holds, then \( c \) cannot be extended to a 5-coloring of \( G \). To prove the converse we will show, by induction on \( |V(G)| \), that if none of (i)–(vi) holds, then \( c \) can be extended to a 5-coloring of \( G \). Since \( c \) extends if \( |V(G)| \leq 4 \), we assume that \( |V(G)| \geq 5 \), and that the lemma holds for all graphs on fewer vertices. We may also assume that \( V(G) \neq V(C) \), and that every vertex of \( G - V(C) \) has
degree at least five, for we can delete a vertex of $G - V(C)$ of degree at most four and proceed by induction. Likewise, we may assume that

(*) the graph $G$ has no cycle of length at most four whose removal disconnects $G$.

This is because if a cycle $C'$ of length at most four separates $G$, then we first delete all vertices and edges drawn in the open disk bounded by $C'$ and extend $c$ to that graph by induction. Then, by another application of the induction hypothesis we extend the resulting coloring of $C'$ to a coloring of the entire graph $G$. Thus we may assume (*).

Let $v$ be a vertex of $G - V(C)$ joined to $m$ vertices of $C$, where $m$ is as large as possible. We claim that if $m \leq 2$, then the lemma holds. To prove the claim assume that every vertex of $G - V(C)$ has at most two neighbors in $C$. We deduce that some vertex of $G - V(C)$ has degree at most five, for otherwise $G - V(C)$ has at least five vertices, contrary to [39, Lemma 3.1], because $k \leq 7$. This shows that $G - V(C)$ has a vertex $u$ of degree five. Since $G$ has no separating triangle we deduce that the vertex $u$ has two neighbors $u_1$, $u_2$, which are not adjacent and not on $C$. Let $J$ be obtained from $G$ by deleting $u$, identifying $u_1$ and $u_2$, and deleting all resulting parallel edges. Since $m \leq 2$ the graph $J$ satisfies none of (i)–(vi), and hence the coloring $c$ can be

Figure 9: Graphs that have non-extendable colorings
extended to a 5-coloring of $J$ by the induction hypothesis. It follows that the coloring $c$ can be extended to $G$, as desired. Thus we may assume that $m \geq 3$.

Since (i) does not hold, the coloring $c$ extends to a 5-coloring $c'$ of the graph $G' := G[V(C) \cup \{v\}]$. Let $D$ be a facial cycle of $G'$ other than $C$, and let $H$ be the subgraph of $G$ consisting of $D$ and all vertices and edges drawn in the disk bounded by $D$. If $c'$ extends to $H$ for every choice of $D$, then $c$ extends to $G$, and the lemma holds. We may therefore assume that $D$ was chosen so that $c'$ does not extend to $H$. By the induction hypothesis $H$ and $D$ satisfy one of (i)–(vi).

If $H$ and $D$ satisfy (i), then there is a vertex $w \in V(H) - V(D)$ that sees five distinct colors on $D$. Thus $w$ has at least four neighbors on $C$, and hence $m \geq 4$. It follows that every bounded face of the graph $G[V(C) \cup \{v, w\}]$ has size at most four, and hence $V(G) = V(C) \cup \{v, w\}$ by $(\ast)$. Since (i) and (ii) do not hold for $G$, we deduce that $c$ can be extended to a 5-coloring of $G$, as desired.

If $H$ and $D$ satisfy (ii), then there are adjacent vertices $v_1, v_2 \in V(H) - V(D)$ that see the same four colors on $D$. It follows that $m \geq 3$, and similarly as in the previous paragraph we deduce that $V(G) = V(C) \cup \{v, v_1, v_2\}$. It follows that $c$ can be extended to a 5-coloring of $G$: if both $v_1$ and $v_2$ are adjacent to $v$ we use that $G$ does not satisfy (i), (ii), or (iii); otherwise we use that $G$ does not satisfy (i), (ii), or (iv).

If $H$ and $D$ satisfy (iii), then there are three pairwise adjacent vertices of $v_1, v_2, v_3 \in V(H) - V(D)$ that see the same three colors on $D$. It follows in the same way as above that $V(G) = V(C) \cup \{v, v_1, v_2, v_3\}$. If $v$ sees at most three colors on $C$, then $c$ extends to a 5-coloring of $G$, because there are at least two choices for $c'(v)$. Thus we may assume that $v$ sees at least four colors. It follows that $m = 4$, because $k \leq 7$. Since $G$ does not satisfy (v) or (vi) we deduce that $c$ extends to a 5-coloring of $G$.

If $H$ and $D$ satisfy (iv), then there are three vertices of $V(H) - V(D)$ forming the first subgraph in Figure 9. But at least one of these vertices has four neighbors on $C$, and hence $c$ can be extended to a 5-coloring of $G$. The lemma holds.
and hence \( m \geq 4 \), contrary to \( k \leq 7 \).

Finally, if \( H \) and \( D \) satisfy (v) or (vi), then \( H \) has a subgraph isomorphic to the second or third graph depicted in Figure 9, and the restriction of \( c' \) to \( D \) is uniquely determined (up to a permutation of colors). Since \( D \) has length seven, it follows that \( m \leq 3 \), and hence \( c'(v) \) can be changed to a different value, contrary to the fact that the restriction of \( c' \) to \( D \) is uniquely determined.

\[ \square \]

**Corollary 3.2.2.** Let \( G \) be a 6-vertex-critical graph on some surface. Let \( H \) be a connected induced subgraph of \( G \) (i.e. \( H = G(H) \)). If each facial walk of \( H \) has length 3 or 4, then \( H = G \).

**Proof.** If \( H \neq G \), then \( H \) is 5-colorable. Any 5-coloring of \( H \) can be extended to a 5-coloring of \( G \) by Lemma 3.2.1. but \( G \) is not 5-colorable, so \( H = G \).

\[ \square \]

The following lemma is shown in [27].

**Lemma 3.2.3.** All 6-regular graphs embeddable on the Klein bottle are 5-colorable.

In order to describe the 6-regular graphs on the Klein bottle, we will need to describe how to draw triangulations on a grid. Notice by Euler’s formula that any 6-regular graph on the torus or Klein bottle must be a triangulation. Following the description of [27], suppose that \( p \) and \( k \) are natural numbers. Let \( R_{p,k} \) be a \( p \) by \( k \) grid, and add additional diagonal edges from bottom left to upper right of each square of the grid. Let \( H_{p,k} \) to be the embedding on the annulus that identifies the top and bottom vertices of the grid as shown in Figure 9. For \( u_{i,j} \in V(H_{p,k}) \) let \( i \) be taken modulo \( p \). Define \( C_j \) to be the cycle of \( H_{p,k} \) that passes in order through the vertices \( u_{j,0}, u_{j,1}, \ldots, u_{j,p} \), for \( j = 0, 1, \ldots, k \). These cycles are called *geodesic cycles* in \( H_{p,k} \).

We can now describe the 6-regular graphs on the Klein bottle via [28]. To produce a 6-regular graph on the Klein bottle, identify each \( u_{0,j} \) with \( u_{k,-j} \) for each \( h \). This produces a 6-regular graph on the Klein bottle of *handle type* and is described by
Figure 10: The graph $H_{p,k}$.

$Kh(p,k)$. We can also obtain a 6-regular graph on the Klein bottle of crosscap type as follows. Let $p$ be even. Identify $u_{0,j}$ with $u_{0,j+p/2}$ and $u_{k,j}$ with $u_{k,j+p/2}$ in $H_{p,k}$ for each $j$, respectively. This produces two cycles of length $p/2$ from $C_0$ and $C_k$. If instead $p$ is odd, then we add a crosscap to each boundary component of $H_{p,k-1}$. For each $j$, draw an edge between $u_{0,j}$ to $u_{0,j+m}$ and $u_{0,j+m+1}$, and $u_{k-1,j}$ to $u_{k-1,j+m}$ and $u_{k-1,j+m+1}$ on the crosscaps added. This graph is denoted $Kc(p,k)$. We can now state Negami’s [28] result on the structure of 6-regular graphs on the Klein bottle.

**Theorem 3.2.4.** A loopless 6-regular Klein bottle graph is equivalent to precisely one of

$$Kh(p,k)(p \geq 2, k \geq 2) \text{ and } Kc(p,k)(p \geq 3, k \geq 2).$$

In particular, $Kh(p,k)$ is simple if and only if $p \geq 3$ and $k \geq 3$ and $Kc(p,k)$ is simple if and only if $p \geq 5$ and $k \geq 2$.

Nakamoto and Sasanuma determine the chromatic numbers of all 6-regular graphs on the Klein bottle.

**Theorem 3.2.5.** Let $G$ be a loopless 6-regular graph on the Klein bottle. Then $\chi(G) =$
4, with the following exceptions:

- \( G = K_c(3l, k) \) for \( l \geq 1, k \geq 2 \) if and only if \( \chi(G) = 3 \).
- \( G = K_c(5, m+1), Kh(3, 2), Kh(4, 2m+1), Kh(2, 2m+3) \) for \( m \geq 1 \) if and only if \( \chi(G) = 5 \).
- \( G = Kh(2, 3) \) if and only if \( \chi(G) = 6 \).

Notice that \( Kh(2, 3) \) is not simple and contains \( K_6 \) as a subgraph. A corollary of this theorem proves Lemma 3.2.3.

The next lemma is an adaptation of [39, Lemma 5.2] for the Klein bottle.

**Lemma 3.2.6.** Let \( G \) be isomorphic to \( C_3 + C_5 \), let \( S \) be a cycle in \( G \) of length three with vertex-set \( \{z_0, z_1, z_2\} \), and let \( u_1 \) be a vertex in \( G \backslash V(S) \) joined to \( z_0 \). Let \( G' \) be obtained from \( G \) by splitting \( z_0 \) into two nonadjacent vertices \( x \) and \( y \) such that \( u_1 \) and at most one more vertex \( u_0 \) in \( G' \) is joined to both \( x \) and \( y \) and such that \( yz_1z_2x \) is a path in \( G' \). Let \( G'' \) be obtained from \( G' \) by adding a vertex \( v_0 \) and joining \( v_0 \) to \( x, y, u_1, z_1, z_2 \). If \( G'' \) is not 5-colorable and can be drawn in the Klein bottle, then it has a subgraph isomorphic to either \( C_3 + C_5 \) or \( L_4 \).

**Proof.** This proof is based on the argument of [39, Lemma 5.2], except that instead of invoking [39, Proposition 2.3] on two occasions we use the fact that in those cases the graph \( G'' \) is isomorphic to \( L_4 \).

Suppose that \( G'' \not\supseteq \{C_3+C_5, L_4\} \). Suppose that one of \( x, y \) has the same neighbors in \( G' \) as \( z_0 \) does in \( G \). Then \( G' \not\supseteq L_4 \) as \( G' \) contains only nine vertices, and \( G' \not\supseteq C_3+C_5 \) by inspection. Thus we can assume that \( z_0 \) contains at least two neighbors in \( G \) such that one is a neighbor of \( x \) but not \( y \) and the other is a neighbor of \( y \) but not \( x \).

We can also assume that each of \( x, y \) has degree at least five in \( G'' \) and hence \( z_0 \) has degree at least six in \( G \). Suppose that \( x \) had degree at most four in \( G'' \). Then \( G'' - \{x, v_0\} \) is a proper subgraph of \( C_3 + C_5 \) as \( y \) is not a neighbor of one of \( x \)'s
neighbors. Since $G'' - \{x, v_0\}$ is a proper subgraph, we can properly 5-color it, and extend this to a 5-coloring of $G''$ by coloring $v_0$, then $x$.

Let $G$ be defined by a 5-cycle $p_1p_2p_3p_4p_5$ and a 3-cycle $q_1q_2q_3q_1$ and the 15 edges $p_iq_j$ where $1 \leq i \leq 3, 1 \leq j \leq 5$. Since the degree of $z_0$ in $G$ is at least 6, then $z_0 \in \{q_1, q_2, q_3\}$.

The remainder of the proof is an analysis based on which vertices are $z_0, z_1, z_2$. First suppose that $z_0, z_1, z_2$ are $q_3, q_1, q_2$, respectively. If both $p_0$ and $p_1$ are in $\{p_1, p_2, p_3, p_4, p_5\}$, then we can color $y, z_1, z_2, x$ with 2, 1, 2, 1, respectively. We can color the remaining vertices with colors 3, 4, 5 as the remaining vertices are $v_0$ and a 5-cycle, and in this case $v_0$ is only adjacent to one of the vertices of the 5-cycle. If $u_1 = p_1$ and $u_0 = z_1$, then we color $y, z_1, z_2, x, u_1$ by 2, 1, 2, 3, 4, respectively. Since $y$ has degree at least five in $G''$, some vertex in $\{p_2, p_3, p_4, p_5\}$ can obtain color 3 and the remaining vertices may be colored with colors 4 and 5.

Now consider the case where $z_0, z_1, z_2$ are $q_1, p_1, p_2$, respectively and $u_0$ is not in $\{z_1, z_2\}$. Color $y, z_1, z_2, x, u_0, u_1$ by 2, 1, 2, 1, 3, 4, respectively. We can extend this to a 5-coloring of $G''$, coloring $v_0$ last except in the following three situations (or equivalent situations). If $u_0 = q_2$ and $u_1 = p_4$, color $q_3$ by the same color as $x$ or $y$ and recolor either $z_1$ or $z_2$ by 4 and color the remaining vertices color 5. If $u_0 = p_3$ and $u_1 = p_4$, then color $q_3$ by color 1 or 2 and recolor $z_1$ or $z_2$ color 4. Then we can color $p_5, q_2$ with colors 3 and 5 respectively. If $u_1 = p_3$ and $u_0 = p_5$, color $q_3$ by 1 or 2 and recolor one of $z_1, z_2$ by 3 and recolor $p_3, p_4, p_5, q_2$ by 4, 3, 4, 5, respectively.

Now suppose that $z_0, z_1, z_2$ are $q_1, p_1, p_2$, respectively and $u_0$ is in $\{p_1, p_2\}$. Without loss of generality let $u_0 = p_2$. Suppose that $u_1 \in \{p_3, p_4, p_5\}$. The we can color $y, z_1, z_2, x$ by 2, 4, 3, 1 and can color $u_1$ by 3 except for the case when $u_1 = p_3$. Then we color $u_1$ by 4. Next, color one of $q_2, q_3$ color 1 or 2. If both $q_2, q_3$ can be colored 1, 2 then the rest of the coloring follows. So we can assume that $q_2, q_3$ are colored by 2, 5, respectively and both $q_2, q_3$ are adjacent to $x$. (The argument is analogous if
\(q_2, q_3\) are both adjacent to \(y\).) Since \(y\) has degree at least four in \(G'\), then at least one vertex in \(\{p_3, p_4, p_5\} \setminus \{u_1\}\) is joined to \(y\) and is colored 1. With possibly a swapping of the colors of \(z_1\) and \(z_2\), we can now complete the 5-coloring.

Suppose that \(z_0, z_1, z_2\) are \(q_1, p_1, p_2\), respectively and \(u_0 = p_2\) and \(u_1 = q_2\). Color \(y, z_1, z_2, x, q_2\) colors 2, 1, 3, 1, 4, respectively. If \(q_3\) can be colored 2, then color \(p_3, p_4, p_5, v_0\) colors 5, 3, 3, 5, respectively. Assume that \(q_3\) is adjacent to \(y\). Then color \(q_3\) by 5. If we can color \(\{p_3, p_4, p_5\}\) by colors \(\{1, 2, 3\}\), then color \(v_0\) with 5. If not, then \(p_3, p_4\) are adjacent to the same vertex in \(\{x, y\}\). Since \(x\) has degree at least four in \(G'\), the vertex must be \(x\). So then \(p_5\) is adjacent to \(y\) since otherwise we color \(p_3, p_4, p_5\) by 2, 3, 2, respectively. We now claim the remaining graph is isomorphic to \(L_4\). To see the isomorphism, we first will label the vertices of the figure of \(L_4\). We will label them from top to bottom, breaking ties from left to right. So the vertex on the top left is labeled \(a\), the vertex on the top right is labeled \(b\). The next row of vertices is labeled \(c\) and \(d\), respectively. The vertices in the third row from the top is labeled \(e\), and the vertices on the bottom row are labeled \(f, g, h, i, j\) from left to right. With this notation, we have the following correspondence: \(a \equiv z_1, b \equiv y, c \equiv q_3, d \equiv z_2, e \equiv q_2, f \equiv p_5, g \equiv p_4, h \equiv p_3, i \equiv x, j \equiv v_0\). Thus \(G''\) is isomorphic to \(L_4\).

Now, consider the case when \(z_0, z_1, z_2\) are \(q_1, q_2, p_1\), respectively. If \(u_0 \notin \{z_1, z_2\}\), then color \(y, z_1, z_2, x, p_2, p_3, p_4, p_5, q_3\) by 2, 1, 2, 1, 3, 4, 3, 4, 5, respectively. If \(u_0 = p_1\), color \(y, z_1, z_2, x\) by 2, 1, 3, 1, respectively. If \(q_3\) is not adjacent to \(y\), then color \(q_3\) by 2 and vertices \(p_2, p_3, p_4, p_5\) colors 4 and 5. If \(q_3\) is adjacent to \(y\), color \(q_3\) by 5. Since \(x\) has degree at least four in \(G'\), some vertex in \(\{p_2, \ldots, p_5\}\) can be colored 2. The other vertices in this set could then be colored with colors 3 and 4. Thus assume that \(u_0 = q_2 = z_1\). Color \(y, z_1, z_2, x, u_1\) by 2, 3, 2, 1, 4 and we now will try and extend this coloring. If \(q_3\) can be colored 1, then color \(p_2, p_3, p_4, p_5\) by colors 4 and 5. So we assume that \(q_3\) is adjacent to \(x\). If \(u_1 = p_3\), then recolor \(z_2\) by color 4 and color \(q_3\)
by 2. Since $y$ also has degree at least four in $G'$, it must be adjacent to at least one of $p_4, p_5$, which we color 1. The remaining vertices of $\{p_1, \ldots, p_5\}$ are colored 5. If $u_1 = q_3$, then we color one of $p_2$ or $p_5$ color 1 if possible and complete the coloring by using 5 for two vertices in $\{p_2, p_3, p_4, p_5\}$. Now assume that both $p_2$ and $p_5$ are joined to $x$. Since $y$ has degree at least four in $G'$ it follows that $y$ is adjacent to $p_3$ and $p_4$. We now claim that $G''$ is not embeddable on the Klein bottle. Notice that if an embedding of this graph exists, it must be that it is a triangulation as it has 10 vertices and 30 edges. Consider the induced embeddings of $G'' - p_2, G'' - p_5$ and $G'' - v_0$, respectively. The face of $G'' - p_2$ containing $p_2$ is bounded by a Hamiltonian cycle $R_1$ of the neighborhood of $p_2$. There exist similarly constructed Hamiltonian cycles around $p_5$ and $v_0$. However, each of these cycles contains the edge $xp_1$. This would mean that $xp_1$ is part of three facial triangles, a contradiction.

Finally, consider the subcase where $z_0, z_1, z_2, u_0, u_1$ are $q_1, q_2, p_1, q_2, p_2$, respectively. Color $y, z_1, z_2, x, u_1, q_3$ by 2, 3, 2, 1, 4, 5, respectively. We may assume that $q_3$ is adjacent to $x$ else we can recolor $q_3$ by 1 and complete the coloring. Also, we can assume that $p_5$ is adjacent to $x$ else we color $p_5, p_4$, by 1, 4 and complete the coloring. Color $p_5$ by 4. The coloring can be completed unless $p_3$ and $p_4$ are both adjacent to the same vertex in $\{x, y\}$. Since $y$ must have degree at least four in $G'$, it follows that $p_3$ and $p_4$ are adjacent to $y$. This graph is isomorphic to $L_4$. Using the notation from above we have the following correspondence: $a \equiv x, b \equiv z_2 = p_1, c \equiv q_3, d \equiv p_2 = u_1, e \equiv q_2 = z_1 = u_0, f \equiv p_5, g \equiv p_4, h \equiv p_3, i \equiv y, j \equiv v_0$. Thus $G''$ is isomorphic to $L_4$.

\[ \square \]

**Lemma 3.2.7.** Let $G$ be a graph drawn in the Klein bottle, and let $c, d \in V(G)$ be such that $G \setminus c$ does not embed in the projective plane, and $G$ does not embed in the torus. Then every closed curve in the Klein bottle intersecting $G$ in a subset of $\{c, d\}$ separates the Klein bottle.
Proof. Let \( \phi \) be a closed curve in the Klein bottle intersecting \( G \) in a subset of \( \{c, d\} \), and suppose for a contradiction that it does not separate the Klein bottle. Then \( \phi \) is either one-sided or two-sided. If \( \phi \) is one-sided, then it intersects \( G \setminus c \) in at most one vertex, and hence the Klein bottle drawing of \( G \setminus c \) can be converted into a drawing of \( G \setminus c \) in the projective plane, a contradiction. Thus \( \phi \) is two-sided, but then the drawing of \( G \) can be converted into a drawing of \( G \) in the torus, again a contradiction. \( \square \)

Lemma 3.2.8. Let \( G \) be \( L_1 \) or \( L_2 \) with its vertices numbered as in Figure 11, and let it be drawn in the Klein bottle. Then

(i) every face is bounded by a triangle, except for exactly one, which is bounded by a cycle of length five with vertices \( c_1, a_i, c_2, b_j, b_k \) in order for some indices \( i, j, k \), and

(ii) for \( i = 0, 1, 2 \) the vertices \( a_1, a_2, a_3 \) appear consecutively in the cyclic order around \( c_i \) (but necessarily in the order listed), and so do the neighbors of \( c_i \) that belong to \( \{b_1, b_2, b_3, b_4\} \).

Proof. Let \( i \in \{1, 2\} \). There are indices \( j, k \) such that \( a_j \) and \( b_k \) are both adjacent to \( c_i \) and are next to each other in the cyclic order around \( c_i \). Let \( f_i \) be the face incident with both the edges \( c_i a_j \) and \( c_i b_k \). We claim that the walk bounding \( f_i \)
includes at most one occurrence of $c_i$ and no occurrence of $c_0$. Indeed, otherwise we can construct a simple closed curve either passing through $f_i$ and intersecting $G$ in $c_i$ only (if $c_i$ occurs at least twice in the boundary walk of $f_i$), or passing through $f_i$ and a neighborhood of the edge $c_i w$ and intersecting $G$ in $c_i$ and $c_0$ (if $c_0$ occurs in the boundary walk of $f_i$). By Lemma 3.2.7 this simple closed curve separates the Klein bottle. It follows from the construction that it also separates $G$, contrary to the fact that $G\backslash\{c_i, c_0\}$ is connected. This proves our claim that the walk bounding $f_i$ includes at most one occurrence of $c_i$ and no occurrence of $c_0$.

Since $f_i$ includes a subwalk from $a_j$ to $b_k$ that does not use $c_i$, we deduce that $c_{3-i}$ belongs to the facial walk bounding $f_i$. But the neighbors of $c_1$ and $c_2$ in $\{b_1, b_2, b_3, b_4\}$ are disjoint, and hence $f_i$ has length at least five. By Euler’s formula $f_1 = f_2$, this face has length exactly five, and every other face is bounded by a triangle. This proves (i). Statement (ii) also follows, for otherwise there would be another face with the same properties as $f_1 = f_2$, and yet we have already shown that this face is unique.

**Lemma 3.2.9.** Let $G$ be $L_5$ or $L_6$ with its vertices numbered as in Figure 12, and let it be drawn in the Klein bottle. Then

(i) every face is bounded by a triangle, except for exactly two, which are bounded by cycles $C_1, C_2$ of length five, each with vertices $c_1, a_i, c_2, b_j, b_k$ in order for some indices $i, j, k$,

(ii) if $G = L_5$, then $C_1 \cap C_2$ consists of the vertices $c_1, c_2$, and if $G = L_6$, then $C_1 \cap C_2$ consists of the vertices $c_1, c_2, b_5$ and the edge $c_2 b_5$, and

(iii) for $i = 1, 2$ the vertices $a_1, a_2, a_3, a_4$ appear consecutively in the cyclic order around $c_i$ (but necessarily in the order listed), and so do the neighbors of $c_i$ that belong to $\{b_1, b_2, b_3, b_4, b_5\}$.

**Proof.** The proof is similar to Lemma 3.2.8. Again, notice for $i = 1, 2$, there are indices $j, k$ such that $a_j$ and $b_k$ are both adjacent to $c_i$ and are next to each other.
in the cyclic order around \( c_i \). Again we make the claim that the walk bounding \( f_i \) includes at most one occurrence of \( c_i \). We will, as in Lemma 3.2.8 construct a simple closed curve that passes through \( f_i \) and intersects \( G \) in \( c_i \) at least twice in the boundary walk of \( f_i \). But since \( L_5 \) and \( L_6 \) are not embeddable in the torus and \( L_5 \setminus \{c_i\} \) and \( L_6 \setminus \{c_i\} \) are not embeddable in the projective plane, we may apply Lemma 3.2.7 and obtain a simple closed curve that separates the Klein bottle. By construction, this curve also separates \( G \), a contradiction, as \( G \setminus \{c_i\} \) is connected. So the walk bounding \( f_i \) contains at most one occurrence of each \( c_i \).

For each of \( L_5 \) and \( L_6 \) there are four pairs of edges that must be parts of faces, call them \( f_1, f_2 \). These edges are, for \( L_5 \), \( b_1 c_1, c_1 a_1; b_3 c_1, c_1 a_4; b_4 c_2, c_2 a_1; b_5 c_2, c_2 a_4 \), and for \( L_6 \), \( b_1 c_1, c_1 a_1; b_4 c_1, c_1 a_4; b_5 c_2, c_2 a_1; b_5 c_2, c_2 a_4 \). Since \( c_1 \) and \( c_2 \) are each used at most once in each face, there are at least two faces. In addition, the neighbors of \( c_1, c_2 \) in \( \{b_1, b_2, b_3, b_4, b_5\} \) are disjoint, so each of \( f_1 \) and \( f_2 \) have length at least five. However, by Euler’s formula there are at most two faces of length exactly five, and every other face is bounded by a triangle. This proves (i).

Statement (ii) follows for \( L_5 \) since edges \( b_4 c_2 \) and \( b_5 c_2 \) are in different faces and there are no multiple edges in \( L_5 \). Similarly (ii) follows for \( L_6 \) because edge \( b_5 c_2 \) and vertex \( c_1 \) must be in each of the two 5-faces of \( L_6 \). Statement (iii) also follows, for

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure12.png}
\caption{The graphs \( L_5 \) and \( L_6 \) with their vertices labeled}
\end{figure}
otherwise there would be another non-triangular face other than $f_1$ and $f_2$, and yet we have already shown that there are at most two of these 5-faces.

\[ \square \]

### 3.3 Reducing to $K_6$

If $v$ is a vertex of a graph $G$, then we denote by $N_G(v)$, or simply $N(v)$ if the graph can be understood from the context, the open neighborhood of the vertex $v$; that is, the subgraph of $G$ induced by the neighbors of $v$. Sometimes we will use $N(v)$ to mean the vertex-set of this subgraph. We say that a vertex $v$ in a graph $G$ embedded in a surface has a *wheel neighborhood* if the neighbors of $v$ form a cycle $C$ in the order determined by the embedding, and the cycle $C$ is null-homotopic.

Let $G_0$ be a graph drawn in the Klein bottle such that $G_0$ is not 5-colorable and has no subgraph isomorphic to any of the graphs listed in Theorem 3.1.2. Let a vertex $v_0 \in V(G_0)$ of degree exactly five be chosen so that each of the following conditions hold subject to all previous conditions:

(i) $|V(G_0)|$ is minimum,

(ii) the clique number of $N(v_0)$, the neighborhood of $v_0$, is maximum,

(iii) the number of largest complete subgraphs in $N(v_0)$ is maximum,

(iv) the number of edges in $N(v_0)$ is maximum,

(v) $|E(G_0)|$ is minimum,

(vi) if possible, $v_0$ has a wheel neighborhood.

In those circumstances we say that the pair $(G_0, v_0)$ is an *optimal pair*. Given an optimal pair $(G_0, v_0)$ we say that a pair of vertices $v_1, v_2$ is an *identifiable pair* if $v_1$ and $v_2$ are non-adjacent neighbors of $v_0$. If $v_1, v_2$ is an identifiable pair, then we define $G_{v_1v_2}$ to be the graph obtained from $G_0$ by deleting all edges incident with $v_0$ except $v_0v_1$ and $v_0v_2$, contracting the edges $v_0v_1$ and $v_0v_2$ into a new vertex $z_0$, and deleting all resulting parallel edges. This also defines a drawing of $G_{v_1v_2}$ in the Klein bottle.
We now introduce notation that will be used throughout the rest of the paper. Let \( G' \) be obtained from \( G \) by deleting all those edges that got deleted during the construction of \( G_{v_1v_2} \). That means all edges incident with \( v_0 \) except \( v_0v_1 \) and \( v_0v_2 \) and all those edges of \( G \) that got deleted because they became parallel to another edge. Thus if a vertex \( v \) of \( G \) is adjacent to both \( v_1 \) and \( v_2 \), then \( G' \) will include exactly one of the edges \( vv_1 \), \( vv_2 \). Thus the edges of \( G' \) may be identified with the edges of \( G_{v_1v_2} \), and in what follows we will make use of this identification. Now if \( J \) is a subgraph of \( G_{v_1v_2} \) with \( z_0 \in V(J) \), then let \( J' \) be the corresponding subgraph of \( G' \); that is, \( J' \) has vertex-set \( \{v_0, v_1, v_2\} \cup V(J) - \{z_0\} \) and edge-set \( \{v_0v_1, v_0v_2\} \cup E(J) \). Let \( \hat{R}_1 \) and \( \hat{R}_2 \) be the two faces of \( J' \) incident with \( v_0 \), and let \( R_1, R_2 \) be the corresponding two faces of \( J \). We call \( R_1, R_2 \) the hinges of \( J \). Finally, let \( \hat{R} \) be the face of \( J' \) containing \( v_0 \).

We now will state and prove a lemma of Thomassen [39, Lemma 5.1] describing some of the chromatic properties of \( C_3 + C_5 \) and \( K_2 + H_7 \).

**Lemma 3.3.1.** Let \( G \) be a copy of \( C_3 + C_5 \) or \( K_2 + H_7 \). Let \( z_0z_1z_2z_3z_0 \) be a 4-cycle in \( G \) such that \( z_0 \) and \( z_2 \) are neighbors. Let \( G_1 \) be obtained from \( G \) by splitting \( z_0 \) into two nonadjacent vertices \( x, y \) such that \( G_1 \) contains the edges \( z_1y, z_3x, z_2y, z_2x \) and such that \( z_2 \) is the only vertex joined to both \( x \) and \( y \). Then \( G_1 \) has 5-colorings \( c, c', c'' \) such that:

(i) \( \{c(x), c(z_3)\} \neq \{c(y), c(z_1)\} \),

(ii) \( c'(y) = c'(z_3) \) or \( c'(x) = c'(z_1) \),

(iii) Either \( c''(y), c''(z_1), c''(z_2), c''(z_3) \) are all distinct or \( c''(y), c''(z_1), c''(z_2), c''(z_3) \) are not distinct.

**Proof.** We first prove case (i). \( G_1 - x \) is a property subgraph of \( G \) and is therefore 5-colorable. If \( x \) is adjacent to only three colors, then there are at least two choices for \( c(x) \) and we can obtain the desired coloring. So suppose that \( x \) is adjacent to vertices...
consisting of at least four different colors. In particular, \(x\) and similarly \(y\) have degree at least four in \(G_1\). Then \(z_0\) has degree at least seven in \(G\), so \(z_0\) is jointed to all other vertices of \(G\). If \(G = C_3 + C_5\), then we can assume that some \(z_i, (i = 1, 2, 3)\) is in the \(C_5\). We then 5-color \(G - z_0\) such that at least one vertex distinct from \(z_i\) has the same color as \(z_i\). We can then extend this 5-coloring to a 5-coloring of \(G\). Note that if \(c(z_1) = c(z_3)\) then \(x\) and \(y\) cannot be adjacent to the same four colors as this would result in a 5-coloring of \(G\).

Now suppose that \(G = K_2 + H_7\). Again, \(x\) and \(y\) each have degree at least four in \(G_1\) and \(z_0\) has degree eight in \(G\). Therefore, we may assume that \(N(y)\) (the neighborhood of \(y\)) and \(N(x)\) have vertex sets \(\{z_1, z_2, w_1, w_2, w_3\}\) and \(\{z_2, z_3, u_1, u_2\}\), respectively. Since \(G - \{z_0, z_3\}\) is 4-colorable, there exists a 5-coloring \(c\) of \(G_1 - x\) such that \(y\) and \(z_3\) are the only vertices of color 1. We may also assume that \(x\) is adjacent to four colors. For instance, let \(u_1, u_2, z_2\) have colors 2, 3, 4, respectively. If \(c(z_1) \neq 5\), we can color \(x\) by color 5 and complete the proof. Thus assume that \(c(z_1) = 5\). Then if \(y\) is not adjacent to colors 2, 3, change \(c(y)\) to 2 or 3 and complete the proof. Thus assume that \(c(w_1) = 2\) and \(c(w_2) = 3\). Now, interchange the colors of \(w_1\) and \(y\) (and change \(c(w_3)\) to 1 if \(c(w_3) = 2\)). This new coloring extends to the desired 5-coloring unless \(z_3\) is adjacent to \(w_1\) (or to \(w_3\) if \(c(w_3) = 2\)). Similarly, \(z_3\) is adjacent to \(u_2\). Thus the degree of \(z_3\) is at least 6. We have shown above that in any 4-coloring of \(G - \{z_0, z_3\}\), \(u_1, u_2, z_1, z_2\) are forced to have distinct colors. So one of these vertices has degree eight (and it must be one of \(z_1, z_2\). So \(z_3\) is the unique vertex in \(G\) of degree six. If we 4-color \(G - \{z_0, z_3\}\) and then delete the vertex in \(G - \{z_0, z_3\}\) which is jointed to all other vertices, then we obtain a 3-coloring of \(H_7\) minus the vertex \(z - 3\) of degree 4 in \(H_7\). In this 3-coloring, the four neighbors of \(z_3\) (namely \(u_1, u_2, w_1, w_2\)) must have three distinct colors. This contradiction proves part (i).

