COLOR-CRITICAL GRAPHS ON SURFACES

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DEDICATION

To my parents

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SUMMARY

A graph is $(t+1)$-critical if it is not t-colorable, but every proper subgraph is. In this thesis, we study the structure of critical graphs on higher surfaces. One major result in this area is Carsten Thomassen's proof that there are finitely many 6-critical graphs on a fixed surface. This proof involves a structural theorem about a precolored cycle C of length q. In general terms, he proves that a coloring ϕ of C can be extended inside the cycle, or there exists a subgraph H with at most 5^{q^3} vertices such that ϕ can not be extended to a 5-coloring of H. In Chapter 2, we provide an alternative proof that reduces the number of vertices in H to be cubic in q. In Chapter 3, we find the nine 6-critical graphs among all graphs embeddable on the Klein bottle. Finally, in Chapter 4, we prove a result concerning critical graphs related to an analogue of Steinberg's conjecture for higher surfaces. We show that if G is a 4-critical graph embedded on surface Σ, with Euler genus g and has no cycles of length four through ten, then $|V(G)| \leq 2442g + 37$.

CHAPTER I

INTRODUCTION

In this chapter, we will provide the graph theoretic context of the results to follow. In Section 1.1, we give descriptions of the basic terminology and structures used for our results. In Section 1.2, we explain how graphs can be drawn on surfaces other than the plane. In Section 1.3, we describe graph-theoretic problems related to finding color-critical graphs on fixed surfaces and state the contributions of this thesis to the research area. In Section 1.4, we describe a conjecture of Steinberg and our results related to this question.

1.1 Graph Theoretic Preliminaries

We follow the exposition of Bondy and Murty in [9]. A *multigraph* is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of *vertices*, a set $E(G)$, disjoint from $V(G)$ of *edges* and an incidence function ψ_G, that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e *joins* u and v, and we say that u and v are *adjacent* and that e is *incident* with u and v. Define a *loop* as an edge whose endpoints are the same. All other edges are called *links*. A multigraph is a *graph* if it has no loops and no two of its links join the same pair of vertices. In this thesis, we will only consider graphs. One class of graphs are *complete graphs* which consist of graphs with vertex set V and an edge joining every pair of distinct vertices in V. Graphs are usually represented in a pictorial manner with vertices appearing as points and edges represented by lines connecting the two vertices associated with the edge.

For a graph $G = (V, E, \psi)$, if $V' \subseteq V$, $E' \subseteq E$, ψ' is the restriction of ψ to E', and for every edge $e' \in E'$ both endpoints are in V', then $G' = (V', E', \psi')$ is a *subgraph* of G. Given a graph $G = (V, E, \psi)$, if X is a subset of vertices, we denote by $G[X]$ the subgraph with vertex set X and edge set containing every edge of G with both endpoints contained in X. Then $G[X]$ is the graph *induced* by X. A graph H is a *subdivision* or *topological minor* of G if $H = G$ or if H is constructed by replacing some edges of G with internally disjoint paths such that each of these paths has only its endpoints in common with G. Let u and v be vertices in graph G. If we *identify* u and v, we delete u and v and create a new vertex w that is adjacent to all vertices that were adjacent to u or v. However, if vertex x is adjacent to both u and v, we only draw one edge between w and x.

Two graphs G and H are *isomorphic* if there exists a bijection f between $V(G)$ and $V(H)$ with the property that any two vertices u and v in G are adjacent if and only if $f(u)$ and $f(v)$ are adjacent in H. In general, the problem of determining whether two graphs are isomorphic has undetermined complexity, and the problem of showing whether one graph has a subgraph isomorphic to another graph is NP-Complete. Some of our results involve lists characterizing graphs by whether or not they have a subgraph isomorphic to a list of graphs.

1.2 Graphs on Surfaces

We define a *surface* as a connected, compact, 2-dimensional manifold with empty boundary. We follow the exposition of Mohar and Thomassen [26] to describe how we view graphs on surfaces and ask the reader to refer to this text for further details. Two surfaces are *homeomorphic* if there exists a bijective continuous mapping between them.

Let X be a topological space. An *arc* in X is the image of a continuous one-to-one function $f : [0, 1] \to X$. We say a graph G is *embedded* in a topological space X if the vertices of G are distinct elements of X and every edge of G is an arc connecting in X the two vertices it joins in G, such that its interior is disjoint from other edges and vertices. An *embedding* of a graph G in topological space X is an isomorphism of G with a graph G' embedded in X. A *planar embedding* of a graph is an embedding of G in the plane. A topological space X is *arcwise connected* if any two elements of X are connected by an arc in X. The existence of an arc between two points of X determines an equivalence relation whose equivalence classes are called the *arcwise connected components*, or the *regions* of X. A *face* of $C \subseteq X$ is an arcwise connected component of $X \setminus C$. A 2-*cell embedding* is an embedding where every face is homeomorphic to an open disk [26].

Graphs on different surfaces have different properties. For instance, it is not possible to construct a planar embedding of K_5 on the plane. However, on the torus, there exist planar embeddings for K_5, K_6 and K_7 (but no larger complete graphs). In order to describe graph-theoretic properties on surfaces, we first must be able to describe the structure of all possible surfaces. We can do this via the classification theorem of surfaces. Before stating this theorem, we must define a few terms.

One way to distinguish between surfaces is by classifying the surface as *orientable* or *nonorientable*. In a nonorientable surface, there exist curves where right and left interchange along the curve. A straightforward example of this is to consider a curve drawn parallel to the boundary on a Möbius strip. This allows for *one-sided curves*, curves which have no inside or outside. This is not the case in an orientable surface, where all closed curves have an inside and an outside.

One way to construct a surface is by taking a collection of pairwise disjoint polygons in the plane such that the sum of the number of edges in the collection of polygons is even. We also require all the polygons to be regular and have the same edge length. So, for instance, a triangle is simply a three-sided polygon whose edge lengths are all the same. Each side of each polygon will be oriented one of two ways. We can do this by associating an initial point and an ending point to each edge and pictorially denote this by an arrow on the edge that points to the ending point. Then we can identify pairs of edges so that initial points agree. So all points in the union of these polygons still have open neighborhoods homeomorphic to the plane. The sides of these polygons and their endpoints determine a connected multigraph G embedded on surface S. If all the polygons are triangles, then G *triangulates* surface S. It can be shown [26] that every surface is homeomorphic to a triangulated surface.

The most basic surface that is considered in the classification theorem of surfaces is the *sphere*, denoted $\mathcal{S}_0$. It can be constructed by taking four equilateral triangles and connecting them to create a regular tetrahedron. Given a surface S, we can perform additional operations to make a more complicated surface. In particular, we can add a *handle*, add a *twisted handle* or add a *crosscap*. First we will describe how to add a handle. Let T_1 and T_2 be two disjoint triangles in S all of whose side lengths are the same. If surface S is orientable, then configure the edges of T_1 and T_2 so that the directions of T_1's edges are the opposite of T_2's when each is viewed in a clockwise direction. Then if we remove the interiors of T_1 and T_2 and identify the edges of T_1 to the edges of T_2, this creates a new surface S'. We say that S' is obtained by *adding a handle* to S. Notice that we can only add a handle to an orientable surface. Suppose that S is orientable and the clockwise orientations of T_1 and T_2 are the same. If we remove the interior of T_1 and T_2 and we identify the edges of T_1 to the edges of T_2 then the resulting surface, call it S'', is the result of *adding a twisted handle* to S. If S is non-orientable, then any handle added is a twisted handle. Finally, suppose that we have a simple closed disk, call it T. Suppose that we delete the interior of T from S and identify diametrically opposite points of T. This adds a *crosscap* to S. It can also be shown that adding two crosscaps is equivalent to adding a twisted handle.

We can now state the classification theorem of surfaces. It states that every surface is homeomorphic to either $\mathcal{S}_g$, the surface obtained from the sphere by adding g handles, or $\mathcal{N}_k$, the surface obtained from the sphere by adding k cross-caps. Using this terminology, $\mathcal{S}_0 = \mathcal{N}_0$ is the sphere, $\mathcal{S}_1$ is the torus, $\mathcal{N}_1$ is the projective plane and $\mathcal{N}_2$ is the Klein bottle. One way that we can distinguish between these surfaces is that graphs embedded on these surfaces have different properties. In particular we have the following:

Theorem 1.2.1. *[26] Let Σ be a surface, and G be a graph that is 2-cell embedded in Σ, with n vertices, q edges and f faces. Then Σ is homeomorphic to either $\mathcal{S}_h$ or $\mathcal{N}_k$ where h and k are defined by the equations*

$$n - q + f = 2 - 2h = 2 - k.$$

With this theorem in mind, define the *Euler characteristic* of a surface Σ to be $\chi(\Sigma) = 2 - 2h$ if $\Sigma = \mathcal{S}_h$ and $\chi(\Sigma) = 2 - k$ if $\Sigma = \mathcal{N}_k$. Also, define the *Euler genus* of surface Σ, $g(\Sigma)$ to be $g(\Sigma) = 2 - \chi(\Sigma)$. We can now state Euler's formula for surfaces.

Theorem 1.2.2. *[26] Let G be a multigraph which is 2-cell embedded in a surface Σ. If G has n vertices, q edges and f faces in Σ, then*

$$n - q + f = \chi(\Sigma).$$

With this theorem in mind we can see that graphs embedded on different surfaces have different structures. Notice that the torus and the Klein bottle have the same Euler characteristic but are somehow different. One way to distinguish between surfaces is by classifying the surface as *orientable* or *nonorientable*. Another way to distinguish between these surfaces is to study the type of cycles that can be drawn on each surface. On the Klein bottle, a nonorientable surface, it is possible to draw a *one-sided cycle*, which is a cycle whose boundary only has one side. It is impossible to do this on the orientable torus, as all cycles on the torus are *two-sided*, that is the boundary of these cycles have two sides. Note that two-sided cycles can also be drawn on a Klein bottle. One reason why the proof of a result we will describe in Section 1.3

for the Klein bottle is more complicated than Thomassen's analogous result for the torus is that the Klein bottle is nonorientable and leads to additional complications that will be described in later chapters.

Another important property of graphs embedded on surfaces is that cycles drawn on the surface may have different properties. A *homotopy* between two functions f and g from a space X to a space Y is a continuous map G from $X \times [0,1] \to Y$ such that $G(x,0) = f(x)$ and $G(x,1) = g(x)$. Two spaces are *homotopic* if there is a homotopy between them. A *contractible* cycle is a cycle C such that some parametrization of C is homotopic to a constant map. We call it contractible because it can be contracted to a point. However, on some surfaces there also exist cycles which are *noncontractible*. The torus is one surface that contains non-contractible cycles. One metric useful in the study of embedded graphs is the property of *face-width*, also called *representativity*. The face-width of a graph G embedded on surface Σ is the smallest number k such that Σ contains a noncontractible closed curve that intersects the graph in k points.

Finally, when studying graphs on surfaces, it is often useful to construct a two-dimensional representation of a surface that is easier work with and visualize than something in a higher dimension. To do this we again use polygons having pairs of edges with a starting and ending point that are identified with each other. As motivating examples we show depictions of the Klein bottle and the torus in two dimensions and an embedding of K_6 on the Klein bottle. Here opposite sides of the quadrilateral are identified.

1.3 Graph Coloring on Surfaces

One of the most well-studied areas of graph theory relates to *graph coloring*. Initially, mathematicians were interested in the coloring of maps. They wanted to determine the fewest number of colors necessary so that regions or countries that touched each

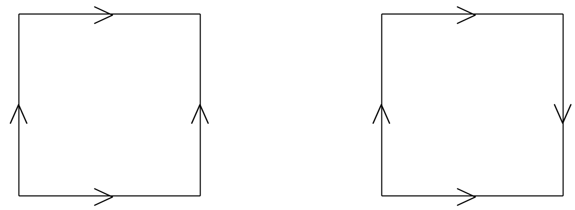


Figure 1: A two-dimensional representation of a torus and Klein bottle. The torus is on the left and the Klein bottle is on the right.

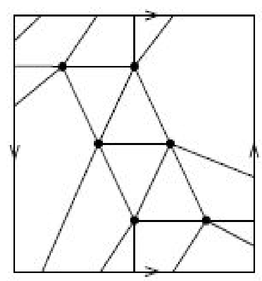


Figure 2: A planar embedding of K_6 on the Klein bottle.

other on the map had different colors. This can be modeled using a graph where each region on a map corresponds to a vertex of a graph and if two regions share a nonzero length boundary segment, there is an edge between them in the graph. A *proper coloring* is a coloring of the vertices such that the vertices of each edge are colored differently. After a few failed attempts, the Four-Color Theorem was proved [3, 4, 32] and we now know that any such map (and therefore planar graph) can be properly colored using at most four colors.

There have been many other variants and generalizations to this graph coloring problem. In the *list coloring* problem, every vertex is given a list of colors that can be used, but unlike the traditional coloring problem, the lists may be different. If the lists of each vertex are at least size k and the graph can be properly colored, we say

G is *k-choosable*. In the *edge coloring* edges are given colors. There are other coloring problems of interest such as *total coloring* and *circular coloring*. Graph theorists are also interested in the colorability of particular classes of graphs. One famous result is that of Grötzsch [18], which states that every triangle free planar graph is 3-colorable.

We are also interested in coloring problems for graphs embedded on surfaces. A fundamental question in topological graph theory is as follows: Given a surface Σ and an integer $t \geq 0$, which graphs drawn in Σ are t-colorable? Heawood proved that if Σ is not the sphere, then every graph in Σ is t-colorable as long as $t \geq H(\Sigma) := \lfloor (7 + \sqrt{24\gamma + 1})/2 \rfloor$, where γ is the *Euler genus of* Σ, defined as twice the genus if Σ is orientable and the cross-cap number otherwise. Ringel and Youngs [29] proved that the bound is best possible for all surfaces except the Klein bottle. Every graph embeddable in the Klein bottle requires only six colors, but Heawood's bound says this bound is seven. This was shown by Franklin [15] in 1934. Dirac [11] and Albertson and Hutchinson [1] improved Heawood's result by showing that every graph in Σ is actually $(H(\Sigma) - 1)$-colorable, unless it has a subgraph isomorphic to the complete graph on $H(\Sigma)$ vertices.

There have been a number of results which, aside from being interesting by themselves have produced useful structural lemmas helpful for other problems in coloring and list coloring. Recall that a graph that is k-colorable is not necessarily k-choosable. For instance, every planar graph is 4-colorable by the Four Color Theorem, but not every planar graph is 4-choosable. The first example of such a graph was given by Voigt [44]. The following is a result of Carsten Thomassen in [37]:

Theorem 1.3.1. *Every planar graph is 5-choosable.*

A common technique when working on coloring problems is to consider a precolored cycle and see whether all precolorings extend. This is precisely the technique that Thomassen used when proving his theorem. In particular, he proves the following lemma:

Lemma 1.3.2. *Suppose that every inner face of graph G is bounded by a triangle and its outer face by a cycle $C = v_1 \cdots v_k v_1$. Suppose further that v_1 has already been colored with color 1 and v_2 has been colored with color 2. Suppose finally that with every other vertex of C a list of at least three colors is associated, and with every vertex of $G - C$ a list of at least five colors. Then the coloring of v_1 and v_2 can be extended to a coloring of G from the given lists.*

Notice that the lemma almost immediately implies the theorem of Thomassen. First, suppose we are given a planar graph where every vertex is given a list of five colors. We will add edges to this graph until it is a maximally planar, that is, until no more edges can be drawn without forcing edges to cross. It is well-known that maximally planar graphs are triangulations, so the outer face of this graph has size three. Precolor two of the vertices of this triangle with colors on their respective lists such that these colors are different. The remaining graph satisfies the hypotheses of Thomassen's lemma, and hence is 5-choosable.