For part (ii), notice that at least one of \(x, y\), (wlog, say \(x\)) has degree at most four.
in $G_1$. As in (i) we can 5-color $G_1 - x$ such that $y$ and $z_3$ are vertices of color 1. This proves (ii).

For part (iii), we may assume that $x$ has degree at most five in $G_1$, else we use the proof of part (ii). If $y$ has degree 2 in $G_1$, then we first 5-color $G_1$ and then modify the color of $y$, if necessary. We can then assume that $y$ has degree at most 3 in $G_1$. So $z_0$ is joined to all the other vertices of $G$. Suppose that $w$ is a vertex in $N(y) - \{z_1, z_2\}$. Since $G - \{w, z_0\}$ is 4-colorable, $G_1 - y$ has a 5-coloring $c''$ such that $x$ and $w$ are the only vertices colored 5. We can extend $c''$ to a 5-coloring of $G_1$. This coloring $c''$ has the desired property.

\[\square\]

**Lemma 3.3.2.** Let $(G_0, v_0)$ be an optimal pair, and let $v_1, v_2$ be an identifiable pair. Then $G_{v_1v_2}$ has no subgraph isomorphic to $C_3 + C_5$ or $K_2 + H_7$.

**Proof.** This follows by using the argument of [39, Theorem 6.1, Claim (9)], using Lemma 3.2.6 instead of [39, Lemma 5.2]. We will follow Thomassen’s exposition throughout this proof.

Suppose for purposes of contradiction that there exists a subgraph $H_{v_1v_2}$, a subgraph of $G_{v_1v_2}$ such that $H_{v_1v_2} = C_3 + C_5$ or $H_{v_1v_2} = K_2 + H_7$. Let $z_0$ be the vertex in $H_{v_1v_2}$ that corresponds to $\{v_1, v_2\}$ in $G_0$. Define $v_1u_1u_2\cdots u_kv_2z_1z_2\cdots z_mv_1$ be the facial walk in the subgraph of $G_0 - v_0$ induced by $(V(H_{v_1v_2})\setminus \{z_0\}) \cup \{v_1, v_2\}$ that bounds the face containing $v_0$. We may assume that $1 \leq k \leq m$, $\{v_1, v_2\} \cap \{u_1, \ldots, u_k\} = \emptyset$ and that $G_0$ is drawn on the Klein bottle such that $k + m$ is minimized. We obtain $G_{v_1v_2}$ by deleting one edge in each double edge of $H'_{v_1v_2}$, where $H'_{v_1v_2}$ is the multigraph induced by $H_{v_1v_2}$.

By our assumption either $H_{v_1v_2} = C_3 + C_5$ or $H_{v_1v_2} = K_2 + H_7$, so $G_{v_1v_2}$ has exactly one face bounded by a 4-cycle. All other faces are bounded by a 3-cycle. This also holds for $H'_{v_1v_2}$, as all other faces here are bounded by either 3-cycles or 2-cycles. So $k \leq 2$ and $m \leq 3$. The vertices $v_2, z_1, \ldots, z_m, v_1$ are distinct. Note that $z_1 \neq v_2$ as
Suppose that $v_2 z_1$ is an edge in $G_0$ and $z_1 \neq v_1$ as $v_1 v_2$ is not an edge in $G_0$. Also $z_2 \neq v_1, v_2$ as $G_0$ does not have multiple edges. Finally, all vertices of $G_{v_1 v_2}$ are either in $H_{v_1 v_2}$ or inside one of the cycles $R_1 : v_0 v_1 u_1 \cdots u_k v_2 v_0$ or $R_2 : v_0 v_2 z_1 \cdots z_m v_1 v_0$ by Corollary 3.2.2. Suppose that $q_i$ is the number of vertices inside $R_i$ for $i = 1, 2$.

First suppose that $m = 3$. Then $H'_{v_1 v_2}$ has no 2-cycle except possibly $v_1 u_1 v_2$ if $k = 1$. Any other 2-cycle would be of the form $v_1 w v_2$, and it would be nonfacial, which contradicts Euler’s formula. It follows that all vertices $v_1, u_1, \ldots, u_k, v_2, z_1, \ldots, z_k$ are distinct except that $z_2$ could be equal to one of $u_1, u_2$.

By Lemma 3.2.1 and Corollary 3.2.2, $q_1 \leq 1$ and $q_2 \leq 3$. If $q_2 = 3$, then by Euler’s formula, $N(v_0)$ contains no $C_3$. Since $H_{v_1 v_2}$ is a $C_3 + C_5$ or $K_2 + H_7$, we can find a vertex $v_1$ of degree five (in both $H_{v_1 v_2}$ and $G_0$) such that $N(v_1)$ has a $C_3$, unless $H_{v_1 v_2} = C_3 + C_5$, $k = 2$ and $z_1, z_2, z_3, u_1, u_2$ are the vertices of the $C_5$. In this case we can 5-color $G_0$ as follows: Color $v_1, v_2$ by 1, 2, respectively. Since $H'_{v_1 v_2}$ has no double edges when $k = 2$, we color the other two vertices of the $C_3$ in $C_3 + C_5$ colors 1 and 3. Color $z_3$ by 2 and color the remaining vertices in the $C_5$ of $C_3 + C_5$ colors 4 or 5. This coloring extends to a coloring of $G_0$ and therefore gives a contradiction when $q_2 = 3$.

Suppose that $m = 3$ and $q_2 = 2$. We also obtain a contradiction as above, unless the interior of $R_2$ contains vertices $w_1, w_2$ and the edges $w_1 w_2, w_1 v_0, w_1 v_2, w_1 z_1, w_1 z_2, w_2 z_2, w_2 z_3, w_2 v_1, w_2 v_0$, and $v_0$ is also adjacent to $z_2 \in \{u_1, u_2\}$. By Corollary 3.2.2, $q_1 = 0$ and $G_0$ is 5-colorable by Lemma 3.3.1, part (i). This shows that $q_2 < 2$ when $m = 3$.

If $m = 3$ and $q_2 = 1$ and the vertex inside $R_2$ is $w_1$, then $w_1$ is joined to at least five vertices of $R_2$. If $w_1$ is joined to both $v_1$ and $v_2$, then $k = 2$ and $v_0$ is adjacent to $x, w_1, y, u_2, u_1$. Then $N(v_0)$ contains at most one $C_3$ (if $z_2 \in \{u_1, u_2\}$). We can then find a vertex $v$ of degree 5 in $H_{v_1 v_2}$ and in $G_0$ such that $N(v)$ has at least two $C_3 - s$ (where $s$ is a vertex in each $C_3$) because $\{z_1, z_2, z_3, u_1, u_2\}$ has less than five vertices.
This contradiction shows that \( w_1 \) is not adjacent to both \( v_1 \) and \( v_2 \). We can define the vertices such that \( w_1 \) is adjacent to \( v_0, v_2, z_1, z_2, z_3 \) and \( v_0 \) is adjacent to \( z_3 \) and \( z_2 \in \{ u_1, u_2 \} \). In particular, \( q_1 = 0 \). Any 5-coloring of \( G_0 - \{ v_0, w_1 \} \) that satisfies the conclusion of Lemma 3.3.1, part (iii) can be extended to a 5-coloring of \( G \). This proves that \( q_2 = 0 \) if \( m = 3 \).

Suppose now that \( m = 3 \) and \( q_2 = 0 \). Then \( v_0 \) has at least one neighbor in \( \{ z_1, z_2, z_3 \} \). Then \( v_0 \) is adjacent to both \( z_1 \) and \( z_3 \) for if \( v_0 \) is adjacent only to \( z_i \), then we can add the edge \( z_iv_1 \) (where \( i = 1 \) or \( 2 \)). This contradicts condition (iv) of the definition of optimal pair for \( v_0 \). (Note that if \( v_0 \) is adjacent to \( z_2 \) and \( z_2 = u_1 \), then \( q_1 = 0 \), since otherwise we add \( z_2v_1 \) and delete \( v_1u_1 \) and this brings us to the case when \( q_2 = 1 \). Then \( v_0 \) is adjacent to \( z_1 \) and edge \( z_1v_1 \) can be3 added.) So the 5-coloring of \( G_0 - v_0 \) in part (ii) of Lemma 3.3.1 can be extended to a 5-coloring of \( G_0 \). This shows that \( 1 \leq k \leq m \leq 2 \). Possibly \( \{ z_1, z_2 \} \cap \{ u_1, \ldots, u_k \} \neq \emptyset \).

We now 5-color \( G_0 \) minus the interior of the walk \( W = v_2z_1z_2v_1u_1 \cdots u_kv_2 \). By Lemma 3.2.1, the interior of \( W \) either contains just one vertex (and this must be \( v_0 \)) adjacent to all five colors, or two vertices \( v_0, v'_0 \) and edges \( v_0v_2, v_0z_1, v_0z_2, v_0v_1, v_0v'_0, v'_0v_1, v'_0u_1, v'_0u_2, v'_0v_2 \). Suppose first that the interior of \( W \) contains both \( v_0 \) and \( v'_0 \). Since \( G_{v_1v_2} \) is \( C_3 + C_5 \) or \( K_2 + h_7 \) with an additional vertex \( v'_0 \), we can find a vertex, \( r_1 \) with degree five in \( G_0 \) such that \( N(r_1) \) has at least one triangle. In addition if \( N(v_0) \) has at least one triangle, then \( r_1 \) can be chosen such that \( N(r_1) \) has at least two triangles. So \( N(v_0) \) has at least two triangles by property (iii) of an optimal pair. By Euler’s formula, \( G_0 \) can not contain both edges \( v_1z_1, v_2z_2 \). So \( k = 2 \) and \( \{ z_1, z_2 \} \cap \{ u_1, u_2 \} \neq \emptyset \).

It cannot be that \( \{ z_1, z_2 \} = \{ u_1, u_2 \} \) because then \( H_{v_1v_2} \) would have two double edges, again contradicting Euler’s formula. So we may assume that \( u_1 = z_1 \) and \( u_2 \neq z_2 \).

By Lemma 3.2.1, \( z_2 \) and \( u_2 \) have the same color in any 5-coloring of \( G_0 - \{ v_0, v'_0 \} \). So \( G_1 = (G_0 - \{ v_0, v'_0 \}) \cup \{ z_2u_2 \} \) is not 5-colorable. Hence \( G_1 \) contains a subgraph, \( G'_1 \) which is one of the graphs on our list, by the minimality of \( G_0 \). If \( z_2u_2 \) is contained
in a facial cycle $z_2u_2qz_2$ of $G'_1$, then either $qz_2v_1u_1u_2q$ or $qz_2z_1v_2u_2q$ is a contractible 5-cycle with more than one vertex in its interior, contradicting Lemma 3.2.1. So $z_2u_2$ is not in a facial 3-cycle of $G'_1$. Thus $G'_1 = K_6$. This $K_6$ contains $z_2$ and $u_2$ but none of $v_1, v_2$ because $H'_{v_1v_2}$ has only one double edge; $v_1$ or $v_2$ has degree at most six in $G_0$, since otherwise $v_1, v_2, v_0, v'_0$ are inside a walk of length six. So $G'_1 = K_6$ can be obtained from $H_{v_1v_2}$ by first deleting a vertex of degree at most six (and some more vertices) and then adding an edge. This is impossible because $H_{v_1v_2} = C_3 + C_5$ or $K_2 + H_7$.

We now return to the case when $W$ has at most one vertex, namely $v_0$, in its interior. If $k = 1$, then there is at most one vertex distinct from $u_1$, say $u_0$ that is adjacent to both $v_1$ and $v_2$, by Euler’s formula. If $k = 2$, we can interchange between $\{z_1, z_2\}$ and $\{u_1, u_2\}$. In any case we can assume that $N(v_0)$ has vertex set $\{x, y, z_1, z_2, u_1\}$ and can apply Lemma thomlem5.2 as well as an analogous lemma on $K_2 + H_7$ to 5-color $G_0$.

\[\square\]

**Lemma 3.3.3.** Let $(G_0, v_0)$ be an optimal pair, let $v_1, v_2$ be an identifiable pair, let $J$ be a subgraph of $G_{v_1v_2}$ isomorphic to $L_1, L_2, L_5$ or $L_6$, and let $R_1, R_2$ be the hinges of $J$. If $R_1$ and $R_2$ share a vertex $u \neq z_0$ and at least one of them has length three, then the other one has length five and there exists an index $i \in \{1, 2\}$ such that $\hat{R}_1 \cup \hat{R}_2 \setminus \{v_0, v_i\}$ is a cycle in $G_0$ that bounds an open disk containing $v_0$ and $v_i$.

**Proof.** By the symmetry we may assume that $R_2$ has length three. Thus $u$ is adjacent to $z_0$ in $J$. Since $R_1$ is an induced cycle, the cycles $R_1, R_2$ share the edge $z_0u$. Thus $\hat{R}_1, \hat{R}_2$ share the edge $v_iu$ for some $i \in \{1, 2\}$, and the second conclusion follows. By Lemma 3.2.1 the cycle $\hat{R}_1 \cup \hat{R}_2 \setminus \{v_0, v_i\}$ has length at least six, and hence $R_1$ has length five, as desired. \[\square\]

We denote by $K_5^-$ the graph obtained from $K_5$ by deleting an edge, and by $K_5 - P_3$.
the graph obtained from $K_5$ by deleting two adjacent edges.

**Lemma 3.3.4.** Let $(G_0, v_0)$ be an optimal pair, let $v_1, v_2$ be an identifiable pair, and let $J$ be a subgraph of $G_{v_1,v_2}$ isomorphic to $L_1$, $L_2$, $L_5$ or $L_6$. Then there exists a vertex $s \in V(G_0) - \{v_0\}$ of degree five such that

(i) $N_{G_0}(s)$ has a subgraph isomorphic to $K_5 - P_3$, and

(ii) if both hinges of $J$ have length five, then $N_{G_0}(s)$ has a subgraph isomorphic to $K^{-}_5$.

**Proof.** We begin by proving the first assertion. Assume that the notation is as in the paragraph prior to Lemma 3.3.2, and suppose first that $J = L_5$. Let the vertices of $J$ be numbered as in Figure 12. It follows from Lemma 3.2.9 that the indices of $a_i$ and $b_j$ can be renumbered so that the faces of $J$ around $c_1$ are $a_1c_1a_2$, $a_2c_1a_3$, $a_3c_1a_4$, $a_4c_1b_3b_5c_2$, $b_3c_1b_2$, $b_2c_1b_1$, $b_1c_1a_1c_2b_4$, in order. Recall that $z_0$ is the vertex of $J$ that results from the identification of $v_1$ and $v_2$. If $z_0 \neq c_1$, then one of the vertices $a_2, a_3, b_2$ is not incident with $\hat{R}_1$ or $\hat{R}_2$, and hence has the same neighbors in $J$ and in $G_0$. It follows that such a vertex satisfies the conclusion of the lemma, as desired. We will use the same argument again later, whereby we will simply say that a certain vertex satisfies the conclusion of the lemma and assume that this means condition (i) in this part of the proof.

Thus we may assume that $z_0 = c_1$, and since we may assume that no vertex satisfies the conclusion of the lemma, we deduce that one of $R_1$ and $R_2$ is the face $a_2c_1a_3$ and the other is $b_1c_1b_2$ or $b_2c_1b_3$. Thus we may assume that $R_1$ is $a_2c_1a_3$ and $R_2$ is $b_1c_1b_2$. We may assume, by swapping $v_1$ and $v_2$, that the neighbors of $v_1$ in $\hat{J}$ are $a_1, a_2, v_0, b_1$ and that the neighbors of $v_2$ are $a_3, a_4, b_3, b_2, v_0$. Hence the face $\hat{R}$ is $v_1a_2a_3v_2b_2b_1$. Now $v_1$ is not adjacent to $a_3$ in $G_0$, for otherwise $a_2$ satisfies the conclusion of the lemma. We shall abbreviate this argument by $a_2 \Rightarrow v_1 \not\sim a_3$. Similarly, we have $b_5 \Rightarrow b_3 \not\sim c_2$ and $b_3 \Rightarrow v_2 \not\sim b_5$. We shall define a 5-coloring $c$ of
Furthermore, we may assume, by swapping bounded by a cycle of length five, and the remaining faces are bounded by triangles. Assume first that \( b_4 \) is adjacent to \( a_1 \). Then \( b_2 \) is not adjacent to \( v_1 \), for otherwise \( b_1 \) satisfies the conclusion of the lemma. Furthermore, there is no vertex of \( G \) in the face of \( \tilde{J} \) bounded by \( b_1 v_1 a_1 c_2 b_4 \). In that case we let \( c(b_4) = 4 \) and \( c(b_2) = 3 \). If \( b_4 \) is not adjacent to \( a_1 \), then we let \( c(b_4) = 3 \) and \( c(b_2) = 4 \). In either case it follows from Lemma 3.2.1 and the fact that \( v_0 \) is adjacent to \( v_1 \) and \( v_2 \) that \( c \) extends to a 5-coloring of \( G_0 \), a contradiction. This completes the case \( J = L_5 \).

If \( J = L_6 \) we proceed analogously. By Lemma 3.2.9 we may assume that the faces around \( c_1 \) are \( a_1 c_1 a_2, a_2 c_1 a_3, a_3 c_1 a_4, a_4 c_1 b_5 c_2, b_4 c_1 b_3, b_3 c_1 b_2, b_2 c_1 b_1 \) and \( b_1 c_1 a_1 c_2 b_5 \). If \( z_0 \neq c_1 \), or if one of \( R_1, R_2 \) is not \( a_1 c_1 a_2 \) or \( b_2 c_1 b_3 \), then one of \( a_2, a_3, b_2, b_3 \) satisfies the conclusion of the lemma. Thus we may assume that \( R_1 \) is \( a_2 c_1 a_3 \) and \( R_2 \) is \( b_2 c_1 b_3 \). We may also assume, by swapping \( v_1 \) and \( v_2 \) that the neighbors of \( v_1 \) in \( \tilde{J} \) are \( a_1, a_2, v_0, b_1 \) and \( b_2 \) and the neighbors of \( v_2 \) in \( \tilde{J} \) are \( a_3, a_4, b_3, b_4 \), and \( v_0 \). Now \( a_1 \Rightarrow v_1 \not\sim y, b_4 \Rightarrow v_2 \not\sim b_5, a_3 \Rightarrow a_2 \not\sim v_2, \) and \( b_2 \Rightarrow b_3 \not\sim v_1 \). With these constraints in mind and recalling that \( v_0 \) is adjacent to \( v_1 \) and \( v_2 \), consider the following coloring:

\[
c(a_4) = c(b_1) = 1, \quad c(a_1) = c(b_2) = 2, \quad c(b_3) = c(v_1) = c(c_2) = c(b_4) = 4, \quad c(b_5) = c(a_2) = c(v_2) = 5
\]

It follows from Lemma 3.2.1 that \( c \) extends to a 5-coloring of \( G_0 \), a contradiction. This completes the case \( J = L_6 \).

We now consider the case \( J = L_1 \). By Lemma 3.2.8 exactly one face of \( J \), say \( F \), is bounded by a cycle of length five, and the remaining faces are bounded by triangles. Furthermore, we may assume, by swapping \( b_1, b_2 \), and by permuting \( a_1, a_2, a_3 \) that the faces around \( c_1 \) in order are \( F, b_2 c_1 b_1, b_1 c_1 c_0, c_0 c_1 a_1, a_3 c_1 a_1, a_1 a_2 c_1 a_3 \). By swapping \( b_3, b_4 \) we may assume that the faces around \( c_2 \) are \( F, b_3 c_2 b_4, b_4 c_2 c_0, c_0 c_2 a_\alpha, a_\beta c_2 a_\alpha, c_2 c_0 a_\alpha \) for some distinct indices \( \alpha, \beta, \gamma \in \{1, 2, 3\} \). Thus the face \( F \) is bounded by the cycle \( c_1 a_2 c_2 b_3 b_2 \), and hence \( \gamma = 2 \). Since \( a_1 c_0 c_1, c_1 c_0 b_1, b_4 c_0 c_2 \) and \( c_2 c_0 a_\alpha \) are faces of \( J \) we deduce that the faces around \( c_0 \) in order are \( a_1 c_0 c_1, c_1 c_0 b_1, b_1 c_0 c_1, b_1 c_0 b_4, b_4 c_0 c_2, c_2 c_0 a_\alpha, a_\alpha c_0 a_\delta, a_\delta c_0 a_1 \) for some integers \( i, j, \delta \) with \( \{i, j\} = \{2, 3\} \) and \( \delta \in \{2, 3\} - \{\alpha\} \).
Since $\gamma = 2$ we have $\alpha \neq 2$, and hence $\alpha = 3$ and $\delta = 2$.

Now if $z_0 \neq c_0$, then one of the vertices $a_1, a_2, a_3, b_1, b_2, b_3, b_4$ satisfies the conclusion of the lemma, and hence we may assume that $z_0 = c_0$. Furthermore, it is not hard to see that one of the above vertices satisfies the conclusion of the lemma unless one of $R_1, R_2$ is $a_1c_0a_2$ or $a_2c_0a_3$ and the other is one of $b_1c_0b_i$, $b_i c_0b_j$, $b_j c_0b_4$. Thus by symmetry we may assume that $R_1$ is $a_1c_0a_2$ and that $R_2$ is one of $b_1c_0b_i$, $b_i c_0b_j$, $b_j c_0b_4$.

We may assume that in $\hat{J}$ the vertex $v_1$ is adjacent to $c_1$ and $v_2$ is adjacent to $c_2$. We see that $a_3 \Rightarrow c_1 \not\sim c_2$ and $a_3 \Rightarrow a_1 \not\sim v_2$. Furthermore, if $R_2$ is the face $b_1c_0b_i$, then $b_4 \Rightarrow b_1 \not\sim v_2$, and if $R_2$ is the face $b_1c_0b_i$, then $b_1 \Rightarrow v_1 \not\sim b_4$. Let $c$ be the coloring of $\hat{J}\backslash v_0$ defined by $c(b_1) = c(v_2) = 1$, $c(b_i) = c(a_1) = 2$, $c(b_j) = c(v_1) = c(a_3) = 3$, $c(b_4) = c(a_2) = 4$, and $c(x) = c(y) = 5$, and let $c'$ be obtained from $c$ by changing the colors of the vertices $v_1, v_2, a_2$ to $4, 2, 1$, respectively. It follows from Lemma 3.2.1 by examining the three cases for $R_2$ separately that one of $c, c'$ extends to a 5-coloring of $G$, a contradiction. This completes the case $G = L_1$.

Finally, let $J = L_2$. We proceed similarly as above, using Lemma 3.2.8. Let $F$ be the unique face of $J$ of size five. By renumbering $a_1, a_2, a_3$ and $b_1, b_2, b_3$ we may assume that the faces around $c_1$ are $F, b_3c_1b_2, b_2c_1b_1, b_1c_1c_0, c_0c_1a_1, a_1c_1a_3, a_3c_1a_2$. Then the faces around $c_2$ are $F, b_4c_2c_0, c_0c_2a_\alpha, a_\alpha c_2a_\beta, a_\beta c_2a_\gamma$ for some distinct integers $\alpha, \beta, \gamma \in \{1, 2, 3\}$. It follows that $\gamma = 2$ and that $F$ is bounded by $c_1b_3b_4c_2a_2$. Since $b_1c_1c_0, c_0c_1a_1, b_4c_2c_0, c_0c_2a_\alpha$ are faces of $J$ we deduce that $\alpha \neq 1$ (and hence $\alpha = 3$ and $\beta = 1$) and that the cyclic order of the neighbors of $c_2$ around $c_2$ is $c_1b_1b_3b_4c_2a_3a_2a_1$ for some distinct integers $i, j \in \{2, 3\}$. (Recall that all faces incident with $c_0$ are triangles.) Since $b_4$ is adjacent to $b_3$ in the boundary of $F$ we deduce that $i = 3$ and $j = 2$.

Similarly as above, it is easy to see that some $a_i$ or $b_j$ satisfies the conclusion of the lemma, unless $z_0 \in \{c_0, c_1\}$. Suppose first that $z_0 = c_1$. We may assume that $R_1$ is $b_1b_2c_1$ and $R_2$ is $a_1a_3c_1$, for otherwise some vertex satisfies the conclusion of the
lemma. We may assume that $v_1$ is adjacent to $a_2, a_3, b_2, b_3$. We have $a_2 \Rightarrow v_1 \not\sim c_2$, $a_1 \Rightarrow a_3 \not\sim v_2$ and $b_2 \Rightarrow v_1 \not\sim b_1$. Let $c(a_2) = c(b_2) = 1, c(a_3) = c(b_4) = c(v_2) = 2, c(a_1) = c(b_3) = 3, c(v_1) = c(b_2) = c(c_1) = 4, $ and $c(c_0) = 5$. It follows from Lemma 3.2.1 that $c$ extends to a 5-coloring of $G_0$, a contradiction. Thus we may assume that $z_0 = c_0$. Similarly as above we may assume that $R_1$ is $b_1b_3c_0$ or $b_3b_2c_0$ and that $R_2$ is $a_1a_2c_0$ or $a_2a_3c_0$. We may assume that $v_1$ is adjacent to $a_1$ and $b_1$. If $R_2$ is $a_1a_2c_0$, then we have $a_3 \Rightarrow c_1 \not\sim c_2$ and $a_3 \Rightarrow a_1 \not\sim v_2$. If $R_2$ is $a_2a_3c_0$, then $a_1 \Rightarrow c_1 \not\sim c_2$ and $a_1 \Rightarrow a_3 \not\sim v_2$. If $R_1$ is $b_1b_3c_0$, then $b_2 \Rightarrow b_1 \not\sim v_2$. Let $c(a_1) = c(b_1) = c(v_2) = 1, c(b_3) = 2, c(a_2) = c(b_2) = 3, c(a_3) = c(b_4) = c(v_1) = 4$ and $c(c_1) = c(c_2) = 5$. It follows from Lemma 3.2.1 that $c$ extends to a 5-coloring of $G_0$, a contradiction.

We now prove the second assertion, using similar techniques as the previous case.

We begin by showing that $z_0$ is not a vertex of degree five, and then proceed by handling each of $L_1, L_2, L_5$ or $L_6$ individually. First suppose that $z_0 = c_2$ in $L_2$ or $L_6$. In this case, one of $b_1, b_2, b_3$ is a vertex that satisfies the condition of the lemma (by which, in this and in subsequent cases we mean condition (ii)). Now, suppose that $z_0$ is a vertex of degree five in $L_1, L_2$ not already handled above. In the original graph, by criticality, $z_0$’s five neighbors are colored 1, 2, 3, 4, 5, respectively. Further, the cycle containing $z_0$, call it $C_1$, must be a cycle of length seven. Cycle $C_1$ can be no larger as $z_0$ is degree five and $L_1, L_2$ consists of a single 5-cycle. Further, if $C_1$ has length less than seven, then after the split operation there are three internal vertices inside $C_1$, not all mutually adjacent, a contradiction. So there are no chords inside the 5-cycle contained within the 7-cycle.

Suppose that $C_1 = d_1d_2d_3d_4d_5f_1f_2d_1$. Here $c(d_i) = i$. Let $c(f_1) = 1, c(f_2) = 5$. We must now check cases (i) - (vi) of Lemma 3.2.1 to ensure that every coloring of the rest of the 6-cycle extends inside it. To do this, first notice that cases (iv)-(vi) do not hold because the vertices colored the same around the 7-cycle are not in the locations.
stated in the lemma. In case (i) for an internal vertex to be adjacent to five vertices on $C_1$, the extra added edges creates a situation where there are two vertices inside a 5-cycle, a contradiction. Similarly in case (ii), where two adjacent vertices must be adjacent to four vertices on $C_1$, this leaves all triangles and a 4-cycle inside $C_1$, but there is an additional internal vertex (as the split creates at least three internal vertices), a contradiction. In case (iii), we must have three internal vertices to be adjacent to each other and to three vertices on $C_1$. But this is impossible as $v_1 \not\sim v_2$ and so there must be an internal vertex in either $R_1, R_2$, but this internal vertex can only see two vertices on $C_1$. This completes the case when $z_0$ is a vertex of degree five in $L_1$ or $L_2$.

Now suppose there is a vertex of degree 5 in $L_5, L_6$ not previously considered above. Notice that if $z_0 = b_i$, then one of $a_2, a_3$ satisfies the condition of the lemma. So we may assume that $z_0 = a_i$ for some $i$. We wish to apply the same argument as the previous paragraph, but now we may not be able to color vertices $f_1, f_2$ in $C_1$ arbitrarily because they may be part of the other 5-cycle in $L_5$ or $L_6$. In this case without loss of generality, the vertices of $C_1$, are, in order $c_2a_4a_3a_2c_1b_1b_5c_2$. Let $c(c_1) = 1, c(c_2) = 5, c(a_2) = 2, c(a_3) = 3, c(a_4) = 4$. Let $C_2$ be the other 5-cycle, defined by $c_1a_3c_2b_ib_jc_1$. The vertices $b_i, b_j$ differ when considering $L_5$ and $L_6$. Suppose first that we are considering $L_5$. Then $b_i = b_4$ and $b_j = \{b_1, b_2, b_3\}$. Suppose there is no chord in $C_2$. Then let $c(b_i) = 3, c(b_5) = 1, c(b_1) = 5$, and color the rest of the $b_i$’s properly. Then the argument for coloring inside $C_1$ is identical to the case for $L_1$ and $L_2$ above. If instead there were chords in $C_2$, the only chord that would force us to color $C_1$ differently is if there was an edge between $b_1$ and $c_2$. In this case, let $c(b_1) = 2$ and let $c(b_5) = 1$. Color the rest of the vertices on $C_1$ as before. Notice that the arguments for cases (i) - (iii) above did not depend on the specific coloring of $C_1$, so these arguments also hold here. Also observe that cases (v) and (vi) do not hold because in our coloring the pairs of vertices colored the same are consecutive.
In case (iv) the only time the pairs of vertices colored the same are consecutive is not in the same configuration as our coloring of $C_1$. This completes the case when $z_0$ is a vertex of degree 5 in $L_5$.

Suppose instead that $z_0$ is a vertex of degree five in $L_6$. Now, vertex $b_5$ is present in both $C_1$ and $C_2$. So $C_2$ is the 5-cycle, defined by $c_1a_3c_2b_5b_jc_1$, where $j = \{2, 3\}$. Suppose there are no chords in $C_2$. Then we may color the vertices of $C_1$ and $C_2$ as in the previous paragraphs. So suppose that $C_2$ has a chord. In particular, the only chord that affects the coloring of $C_1$ is edge $c_1b_5$. In this case, let $c(b_5) = 4, c(b_1) = 5, c(b_2) = 2$. The same arguments concerning cases (i) - (vi) of Lemma 3.2.1 now apply to this situation. This concludes the case when $z_0$ is a vertex of degree 5 in $L_6$.

So we may now assume that $z_0$ has degree at least six. Further, we may assume that for $L_6$, $z_0 = c_1$ and we first consider this case. Notice that one hinge must use vertices of the form $a_i$ and the other must use vertices of the form $b_i$. Otherwise, one of vertices $b_2, b_3, a_2, a_3$ satisfy the conditions of the lemma. Suppose that $c_1$ is split into vertices $v_1, v_2$ such that without loss of generality, vertex $v_1$ is adjacent to at least one vertex of the form $b_i$. Let $c(c_2) = 2$ and $c(b_5) = 3$. If there is an edge between $b_5$ and $v_1$, then $c(v_1) = 1$ and $c(a_1) = 3$. Otherwise let $c(v_1) = 3$ and $c(a_1) = 1$. If there is an edge between $v_2$ and $c_2$, then $c(v_2) = 5$ and $c(b_4) = 2$. Otherwise let $c(v_2) = 2$ and $c(b_4) = 1$. Let $b_\alpha$ be the vertex of the form $b_i$ of highest index not adjacent to $v_2$ via the split, and let $a_\alpha$ be the vertex of lowest index not adjacent to $v_1$ via the split. If there is no edge in $R_1$ between $v_2$ and $b_\alpha$, let $c(b_\alpha) = c(v_2)$. Otherwise, color $b_\alpha$ properly. Similarly if there is no edge in $R_2$ between $v_1$ and $a_\beta$, let $c(a_\beta) = c(v_1)$. Otherwise, color $a_\beta$ properly. Notice that both edges are not present else $v_0$ is contained in a 4-cycle. If one such edge is present, notice that $v_0$ is contained in a 5-cycle but two vertices are colored the same. The rest of the vertices of the form $a_i$ and $b_i$ may now be properly colored, and so the graph can be properly colored.

We now consider the case of $L_5$. 


First suppose that \( z_0 = c_2 \), the vertex of degree six. Notice that if the split uses two triangles containing vertices of the form \( a_i \), but not \( b_i \), then \( a_1a_2a_3a_4v_2a_1 \) is a 5-cycle that contains two internal vertices, namely \( v_0 \) and \( v_1 \) from the split construction, a contradiction. So it follows, up to symmetry, that the split is between \( b_4c_2b_5 \) and \( a_1c_2a_2 \) or \( b_4c_2b_5 \) and \( a_2c_2a_3 \). Also, assume without loss of generality that the 5-cycles in this graph are \( a_1c_1b_1v_1 \) and \( a_4c_1b_2v_2a_2c_2a_3 \). First suppose that the split is between \( b_4c_2b_5 \) and \( c_2a_1a_2 \). Let \( c(a_i) = i \) for \( i \in \{1, 2, 3, 4\} \). Let \( c(c_1) = 5 \). Let \( c(v_1) = 2, c(v_2) = 1 \). Here \( v_1 \) is adjacent to \( a_1 \) and \( b_4 \), and \( v_2 \) is adjacent to \( a_2, a_3, a_4, b_5 \). Edge \( a_1v_2 \) is not present else \( a_3 \) is a vertex that satisfies the conditions of the lemma. Edge \( b_1c_5 \) is not present else either vertex \( b_2, b_3 \) is one we desire. Also, \( b_4a_1 \) is not present else there are two vertices in 5-cycle \( b_4a_1a_2v_2b_5 \). Let \( c(b_1) = 2, c(b_2) = 3, c(b_3) = 4, c(b_4) = 1, c(b_5) = 5 \). If edge \( v_1b_1 \) or \( v_1a_2 \) is present color \( v_2 \) color 3. If \( b_4v_2 \) is present, let \( c(b_4) = 4, c(b_3) = 1 \). These colorings do not allow for cases (i) - (iii) in 6-cycle \( a_1a_2v_2c_2c_3v_1 \), and so we are finished with this case. Now suppose the split is between \( b_4c_2b_5 \) and \( a_2c_2a_3 \). Suppose that \( c(a_1) = 1, c(a_2) = 2, c(a_3) = 3, c(a_4) = 4, c(c_1) = 5, c(b_1) = 2, c(b_2) = 3, c(b_3) = 4, c(b_4) = 1, c(b_5) = 5, c(v_1) = 3, c(v_2) = 2 \). Now, if \( a_2v_2 \) is present let \( c(v_2) = 1 \). If \( a_3v_1 \) is present let \( c(v_1) = 4 \). Notice that edges \( b_4a_1 \) and \( b_5c_1 \) can not occur as in the argument above. So all we must do is ensure that conditions (i)-(iii) of Lemma 3.2.1 do not hold in the 6-cycle \( a_2a_3v_2b_5b_4v_1 \). If neither \( a_3v_1 \) nor \( b_4a_1 \) were present, then none of these conditions hold. The same is true if edge \( a_2v_1 \) was present. However, if edge \( a_3v_1 \) was present then let \( c(v_2) = 1 \), and then none of the conditions of Lemma 3.2.1. This completes the proof that \( z_0 \neq c_2 \) in \( L_5 \).

Now suppose that \( z_0 = c_1 \) in \( L_5 \). As before, notice that if the split uses two triangles containing vertices of the form \( a_i \), but not \( b_i \), then \( a_1a_2a_3a_4v_2a_1 \) is a 5-cycle that contains two internal vertices, namely \( v_0 \) and \( v_1 \) from the split construction, a contradiction. Also, assume without loss of generality that the 5-cycles in this graph are \( a_1c_1b_1b_4v_1 \) and \( a_4c_1b_2b_5v_2 \). Up to symmetry there are two situations. First
suppose that we split between triangles $b_1c_1b_3$ and $a_1c_1a_2$. Now, suppose that edge $v_2b_1$ is not present. Then let $c(a_1) = 1$, $c(a_2) = 2$, $c(a_3) = 3$, $c(a_4) = 4$, $c(c_2) = 5$, $c(b_1) = 1$, $c(b_2) = 2$, $c(b_3) = 5, c(b_4) = 3$, $c(b_5) = 1$, $c(v_1) = 3$, $c(v_2) = 1$. Here $v_1$ is adjacent to $b_1, a_1$ and $v_2$ is adjacent to $a_2, a_3, a_4, b_2, b_3$. Edge $a_1b_1$ is not present else there are two vertices inside 5-cycle $a_1b_1b_3v_2a_2a_1$. Also, edge $a_1v_2$ is not present else one of $a_2, a_3$ is a vertex that satisfies the conditions of this lemma. If edge $b_4a_4$ is present the given coloring is a proper 5-coloring. If this edge is not present, then there may be a vertex in 5-cycle $a_4c_2b_4b_2v_2a_4$, and if this is the case, then let $c(b_5) = 3, c(v_1) = 4, c(b_4) = 4$. Thus, we may assume that $v_2b_1$ is present. Then edge $v_2b_4$ is not present else either $b_2, b_3$ satisfies the condition of the lemma. Again, edge $a_1b_1$ is not present else there are two vertices inside 5-cycle $a_1b_1b_3v_2a_2a_1$. Also, edge $a_1v_2$ is not present else an internal vertex is inside a 4-cycle. Now, let $c(a_1) = 1$, $c(a_2) = 2$, $c(a_3) = 3$, $c(a_4) = 4$, $c(c_2) = 5$, $c(b_1) = 1$, $c(b_2) = 2$, $c(b_3) = 5, c(b_4) = 4$, $c(b_5) = 3$, $c(v_1) = 4$, $c(v_2) = 1$. This coloring gives a proper 5-coloring if there is a vertex of degree 5 in cycle $a_1c_1b_1b_4v_2a_1$. In this case, let $c(b_1) = 3, c(b_2) = 2$. This finishes the case of the first split for $c_1$ in $L_5$.

Now, suppose that we split between triangles $b_1c_1b_3$ and $a_2c_1a_3$. Suppose that edge $a_2v_2$ was present. Then let $c(a_1) = 1$, $c(a_2) = 2$, $c(a_3) = 3$, $c(a_4) = 4$, $c(c_2) = 5$, $c(b_1) = 1$, $c(b_2) = 2$, $c(b_3) = 5, c(b_4) = 4$, $c(b_5) = 3$, $c(v_1) = 4$, $c(v_2) = 1$. Notice that edge $a_1b_1$ is not present as the 5-cycle $a_1b_1b_3v_2a_2a_1$ contains two internal vertices. This coloring is a proper 5-coloring if edge $a_4b_4$ does not exist. If $a_4b_4$ is present, then let $c(b_4) = 3, c(b_5) = 4$. So we may suppose $a_2v_2$ is not present. Similarly, suppose edge $b_1v_2$ is not present. Then let $c(a_1) = 1$, $c(a_2) = 2$, $c(a_3) = 3$, $c(a_4) = 4$, $c(c_2) = 5$, $c(b_1) = 1$, $c(b_2) = 3$, $c(b_3) = 5, c(b_4) = 4$, $c(b_5) = 2$, $c(v_1) = 4$, $c(v_2) = 2$. Again, $a_1b_1$ is not present. This is a valid coloring unless edge $b_4v_4$ is present. If so, then let $c(b_4) = 5, c(b_3) = 4$. This gives a proper 5-coloring. Thus we may assume edges $a_2v_2$ and $b_1v_2$ are not present.
Now we will condition on whether chords $b_5v_1$ and $a_4b_4$ are present. In this series of four colorings, let $c(a_1) = 1$, $c(a_2) = 2$, $c(a_3) = 3$, $c(a_4) = 4$, $c(c_2) = 5$, $c(v_2) = 2$. Now suppose that neither chord is present. Then let $c(b_1) = 2$, $c(b_2) = 1$, $c(b_3) = 5, c(b_4) = 4$, $c(b_5) = 3$, $c(v_1) = 3$. If chord $a_4b_4$ is present but $b_5v_1$ is not, then let $c(b_1) = 2$, $c(b_2) = 5$, $c(b_3) = 1, c(b_4) = 3$, $c(b_5) = 4$, $c(v_1) = 4$. If chord $b_5v_1$ is present but $a_4b_4$ is not, then let $c(b_1) = 2$, $c(b_2) = 1$, $c(b_3) = 3, c(b_4) = 4, c(b_5) = 3, c(v_1) = 4$. If both chords $b_5v_1$, $a_4b_4$ are present, then let $c(b_1) = 2$, $c(b_2) = 1, c(b_3) = 5, c(b_4) = 3, c(b_5) = 4, c(v_1) = 3$. Notice that in the 6-cycle $a_2a_3v_2b_5b_5v_1a_2$, there are three vertices colored the same. Thus, the coloring extends inside this 6-cycle and each of these colorings are valid. This completes the case when $z_0$ is a vertex in $L_5$.

Suppose that $z_0$ is a vertex in $L_2$. Consider the case when $z_0 = c_1$ of $L_2$. Consider the following cycle of vertices around $a_1$: $b_1b_2b_3b_4c_2a_3a_2a_1c_0b_1$. Notice that the split must use one triangle that involves vertices of the form $b_i$ and one triangle that involves vertices of the form $a_i$, else either $a_2$ or $b_2$ is a vertex that satisfies the conditions of the lemma. Further, we know that the 5-cycle that includes $c_1$ in $L_2$ is $b_3b_4c_2a_3c_1b_3$. Also note that around $c_1$, the two triangles that are split must be at least three edges apart. In general let $v_1$ be the vertex of the split that is adjacent to $c_0$. Up to symmetry, this means we must split along the following pairs of triangles: $(b_2c_1b_3, a_2c_1a_3), (b_2c_1b_3, a_1c_1a_2), (b_2c_1b_3, c_0c_1a_1), (b_1c_1b_2, a_1c_1a_2)$.

First suppose that we split between triangles $b_2c_1b_3$ and $a_2c_1a_3$. In this case notice that $b_3 \not\sim a_3$ else two vertices are contained in the 5-cycle $b_2b_32a_3a_2v_1b_2$. Also, $c_1 \not\sim a_3$ and $b_3 \not\sim c_1$ else $a_2$ or $b_2$, respectively would be a vertex that satisfies the conditions of the lemma. Now let $c(b_1) = 1, c(b_2) = 2, c(b_3) = 3, c(b_4) = 4, c(c_0) = 5, c(v_1) = 3, c(c_2) = 1, c(a_1) = 2, c(a_2) = 4, c(a_3) = 3$. Notice that three vertices are colored the same in the 7-cycle $b_2b_3b_4c_0a_3a_2v_1b_2$. This eliminates cases (iv) - (vi) of Lemma 3.2.1. Further, condition (i) does not hold since the 7-cycle contains only four colors. Condition (iii) does not hold since two colors are only used once, and

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condition (ii) does not hold because there does not exist a pair of internal vertices that can each see four different colors.

Now suppose the split was between triangles \( b_2c_1b_3 \) and \( a_1c_1a_2 \). If there is an edge between \( c_1a_2 \) then we argue as in the case above. Further, \( c_1 \not\sim b_3 \) else \( b_2 \) satisfies the conditions of the lemma. So let \( c(b_1) = 1, c(b_2) = 2, c(b_3) = 3, c(b_4) = 4, c(c_0) = 5, c(v_1) = 3, c(a_1) = 4, c(a_2) = 3, c(a_3) = 1, c(v_2) = 5, c(c_2) = 2 \). This coloring is a proper 5-coloring unless there is a vertex adjacent to all five vertices of the 5-cycle \( a_3c_2b_1b_3v_2a_3 \). In this case, let \( c(c_2) = 2 \) or 4, depending upon whether there edge \( b_2v_2 \) or \( a_1v_2 \) is present.

Next, suppose the split was between triangles \( b_2c_1b_3 \) and \( c_0c_1a_1 \). In this case edge \( c_0v_2 \) is not present else \( a_2 \) satisfies the conditions of the lemma. Also, \( b_3 \not\sim v_1 \) else vertex \( b_2 \) satisfies the conditions of the lemma. Then let \( c(b_1) = 1, c(b_2) = 2, c(b_3) = 3, c(b_4) = 4, c(c_0) = 5, c(v_1) = 3, c(a_1) = 4, c(a_2) = 3, c(a_3) = 1, c(v_2) = 5, c(c_2) = 2 \). This works unless there is a vertex adjacent to all five vertices of the 5-cycle \( a_3c_2b_1b_3v_2a_3 \). In this case, let \( c(a_1) = 1 \) and \( c(a_3) = 4 \).

Finally suppose the split was between triangles \( b_1c_1b_2 \) and \( a_1c_2a_2 \). Notice that edges \( a_2v_1 \) and \( b_2v_1 \) are not present else we can use arguments from the last three cases to satisfy the lemma. Now, let \( c(b_1) = 1, c(b_2) = 2, c(b_3) = 3, c(b_4) = 4, c(c_0) = 5, c(v_1) = 2, c(a_1) = 3, c(a_2) = 2, c(a_3) = 4, c(v_2) = 5, c(c_2) = 1 \). This coloring holds unless there is a chord between \( b_4 \) and \( a_3 \). If this is the case, let \( c(c_2) = 3, c(a_3) = 1, c(a_1) = 4 \). Notice that this coloring holds unless there is a chord between \( b_3 \) and \( c_2 \). But both these chords may not be present simultaneously, so this completes the proof of this split and the proof when \( z_0 = c_1 \) in \( L_2 \).

Now suppose that \( z_0 = c_0 \) in \( L_2 \). In this case, observe that the neighborhood around \( c_0 \) in \( L_2 \) is entirely triangles, but there is a 5-cycle that is defined by \( c_1a_2c_2b_4b_2c_1 \). Notice that triangle \( c_0c_2b_4 \) can not be used for any split else either \( a_2 \) or \( b_2 \) satisfies the conditions of the lemma. Further, one triangle must contain a
vertex of the form $a_i$ and another triangle must contain a vertex of the form $b_i$, else either $a_2$ or $b_2$ is a vertex that satisfies the conditions of the lemma. Also note that around $c_0$, the two triangles that are split must be at least three edges apart around the neighborhood of $c_0$. Up to symmetry, this gives seven pairs of triangles which must be analyzed, $(a_3c_0c_2, b_2c_0b_3), (a_1c_0a_2, b_3c_0b_4), (a_1c_0a_1, b_3c_0b_4), (a_2c_0a_3, b_2c_0b_3), (a_1c_0a_2, b_2c_0b_3), (c_1c_0a_1, b_2c_0b_3), (a_1c_0a_2, b_1c_0b_2)$. Without loss of generality, let $v_1$ be the vertex of the split adjacent to $c_1$. In all cases, assume that the vertices around $c_0$ are, in order, $c_1b_1b_2b_3b_4c_2a_3a_2a_1c_1$. Further, suppose that the 5-cycle in $L_2$ is defined by, $b_2b_4c_2a_3c_1b_2$. In all the colorings when $z_0 = c_0$, let $c(b_1) = 1, c(b_2) = 2, c(b_3) = 3, c(b_4) = 4, c(c_1) = 5$.

First consider the case when triangles $a_3c_0c_2$ and $b_2c_0b_3$ are split. Notice that $v_1 \not\sim c_2$ else there are two vertices in the 5-cycle $v_1c_2b_4b_3b_2$. Also $v_1 \not\sim b_3$, else there are two vertices in the 5-cycle $c_1b_3b_4c_2a_3c_1$. With this in mind, let $c(v_1) = 3, c(v_2) = 2, c(c_2) = 3, c(a_1) = 4, c(a_2) = 1, c(a_3) = 2$. This coloring holds unless there is vertex inside the 5-cycle in $L_2$. In this case, let $c(a_1) = 1, c(a_2) = 4$. Observe that this coloring does not allow for conditions (i)-(iii) of Lemma 3.2.1 to hold around the 6-cycle $b_2b_3v_2c_2a_3v_1b_2$. This completes the case, call it Case 1.

Now suppose that triangles $a_1c_0a_2$ and $b_3c_0b_4$ are split. First suppose that $v_2 \not\sim a_1$. Now, $v_1 \not\sim b_4$, else $b_2$ is a vertex that satisfies the conditions of the lemma. Vertex $v_1 \not\sim a_2$ as this is symmetric to Case 1 of this series of arguments. Let $c(v_1) = 4, c(v_2) = 1, c(c_2) = 3, c(a_1) = 1, c(a_2) = 4, c(a_3) = 2$. This coloring admits a proper 5-coloring unless edge $a_2b_4$ exists. In this case let $c(c_2) = 5, c(a_2) = 3$. This coloring holds unless edge $c_1c_2$ exists, but both $c_1c_2$ and $a_2b_4$ can not be present simultaneously. Now, suppose that $v_2 \sim a_1$. Then let $c(v_1) = 4, c(v_2) = 1, c(c_2) = 5, c(a_1) = 2, c(a_2) = 3, c(a_3) = 4$. Notice that if $c_1c_2$ exists then $a_3$ is a vertex we desire, so this coloring admits a proper 5-coloring. Call this Case 2.

Next, suppose that triangles $c_1c_0a_1$ and $b_3c_0b_4$ are split together. Then notice
that \( c_1 \not\sim v_2 \) and \( c_1 \not\sim b_4 \) else \( a_2 \) is a vertex that satisfies the conditions of the lemma. Further \( v_1 \not\sim a_1 \) as this would give a situation equivalent to Case 2. Let \( c(v_1) = 4, c(v_2) = 5, c(c_2) = 1, c(a_1) = 4, c(a_2) = 3, c(a_3) = 2. \) This coloring admits a proper 5-coloring unless there is a vertex in 5-cycle \( b_2b_4c_2a_2c_1b_2 \). In this case, let \( c(a_2) = 2 \) and \( c(a_3) = 3. \) Call this Case 3.

Suppose that triangles \( a_2c_0a_3 \) and \( b_2c_0b_3 \) are split. Notice that \( v_1 \not\sim a_3 \), else we are in Case 1. Further \( v_1 \not\sim b_3 \) else we are in a situation that is symmetric to Case 3. So let \( c(v_1) = 3, c(v_2) = 5, c(c_2) = 1, c(a_1) = 2, c(a_2) = 4, c(a_3) = 3. \) This coloring admits a proper 5-coloring unless edge \( a_2b_4 \) is present in 5-cycle \( b_2b_4c_2a_2c_1b_2 \). In this case, let \( c(a_2) = 1 \) and \( c(c_2) = 2. \) Notice that edges \( b_2c_2 \) and \( a_2b_4 \) can not exist simultaneously. This allows us to complete a proper 5-coloring. Call this Case 4.

Suppose now that triangles \( a_1c_0a_2 \) and \( b_2c_0b_3 \) are split. In this case, notice that \( v_1 \not\sim a_2 \) else we are in the situation defined by Case 4, and \( v_1 \not\sim b_3 \) else we are in Case 2. Now, let \( c(v_1) = 3, c(v_2) = 5, c(c_2) = 2, c(a_1) = 1, c(a_2) = 3, c(a_3) = 4. \) This coloring admits a proper 5-coloring unless edge \( b_2c_2 \) is present in 5-cycle \( b_2b_4c_2a_2c_1b_2 \). In this case, let \( c(a_1) = 2 \) and \( c(c_2) = 1. \) This allows us to complete a proper 5-coloring. Call this Case 5.

Next, suppose that triangles \( c_1c_0a_1 \) and \( b_2c_0b_3 \) are split. In this case, notice that \( c_1 \not\sim v_2 \) else there are two vertices in 5-cycle \( c_1v_2b_3b_2b_1c_1 \) and \( b_2 \not\sim v_2 \) else there are two vertices in 5-cycle \( c_1a_1v_2b_2b_1c_1 \). Further, notice that if there is a chord between \( c_1c_2 \) in 5-cycle \( b_2b_4c_2a_2c_1b_2 \), then \( a_2 \) is a vertex that satisfies the conditions of the lemma. Now, let \( c(v_1) = 4, c(v_2) = 2, c(c_2) = 5, c(a_1) = 3, c(a_2) = 1, c(a_3) = 4. \) This coloring allows us to complete a 5-coloring. Call this Case 6.

Finally, suppose that triangles \( a_1c_0a_2 \) and \( b_1c_0a_2 \) were split. Then notice that \( a_1 \not\sim v_2 \) else there are two vertices in 5-cycle \( a_1v_2b_2b_1c_1a_1 \), \( b_1 \not\sim v_2 \) else there are two vertices in 5-cycle \( b_1v_2a_2a_1c_1b_1 \). Further, \( v_1 \not\sim b_2 \) and \( v_1 \not\sim a_2 \) else we are in Case 5. Let \( c(v_1) = 2, c(v_2) = 1, c(c_2) = 5, c(a_1) = 1, c(a_2) = 3, c(a_3) = 2. \) This coloring
admits a proper 5-coloring unless there is an edge between \( c_1 \) and \( c_2 \) in the 5-cycle \( b_2b_4c_2a_2c_1b_2 \). In this case let \( c(c_2) = 1, c(a_1) = 4 \). This completes the final case when \( z_0 = c_0 \) in \( L_2 \). This completes the proof for \( L_2 \).

Suppose that \( z_0 \) is a vertex in \( L_1 \). We have already considered vertices of degree five. This leaves \( c_0, c_1, c_2 \). In this case, observe that \( c_1 \) and \( c_2 \) are identical up to symmetry so without loss of generality, consider the case when \( z_0 = c_1 \). The vertices forming an 8-cycle around \( c_1 \) are \( c_0b_1b_2b_3c_2a_3a_2a_1c_0 \). Notice that this includes the neighbors of \( c_1 \) and the 5-cycle that contains \( c_1, c_1b_2b_3c_2a_3c_1 \). Note that around \( c_1 \), the two triangles that are split must be at least three edges apart around the neighborhood of \( c_0 \). Up to symmetry, this gives three pairs of triangles which must be analyzed, namely \((a_1c_1a_2, b_1c_1b_2)\), \((a_2c_1a_3, b_1c_1b_2)\), and \((a_2c_1a_3, c_0c_1b_1)\). In all the colorings when \( z_0 = c_1 \), let \( c(b_1) = 1, c(b_2) = 2, c(b_3) = 3, c(b_4) = 4, c(c_1) = 5 \).

Now, first suppose we are in the case when triangles \( a_1c_1a_2 \) and \( b_1c_1b_2 \) are split. Notice that \( b_1 \not\sim v_2 \) else there are two vertices contained in 5-cycle \( b_1v_2a_2a_1c_0b_1 \). Also \( a_1 \not\sim v_2 \) else there are two vertices contained in 5-cycle \( a_1v_2b_2b_1c_0a_1 \). Let \( c(v_2) = 5, c(c_2) = 1, c(a_1) = 3, c(a_2) = 2, c(a_3) = 4 \). This coloring, unless there is a vertex of degree 5 in 5-cycle \( b_2b_3c_2a_3v_2 \) gives a valid 5-coloring as the 6-cycle \( b_1b_2v_2a_2a_1c_0b_1 \) does not satisfy conditions (i)-(iii) in Lemma 3.2.1. If such a vertex of degree five exists, then let \( c(a_2) = 4, c(a_3) = 2 \). This completes Case 1 of the situation when \( z_0 = c_1 \) of \( L_1 \).

Suppose that triangles \( a_2c_1a_3 \) and \( b_1c_1b_2 \) are split. If both \( v_1b_2 \) and \( v_1a_3 \) were present then two vertices are inside the 5-cycle \( v_1b_2b_3c_2a_3v_1 \). Then suppose that neither edge \( v_1b_2 \) nor \( v_1a_3 \) were present. In this case, the coloring where \( c(v_1) = 2, c(v_2) = 4, c(c_2) = 1, c(a_1) = 4, c(a_2) = 3, c(a_3) = 2 \) admits a valid 5-coloring. Notice that edge \( b_3a_3 \) is not present else there would be two vertices inside the 5-cycle \( b_2a_3a_2v_1b_1b_2 \). Suppose edge \( v_1b_2 \) was present but \( v_1a_3 \) was not. Let \( c(a_2) = 2, c(v_1) = 3, c(a_3) = 3, c(c_2) = 2 \). Notice that if one of edges \( b_3a_3 \) or \( b_2c_2 \) was present, two vertices
would again be inside a 5-cycle. So this coloring admits a valid 5-coloring. finally suppose that $v_1a_3$ was present. Then let $c(a_2) = 1, c(v_1) = 2, c(a_3) = 3, c(c_2) = 2$. For the same reasons as above in this paragraph, this coloring also admits a valid 5-coloring. This completes Case 2 of the situation when $z_0 = c_1$ of $L_1$.

Finally, suppose that triangles $a_2c_1a_3$ and $c_0c_1b_1$ are part of the split. In this case $c_0 \not\sim v_2$, else there are two vertices in the 5-cycle $c_0v_2a_3a_2a_1c_0$. Also, $v_1 \not\sim b_1$ else we are in Case 2 of this analysis. Vertex $b_2 \not\sim a_3$ as there are three non-mutually adjacent vertices in cycle $b_2a_3a_2a_1c_0b_1b_2$. Also $v_1 \not\sim a_3$, else $a_2$ is a vertex that satisfies the conditions of the lemma. With this in mind, let $c(a_1) = 2, c(a_2) = 3, c(a_3) = 4, c(c_2) = 1, c(v_2) = 1$. Notice that the 6-cycle $c_0a_1a_2a_3v_2b_1c_0$ does not satisfy conditions (i)-(iii) in Lemma 3.2.1. So this gives a valid coloring unless there is a vertex of degree five in the 5-cycle $c_1a_3c_2b_3b_2c_1$. In this case, let $c(a_2) = 3, c(a_3) = 4$. This completes the proof when $z_0 = c_1$ of $L_1$.

The final vertex to check is when $z_0 = c_0$ of $L_1$. In this case, observe that the neighborhood around $c_0$ in $L_1$ is entirely triangles, but there is a 5-cycle that is defined by $c_1a_2c_2b_3b_2c_1$. Further, one triangle must contain a vertex of the form $a_i$ else $a_2$ is a vertex that satisfies the conditions of the lemma. Also note that around $c_0$, the two triangles that are split must be at least three edges apart around the neighborhood of $c_0$. Up to symmetry, this gives eight pairs of triangles which must be analyzed, $(c_1c_0a_1, b_3c_0b_2), (c_1c_0a_1, b_2c_0b_4), (c_1c_0a_1, b_4c_0c_2), (c_1c_0a_1, c_2c_0c_3), (a_1c_0a_2, b_1c_0b_3), (a_1c_0a_2, b_3c_0b_2), (a_1c_0a_2, b_2c_0b_4)$ and $(a_1c_0a_2, b_4c_0c_2)$. Without loss of generality, let $v_1$ be the vertex of the split adjacent to $c_1$. In all cases, assume that the vertices around $c_0$ are, in order, $c_1b_1b_3b_2b_4c_2a_3a_2a_1c_1$. In all the colorings when $z_0 = c_0$, let $c(b_1) = 1, c(b_2) = 2, c(c_2) = 3, c(c_4) = 4$.

First suppose that $c_0$ is split between triangles $a_1c_0a_2$ and $b_4c_0c_2$. Note that $c_2 \not\sim v_1$ as there would be two vertices in 5-cycle $a_1v_1c_2a_3a_2a_1$. Also, $v_1 \not\sim a_2$, else two vertices are in 5-cycle $v_1a_2a_3c_2b_4$ In this case, let $c(c_1) = 3, c(v_1) = 5, c(a_1) = 2, c(a_2) = 5,$
\(c(a_3) = 4, c(c_2) = 1, c(v_2) = 3\). Notice that if edge \(c_1 \sim b_3\) then \(b_2\) is a vertex that satisfies the conditions of the lemma. This coloring admits a proper 5-coloring. Call this Case 1.

Now suppose that \(c_0\) is split between triangles \(a_1c_0a_2\) and \(b_2c_0b_4\). Notice \(v_1 \not\sim b_4\) else we are in Case 1. Now, let \(c(c_1) = 5, c(v_1) = 4, c(a_1) = 2, c(a_2) = 4, c(a_3) = 3, c(c_2) = 1, c(v_2) = 5\). This coloring holds if there is not a vertex of degree 5 inside the cycle \(c_1b_2b_3c_2a_2c_1\) or edge \(v_1a_2\) is not present. if \(v_1a_2\) is present, let \(c(a_2) = 3, c(a_3) = 4\). This coloring holds unless there edge \(a_2b_3\) is present. Then let \(c(c_2) = 5, c(a_2) = 1\). This completes Case 2.

Next, suppose that \(c_0\) is split between triangles \(a_1c_0a_2\) and \(b_1c_0b_3\). Observe that \(b_1 \not\sim v_2\), else 5-cycle \(b_1v_2a_2a_1c_1b_1\) contains two vertices. Similarly, \(a_1 \not\sim v_2\) else 5-cycle \(a_1v_2b_3b_1c_1a_1\) contains two vertices. Now, let \(c(c_1) = 5, c(a_1) = 2, c(a_2) = 4, c(a_3) = 3, c(c_2) = 1, c(v_2) = 5\). Notice that \(b_1b_3v_2a_2a_1c_1b_1\) form a 6-cycle that satisfies conditions (i)-(iii) in Lemma 3.2.1. This coloring holds unless there is a vertex adjacent to all five vertices of cycle \(a_2c_1b_2b_3c_2a_2\). In this case let \(c(a_1) = 4, c(a_2) = 2\). This completes Case 3.

Now, suppose that \(c_0\) is split between triangles \(a_1c_0a_2\) and \(b_3c_0b_2\). Notice \(v_1 \not\sim b_2\) else we are in Case 2, and \(v_2 \not\sim b_3\) else we are in Case 3. First suppose that edge \(a_1v_2\) does not exist. Then let \(c(c_1) = 4, c(a_1) = 3, c(a_2) = 5, c(a_3) = 2, c(c_2) = 1, c(v_2) = 3, c(v_1) = 2\). This holds unless there is a vertex adjacent to all vertices of the 5-cycle \(a_2c_1b_2b_3c_2a_2\). In this case, let \(c(c_2) = 2, c(a_3) = 1\). Finally, if \(a_1v_2\) was present, let \(c(a_1) = 1, c(a_2) = 4, c(a_3) = 2, c(c_2) = 5\). In this situation, notice that \(c_1 \not\sim c_2\) as then \(a_2\) is a vertex that satisfies the conditions of the lemma. This completes Case 4.

Suppose next that \(c_0\) is split between triangles \(c_1c_0a_1\) and \(a_3c_0c_2\). Note that \(a_1 \not\sim v_1\) else \(a_1v_1c_2a_3a_2a_1\) contains two vertices. Also, \(c_1 \not\sim v_2\) as then 5-cycle \(c_1v_2a_3c_2v_1c_1\) also contains two vertices. Similarly \(a_3 \not\sim v_1\) and \(c_2 \not\sim v_2\), else two vertices are in 5-cycles \(a_3v_1c_2a_1d_2a_3\) and \(c_2a_2a_1c_1v_1c_2\), respectively. Now, let \(c(c_1) = 4, c(a_1) = 2, c(a_2) = 5\),
$c(a_3) = 3$, $c(c_2) = 1$, $c(v_2) = 4$, $c(v_1) = 5$. This coloring holds unless there is a vertex of degree five in $c_2 b_3 b_2 c_1 a_2 c_2$. In this case let $c(a_2) = 2$, $c(a_1) = 5$. This gives a proper 5-coloring of the graph. Call this Case 5.

Now, suppose that $c_0$ is split between triangles $c_1 c_0 a_1$ and $b_3 c_0 b_2$. Notice that $v_1 \not\sim c_2$ else we are in Case 5. Similarly, $a_1 \not\sim v_1$ else we are in Case 1. Let $c(c_1) = 4$, $c(a_1) = 2$, $c(a_2) = 5$, $c(a_3) = 3$, $c(c_2) = 1$, $c(v_2) = 4$, $c(v_1) = 5$. This admits a proper 5-coloring unless there is a vertex of degree five in $c_2 b_3 b_2 c_1 a_2 c_2$. In this case let $c(a_2) = 3$, $c(a_3) = 5$. This completes Case 6.

Suppose that $c_0$ is split between triangles $c_1 c_0 a_1$ and $b_3 c_0 b_2$. Then it follows that $c_1 \not\sim v_2$ else $c_1 v_2 b_3 b_2 c_1 c_0 a_1$ is a 5-cycle that contains two vertices. Similarly, $a_3 \not\sim v_1$, $b_3 \not\sim v_2$, else 5-cycles $a_3 v_1 b_3 b_2 v_2 a_1$ and $b_3 v_2 a_1 c_1 b_1$, respectively, contain two vertices. Let $c(c_1) = 5$, $c(a_1) = 4$, $c(a_2) = 1$, $c(a_3) = 2$, $c(c_2) = 5$, $c(v_2) = 3$, $c(v_1) = 4$. Notice that $c_1 \not\sim c_2$ else $a_2$ is a vertex that satisfies the conditions of the lemma. The coloring given admits a proper 5-coloring. This completes Case 7.

Finally, suppose that $c_0$ is split between $b_2 c_0 b_4$ and $c_1 c_0 a_1$. Then notice that $v_1 \not\sim b_4$ else we are in Case 6, $v_2 \not\sim b_2$ else we are in Case 7 and $v_1 \not\sim a_1$ else we are in Case 2. Now, let $c(c_1) = 5$, $c(a_1) = 1$, $c(a_2) = 4$, $c(a_3) = 2$, $c(c_2) = 5$, $c(v_2) = 3$, $c(v_1) = 4$. Notice this coloring admits a proper 5-coloring unless there is an edge between $c_1$ and $c_2$. However, in this case, if $c_1 \sim c_2$, then $a_2$ is a vertex that satisfies the conditions of the lemma. This completes the case when $z_0 = c_0$, the case of $L_1$ and the proof.

\[ \Box \]

**Lemma 3.3.5.** Let $(G_0, v_0)$ be an optimal pair, let $v_1, v_2$ be an identifiable pair, and let $J$ be a subgraph of $G_{v_1 v_2}$. Then $J$ is not isomorphic to $L_1$, $L_2$, $L_5$ or $L_6$.