Another characterization fundamental in the theory of graph coloring is that of critical graphs. We say that a graph is $(t+1)$-*critical* if it is not t-colorable, but every proper subgraph is. We follow the notation as described in [26]. Dirac [12] proved that for every $t \geq 8$ and every surface Σ there are only finitely many t-critical graphs on Σ. By a result of Gallai [16] this can be extended to $t = 7$. Finally, Thomassen [36] proved a deeper result.

Theorem 1.3.3. *For every surface Σ, there are finitely many 6-critical graphs that embed in Σ.*

In addition, he does not give an explicit upper bound for the number of vertices of 6-critical graphs on a fixed surface, but an analysis of his proof shows this bound is at least doubly exponential. This automatically gives a linear time algorithm to decide 5-coloring, because we can test whether a graph in a fixed surface has a particular subgraph of bounded size in linear time [14]. However, the proof of Thomassen's theorem [36] is rather long. In this paper, Thomassen proves a theorem that bounds the size of a minimal subgraph H such that a precoloring of a cycle of length q does not extend to H. In particular, he proves the following theorem:

Theorem 1.3.4. *Let G be a 2-connected plane graph with no separating triangle and with outer cycle C of length q. Let ϕ be a 5-coloring of the vertices of C. Then G contains a connected subgraph H with at most 5^{q^3} vertices such that either (i) or (ii) holds:*
(i) ϕ cannot be extended to a 5-coloring of H,
(ii) ϕ can be extended to a 5-coloring of H such that each vertex of $G - H$ which is joined to more than two colors of H either has degree at most 4 or has degree 5 and is joined to two distinct vertices of H of the same color. The coloring of H in (ii) can be extended to a 5-coloring in G.

In Chapter 2, we improve this theorem by reducing the constant 5^{q^3} to a bound that is cubic in q by proving an analogous theorem with a slightly different outcome (ii) which is based upon our analysis. This is work that grew out of ongoing research with Ken-ichi Kawarabayashi which attempts to give a shorter proof of Thomassen's theorem that there are finitely many 6-critical graphs on a fixed surface. We prove the following theorem that grew out of the methods we have developed so far:

Theorem 1.3.5. *Let G be a 2-connected plane graph with no separating triangle and with outer cycle C of length q. Let ϕ be a 5-coloring of the vertices of C. Then G contains a connected subgraph H, which includes the vertices of C, with at most $7(q-2)^3$ vertices such that either (i) or (ii) holds:*
(i) ϕ can not be extended to a 5-coloring of H, or
(ii) there exists a 5-coloring ϕ' of the subgraph of G induced by H that extends ϕ such that for all $v \in V(G) - V(H)$, the following holds: either v sees at most two colors

of ϕ', or the number of colors in ϕ' seen by v plus the number of neighbors of v in $G \setminus V(H)$ is at most four. In this case the 5-coloring ϕ' extends to a 5-coloring of G.

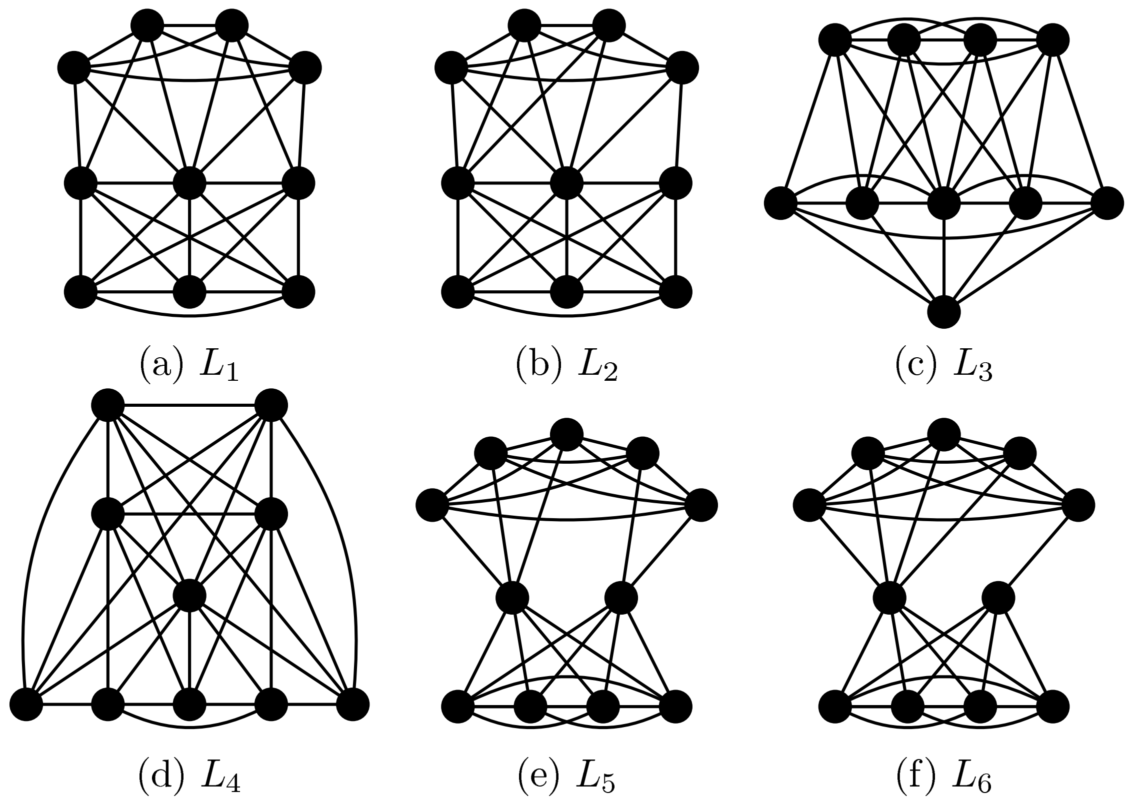
Aside from Thomassen's theorem that says there are finitely many 6-critical graphs on a fixed surface, for some small surfaces, we know the precise list of the 6-critical graphs. For instance, there are no 5-critical graphs on the plane by the Four-Color Theorem [3, 4, 32]. Hence there are also no 6-critical graphs on the plane. Albertson and Hutchinson [1] proved that a graph in the projective plane is 5-colorable if and only if it has no subgraph isomorphic to K_6, the complete graph on six vertices.

Thomassen characterized [39] the 6-critical graphs on the torus and found there are four. We now give a description of this result. First, if K, L are graphs, then define $K + L$ as the graph obtained from the union of a copy of K with a disjoint copy of L by adding all edges between K and L. The graph H_7 is obtained by taking two copies of K_4, deleting edge xy in one of the copies and deleting edge uv in the other copy, adding edge yv and identifying vertices x and u. The graph T_{11} is obtained from a cycle of length 11 by adding edges joining all pairs of vertices at distance two or three.

Theorem 1.3.6. *A graph in the torus is 5-colorable if and only if it has no subgraph isomorphic to K_6, $C_3 + C_5$, $K_2 + H_7$ or T_{11}.*

In addition, one structural result useful in Thomassen's paper is another statement about precolored outer cycles.

Lemma 1.3.7. *Let G be a planar graph with outer cycle $S : x_1x_2 \cdots x_kx_1, k \leq 6$. Let c be a 5-coloring of $G[S]$. Then c can be extended to a 5-coloring of G if and only if none of (i), (ii), (iii) below hold:*
(i) S has five colors and $G - S$ has a vertex joined to all five colors of S,
(ii) $k = 6$, and S has precisely four colors. The graph $G - S$ contains two adjacent vertices each joined to all four colors of S, and



(a) L_1 (b) L_2 (c) L_3

(d) L_4 (e) L_5 (f) L_6

Figure 3: The graphs $L_1, L_2, \ldots, L_6$

(iii) $k = 6$*, and* S *has precisely three colors.* $G - S$ *contains three pairwise adjacent vertices each of which is joined to all three colors of* S*.*

In this thesis, we find the nine 6-critical graphs on the Klein bottle. This was joint work with Chenette, Postle, Streib and Thomas. It should be noted that another group [19] found these nine 6-critical graphs on the Klein bottle, but their proof relies upon a computer search. Our result is entirely self-contained and does not require the use of a computer.

Theorem 1.3.8. *A graph in the Klein bottle is 5-colorable if and only if it has no subgraph isomorphic to* K_6*,* $C_3 + C_5$*,* $K_2 + H_7$ *or any of the graphs* $L_1, L_2, \ldots, L_6$*.*

The graphs $L_1, L_2, \ldots, L_6$ are defined in Figure 3. It should be noted that three of the graphs on this list are also 6-critical graphs on the torus. The reason that T_{11} is not on this list is that it is not embeddable on the Klein bottle. In fact, there is a result of Lawrencenko and Negami [22] which says there are no 6-regular graphs embeddable on both the torus and the Klein bottle. Part of our result involves extending the preceding lemma of Thomassen to the case of precolored 7-cycles.

Our result settles Problem 3 of Thomassen in [36]. This result also implies that in order to test 5-colorability of a graph G drawn in the Klein bottle, all we must check is whether the graph has a subgraph isomorphic to one of these nine graphs. By the algorithms of [14] and [24], we obtain the following.

Corollary 1.3.9. *There exists an explicit linear-time algorithm to decide whether an input graph embeddable in the Klein bottle is 5-colorable.*

Another corollary of our theorem involves a complicated result concerning Eulerian triangulations of Kral, Mohar, Nakamoto, Pangrac and Suzuki [21]. An *Eulerian* triangulation is a triangulation where all its vertices have even degree. Here is the corollary.

Corollary 1.3.10. *An Eulerian triangulation of the Klein bottle is 5-colorable if and only if it has no subgraph isomorphic to* K_6*.*

By inspection, each of the graphs from our main theorem has a subgraph isomorphic to a subdivision of K_6. So we can deduce the following corollary.

Corollary 1.3.11. *If a graph in the Klein bottle is not 5-colorable, then it has a subgraph isomorphic to a subdivision of* K_6*.*

Our result is also related to Hajos' conjecture, which states that for every integer $k \geq 1$, if a graph G is not k-colorable, then it has a subgraph isomorphic to a subdivision K_{k+1}. Hajos' conjecture is known to be true for $k = 1, 2, 3$ but false for all $k \geq 6$. The cases $k = 4$ and $k = 5$ remain open. Further, in [41] (Conjecture 6.3), Thomassen conjectured that Hajos' conjecture holds for every graph in the projective plane or the torus. His results imply that it suffices to prove this conjecture for $k = 4$, but this is still open. One may also try to extend Thomassen's conjecture to graphs in the Klein bottle. The previous corollary then implies it would suffice to prove the conjecture for $k = 4$.

Another related conjecture of Thomassen [41] (Conjecture 6.2) states that every graph which triangulates some surface satisfies Hajos' conjecture. He also noted that this holds for $k \leq 4$ for every surface by a deep theorem of Mader [23] and that it holds for the projective plane and the torus by [39]. So the previous corollary implies that Thomassen's second conjecture holds for graphs in the Klein bottle. This conjecture was disproved for general surfaces by Mohar [25] and qualitatively stronger counterexamples were found by Rodl and Zich [31].

1.4 Steinberg's conjecture and higher surfaces

A famous result in graph coloring is that of Grötzsch's theorem [18], which states that every triangle-free planar graph is 3-colorable. A shorter and more general proof of this statement was shown by Thomassen, who also proved an analogous statement for list coloring. The *girth* of a graph is the length of its smallest cycle. Thomassen showed [42] that every planar graph of girth at least five is also 3-list-colorable, meaning that it is possible to properly color the vertices of graph G where $|L(v)| = 3$ for each $v \in G$.

In addition, there are results related to Grötzsch's theorem for higher surfaces. In [40] Thomassen showed that every graph on the torus with girth at least five is 3-colorable. In addition, he showed that a graph embeddable on the projective plane with no contractible 3-cycle nor 4-cycle is 3-colorable. This work was extended by Gimbel and Thomassen [17] who proved that a triangle-free graph in the projective plane is 3-colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane. A quadrangulation is an embedded graph where every face is of length four. In addition, they show that every graph of girth at least six and genus two is 3-colorable. The *girth* of a graph is the length of its smallest cycle.

A famous related conjecture is due to Steinberg [35], which states that any planar graph without 4-cycles or 5-cycles is 3-colorable. One paradigm suggested by Erdős for Steinberg's conjecture is to find the minimum k such that a planar graph that excludes cycles from length 4 to length k can be 3-colored. Steinberg's conjecture is that $k = 5$. There have been a series of papers that have tightened the value of k. A number of these papers are based on *discharging* arguments. In a discharging argument particular structures of a graph (such as faces, edges or vertices) are assigned an initial charge. Based on discharging rules germane to the problem, charge is transferred between graph structures, but the total charge is preserved. This often leads to a contradiction after the sum of the charges is counted in a different way. One basic result that can be shown using discharging is that if the minimum degree of a planar graph is five, then either it has an edge with endpoints both of degree five or one with endpoints of degrees five and six. For a more detailed explanation of discharging see [9].

In 1995, Sanders and Zhao [34] showed that $k \leq 9$. Borodin [5] also proves this same result in a separate paper. Both employ similar discharging arguments. Later, Salavatipour [33] showed that $k \leq 8$, and the most current result is of Borodin et al [6] who showed that $k \leq 7$. In this paper, the authors are able to show the stronger theorem that every proper 3-coloring of a cycle containing eight to eleven vertices can be extended to a 3-coloring of the entire graph.

Aside from the Erdős-type results, there have been other papers that describe more structural properties. One such structural characteristic that has been investigated is that of adjacent (also called intersecting) triangles. By only including triangles spaced far apart, we can reduce the cycle restrictions dramatically. In fact, for plane graphs, if we only exclude 5-cycles and triangles, Borodin and Raspaud [8], with a correction from Xu [45] prove that if the triangles are at least distance four from each other, then the graph is 3-colorable. Distance between triangles is counted as the minimum distance between vertices of different triangles. Borodin and Raspaud also conjecture that the distance restriction could be strengthened to no intersecting 3-cycles. This conjecture is best possible because there exist planar graphs without 5-cycles that require four colors, as well as planar 4-chromatic graphs without intersecting triangles.

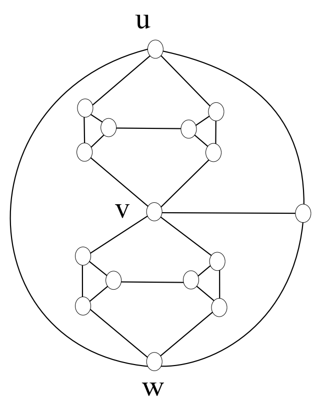


Figure 4: A planar 4-chromatic graph that has no 4-cycles.

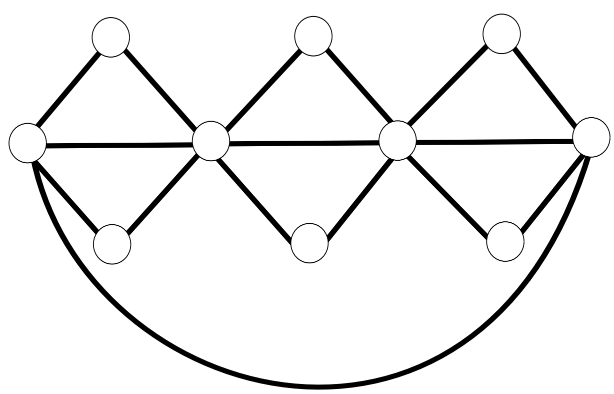


Figure 5: A planar 4-chromatic graph that has no 5-cycles.