**Proof.** Let $G$, $v_0$, $v_1$, $v_2$ and $J$ be as stated, and suppose for a contradiction that $J$ is isomorphic to $L_1$, $L_2$, $L_5$ or $L_6$. Let $R_1, R_2$ be the hinges of $J$, and let $\hat{J}$, $\hat{R}_1$ and
\( \hat{R}_1 \) be as prior to Lemma 3.3.2. From Lemma 3.3.4 and conditions (ii)–(iv) in the definition of an optimal pair we deduce that

(1) \( N_{G_0}(v_0) \) has a subgraph isomorphic to \( K_5 - P_3 \),

and

(2) if both \( R_1 \) and \( R_2 \) have length five, then \( v_1, v_2 \) is the only non-adjacent pair of vertices in \( N_{G_0}(v_0) \).

Let \( v_3, v_4, v_5 \) be the remaining neighbors of \( v_0 \) in \( G_0 \). If at least two of them belong to the interior of \( \hat{R}_1 \) or \( \hat{R}_2 \), then they belong to the interior of the same face, say \( R_1 \), by (1). But then \( \hat{R}_1 \) is bounded by a cycle of length seven, and that again contradicts (1) by inspecting the outcomes of Lemma 3.2.1. Thus at most one of \( v_3, v_4, v_5 \) belongs to the interior of \( \hat{R}_1 \) or \( \hat{R}_2 \).

From the symmetry we may assume that the edges \( v_0v_4 \) and \( v_0v_5 \) belong to the face \( \hat{R}_1 \). We may also assume that \( v_5 \) belongs to the boundary of \( \hat{R}_1 \), and that if \( v_4 \) does not belong to the boundary of \( \hat{R}_1 \), then the edge \( v_0v_3 \) belongs to \( \hat{R}_2 \). We claim that \( v_4 \) belongs to the boundary of \( \hat{R}_1 \). To prove this suppose to the contradiction that \( v_4 \) belongs to the interior of \( \hat{R}_1 \). Then one of the edges \( v_1v_4, v_2v_4 \) does not belong to \( G_0 \), and so we may assume \( v_2v_4 \) does not. By (1) \( v_1, v_2 \) and \( v_2, v_4 \) are the only non-adjacent pairs of vertices in \( N_{G_0}(v_0) \), and by (2) at least one of \( R_1, R_2 \) has length three. It follows that \( v_3 \) belongs to the boundary of \( \hat{R}_1 \), and the choice of \( v_4, v_5 \) implies that the edge \( v_0v_3 \) lies in the face \( \hat{R}_2 \). Thus \( v_3 \) belongs to the boundary of \( \hat{R}_2 \).

By Lemma 3.3.3 there exists an index \( i \in \{1, 2\} \) such that the cycle \( R_1 \cup R_2 \setminus \{v_0, v_i\} \) bounds a disk containing \( v_0, v_i \) in its interior. By shortcutting this cycle through \( v_0 \) we obtain a cycle of \( G_0 \) of length at most four bounding a disk that contains the vertex \( v_i \) in its interior, contrary to Lemma 3.2.1. This proves our claim that \( v_4 \) belongs to the boundary of \( \hat{R}_1 \). We may assume that \( v_0, v_1, v_4, v_5, v_2 \) occur on the boundary of \( \hat{R}_1 \) in the order listed.

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Let $e \in E(G_0)$ have ends either $v_1, v_5$, or $v_2, v_4$. Then $e \not\in E(\hat{J})$, because the boundary of $\hat{R}_1$ is an induced cycle of $\hat{J}$. Moreover, $e$ does not belong to the face $\hat{R}_1$, because the edges $v_0v_4, v_0v_5$ belong to that face. Thus $e$ belongs to $\hat{R}_2$ or a face of $\hat{J}$ of length five. We claim that $e$ does not belong to $\hat{R}_2$. To prove the claim suppose to the contrary that it does, and from the symmetry we may assume that $e = v_2v_4$. We now argue that not both $R_1, R_2$ are pentagons. Indeed, otherwise $v_1$ is adjacent to $v_5$ by (2), and the edge $v_1v_5$ belongs to $\hat{R}_2$, because there is no other face of length at least five to contain it. In particular, $v_4, v_5$ belong to the boundary of $\hat{R}_2$, and because the edges $v_1v_5, v_2v_4$ do not cross inside $\hat{R}_2$, the vertices $v_1, v_0, v_2, v_4, v_5$ occur on the boundary of $\hat{R}_2$ in the order listed. It now follows by inspecting the 5-cycles of $L_5$ and $L_6$ that this is impossible. Thus not both $R_1, R_2$ are pentagons. By Lemma 3.3.3 the cycle $\hat{R}_1 \cup \hat{R}_2 \setminus \{v_0, v_1\}$ bounds a disk with $v_0, v_1$ in its interior. By shortcutting this cycle using the chord $v_2v_4$ we obtain a cycle in $G_0$ of length at most five bounding a disk with at least two vertices in its interior, contrary to Lemma 3.2.1. This proves our claim that $v_1v_5$ and $v_2v_4$ do not lie in the face $\hat{R}_2$.

By (1) and the symmetry we may assume that $v_2v_4 \in E(G_0)$, and hence the edge $v_2v_4$ belongs to a face $\hat{F}$ of $\hat{J}$ such that $\hat{F} \neq \hat{R}_1, \hat{R}_2$. Let $F$ be the corresponding face of $J$. Since $F$ is bounded by an induced cycle, we deduce that $v_4$ is not adjacent to $z_0$ in $J$. Consequently, $R_1$ has length five. Thus $R_1$ and $F$ have length five, and all other faces of $J$, including $R_2$, are triangles. In particular, $J = L_5$ or $J = L_6$, and $v_1, v_5$ are not adjacent in $G_0$ (because no face of $\hat{J}$ can contain the edge $v_1v_5$). By (1) $v_1, v_2$ and $v_1, v_5$ are the only non-adjacent pairs of vertices in $N_{G_0}(v_0)$. Condition (1) also implies that $v_3$ belongs to the boundary of $\hat{R}_2$. Using that and the fact that $v_3$ is adjacent to $v_1$ and $v_2$ in $G_0$, it now follows that there exists a vertex of $G_0 \setminus v_0$ whose neighborhood in $G_0$ has a subgraph isomorphic to $K_5 - P_3$. Finding such a vertex requires a case analysis reminiscent of but simpler than the proof of Lemma 3.3.4. We omit further details. The existence of such a vertex contradicts the fact that $(G_0, v_0)$
is an optimal pair.

\textbf{Lemma 3.3.6.} Let \((G_0, v_0)\) be an optimal pair, let \(v_1, v_2\) be an identifiable pair, and let \(J\) be a subgraph of \(G_{v_1v_2}\). Then \(J\) is not isomorphic to \(L_3\) or \(L_4\).

Proof. Let \(G_0, v_0, v_1, v_2\) and \(J\) be as stated, and suppose for a contradiction that \(J\) is isomorphic to \(L_3\) or \(L_4\). Let \(R_1, R_2\) be the hinges of \(J\), and let \(\hat{J}, \hat{R}_1, \hat{R}_2\) be as prior to Lemma 3.3.2. Since by Euler’s formula \(J\) triangulates the Klein bottle, we deduce that the faces \(\hat{R}_1, \hat{R}_2\) have size five, and every other face of \(\hat{J}\) is a triangle. Let the boundaries of \(\hat{R}_1\) and \(\hat{R}_2\) be \(v_1v_0v_2a_1b_1\) and \(v_1v_0v_2cb_1\), respectively. Let the neighbors of \(v_1\) in \(\hat{J}\) in cyclic order be \(v_0, b_1, b_2, \ldots, b_l\), and let the neighbors of \(v_2\) in \(\hat{J}\) be \(v_0, a_1, a_2, \ldots, a_k, c\). Then \(\deg_J(z_0) = k + l + 1\). Since \(J\) has no parallel edges the vertices \(a_1, a_2, \ldots, a_k, c, b_1, b_{l-1}, \ldots, b_1\) are distinct, and since \(J\) is a triangulation they form a cycle, say \(C\), in the order listed. Since \(v_1\) is not adjacent to \(v_2\) in \(G_0\), Lemma 3.2.1 implies that \(|V(C)| \geq 7\).

Let us assume that \(|V(C)| = 7\). Then \(z_0\) has degree seven, and hence \(J = L_4\), because \(L_3\) has no vertices of degree seven. Let \(X\) be the set of neighbors of \(z_0\) in \(J\). By inspecting the graph obtained from \(L_4\) by deleting a vertex of degree seven, we find that for every \(x \in X\), there exists a 5-coloring of \(J \setminus z_0\) such that no vertex of \(X \setminus \{x\}\) has the same color as \(x\). But this contradicts Lemma 3.2.1 applied to the subgraph of \(G_0\) consisting of all vertices and edges drawn in the closed disk bounded by \(C\), because \(X = V(C)\). This completes the case when \(|V(C)| = 7\).

Since \(L_3\) and \(L_4\) have no vertices of degree eight, it follows that \(|V(C)| = 9\), and hence \(z_0\) is the unique vertex of \(J\) of degree nine. From the symmetry between \(v_1\) and \(v_2\), we may assume that \(\deg_J(v_1) \leq 5\); in other words \(l \leq 4\). The graph \(J\) is 6-critical. Since \(z_0\) is adjacent to every other vertex of \(J\), we deduce that \(J \setminus z_0 \setminus x\) is 4-colorable for every vertex \(x \in V(J) \setminus \{z_0\}\), and hence

\begin{enumerate}
\item for every vertex \(x \in V(J) \setminus \{z_0\}\), the graph \(J \setminus z_0\) has a 5-coloring such that \(x\)
is the only vertex colored 5.

From Lemma 3.2.1 applied to the boundary of the face \( \hat{R} \) of \( \hat{J} \setminus v_0 \), we deduce that one of \( \hat{R}_1, \hat{R}_2 \) contains no vertex of \( G_0 \) in its interior, and the other contains at most one. Since \( v_0 \) has degree five, we may assume from the symmetry between \( \hat{R}_1 \) and \( \hat{R}_2 \) that \( v_0 \) is adjacent to \( a_1 \) and \( b_1 \) (and hence \( \hat{R}_1 \) includes no vertices of \( G_0 \) in its interior). We claim that \( l = 4 \) and \( v_1 \) is adjacent to \( c \). To prove the claim suppose to the contrary that either \( l \leq 3 \) or \( v_1 \) is not adjacent to \( c \). Then \( \text{deg}_{J}(v_1) \leq 5 \). By (1) there exists a coloring of \( J \setminus z_0 = \hat{J} \setminus \{v_0, v_1, v_2\} \) such that \( b_1 \) is the only vertex colored 5. We give \( v_2 \) the color 5, then we color \( v_1 \), then we color the unique vertex in the interior of \( \hat{R}_2 \) if there is one, and finally color \( v_0 \). The last three steps are possible, because each vertex being colored sees at most four distinct colors. Thus we obtain a 5-coloring of \( G_0 \), a contradiction. This proves our claim that \( l = 4 \) and \( v_1 \) is adjacent to \( c \). It follows that \( k = 4 \) and \( V(G_0) = \{v_0, v_1, v_2, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c\} \). We have \( \text{deg}_{G_0}(v_1) = \text{deg}_{G_0}(v_2) = 6 \), and since \( \text{deg}_{J}(c) \leq \text{deg}_{G_0}(c) - 2 \), we deduce that \( \text{deg}_{G_0}(c) \geq 7 \). Thus we have shown that

\[
(2) \text{ if } x_1, x_2, x_3, x_4, x_5 \text{ are the neighbors of } v_0 \text{ in } G_0 \text{ listed in their cyclic order around } v_0, \text{ the vertex } x_1 \text{ is not adjacent to } x_3 \text{ in } G_0 \text{ and } G_{x_1,x_3} \text{ has a subgraph isomorphic to } L_3 \text{ or } L_4, \text{ then } \text{deg}_{G_0}(x_1) = \text{deg}_{G_0}(x_3) = 6 \text{ and } \text{deg}_{G_0}(x_2) \geq 7.
\]

It also follows that \( v_1 \) is not adjacent to \( a_1 \) in \( G_0 \) and that \( v_2 \) is not adjacent to \( b_1 \) in \( G_0 \). Not both \( G_{v_1a_1} \) and \( G_{v_2b_1} \) have a subgraph isomorphic to \( L_3 \) or \( L_4 \) by (2), and so from the symmetry we may assume that \( G_{v_1a_1} \) does not. By the optimality of \((G_0, v_0)\) and Lemmas 3.3.2 and 3.3.5, it follows that \( G_{v_1a_1} \) has a subgraph isomorphic to \( K_6 \). Thus \( G \setminus \{v_0, v_1, v_2\} \) has a subgraph \( K \) isomorphic to \( K_5 \). If \( v_2 \notin V(K) \), then \( V(K) \cup \{z_0\} \) induces a \( K_6 \) subgraph in \( J \), a contradiction. Thus \( v_2 \in V(K) \), and hence \( V(K) = \{v_2, a_2, a_3, a_4, c\} \). Let \( i \in \{3, 4\} \). If \( a_1 \) is not adjacent to \( a_i \) in \( G_0 \), then we 5-color \( G_0 \) as follows. By (1) there is a 5-coloring of \( G_0 \setminus \{v_0, v_1, v_2\} \) such
that \(a_1\) and \(a_i\) are the only two vertices colored 5. We give \(v_1\) color 5, then color \(v_2\)
and finally \(v_0\). Similarly as before, this gives a valid 5-coloring of \(G_0\) a contradiction.
Thus, \(a_1\) is adjacent to \(a_3\) and \(a_4\) and hence \(a_1\) is not adjacent to \(c\), for otherwise
\(\{a_1, a_2, a_3, a_4, v_2, c\}\) induces a \(K_6\) subgraph in \(G_0\).

Since \(\text{deg}_{G_0}(v_2) = 6\), it follows from (2) that \(G_{ca_1}\) has no subgraph isomorphic
to \(L_3\) or \(L_4\). By the optimality of \((G_0, v_0)\) and Lemmas 3.3.2 and 3.3.5 it follows
that \(G_{ca_1}\) has a subgraph isomorphic to \(K_6\). By an analogous argument as above we
deduce that \(\{v_1, b_1, b_2, b_3, b_4\}\) is the vertex-set of a \(K_5\) subgraph of \(G_0\). The existence
of the two \(K_5\) subgraphs implies that \(a_2, a_3, a_4, b_2, b_3, b_4\) have \(K_4\) subgraphs in their
neighborhoods, and the optimality of \((G_0, v_0)\) implies that \(a_2, a_3, a_4, b_1, b_2, b_3\) all have
degree at least six in \(G_0\), and hence in \(J\). Thus \(a_1, b_1, c\) are the only vertices of \(J\) of
degree five. Thus, \(J = L_3\) and \(a_1, b_1, c\) are pairwise adjacent, a contradiction, because
we have shown earlier that \(a_1\) is not adjacent to \(c\).

The results of this section may be summarized as follows.

**Lemma 3.3.7.** Let \((G_0, v_0)\) be an optimal pair, and let \(v_1, v_2\) be an identifiable pair.
Then \(G_{v_1v_2}\) has a subgraph isomorphic to \(K_6\).

**Proof.** Every 5-coloring of \(G_{v_1v_2}\) may be extended to a 5-coloring of \(G_0\), and hence
\(G_{v_1v_2}\) is not 5-colorable. By the choice of \(G_0\) the graph \(G_{v_1v_2}\) has a subgraph isomor-
phic to one of the graphs listed in Theorem 3.1.2. By Lemmas 3.3.2, 3.3.5 and 3.3.6
that subgraph is \(K_6\), as desired.

### 3.4 Using \(K_6\)

**Lemma 3.4.1.** Let \((G_0, v_0)\) be an optimal pair. Then \(G_0\) has at least 10 vertices, and
if it has exactly 10, then it has a vertex of degree nine.

**Proof.** First, suppose that \(|V(G_0)| \leq 10\). We now employ a result of Gallai [16],
which states that if \(G\) is a \(k\)-critical graph with at most \(2k - 2\) vertices, then \(G\) is
of the form $G = G_1 + G_2$, where $G_i$ is $k_i$-critical for $i = 1, 2$, and $k_1 + k_2 = k$. By this result, it follows that $G_0$ is of the form $H_1 + H_2$, where $H_i$ is $k_i$-vertex-critical, $k_1 \leq k_2$, and $k_1 + k_2 = 6$. If $k_1 = k_2 = 3$, then we obtain either $K_6$ or $C_3 + C_5$ for $G_0$, a contradiction. So $k_1 \leq 2$ and therefore $G_0$ has a vertex adjacent to all other vertices. Now, suppose for purposes of contradiction that $|V(G_0)| \leq 9$. Let $k_1 = 1$. Then $|V(H_2)| \leq 8$ and so $H_2$ is of the form $H'_2 + H''_2$, where $H'_2 = K_2$ or $K_1$. So $k_1 = 2$ and that $H_2$ is 4-vertex-critical. By results of [16] and [43], the only 4-vertex-critical graphs with at most seven vertices are $K_4, K_1 + C_5, H_7$ and $M_7$, where $M_7$ is obtained from a 6-cycle, $x_1x_2 \cdots x_6x_1$ by adding an additional vertex $v$ and edges $x_1x_3, x_3x_5, x_5x_1, vx_2, vx_4, vx_6$. However, $G_0 \not\supseteq K_2 + (K_1 + C_5) = K_3 + C_5$ and $G_0 \not\supseteq K_2 + H_7$. This implies that $G_0 \supseteq K_2 + M_7$. The graph $G_0$ has nine vertices and at least 27 edges, so $G_0 = K_2 + N_7$. However, $N(v)$ is not a wheel neighborhood, a contradiction. This proves the lemma.

**Lemma 3.4.2.** Let $(G_0, v_0)$ be an optimal pair. Then there are at least two identifiable pairs.

**Proof.** Since $G_0$ has no subgraph isomorphic to $K_6$ there is at least one identifiable pair. Suppose for a contradiction that $v_1, v_2$ is the only identifiable pair. Thus the subgraph of $G_0$ induced by $v_0$ and its neighbors is isomorphic to $K_6$ with one edge deleted. By Lemma 3.3.7 the graph $G_0 \setminus \{v_0, v_1, v_2\}$ has a subgraph $K$ isomorphic to $K_5$, and every vertex of $K$ is adjacent to $v_1$ or $v_2$. Let $t$ be the number of neighbors of $v_0$ in $V(K)$. If $t = 0$, then $G_0$ has a subgraph isomorphic to $L_5$ or $L_6$; if $t = 1$, then $G_0$ has a subgraph isomorphic to $L_1$ or $L_2$; if $t = 2$, then $G_0$ has a subgraph isomorphic to $K_2 + H_7$; and if $t = 3$, then $G_0$ has a subgraph isomorphic to $C_3 + C_5$.

**Lemma 3.4.3.** Let $(G_0, v_0)$ be an optimal pair. Then $v_0$ has a wheel neighborhood.

**Proof.** Let us say that a vertex $v \in V(G_0)$ is a fan if its neighbors form a cycle in the order determined by the embedding of $G_0$. We remark that if $v$ is a fan, then the
embedding of $G_0$ can be modified so that $v$ will have a wheel neighborhood. Thus it suffices to show that $v_0$ is a fan. Suppose for a contradiction that there exist non-adjacent vertices $a, b \in N(v_0)$ that are consecutive in the cyclic order of the neighbors of $v_0$. By condition (iv) in the definition of an optimal pair, the graph $G' = G_0 + ab$ has a subgraph $M$ isomorphic to one of the graphs from Theorem 3.1.2. Assume, for a contradiction, that $v_0 \notin V(M)$. By optimality condition (i), $G_0 \setminus v_0$ has a 5-coloring $c$. Since $c$ is not a 5-coloring of $M$ it follows that $c(a) = c(b)$. But then $c$ can be extended to a 5-coloring of $G_0$, a contradiction. Thus $v_0 \in V(M)$. Since $\deg(v_0) = 5$, we get that $N_{G_0}(v_0) \subseteq V(M)$. Further note that $a, b$ are adjacent in $M$, because $M$ is not a subgraph of $G_0$.

First, assume $M = K_6$. Then $V(M) = \{v_0\} \cup N(v_0)$. This implies that there is at most one identifiable pair, contrary to Lemma 3.4.2. Second, assume $M = L_3$ or $L_4$. As each is a triangulation, Lemma 3.2.1 implies that $G_0 = M \setminus ab$. But $M$ is 6-critical, so $G_0$ has a 5-coloring, a contradiction.

Third, assume that $M = C_3 + C_5$ or $K_2 + H_7$. Because $M$ is one edge short of being a triangulation, there is a unique face in $M$ of length four. As $ab \in E(M)$, the embedding of $M \setminus ab$ has at most two faces of size strictly bigger than three, and if it has two, then they both have size four. Since $G_0$ has at least 10 vertices by Lemma 3.4.1, Lemma 3.2.1 implies that $M \setminus ab$ has a face $f$ of size five whose interior includes a vertex of degree five. However, $f$ is the only face of $M \setminus ab$ of size at least four, and hence it also includes the edge $ab$, but that is impossible. This completes the case when $M = C_3 + C_5$ or $K_2 + H_7$.

Fourth, suppose that $M$ is either $L_5$ or $L_6$, and let the notation be as in the proof of Lemma 3.3.4. In particular, every face incident with $a_2$ or $b_2$ is a triangle. At least one of $a_2, b_2$, say $s$, is not equal to $v_0$ and does not include both $a, b$ in its neighborhood. But then the neighborhoods of $s$ in $G$ and in $M$ are the same, and hence $s$ satisfies conditions (ii)-(iv) in the definition of an optimal pair by Lemma 3.4.2. But $s$ is a
fan in $M$, and hence has a wheel neighborhood in some embedding of $G_0$, contrary to condition (vi) in the definition of optimal pair.

If $M = L_1$, then we apply the argument of the previous paragraph to the vertices $a_1, b_1, b_4$, using the notation of Lemma 3.3.4. Finally, suppose that $M$ is $L_2$, and let the notation be again as in the proof of Lemma 3.3.4. Every face incident with one of the vertices $a_3, b_2$ is a triangle, and at least one of those vertices, say $s$, has the property that $s \neq v_0$ and if the neighborhood of $s$ includes both $a$ and $b$, then $a, b$ are not consecutive in the cyclic ordering around $s$ and $\{a, b\} \cap \{x, y\} \neq \emptyset$ for every pair of distinct non-adjacent vertices $x, y \in N_M(v_0)$. Since $s$ is a fan in $M$ its choice implies that it is a fan in $G_0$, and hence has a wheel neighborhood in some embedding of $G_0$, contrary to condition (vi) in the definition of optimal pair. Furthermore, in $G_0$ there are at most two pairs of non-adjacent vertices in the neighborhood of $s$, and if there are two, then they are not disjoint. Thus $s$ satisfies conditions (ii)-(iv) in the definition of an optimal pair by Lemma 3.4.2, contrary to condition (vi) in the definition of an optimal pair. 

A drawing of a graph $G$ in a surface is 2-cell if every face of $G$ is homeomorphic to an open disk.

**Lemma 3.4.4.** Let $(G_0, v_0)$ be an optimal pair, and let $v_1, v_2$ be an identifiable pair, and let $J$ be a subgraph of $G_{v_1v_2}$ isomorphic to $K_6$. Then the drawing of $J$ in the Klein bottle is 2-cell.

**Proof.** Let $v_0, R_1, R_2, \hat{R}_1, \hat{R}_2$ be as before, and suppose for a contradiction that the drawing of $J$ is not 2-cell. Since $K_6$ has a unique drawing in the projective plane [20, page 364], it follows that every face of $J$ is bounded by a triangle, and exactly one face, say $F$, is homeomorphic to the Möbius strip. If $F$ is not $R_1$ or $R_2$, then the boundary of $F$ is a separating triangle of $G_0$, a contradiction, because no 6-critical graph has a separating triangle. Thus we may assume that $F = R_2$. 

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Since both $R_1$ and $R_2$ are triangles, and they share at least one vertex, there exists a vertex $s \in V(J)$ not incident with $R_1$ or $R_2$. Thus in $\hat{J}$ all the faces incident with $s$ are triangles, and hence $\deg_{G_0}(s) = \deg_{\hat{J}}(s) = 5$ by Lemma 3.2.1. Furthermore, if $R_1$ and $R_2$ share an edge, then $N_{G_0}(s)$ has a subgraph isomorphic to $K_5^-$, the complete graph on five vertices with one edge deleted. This implies, by the optimality of $(G_0, v_0)$, that $N_{G_0}(v_0)$ has a subgraph isomorphic to $K_5^-$, contradicting Lemma 3.4.2.

So we may assume that $R_1$ and $R_2$ have no common edge. Let the facial walk incident with $\hat{R}_1$ be $v_0, v_1, z_1, z_2, v_2, v_0$, and the facial walk incident with $\hat{R}_2$ be $v_0, v_1, z_3, z_4, v_2, v_0$. Notice, from the embedding of $J$, that the $z_i$ are distinct. Also notice that $s$ is the lone vertex in $\hat{J}$ not incident with either $\hat{R}_1$ or $\hat{R}_2$, and $N_{G_0}(s)$ includes no two disjoint pairs of non-adjacent vertices. This implies, by the optimality of $(G_0, v_0)$, that $N_{G_0}(v_0)$ includes no two disjoint pairs of non-adjacent vertices. We shall refer to this as the DP property.

Let $N(v_0) = \{v_1, v_2, v_3, v_4, v_5\}$. Assume that some neighbor of $v_0$, say $v_3$, belongs to $\hat{R}_1$. By Lemma 3.2.1, $v_3$ is adjacent to all vertices incident with $\hat{R}_1$. Thus $v_4$ and $v_5$ belong to the closure of $R_2$. In either case, $v_3$ and $v_4$ are not adjacent in $G_0$. Since $v_1$ and $v_2$ are also not adjacent, this contradicts the DP property.

Since $v_1$ is not adjacent to $v_2$ in $G_0$ it follows from Lemma 3.4.3 that at least one of $v_3, v_4, v_5$ belongs to the closure of $\hat{R}_1$. Thus there remain two cases, depending on whether one or two of those vertices are incident with $\hat{R}_1$. If it is two vertices, then we may assume without loss of generality that $v_3 = z_1$ and $v_4 = z_2$. As $z_1$ and $z_2$ are not incident to $\hat{R}_2$, $v_3, v_2$ and $v_4, v_1$ are not adjacent in $G_0$, contrary to the DP property. Thus we may assume that $v_3 = z_1$ and $v_4$ and $v_5$ belong to the closure of $\hat{R}_2$. By the DP property $v_3, v_4$ and $v_3, v_5$ are adjacent in $G_0$. Thus, without loss of generality, $v_4 = z_3$ and $v_5 = z_4$. Furthermore, it follows from the DP property that either $v_1, v_5$ or $v_2, v_4$ are adjacent in $G_0$. Thus the subgraph $L$ of $\hat{J}$ consisting of the vertices $v_0, v_1, v_2, v_4, v_5$ and edges between them that belong to the closure of $\hat{R}_2$ has
five vertices and at least eight edges. We can regard $L$ as drawn in the Möbius band with the cycle $v_1v_0v_2v_5v_4$ forming the boundary of the Möbius band. As such the graph $L$ has at least three faces. Since the sum of the lengths of the faces is at least 11, at most one of them has length at least five. That face of $L$ includes at most one vertex of $G_0$ by Lemma 3.2.1, and the other faces of $L$ include none. Thus $G_0$ has at most nine vertices, contrary to Lemma 3.4.1.

Lemma 3.4.5. Let $(G_0, v_0)$ be an optimal pair, let $v_1, v_2$ be an identifiable pair, and let $J$ be a subgraph of $G_{v_1v_2}$ isomorphic to $K_6$. Then some face of $J$ has length six.

Proof. Let $\tilde{J}$ denote the graph consisting of $\hat{J}$ and edges of $G_0$ not in $\hat{J}$ from $v_1$ or $v_2$ to the boundary of $\hat{R}_1$ or $\hat{R}_2$ that are drawn inside $\hat{R}_1$ or $\hat{R}_2$. Let $\hat{R}_1$ be the face in $\tilde{J}$ that contains $v_0$ and is contained in $R_1$, and similarly for $\hat{R}_2$. We assume for a contradiction that no face of $J$ has length six. By Lemma 3.4.4 the embedding of $J$ is 2-cell, and so, by Euler’s formula, all faces of $J$ are bounded by triangles, except for either three faces of length four, or one face of length four and one face of length five. Each face of $\tilde{J}$ other than $\hat{R}_1$ and $\hat{R}_2$ will be called special if it has length at least four. Thus there are at most three special faces, and if there are exactly three, then they have length exactly four.

Let us denote the vertices on the boundary of $\hat{R}_1$ as $v_1, v_0, v_2, u_1, \ldots, u_k$ in order, and let the vertices on the boundary of $\hat{R}_2$ be $v_2, v_0, v_1, z_1, \ldots, z_l$ in order. Note that $u_1, u_2, \ldots, u_k$ are pairwise distinct, and similarly for $z_1, z_2, \ldots, z_l$. A special face of length five may include a vertex of $G_0$ in its interior; such vertex will be called special. It follows that there is at most one special vertex. An edge of $G_0$ is called special if it has both ends in $\hat{J}\setminus v_0$, but does not belong to $\hat{J}$, and is not $v_1z_1$ or $v_2z_1$ if $l = 1$, and is not $v_1u_1$ or $v_2u_1$ if $k = 1$. It follows that every special edge is incident with $v_1$ or $v_2$. Furthermore, the multigraph obtained from $G_0$ by deleting all vertices in the faces $\hat{R}_1$ and $\hat{R}_2$ and contracting the edges $v_0v_1$ and $v_0v_2$ has $J$ as a spanning subgraph,
and each special edge belongs to a face of $J$ of length at least four. It follows that there are at most three special edges. Furthermore, if there is a special vertex, then there is at most one special edge, and each increase of $k$ or $l$ above the value of two decreases the number of special edges by one.

Since $R_1$ and $R_2$ have length three, four, or five, we deduce that $k, l \in \{1, 2, 3, 4\}$. The graph $\hat{J}\{v_0, v_1, v_2\} = J\{z_0\}$ is isomorphic to $K_5$, and $u_1, u_2, \ldots, u_k$ are its distinct vertices; let $u_{k+1}, \ldots, u_5$ be the remaining vertices of this graph. It follows that if $c$ is a 5-coloring of $\hat{J}$ and $c(u_i) = c(z_j)$, then $u_i = z_j$. We will refer to this property as injectivity. From the symmetry we may assume that $k \geq l$. Since $J$ has at most one face of length five, it follows that $l \leq 3$. We distinguish three cases depending on the value of $l$.

**Case 1: $l = 1$**

By Lemma 3.4.3 the vertex $v_0$ is adjacent to $z_1$. Also notice then that $v_1 z_1 v_2 u_1 u_2 \ldots u_k$ is a null-homotopic walk $W$ of length at most seven. Since $G_0$ is 6-critical, the graph $G\{v_0\}$ has a 5-coloring, say $c$. By Lemma 3.2.1 applied to the subgraph $L$ of $G_0$ drawn in the disk bounded by $W$ and the coloring $c$, the graph $L$ satisfies one of (i)–(vi) of that lemma. We discuss those cases separately.

Case (i): There are eight vertices in $\hat{J}$ and none in the interior of $\hat{R}_1$ and $\hat{R}_2$, and at most one special vertex. Thus $|V(G_0)| \leq 9$, contradicting Lemma 3.4.1.

Case (ii): As before $|V(G_0)| \leq 9$, a contradiction, unless there exists a special vertex $v_0'$. This implies that $|\hat{R}_1| = |\hat{R}_1| = 6$. Without loss of generality suppose $v_0$ is adjacent to $u_3, v_1, z_1, v_2$ and a vertex $v_3$ which is adjacent to $v_0, v_2, u_1, u_2, u_3$. Notice that $v_0'$ must have degree five in $G_0$ and its neighborhood must contain a subgraph isomorphic to $K_5 - P_3$, since four of its neighbors are in $J\{z_0\}$ and thus form a clique. Meanwhile the neighborhood of $v_0$ is missing the edges $v_1 v_2, v_1 v_3$, and $v_2 u_3$. The last one does not belong to $\hat{J}$, does not lie in $\hat{R}_1$ (because we have already described the graph therein), and is not special, because all special edges have been accounted for.
Thus the pair \((G_0, v'_0)\) contradicts the optimality of \((G_0, v_0)\).

Case (iii): The graph \(L \setminus W\) consists of three pairwise adjacent vertices, and \(v_0\) is one of them. Let \(v_3, v_4\) be the remaining two. By Lemma 3.4.3 we may assume, using the symmetry that exchanges \(v_1, v_4, u_1, u_2\) with \(v_2, v_3, u_k, u_{k-1}\), that \(v_3\) has neighbors \(v_0, v_2, u_1, u_2, v_4\) and \(v_4\) is adjacent to \(v_1, v_0, v_3, u_2\) and either \(u_3\) or \(u_4\). In either case \(z_1\) and \(u_2\) are colored the same, and hence they are equal by injectivity. To be able to treat both cases simultaneously, we swap \(u_3\) and \(u_4\) if necessary; thus we may assume that \(v_4\) is adjacent to \(u_3\). We can do this, because we will no longer use the order of \(u_1, u_2, \ldots, u_k\) for the duration of case (iii). The vertex \(v_1\) is adjacent to \(u_2, u_3, u_4, u_5\), for otherwise its color can be changed, in which case the coloring \(c\) could be extended to \(L\), contrary to the fact that \(G_0\) has no 5-coloring. Similarly, \(v_2\) is adjacent to \(u_1, u_2, u_4, u_5\). It follows that \(G_0\) has a subgraph isomorphic to \(L_3\), a contradiction.

To describe the isomorphism, the vertices corresponding to the top row of vertices in Figure 7(c) in left-to-right order are \(u_1, u_4, u_5, u_3, u_2\), the vertices corresponding to the middle row are \(v_3, v_2, u_2 = z_1, v_1, v_4\), and the bottom vertex is \(v_0\). This completes case (iii).