These graphs are relatively small (compared to similar counterexamples we will describe for similar choosability problems) as they contain 16 and 10 vertices, respectively. Baogang Xu has strengthened this result so that the distance between intersecting triangles can be reduced to three [46]. In addition, Borodin [7] et al showed that planar graphs without 5- and 7-cycles without adjacent triangles are 3-colorable.

This condition given by Erdős actually also holds for arbitrary surfaces. Zhao [49] showed that for every surface Σ, there exists some k so that if G is a graph embedded on a surface Σ, and there are no cycles of length 4 up to length k, then G is 3-colorable. However, this bound is far from tight for graphs of higher genus. In particular, the bound is $11 - 12\chi(\Sigma)$. So for a double torus, say, this requires the exclusion of cycles up to length 23. Zhao also shows that by using a method similar to his $k \leq 9$ paper, for surfaces with characteristic zero and above, we also have that $k \leq 9$. This bound could be far from tight, and it is still open whether k could be reduced further or if there are any counterexamples to Steinberg's conjecture for graphs with Euler characteristic at least zero.

For finitely many exceptions, we obtain a stronger bound than Zhao using as our guide the stronger statement proved in [6]. In particular, we prove the following theorem.

Theorem 1.4.1. *Let Σ be a surface of Euler genus g. If G is 4-critical and has no cycles of length four through ten, then $|V(G)| \leq 2442g + 37$.*

The proof of this theorem involves an inductive argument coupled with a discharging process. This is the main objective of Chapter 4.

CHAPTER II

SIX-CRITICAL GRAPHS ON A PRECOLORED CYCLE

2.1 Introduction

As stated in the introduction, Carsten Thomassen, in [36] shows that there are finitely many 6-critical graphs on a fixed surface. One step of his proof involves a structural theorem about a precolored cycle C of length q. In general terms, he proves that the coloring ϕ of C can be extended inside the cycle, or there exists a subgraph H with at most 5^{q^3} vertices such that ϕ can not be extended to a 5-coloring of H. In this chapter, we provide an alternative proof that reduces the number of vertices in H to be cubic in q.

2.2 The Proof

We begin this section with definitions related to list coloring that will help us prove the main result of this chapter.

Definition 2.2.1. Let G be a graph. A *list-assignment* is a function L which assigns to every vertex $v \in V(G)$ a set $L(v)$ of natural numbers, which are called *admissible colors* for that vertex. An *L-coloring* of G is an assignment of admissible colors to all vertices of G, i.e., a function $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for every $v \in V(G)$, such that for every edge uv we have $c(u) \neq c(v)$. If k is an integer and $|L(v)| \geq k$ for every $v \in V(G)$, then L is a *k-list-assignment*. The graph is *k-list-colorable* (or *k-choosable*) if it admits an L-coloring for every k-list-assignment L. If $L(v) = \{1, 2, \ldots, k\}$ for every v, then every L-coloring is referred to as a *k-coloring* of G. If G admits an L-coloring (k-coloring), then we say that G is *L-colorable* (*k-colorable*).

We say that a vertex *sees* a color if it is adjacent to at least one precolored vertex of that color. Further, a coloring ϕ of a subset H of G *extends* to G if there exists a proper coloring of G after all the vertices in H are precolored by ϕ. Given a cycle C, and a vertex v adjacent to three consecutive vertices of C, called, in order, x_1, x_2, x_3, define the operation $*$ such that $C * v$ defines a new cycle C' where the path $x_1x_2x_3$ is substituted by x_1vx_3. In this case, we say that C' is a *rerouting* of C via v.

We now state the following lemma, which will be useful in the proof of Theorem 2.2.4. Notice that in the statement of the lemma and its proof, coloring refers to 5-coloring.

Lemma 2.2.2. *Let G be a plane graph with outer cycle C_1 that has no separating triangles. Let L be a set of vertices in C_1. Let ϕ be a coloring of the subgraph of C_1 induced by L. Then there exists a cycle C_1' obtained from C_1 via a series of rerouting operations via only vertices distance at most two from C_1 such that either:*
(i) C_1' has a chord, or some vertex v in the open disk bounded by C_1' has two neighbors on C_1' that are at least distance three apart on C_1', or
(ii) there exists a set L' of G that includes L, and a coloring ϕ' of the subgraph of G induced by L' that extends ϕ such that for all $v \in V(G) - L'$, the following holds: If v does not lie in the closed disk bounded by C_1', then v is a vertex of $G \setminus (V(C_1') \cup L')$ and the number of colors in ϕ' seen by v plus the number of neighbors of v in $G \setminus V(L')$ is at most four. Further, either $|V(C_1) \cap L'| < |L|$ or every v not in L' in the closed disk bounded by C_1' sees at most two colors of ϕ'. We also require that the subgraph induced by L' is connected.

Proof. If there are no vertices in $G - C_1$ that are adjacent to three or more vertices of L, then we are finished because C_1 and L satisfy the conclusion of (ii) as C_1' and L', respectively.

Before continuing, we introduce some notation. Let D be a cycle of vertices that are all distance at most two from C_1. Define a *tripod on D* to be a vertex, y, that is adjacent to at least three vertices on D. For the purposes of this lemma, we may assume that y must be adjacent to exactly three consecutive vertices on D, else condition (i) of the lemma is satisfied. Let y be a tripod that is adjacent to x_1, x_2, x_3, in that order on some cycle D. We say that y is an *initial tripod* if vertex x_2 is a member of C_1 and if any of the x_i's belong to C_1, then they are members of L. Vertices x_1 and x_3 need not be members of C_1.

We now describe how to construct C_1'. Start with C_1, and then if there exists an initial tripod v, construct C_2 by rerouting C_1 via v. Continue this process if there are additional initial tripods, creating cycles $C_3, \ldots$ until there are no more initial tripods. Call this graph C'. Notice that by construction, every vertex in C' is either on C_1 or distance one from C_1. Define a *candidate vertex* to be a vertex of C' not on C_1. The reason we call this vertex a candidate vertex is because it is a candidate for being a member of L'. Suppose that E is the set of candidate vertices associated with the cycle C' in graph G. Notice there are at most q candidate vertices and hence at most q initial tripod reroutings. If there are no vertices (that are neither candidate vertices nor in C_1) that are adjacent to three candidate vertices or vertices in L, then let C_1' be C' and let L' be equal to $L \cup E$. Color the vertices of E properly to obtain ϕ' in this case. Then L' and ϕ' satisfy proposition (ii). Further, suppose there are no vertices w that satisfy condition (i) where C_1' is C'.

Before we continue with the proof, we make some general observations about the structure of candidate vertices which will be useful later. If a vertex is a candidate vertex, then it is on a path of candidate vertices such that one of these vertices is adjacent to three vertices of C_1, which we label v_1, v_2, v_3 in that order. To see this, consider how initial tripods are generated. The first initial tripod, with candidate vertex x_1, must be adjacent to three vertices of C_1. The next initial tripod has a candidate vertex, x_2 which either is also adjacent to three vertices of C_1 or adjacent to x_1 and two vertices of C_1. Notice that if x_2 was adjacent to exactly two vertices of C_1, then these two vertices must be either v_3 and v_4 or v_0 and v_1, where v_0 is a member of C_1 adjacent to v_1 and v_4 is a member of C_1 adjacent to v_3. Otherwise the rerouting $C_1' = C_1 * x_1 * x_2$ satisfies condition (ii) of this lemma after properly coloring x_1, x_2 and adding them to L'. Subsequent candidate vertices, say $x_3, \ldots$ must also be adjacent to at least two vertices of C_1 unless they are adjacent to two separate paths of candidate vertices (each of which must contain a candidate vertex adjacent to three vertices of C_1) or both ends of a single path of candidate vertices. The only way this latter case could occur is then if the single path of candidate vertices traverses around the entire cycle. Even in these special cases, these candidate vertices are still adjacent to one vertex of C_1.

We may assume there exists a vertex v, not on C_1 adjacent to three consecutive vertices of C', call them x_1, x_2, x_3, such that if any of these vertices are on C_1 then these vertices are in L. We first consider the situation when at least one of x_1, x_2, x_3 is a member of C_1, and hence a member of L.

Suppose that x_1 and x_3 and both are on C_1, but x_2 is not (else v would be an initial tripod). Then the rerouting of C_1, $C_1' = C_1 * v$ satisfies the condition of proposition (ii). Notice that v does not receive a color, so the condition $|C_1 \cap L'| < |L|$ is satisfied. This completes the case when both $x_1, x_3 \in C_1$.

Suppose that x_1 is a member of C_1. Then x_2 is not a member of C_1 as then v would have been an initial tripod. Suppose that x_3 is not on C_1. Without loss of generality, let $x_1 = v_1$. Suppose that x_2 is adjacent to exactly two vertices on C_1, call them v_1, v_2. Also suppose that x_3 is adjacent to exactly three vertices of C_1. Then x_2 is an vertex of degree four. We can construct C_1' by $C_1' = C_1 * x_3 * x_2 * v$. Also, color vertex x_3 properly and include it in L'. Then C_1' and L' satisfy condition (ii). Suppose instead that x_3 is adjacent to exactly two vertices of C_1, call them v_2 and v_3. In this case, x_3 must be adjacent to another candidate vertex, which is part of a path of candidate vertices $x_4, \ldots$ that includes a vertex, call it x_k that is adjacent to three vertices of C_1. In this case, let C_1' be the path $x_1 v x_3 x_4 \cdots x_k$, edge $x_k v_l$, where v_l is the vertex furthest from x_1 on C_1 in the direction of v_3 that traverses v_1, v_2, v_3 in that order, and the path $v_l v_{l+1} \cdots x_1$. Observe that $x_4 \neq x_2$ since x_2 has two neighbors of C_1 and it is not adjacent to any other candidate vertices. In addition, properly color vertices $x_3, x_4, \cdots, x_k$ and make them part of L'. Again condition (ii) is satisfied as $|L' \cup C_1| < |L|$. Now, suppose that x_3 is adjacent to one vertex of C_1. This is a contradiction to our assumption that x_2 is adjacent to two vertices of C_1. Observe that x_2 must have been an initial tripod before x_3, but x_2 is not adjacent to any other candidate vertices. If x_2 were adjacent to other candidate vertices, then v would satisfy condition (i) as it would be adjacent to at least four vertices on a cycle, say C_i, constructed in the process of creating C'. So x_2 must be adjacent to three vertices on C_1. This contradicts our assumption that x_2 is adjacent to exactly two vertices of C_1. This concludes the case when $x_1 \in C_1$, $x_3 \notin C_1$ and x_2 is adjacent to exactly two vertices of C_1.

Now suppose that $x_1 \in C_1$, $x_3 \notin C_1$ and x_2 is adjacent to exactly three vertices of C_1. Then we must give x_3 an explicit color so that x_2 can satisfy the conditions for an isolated vertex stated in condition (ii) of the lemma. Now, without loss of generality, say that the neighbors of x_2 on C_1, labeled $x_1 = v_1, v_2, v_3$ in that order are colored $\alpha, 1, 2$, respectively. Notice that x_3 is also adjacent to v_3. If $\alpha \notin \{1, 2\}$ then let $\phi'(x_3)$ be 1 or α. If $\alpha \in \{1, 2\}$, then let $\phi'(x_3)$ be an arbitrary color that keeps the coloring proper at this point. Set $x_3 \in L'$. If x_3 is adjacent to three vertices of C_1, then at the time of either x_2 or x_3's addition as an initial tripod, one of x_2 and x_3 was adjacent to four vertices of the cycle (depending upon whether x_2 or x_3 was added first). Then condition (i) of the lemma is obtained. Now suppose that x_3 is adjacent to exactly two vertices of C_1, called v_3 and v_4. Let $C_1' = C_1 * x_2 * x_3 * v$. Let $L' = L \cup \{x_3\}$. If x_3 is adjacent to only one vertex of C_1, then it must be adjacent to a candidate vertex other than x_2, which is part of a path of candidate vertices $x_4, \ldots$

that includes a vertex, call it x_k that is adjacent to three vertices of C_1. In this case, let C_1' be the path $x_1 v x_3 x_4 \cdots x_k$, edge $x_k v_l$, where v_l is the vertex furthest from x_1 on C_1 in the direction of v_3, that traverses vertices v_1, v_2, v_3 in that order, and the path $v_l v_{l+1} \cdots x_1$. In addition, properly color vertices $x_4, \cdots, x_k$ and make them part of L'. Again condition (ii) is satisfied as $|L' \cup C_1| < |L|$.

Finally, suppose that $x_1 \in C_1$, $x_3 \notin C_1$ and x_2 is adjacent to exactly one vertex of C_1 and this vertex must be x_1. Notice that x_1, x_2, x_3 are consecutive vertices of C'. Also notice that x_3 must be adjacent to at least two vertices of C', as it is a candidate vertex adjacent to a candidate vertex with only one neighbor on C'. Now x_2 must be adjacent to another candidate vertex, call it x_0. This vertex must also be adjacent to at least two vertices of C'. But then vertex v, adjacent to x_1, x_2, x_3 can not be drawn.

An analogous set of arguments hold by symmetry if x_3 was the member of C_1 initially identified instead of x_1.

From the results of the previous paragraphs, we may assume that vertex v is adjacent to three candidate vertices. We now describe how to color candidate vertices and reroute C' to construct the cycle C_1', the set L' and associated coloring ϕ' that we desire. For each of these situations, here is the general set-up. Let v be a vertex adjacent to three candidate vertices of C', which we call, in order, x_1, x_2, x_3. For the purposes of this proof, let (a, b, c) be the number of vertices x_1, x_2, x_3, respectively are adjacent to in C_1.

Case 1: (2,3,2) Suppose that x_2's neighbors on C_1, denoted v_1, v_2, v_3 are colored α, 1 and 2, respectively. First suppose that $\alpha \neq \{1, 2\}$. Then let $\phi'(x_1)$ equal 1 or 2 and let $\phi'(x_3)$ 1 or α. Similarly, if $\alpha = 2$, then let $\phi'(x_1)$ and $\phi'(x_3)$ be colored with 1 or 2, with the additional flexibility that an additional color (namely, 3, 4, or 5) can be used amongst $\phi'(x_1)$ and $\phi'(x_3)$ as the argument still applies if x_2 sees three colors of ϕ'. Let the two neighbors of x_1 on C_1 be denoted v_0, v_1, and the two neighbors of x_3 on C_1 be denoted v_3, v_4. Notice that vertex v and x_2 are not members of L', but x_1 and x_3 are now members of L'. Let $C_1' = C_1 * x_2 * x_3 * x_1 * v$. Observe that condition (ii) is satisfied as $|L' \cup C_1| < |L|$.

Case 2: (3,2,2) Observe that the (2,2,3) case is identical. Suppose that vertex x_1 is adjacent to v_1, v_2, v_3 in C_1. Again, we want to color via ϕ' vertices x_1 and x_3 such that, under ϕ', x_2 sees at most three colors. So suppose that without loss of generality v_3 and v_4 are colored 1 and 2, respectively. Now, let $\phi'(x_3)$ be 1 or a color in $\{3, 4, 5\}$ that is not the color of v_1 nor the color of v_2. We now define $\phi'(x_1)$. If $\phi'(x_3) = 1$, then color x_1 with a proper color. If $\phi'(x_3) \neq 1$, then let $\phi'(x_1) = \phi'(x_3)$. Notice that vertex v and x_2 are not members of L'. So $C_1' = C_1 * x_1 * x_2 * x_3 * v$. Observe that condition (ii) is satisfied as $|L' \cup C_1| < |L|$.

Case 3: (2,2,2) Again, we want to color via ϕ' x_1 and x_3 such that, under ϕ', x_2 sees at most three colors. So suppose that without loss of generality x_2 is adjacent to v_1, v_2 on C_1 and v_1, v_2 are colored 1 and 2 respectively. Suppose that x_1 is adjacent to v_0, v_1 and x_3 is adjacent to v_2, v_3. Now, vertex x_1 can be colored 2 or 3 (as it is adjacent to a vertex on C_1' colored 1, and vertex x_3 can be colored 1 or 3, as it is adjacent to the vertex on C_1 colored 2. Suppose, without loss of generality that there exists a vertex adjacent to three vertices of C_1 along the path of candidate vertices $x_3, x_4, \ldots$. Suppose this vertex is x_k, which is adjacent to vertices v_{l-2}, v_{l-1}, v_l on C_1. Here v_l is furthest from v_1 on C_1 in the direction of C_1 that traverses v_1, v_2, v_3 in that order. Notice that v and x_2 are not members of L'. In addition, properly color vertices $x_4, \ldots, x_k$ and make them part of L'. So C_1' is the cycle $v_0 x_1 v x_3 x_4 \cdots x_k v_l v_{l+1} \cdots v_0$. Observe that condition (ii) is satisfied as $|L' \cup C_1| < |L|$.

Observe that cases involving two threes in (a, b, c) except (3,1,3) are impossible because the process of generating C' would have constructed a vertex v adjacent to more than three vertices on C_k, for some k. So the remaining cases include at least one vertex that has a 1 in (a, b, c). Also let d/e denote the situation where either that vertex is adjacent to either d or e vertices on C_1'.

Case 4: (2/3,1,2/3) Observe that vertex x_2 has degree four, so we may color x_1, x_3 arbitrarily but properly. In $(3,1,3)$, let $C_1' = C_1 * x_1 * x_3 * x_2 * v$ and add x_1, x_3 to L'. This satisfies condition (ii). For cases when $a = 2, c = 2$ or when $a, c = 2$, we will have to travel along a path of candidate vertices moving away from x_2 starting with x_1 (when $a = 2$) or x_3 (when $c = 2$), or both when $a, c = 2$ before C_1' returns to C_1. Let P_1 be the path obtained when $a = 2$, starting at x_1 in the direction away from x_2 and continuing until a vertex of degree three is reached along this path of candidate vertices. Similarly define P_2 to be the path obtained when $c = 2$, starting at x_3 in the direction away from x_2 and continuing until a vertex of degree three is reached along this path of candidate vertices. Color the vertices of P_1 or P_2 or both if they are constructed and include these vertices in L'. Again, condition (ii) is satisfied.

Case 5: (1,2,2/3) or (2/3,2,1) Without loss of generality, suppose we are in the case (1,2,2/3). The arguments for Cases 2 and 3 hold for this case, except that vertex x_1 has one less restriction in its set of possible colors. When C_1' is constructed, instead of directly traveling to C_1 from x_1, construct a path of candidate vertices P_1 as described in Case 4 so that C_1' returns to C_1 once a vertex on P_1 is reached that is adjacent to three vertices of C_1'. Properly color the vertices of P_1 and include them in L'. Further, if $c = 2$ we will create an analogous path of candidate vertices P_2 starting at x_3 and ending at a candidate vertex adjacent to three vertices of C_1. Now C_1' travels along P_2 before returning to C_1'. Properly color the vertices of P_2 and include them in L'. Again, condition (ii) is satisfied.

Case 6: (1,3,1/2) or (1/2,3,1) Without loss of generality, suppose we are in the case (1,3,1/2). The arguments for Case 1 hold in for this case in the coloring of x_1 and x_3, except that vertex x_1 or (both x_1 and x_3) has one less restriction in its set of possible colors. When C_1' is constructed, instead of directly traveling to C_1 from x_1 or x_3, if x_1 or x_3 (or both) is a vertex adjacent to one vertex of C_1', then construct paths P_1 and P_2, for x_1, x_3 respectively as in Cases 4 and 5. If P_1 and P_2 are used, properly color these paths of candidate vertices and include them in L'. Again, condition (ii) is satisfied.

Case 7: (1, 1/2, 1) This case is impossible because (as described in the observations about candidate vertices) in every path of candidate vertices containing a pair of candidate vertices adjacent to one vertex of C_1, say x_i and x_j, there must be a candidate vertex adjacent to three vertices of C_1 between x_i and x_j along this path of candidate vertices.

Finally, notice that L' is connected because all vertices that are added to L' are adjacent to a vertex in L. Further notice that L' does not include any vertices inside the open disk bounded by C_1'. □

Lemma 2.2.3. *Let G be a plane graph with outer cycle C_1 of length q that has no separating triangles. Let ϕ be a coloring of C_1. Then there exists a cycle C_1' obtained from C_1 via a series of rerouting operations, where there are at most $3q^2$ vertices in G outside of the open disk bounded by C_1', such that either:*
(i) C_1' has a chord, or some vertex v in the open disk bounded by C_1' has two neighbors on C_1' that are at least distance three apart on C_1', or
(ii) there exists a set L' of G, consisting only of vertices outside the open disk bounded by C_1' that includes C_1 and a coloring ϕ' of the subgraph of G induced by L' that extends ϕ such that for all $v \in V(G) - L'$ the following holds: If v lies in the closed disk bounded by C_1', then v sees at most two colors of ϕ'. Otherwise, the number of colors in ϕ' seen by v plus the number of neighbors of v in $G \setminus V(L')$ is at most four.

Proof. Suppose G, C_1, ϕ are described as in the statement of the lemma. By Lemma 2.2.2 there exists a cycle C_1' that either satisfies condition (i) or condition (ii) of Lemma 2.2.2.

First suppose that C_1' satisfies condition (i) of Lemma 2.2.2. Notice that by Lemma 2.2.2, C_1' uses vertices distance at most two from C_1. At most q vertices are distance one from C_1 are generated and at most q vertices distance two from C_1 are generated from the process in Lemma 2.2.2. So this satisfies the condition that the graph inside C_1' excludes at most $3q^2$ vertices from G. Thus C_1' satisfies condition (i) of this lemma.

Suppose instead that C_1' satisfies condition (ii) of Lemma 2.2.2. If every v in the closed disk bounded by C_1' sees at most two colors of ϕ', then condition (ii) of this lemma is satisfied. Call this condition (ii)(a) of Lemma 2.2.2. The lemma is satisfied because all the vertices of G not inside C_1' are either in the L' given in Lemma 2.2.2 or are vertices that satisfy the conditions in condition (ii) for vertices in $G \setminus (V(C_1') \cup L')$. Note that L' is connected. Observe that in this iteration of Lemma 2.2.2, the graph inside C_1' excludes at most $2q$ vertices from G. If the other conclusion of condition (ii) from Lemma 2.2.2, $|C_1 \cap L'| < |L|$, holds (call this condition (ii)(b)), then we will again apply Lemma 2.2.2 but now C_1 is the C_1' generated by the lemma. Notice that this time $|L|$ is at most $q-1$. Also observe that there are at most $2q$ vertices of G outside of this new C_1.

So we will apply Lemma 2.2.2 repeatedly until either condition (i) of Lemma 2.2.2 holds or condition (ii)(a) of Lemma 2.2.2 holds. Notice that condition (ii)(b) may only be satisfied q times in a row because in the first iteration of Lemma 2.2.2, $|L| = q$ and the number of precolored vertices decreases at least by one after each iteration of Lemma 2.2.2. Notice after q iterations of Lemma 2.2.2 at most $2q(q)$ vertices of G have been excluded from the given C_1' and its interior. Thus, after $q+1$ iterations of Lemma 2.2.2, one of the two conclusions that satisfies this lemma is obtained and at most $2q(q+1) \leq 3q^2$ vertices are excluded from G in the provided C_1'. This concludes the proof of the lemma. □

Theorem 2.2.4. *Let G be a 2-connected plane graph with no separating triangle and with outer cycle C of length q. Let ϕ be a 5-coloring of the vertices of C. Then G contains a connected subgraph H, which includes the vertices of C, with at most $7(q-2)^3$ vertices such that either (i) or (ii) holds:*

(i) ϕ can not be extended to a 5-coloring of H, or

(ii) there exists a 5-coloring ϕ' of the subgraph of G induced by H that extends ϕ such that for all $v \in V(G) - V(H)$, the following holds: either v sees at most two colors of ϕ', or the number of colors in ϕ' seen by v plus the number of neighbors of v in $G \setminus V(H)$ is at most four. In this case the 5-coloring ϕ' extends to a 5-coloring of G.

Proof. This proof is by induction on the size of q. Let f_q be the maximum size of a graph H, described in the theorem when cycle C is length q. In either case assume that cycle C is length q. First, suppose that ϕ can be extended to a 5-coloring of G. We will now apply Lemma 2.2.3 to this situation. If condition (ii) of Lemma 2.2.3 is obtained, then the subgraph induced by L' given in Lemma 2.2.3 is an H, with associated coloring ϕ' that satisfies conclusion (ii) of this theorem. Observe that by Lemma 2.2.3, the size of the subgraph induced by L' is at most $3q^2$, which is less than $7(q-2)^3$. Further, we show that the 5-coloring ϕ' extends to a 5-coloring of G. To do this, observe that by Lemma 2.2.3, we obtain a cycle C_1' and coloring ϕ' of a set of at most $3q^2$ vertices such that every vertex contained in C_1' sees at most two colors of ϕ'. Notice that there exists some cycle C_1'' that contains all vertices inside C_1' but not necessarily vertices of C_1' such that each of these vertices has at least three colors available. Let M be the graph contained in the closed disk bounded by C_1''. Triangulate each component of M. We can then apply Lemma 1.3.2 to show that there exists a 5-coloring that extends inside C_1''. By construction of C_1', the vertices on C_1' and outside of C_1', but not in C_1'' may now be properly colored. Thus the coloring of ϕ' extends to G.

Suppose instead case (i) of Lemma 2.2.3 occurs. This condition gives a cycle C_1' in G. Let K be the set of vertices of G outside or on C_1'. Notice that by Lemma 2.2.3 that the number of vertices of K is at most $3q^2$. The lemma gives ϕ', a coloring of L', a subset of K. Further, the lemma gives that there exists either a chord or a path of length two that separates the graph inside C_1' into two graphs G_1, G_2, with outer cycles D_1, D_2, respectively whose lengths are each smaller than q. Since ϕ extends to G, there exists a coloring ϕ_0 that extends ϕ to a 5-coloring of G. Let coloring ϕ_1 be coloring ϕ_0 restricted to $G \setminus G_1$, and let ϕ_2 be the coloring ϕ_0 restricted to $G \setminus G_2$. We may then apply induction on G_1 and G_2, using colorings ϕ_1 and ϕ_2 respectively. Let the length of D_1 be l_1 and the length of D_2 be l_2. If a chord was used to partition G_1, G_2, then it follows that f_q satisfies the recurrence relation $f_q \leq f_{l_1+1} + f_{l_2+1} + 3q^2$, $q \geq 4$. Similarly, if a path of length two was used to partition G_1, G_2, then f_q satisfies the recurrence relation $f_q \leq f_{l_1+2} + f_{l_2+2} + 3q^2$, $q \geq 6$. We will obtain a bound for f_q after we consider the case when ϕ does not extend to a 5-coloring of G. This completes the case when ϕ extends to a 5-coloring of G.

Now suppose that ϕ does not extend to a 5-coloring of G. Let H be a minimal subgraph that does not extend ϕ. So we may assume that $M = C \cup H$ is a C-critical graph. Then when we apply Lemma 2.2.3, condition (ii) can not occur, as this would provide for a 5-coloring of G. This is because, after triangulating the graph inside the open disk bounded by C_1', we can use Lemma 1.3.2. Thus, we may assume that condition (i) of Lemma 2.2.3 occurs. This condition gives a cycle C_1' in G. Let K be the set of vertices of G outside or on C_1'. Notice that by Lemma 2.2.3 the number of vertices of K is at most $3q^2$. The lemma gives ϕ', a coloring of L', a subset of K. Further, the lemma gives that there exists either a chord or a path of length two that separates the graph inside C_1' into two graphs G_1, G_2, with outer cycles D_1, D_2, respectively whose lengths, denoted l_1, l_2, respectively, are each smaller than q. Notice that ϕ' can be extended to K because every vertex not in L' in K satisfies condition (ii) of Lemma 2.2.3. Call this coloring ϕ''.

Notice that graphs G_1 and G_2 are D_1-critical and D_2-critical, respectively and so we may apply induction to each of these graphs. If there is a chord that partitions G_1 and G_2, then $f_q \leq f_{l_1+1} + f_{l_2+1} + 3q^2$, for $q \geq 4$. If instead there is a path of length two that partitions G_1 and G_2, then f_q satisfies the recurrence relation

$f_q \leq f_{l_1+2} + f_{l_2+2} + 3q^2$, for $q \geq 6$.

We now must give a bound for f_q. To do this we must ensure that f_q is larger than the inductive bound given from the recurrence relations described above and give appropriate base cases. To begin, we describe some bounds for base cases. First, observe that $f_3 = 4$. To see this, observe that the first vertex inside the triangle must be adjacent to all three vertices inside the triangle. Then the next vertex added to this graph will create a separating triangle, a contradiction. We now will estimate f_4. If we begin with a 4-cycle, we may have to proceed through Lemma 2.2.2 adding, by Lemma 2.2.3, at most $3q^2 = 48$ additional vertices to H before we find a chord that separates the graph into two 3-cycles. We know $f_3 = 4$, so $f_4 \leq 56$. We can make a similar argument to bound f_5. Lemma 2.2.3 implies that at most $3q^2 = 75$ additional vertices are added to H before we find a chord that separates the graph into a 4-cycle and a 3-cycle. So then $f_5 \leq 75 + f_4 + f_3 \leq 135$.