Cases (iv)-(vi): We have \(k = 4\). Hence \(R_1\) has length five, and therefore there is at most one special edge. Consequently, one of \(v_1, v_2\) is not adjacent in \(\tilde{J}\) to at least two vertices among \(u_1, u_2, u_3, u_4\). Since every face of \(\tilde{J}\) except \(\tilde{R}_1\) and one other face of length four is bounded by a triangle this implies that in the coloring \(c\), one of \(v_1, v_2\) sees at most three colors. From the symmetry we may assume that \(v_2\) has this property. Thus \(c(v_2)\) may be changed to a different color.

By using this fact and examining the cases (iv)-(vi) of Lemma 3.2.1 we deduce that \(L\) is isomorphic to the graph of case (iv). Let the vertices of \(L\) be numbered as in Figure 9(iv). It further follows that \(v_2 = x_4\) or \(v_2 = x_5\), and so from the symmetry we may assume the former. Since \(z_1\) has a unique neighbor in \(L \setminus W\) we deduce that \(z_1 = x_3, v_1 = x_2, u_4 = x_1\) and so on. Notice that \(x_8\) has degree five in \(G_0\) and that
its neighborhood is isomorphic to $K_5 - P_3$. Meanwhile, the neighborhood of $v_0$ is
certainly missing the edges $v_1v_2$ and $v_1x_9$. Now if $x_3 \neq x_5$ then $x_3$ is not adjacent
to $x_9$ and $N(v_0)$ is missing at least three edges, a contradiction to the optimality of
$(G_0, v_0)$, given the existence of $x_8$. So $x_3 = x_5$, but then the edges $x_3v_2$, $x_5v_2$ are
actually the same edge, because $\tilde{J}$ does not have parallel edges. It follows that $v_2$ has
degree at most four in $G_0$, a contradiction.

Case 2: $l = 2$

By Lemma 3.4.3 either $v_0$ is adjacent to both $z_1$ and $z_2$, in which case we define
$v_0 := v_0$, or there exists a vertex $\overline{v}_0$ in $\tilde{R}_2$ adjacent to $v_0, v_1, v_2, z_1, z_2$. Let $W$ denote
the walk $v_1\overline{v}_0v_2u_1 \ldots u_k$ of length at most seven, and let $X$ denote the set of vertices
of $G_0$ drawn in the open disk bounded by $W$. We claim that $X \neq \emptyset$. This is clear if
$\overline{v}_0 \neq v_0$, and so we may assume that $\overline{v}_0 = v_0$. But then $X = \emptyset$ implies $|V(G_0)| \leq 9,$
contrary to Lemma 3.4.1. Thus $X \neq \emptyset$. Let $x \in X$ have the fewest number of
neighbors on $W$. Since $G_0$ is 6-critical, the graph $G_0 \setminus x$ has a 5-coloring, say $c$. By
Lemma 3.2.1 applied to the subgraph $L$ of $G_0$ drawn in the disk bounded by $W$ and
the coloring $c$, the graph $L$ and coloring $c$ satisfy one of (i)−(vi) of that lemma.

Suppose first that $L$ and $c$ satisfy (i). Then $|X| = 1$ by the choice of $x$. As before
$|V(G_0)| \leq 9$, contradicting Lemma 3.4.1, unless there is a special vertex. Hence $k \leq 3$.
If $k = 3$, then $R_1$ has length four, and all special faces have been accounted for. In
particular, $\tilde{J} = \hat{J}$. The fact that the coloring $c$ cannot be extended to $L$ implies that
$\{c(z_1), c(z_2)\} \subseteq \{c(u_1), c(u_2), c(u_3)\}$, and hence $\{z_1, z_2\} \subseteq \{u_1, u_2, u_3\}$ by injectivity.
Thus $u_1$ or $u_3$ is equal to one of $z_1, z_2$. Since there are no special edges, either $u_1v_2$
and $z_2v_2$, or $u_kv_1$ and $z_1v_1$ are the same edge, but then $v_1$ or $v_2$ has degree at most
four, a contradiction. If $k = 2$ we reach the same conclusion, using the fact that in
that case there is at most one special edge. It follows that $L$ and $c$ do not satisfy (i).

Next we dispose of the case $k \leq 3$. To that end assume that $k \leq 3$. Then $W$
has length at most six. Thus $L$ and $c$ satisfy either (ii) or (iii) of Lemma 3.2.1,
and so $W$ has length exactly six and $k = 3$. In particular, $R_1$ has length four, and so there is either at most one special vertex, or at most two special edges, and not both. It follows that either $c(v_1)$ or $c(v_2)$ can be changed without affecting the colors of the other vertices of $G_0 \setminus X$. That implies that $L$ and $c$ satisfy (ii).

Let $v_3$ be the unique neighbor of $\overline{v}_0$ in $X$, and let $v_4$ be the other vertex of $X$. From the symmetry we may assume that $v_3$ is adjacent to $\overline{v}_0, v_1, v_2, u_1, v_4$, and $v_4$ is adjacent to $v_1, v_3, u_1, u_2, u_3$. By considering the walk $u_1 u_2 u_3 v_1 z_1 z_2 v_2$ and the subgraph drawn in the disk it bounds, and by applying Lemma 3.2.1 to this graph and the coloring $c$ we deduce that $|\{c(u_1), c(u_2), c(u_3)\} \cap \{c(z_1), c(z_2)\}| = 1$. That implies $|\{u_1, u_2, u_3\} \cap \{z_1, z_2\}| = 1$ by injectivity, and so we may assume that $u_5$ is not equal to $z_1$ or $z_2$. It follows that the neighborhood of $u_5$ has a subgraph isomorphic to $K_5 - P_3$. However, the neighborhood of $\overline{v}_0$ is missing $v_1 v_2$ and at least one of the edges $v_3 z_1$ and $v_3 z_2$, contrary to the optimality of $(G_0, v_0)$ if $v_0 = \overline{v}_0$. Similarly, the neighborhood of $v_3$ is missing $v_1 v_2$ and $\overline{v}_0 v_4$, a contradiction if $v_0 = v_3$. This completes the case $k \leq 3$.

Thus we may assume that $k = 4$. It follows that $R_1$ has length five, and hence there is at most one special edge. Let $i \in \{1, 2\}$. If $v_i$ is adjacent to both $z_1$ and $z_2$, then one of the edges $v_i z_1, v_i z_2$ is special. It follows that in $G_0$, either $v_1$ is not adjacent to $z_2$, or $v_2$ is not adjacent to $z_1$. But $z_2$ is the only neighbor of $v_1$ in $G_0 \setminus X$ colored $c(z_2)$, because $G_0 \setminus (X \cup \{v_0, v_1, v_2\})$ is isomorphic to $J \setminus z_0$, which, in turn, is isomorphic to $K_5$. Thus there is a (proper) 5-coloring $c_1$ of $G_0 \setminus X$ obtained by changing the color of at most one of the vertices $v_1, v_2$ such that either $c_1(v_1) = c_1(z_2)$ or $c_1(v_2) = c_1(z_1)$. Now $c_1(\overline{v}_0)$ can be changed to another color, thus yielding a coloring $c_2$ of $G_0 \setminus X$.

If $L$ and $c$ satisfy one of the cases (iii)-(vi), then one of the colorings $c_1, c_2$ extends into $L$, a contradiction. Thus $L$ and $c$ satisfy (ii) of Lemma 3.2.1. Let $v_3 \in X$ be the unique vertex of $X$ adjacent to $\overline{v}_0$, and let $v_4$ be the other vertex in $X$. If both $v_3$ and $v_4$ have degree five in $G_0$, then one of the colorings $c_1, c_2$ extends into $L$, a contradiction.
Thus one of \( v_3, v_4 \) has degree five, and the other has degree six. It follows that \( v_3 \) is adjacent to \( v_1, v_2 \), and either \( u_1 \) or \( u_4 \), and so from the symmetry we may assume it is adjacent to \( u_1 \). If \( c_1(v_1) = c_1(u_1) \), then we can extend one of the colorings \( c_1, c_2 \) into \( L \) by first coloring \( v_4 \) and then \( v_3 \). Thus \( c_1(v_1) \neq c_1(u_1) \). If \( v_4 \) is not adjacent to \( u_1 \), then we can extend \( c_1 \) or \( c_2 \) by giving \( v_4 \) the color \( c_1(u_1) \), and then coloring \( v_3 \). Thus \( v_4 \) is adjacent to \( v_1 \). If \( v_4 \) has degree five, then its neighbors are \( u_1, u_2, u_3, u_4, v_3 \), and the neighbors of \( v_3 \) are \( \overline{v}_0, v_1, v_2, u_1, u_4, v_4 \). Let \( d \) a 5-coloring of \( G_0 \setminus \overline{v}_0 \). Since the coloring \( d \) cannot be extended to \( \overline{v}_0 \), it follows that the neighbors of \( \overline{v}_0 \) receive different colors. Now similarly as in the construction of \( c_1 \) above, we can change either the color of \( v_1 \), or the color of \( v_2 \). The resulting coloring then extends to \( \overline{v}_0 \), a contradiction.

This completes the case when \( v_4 \) has degree five, and hence \( v_4 \) has degree six. It follows that the neighbors of \( v_4 \) are \( u_1, u_2, u_3, u_4, v_1, v_3 \) and the neighbors of \( v_3 \) are \( \overline{v}_0, v_1, v_2, u_1, v_4 \). Let \( d_1 \) be a 5-coloring of the graph \( G_0 \setminus \{ \overline{v}_0, v_3 \} \). Since the coloring \( d_1 \) does not extend into \( \overline{v}_0, v_3 \), we deduce that \( \{d_1(z_1), d_1(z_2)\} = \{d_1(v_4), d_1(u_1)\} \). By injectivity this implies that \( u_1 = z_1 \) or \( u_1 = z_2 \). If \( u_1 = z_2 \), then one of the edges \( v_2u_1, v_2z_2 \) is special, because they cannot be the same edge, given that \( v_2 \) has degree at least five in \( G_0 \). Thus all special edges have been accounted for, and so \( z_1 \) is not adjacent to \( u_1 \). Thus \( d_1(v_1) \) can be changed to \( d_1(u_1) \), and the new coloring extends to all of \( G_0 \), a contradiction. Thus \( u_1 = z_1 \). It follows that \( G_0 \) is isomorphic to \( L_3 \).

First of all, the vertex \( v_1 \) is not adjacent to both \( u_2 \) and \( u_3 \), for otherwise the vertices \( v_1, v_4, u_1, u_2, u_3, u_4 \) form a \( K_6 \) subgraph in \( G_0 \). If \( v_1 \) is adjacent to neither \( u_2 \) nor \( u_3 \), then \( v_2 \) is adjacent to these vertices, and an isomorphism between \( G_0 \) and \( L_3 \) is given by mapping the vertices in the top row in Figure 7(c), in left-to-right order, to \( u_4, u_2, u_3, u_5 \), the middle row to \( v_1, v_4, u_1 = z_1, v_2, \overline{v}_0 \) and the bottom vertex to \( v_3 \). If \( v_1 \) is adjacent to exactly one of \( u_2, u_3 \), then due to the symmetry in the forthcoming argument we may assume that \( v_1 \) is adjacent to \( u_3 \), and hence \( v_2 \) is adjacent to \( u_2 \). Then an isomorphism is given by mapping the top row to \( v_4, u_4, u_3, u_2 \), the middle
row to $v_3, v_1, u_1 = z_1, u_5, v_2$, and mapping the bottom vertex to $\overline{v}_0$. This completes the case $l = 2$.

**Case 3: $l = 3$**

Lemma 3.4.3 implies that $v_0$ has at most one neighbor among $\{z_1, z_2, z_3, u_1, u_2, \ldots, u_k\}$, and such neighbor must be $u_1$, $u_k$, $z_1$, or $z_3$.

We claim that either $v_0$ is adjacent to $z_1$ or $z_3$, or $k = 3$ and $v_0$ is adjacent to $u_1$ or $u_3$. To prove this claim let us assume that $v_0$ has no neighbor among $\{z_1, z_2, z_3\}$. Let $C$ be the cycle $v_1 z_1 z_2 z_3 v_2 v_0$, and let $X$ denote the set of vertices of $G_0$ drawn in the open disk bounded by $C$. We have $X \neq \emptyset$ by Lemma 3.4.3. Let $c$ be a coloring of $G \setminus X$, and let $L$ denote the subgraph of $G_0$ consisting of all vertices and edges drawn in the closed disk bounded by $C$. By Lemma 3.2.1 the graph $L$ satisfies one of the conditions (i), (ii), (iii) of that lemma. The vertices $z_1$ and $z_3$ are adjacent, because the graph obtained from $\hat{J}$ by deleting $v_0$, $v_1$, $v_2$ and the vertices drawn in the faces $\tilde{R}_1$ or $\tilde{R}_2$ is isomorphic to $K_5$. We may also assume, by the symmetry between $v_1$ and $v_2$, that $v_1$ is adjacent to $z_2$. We claim that we may assume that the neighborhood of $v_0$ is a 5-cycle. This is clear if $v_0$ has no neighbor in $\{u_1, u_2, u_3, u_4\}$, and so we may assume that $v_0$ is adjacent to $u_1$. Then we may assume that $k = 4$, for otherwise the claim we are proving holds. Thus there is no special edge. By Lemma 3.4.3 there exists a vertex inside $\tilde{R}_1$ adjacent to $v_0, v_1, u_1$. Since there is no special edge the vertex $v_1$ is not adjacent to $u_1$, and $u_1$ is not adjacent to $z_1$, because $v_2$ has degree at least five in $G_0$. It follows that the neighborhood of $v_0$ is indeed a 5-cycle. If $|X| \geq 2$, then there exists a vertex in $X$ whose neighborhood has a subgraph that is a 5-cycle plus at least one additional edge, namely $z_1 z_3$ or $v_1 z_2$. That contradicts the optimality of $(G_0, v_0)$. Thus $|X| = 1$. Let $x$ denote the unique element of $X$, and let us assume first that $k = 4$. Then there are no special edges, and so $v_1$ is not adjacent to $z_3$ and $v_2$ is not adjacent to $z_1$. Let $C'$ denote the cycle $v_1 x v_2 u_1 u_2 u_3 u_4$, and let $X'$ be the set of vertices of $G_0$ drawn in the open disk bounded by $C'$. Then $G_0 \setminus (X' \cup \{x\})$ has...
a 5-coloring $c'$ such that $c'(v_1) = c'(z_3)$ and $c'(v_2) = c'(z_1)$. Then $c'$ can be extended to $x$ in at least two different ways. By Lemmas 3.2.1 and 3.4.3 the coloring $c'$ can be extended to all of $G_0$, unless (up to symmetry between $v_1$ and $v_2$) $v_0$ is adjacent to $u_1$, there exists a vertex $y$ adjacent to $u_1, u_2, u_3, u_4$ and $c'(v_1) = c'(u_5)$. But $v_1$ is not adjacent to $u_1$ (because $v_2$ is and there are no special edges), and hence the color of $v_1$ can be changed to $c'(u_1)$, and the resulting coloring extends to all of $G_0$, a contradiction. This completes the case $k = 4$. Thus $k = 3$, and so there is at most one special edge. Let $c''$ be a 5-coloring of $G_0 \setminus X'$. It follows that the color of at least one of the vertices $v_1, v_2$ can be changed to a different color, without affecting the colors of the other vertices of $G \setminus X'$. It follows from Lemma 3.2.1 that $|X'| \leq 2$. That, in turn, implies that $v_0$ is adjacent to $u_1$ or $u_3$, and hence proves our claim from the beginning of this paragraph.

Thus we may assume that $v_0$ is adjacent to $z_3$. By Lemma 3.4.3 there exists a vertex $v_3$ adjacent to $v_0, v_1, z_1, z_2, z_3$ and a vertex $v_4$ in $\tilde{R}_1$ that is adjacent to $v_0, v_1, v_2$. The neighborhood of $v_3$ includes the edge $z_1z_3$, and so by the optimality of $(G_0, v_0)$ the neighborhood of $v_0$ includes the edge $v_4z_3$. Thus $z_3 \in \{u_1, u_2, u_3, u_4\}$. Assume first that $k = 4$. Then there are no special edges, and hence $z_3 \neq u_4$. Next we deduce that $z_3 \neq u_1$, for otherwise $v_2u_1$ and $v_2z_3$ are the same edge, which implies (given that $z_3 = u_1$ is adjacent to $v_4$) that $v_2$ has degree at most three, a contradiction. Thus $z_3 \in \{u_2, u_3\}$. Let $Y$ consist of $v_0$ and all vertices in $\tilde{R}_1$ or $\tilde{R}_2$. Since $z_3$ is adjacent to $v_4$ we deduce that $|Y| \leq 4$. Since there are no special edges, $z_3$ is not adjacent to $v_1$, and $v_2$ is not adjacent to $u_4$. Thus $G_0 \setminus Y$ has a coloring $d$ such that $d(v_1) = d(z_3)$ and $d(v_2) = d(u_4)$. Since $z_3 \in \{u_2, u_3\}$ this coloring can be extended to the vertices drawn in $\tilde{R}_1$, and since $d(v_1) = d(z_3)$ it can be further extended to $v_0$ and $v_3$, a contradiction.

Thus $k = 3$. Let $W$ denote the walk $v_1v_3z_3v_2u_1u_2u_3$, and let $d'$ be a 5-coloring of $G_0 \setminus (Y \setminus \{v_3\})$. We now apply Lemma 3.2.1 to the graph drawn in the closed disk bounded by $W$ and coloring $d'$, and note that either the color of each of $v_1, v_2$ can
be changed to a different color, independently of each other and independently of the colors of other vertices, except possibly \( v_3 \), or the color of one of \( v_1, v_2 \) can be changed to two different values. In either case, one of the resulting colorings extends to \( G_0 \), a contradiction.

\[ \square \]

**Lemma 3.4.6.** Let \((G_0, v_0)\) be an optimal pair, let \(v_1, v_2\) be an identifiable pair, and let \(J\) be a subgraph of \(G_{v_1v_2}\) isomorphic to \(K_6\). Then the drawing of \(J\) in the Klein bottle does not have a facial walk of length six.

**Proof.** Suppose for a contradiction that there exists a subgraph \(J\) of \(G_{v_1v_2}\) isomorphic to \(K_6\) such that the drawing of \(J\) in the Klein bottle has a face \(F_0\) bounded by a walk \(W\) of length six. Let the vertices of \(J\) be \(z_1, z_2, \ldots, z_6\). Since \(K_7\) cannot be embedded in the Klein bottle, it follows that \(W\) has a repeated vertex. If \(W\) has exactly one repeated vertex, then (since \(J\) is simple) we may assume that the vertices on \(W\) are \(z_6, z_2, z_4, z_6, z_3, z_5\), in order. There exists a closed curve \(\phi\) passing through \(z_6\) and otherwise confined to \(F_0\) such that there is an edge of \(J\) on either side of \(\phi\) in a neighborhood of \(z_6\). The curve \(\phi\) cannot be separating, because \(G_0 \setminus v_0\) is connected, and it cannot be 2-sided, because \(G_0 \setminus v_0\) is not planar. It follows that \(\phi\) is 1-sided.

By Euler’s formula every face of \(J\) other than \(F_0\) is bounded by a triangle. It follows that the triangles \(z_4z_5z_6\), \(z_1z_6z_3\), and \(z_1z_6z_2\) bound faces of \(J\). Furthermore, either \(z_3z_5z_2\) or \(z_3z_5z_4\) is a face, but since \(J\) is simple we deduce that it is the former. It follows that \(z_1z_3z_4\), \(z_2z_3z_4\), \(z_1z_2z_5\) and \(z_1z_4z_5\) are faces of \(J\), and those are all the faces of \(J\). The drawing of \(J\) is depicted in Figure 13, where diagonally opposite vertices and edges are identified, and the asterisk indicates another cross-cap.

Similarly, if \(W\) has at least two repeated vertices, then it has exactly two, and we may assume that the vertices of \(W\) are \(z_6z_5z_4z_6z_2z_4\). Similarly as in the previous paragraph, the embedding is now uniquely determined, and is depicted in Figure 14.

In either case let \(R_1\) and \(R_2\) be the hinges of \(J\), and let \(F_{ijk}\) denote the facial
Figure 13: An embedding of $K_6$ with a facial walk on five vertices

Figure 14: An embedding of $K_6$ with a facial walk on four vertices
triangle incident with $z_i, z_j, z_k$ if it exists. We should note that specifying the hinges does not uniquely determine the graph $\hat{J}$, because the face $F_0$ has multiple incidences with some vertices. For instance, if $W$ has five vertices, $z_0 = z_6$, and $R_1 = F_0$, then it is not clear whether the split occurs in the “angle” between the edges $z_3z_6$ and $z_4z_6$, or in the angle between $z_5z_6$ and $z_2z_6$. To overcome this ambiguity we will write $R_1 = F_{364}$ in the former case, and $R_1 = F_{265}$ in the latter case. Notice that this is just a notational device; there is no face bounded by $z_3z_6z_5$ or $z_2z_6z_4$. We proceed in a series of claims.

(1) *Not both $R_1$ and $R_2$ are bounded by triangles.*

To prove (1) suppose for a contradiction that $R_1$ and $R_2$ are both facial triangles. Let us recall that $z_0$ is the vertex of $G_{v_1v_2}$ that results from the identification of $v_1$ and $v_2$. Suppose first that $R_1$ and $R_2$ are consecutive in the cyclic order around $z_0$. Then $v_0$ and one of $v_1$ or $v_2$ is in the interior of a 4-cycle in $G_0$, contrary to Lemma 3.2.1. Similarly, if the cyclic order around $z_0$ has $R_1$ followed by a facial triangle, followed by $R_2$, then there would be two vertices in the interior of a 5-cycle in $G_0$, contrary to Lemma 3.2.1. In addition, if the cyclic order has $R_1$, followed by two facial triangles, followed by $R_2$, then there are two vertices inside a 6-cycle. Hence, we are in either case (ii) or (iii) of Lemma 3.2.1. However, the boundary has five vertices that form a clique. So 5-color all but the interior of this 6-walk (using that $G_0$ is 5-critical); the boundary must have five colors, contrary to Lemma 3.2.1. We conclude that $R_1$ and $R_2$ must have $F_0$ in between them in the cyclic order around $z_0$, on both sides. In particular, $W$ has five vertices.

Thus the only case remaining is that $z_0 = z_6$, where $J$ is embedded with a facial 6-walk on five vertices. Suppose without loss of generality that $R_1 = F_{126}$ and $R_2 = F_{456}$, and that $v_2$ is adjacent to $z_1, z_3$ and $z_4$. Then the faces of the subgraph induced by $v_1, v_2, z_1, z_2, z_3, z_4, z_5$ are all triangles but perhaps for two six-cycles: $v_1, z_2, z_1, v_2, z_4, z_5$. 

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and $v_1, z_5, z_3, v_2, z_4, z_2$. Since $v_0$ is adjacent to $v_1$ and $v_2$ it follows from Lemma 3.2.1 that the only vertex in $G_0$ in the interior of the first six-cycle is $v_0$. Hence there must be at least two vertices in the interior of the other six-cycle, else $|V(G_0)| \leq 9$, a contradiction. Thus we are in either case (ii) or (iii) of Lemma 3.2.1. Note that the disk bounded by the second cycle includes no chord. So $v_1$ is not adjacent to $z_3$. Now if $v_1$ is not adjacent to $z_1$, we color $G_0$ as follows. Let the color of $z_i$ be $i$. Color $v_1$ with color 1. Then color $v_0$ and $v_2$, and extend the coloring to the interior of the second six-cycle by Lemma 3.2.1. Hence we may assume that $v_1$ is adjacent to $z_1$. But then $v_0$ is adjacent to $z_1, z_4, z_5$ while $v_1$ is not adjacent to $z_4$. Now $v_1$ may be colored either 3 or 4. One of these options extends to the interior of the second six-cycle after we color $v_1, v_0, v_2$ in that order. This proves claim (1).

In light of (1) we may assume that $R_1 = F_0$.

(2) If $R_2$ is bounded by a triangle, then it is not consecutive with $F_0$ in the cyclic order around $z_0$ in $J$.

To prove (2) suppose for a contradiction that $R_2$ is bounded by a triangle and that it is consecutive with $F_0$ in the cyclic order around $z_0$ in $J$. It follows that one of $v_1, v_2$ has degree two in $\hat{J}$, and so we may assume that it is $v_1$ and that its neighbors are $v_0$ and $z_j$. Consider the subgraph $\hat{J} \setminus \{v_0, v_1\}$. All of its faces are triangles but for a 7-walk. We 5-color this subgraph, which is isomorphic to $K_6$ minus an edge. Thus $v_2$ must receive the same color as $z_j$. Since this subgraph only has six vertices, the interior of the 7-walk must be as in case (v) or (vi) of Lemma 3.2.1, for otherwise there would be at most nine vertices in $G_0$, contrary to Lemma 3.4.1. Consider the edge $z_0 z_j$ in $J$, which must be on the boundary of $F_0$. Now the vertex or vertices not on the boundary of $F_0$ must be on the boundary of $R_2$, for otherwise the 7-walk would only have four colors and we could extend the 5-coloring to its interior, a contradiction. Since $R_2$ is a facial triangle this means that either $z_0$ or $z_j$ is $z_6$ and that $W$ has five
vertices. However, then the color of $z_0$ and $z_j$ appears three times on the boundary of the 7-walk. So the 5-coloring may also be extended, a contradiction. This proves (2).

By an s-vertex we mean a vertex $s \in V(G_0)$ of degree five such that $N_{G_0}(s)$ has a subgraph isomorphic to $K_5 - P_3$. If $G_0$ has an s-vertex, then the optimality of $(G_0, v_0)$ implies that $N_{G_0}(v_0)$ does not include two disjoint pairs of non-adjacent vertices.

(3) Let $R_2$ be bounded by a triangle; then $\hat{R}_2$ is bounded by a pentagon, say $v_0v_1r_1r_2v_2$.

Assume further that $G_0$ has an s-vertex. Then either

(a) $\hat{R}_2$ includes a unique vertex $v$ of $G$, and $v$ is adjacent to $v_0, r_1, r_2$ and all neighbors of $v_0$ other than $v$, or

(b) $v_0$ is adjacent to $r_1, r_2$, and $r_1, r_2$ are adjacent to the neighbor of $v_0$ other than $v_1, v_2, r_1, r_2$, or

(c) $v_0, v_1, v_2$ are all adjacent to $r_i$ for some $i \in \{1, 2\}$, and $r_i$ is adjacent to the two neighbors of $v_0$ other than $v_1, v_2, r_i$.

To prove (3) we first notice that $\hat{R}_2$ includes at most one vertex of $G_0$ by Lemma 3.2.1. If it includes exactly one vertex, then (a) holds by the existence of an s-vertex, and the optimality of $(G_0, v_0)$. If $\hat{R}_2$ includes no vertex of $G_0$, then by Lemma 3.4.3 either $v_0$ is adjacent to both $r_1$ and $r_2$, or $v_0, v_1, v_2$ are all adjacent to $r_i$ for some $i \in \{1, 2\}$. We deduce from the existence of an s-vertex and the optimality of $(G_0, v_0)$ that either (b) or (c) holds. This proves (3).

(4) The walk $W$ has five vertices.

To prove (4) we suppose for a contradiction that $W$ has four vertices. Suppose first that $z_0 = z_2$. Then by (2) and the symmetry we may assume that $R_2 = F_{125}$. It follows that $z_3$ is an s-vertex, and so we may apply (3). But (a) does not hold, because
in that case \( v_0 \) has four neighbors in \( \hat{R}_1 \) or on its boundary, and not all of them can be adjacent to the neighbor of \( v_0 \) in \( \hat{R}_2 \). If (b) holds, then \( v_0 \) is adjacent to \( z_1 \) and \( z_5 \), and \( v \) is adjacent to \( z_1 \), where \( v \) is the neighbor of \( v_0 \) other than \( v_1, v_2, z_1, z_5 \). Now \( v \neq z_5 \), because otherwise both \( \hat{R}_1 \) and \( \hat{R}_2 \) include an edge joining \( v_0 \) and \( z_5 \), contrary to the fact that \( G_0 \) is simple. Since \( v \) is adjacent to \( z_1 \) we deduce that \( v = z_4 \) or \( v = z_6 \). In either case Lemma 3.4.3 implies that \( v_1 \) or \( v_2 \) has degree at most four, a contradiction.

Thus we may assume that (c) holds, and so \( v_0, v_1, v_2 \) are all adjacent to \( z_1 \) or \( z_5 \). In the former case we can change notation so that \( R_2 = F_{126} \), contrary to (2). Thus \( v_0, v_1, v_2 \) are all adjacent to \( z_5 \). Let \( v_1 \) be adjacent to \( z_3, z_4, z_5 \); then \( v_2 \) is adjacent to \( z_1, z_3, z_6 \). Let the vertices \( v_2, z_5, v_1, v_4, v_5 \) form the wheel neighborhood of \( v_0 \), in order. Since an s-vertex exists, the optimality of \( (G_0, v_0) \) implies that either \( v_1 \) is adjacent to \( v_5 \), or \( v_2 \) is adjacent to \( v_4 \), or both. We may assume from the symmetry that \( v_1 \) is adjacent to \( v_5 \). Since \( v_5 \) is adjacent to \( z_5 \) by (c), we deduce that \( v_5 = z_4 \) or \( v_5 = z_6 \), because \( v_5 \neq z_5 \) for the same reason as above. If \( v_5 = z_6 \), then \( v_2 z_6 \) and \( v_2 v_5 \) are the same edge, and it follows from Lemma 3.4.3 that \( v_2 \) has degree at most four. Thus \( v_5 = z_4 \). It follows that \( v_2 \) is adjacent to \( z_4 \), and hence the neighborhood of \( z_1 \) has a subgraph isomorphic to \( K_5^- \), contrary to Lemma 3.4.2. This completes the case \( z_0 = z_2 \).

Thus by symmetry we may assume that \( z_0 = z_4 \). Again by symmetry we may assume that \( R_1 = F_{246} \) and \( R_2 \) is either \( F_{134} \) or \( F_{145} \). Assume first that \( R_2 = F_{145} \). Let \( v_1 \) be adjacent to \( z_1, z_2, z_3 \). Then \( z_3 \) is an s-vertex, and so we may use (3). If (a) holds, and \( v \) is as in (a), then it is not possible for \( v \) to be adjacent to all neighbors of \( v_0 \) other than \( v \), a contradiction. If (b) holds, then \( v_2 \) is not adjacent to \( z_1 \), and hence \( v_1 \) is adjacent to \( z_5 \), by the optimality of \( (G_0, v_0) \), because an s-vertex exists. Thus the neighborhood of \( z_3 \) in \( G_0 \) has a subgraph isomorphic to \( K_5^- \), contrary to Lemma 3.4.2. Thus (c) holds. If \( v_0, v_1, v_2 \) are adjacent to \( z_5 \), then \( N_{G_0}(z_3) \) has a
subgraph isomorphic to $K_{5}^{−}$, contrary to Lemma 3.4.2. Hence $v_{0}, v_{1}, v_{2}$ are adjacent to $z_{1}$. By (c) the vertex $z_{1}$ is adjacent to $v_{4}, v_{5}$, the two neighbors of $v_{0}$ other than $v_{1}, v_{2}, z_{1}$. It follows that $\{v_{4}, v_{5}\} \subseteq \{z_{2}, z_{5}, z_{6}\}$. However, if $v_{0}$ is adjacent to $z_{2}$, then $v_{1}$ would be of degree at most four in $G_{0}$, a contradiction. Thus $v_{0}$ is adjacent to $z_{5}$ and $z_{6}$; hence $v_{1}$ is adjacent to $z_{5}$ by Lemma 3.4.3. Now the graph has eight vertices and perhaps one more inside the 5-cycle $v_{1}z_{2}z_{6}v_{2}z_{5}$. Hence $G_{0}$ has at most nine vertices, contrary to Lemma 3.4.1. This completes the case $R_{2} = F_{145}$.

We may therefore assume that $R_{2} = F_{246}$. From the symmetry we may assume that $v_{1}$ is adjacent to $z_{2}$ and $z_{3}$. If $\hat{R}_{2}$ includes an edge incident with $v_{1}$ or $v_{2}$, then Lemma 3.4.3 implies that $v_{0}, v_{1}, v_{2}$ are all adjacent to $z_{1}$ or $z_{3}$. Then we may change our notation so that either $R_{2} = F_{145}$ or $R_{2} = F_{234}$. In the former case we get a contradiction by the result of the previous paragraph, and in the latter case we get a contradiction by (2). Thus $\hat{R}_{2}$ includes no edge incident with $v_{1}$ or $v_{2}$. Hence either $v_{0}$ is adjacent to $z_{1}$ and $z_{3}$, or $v_{0}$ is adjacent to an internal vertex $v_{3}$ of degree five which is adjacent to $z_{1}$ and $z_{3}$. In either case there is a vertex of degree five in $G_{0}$ adjacent to $v_{1}, z_{3}, z_{1},$ and $v_{2}$. For this vertex, $z_{3}, v_{2}$ is an identifiable pair. Note that $G_{z_{3}v_{2}}$ is not 5-colorable. We 5-color the vertices $z_{1}, z_{2}, v_{2} = z_{3}, z_{5}, z_{6}$ so that each gets a unique color. Then this coloring extends to $G_{z_{3}v_{2}}$, unless we are in case (ii) of Lemma 3.2.1 for the following walk on six vertices: $z_{5}, v_{2} = z_{3}, z_{6}, z_{2}, v_{2} = z_{3}, z_{6}$ in $G_{z_{3}v_{2}}[\{z_{1}, z_{2}, v_{2} = z_{3}, z_{4}, z_{5}, z_{6}\}]$. This implies that there are two adjacent vertices $w_{1}$ and $w_{2}$ such that, in $G_{0}$, $w_{1}$ is adjacent to $z_{2}, z_{6}, v_{2},$ and $z_{5}$, while $w_{2}$ is adjacent to $z_{6}, z_{5}, z_{2},$ and one of $v_{2}, z_{3}$. But then the subgraph induced by the eight vertices: $z_{1}, z_{2}, z_{3}, z_{5}, z_{6}, v_{2}, w_{1}, w_{2}$ has all facial triangles except for perhaps one 5-cycle. Yet there can be at most one vertex in the interior of that 5-cycle. Thus $G_{0}$ has at most nine vertices, a contradiction. This proves (4).

(5) $z_{0} \neq z_{2}, z_{3}$.

We may assume to a contradiction that $z_{0} = z_{2}$ since the case where $z_{0} = z_{3}$ is
symmetric. By (2) \( R_2 = F_{125} \) or \( F_{235} \). Suppose first that some edge of \( G_0 \) is incident with \( v_1 \) or \( v_2 \) and lies inside \( \hat{R}_2 \). Then \( v_0, v_1, \) and \( v_2 \) are all adjacent to \( z_5 \), for otherwise we may change our notation so that \( \hat{R}_2 = F_{126} \), contrary to (2). Let \( v_4 \) and \( v_5 \) be neighbors of \( v_0 \) such that the cyclic order around \( v_0 \) is \( v_1, z_5, v_2, v_5, v_4 \). Now notice that \( z_1 \) is degree five in \( G_0 \) and \( N_{G_0}(z_1) \) has a subgraph isomorphic to \( K_5 - P_3 \). Since \( N_{G_0}(z_0) \) is missing the edge \( v_1v_2 \), one of the edges \( v_1v_5, v_2v_4 \) must be present or \( z_1 \) would contradict the choice of \( v_0 \). This implies that \( v_1 \) and \( v_2 \) are both adjacent to \( v_j \) for some \( j \in \{4, 5\} \). Thus the edges \( v_1v_j, v_2v_j \) must go to a repeated vertex on the boundary of \( R_1 \) or \( v_0 \) would be in a four-cycle in \( G_0 \), a contradiction. Thus \( v_j = z_6 \) and the edge \( v_2z_6 \) is already present. The edge \( v_1z_6 \) then implies that \( z_4 \) is degree five in \( G_0 \) and that \( N_{G_0}(z_4) \) has a subgraph isomorphic to \( K_5^− \), contrary to Lemma 3.4.2. Thus \( \hat{R}_2 \) includes no edge of \( G_0 \) incident with \( v_1 \) or \( v_2 \).

Now suppose that \( R_2 = F_{125} \). We may assume that \( v_1 \) is adjacent to \( z_3, z_4, z_5 \). Then either the cyclic order around \( v_0 \) is \( v_1, z_5, z_1, v_2, v_0 \), and an unspecified vertex \( v_3 \), or \( v_0 \) is adjacent to a vertex \( v_3 \) of degree five with cyclic order: \( v_1, z_5, z_1, v_2, v_0 \). In either case, \( z_1v_1 \) is an identifiable pair for a vertex of degree five in \( G_0 \). Note that \( G_{v_1z_1} \) is not 5-colorable. We 5-color the vertices \( v_1 = z_1, z_3, z_4, z_5, z_6 \) of \( G_{v_1z_1} \) such that each gets a unique color. Since this coloring does not extend to \( G_{v_1z_1} \) we deduce from Lemma 3.2.1 applied to the walk \( z_6, v_1 = z_1, z_4, z_6, z_3, z_5 \) on six vertices that case (i) of that lemma holds. That implies there exists a vertex \( w_1 \) in \( G_0 \) that is adjacent to \( v_1, z_4, z_6, z_3 \) and \( z_5 \). Let \( H := G[\{z_1, z_3, z_4, z_5, z_6, v_1, w_1\}] \). The edge \( w_1z_6 \) may be embedded in two different ways. In one way of embedding the edge the graph \( H \) has all faces bounded by triangles, except for one bounded by a 4-cycle and one bounded by a 5-cycle. But then \( G_0 \) has at most eight vertices by Lemma 3.2.1, contrary to Lemma 3.4.1. It follows that the edge \( w_1z_6 \) is embedded in such a way that all faces of \( H \) are bounded by triangles, except for one face bounded by the walk \( z_6z_1z_5v_1w_1z_5 \) of length six. Since \( G_0 \) has at least ten vertices by Lemma 3.4.1, we must be in case
(iii) of Lemma 3.2.1 when applied to said walk. This can happen in two ways. In the first case there are pairwise adjacent vertices $a, b, c \in V(G_0)$ such that $a$ is adjacent to $z_1, z_5, z_6$, the vertex $b$ is adjacent to $z_5, v_1, w_1$ and $c$ is adjacent to $w_1, z_5, z_6$. Now $G_0$ is isomorphic to $L_4$ by an isomorphism that maps $z_3$ and $z_4$ to the top two vertices in Figure 7(d) (in left-to-right order), $z_6$ and $w_1$ to the vertices in the second row, $z_5$ to the unique vertex of degree nine, and $z_1, a, c, b, v_1$ to the last row of vertices in that figure. In the second case there are pairwise adjacent vertices $a, b, c \in V(G_0)$ such that $a$ is adjacent to $z_1, z_5, v_1$, the vertex $b$ is adjacent to $z_5, v_1, w_1$ and $c$ is adjacent to $z_1, z_5, z_6$. Now $G_0$ is isomorphic to $L_3$ by an isomorphism that maps the top row of vertices in Figure 7(c) to $z_6, z_3, z_4, w_1$ (again in left-to-right order), the middle row to $c, z_1, z_5, v_1, b$ and the bottom vertex to $a$. Since either case leads to a contradiction, this completes the case $R_2 = F_{125}$.

It follows that $R_2 = F_{235}$. We may assume that $v_2$ is adjacent to $z_1, z_5, z_6$. Then either the cyclic order around $v_0$ is $v_1, z_3, z_5, v_2$, and an unspecified vertex $v_3$, or $v_0$ is adjacent to a vertex $v_3$ of degree five with cyclic order: $v_1, z_3, z_5, v_2, v_0$. Note that $z_1$ is degree five in $G_0$ and $N_{G_0}(z_1)$ has a subgraph isomorphic to $K_5 - P_3$. Thus in either case, $v_2v_3$ is an identifiable pair for a vertex of degree five in $G_0$, for otherwise $N_{G_0}(z_1)$ has a subgraph isomorphic to $K_5^-$, a contradiction. Note that $G_{v_2v_3}$ is not 5-colorable. We 5-color the vertices $z_1, v_2 = z_3, z_4, z_5, z_6$ of $G_{v_2v_3}$ such that each gets a unique color. Since this coloring does not extend to $G_{v_2v_3}$, we deduce that the 6-walk $z_6v_2 = z_3z_4z_6v_2 = z_3z_5$ satisfies (ii) of Lemma 3.2.1. Thus, in $G_0$, there exists two adjacent vertices $w_1$ and $w_2$ such that $w_1$ is adjacent to $z_4, z_6, z_3$, and $z_5$, while $w_2$ is adjacent to $z_4, z_5, z_6$ and $v_2$. But then $w_1$ is degree five in $G_0$ and $N_{G_0}(w_1)$ has a subgraph isomorphic to $K_5^-$, a contradiction. This proves (5).

(6) $z_0 \neq z_4, z_5$.

To prove (6) we may assume for a contradiction that $z_0 = z_4$ since the case where $z_0 = z_5$ is symmetric. Thus $R_2 = F_{134}$ or $F_{145}$ by (2). Assume first that $R_2 = F_{145}$,
and that $\hat{R}_2$ includes no edges incident with $v_1$ or $v_2$. Then either the cyclic order around $v_0$ is $v_1, z_1, z_5, v_2$, and an unspecified vertex $v_3$, or $v_0$ is adjacent to a vertex $v_3$ of degree five with cyclic order: $v_1, z_1, z_5, v_2, v_0$. If the edge $v_1z_5$ is present, then in the subgraph of $G_0$ induced by $z_1, z_2, z_3, z_5, z_6$ and $v_2$, there is only one face that is not bounded by a triangle or 4-cycle—the following walk on six vertices: $z_5, z_3z_6z_5z_1v_2$. Thus there are at most nine vertices in $G_0$ by Lemma 3.2.1, contrary to Lemma 3.4.1. Hence, in either case $v_1z_5$ is an identifiable pair for a vertex of degree five in $G_0$. Note that $G_{v_1z_5}$ is not 5-colorable. We 5-color the vertices $z_1, z_2, z_3, v_1 = z_5, z_6$ of $G_{v_1z_5}$ such that each gets a unique color. Since this 5-coloring does not extend to a 5-coloring of $G_{v_1z_5}$ we deduce that case (ii) of Lemma 3.2.1 holds for the following walk on six vertices: $z_6, z_2, v_1 = z_5, z_6, z_3, v_1 = z_5$. Thus, in $G_0$, there are two adjacent vertices $w_1$ and $w_2$ such that $w_1$ is adjacent to $z_2, z_6, z_5$, and $z_3$, while $w_2$ is adjacent to $z_2, z_6, z_3$ and $v_1$. But then $w_1$ is degree five in $G_0$ and $N_{G_0}(w_1)$ has a subgraph isomorphic to $K_5^-$, contrary to Lemma 3.4.2. This completes the case when $R_2 = F_{145}$ and $\hat{R}_2$ includes no edges incident with $v_1$ or $v_2$.

For the next case assume that $R_2 = F_{134}$, and again that $\hat{R}_2$ includes no edges incident with $v_1$ or $v_2$. Then either the cyclic order around $v_0$ is $v_1, z_3, z_1, v_2$, and an unspecified vertex $v_3$, or $v_0$ is adjacent to a vertex $v_3$ of degree five with cyclic order: $v_1, z_3, z_1, v_2, v_0$. Next we dispose of the case that $v_2$ is adjacent to $z_3$. In that case we consider the subgraph of $G_0$ induced by $z_1, z_2, z_3, z_5, z_6$ and $v_2$. There is only one face that is not bounded by a triangle or 4-cycle—the following walk on seven vertices: $z_5z_3v_2z_1z_3z_2z_6$. We 5-color the subgraph as follows: $c(z_i) = i$ for $i = 1, 2, 3, 5$, $c(z_6) = 4$, and $c(v_2) = 2$ and apply Lemma 3.2.1. By Lemma 3.4.1 cases (v) or (vi) of Lemma 3.2.1 hold. Since $z_2$ and $v_2$ have the same color and $z_3$ is a repeated vertex it follows from Lemma 3.2.1 that $G_0$ has four vertices $a, b, c, d$ such that $d$ is adjacent to $z_2, z_3, z_5, z_6$, the vertices $a, b, c$ form a triangle and either $a$ is adjacent to $z_1, v_2, z_3$, the vertex $b$ is adjacent to $z_1, z_2, z_3$, and $c$ is adjacent to $z_2, z_3, d$ (case (v) of
Lemma 3.2.1), or \(a\) is adjacent to \(z_1, v_2, z_3\), the vertex \(b\) is adjacent to \(v_2, z_3, d,\) and \(c\) is adjacent to \(z_2, z_3, d\) (case (vi) of Lemma 3.2.1). In the former case \(d\) is an \(s\)-vertex, and yet \(v_0 = a, c\) is not adjacent to \(z_1\) and \(b\) is not adjacent to \(v_2\), contrary to the optimality of \((G_0, v_0)\). In the latter case \(G_0\) is isomorphic to \(L_3\) by a mapping that sends the top row of vertices in Figure 7(c) to \(z_1, z_6, z_5, z_2\) (in left-to-right order), the middle row to \(a, v_2, z_3, d, c\) and the bottom vertex to \(b\), a contradiction. Thus \(v_2\) is not adjacent to \(z_3\), and hence \(v_2 z_3\) is an identifiable pair for a vertex of degree five in \(G_0\). Note that \(G_{v_2 z_3}\) is not 5-colorable. We 5-color the vertices \(z_1, z_2, v_2 = z_3,\) \(z_5, z_6\) of \(G_{v_2 z_3}\) such that each gets a unique color. Since this coloring not extend to \(G_{v_2 z_3}\) we deduce that case (ii) of Lemma 3.2.1 holds for the following 6-walk: \(v_0, v_1, v_2\). However this would imply that there are two internal vertices \(w_1\) and \(w_2\), both adjacent to \(z_2\) and both adjacent to \(z_5\). But then one of them is not adjacent to \(z_3 = v_2\), a contradiction. This completes both cases when \(\tilde{R}_2\) includes no edges incident with \(v_1\) or \(v_2\).

Thus in the proof of (6) we have shown that \(\tilde{R}_2\) includes an edge incident with \(v_1\) or \(v_2\). Then \(v_0, v_1, v_2\) are all adjacent to \(z_1, z_3\) or \(z_5\). However, if they are all adjacent to \(z_3\), then we can change notation so that \(R_2 = F_{234}\), contrary to (2), and if they are all adjacent to \(z_5\), then we can change notation so that \(R_2 = F_{456}\), again contrary to (2). Thus \(v_0, v_1, v_2\) are all adjacent to \(z_1\). We may assume that the notation is chosen so that \(v_1\) is adjacent to \(z_2\) and \(z_3\) while \(v_2\) is adjacent to \(z_5\) and \(z_6\). Let \(v_4\) and \(v_5\) be neighbors of \(v_0\) numbered so that the cyclic order around \(v_0\) is \(v_2, z_1, v_1, v_4, v_5\).

Next we claim that \(v_1\) is not adjacent to \(z_6\). Suppose it were. The triangle \(z_2 v_1 z_6\) is null-homotopic in \(G_0\) by Lemma 3.2.1 applied to the 4-cycle \(z_1 z_5 z_6 v_1\). Now consider the subgraph induced by the vertices \(z_1, z_2, z_3, z_5, z_6,\) and \(v_1\). All of its faces are triangles but for the 7-walk \(z_1 z_5 z_6 z_3 z_5 z_6 v_1\). We 5-color these vertices as follows: \(c(z_i) = i\) for \(i = 1, 3, 5, c(z_6) = 4,\) and \(c(v_1) = 5\). Now we must be in case (v) or (vi) of Lemma 3.2.1, for otherwise \(|V(G_0)| \leq 9,\) contrary to Lemma 3.4.1. Yet, since the
fifth color would appear three times on the boundary, we can extend this coloring to all of $G_0$, a contradiction. Thus $v_1$ is not adjacent to $z_6$.

Now we claim that $v_4, v_5 \not\in \{z_1, z_2, \ldots, z_6\}$. To prove this claim we suppose the contrary. Then $v_0$ is adjacent to $z_2, z_3, z_5$ or $z_6$. If $v_0$ is adjacent to $z_2$, then $v_1$ has degree at most four in $G_0$. If $v_0$ is adjacent to $z_6$, then either $v_2$ is degree four in $G_0$, a contradiction, or $v_1$ is adjacent to $z_6$, a contrary to the previous paragraph. If $v_4 = z_3$, then the 5-cycle $v_1z_3z_6z_5z_1$ has the vertices $v_0$ and $v_2$ in its interior, contrary to Lemma 3.2.1. Let us assume that $v_5 = z_3$. Then $v_2$ is degree five and $N(v_2)$ is missing at most the edges $v_0z_5$ and $v_0z_6$. Yet these edges must not be present, for otherwise $N(v_2)$ has a subgraph isomorphic to $K_5 - e$, contrary to Lemma 3.4.2. Hence $v_4 \not\in \{z_1, z_2, \ldots, z_6\}$, but then it is not adjacent to $z_1$. Thus $N_{G_0}(v_0)$ includes two disjoint edges. However, $N_{G_0}(v_2)$ has a subgraph isomorphic to $K_5 - P_3$, contradicting the optimality of $(G, v_0)$. Thus we may assume that $v_0$ is adjacent to $z_5$. This implies, by Lemma 3.4.3, that $v_4 = z_5$, because $v_2$ is already adjacent to $z_5$ and $v_5 = z_5$ would imply the existence of another edge from $v_2$ to $z_5$, not homotopic to the existing one.

Then the subgraph of $G_0$ induced by $z_1, z_2, z_3, z_5, z_6, v_1$ has only one face—that can have vertices in its interior. But then there are at most nine vertices in $G_0$ by Lemma 3.2.1, contrary to Lemma 3.4.1. This proves our claim that $v_4, v_5 \not\in \{z_1, z_2, \ldots, z_6\}$.

Continuing with the proof of (6), we note that $v_2$ is not adjacent to $v_4$, for otherwise $v_5$ is of degree four in $G_0$, a contradiction. Similarly $v_1$ is not adjacent to $v_5$. Since $z_1$ is not adjacent to $v_4$ or $v_5$, the neighborhood of $v_0$ in $G_0$ is a cycle of length five. The vertex $v_2$ is not adjacent to $z_2$, for otherwise the 4-cycle $z_2v_2v_0v_1$ includes the vertices $v_4$ and $v_5$ in its interior, contrary to Lemma 3.2.1. Furthermore, the vertex $v_4$ is not adjacent to $z_2$, for otherwise the neighborhood of $v_1$ in $G_0$ has a subgraph isomorphic to a 5-cycle plus one edge, contrary to the optimality of $(G_0, v_0)$. We now consider the graph $G_{v_2,v_4}$. It has a subgraph $H$ isomorphic to $K_6$, and the new vertex $w$ of $H$
obtained by identifying $v_2$ and $v_4$ belongs to $H$. Let $\Delta$ denote the open disk bounded by the walk $z_1 z_5 z_6 z_3 z_5 z_6 z_2 z_3$ of $G_{v_2v_4}$. Since $w$ belongs to $\Delta$, all vertices of $H$ belong to the closure of $\Delta$. However, $z_2 \not\in V(H)$, because $z_2$ is not adjacent to $v_2$ or $v_4$ in $G_0$. Since $v_1$ is not adjacent to $z_6$ as shown two paragraphs ago, we deduce that not both $z_6$ and $v_1$ belong to $H$. That implies that $z_1 \not\in V(H)$, because at most six neighbors of $z_1$ in $G_{v_2v_4}$ (including $z_2 \not\in V(H)$) belong to the closure of $\Delta$. If $v_1 \not\in V(H)$, then no edge incident with one of the two occurrences of $z_3$ on the boundary of $\Delta$ belongs to $H$. Thus regardless of which of $v_1$, $z_0$ does not belong to $H$, there is a planar graph $H'$ obtained from $H$ by splitting at most two vertices, and a drawing of $H'$ in the unit disk with vertices $p, q, r, s$ drawn on the boundary in order such that $H$ is obtained from $H'$ by identifying $p$ with $r$, and $q$ with $s$. It follows that $H$ can be made planar by deleting one vertex, contrary to the fact that it is isomorphic to $K_6$. This proves (6).

Since $R_1 = F_0$ it follows that $z_0 \neq z_1$. Thus $z_0 = z_6$ by (5) and (6).

(7) We may assume that $R_2 \neq F_{136}$ and $R_2 \neq F_{126}$.

To prove (7) we may assume for a contradiction by symmetry that $R_2 = F_{136}$. Then by (2) we have $R_1 = F_{264}$. We may assume that $v_1$ and $v_2$ are numbered so that $v_1$ is adjacent to $z_1$ and $z_2$. We may assume that $\hat{R}_2$ includes no edge incident with $v_1$ or $v_2$; for if it includes the edge $v_2 z_1$, then we can change notation so that $R_2 = F_{126}$, contrary to (2), and if it includes the edge $v_1 z_3$, then we can change notation and reduce to the case when $R_2 = F_0$, which is handled below. Then either the cyclic order around $v_0$ is $v_1, z_1, z_3, v_2$, and an unspecified vertex $v_3$, or $v_0$ is adjacent to a vertex $v_3$ of degree five with cyclic order: $v_1, z_1, z_3, v_2, v_0$. In either case, $z_1, v_2$ is an identifiable pair for a vertex of degree five in $G_0$. Note that $G_{v_2z_1}$ is not 5-colorable. We 5-color the vertices $z_1, z_2, z_3, z_4, z_5$ of $G_{v_2z_1}$ such that each gets a unique color. Since this coloring does not extend to the rest of $G_{v_2z_1}$ we deduce that case (i)
of Lemma 3.2.1 holds for the following 6-walk on five vertices: $z_1 v_2$, $z_2$, $z_4$, $z_1 v_2$, $z_3$, $z_5$. This implies that there exists a vertex $w_1$ in $G_0$ such that $w_1$ is adjacent to $z_2$, $z_4$, $v_2$, $z_3$, and $z_5$ in $G_0$. In the subgraph of $G_0$ induced by those six vertices and $z_1$, all the faces are triangles but for the face bounded by the cycle $z_1 z_3 v_2 z_5 w_1 z_2$. Since $G_0$ must have at least ten vertices, we must be in case (iii) of Lemma 3.2.1. Now 5-color the subgraph induced by those six vertices and $z_4$ such that $c(z_i) = i$ for $i = 1, 2, 3, 5$, $c(w_1) = 1$, and $c(v_2) = 2$. The above-mentioned cycle is colored using four colors, and hence the 5-coloring may be extended to $G_0$, a contradiction. This proves (7).

In light of (7) we may assume that both $R_1$ and $R_2$ are equal to $F_0$. Thus we may assume that $R_1 = F_{264}$ and $R_2 = F_{365}$. We may assume that $v_1$ and $v_2$ are numbered so that $v_1$ is adjacent to $z_1$, $z_2$, and $z_3$. Let the remaining neighbors of $v_0$ be $v_3, v_4, v_5$ numbered so that the cyclic order around $v_0$ is $v_1, v_3, v_2, v_5, v_4$. This specifies the cyclic order uniquely up to reversal, and so we may assume by symmetry that the cyclic order around $v_1$ (of a subset of the neighbors of $v_1$) is $z_1, z_3, v_3, v_0, v_4, z_2$, where possibly $v_3 = z_3$ and $v_2 = v_4$.

(8) *The vertex $v_1$ is not adjacent to $z_4$ or $z_5$.*

To prove (8) we note that $z_1$ has degree five in $G_0$ and that its neighborhood has a subgraph isomorphic to $K_5 - P_3$. If $v_1$ was adjacent to $z_4$ or $z_5$, then the neighborhood of $z_1$ would have a subgraph isomorphic to $K_5 - P_3$, contrary to Lemma 3.4.2 and the optimality of $(G_0, v_0)$. This proves (8).

Since $z_1$ has degree five in $G_0$ and its neighborhood has a subgraph isomorphic to $K_5 - P_3$, we deduce from the optimality of $(G_0, v_0)$ and Lemma 3.4.2 that the neighborhood of $v_0$ is isomorphic to $K_5 - P_3$. It follows that

(9) *the vertex $v_3$ is adjacent to $v_4$ or $v_5$* and
either \( v_1 \) is adjacent to \( v_5 \), or \( v_2 \) is adjacent to \( v_4 \), and not both.

(11) The vertex \( v_2 \) is adjacent to \( v_4 \).

To prove (11) suppose for a contradiction that \( v_2 \) and \( v_4 \) are not adjacent. We will consider \( G_{w_2v_4} \) and its new vertex \( w \) formed by identifying \( v_2 \) and \( v_4 \). Let us note that all faces of the subgraph of \( G_{v_2v_4} \) induced by \( z_1, z_2, z_3, z_4, z_5, v_1, w \) are bounded by triangles except for a face bounded by the 8-walk \( W_1 = v_1wz_5z_3v_1wz_4z_2 \). Let \( D_1 \) be the open disk bounded by \( W_1 \), let \( W_0 = v_1v_4v_5v_2z_5z_3v_1v_3v_2z_4z_2 \) be a corresponding walk in \( G_0 \), and let \( D_0 \) be the open disk bounded by \( W_0 \). By Lemma 3.3.7 the graph \( G_{v_2v_4} \) has a subgraph \( H \) isomorphic to \( K_6 \). Since \( G \) has no \( K_6 \) subgraph it follows that \( w \in V(H) \). If \( z_1 \in V(H) \), then, since \( z_1 \) has degree five in \( G_0 \), all neighbors of \( z_1 \) belong to \( V(H) \), contrary to (8). Thus all vertices of \( H \) belong to \( W_1 \) or \( D_1 \), and by Lemma 3.4.3 each vertex of \( H \setminus w \) (when regarded as a vertex of \( G_0 \)) belongs to \( W_0 \) or \( D_0 \). Assume for a moment that all but possibly one vertex of \( H \) belong to \( W_1 \). Then \( z_4 \) or \( z_5 \) belongs to \( V(H) \), and so \( v_1 \notin V(H) \) by (8). Thus exactly one vertex of \( H \), say \( w_1 \), belongs to \( D_1 \) and \( V(H) = \{w, w_1, z_2, z_3, z_4, z_5\} \). It follows that \( v_1 \notin \{w_1, z_2, z_3, z_4, z_5\} \). Thus \( v_4 \) is not adjacent to \( z_3 \) in \( G_0 \), because the edge \( z_3v_4 \) would have to lie in \( D_0 \), where it would have to cross the path \( z_4w_1z_5 \). But \( w \) is adjacent to \( z_3 \) in \( H \), and so \( v_2 \) is adjacent to \( z_3 \) in \( G_0 \). It follows that the the 4-cycle \( v_1v_0v_2z_3 \) is null-homotopic, for otherwise the edge \( v_2z_3 \) and path \( z_2w_1z_5 \) would cross in \( D_0 \). We deduce from Lemma 3.2.1 applied to the 4-cycle \( v_1v_0v_2z_3 \) that \( v_3 = z_3 \).

But \( v_3 \) is adjacent to \( v_4 \) by (9), and yet \( z_3 \) is not adjacent to \( v_4 \), a contradiction. This completes the case when at most one vertex of \( H \) belongs to \( D \).

Thus at least two vertices of \( H \), say \( w_1 \) and \( w_2 \) belong to of \( D_1 \). Since \( W_1 \) has exactly two repeated vertices, the argument used at the end of the proof of (6) shows that \( w_1 \) and \( w_2 \) are the only two vertices of \( H \) in \( D_1 \). Also, it follows that \( w, v_1 \), the two repeated vertices of \( W_0 \), belong to \( H \). Since \( v_1 \) is in \( H \), (8) implies that \( z_4, z_5 \notin V(H) \).
It follows that $z_2, z_3 \in V(H)$, and consequently $v_4 \not\in \{z_2, z_3\}$. Thus each of $w_1, w_2$ is adjacent in $G_0$ to $v_1, z_2, z_3$ and to $v_2$ or $v_4$. It follows from considering the drawing of $G_0$ inside $D_0$ that one of $w_1, w_2$, say $w_1$, is adjacent to $v_2$ and the 4-cycle $v_1v_0v_2w_1$ is null-homotopic. By Lemma 3.2.1 applied to this 4-cycle we deduce that $w_1 = v_3$. Thus the edge $v_3v_4$ belongs to $D_0$. But $w_2 \not= v_4$, because $v_4$ is not a vertex of $H$, and yet the edge $v_3v_4$ intersects the path $z_3w_2z_2$ inside $D_0$, a contradiction. This proves (11).

(12) *The vertex $v_5$ is adjacent to $v_1$.*

We prove (12) similarly as the previous claim. Suppose for a contradiction that $v_1$ and $v_5$ are not adjacent, and consider $G_{v_1v_5}$ and its new vertex $w$. The subgraph of $G_{v_1v_5}$ induced by $z_1, z_2, z_3, z_4, z_5, w, v_2$ has all faces bounded by triangles except for one bounded by the 8-walk $W_1 = wv_2z_5z_3wv_2z_4z_2$. Let $D_1$ be the open disk bounded by $W_1$, and let $W_0, D_0$ be as in (11). Similarly as in the proof of (11) the graph $G_{v_1v_5}$ has a subgraph $H$ isomorphic to $K_6$ with $w \in V(H)$. We claim that $z_4 \not\in V(H)$. Indeed, if $z_4$ is in $H$, then it is adjacent to $w$ in $H$; but $z_4$ is not adjacent in $G_0$ to $v_1$ by (8), and hence $z_4$ is adjacent to $v_5$ in $G_0$. Yet $v_2$ is adjacent to $v_4$ by (10). Since $v_4 \not\in \{z_4, z_5\}$ by (8), the edges $v_2v_4$ and $z_4v_5$ must cross inside $D_0$, a contradiction. This proves our claim that $z_4 \not\in V(H)$. It follows that $z_1 \not\in V(H)$, because $z_1$ has degree five in $G_{v_1v_5}$, and $z_4$ is one of its neighbors.

If $D_1$ includes at most one vertex of $H$, then $w, v_2, z_2, z_3, z_5 \in V(H)$, and exactly one vertex of $H$, say $w_1$, belongs to $D_1$. Thus $w_1$ is adjacent to $z_2$ and $z_5$ in $G_0$, and that implies that the edges $v_3v_4$ and $v_3v_5$ do not lie in $D_1$. Therefore $v_3, v_4, v_5 \in \{z_2, z_3, z_4, z_5\}$, but that is impossible, given the existence of $w_1$. This completes the case that $D_1$ includes at most one vertex of $H$. Thus, similarly as in (11), it follows that $D_1$ includes exactly two vertices of $H$, say $w_1$ and $w_2$. Now $V(H)$ includes $w, v_2$ and exactly two of $\{z_2, z_3, z_5\}$. But it cannot include $z_5$ and $z_3$, because otherwise for
some \( j \in \{1, 2\} \) the paths \( z_5 w_j v_2 \) and \( z_3 w_{3-j} v_2 \) cross inside \( D_0 \). Thus \( V(H) \) includes \( z_2 \) and \( z_i \) for some \( i \in \{3, 5\} \). Choose \( j \in \{1, 2\} \) such that \( w_j \neq v_3 \). Then the path \( z_2 w_j z_i \) is not disjoint from the edges \( v_3 v_4, v_3 v_5 \) (because they cross inside \( D_0 \)), and so it follows that \( i = 3 \) and \( v_3 = z_3 \). Since there is no crossing in \( D_0 \) and \( w_1 \) and \( w_2 \) are adjacent to \( z_2 \) and \( z_3 \), they are not both adjacent to \( v_5 \). Thus we may assume that \( w_1 \) is adjacent to \( v_1 \). This argument shows, in fact, that the cycle \( v_1 v_0 v_2 w_1 \) is null-homotopic, and so it follows from Lemma 3.2.1 that \( v_3 = w_1 \), a contradiction, because \( w_1 \) lies in \( D_1 \) and \( v_3 = z_3 \) does not. This proves (12).

Now claims (10), (11), and (12) are contradictory. This completes the proof of Theorem 3.4.6.

**Proof of Theorem 3.1.2.** It follows by direct inspection that none of the graphs listed in Theorem 3.1.2 is 5-colorable. Conversely, let \( G_0 \) be a graph drawn in the Klein bottle that is not 5-colorable. We may assume, by taking a subgraph of \( G_0 \), that \( G_0 \) is 6-critical. Then \( G_0 \) has minimum degree at least five. By Lemma 3.2.3 the graph \( G_0 \) has a vertex of degree exactly five, and so we may select a vertex \( v_0 \) of \( G_0 \) such that \((G_0, v_0)\) is an optimal pair. If there is no identifiable pair, then \( G_0 \) has a \( K_6 \) subgraph, as desired. Thus we may select an identifiable pair \( v_1, v_2 \). Let \( G' := G_{v_1 v_2} \). By Lemma 3.3.7 the graph \( G' \) has a subgraph \( H \) isomorphic to \( K_6 \). By Lemma 3.4.4 the drawing of \( H \) is 2-cell, and by Lemma 3.4.5 some face of \( H \) has length six, contrary to Lemma 3.4.6.
CHAPTER IV

AN ANALOGUE TO STEINBERG’S CONJECTURE FOR SURFACES

4.1 Introduction

In this chapter, we will investigate the structure of 3-colorable graphs on fixed surfaces that exclude certain structures. In particular, we will investigate graphs embedded on surface Σ that exclude contractible cycles of length four through ten and show that we can bound the size of $C$-critical graphs (which we will define later) on any fixed surface. The main technique of our proof is discharging and Chapter 4.2 describes this discharging process. With this structure in hand, in Chapter 4.3, we will prove some inductive results that allow us to bound a particular invariant quantity. We state and prove the main theorem in Chapter 4.4.

4.2 The Discharging Process

We will employ a discharging technique similar to that used by Borodin et al [6]. Given a graph $G$, if it contains a reducible configuration then we will reduce the graph to a smaller graph and apply induction to a carefully selected statement. In particular the general argument is as follows: Give each vertex and each face an initial charge. If there are no reducible configurations, then the result follows by Euler’s formula using a discharging argument.

Definition 4.2.1. Let $C$ be the set of facial cycles $\{C_1, C_2, \ldots, C_k\}$ in a graph $G$ drawn in a surface Σ. We say that $G$ is $C$-critical if $G \neq C_1 \cup C_2 \cup \cdots \cup C_k$ and for every subgraph $G'$ of $G$ that includes all the cycles in $C$, some 3-coloring of $C_1 \cup C_2 \cup \cdots \cup C_k$ extends to a 3-coloring of $G'$ but not to a 3-coloring of $G$. 
Let $G$ be a graph drawn in a surface $\Sigma$, and let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a set of facial cycles. If $v$ is a cutvertex, define the initial charge of vertex $v$ to be $d(v)$, where $d(v)$ is the degree of vertex $v$. If $v$ is not a cutvertex, let the initial charge of $v$ be $d(v) - 4$. For a face $f$ in $G$, the length of $f$ is defined to be the sum of the lengths of the walks that form the boundary of $f$. For all faces except the faces bounded by cycles in $\mathcal{C}$, let the initial charge be defined by $|f| - 4$. Define $|f|$ to be the length of face $f$. Give each face bounded by $C_i$ charge $4|V(C_i)|/3 - 2/3$. Denote the initial charge of a vertex or face to be $\text{ch}(v)$ and $\text{ch}(f)$, respectively. Let $\text{ch}'(v)$ and $\text{ch}'(f)$ be the final charge of the specified vertex or face after the discharging rules listed below are applied. The additional charge based upon the lengths of the precolored cycles contributes an additional charge of $\sum_{C_i \in \mathcal{C}} (|V(C_i)|/3 + 10/3)$. The idea of our proof is that for each fixed surface, we can limit the number of graphs which have charge bounded by $-4g + \sum_i (|V(C_i)|/3 + 10/3)$ and use this to show that our invariant is bounded by a function linear in $|\mathcal{C}|$. This implies that the number of $\mathcal{C}$-critical graphs have bounded size.