The inductive argument gives that for every $q \geq 4$, either $f_q \leq 3q^2$, or there exist l_1, l_2 such that $2 \leq l_1, l_2 \leq q-2$, $l_1 + l_2 = q$ and

$$f_q \leq f_{l_1+1} + f_{l_2+1} + 3q^2,$$

or there exist l_1, l_2 such that $2 \leq l_1, l_2 \leq q-2$, $l_1 + l_2 = q$ and

$$f_q \leq f_{l_1+2} + f_{l_2+2} + 3q^2.$$

We will show that $f_q \leq 7(q-2)^3$. For $q \leq 6$, we will prove this directly. Two paragraphs above, we have shown that $f_3 = 4, f_4 \leq 56$ and $f_5 \leq 135$. Notice that the claimed bound for f_q in these cases gives $f_3 \leq 5, f_4 \leq 56, f_5 \leq 189$. Suppose that $q = 6$, then f_6 must satisfy the inequalities $f_6 \leq f_5 + f_5 + 3q^2$, $f_6 \leq f_5 + f_3 + 3q^2$, and $f_6 \leq f_4 + f_4 + 3q^2$. Notice that the first of these three inequalities dominates the other two since $f_5 \geq f_4 \geq f_3$. So, $f_6 \leq 135 + 135 + 3(6)^2 = 378$. This satisifes $f_6 \leq 7(4)^3 = 448$. Thus, our bound holds for f_6. For $q \geq 7$ we have

$$\begin{aligned} f_q &\leq f_{l_1+2} + f_{l_2+2} + 3q^2 \leq 7l_1^3 + 7l_2^3 + 3q^2 \\ &\leq 7(q-3)^3 + 7(3)^3 + 3q^2 \leq 7(q-2)^3. \end{aligned}$$

Notice that the last inequality holds for all $q \geq 7$ as the inequality $-63q^2 + 189q + 3q^2 \leq -42q^2 + 84q - 56$ holds. Thus, we have obtained a cubic bound for $|V(H)|$. In particular, we have shown that if ϕ can not be extended to a 5-coloring of H then $|V(H)| \leq 7(q-2)^3$. □

Theorem 2.2.5. *Let G be a plane graph with outer cycle C of length q. Let ϕ be a 5-coloring of the vertices of C. If the coloring of ϕ does not extend to G, then there exists a connected subgraph H of G that includes C, of size at most $7(q-2)^3$, for which ϕ can not be extended.*

Proof. By Theorem 2.2.4, we know that either ϕ can not be extended to a 5-coloring of H, where H has at most $7(q-2)^3$ vertices and so the theorem holds, or condition (ii) of Theorem 2.2.4 is obtained. If the latter case holds, observe that by Theorem 2.2.4, the 5-coloring ϕ' obtained from Theorem 2.2.4 can be extended to a 5-coloring of G. □

CHAPTER III

FIVE-COLORING GRAPHS ON THE KLEIN BOTTLE

3.1 Introduction

In this chapter we turn our attention to 5-coloring graphs on relatively simple surfaces. Even for graphs of low genus, the most interesting value of t for the t-colorability problem on a fixed surface seems to be $t = 5$. By the Four-Color Theorem every graph in the sphere is 4-colorable, but on every other surface there are graphs that cannot be 5-colored. Albertson and Hutchinson [1] proved that a graph in the projective plane is 5-colorable if and only if it has no subgraph isomorphic to K_6, the complete graph on six vertices. Thomassen [39] proved the analogous (and much harder) result for the torus, as follows. If K, L are graphs, then by $K + L$ we denote the graph obtained from the union of a copy of K with a disjoint copy of L by adding all edges between K and L. The graph H_7 is depicted in Figure 6 and the graph T_{11} is obtained from a cycle of length 11 by adding edges joining all pairs of vertices at distance two or three.

Theorem 3.1.1. *A graph in the torus is* 5*-colorable if and only if it has no subgraph*

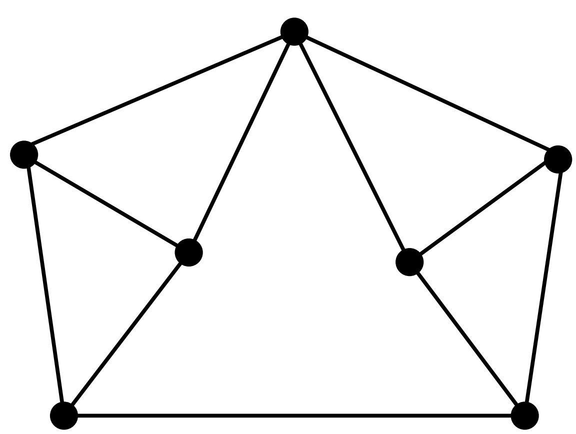


Figure 6: The graph H_7

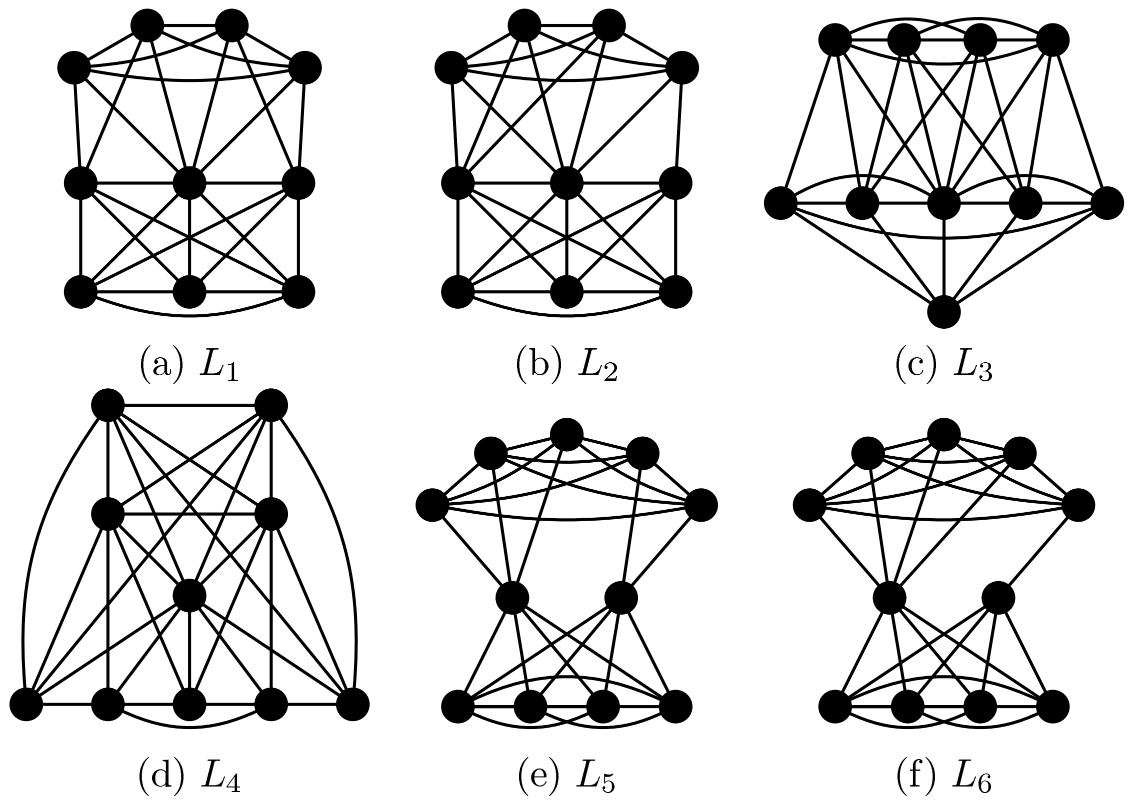


Figure 7: The graphs $L_1, L_2, \ldots, L_6$

isomorphic to K_6*,* $C_3 + C_5$*,* $K_2 + H_7$*, or* T_{11}*.*

Our objective in this chapter to prove the analogous result for the Klein bottle, the following. The graphs $L_1, L_2, \ldots, L_6$ are defined in Figure 7. Lemma 3.4.2 explains the relevance of most of these graphs.

Theorem 3.1.2. *A graph in the Klein bottle is* 5*-colorable if and only if it has no subgraph isomorphic to* K_6*,* $C_3 + C_5$*,* $K_2 + H_7$*, or any of the graphs* $L_1, L_2, \ldots, L_6$*.*

Theorem 3.1.2 settles a problem of Thomassen [36, Problem 3]. It also implies that in order to test 5-colorability of a graph G drawn in the Klein bottle it suffices to test subgraph isomorphism to one of the graphs listed in Theorem 3.1.2. In order to describe aspects of a linear-time algorithm to test subgraph isomorphism, we must first define the notion of *treewidth*. To obtain the treewidth of a graph, we must first transform the graph into a *tree-decomposition*. A tree-decomposition of a graph G is a pair $(T, \mathcal{W})$, where T is a tree and $\mathcal{W}$ is a set of subsets W_t of $V(G)$ (sometimes called "bags"), one for each vertex t of T such that:

- $\cup_{t \in V(T)} W_t = V(G)$.
- For each edge $e = uv$ of G, there is a $t \in V(T)$ such that $u, v \in W_t$.
- For each vertex $u \in V(G)$, the subtree of T induced by $\{t | u \in W_t\}$ is connected.

In words, these conditions mean that every vertex is in at least one bag. They also imply that for each edge of G, there exists some bag that contains both vertices of the edge. Further, for each particular vertex of G, the subgraph consisting of the bags containing a particular vertex is connected.

For some tree-decomposition $(T, \mathcal{W})$, $w(T, \mathcal{W})$ is the maximum of $|W_t|$ over all vertices of T. The *treewidth* of a graph G is defined to be the minimum value of $w(T, \mathcal{W})$ over all tree-decompositions $(T, \mathcal{W})$ of G. The standard definition of treewidth calls for the treewidth of a graph to actually be $(\max |W_t|) - 1$ so that trees have treewidth 1.

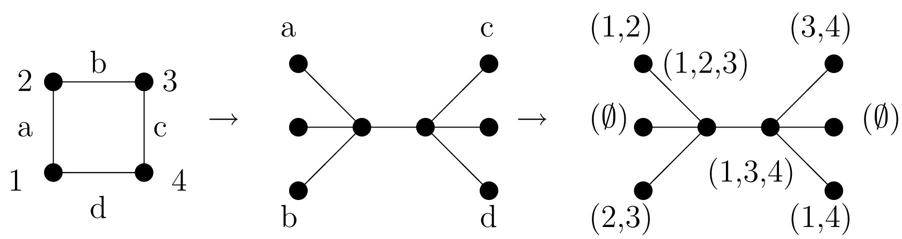


Figure 8: An example of a graph, a decomposition and a tree-decomposition derived from this decomposition.

A family $\mathcal{F}$ of graphs has the *diameter-treewidth* property if there is some function $f(D)$ such that every graph in $\mathcal{F}$ with diameter at most D has treewidth $f(D)$. An algorithm of Eppstein shows that

Theorem 3.1.3. *[14] [13] There exists a linear-time algorithm for the subgraph isomorphism decision problem for all families of graphs that have the diameter-treewidth property.*

Initially the result was proved only for planar graphs but the diameter-treewidth property result allows us to extend this algorithm to other families of graphs. The idea of the algorithm is that the graph is partitioned into pieces with small treewidth and then dynamic programming is applied to each piece.

A graph H is a *minor* of G if it can be obtained from G via edge contractions and edge deletions. We say that a family is *minor-closed* if it is closed under the minor operations of edge deletion and edge contraction. An *apex graph* is a graph G such that for some vertex v, after v is deleted from G, the remaining graph $G - v$ is planar.

In [13] Eppstein proves the following:

Theorem 3.1.4. *If $\mathcal{F}$ is a minor-closed family of graphs, then $\mathcal{F}$ has the diameter-treewidth property if and only if $\mathcal{F}$ does not contain all apex graphs.*

Notice that the family of graphs embeddable on a surface Σ does not contain all apex graphs since maximal apex graphs have $4n$ - 10 edges and by Euler's formula, graphs embeddable on Σ with genus g has at most $3n + O(g)$ edges.

Another useful algorithm is one of [24]. He shows that for an arbitrary fixed surface Σ, there is a linear-time algorithm that, for each graph G, either finds an embedding of G in Σ or finds a subgraph of G that is homeomorphic to a minimal forbidden subgraph for embeddability in Σ. In the algorithm, the minimal forbidden subgraph is one of a bounded number of edges, so this gives a constructive proof of a result of Robertson and Seymour [30] that for each closed surface, there are only finitely many minimal forbidden subgraphs.

Using the algorithms of [14], [13] and [24] above we obtain the following corollary.

Corollary 3.1.5. *There exists an explicit linear-time algorithm to decide whether an* *input graph is embeddable in the Klein bottle, and if it is embeddable in the Klein bottle, whether it is* 5*-colorable.*

Proof. First, we use Mohar's algorithm to determine whether a graph G is embedabble on the Klein bottle. We then use the algorithm of Eppstein to see if G contains a subgraph isomorphic to one of the nine subgraph-minimal 6-critical graphs on the Klein bottle. If all subgraphs of G are not isomorphic to any of the nine graphs, then G is 5-colorable. Otherwise, it is not 5-colorable. □

We will show that with the sole exception of K_6, none of the graphs listed in Theorem 3.1.2 can be a subgraph of an Eulerian triangulation of the Klein bottle. Thus we deduce the following theorem of Král', Mohar, Nakamoto, Pangrác and Suzuki [21].

Corollary 3.1.6. *An Eulerian triangulation, G, of the Klein bottle is* 5*-colorable if and only if it has no subgraph isomorphic to K_6.*

Before proving this corollary, we require an additional lemma about *defective Eulerian triangulation.* A defective triangulation is a triangulation where all the vertices have even degree except for exactly two adjacent vertices.

Lemma 3.1.7. *There are no defective Eulerian triangulations of the plane.*

Proof. Suppose for purposes of contradiction there was a defective Eulerian triangulation, G, with adjacent vertices u and v, each of odd degree. Delete edge uv from G, creating a unique face bounded by a 4-walk. Every vertex of this new graph, G', has even degree and hence G' is bipartite. Consider the geometric dual of G', call it G''. In a geometric dual, faces of G' become vertices of G'' and there is an edge between two vertices of G'' if they share an edge in G'. In our case, G' is 3-regular except for a single vertex of degree four. This is a contradiction because G'' is also bipartite and the number of edges leaving each part of the bipartition is a multiple of three. □

We can now prove the corollary.

Proof. Let G be an Eulerian triangulation of the Klein bottle. If $\chi(G) \geq 7$, then G contains some 7-critical subgraph H, which must have minimum degree six. But by Euler's formula, any graph on the Klein bottle that has minimum degree six is 6-regular. But by a result of Nakamoto and Sasanuma [27], all 6-regular graphs on the Klein bottle are 5-colorable. Thus $\chi(G) \leq 6$.

We now will show that $\chi(G) = 6$ if and only if it has a K_6 subgraph. If G contains K_6, then it follows that $\chi(G) = 6$ by the first paragraph.

Now, suppose that $\chi(G) = 6$ but G does not contain a subgraph of K_6. By Theorem 3.1.2, G must contain M, one of the other eight 6-critical graphs on the Klein bottle. Fix an embedding of G. Consider the embedding of M inside the embedding of G. Let T be any 3-face of M with vertices v_1, v_2, v_3. Let G_T be the subgraph of G induced by T and the vertices in the interior of T.

We now claim that the parity of $deg_M(v_i)$ is equal to $deg_{G_T}(v_i)$ for each of v_1, v_2, v_3. We first show that the number of i such that $deg_M(v_i) \not\equiv deg_{G_T}(v_i)$ modulo 2 must be even. Consider $G[G_T \setminus T]$, the graph of G induced by $G_T \setminus T$. All of the vertices in this graph must have even degree. But all edges with one end in T and the other in $G[G_T \setminus T]$ are only counted once toward the sum of degrees in $G[G_T \setminus T]$. So if there are an odd number of edges leaving $G[G_T \setminus T]$, the sum of the degrees in $V(G[G_T \setminus T])$ would be odd, a contradiction. This proves the statement after the claim. So the parity may change only one zero or two of the vertices of T. However, if the parity of two of the vertices change, then $G[G_T]$ is a defective Eulerian triangulation. This contradicts the previous lemma as T is embeddable in the plane. Thus, the claim follows and so we may assume the parity of $deg_M(v_i)$ is equal to $deg_{G_T}(v_i)$ for each of v_1, v_2, v_3.

To finish the proof, we will investigate each of the remaining eight graphs individually. First suppose that M is isomorphic to L_3 or L_4. Each are triangulations, so it follows by the claim that $G = M$. However, both L_3 and L_4 contain vertices of odd degree, a contradiction.

Now suppose that M is isomorphic to $C_3 + C_5$ or $K_2 + H_7$. Each of these graphs has a unique 4-cycle, and at least five vertices of odd degree, so there at least one vertex, v, of odd degree not on the unique 4-cycle. This contradicts the claim as the parity of $deg_M(v)$ is different from $deg_{G_T}(v)$.

Suppose that M is isomorphic to L_1 or L_2. Both L_1 and L_2 contain all 3-faces except for one 5-face. Both graphs contain seven vertices of degree five. Again we obtain a contradiction to the claim.

Finally, suppose that M is isomorphic to L_5 or L_6. This time, each graph contains all 3-faces except for two 5-faces. In addition, these 5-faces share two vertices. This means there are at most a total of eight distinct vertices on the two 5-faces. However, each graph has at least nine vertices of degree five. We again obtain a contradiction to the claim. This completes the proof of the corollary.

□

It follows by inspection that each of the graphs from Theorem 3.1.2 has a subgraph isomorphic to a subdivision of K_6. Thus we deduce the following corollary.

Corollary 3.1.8. *If a graph in the Klein bottle is not 5-colorable, then it has a subgraph isomorphic to a subdivision of K_6.*

This is related to Hajós' conjecture, which states that for every integer $k \geq 1$, if a graph G is not k-colorable, then it has a subgraph isomorphic to a subdivision K_{k+1}. Hajós' conjecture is known to be true for $k = 1, 2, 3$ and false for all $k \geq 6$. The cases $k = 4$ and $k = 5$ remain open. In [41, Conjecture 6.3] Thomassen conjectured that Hajós' conjecture holds for every graph in the projective plane or the torus. His results [39] imply that it suffices to prove this conjecture for $k = 4$, but that is still open. Likewise, one might be tempted to extend Thomassen's conjecture to graphs in the Klein bottle; Corollary 3.1.8 then implies that it would suffice to prove this extended conjecture for $k = 4$ as Corollary 3.1.8 handles the $k = 5$ case and the results for all other values of k are described above.

Thomassen proposed yet another related conjecture [41, Conjecture 6.2] stating that every graph which triangulates some surface satisfies Hajós' conjecture. He also pointed out that this holds for $k \leq 4$ for every surface by a deep theorem of Mader [23], and that it holds for the projective plane and the torus by [39]. Thus Corollary 3.1.8 implies that Thomassen's conjecture described at the beginning of this paragraph holds for graphs in the Klein bottle. For general surfaces the conjecture was disproved by Mohar [25] with a relatively small counterexample. Qualitatively stronger counterexamples were found by Rödl and Zich [31].

Our proof of Theorem 3.1.2 follows closely the argument of [39], and therefore we assume some familiarity with that paper, but include in this thesis the results of anything taken from [39]. We proceed as follows. First we show, using the description of all 6-regular graphs in the Klein bottle, that every 6-regular graph in the Klein bottle is 5-colorable. That allows us to select a minimal counterexample G_0 and a suitable vertex $v_0 \in V(G_0)$ of degree five. If every two neighbors of v_0 are adjacent, then G_0 has a K_6 subgraph and the result holds. We may therefore select two non-adjacent neighbors x and y of v_0. Let G_{xy} be the graph obtained from G_0 by deleting v_0, identifying x and y and deleting all resulting parallel edges. If G_{xy} is 5-colorable, then so is G_0, as is easily seen. Thus we may assume that G_{xy} has a subgraph isomorphic to one of the nine graphs on our list, and it remains to show that either G_0 can be 5-colored, or it has a subgraph isomorphic to one of the nine graphs on the list. In Chapter 3.2, we prove a series of structural lemmas that will be useful to our proof. In Chapter 3.3, we show that G_{xy} must have a subgraph to K_6. In Chapter 3.4, we handle the case when $G_{xy} = K_6$.

3.2 Lemmas

Our first lemma is an extension of [39, Lemma 4.1], which proves the same result for cycles of length at most six. If C is a subgraph of a graph G and c is a coloring of C, then we say that a vertex $v \in V(G) - V(C)$ *sees a color* α *on* C if v has a neighbor $u \in V(C)$ such that $c(u) = \alpha$.

Lemma 3.2.1. *Let G be a plane graph with an outer cycle C of length $k \leq 7$, and let c be a 5-coloring of $G[V(C)]$. Then c cannot be extended to a 5-coloring of G if and only if $k \geq 5$ and the vertices of C can be numbered $x_1, x_2, \ldots, x_k$ in order such that one of the following conditions hold:*

(i) *some vertex of $G - V(C)$ sees five distinct colors on C,*

(ii) *$G - V(C)$ has two adjacent vertices that both see the same four colors on C,*

(iii) *$G - V(C)$ has three pairwise adjacent vertices that each see the same three colors on C,*

(iv) *G has a subgraph isomorphic to the first graph shown in Figure 9, and the only pairs of vertices of C colored the same are either $\{x_5, x_2\}$ or $\{x_5, x_3\}$, and either $\{x_4, x_6\}$ or $\{x_4, x_7\}$,*

(v) *G has a subgraph isomorphic to the second graph shown in Figure 9, and the only pairs of vertices of C colored the same are exactly $\{x_2, x_6\}$ and $\{x_3, x_7\}$,*

(vi) *G has a subgraph isomorphic to the third graph shown in Figure 9, and the only pairs of vertices of C colored the same are exactly $\{x_2, x_6\}$ and $\{x_3, x_7\}$.*

Proof. Clearly, if one of (i)–(vi) holds, then c cannot be extended to a 5-coloring of G. To prove the converse we will show, by induction on $|V(G)|$, that if none of (i)–(vi) holds, then c can be extended to a 5-coloring of G. Since c extends if $|V(G)| \leq 4$, we assume that $|V(G)| \geq 5$, and that the lemma holds for all graphs on fewer vertices. We may also assume that $V(G) \neq V(C)$, and that every vertex of $G - V(C)$ has

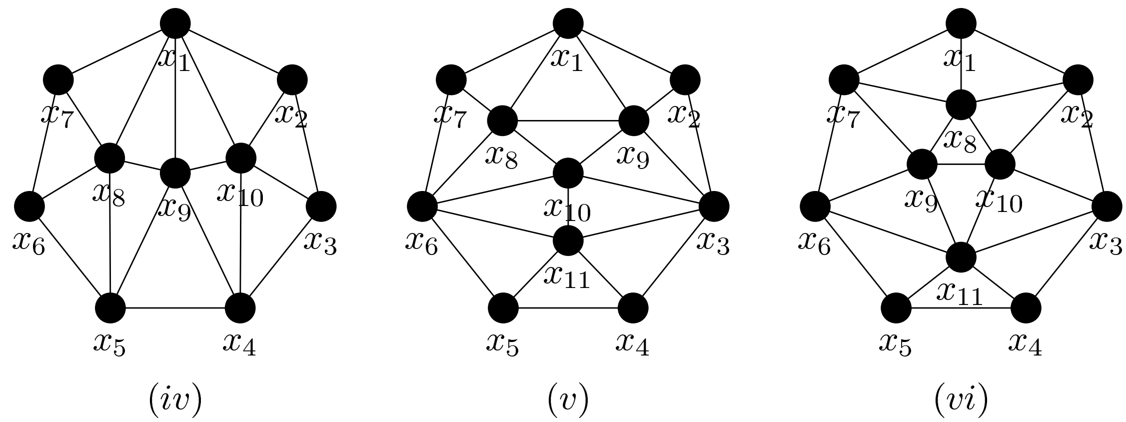


Figure 9: Graphs that have non-extendable colorings

degree at least five, for we can delete a vertex of $G - V(C)$ of degree at most four and proceed by induction. Likewise, we may assume that

(*) *the graph G has no cycle of length at most four whose removal disconnects G.*

This is because if a cycle C' of length at most four separates G, then we first delete all vertices and edges drawn in the open disk bounded by C' and extend c to that graph by induction. Then, by another application of the induction hypothesis we extend the resulting coloring of C' to a coloring of the entire graph G. Thus we may assume (∗).

Let v be a vertex of $G - V(C)$ joined to m vertices of C, where m is as large as possible. We claim that if $m \leq 2$, then the lemma holds. To prove the claim assume that every vertex of $G - V(C)$ has at most two neighbors in C. We deduce that some vertex of $G - V(C)$ has degree at most five, for otherwise $G - V(C)$ has at least five vertices, contrary to [39, Lemma 3.1], because $k \leq 7$. This shows that $G - V(C)$ has a vertex u of degree five. Since G has no separating triangle we deduce that the vertex u has two neighbors u_1, u_2, which are not adjacent and not on C. Let J be obtained from G by deleting u, identifying u_1 and u_2, and deleting all resulting parallel edges. Since $m \leq 2$ the graph J satisfies none of (i)–(vi), and hence the coloring c can be extended to a 5-coloring of J by the induction hypothesis. It follows that the coloring c can be extended to G, as desired. Thus we may assume that $m \geq 3$.

Since (i) does not hold, the coloring c extends to a 5-coloring c' of the graph $G' := G[V(C) \cup \{v\}]$. Let D be a facial cycle of G' other than C, and let H be the subgraph of G consisting of D and all vertices and edges drawn in the disk bounded by D. If c' extends to H for every choice of D, then c extends to G, and the lemma holds. We may therefore assume that D was chosen so that c' does not extend to H. By the induction hypothesis H and D satisfy one of (i)–(vi).

If H and D satisfy (i), then there is a vertex $w \in V(H) - V(D)$ that sees five distinct colors on D. Thus w has at least four neighbors on C, and hence $m \geq 4$. It follows that every bounded face of the graph $G[V(C) \cup \{v, w\}]$ has size at most four, and hence $V(G) = V(C) \cup \{v, w\}$ by $(*)$. Since (i) and (ii) do not hold for G, we deduce that c can be extended to a 5-coloring of G, as desired.

If H and D satisfy (ii), then there are adjacent vertices $v_1, v_2 \in V(H) - V(D)$ that see the same four colors on D. It follows that $m \geq 3$, and similarly as in the previous paragraph we deduce that $V(G) = V(C) \cup \{v, v_1, v_2\}$. It follows that c can be extended to a 5-coloring of G: if both v_1 and v_2 are adjacent to v we use that G does not satisfy (i), (ii), or (iii); otherwise we use that G does not satisfy (i), (ii), or (iv).

If H and D satisfy (iii), then there are three pairwise adjacent vertices of $v_1, v_2, v_3 \in V(H) - V(D)$ that see the same three colors on D. It follows in the same way as above that $V(G) = V(C) \cup \{v, v_1, v_2, v_3\}$. If v sees at most three colors on C, then c extends to a 5-coloring of G, because there are at least two choices for $c'(v)$. Thus we may assume that v sees at least four colors. It follows that $m = 4$, because $k \leq 7$. Since G does not satisfy (v) or (vi) we deduce that c extends to a 5-coloring of G.

If H and D satisfy (iv), then there are three vertices of $V(H) - V(D)$ forming the first subgraph in Figure 9. But at least one of these vertices has four neighbors on C, and hence $m \geq 4$, contrary to $k \leq 7$.

Finally, if H and D satisfy (v) or (vi), then H has a subgraph isomorphic to the second or third graph depicted in Figure 9, and the restriction of c' to D is uniquely determined (up to a permutation of colors). Since D has length seven, it follows that $m \leq 3$, and hence $c'(v)$ can be changed to a different value, contrary to the fact that the restriction of c' to D is uniquely determined. □

Corollary 3.2.2. *Let G be a 6-vertex-critical graph on some surface. Let H be a connected induced subgraph of G (i.e. $H = G(H)$). If each facial walk of H has length 3 or 4, then $H = G$.*

Proof. If $H \neq G$, then H is 5-colorable. Any 5-coloring of H can be extended to a 5-coloring of G by Lemma 3.2.1. but G is not 5-colorable, so $H = G$. □

The following lemma is shown in [27].

Lemma 3.2.3. *All* 6*-regular graphs embeddable on the Klein bottle are* 5*-colorable.*

In order to describe the 6-regular graphs on the Klein bottle, we will need to describe how to draw triangulations on a grid. Notice by Euler's formula that any 6-regular graph on the torus or Klein bottle must be a triangulation. Following the description of [27], suppose that p and k are natural numbers. Let $R_{p,k}$ be a p by k grid, and add additional diagonal edges from bottom left to upper right of each square of the grid. Let $H_{p,k}$ to be the embedding on the annulus that identifies the top and bottom vertices of the grid as shown in Figure 9. For $u_{i,j} \in V(H_{p,k})$ let i be taken modulo p. Define C_j to be the cycle of $H_{p,k}$ that passes in order through the vertices $u_{j,0}, u_{j,1}, \ldots, u_{j,p}$, for $j = 0, 1, \ldots, k$. These cycles are called *geodesic cycles* in $H_{p,k}$.

We can now describe the 6-regular graphs on the Klein bottle via [28]. To produce a 6-regular graph on the Klein bottle, identify each $u_{0,j}$ with $u_{k,-j}$ for each h. This produces a 6-regular graph on the Klein bottle of *handle type* and is described by

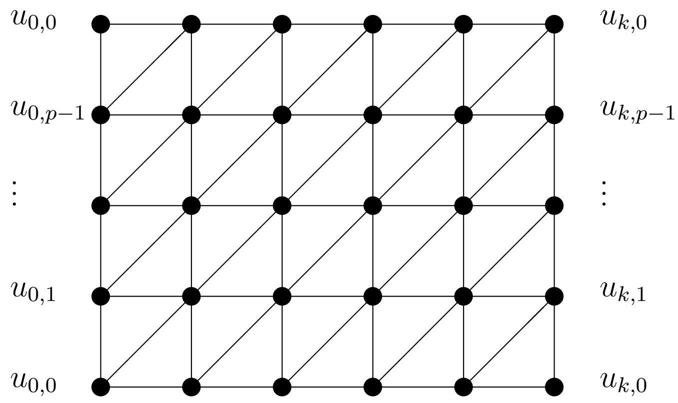


Figure 10: The graph $H_{p,k}$.

$Kh(p,k)$. We can also obtain a 6-regular graph on the Klein bottle of *crosscap type* as follows. Let p be even. Identify $u_{0,j}$ with $u_{0,j+p/2}$ and $u_{k,j}$ with $u_{k,j+p/2}$ in $H_{p,k}$ for each j, respectively. This produces two cycles of length $p/2$ from C_0 and C_k. If instead p is odd, then we add a crosscap to each boundary component of $H_{p,k-1}$. For each j, draw an edge between $u_{0,j}$ to $u_{0,j+m}$ and $u_{0,j+m+1}$, and $u_{k-1,j}$ to $u_{k-1,j+m}$ and $u_{k-1,j=m+1}$ on the crosscaps added. This graph is denoted $Kc(p,k)$. We can now state Negami's [28] result on the structure of 6-regular graphs on the Klein bottle.

Theorem 3.2.4. *A loopless 6-regular Klein bottle graph is equivalent to precisely one of*

$$Kh(p,k)(p \geq 2, k \geq 2) \text{ and } Kc(p,k)(p \geq 3, k \geq 2).$$

In particular, $Kh(p,k)$ *is simple if and only if* $p \geq 3$ *and* $k \geq 3$ *and* $Kc(p,k)$ *is simple if and only if* $p \geq 5$ *and* $k \geq 2$.

Nakamoto and Sasanuma determine the chromatic numbers of all 6-regular graphs on the Klein bottle.

Theorem 3.2.5. *Let* G *be a loopless 6-regular graph on the Klein bottle. Then* $\chi(G) =$

4, *with the following exceptions:*

- $G = Kc(3l, k)$ *for* $l \geq 1, k \geq 2$ *if and only if* $\chi(G) = 3$.
- $G = Kc(5, m+1), Kh(3, 2), Kh(4, 2m+1), Kh(2, 2m+3)$ *for* $m \geq 1$ *if and only if* $\chi(G) = 5$.
- $G = Kh(2, 3)$ *if and only if* $\chi(G) = 6$.

Notice that $Kh(2, 3)$ is not simple and contains K_6 as a subgraph. A corollary of this theorem proves Lemma 3.2.3.

The next lemma is an adaptation of [39, Lemma 5.2] for the Klein bottle.