We say that a face is internal if it is not bounded by a cycle contained in $\mathcal{C}$. A vertex is bad if it has degree 3 and is incident to a 3-face. These rules are a slight modification to those used in Borodin [6] and are defined as follows:

1. Each face of length three receives charge of $\frac{1}{3}$ from each of its incident vertices.

2. Each internal non-triangular face $f$ sends to each incident vertex $v$ charge $\frac{2}{3}$ if $v$ is degree 2 or $v$ is bad. Also, $f$ sends to vertex $v$ charge $\frac{1}{3}$ if $v$ is degree 3 and $v$ is not bad, or $v$ is degree 4 and incident with a 3-face not adjacent to $f$ or is incident with two 3-faces both adjacent to $f$.

3. Each vertex $v \in C_i \in \mathcal{C}$ receives charge $\frac{4}{3}$ from face $C_i$ if the degree of $v$ is two and charge $\frac{2}{3}$ otherwise.

4. For any cutvertex $v$ and every face $f$ incident with $v$, if $f$ is not bounded by a
triangle, then \( v \) sends \( 1/2 \) to \( f \); otherwise let \( e \) be the edge incident with \( f \) but not \( v \), and let \( g \) be the other face incident with \( e \). Then \( v \) sends \( 1/2 \) to \( g \).

Notice that total charge is preserved via these discharging rules. We show that after discharging every vertex has nonnegative charge. In addition we show that every 3-face has nonnegative charge and every other face \( f \) has charge at least \( \frac{1}{3}(|f| - 9) \).

We now prove a series of lemmas that show conclusions of the discharging rules for graphs that adhere to the properties of graphs that are not reducible.

Let \( G \) be a graph drawn in a surface \( \Sigma \), and let \( \mathcal{C} \) be a set of facial cycles of \( G \). A cycle \( C \) of \( G \) is \( \mathcal{C} \)-admissible if either \( |V(C)| = 3 \) or \( |V(C)| \geq 11 \), or both of the following conditions hold:

(i) if \( C \) bounds a closed disk \( \Delta \), then \( \Delta \) includes a member of \( \mathcal{C} \), and

(ii) if \( C \) separates \( \Sigma \) into surfaces \( \Sigma_1, \Sigma_2 \) with boundary \( C \), then there exist \( v_1, v_2 \in V(C) \) such that for \( i = 1, 2 \), the surface \( \Sigma_i \) does not include an edge of \( G \setminus E(C) \) incident with \( v_i \).

We define \( S_{10}(\Sigma) \) to be the set of all pairs \((G, \mathcal{C})\) as above, such that

(i) every vertex of \( G \) of degree at most two belongs to a cycle in \( \mathcal{C} \),

(ii) every cycle in \( G \) is \( \mathcal{C} \)-admissible, and

(iii) if a cycle \( C \) in \( G \) bounds a closed disk that includes no member of \( \mathcal{C} \), then either \( |V(C)| = 3 \) or \( |V(C)| \geq 11 \).

**Lemma 4.2.2.** Let \( G, \mathcal{C}, \Sigma \) be as above, let \( C \) be a \( \mathcal{C} \)-admissible cycle in \( G \), and let \( H \) be a subgraph of \( G \) containing \( C \) and every cycle in \( \mathcal{C} \). Then \( C \) is \( \mathcal{C} \)-admissible in \( H \).

**Proof.** This is apparent as properties (i) - (ii) in the definition of \( \mathcal{C} \)-admissible are preserved under the subgraph operation.

**Lemma 4.2.3.** Let \( \Sigma \) be a surface, let \((G, \mathcal{C}) \in S_{10}(\Sigma)\), let \( G \) be \( \mathcal{C} \)-critical, and let \( C \) be a cycle forming a subset of the boundary of a face of a component of \( G \). Then
\(|V(C)| \geq 11\).

Proof. By definition of \(S_{10}\), it follows that cycle \(C\) is \(C\)-admissible. As a result, if \(4 \leq |V(C)| < 11\), by condition (ii) of the definition of \(C\)-admissible, there exist \(v_1, v_2 \in V(C)\) such that for \(i = 1, 2\), the surface \(\Sigma_i\) does not include an edge of \(G \setminus E(C)\) incident with \(v_i\). Since \(C\) is a subset of the boundary of a face, one of \(v_1, v_2\) must have degree two. This contradicts the assumption that \(G\) is \(C\)-critical.

Lemma 4.2.4. Let \(\Sigma\) be a surface and let \((G, C) \in S_{10}(\Sigma)\). If \(v \in V(G)\), then \(ch'(v) \geq 0\).

Proof. Suppose \(d(v) = 2\). Suppose first that \(v\) is not a cutvertex. Then \(v \in C_i\). Then it receives \(4/3\) units of charge from \(C_i\) and \(2/3\) from its other incident face. Thus \(ch'(v) = -2 + 4/3 + 2/3 = 0\). If \(v\) is a cutvertex, then \(ch'(v) = 2 + 4/3 + 2/3 - 2(1/3) = 10/3 \geq 0\). Suppose \(d(v) = 3\). First suppose that \(v\) is not a cutvertex. If \(v \notin C_i\), then \(v\) receives \(2/3\) from \(C_i\) and \(2/3\) from one (or possibly two) additional non-triangular face. In addition, it may lose charge \(1/3\) if it is part of a \(3\)-face. Observe that \(v\) can be part of only one \(3\)-face else \(G\) contains a \(4\)-cycle. If \(v \notin C_i\), then \(ch'(v) = 3 - 4 + (1/3 \cdot 3) = 0\). If instead \(v\) is a cutvertex, then the above rules apply, but \(v\) also sends \(3(1/3) = 1\) unit of charge from rule 4. So \(ch'(v) \geq 3 + 3(1/3) - 1 = 3 \geq 0\).

Now suppose that \(d(v) = 4\). Suppose that \(v\) is not a cutvertex. First, if \(v \in C_i\) then \(v\) receives \(2/3\) from \(C_i\) and distributes at most \(1/3\) units of charge to two incident \(3\)-faces. Suppose that \(v \notin C_i\). If \(v\) is not incident to a \(3\)-face, then \(ch(v) = ch'(v) = 0\). If \(v\) is incident to one \(3\)-face, then \(v\) loses charge \(1/3\) by rule 1 and gains charge \(1/3\) by rule 2. Similarly, if \(v\) is incident to two \(3\)-faces, then \(v\) loses \(1/3\) units of charge to each of the two incident triangles but gains \(1/3\) units of charge from both its other incident faces. If instead \(v\) is a cutvertex, then aside from rule 4, the charge \(v\) receives and gives is the same. So including rule 4, \(ch'(v) \geq 4 - 4(1/3) = 8/3 \geq 0\).
If \( d(v) \geq 5 \), then if \( v \) is not a cutvertex then \( ch'(v) = d(v) - 4 \geq 0 \). If instead \( v \) is a cutvertex, then \( ch'(v) = d(v) - 1/3(d(v)) \geq 0 \).

\[ \square \]

**Definition 4.2.5.** Given a graph \( G \) and a collection of cycles \( C \), a tetrad is a subgraph isomorphic to the graph in Figure 1 such that \( d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3 \). In particular, where \( xv_1v_2v_3v_4x' \) is contained in a boundary of a face, \( t' \neq x \) and \( t \neq x' \). Also, vertices \( v_1, v_2, v_3 \) and \( v_4 \) are not contained in \( C \).

![Figure 15: A tetrad.](image)

We will show that if graph \( G \) contains a tetrad that does not use any vertices in \( C \), then \( G \) can be reduced to a smaller graph while preserving 3-colorability.

**Proposition 4.2.6.** Let \( G \) be a graph with a tetrad \( T \). Let \( G' \) be the multigraph obtained after deleting \( v_1, v_2, v_3, v_4 \) and identifying \( x \) with \( t \). Call this multigraph \( G' \). Every 3-coloring of \( G' \) extends to a 3-coloring of \( G \).

**Proof.** Let \( \phi \) be a 3-coloring of \( G' \). We will describe how to modify it to produce a 3-coloring of \( G \). In particular, we must color vertices \( v_1, v_2, v_3 \) and \( v_4 \). The coloring \( \phi \) gives rise to a 3-coloring \( \psi \) of \( G \setminus \{v_1, v_2, v_3, v_4\} \) with \( \psi(x) = \psi(t) \) and \( \psi(v) = \phi(v) \) for every \( v \in V(C) \). Since \( v_4 \) is adjacent to \( t \) and \( x' \), it has at least one color available, so we can extend \( \psi \) to \( v_4 \). Now, the only colored vertices \( v_3 \) is adjacent to are \( t \) and \( v_4 \), so it can be colored. If \( \psi(v_3) = \psi(t') \), then let \( \psi(v_2) = \psi(t) \) and color \( v_1 \) with its lone remaining color. If \( \psi(v_3) \neq \psi(t') \) then let \( \psi(v_1) = \psi(v_3) \). As a result, \( v_2 \) has one color available and thus can be colored. \[ \square \]
Define a tetrad to be safe to mean that none of \(x, t, t', x'\) is a cutvertex.

**Lemma 4.2.7.** Let \(\Sigma\) be a surface and let \((G, \mathcal{C}) \in S_{10}(\Sigma)\). Suppose \(G\) contains no safe tetrads. Let \(f\) be a face of \(G\) not in \(\mathcal{C}\). Then if \(|f| = 3\), \(ch'(f) = 0\) and if \(|f| > 3\), then \(ch'(f) \geq \frac{1}{3}(|f| - 9)\).

**Proof.** If \(|f| = 3\), then \(ch'(f) = -1 + 3 \cdot 1/3 = 0\). Now, suppose that \(|f| \neq 3\). By our hypotheses, \(|f| \geq 11\). Observe that if there are five consecutive bad vertices on \(f\), then there exists either a safe tetrad or a cutvertex that sends charge 1/3 into \(f\). However, we are assuming there are no safe tetrads, so for every set of five consecutive vertices, either one of them must be not bad, or one of them must be a cutvertex that sends charge of 1/3 into \(f\). and hence only receives charge at most 1/3 from \(f\).

First suppose there are no cutvertices that send charge into \(f\). Then since there are no safe tetrads, there must be at least three vertices that are not bad. Then \(ch'(f) \geq |f| - 4 - \frac{2}{3}(|f| - 3) + 3 \cdot \frac{1}{3} = \frac{1}{3}(|f| - 9)\). Suppose instead there was one cutvertex. Around this cutvertex, there are still two disjoint sets of five vertices, so there are at least two vertices that are not bad. Then \(ch'(f) \geq |f| - 4 - \frac{2}{3}(|f| - 2) + 2 \cdot \frac{1}{3} + \frac{1}{3} = \frac{1}{3}(|f| - 9)\). If there were two cutvertices, then regardless of how they are placed on a cycle of length at least 11, there still exists a set of five consecutive vertices on the cycle which does not contain a cutvertex. Hence, there is at least one vertex that is not bad. This gives, \(ch'(f) \geq |f| - 4 - \frac{2}{3}(|f| - 1) + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = \frac{1}{3}(|f| - 9)\). Finally, if there are at least three cutvertices that contribute charge to \(f\), then we obtain the lemma, even if all the other vertices in \(f\) were bad. This is because \(ch'(f) \geq |f| - 4 - \frac{2}{3}(|f|) + 3 \cdot \frac{1}{3} = \frac{1}{3}(|f| - 9)\). This completes the proof.

**Lemma 4.2.8.** Let \(\Sigma\) be a surface and let \((G, \mathcal{C}) \in S_{10}(\Sigma)\). Let \(f\) be a face bounded by a cycle in \(\mathcal{C}\). Then \(ch'(f) \geq 0\).

**Proof.** Observe that there is at least one vertex in each \(C_i \in \mathcal{C}\) with degree at least three. As a result, this vertex receives at most charge 2/3 from \(f\). So \(ch'(C_i) \geq 0\).
4/3|V(C_i)| − 2/3 − 4/3(|V(C_i)| − 1) − 2/3 ≥ 0 as desired. \hfill \square

**Lemma 4.2.9.** Let $\Sigma$ be a surface, let $(G, C) \in \mathcal{S}_{10}(\Sigma)$ with Euler genus $g$, and let every face of $G$ be homeomorphic to a disk. Then the sum of the charges of the vertices and faces in $G$ is given by

$$-4\chi(g) + \sum_{i=1}^{k} \left( \frac{10}{3} + \frac{|V(C_i)|}{3} \right) + 4x,$$

where $x$ is the number of cutvertices.

**Proof.** Recall that we give each vertex charge $d(v) - 4$ and each face not in $C$ charge $|f| - 4$. Each face in $C$ is given charge $4/3|V(C)| - 2/3$. By Lemmas 4.2.4, 4.2.7, and 4.2.8, we know that every vertex has nonnegative charge after discharging. In addition, by Lemma 4.2.7, if $|f| > 3$, then the amount of charge face $f$ contains after discharging is at least $\frac{1}{3}(|f| - 9)$. Notice that if there were no faces bounded by cycles in $C$ or cutvertices, our discharging rules give

$$\sum_{v \in V} (d(v) - 4) + \sum_{f \notin C} (|f| - 4) = -4\chi(g).$$

However, instead of giving face $C_i$ charge $|V(C_i)| - 4$, we give it charge $4/3|V(C_i)| - 2/3$, which is $(10/3 + \frac{|V(C_i)|}{3})$ more than the uncolored faces. In addition, we give each cutvertex four more units of charge than vertices that are not cutvertices. As a result, the sum of the charges of the vertices and faces in $G$ is given by $-4\chi(g) + \sum_{i=1}^{k} \left( \frac{10}{3} + \frac{|V(C_i)|}{3} \right) + 4x$, completing the proof. \hfill \square

**Proposition 4.2.10.** Let $\Sigma$ be the sphere, let $(G, C) \in \mathcal{S}_{10}(\Sigma)$ where $C = \{C\}$, suppose that $G$ has no tetrads and that $G$ is 2-connected. Then

$$\sum_{f \in \mathcal{F}(G, C)} (|f| - 9) \leq |V(C)| - 14$$

**Proof.** By Lemma 4.2.9, the sum of the charges (before discharging) is $-8 + (10/3 + \frac{|V(C)|}{3})$, because every face is homeomorphic to a disk and there are no cutvertices. By
our discharging rules, we know that the \( ch'(v) \geq 0 \), \( ch'(t) = 0 \), where \( t \) is a triangle, \( ch'(C) \geq 0 \) and \( ch'(f) \geq \frac{1}{3}(|f| - 9) \) for all other faces \( f \). As a result we have

\[
\frac{1}{3} \sum_{f \in \mathcal{F}(G, C)} (|f| - 9) \leq \sum_{f \in \mathcal{F}(G, C)} ch'(f) \leq \sum_{v} ch'(v) + \sum_{f} ch'(f) - ch'(C) \leq \sum_{v} ch(v) = \frac{|V(C)|}{3} + \frac{10}{3} - 8.
\]

Multiplying the left and right hand sides of the series of inequalities above by three gives

\[
\sum_{f \in \mathcal{F}(G, C)} (|f| - 9) \leq |V(C)| - 14.
\]

\( \square \)

4.3 Inductive Lemmas

Let \( S \) be a cycle of graph \( G \) embedded in a surface \( \Sigma \) where \( S \) bounds an open disk \( \Delta \). Let \( \text{Int}(S) \) be defined as the set of vertices lying inside \( \Delta \) and let \( \text{Out}(S) \) be defined as the set of vertices lying outside the closure of \( \Delta \). For a graph \( G \), let \( \mathcal{F}(G, C) \) be the set of faces \( f \in G \) such that \( |f| > 3 \), \( f \in G \), \( f \neq C \).

**Lemma 4.3.1.** Let \( G \) be a plane graph with no cycles of length 4, 5, \ldots, 10, excluding the induced cycle \( C \) that bounds the infinite face in \( G \), so that \( C = \{C\} \). Suppose that \( G \) is \( C \)-critical. Then

\[
\sum_{f \in \mathcal{F}(G, C)} (|f| - 9) \leq |V(C)| - 14.
\]

**Proof.** We prove this result by induction on \( |V(G)| \). The lemma is vacuously true if \( |V(G)| \leq 3 \), and we may assume that \( |V(G)| \geq 4 \) and that the lemma holds for all graphs on fewer than \( |V(G)| \) vertices.

Now suppose there is a tetrad in \( G \) with its vertices labeled as in Figure 1. We will perform the operation used by Borodin et al [6] to remove the tetrad. In particular, delete \( v_1, v_2, v_3, v_4 \) and identify \( x \) with \( t \) as shown in Figure 1. Call this multigraph \( G' \). By Proposition 4.2.6, every precoloring of \( C \) that extends to \( G' \) also extends to \( G \).
We must ensure that after the tetrad operation is completed that \((G', \{C\}) \in \mathcal{S}_{10}(\mathbb{S}^2)\) and \(G'\) has no loops. In addition we must not identify two vertices of \(C\) nor can we create a chord of \(C\).

We first show that \((G', \{C\}) \in \mathcal{S}_{10}(\mathbb{S}^2)\). Suppose that after \(x\) is identified with \(t\) there is a cycle, \(B'\), such that \(B'\) has length between four and ten. Let \(B\) be the corresponding cycle of \(G\). If \(B\) does not pass through \(t'\), then \(B'\) separates \(G\), and so we may apply induction on the number of vertices of \(G\) to the graph \(G - \text{Out}(B)\) and the graph contained inside cycle \(B\). Since \(B\) separates \(G\), then both \(G - \text{Out}(B)\) and \(G - \text{Int}(B)\) are nonempty so we may apply the lemma to each side side of \(B\) separately. However, it may be that \(B\) passes through vertex \(t'\). In this case, in \(G\) this means there is a path, \(P_1\), of length \(k\) from \(x\) to \(t'\) and a path, \(P_2\) of length \(l\) from \(t'\) to \(t\) such that \(k + l \leq 10\). So the length of \(P_1\) or \(P_2\) is at most five. Suppose, without loss of generality that the length of \(P_2\) is at most five. In this case, \(t'v_2v_3tP_2\) is a cycle of length between four and nine, a contradiction as \((G, \{C\}) \in \mathcal{S}_{10}(\mathbb{S}^2)\). Thus \((G', \{C\}) \in \mathcal{S}_{10}(\mathbb{S}^2)\).

Notice that \(G'\) is in fact a graph. Suppose there was a multiple edge in \(G'\). This means a vertex, \(r\), is adjacent to both \(x\) and \(t\) and so in \(G\) we have a 6-cycle, \(xv_1v_2v_3tx\), a contradiction. If a loop exists in \(G'\), then \(x\) would be adjacent to \(t\) in \(G\) and would construct a 5-cycle, \(xv_1v_2v_3tx\), a contradiction.

We now consider the case when there is a path \(P\), consisting of vertices \(u_1, \ldots, u_k\), in order, of length at most five with both ends on \(C\) not forming a triangle. Let \(C, C_1, C_2\) be the cycles of \(C \cup P\). In order to apply induction on cycle \(C_i\), it must be induced. So suppose that \(C_i\) was not induced. Notice that any chord of \(C_i\) must include at least one vertex of \(P\), else \(C\) is not induced, a contradiction. If a chord does not use \(u_1\) or \(u_k\), then we can reroute \(P\) to make it shorter. So any chords are adjacent to \(u_1\) and \(u_k\) only. Thus, if there exists a chord in \(C_1\) or \(C_2\), then the length of \(P\) is at most three.
First, suppose that the length of $P$ is two. Then $C$ is divided into $d \geq 2$ regions. Let $l_1, l_2, \ldots, l_d$ be the length of the outer cycle bounding each of the $d$, regions, respectively. If region $i$ is facial, then it contributes $(l_i + 2) - 9$ to the sum, and if region $i$ is not facial, then by induction, it contributes $(l_i + 2) - 14$. So a upper bound for this sum is to assume each of these regions are facial. This gives:

$$
\sum_{f \in \mathcal{F}(G,C)} (|f| - 9) \leq \sum_{i=1}^{d} (l_i + 2) - 9 \leq |V(C)| - 14.
$$

Now, suppose the length of $P$ is three. Now, each vertex of $P$ must have degree at least three, and so each internal vertex, $v_1, v_2$ of $P$ is either part of at least two faces of length at least 11, and if $v_i$ is adjacent to only two faces of length at least 11, then it is also adjacent to at least one triangle that has an edge on $C$. Otherwise, $v_i$ is incident to at least three faces of length 11. First suppose that each vertex of $P$ not on $C$ is adjacent to two large faces and a triangle. Then,

$$
\sum_{f \in \mathcal{F}(G,C)} (|f| - 9) \leq \sum_{i=1}^{d} d((l_i + 3) - 9) \leq \sum_{i=1}^{d} (l_i) + 2 - 14 \leq |V(C)| - 14.
$$

If instead there are at least three faces of length 11 inside $C$, then $d \geq 3$ and we obtain

$$
\sum_{f \in \mathcal{F}(G,C)} (|f| - 9) \leq \sum_{i=1}^{d} (l_i + 3) - 9 \leq |V(C)| - 14.
$$

Thus, we may assume that $C_1$ and $C_2$ have no chords. First suppose that neither $C_1$ nor $C_2$ is facial. Then, let $G_i$ be the subgraph of $G$ consisting of vertices and edges drawn in the closed disk bounded by $C_i$. Then by induction

$$
\sum_{f \in \mathcal{F}(G,C)} (|f| - 9) = \sum_{f \in \mathcal{F}(G_1,C)} (|f| - 9) + \sum_{f \in \mathcal{F}(G_2,C)} (|f| - 9)
$$

$$
\leq |V(C_1)| - 14 + |V(C_2)| - 14 = |V(C)| + 2|E(P)| - 28 \leq |V(C)| - 14.
$$

This shows that if $C_1$ and $C_2$ are not facial, that there can be no path of length at most seven between $C_1$ and $C_2$. Notice that we can only apply induction when a cycle is not facial and is induced.
Now, suppose that path $P$ has at least one internal vertex. It follows that at least one of $C_1$ and $C_2$ are not facial as internal vertices have degree at least three. Without loss of generality suppose that $C_1$ is not facial and $C_2$ is facial (as if both were not facial the result follows from the computation above). Now, our computation is

$$\sum_{f \in \mathcal{F}(G,C)} (|f|-9) = \sum_{f \in \mathcal{F}(G_1,C)} (|f|-9) + |V(C_2)| - 9 \leq |V(C_1)| - 14 + |V(C_2)| - 9 = |V(C)| + 2|V(P)| - 1| - 23 \leq |V(C)| - 14.$$

Notice this equation holds only if $|V(P)| \leq 5$, and this means that if there does not exist a path $P$ of length at most four where vertices of $P$ are only adjacent to vertices in the interior of one of $C_1$ and $C_2$.

Thus we can conclude that there is no path $P$ of length at most four with both ends in $C$, and there is no path $P$ with both ends in $C$ of length at most five when both $C_1$ and $C_2$ are not facial cycles.

Now suppose that both vertices $x$ and $t$ from the tetrad reduction were elements of $C$. This contradicts the above claim as the path $xv_1v_2v_3t$ is a path of length four with both ends on $C$. Similarly, suppose that $x \in C$ and $t \notin C$ is adjacent to a vertex, $v$ in $C$. After $x$ is identified with $t$ it may be that $v$ and $x$ have the same color. However, the path $xv_1v_2v_3tv$ is a path of length five with both ends on $C$. This also contradicts the above claim as this path partitions $C$ into two non-facial cycles. Suppose instead that $x \notin C$ is adjacent to a vertex $v \in C$ and $t \in C$. After $x$ is identified with $t$ it may be that $v$ and $t$ have the same color. However, the path $vxv_1v_2v_3t$ is a path of length five with both ends on $C$. This contradicts the above claim as this path partitions $C$ into two non-facial cycles.

Let $G''$ be a minimal subgraph of $G'$ such that any 3-coloring of $C$ that extends to $G''$ also extends to $G'$ and hence to $G$. Then $G'' \neq C$ and $G''$ is $C$-critical. Observe
that for every face \( f'' \) in \( G'' \) there is a corresponding cycle \( D_{f''} \) in \( G \), which may include several faces of \( G \) in its interior and therefore may not be facial. However if we let \( D_{f''} \) act as \( C \) in the statement of this lemma, we can apply the induction hypothesis to \( D_{f''} \) to give a bound for \( \sum_{f \in \mathcal{F}(G', C), f \subseteq D_{f''}} (|f| - 9) \), where \( f \subset D \) means that the face \( f \) is a subset of the open disk bounded by \( D \). We can make an analogous estimate for each face in \( G'' \).

So for \( G'' \), \( \sum_{f \in \mathcal{F}(G'', C)} |f| - 9 \leq |V(C)'| - 14 \) by the induction hypothesis. If there is nothing inside cycle \( D \) in \( G \), the argument is the same as for \( G \), which is described in the next paragraph. If \( D \) is not a facial cycle, then we will apply induction to the subgraph of \( G \) induced by \( D \) and \( \text{Int}(D) \).

Now, we will reindex the sum so that each face in \( G'' \) is accounted for individually. For each one of these faces, call one such face \( D_{f''} \), we can apply induction to this face, noting that in the original graph \( G \), the size of the face may have increased by at most four. So we know that \( |D_{f''}| \leq |f''| + 4 \). So we have

\[
\sum_{f \in \mathcal{F}(G, C)} |f| - 9 = \sum_{f'' \in \mathcal{F}(G'', C)} \left( \sum_{f \in \mathcal{F}(G', C), f \subseteq D_{f''}} |f| - 9 \right) \leq \sum_{f'' \in \mathcal{F}(G'', C)} (|D_{f''}| - 14) \leq \sum_{f'' \in \mathcal{F}(G'', C)} (|f''| - 9) \leq |C| - 14, \tag{2}
\]

where the second inequality of (2) follows from \( |D_{f''}| \leq |f''| + 4 \), and the last inequality of (2) follows by induction applied to \( G'' \), as desired. Finally, suppose there are no tetrads in \( G \). By Proposition 4.2.10, we obtain the desired result, because \( G \) is 2-connected, as is easily seen.

A proposition that will be useful is a minor corollary of Lemma 4.3.1 that gives a stronger bound on the minimum size of the precolored cycle \( C \).

**Proposition 4.3.2.** Let \( G \) be a planar graph with a precolored induced outer cycle \( C \). If \( G \) is \( C \)-critical, then \( |V(C)| \geq 18 \).
Proof. We may assume that $G$ consists of more than just the precolored cycle. In addition, by Lemma 4.3.1, we know that $\sum_{f \in \mathcal{F}(G, C)}(|f| - 9) \leq |V(C)| - 14$. There are at least three faces in $G$, as $G$ consists more than $C$, and at least two of these faces is at least length 11. This implies that the sum on the left in the statement of Lemma 4.3.1 is at least four. As a result, $|V(C)| \geq 18$.

In subsequent lemmas, there are complications that arise if $G''$ is not connected or has connectivity one after a tetrad reduction. One tool to assist this is Proposition 4.3.4, but first we need the following result.

**Proposition 4.3.3.** Let $(G, C)$ be in $S_{15}(\Sigma)$, where $\Sigma$ is the sphere, and let $C = abP_1xyP_2a$ bounding the infinite face. We also assume that $a \not\sim x$, $b \not\sim y$. Here, $P_1$ and $P_2$ are paths, vertices $a, x$ are precolored the same and vertices $b, y$ are precolored the same but using a different color than the color of $a$ and $x$. Then a precoloring of $C$ with these properties extends to $G$.

*Proof.* Suppose that the distance between vertices $b$ and $x$ along $P_1$ is equal to $\alpha$ and the distance between vertices $y$ and $a$ along $P_2$ is equal to $\beta$. If $\alpha > 7$, then draw another path, $P_3$, of length three, that connects $b$ and $x$ and lies in the face bounded by $C$. Notice that adding $P_3$ does not create a cycle of length less than 11 because the minimum distance between $b$ and $x$ is eight and $P_3$ is a path of length three, so any cycle that uses $P_3$ must involve vertices $b$ and $x$ and must have length at least 11. Similarly, if $\beta > 8$, then draw a path $P_4$ of length three that connects $a$ and $y$ which also lies in the face bounded by $C$. Again, adding $P_4$ does not create a new cycle of length less than eleven because any new cycle involves vertices $a$ and $x$, the distance between $a$ and $x$ is at least nine and the length of $P_4$ is three. After these two possible constructions, the infinite face is bounded by a cycle of length at most 17. By Proposition 4.3.2, it follows that $G$ is not $C$-critical and so every precoloring of $G[a, b, x, y]$ extends to $G$. 

\[\square\]
Proposition 4.3.4. Suppose that \((G, C) \in S_{10}(\Sigma)\) where vertices \(v_1\) and \(v_2\) are pre-colored. The coloring extends unless \(v_1\) and \(v_2\) are colored the same and are adjacent.

Proof. Suppose that there is an edge, \(e\) between \(v_1\) and \(v_2\), but that \(v_1\) and \(v_2\) are colored differently. Then we will consider instead the graph \(G'\) which is \(G\) after edge \(e\) was removed. If this graph extends, then so does \(G\).

So now consider the case when \(v_1\) and \(v_2\) are not adjacent. Suppose that \(v_1\) and \(v_2\) are connected by a path \(P\) of length at most five. We will split open the internal vertices of this path to create a cycle \(D\) of size at most ten. Two of the vertices in \(D\) are already precolored, and now we will precolor the rest of \(D\). We can then apply Proposition 4.3.3, where cycle \(D\) is the cycle in the statement of the proposition. Thus we may assume there is no path of length at most five between \(v_1\) and \(v_2\). If there is a tetrad, then we reduce \(G\) using our usual tetrad reduction and decrease the size of \(G\).

If there is no tetrad, then we will again use a discharging argument with a modified discharging rule for the two precolored vertices and obtain a contradiction. The idea is that we can not use these precolored vertices in a tetrad reduction, and as a result these vertices can be additional bad vertices that require \(2/3\) units of charge from large faces incident to these vertices. As such, suppose that vertex \(v\) is precolored, has degree three and is part of a 3-face. Then give any face incident to \(v\) and of length at least eleven an additional \(1/3\) unit of charge. Since there are two precolored vertices, we may have added and extra \(4/3\) units of charge. By Lemma 4.2.9, the sum of the charges in \(G\) is at most \(-8 + 10/3 + 4/3 = -10/3\). By our discharging rules, after the discharging process, the final charge of every vertex and face is nonnegative. Since charge is preserved during the discharging process, the sum of the charges must be nonnegative. This is a contradiction.

\(\square\)
If there are more than two precolored cycles, it may be that there is more than one face whose length decreases by eight after a tetrad reduction. The following definitions and proposition is useful to better describe this situation.

**Definition 4.3.5.** Given a face \( f \), define an \( f \)-circle as a set \( X \subset f \cup \{v\} \) for some vertex \( v \) incident with \( f \) such that \( X \) is homeomorphic to a unit circle and null-homotopic. Let \( w_2(f) \) be the maximum number of \( f \)-circles such that they are pairwise non-homotopic after we remove the faces bounded by cycles in \( C \) and pairwise share no points of \( f \).

**Proposition 4.3.6.** Suppose that \( G \) is a graph drawn in surface \( \Sigma \) that contains \( k \geq 2 \) disjoint closed disks \( \Delta_1, \Delta_2, \ldots, \Delta_k \). Let \( \tilde{\Sigma} = \Sigma - (\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k) \). Then the number of closed curves in \( \tilde{\Sigma} \) that are pairwise not homotopic in \( \tilde{\Sigma} \) and each is null homotopic in \( \Sigma \) is \( 2k - 3 \) if \( \Sigma \) is the sphere and \( 2k - 1 \) otherwise.

**Proof.** We prove this via induction on the number of precolored cycles. First suppose that \( \Sigma \) is the sphere. Let \( f_n \) denote the maximum number of circles when there are \( n \) precolored cycles. Notice that \( f_1 = 0 \), \( f_2 = 1 \) and \( f_3 = 3 \).

We claim that \( f_{n_1+n_2} = f_{n_1+1} + f_{n_2+1} - 1 \). To see this notice that given a configuration of circles around a set of precolored cycles, we could cut open one circle to get two groups of precolored cycles, one containing \( n_1 + 1 \) precolored cycles and one with \( n_2 + 1 \) precolored cycles. However, we double-count the cycle that was cut open as it is part of the count in \( f_{n_1} \) and \( f_{n_2} \) and so this is why one is subtracted from the right hand side of the claim.

To prove the proposition, we apply induction. First notice that \( f_3 = 3 \) satisfies the \( 2k - 3 \) bound. Now, suppose that \( f_m \leq 2m + 3 \) for \( 3 \leq m \leq k \). Let \( k = m_1 + m_2 \)

So \( f_{m_1+m_2} = 2(m_1 + 1) - 3 + 2(m_2 + 1) - 3 - 1 = 2(m_1 + m_2) - 3 \), as desired.

If \( \Sigma \) is not the sphere, the argument is identical, except that the base cases are \( f_1 = 1 \), \( f_2 = 3 \) and \( f_3 = 5 \).