Lemma 3.2.6. *Let* G *be isomorphic to* $C_3 + C_5$*, let* S *be a cycle in* G *of length three with vertex-set* $\{z_0, z_1, z_2\}$*, and let* u_1 *be a vertex in* $G \backslash V(S)$ *joined to* z_0*. Let* G' *be obtained from* G *by splitting* z_0 *into two nonadjacent vertices* x *and* y *such that* u_1 *and at most one more vertex* u_0 *in* G' *is joined to both* x *and* y *and such that* yz_1z_2x *is a path in* G'*. Let* G'' *be obtained from* G' *by adding a vertex* v_0 *and joining* v_0 *to* x, y, u_1, z_1, z_2*. If* G'' *is not* 5*-colorable and can be drawn in the Klein bottle, then it has a subgraph isomorphic to either* $C_3 + C_5$ *or* L_4*.*

Proof. This proof is based on the argument of [39, Lemma 5.2], except that instead of invoking [39, Proposition 2.3] on two occasions we use the fact that in those cases the graph G'' is isomorphic to L_4.

Suppose that $G'' \not\supseteq \{C_3 + C_5, L_4\}$. Suppose that one of x, y has the same neighbors in G' as z_0 does in G. Then $G' \not\supseteq L_4$ as G' contains only nine vertices, and $G' \not\supseteq C_3 + C_5$ by inspection. Thus we can assume that z_0 contains at least two neighbors in G such that one is a neighbor of x but not y and the other is a neighbor of y but not x.

We can also assume that each of x, y has degree at least five in G'' and hence z_0 has degree at least six in G. Suppose that x had degree at most four in G''. Then $G'' - \{x, v_0\}$ is a proper subgraph of $C_3 + C_5$ as y is not a neighbor of one of x's neighbors. Since $G'' - \{x, v_0\}$ is a proper subgraph, we can properly 5-color it, and extend this to a 5-coloring of G'' by coloring v_0, then x.

Let G be defined by a 5-cycle $p_1p_2p_3p_4p_5p_1$ and a 3-cycle $q_1q_2q_3q_1$ and the 15 edges p_iq_j where $1 \leq i \leq 3, 1 \leq j \leq 5$. Since the degree of z_0 in G is at least 6, then $z_0 \in \{q_1, q_2, q_3\}$.

The remainder of the proof is an analysis based on which vertices are z_0, z_1, z_2. First suppose that z_0, z_1, z_2 are q_3, q_1, q_2, respectively. If both p_0 and p_1 are in $\{p_1, p_2, p_3, p_4, p_5\}$, then we can color y, z_1, z_2, x with $2, 1, 2, 1$, respectively. We can color the remaining vertices with colors $3, 4, 5$ as the remaining vertices are v_0 and a 5-cycle, and in this case v_0 is only adjacent to one of the vertices of the 5-cycle. If $u_1 = p_1$ and $u_0 = z_1$, then we color y, z_1, z_2, x, u_1 by $2, 1, 2, 3, 4$, respectively. Since y has degree at least five in G'', some vertex in $\{p_2, p_3, p_4, p_5\}$ can obtain color 3 and the remaining vertices may be colored with colors 4 and 5.

Now consider the case where z_0, z_1, z_2 are q_1, p_1, p_2, respectively and u_0 is not in $\{z_1, z_2\}$. Color y, z_1, z_2, x, u_0, u_1 by $2, 1, 2, 1, 3, 4$, respectively. We can extend this to a 5-coloring of G'', coloring v_0 last except in the following three situations (or equivalent situations). If $u_0 = q_2$ and $u_1 = p_4$, color q_3 by the same color as x or y and recolor either z_1 or z_2 by 4 and color the remaining vertices color 5. If $u_0 = p_3$ and $u_1 = p_4$, then color q_3 by color 1 or 2 and recolor z_1 or z_2 color 4. Then we can color p_5, q_2 with colors 3 and 5 respectively. If $u_1 = p_3$ and $u_0 = p_5$, color q_3 by 1 or 2 and recolor one of z_1, z_2 by 3 and recolor p_3, p_4, p_5, q_2 by $4, 3, 4, 5$, respectively.

Now suppose that z_0, z_1, z_2 are q_1, p_1, p_2, respectively and u_0 is in $\{p_1, p_2\}$. Without loss of generality let $u_0 = p_2$. Suppose that $u_1 \in \{p_3, p_4, p_5\}$. The we can color y, z_1, z_2, x by $2, 4, 3, 1$ and can color u_1 by 3 except for the case when $u_1 = p_3$. Then we color u_1 by 4. Next, color one of q_2, q_3 color 1 or 2. If both q_2, q_3 can be colored 1, 2 then the rest of the coloring follows. So we can assume that q_2, q_3 are colored by $2, 5$, respectively and both q_2, q_3 are adjacent to x. (The argument is analogous if

q_2, q_3 are both adjacent to y.) Since y has degree at least four in G', then at least one vertex in $\{p_3, p_4, p_5\} \setminus \{u_1\}$ is joined to y and is colored 1. With possibly a swapping of the colors of z_1 and z_2, we can now complete the 5-coloring.

Suppose that z_0, z_1, z_2 are q_1, p_1, p_2, respectively and $u_0 = p_2$ and $u_1 = q_2$. Color y, z_1, z_2, x, q_2 colors $2, 1, 3, 1, 4$, respectively. If q_3 can be colored 2, then color p_3, p_4, p_5, v_0 colors 5, 3, 3, 5, respectively. Assume that q_3 is adjacent to y. Then color q_3 by 5. If we can color $\{p_3, p_4, p_5\}$ by colors $\{1, 2, 3\}$, then color v_0 with 5. If not, then p_3, p_4 are adjacent to the same vertex in $\{x, y\}$. Since x has degree at least four in G', the vertex must be x. So then p_5 is adjacent to y since otherwise we color p_3, p_4, p_5 by 2, 3, 2, respectively. We now claim the remaining graph is isomorphic to L_4. To see the isomorphism, we first will label the vertices of the figure of L_4. We will label them from top to bottom, breaking ties from left to right. So the vertex on the top left is labeled a, the vertex on the top right is labeled b. The next row of vertices is labeled c and d, respectively. The vertices in the third row from the top is labeled e, and the vertices on the bottom row are labeled f, g, h, i, j from left to right. With this notation, we have the following correspondence: $a \equiv z_1, b \equiv y, c \equiv q_3, d \equiv z_2, e \equiv q_2, f \equiv p_5, g \equiv p_4, h \equiv p_3, i \equiv x, j \equiv v_0$. Thus G'' is isomorphic to L_4.

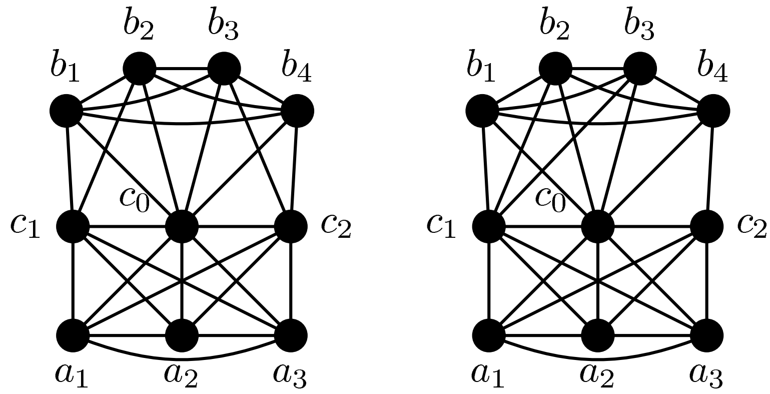
Now, consider the case when z_0, z_1, z_2 are q_1, q_2, p_1, respectively. If $u_0 \notin \{z_1, z_2\}$, then color $y, z_1, z_2, x, p_2, p_3, p_4, p_5, q_3$ by $2, 1, 2, 1, 3, 4, 3, 4, 5$, respectively. If $u_0 = p_1$, color y, z_1, z_2, x by $2, 1, 3, 1$, respectively. If q_3 is not adjacent to y, then color q_3 by 2 and vertices p_2, p_3, p_4, p_5 colors 4 and 5. If q_3 is adjacent to y, color q_3 by 5. Since x has degree at least four in G', some vertex in $\{p_2, \ldots, p_5\}$ can be colored 2. The other vertices in this set could then be colored with colors 3 and 4. Thus assume that $u_0 = q_2 = z_1$. Color y, z_1, z_2, x, u_1 by $2, 3, 2, 1, 4$ and we now will try and extend this coloring. If q_3 can be colored 1, then color p_2, p_3, p_4, p_5 by colors 4 and 5. So we assume that q_3 is adjacent to x. If $u_1 = p_3$, then recolor z_2 by color 4 and color q_3

by 2. Since y also has degree at least four in G', it must be adjacent to at least one of p_4, p_5, which we color 1. The remaining vertices of $\{p_1, \ldots, p_5\}$ are colored 5. If $u_1 = q_3$, then we color one of p_2 or p_5 color 1 if possible and complete the coloring by using 5 for two vertices in $\{p_2, p_3, p_4, p_5\}$. Now assume that both p_2 and p_5 are joined to x. Since y has degree at least four in G' it follows that y is adjacent to p_3 and p_4. We now claim that G'' is not embeddable on the Klein bottle. Notice that if an embedding of this graph exists, it must be that it is a triangulation as it has 10 vertices and 30 edges. Consider the induced embeddings of $G'' - p_2, G'' - p_5$ and $G'' - v_0$, respectively. The face of $G'' - p_2$ containing p_2 is bounded by a Hamiltonian cycle R_1 of the neighborhood of p_2. There exist similarly constructed Hamiltonian cycles around p_5 and v_0. However, each of these cycles contains the edge xp_1. This would mean that xp_1 is part of three facial triangles, a contradiction.

Finally, consider the subcase where z_0, z_1, z_2, u_0, u_1 are q_1, q_2, p_1, q_2, p_2, respectively. Color y, z_1, z_2, x, u_1, q_3 by $2, 3, 2, 1, 4, 5$, respectively. We may assume that q_3 is adjacent to x else we can recolor q_3 by 1 and complete the coloring. Also, we can assume that p_5 is adjacent to x else we color p_5, p_4, by $1, 4$ and complete the coloring. Color p_5 by 4. The coloring can be completed unless p_3 and p_4 are both adjacent to the same vertex in $\{x, y\}$. Since y must have degree at least four in G', it follows that p_3 and p_4 are adjacent to y. This graph is isomorphic to L_4. Using the notation from above we have the following correspondence: $a \equiv x, b \equiv z_2 = p_1, c \equiv q_3, d \equiv p_2 = u_1, e \equiv q_2 = z_1 = u_0, f \equiv p_5, g \equiv p_4, h \equiv p_3, i \equiv y, j \equiv v_0$. Thus G'' is isomorphic to L_4.

□

Lemma 3.2.7. *Let G be a graph drawn in the Klein bottle, and let $c, d \in V(G)$ be such that $G \backslash c$ does not embed in the projective plane, and G does not embed in the torus. Then every closed curve in the Klein bottle intersecting G in a subset of $\{c, d\}$ separates the Klein bottle.*



(a) L_1 (b) L_2

Figure 11: The graphs L_1 and L_2 with their vertices labeled

Proof. Let ϕ be a closed curve in the Klein bottle intersecting G in a subset of $\{c, d\}$, and suppose for a contradiction that it does not separate the Klein bottle. Then ϕ is either one-sided or two-sided. If ϕ is one-sided, then it intersects $G\backslash c$ in at most one vertex, and hence the Klein bottle drawing of $G\backslash c$ can be converted into a drawing of $G\backslash c$ in the projective plane, a contradiction. Thus ϕ is two-sided, but then the drawing of G can be converted into a drawing of G in the torus, again a contradiction. □

Lemma 3.2.8. *Let G be L_1 or L_2 with its vertices numbered as in Figure 11, and let it be drawn in the Klein bottle. Then*

(i) every face is bounded by a triangle, except for exactly one, which is bounded by a cycle of length five with vertices c_1, a_i, c_2, b_j, b_k in order for some indices i, j, k, and

(ii) for $i = 0, 1, 2$ the vertices a_1, a_2, a_3 appear consecutively in the cyclic order around c_i (but necessarily in the order listed), and so do the neighbors of c_i that belong to $\{b_1, b_2, b_3, b_4\}$.

Proof. Let $i \in \{1, 2\}$. There are indices j, k such that a_j and b_k are both adjacent to c_i and are next to each other in the cyclic order around c_i. Let f_i be the face incident with both the edges c_ia_j and c_ib_k. We claim that the walk bounding f_i

includes at most one occurrence of c_i and no occurrence of c_0. Indeed, otherwise we can construct a simple closed curve either passing through f_i and intersecting G in c_i only (if c_i occurs at least twice in the boundary walk of f_i), or passing through f_i and a neighborhood of the edge c_iw and intersecting G in c_i and c_0 (if c_0 occurs in the boundary walk of f_i). By Lemma 3.2.7 this simple closed curve separates the Klein bottle. It follows from the construction that it also separates G, contrary to the fact that $G\backslash\{c_i, c_0\}$ is connected. This proves our claim that the walk bounding f_i includes at most one occurrence of c_i and no occurrence of c_0.

Since f_i includes a subwalk from a_j to b_k that does not use c_i, we deduce that c_{3-i} belongs to the facial walk bounding f_i. But the neighbors of c_1 and c_2 in $\{b_1, b_2, b_3, b_4\}$ are disjoint, and hence f_i has length at least five. By Euler's formula $f_1 = f_2$, this face has length exactly five, and every other face is bounded by a triangle. This proves (i). Statement (ii) also follows, for otherwise there would be another face with the same properties as $f_1 = f_2$, and yet we have already shown that this face is unique. □

Lemma 3.2.9. *Let G be L_5 or L_6 with its vertices numbered as in Figure 12, and let it be drawn in the Klein bottle. Then*

(i)every face is bounded by a triangle, except for exactly two, which are bounded by cycles C_1, C_2 of length five, each with vertices c_1, a_i, c_2, b_j, b_k in order for some indices i, j, k,

(ii) if $G = L_5$, then $C_1 \cap C_2$ consists of the vertices c_1, c_2, and if $G = L_6$, then $C_1 \cap C_2$ consists of the vertices c_1, c_2, b_5 and the edge c_2b_5, and

(iii) for $i = 1, 2$ the vertices a_1, a_2, a_3, a_4 appear consecutively in the cyclic order around c_i (but necessarily in the order listed), and so do the neighbors of c_i that belong to $\{b_1, b_2, b_3, b_4, b_5\}$.

Proof. The proof is similar to Lemma 3.2.8. Again, notice for $i = 1, 2$, there are indices j, k such that a_j and b_k are both adjacent to c_i and are next to each other

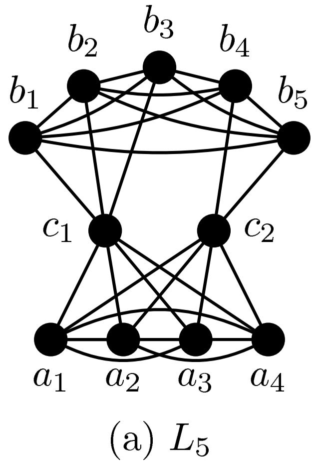


Figure 12: The graphs L_5 and L_6 with their vertices labeled

in the cyclic order around c_i. Again we make the claim that the walk bounding f_i includes at most one occurrence of c_i. We will, as in Lemma 3.2.8 construct a simple closed curve that passes through f_i and intersects G in c_i at least twice in the boundary walk of f_i. But since L_5 and L_6 are not embeddable in the torus and $L_5 \setminus \{c_i\}$ and $L_6 \setminus \{c_i\}$ are not embeddable in the projective plane, we may apply Lemma 3.2.7 and obtain a simple closed curve that separates the Klein bottle. By construction, this curve also separates G, a contradiction, as $G \setminus \{c_i\}$ is connected. So the walk bounding f_i contains at most one occurrence of each c_i.