\( \square \)
Let $\Sigma$ be a surface, and let $(G, C) \in S_{10}(\Sigma)$. By a $\theta$-graph in $G$, we mean a subgraph of $G$ of the form $P_1 \cup P_2 \cup P_3$, where $P_1, P_2, P_3$ are three internally disjoint paths with the same ends. We say that a $\theta$-graph $H$ is triseparating if each face of $H$ that is homeomorphic to a disk includes a member of $C$.

**Lemma 4.3.7.** Let $\Sigma$ be a surface, let $(G, C) \in S_{10}(\Sigma)$, and let $G$ have a tetrad with vertices labeled as in Figure 1. Let $G_1$ be obtained from $G \setminus \{v_1, v_2, v_3, v_4\}$ by identifying $x$ and $t$ and let $G_2$ be obtained from $G \setminus \{v_1, v_2, v_3, v_4\}$ by identifying $x'$ and $t'$. If there is no triseparating $\theta$-graph on at most 29 edges, then there is $i \in \{1, 2\}$ such that $G_i$ is loopless and every cycle in $G_i$ is $C$-admissible.

**Proof.** Suppose the lemma does not hold. Then in each of $G_1$ and $G_2$, there exists a cycle that violates looplessness or $C$-admissibility. This Let $P_1$ be the path from $x$ to $t$ in $G$ that will be identified to make a cycle, $D_1$, of length at most ten in $G_1$. Similarly, let $P_2$ be the path from $t'$ to $x'$ that will be identified to make a cycle, $D_2$, of length at most ten in $G_2$. Notice that $P_1$ and $P_2$ must share a vertex. Call this vertex $v$. Using $P_1, P_2$ and the tetrad, $G$ is partitioned into four non-triangular regions. Call the region that includes the unbounded face $R_4$. Call the other three disks $R_1, R_2, R_3$, respectively such that $x$ and $t'$ are on the boundary of $R_1$, $t'$ and $t$ are on the boundary of $R_2$ and $x'$ are on the boundary of $R_3$.

By hypothesis, there is no triseparating graph on 29 edges. In addition, there must be a cycle of $C$ inside and outside $D_1$ and $D_2$. As a result, the two disks amongst $R_1, R_2, R_3, R_4$ that contain no member of $C$ are either $\{R_1, R_3\}$ or $\{R_2, R_4\}$. The sum of the lengths of the boundaries of these disks are at most $10 + 10 + 3 + 5 = 28$. But each of $D_1$ and $D_2$ must be length at least 11, so both $D_1$ and $D_2$ have length at most 17. As a result, by Proposition 4.3.2, both these disks have no vertices in their interior. But then we can find vertices $v_1, v_2$ along those disks that satisfy assumption (ii) of the definition of $C$ admissible for the other two $R_i$’s. \hfill $\square$
Proposition 4.3.8. Suppose that \( G \) is a graph drawn in surface \( \Sigma \) that contains \( k \geq 2 \) disjoint closed disks \( \Delta_1, \Delta_2, \ldots, \Delta_k \). Let \( \tilde{\Sigma} = \Sigma - (\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k) \). Then in a \( \mathcal{C} \)-critical graph, the number of closed curves in \( \tilde{\Sigma} \) in each homotopy class that hit \( G \) exactly once is at most two.

Proof. Suppose there are two closed curves \( C_1, C_2 \) in \( \tilde{\Sigma} \) that are homotopic in \( \tilde{\Sigma} \). Look at the graph \( H \) contained in the region between curves \( C_1, C_2 \). This is an annular region. Let \( G' = G \setminus H \). Now, since \( G \) is \( \mathcal{C} \)-critical there exists a proper 3-coloring of \( G' \). Let \( x \) and \( y \) be the vertices of \( G \) that hit curves \( C_1, C_2 \), respectively. By Proposition 4.3.4 it follows that \( x \) and \( y \) must be adjacent, else the 3-coloring of \( G' \) may be extended to \( H \). Therefore if \( x \) and \( y \) were not adjacent or if there were more than two closed curves in this homotopy class, then \( G \) would not be \( \mathcal{C} \)-critical.

Definition 4.3.9. Let \( \alpha, \beta, \gamma \) be constants whose value will be determined later. Let \( G \) be a graph embedded on a surface \( \Sigma \). For each face \( f \) of \( G \), if the boundary of \( f \) is connected, let

\[
w_1(f) = |f| - 9 + \alpha(2 - \chi(f)).
\]

If the boundary of \( f \) is disconnected with \( k \) components, let

\[
w_1(f) = \alpha(2 - \chi(R_f)) + \beta(k) + \sum_{C_i \in R_f} |E(C_i)| - \gamma.
\]

In either case, let

\[
w(f) = w_1(f) + w_2(f).
\]

Proposition 4.3.10. Let \( \Sigma \) be a surface, let \( (G, \mathcal{C}) \in \mathcal{S}_{10}(\Sigma) \). Suppose that every face is homeomorphic to a disk, the cycles in \( \mathcal{C} = \{C_1, C_2, \ldots, C_n\} \) are induced, \( n \geq 2 \), \( G \) is \( \mathcal{C} \)-critical and there are no safe tetrads in \( G \). If \( \alpha \geq 10\beta + 6, \beta \geq 66, 2\gamma \geq 3\beta + 50 \)

then

\[
\sum_{f \in \mathcal{F}(G, \mathcal{C})} w(f) \leq \alpha(2 - \chi(\Sigma)) + \beta|\mathcal{C}| + \left( \sum_{i=1}^{n} |V(C_i)| \right) - \gamma - 4.
\]
Proof. Suppose there are no safe tetrads in $G$. By Lemma 4.2.9, the sum of the charges (before discharging) is $-4\chi(\Sigma) + \sum_{i=1}^{n}(10n/3 + \frac{|V(C_i)|}{3}) + 4w_2(f)$. By our discharging rules, we know that $ch'(v) \geq 0$, $ch'(t) = 0$, where $t$ is a triangle, $ch'(C_i) \geq 0$ and $ch'(f) \geq \frac{1}{3}(|f| - 9)$ for all other faces $f$. As a result, we have

$$\frac{1}{3} \sum_{f \in F(G, C)} w(f) \leq \sum_{f \in F(G, C)} ch'(f) + \frac{4}{3}w_2(f) \leq$$

$$\sum_v ch'(v) + \sum_f ch'(f) + \sum_{i=1}^{n} ch'(C_i) + \frac{4}{3}w_2(f) \leq \sum_f ch(f) + \sum_v ch(v) + \frac{4}{3}w_2(f)$$

$$\leq \left( \sum_{i=1}^{n} \frac{|V(C_i)|}{3} \right) + \frac{10n}{3} - 4\chi(\Sigma) + 4x + \frac{8}{3}n.$$ 

Here $x$ denotes the number of cutvertices of $G$. Notice that $x \leq 4n$ by Proposition 4.3.7 and Proposition 4.3.8. Multiplying the left and right hand sides of the series of inequalities above by three and estimating for $x$ gives

$$\sum_{f \in F(G, C)} w(f) \leq \left( \sum_{i=1}^{n} |V(C_i)| \right) + 10n + 12(2 - \chi(\Sigma)) + 48n + 8n - 24.$$ 

Since $n \geq 2$, $\alpha \geq 10\beta + 6$, $\beta \geq 66$, $2\gamma \geq 3\beta + 50$, it follows that

$$\sum_{f \in F(G, C)} w(f) \leq \alpha(2 - \chi(\Sigma)) + \beta|C| + \left( \sum_{i=1}^{n} |V(C_i)| \right) - \gamma - 4.$$ 

Lemma 4.3.11. Let $\Sigma$ be a surface, let $(G, C) \in S_{10}(\Sigma)$, let $n \geq 2$ and suppose that $G$ is $C$-critical where $C = \{C_1, C_2, C_3, \ldots C_n\}$ and each $C_i$ is induced. If $\alpha, \beta, \gamma$ satisfy $\alpha \geq 10\beta + 6$, $\beta \geq 66$, $2\gamma \geq 3\beta + 50$, then:

$$\sum_{f \in F(G, C)} w(f) \leq \alpha(2 - \chi(\Sigma)) + \beta|C| + \left( \sum_{i=1}^{n} |V(C_i)| \right) - \gamma - 4.$$ 

Proof. As before, we prove this result by induction on $|V(G)|$. The lemma is again vacuously true if $|V(G)| \leq 10$, and so we may assume that $|V(G)| \geq 11$ and that the lemma holds for all graphs on fewer than $|V(G)|$ vertices. First, recall from the
computation in Proposition 4.3.6 that \( \sum w_2(f) \leq 4|C| \). Now, suppose that \( G \) has a non-null-homotopic closed curve that does not use any vertices of \( G \). Then we can reduce the genus of the graph we are analyzing, so the result holds if \( \alpha \geq 1 \). Suppose instead that \( G \) has a non-null-homotopic closed curve, \( D \) that uses one vertex of \( G \), call this vertex \( v \). We will cut open along this curve, thus reducing genus by at least one and split vertex \( v \) into two vertices, \( v_1 \) and \( v_2 \). In addition, we will construct two new cycles of \( C \), call them \( D_1, D_2 \), which includes \( v_1 \) and \( v_2 \), respectively and are each length three. In addition we may have increased \( w_2(f) \) by at most \( 4|C| \).

Construct a new graph \( H \), embedded on the new surface obtained by \( \Sigma_1 \), and observe that \( \chi(\Sigma_1) \geq \chi(\Sigma) + 1 \). So this gives:

\[
\sum_{f \in \mathcal{F}(H, \{C \cup \{D_1, D_2\}\})} w(f) = \alpha(2 - (\chi(\Sigma) + 1)) + \beta(|C| + 2) + 8|C|
\]

\[
+ \left( \sum_{i=1}^{n} |V(C_i)| + 6 \right) - \gamma - 4
\]

\[
\leq \alpha(2 - \chi(\Sigma)) + \beta|C| + \sum_{i=1}^{n} |V(C_i)| - \gamma - 4.
\]

This holds because \( \alpha \geq 10\beta + 6 \).

Now, suppose that \( G \) is disconnected. Thus \( G \) has a face \( f_0 \) of disconnected boundary, and let \( G_1', G_2', \ldots, G_l \) be subgraphs of \( G \) with union \( G \), each incident with one boundary component of \( G \). For \( i = 1, 2, \ldots, l \) let \( f_i' \) be the face of \( G_i' \) containing the rest of \( G \), and let \( G_i \) denote the embedded graph obtained from \( G_i' \) by capping off \( f_i' \) by a disk, resulting in a face \( f_i \). By Lemma 4.2.3, it follows that \( G_i \) satisfies condition (iii) of the definition of \( S_{10}(\Sigma) \). Further, suppose that the graphs \( G_1, \ldots, G_k \) contain at least two precolored cycles, and \( G_{k+1}, \ldots G_l \) contain exactly one precolored cycle. Let \( C_i \) denote the collection of precolored cycles in graph \( G_i \).
We now have by Lemma 4.3.1, and induction,

\[
\sum_{f \in \mathcal{F}(G, C)} w(f) = \sum_{i=1}^{k} \sum_{f \in \mathcal{F}(G_i, C_i)} w(f) + \sum_{i=k+1}^{l} \sum_{f \in \mathcal{F}(G_i, C_i)} w(f) + w(f_0) - \sum_{i=1}^{l} w(f_i)
\]

\[
\leq \sum_{i=1}^{k} \left( \alpha(2 - \chi(\Sigma_i)) + \beta|C_i| + \sum_{C \in C_i} E(C) - \gamma - 4 \right) + \sum_{i=k+1}^{l} (\alpha(2 - \chi(\Sigma_i)) + |E(C_i)| - 13) + \alpha(2 - \chi(\Sigma_0))
\]

\[
+ (l)\beta + \sum_{i=1}^{l} |f_i| + 4w_2(f_0) - \gamma - \sum_{i=1}^{l} w(f_i) - 4 \sum_{i=1}^{k} w_2(f_i)
\]

\[
\leq \alpha(2 - \chi(\Sigma)) + \beta|C| + \sum_{C_i \in C} |E(C_i)| - \gamma - 4.
\]

This completes the proof when \( G \) is disconnected.

Suppose there exists a non-contractible cycle \( D \) of length at most 14. We may assume that \( D \) is induced. Either it is two-sided and separating, two-sided and non-separating or one-sided. First, suppose that \( D \) was two-sided and non-separating. Cut \( D \) from surface \( \Sigma \) and construct a new graph \( H \), embedded on \( \Sigma' \) where \( \chi(\Sigma') \geq \chi(\Sigma) + 1 \) with two copies of \( D \), call them \( D_1 \) and \( D_2 \). Let \( C' = C \cup \{D_1, D_2\} \). Then \( H \) is \( C' \)-critical and

\[
\sum_{f \in \mathcal{F}(H, \{C'\})} w(f) = \alpha(2 - (\chi(\Sigma) + 1)) + \beta(|C| + 2) + \left( \sum_{i=1}^{n} |V(C_i)| + 28 \right) - \gamma - 4
\]

\[
\leq \alpha(2 - \chi(\Sigma)) + \beta|C| + \sum_{i=1}^{n} |V(C_i)| - \gamma - 4,
\]

because \( \alpha \geq 2\beta + 28 \).

Suppose instead that \( D \) was two-sided and separating. After cutting open along \( D \), this produces two separate surfaces \( \Sigma_1 \) and \( \Sigma_2 \) and graphs \( H_1 \) and \( H_2 \) where \( H_1 \) is embedded on \( \Sigma_1 \) and \( H_2 \) is embedded on \( \Sigma_2 \). Graphs \( H_1, H_2 \) each contain a copy of \( D \), called \( D_1, D_2 \), respectively. Here, \( \chi(\Sigma_1) + \chi(\Sigma_2) = \chi(\Sigma) - 2 \). Let \( C_1 \) and \( C_2 \) be the set of original precolored cycles on each surface. Let \( C_1' = C \cup \{D_1\} \), and let
\( C_2' = C \cup \{ D_2 \} \). Then \( H_i \) is \( C_i' \)-critical for \( i \in \{ 1, 2 \} \) and

\[
\sum_{f \in \mathcal{F}(H_1, \{ C_1' \})} w(f) + \sum_{f \in \mathcal{F}(H_2, \{ C_2' \})} w(f) \\
= \alpha (2 - \chi(\Sigma_1)) + \beta(|C| + 1) + \left( \sum_{i=1}^{n} |V(C_i)| + 14 \right) - \gamma - 4 \\
+ \alpha (2 - \chi(\Sigma_2)) + \beta(|C| + 1) + \left( \sum_{i=1}^{n} |V(C_i)| + 14 \right) - \gamma - 4 \\
\leq \alpha (2 - \chi(\Sigma)) + \beta|C| + \sum_{i=1}^{n} |V(C_i)| - \gamma - 4,
\]

because \( 2\alpha \geq 2\beta + 28 - \gamma - 4 \).

Finally, suppose that \( D \) was one-sided. Cutting open \( D \) produces a single cycle of length at most 28, and we will call this cycle \( D_1 \). This produces a graph \( H \) embedded on surface \( \Sigma_1 \) where \( \chi(\Sigma_1) = \chi(\Sigma) + 1 \). Let \( C_1' = C \cup \{ D_1 \} \). Then \( H \) is \( C_1' \)-critical, and we have

\[
\sum_{f \in \mathcal{F}(H, \{ C, D_1 \})} w(f) = \alpha (2 - \chi(\Sigma_1)) + \beta(|C| + 1) + \left( \sum_{i=1}^{n} |V(C_i)| + 10 \right) - \gamma - 4 \\
\leq \alpha (2 - \chi(\Sigma)) + \beta|C| + \sum_{i=1}^{n} |V(C_i)| - \gamma - 4,
\]

because \( \alpha \geq \beta + 10 \).

From these three results, we can assume there are no non-contractible cycles of length at most 14.

Next, suppose that there exists a triseparating \( \theta \)-graph, \( H \) on at most 29 edges. Suppose that \( C_1, C_2, C_3 \) are the cycles of \( H \) and \( C_i \) is the set of precolored cycles in the face of \( C_i \). Let \( H_i \) be the subgraph of \( H \) in face \( i = \{ 1, 2, 3 \} \). Further, suppose that \( H_1, H_2, H_3 \) are embedded on surfaces \( \Sigma_1, \Sigma_2, \Sigma_3 \) so that \( \chi(\Sigma_1) + \chi(\Sigma_2) + \chi(\Sigma_3) \geq \chi(\Sigma) + 4 \). Then
\[
\sum_{f \in \mathcal{F}(H, \{C \cup C_1 \cup C_2 \cup C_3\})} w(f) = \sum_{i=1}^{3} \left( \sum_{f \in \mathcal{F}(H, \{C \cup C_i\})} w(f) \right)
\]

\[
= \sum_{i=1}^{3} (\alpha(2 - \chi(\Sigma_i)) + \beta(|C_i| + 1) + |E(C_i)|) - \gamma - 4)
\]

\[
\leq \alpha(2 - \chi(\Sigma)) + \beta|C| + \sum_{i=1}^{n} |V(C_i)| - \gamma - 4,
\]

because, \(3\beta + 2|E(H)| \leq 3\beta + 58 \leq 2(\gamma + 4)\).

Suppose that there exists a path \(P\) of distance at most six between \(C_i\) and \(C_j\). In this case, let \(P\) be a shortest path between cycles \(C_i\) and \(C_j\) with \(k \leq 5\) vertices in this path not on \(C_i\) and \(C_j\) for \(i \neq j, i, j \in \{1, 2, \ldots, n\}\). Construct a new graph \(H\) that consists of a single precolored cycle containing \(C_i, C_j\) and two copies of \(P\). Let \(x_1\) be the vertex in \(P\) on \(C_i\) and let \(x_2\) be the vertex in \(P\) on \(C_j\). Let \(p_1, \ldots, p_k\) be the vertices of \(P\) starting closest to \(C_1\). The graph \(H\), consists of a single precolored cycle \(C_i^*\) defined by \(x_1p_1 \cdots p_kx_2C_2x_2p_k \cdots p_1x_1C_1\). Edges incident to vertices on \(P, x_1\) and \(x_2\) are now incident to only one copy of \(P\) depending on their orientation in \(G\). We can then apply induction on this lemma and the fact that \(\beta \geq 66\) and \(\sum 4w_2(f) \leq 16|C|\) to get

\[
\sum_{f \in \mathcal{F}(G, \mathcal{C})} w(f) \leq \alpha(2 - \chi(\Sigma)) + \beta(|C| - 1) + 2|V(P)| - 1|
\]

\[
+ \sum_{i=1}^{n} |V(C_i)| + \sum_{i=1}^{n} 4w_2(f) - \gamma - 4
\]

\[
\leq \alpha(2 - \chi(\Sigma)) + \beta|C| + \sum_{i=1}^{n} |V(C_i)| - \gamma - 4.
\]

Again we must show that \((G', \mathcal{C}) \in S_{10}(\Sigma)\), has no loops or multiple edges, and that every coloring of \(\mathcal{C}\) that extends to \(G'\) also extends to \(G\). In addition we must not identify two vertices of \(\mathcal{C}\) nor can we create a chord of \(\mathcal{C}\). As a result of the above usage of the above argument, it follows that the two identified vertices of any tetrad must come from the same \(C_i \in \mathcal{C}\) if they are in \(\mathcal{C}\).
Suppose that there exists a path \( P \) of length at most six (where \( P \) has five vertices not on \( C_i \)) with both ends on the same \( C_i \in \mathcal{C} \). Define \( D_1, D_2 \) to be the cycles in \( C_i \cup P \) other than \( C_i \).

First suppose that \( D \) bounds a disk that includes no member of \( \mathcal{C} \). Let \( G_1 \) be the part of the graph contained in \( R \) including, cycle \( D \) and let \( G_2 \) be the remaining graph. If there are no cycles \( C_j \in \mathcal{C} \) within \( D \), then we have that

\[
\sum_{f \in \mathcal{F}(G, \mathcal{C})} w(f) = \sum_{f \in \mathcal{F}(G_1, \{C_i'' \cup P\})} w(f) + \sum_{f \in \mathcal{F}(G_2, (\mathcal{C} \setminus C_i) \cup (C_i' \cup P))} w(f)
\]

\[
\leq \sum_{i=1}^{n-1} |V(C_j)| + \sum_{j=i+1}^{n} |V(C_j)| + |E(C_i')| + |V(P)| - 1 - \gamma - 4
\]

\[
\leq \alpha(2 - \chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{i=1}^{n} |V(C_i)| - \gamma - 4.
\]

This holds because \(|V(P)| - 1| \leq 7\).

Now, suppose that \( D_1 \) and \( D_2 \) are surface separating. Then \( \Sigma = \Sigma_1' \cup \Sigma_2' \cup \Sigma_i \), where \( f_i \) is the face bounded by \( C_i \) and for \( j = \{1, 2\} \), \( \Sigma_j' \) is a surface with boundary \( D_i \).

Let \( \Sigma_j \) be obtained from \( \Sigma_j' \) by capping off \( D_i \) by a disk. By the result of the previous paragraph, we may assume each \( \Sigma_j \) either has positive genus or includes a member of \( \mathcal{C} \). Suppose that the cycles in \( \mathcal{C} \) contained in \( \Sigma \) are labeled \( \mathcal{E}' = \{E_1, \ldots, E_k\} \). Suppose the cycles in \( \mathcal{C} \) not in \( \Sigma \) are labeled \( \mathcal{E}'' = \{F_1, \ldots, F_l\} \). In this case we have
\[ \sum_{f \in \mathcal{F}(G, \mathcal{C})} w(f) = \sum_{f \in \mathcal{F}(G_1, \mathcal{E}' \cup \mathcal{C}' \cup P)} w(f) + \sum_{f \in \mathcal{F}(G_2, \mathcal{E}'' \cup \mathcal{C}' \cup P)} w(f) \]
\[ \leq \beta(|\mathcal{E}'| + 1) + \sum_{j=1}^{k} (|E(E_j)|) + |E(C_j')| + |V(P)| - 1 - \gamma - 4 \]
\[ + \beta(|\mathcal{E}''| + 1) + \sum_{j=1}^{l} |V(C_j)| + |E(C_i'')| + |V(P)| - 1 - \gamma - 4 \]
\[ + \alpha(2 - \chi(\Sigma)) \]
\[ = \alpha(2 - \chi(\Sigma)) + \beta(|\mathcal{C}| + 1) + \beta(|\mathcal{E}''| + 1) \]
\[ + \sum_{j=1}^{n} |V(C_j)| + 2(|V(P)| - 1) - 2\gamma - 8 \]
\[ \leq \alpha(2 - \chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{j=1}^{n} |V(C_j)| - \gamma - 4. \]

This holds because \( \gamma \geq \beta + 6. \)

Finally, suppose that \( D \) does not separate \( \Sigma \). Cut open along \( D \), apply induction to the resulting graph and \( (\mathcal{C} - \{C_i\} \cup \{D_1, D_2\}) \). We now have
\[ \sum_{f \in \mathcal{F}(G, \mathcal{C})} w(f) = \sum_{f \in \mathcal{F}(G_1, \mathcal{E}' \cup \mathcal{C}' \cup P)} w(f) + \sum_{f \in \mathcal{F}(G_2, \mathcal{E}'' \cup \mathcal{C}' \cup P)} w(f) \]
\[ \leq \alpha(2 - \chi(\Sigma_1)) + \beta(|\mathcal{E}'| + 1) + \sum_{j=1}^{k} (|E(E_j)|) + |E(C_j')| + |V(P)| - 1 \]
\[ - \gamma - 4 + \alpha(2 - \chi(\Sigma_2)) + \beta(|\mathcal{E}''| + 1) \]
\[ + \sum_{j=1}^{l} |V(C_j)| + |E(C_i'')| + |V(P)| - 1 - \gamma - 4 \]
\[ = \alpha(2 - \chi(\Sigma)) + \beta(|\mathcal{C}| + 1) + \beta(|\mathcal{E}''| + 1) \]
\[ + \sum_{j=1}^{n} |V(C_j)| + 2(|V(P)| - 1) - 2\gamma - 8 \]
\[ \leq \alpha(2 - \chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{j=1}^{n} |V(C_j)| - \gamma - 4. \]

Again, this holds because \( \gamma \geq \beta + 6. \)

These arguments prove that there exists no \( P \) between two vertices of some \( C_i \) such that \( |E(P)| \leq 5. \)
Suppose there is no safe tetrad. At this point in the proof, we know that every face is homeomorphic to a disk. As a result, the lemma follows from Proposition 4.3.10. So suppose for the rest of the proof, there is a safe tetrad. Let $T$ be a safe tetrad with vertices labeled as in Figure 4.2, and let $G'$ be obtained from $G \setminus \{v_1, v_2, v_3, v_4\}$ by identifying $x$ and $t$ into a vertex $w$. By Lemma 4.3.7 and the fact that $G$ has no triseparating subgraph on at most 29 edges, we may assume that $G'$ is loopless and that every cycle in $G'$ is $C$-admissible. As before, we will perform the operation used by Borodin et al [6] to remove the tetrad as described in the proof of Lemma 4.3.1. Let the resulting multigraph be called $G'$.

Now suppose that both vertices $x$ and $t$ from the tetrad reduction were elements of some $C_i$ or that one of $x$ and $t$ was on $C_i$ and the other was adjacent to a vertex of $C_i$. Then either there is a path $xv_1v_2v_3t$ of length four with both ends on $C_i$ or a path of length five $xv_1v_2v_3tv$ between two vertices of $C_i$, where vertex $v$ is without loss of generality a vertex of $C_i$ adjacent to $t$. Both of these paths contradict earlier claims.

By Proposition 4.2.6, any precoloring of $C$ that extends to $G'$ also extends to $G$. Now, let $G''$ be a minimal subgraph of $G'$ such that any 3-coloring of $C$ that extends to $G''$ also extends to $G'$ and hence to $G$. By Lemma 4.2.2, it follows that $(G'', C) \in S_{10}(\Sigma)$. Then $G''$ is $C$-critical. This is because $G''$ is a subgraph of $G$ and because any 3-coloring of $C$ that extends to $G''$ also extends to $G'$ and hence to $G$. So if every 3-coloring of $C$ extends in $G''$, then it would extend in $G$, a contradiction. Notice that $G'' \neq C_1 \cup \cdots \cup C_k$ because this graph is always 3-colorable.

Let $f''$ be a face of $G''$ and let $J$ denote its boundary. Then there is a canonical subgraph $D_{f''}$ of $G$ that corresponds to $J$, defined as follows. If $w \not\in V(J)$, then $D_{f''} = J$. We may therefore assume that $w \in V(J)$. Let $E_1$ be the set of edges of $G''$ incident with $w$ that correspond to edges of $G$ incident with $x$, and let $E_2$ be defined similarly with $x$ replaced by $t$. If one of $E_1, E_2$ is disjoint from $E(J)$, then
again $D_{f''} := J$; otherwise $D_{f''} := J \cup P$, where $P$ is the path $xv_1v_2v_3t$. Let $\Delta_{f''}$ be the face of $D_{f''}$ that uniquely corresponds to $f''$.

We can apply the induction hypothesis to each $D_{f''}$ individually to give a bound for $\sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta_{f''}} w(f)$. We can make an analogous estimate for each face in $G''$.

So for $G''$, $\sum_{f \in \mathcal{F}(G'', C)} w(f) \leq \alpha(2 - \chi(\Sigma)) + \beta(|C|) + \sum_{j=1}^{n} |V(C_j)| - \gamma - 4$ by the induction hypothesis.

We claim that for every $f'' \in \mathcal{F}(G'', C)$,

$$\sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta_{f''}} w(f) \leq w(f'').$$

By induction, we have

$$\sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta_{f''}} w^*(f) \leq w_1(\Delta_{f''}) - 4,$$

where $w^*$ is defined as follows. In order to apply induction, we need to split vertices so that the boundary components will become cycles; in the process the quantity $w_2(f)$ may decrease. Thus, $w^*(f) = w_1^*(f) + w_2^*(f)$, where $w_1^*(f) = w_1(f)$ and $w_2^*(f)$ is interpreted after the split. We have

$$w_1(\Delta_{f''}) \leq \begin{cases} w_1(f''), & \text{if } P \text{ is not incident with } \Delta_{f''}; \\
1(w''') + 4, & \text{if } P \text{ is incident with } \Delta_{f''} \text{ on one side only}; \\
w_1(f'') + 8, & \text{if } P \text{ is not incident with } \Delta_{f''} \text{ on both sides}. \end{cases}$$

and

$$\sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta_{f''}} w_2(f) \leq w_2(f'') + \sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta_{f''}} w_2^*(f),$$

where the inequality is strict if $P$ is incident with $\Delta_{f''}$ on both sides, because $T$ was chosen to be a safe tetrad. Thus letting $\epsilon = 4$ if $P$ is incident with $\Delta_{f''}$ on both sides and $\epsilon = 0$ otherwise, we have
\[
\sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta f''} w(f) = \sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta f''} w_1(f) + 4 \sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta f''} w_2(f)
\]
\[
\leq 4 \sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta f''} w_2(f) + 4 \sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta f''} w^*(f) - \epsilon
\]
\[
\leq w_1(f'') + 4w_2(\Delta f'') = w(f''),
\]
as claimed.

This gives
\[
\sum_{f \in \mathcal{F}(G, C)} w(f) = \sum_{f'' \in \mathcal{F}(G'', C)} \sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta f''} w(f)
\]
\[
\leq \sum_{f'' \in \mathcal{F}(G'', C)} \sum_{f \in \mathcal{F}(G, C), f \subseteq \Delta f''} w(f'') \leq \alpha(2 - \chi(\Sigma)) + \beta|C| + \sum_{j=1}^{n} |V(C_j)| - \gamma.
\]

We can now prove the main theorem of this chapter.

**Theorem 4.3.12.** Let \( \Sigma \) be a surface and let \( G \) be a \( 4 \)-critical graph with no cycles of length four through ten. Then \( |V(G)| \leq 2442g(\Sigma) + 37 \).

**Proof.** The theorem follows from Lemma 4.3.11 as soon as the constants for \( \alpha, \beta, \gamma \) are determined. From the arguments above, the following inequalities must be satisfied

- \( \beta \geq 66 \)
- \( 2\gamma \geq 3\beta + 50 \)
- \( \alpha \geq 10\beta + 6 \)

The solution to these inequalities that minimizes each of \( \alpha, \beta, \gamma \) is \( \alpha = 666, \beta = 66, \gamma = 124 \). By Lemma 4.3.11 we now have a bound for

\[
\sum_{f \in \mathcal{F}(G, C)} w(f).
\]
However, this lemma only holds if there are at least two cycles in \( C \). Let \( G \) be a 4-critical graph with no cycles of length four through ten. So, we will add two disjoint triangles, call them \( T_1, T_2 \) to \( G \) and let \( C = \{T_1, T_2\} \). We can then bound \( w(f) \).

Let \( g(\Sigma) \) be the Euler genus of surface \( \Sigma \). Lemma 4.3.11 then states that

\[
\sum_{f \in \mathcal{F}(G,C)} w(f) \leq 666g(\Sigma) + 132 + 6 - 128 = 666g(\Sigma) + 10.
\]

We need to convert this result into a bound on the number of vertices of \( G \). Let \( e \) be an edge of \( G \) that is not part of \( C \). Let \( f_1, f_2 \) be the faces (which may be the same) that edge \( e \) bounds. Define the *edge weight* of \( G \) to be the quantity \( w(f_1)/|f_1| + w(f_2)/|f_2| \).

Assume that \( w(f_i) = 0 \) if \( f_i \) is a face of length three. The sum of the edge weights of \( G \) is equivalent to \( \sum_{f \in \mathcal{F}(G,C)} w(f) \). Since each edge in \( G \) is not part of two 3-cycles, else we obtain a cycle of length four, we may assume that at least one \( f_i \) has length at least 11 if \( f_i \) is connected, in which case \( w(f_i) \geq 2 \). If \( f_i \) is disconnected, then \( f_i \) has length at least three, and also \( w(f_i) \geq 10 \). Further, for every edge that \( f_i \) is longer than these minimum lengths, \( w(f_i) \) increases by one. So a lower bound for the edge weight of an edge in this graph is an edge where \( f_1 \) is a triangle and \( f_2 \) is a connected face of length 11. In this case, the edge weight is \( 2/11 \). As a result, we may conclude that there are at most

\[
A = (666g(\Sigma) + 10) \left( \frac{11}{2} \right) = 3663g(\Sigma) + 55
\]

edges in \( G \). Now, every vertex in \( G \) has degree at least three. So there are at most \( B = \lceil \frac{2A}{3} \rceil \) vertices in \( G \). In our case, \( B = 2442g(\Sigma) + 37 \). Notice that \( B \) is a function that is linear in the genus.

\[\square\]

**Remark 4.3.13.** A natural question to ask is whether there exist non-3-colorable graphs with the specifications of Theorem 4.3.12. It is well known that there exist graphs of arbitrarily large chromatic number and girth, so such graphs do exist. In
fact, using the *Hajos construction* (or Hajos sum), we can also generate a sequence of such 4-critical graphs, such that each graph in the sequence embeds on a surface of successively higher genus. Suppose there exists a graph $G$ that is 4-critical and embeds on a surface of genus $g$. Then take another copy of $G$, call it $H$. Let $ab$ be an edge of $G$ and $uv$ be an edge of $H$. Contract vertices $a$ and $u$ into a single vertex, delete edges $ab$ and $uv$ and add the edge $bv$. If $G$ is 4-critical, then this new graph is 4-critical and it embeds on a surface of genus $2g$. We could repeat this procedure by adding another copy of $G$ to this new graph to create a 4-critical graph that embeds on a surface of genus $3g$. This shows that there exists a sequence of 4-critical graphs whose number of vertices increases linearly with genus.
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