For each of L_5 and L_6 there are four pairs of edges that must be parts of faces, call them f_1, f_2. These edges are, for L_5, $b_1c_1, c_1a_1; b_3c_1, c_1a_4; b_4c_2, c_2a_1; b_5c_2, c_2a_4$, and for L_6, $b_1c_1, c_1a_1; b_4c_1, c_1a_4; b_5c_2, c_2a_1; b_5c_2, c_2a_4$. Since c_1 and c_2 are each used at most once in each face, there are at least two faces. In addition, the neighbors of c_1, c_2 in $\{b_1, b_2, b_3, b_4, b_5\}$ are disjoint, so each of f_1 and f_2 have length at least five. However, by Euler's formula there are at most two faces of length exactly five, and every other face is bounded by a triangle. This proves (i).

Statement (*ii*) follows for L_5 since edges b_4c_2 and b_5c_2 are in different faces and there are no multiple edges in L_5. Similarly (*ii*) follows for L_6 because edge b_5c_2 and vertex c_1 must be in each of the two 5-faces of L_6. Statement (iii) also follows, for otherwise there would be another non-triangular face other than f_1 and f_2, and yet we have already shown that there are at most two of these 5-faces. $\square$

3.3 Reducing to K_6

If v is a vertex of a graph G, then we denote by $N_G(v)$, or simply $N(v)$ if the graph can be understood from the context, the open neighborhood of the vertex v; that is, the subgraph of G induced by the neighbors of v. Sometimes we will use $N(v)$ to mean the vertex-set of this subgraph. We say that a vertex v in a graph G embedded in a surface has a *wheel neighborhood* if the neighbors of v form a cycle C in the order determined by the embedding, and the cycle C is null-homotopic.

Let G_0 be a graph drawn in the Klein bottle such that G_0 is not 5-colorable and has no subgraph isomorphic to any of the graphs listed in Theorem 3.1.2. Let a vertex $v_0 \in V(G_0)$ of degree exactly five be chosen so that each of the following conditions hold subject to all previous conditions:

(i) $|V(G_0)|$ is minimum,

(ii) the clique number of $N(v_0)$, the neighborhood of v_0, is maximum,

(iii) the number of largest complete subgraphs in $N(v_0)$ is maximum,

(iv) the number of edges in $N(v_0)$ is maximum,

(v) $|E(G_0)|$ is minimum,

(vi) if possible, v_0 has a wheel neighborhood.

In those circumstances we say that the pair (G_0, v_0) is an *optimal pair*. Given an optimal pair (G_0, v_0) we say that a pair of vertices v_1, v_2 is an *identifiable pair* if v_1 and v_2 are non-adjacent neighbors of v_0. If v_1, v_2 is an identifiable pair, then we define $G_{v_1v_2}$ to be the graph obtained from G_0 by deleting all edges incident with v_0 except v_0v_1 and v_0v_2, contracting the edges v_0v_1 and v_0v_2 into a new vertex z_0, and deleting all resulting parallel edges. This also defines a drawing of $G_{v_1v_2}$ in the Klein bottle.

We now introduce notation that will be used throughout the rest of the paper. Let G'_0 be obtained from G_0 by deleting all those edges that got deleted during the construction of $G_{v_1v_2}$. That means all edges incident with v_0 except v_0v_1 and v_0v_2 and all those edges of G_0 that got deleted because they became parallel to another edge. Thus if a vertex v of G_0 is adjacent to both v_1 and v_2, then G'_0 will include exactly one of the edges vv_1, vv_2. Thus the edges of $G'_0\backslash v_0$ may be identified with the edges of $G_{v_1v_2}$, and in what follows we will make use of this identification. Now if J is a subgraph of $G_{v_1v_2}$ with $z_0 \in V(J)$, then let $\hat{J}$ be the corresponding subgraph of G'_0; that is, $\hat{J}$ has vertex-set $\{v_0, v_1, v_2\} \cup V(J) - \{z_0\}$ and edge-set $\{v_0v_1, v_0v_2\} \cup E(J)$. Let $\hat{R}_1$ and $\hat{R}_2$ be the two faces of $\hat{J}$ incident with v_0, and let R_1, R_2 be the corresponding two faces of J. We call R_1, R_2 the *hinges* of J. Finally, let $\hat{R}$ be the face of $\hat{J}\backslash v_0$ containing v_0.

We now will state and prove a lemma of Thomassen [39, Lemma 5.1] describing some of the chromatic properties of $C_3 + C_5$ and $K_2 + H_7$.

Lemma 3.3.1. *Let G be a copy of $C_3 + C_5$ or $K_2 + H_7$. Let $z_0z_1z_2z_3z_0$ be a 4-cycle in G such that z_0 and z_2 are neighbors. Let G_1 be obtained from G by splitting z_0 into two nonadjacent vertices x, y such that G_1 contains the edges z_1y, z_3x, z_2y, z_2x and such that z_2 is the only vertex joined to both x and y. Then G_1 has 5-colorings c, c', c'' such that:*

(i) $\{c(x), c(z_3)\} \neq \{c(y), c(z_1)\}$,

(ii) $c'(y) = c'(z_3)$ *or* $c'(x) = c'(z_1)$,

(iii) *Either* $c''(y), c''(z_1), c''(z_2), c''(z_3), c''(x)$ *are all distinct or* $c''(y), c''(z_1), c''(z_2), c''(z_3)$ *are not distinct.*

Proof. We first prove case (i). $G_1 - x$ is a property subgraph of G and is therefore 5-colorable. If x is adjacent to only three colors, then there are at least two choices for $c(x)$ and we can obtain the desired coloring. So suppose that x is adjacent to vertices consisting of at least four different colors. In particular, x and similarly y have degree at least four in G_1. Then z_0 has degree at least seven in G, so z_0 is jointed to all other vertices of G. If $G = C_3 + C_5$, then we can assume that some $z_i, (i = 1, 2, 3)$ is in the C_5. We then 5-color $G - z_0$ such that at least one vertex distinct from z_i has the same color as z_i. We can then extend this 5-coloring to a 5-coloring of G. Note that if $c(z_1) = c(z_3)$ then x and y cannot be adjacent to the same four colors as this would result in a 5-coloring of G.

Now suppose that $G = K_2 + H_7$. Again, x and y each have degree at least four in G_1 and z_0 has degree eight in G. Therefore, we may assume that $N(y)$ (the neighborhood of y) and $N(x)$ have vertex sets $\{z_1, z_2, w_1, w_2, w_3\}$ and $\{z_2, z_3, u_1, u_2\}$, respectively. Since $G - \{z_0, z_3\}$ is 4-colorable, there exists a 5-coloring c of $G_1 - x$ such that y and z_3 are the only vertices of color 1. We may also assume that x is adjacent to four colors. For instance, let u_1, u_2, z_2 have colors 2, 3, 4, respectively. If $c(z_1) \neq 5$, we can color x by color 5 and complete the proof. Thus assume that $c(z_1) = 5$. Then if y is not adjacent to colors $2, 3$, change $c(y)$ to 2 or 3 and complete the proof. Thus assume that $c(w_1) = 2$ and $c(w_2) = 3$. Now, interchange the colors of w_1 and y (and change $c(w_3)$ to 1 if $c(w_3) = 2$). This new coloring extends to the desired 5-coloring unless z_3 is adjacent to w_1 (or to w_3 if $c(w_3) = 2$). Similarly, z_3 is adjacent to u_2. Thus the degree of z_3 is at least 6. We have shown above that in any 4-coloring of $G - \{z_0, z_3\}$, u_1, u_2, z_1, z_2 are forced to have distinct colors. So one of these vertices has degree eight (and it must be one of z_1, z_2. So z_3 is the unique vertex in G of degree six. If we 4-color $G - \{z_0, z_3\}$ and then delete the vertex in $G - \{z_0, z_3\}$ which is joined to all other vertices, then we obtain a 3-coloring of H_7 minus the vertex $z - 3$ of degree 4 in H_7. In this 3-coloring, the four neighbors of z_3 (namely u_1, u_2, w_1, w_2) must have three distinct colors. This contradiction proves part (i).

For part (ii), notice that at least one of x, y, (wlog, say x) has degree at most four in G_1. As in (i) we can 5-color $G_1 - x$ such that y and z_3 are vertices of color 1. This proves (ii).

For part (iii), we may assume that x has degree at most five in G_1, else we use the proof of part (ii). If y has degree 2 in G_1, then we first 5-color G_1 and then modify the color of y, if necessary. We can then assume that y has degree at most 3 in G_1. So z_0 is joined to all the other vertices of G. Suppose that w is a vertex in $N(y) - \{z_1, z_2\}$. Since $G - \{w, z_0\}$ is 4-colorable, $G_1 - y$ has a 5-coloring c'' such that x and w are the only vertices colored 5. We can extend c'' to a 5-coloring of G_1. This coloring c'' has the desired property. □

Lemma 3.3.2. *Let (G_0, v_0) be an optimal pair, and let v_1, v_2 be an identifiable pair. Then $G_{v_1v_2}$ has no subgraph isomorphic to $C_3 + C_5$ or $K_2 + H_7$.*

Proof. This follows by using the argument of [39, Theorem 6.1, Claim (9)], using Lemma 3.2.6 instead of [39, Lemma 5.2]. We will follow Thomassen's exposition throughout this proof.

Suppose for purposes of contradiction that there exists a subgraph $H_{v_1v_2}$, a subgraph of $G_{v_1v_2}$ such that $H_{v_1v_2} = C_3 + C_5$ or $H_{v_1v_2} = K_2 + H_7$. Let z_0 be the vertex in $H_{v_1v_2}$ that corresponds to $\{v_1, v_2\}$ in G_0. Define $v_1u_1u_2 \cdots u_kv_2z_1z_2 \cdots z_mv_1$ be the facial walk in the subgraph of $G_0 - v_0$ induced by $(V(H_{v_1v_2}) \setminus \{z_0\}) \cup \{v_1, v_2\}$ that bounds the face containing v_0. We may assume that $1 \leq k \leq m$, $\{v_1, v_2\} \cap \{u_1, \ldots, u_k\} = \emptyset$ and that G_0 is drawn on the Klein bottle such that $k + m$ is minimized. We obtain $G_{v_1v_2}$ by deleting one edge in each double edge of $H'_{v_1v_2}$, where $H'_{v_1v_2}$ is the multigraph induced by $H_{v_1v_2}$.

By our assumption either $H_{v_1v_2} = C_3 + C_5$ or $H_{v_1v_2} = K_2 + H_7$, so $G_{v_1v_2}$ has exactly one face bounded by a 4-cycle. All other faces are bounded by a 3-cycle. This also holds for $H'_{v_1v_2}$, as all other faces here are bounded by either 3-cycles or 2-cycles. So $k \leq 2$ and $m \leq 3$. The vertices $v_2, z_1, \ldots z_m, v_1$ are distinct. Note that $z_1 \neq v_2$ as

v_2z_1 is an edge in G_0 and $z_1 \neq v_1$ as v_1v_2 is not an edge in G_0. Also $z_2 \neq v_1, v_2$ as G_0 does not have multiple edges. Finally, all vertices of $G_{v_1v_2}$ are either in $H_{v_1v_2}$ or inside one of the cycles $R_1 : v_0v_1u_1 \cdots u_kv_2v_0$ or $R_2 : v_0v_2z_1 \cdots z_mv_1v_0$ by Corollary 3.2.2. Suppose that q_i is the number of vertices inside R_i for $i = 1, 2$.

First suppose that $m = 3$. Then $H'_{v_1v_2}$ has no 2-cycle except possibly $v_1u_1v_2$ if $k = 1$. Any other 2-cycle would be of the form v_1wv_2, and it would be nonfacial, which contradicts Euler's formula. It follows that all vertices $v_1, u_1 \ldots, u_k, v_2, z_1, \ldots, z_k$ are distinct except that z_2 could be equal to one of u_1, u_2.

By Lemma 3.2.1 and Corollary 3.2.2, $q_1 \leq 1$ and $q_2 \leq 3$. If $q_2 = 3$, then by Euler's formula, $N(v_0)$ contains no C_3. Since $H_{v_1v_2}$ is a $C_3 + C_5$ or $K_2 + H_7$, we can find a vertex v_1 of degree five (in both $H_{v_1v_2}$ and G_0) such that $N(v_1)$ has a C_3, unless $H_{v_1v_2} = C_3 + C_5$, $k = 2$ and z_1, z_2, z_3, u_1, u_2 are the vertices of the C_5. In this case we can 5-color G_0 as follows: Color v_1, v_2 by 1, 2, respectively. Since $H'_{v_1v_2}$ has no double edges when $k = 2$, we color the other two vertices of the C_3 in $C_3 + C_5$ colors 1 and 3. Color z_3 by 2 and color the remaining vertices in the C_5 of $C_3 + C_5$ colors 4 or 5. This coloring extends to a coloring of G_0 and therefore gives a contradiction when $q_2 = 3$.

Suppose that $m = 3$ and $q_2 = 2$. We also obtain a contradiction as above, unless the interior of R_2 contains vertices w_1, w_2 and the edges $w_1w_2, w_1v_0, w_1v_2, w_1z_1, w_1z_2, w_2z_2, w_2z_3, w_2v_1, w_2v_0$, and v_0 is also adjacent to $z_2 \in \{u_1, u_2\}$. By Corollary 3.2.2, $q_1 = 0$ and G_0 is 5-colorable by Lemma 3.3.1, part (i). This shows that $q_2 < 2$ when $m = 3$.

If $m = 3$ and $q_2 = 1$ and the vertex inside R_2 is w_1, then w_1 is joined to at least five vertices of R_2. If w_1 is joined to both v_1 and v_2, then $k = 2$ and v_0 is adjacent to x, w_1, y, u_2, u_1. Then $N(v_0)$ contains at most one C_3 (if $z_2 \in \{u_1, u_2\}$). We can then find a vertex v of degree 5 in $H_{v_1v_2}$ and in G_0 such that $N(v)$ has at least two $C_3 - s$ (where s is a vertex in each C_3) because $\{z_1, z_2, z_3, u_1, u_2\}$ has less than five vertices.

This contradiction shows that w_1 is not adjacent to both v_1 and v_2. We can define the vertices such that w_1 is adjacent to v_0, v_2, z_1, z_2, z_3 and v_0 is adjacent to z_3 and $z_2 \in \{u_1, u_2\}$. In particular, $q_1 = 0$. Any 5-coloring of $G_0 - \{v_0, w_1\}$ that satisfies the conclusion of Lemma 3.3.1, part (iii) can be extended to a 5-coloring of G. This proves that $q_2 = 0$ if $m = 3$.

Suppose now that $m = 3$ and $q_2 = 0$. Then v_0 has at least one neighbor in $\{z_1, z_2, z_3\}$. Then v_0 is adjacent to both z_1 and z_3 for if v_0 is adjacent only to z_i, then we can add the edge $z_i v_1$ (where $i = 1$ or 2). This contradicts condition (iv) of the definition of optimal pair for v_0. (Note that if v_0 is adjacent to z_2 and $z_2 = u_1$, then $q_1 = 0$, since otherwise we add $z_2 v_1$ and delete $v_1 u_1$ and this brings us to the case when $q_2 = 1$. Then v_0 is adjacent to z_1 and edge $z_1 v_1$ can be3 added.) So the 5-coloring of $G_0 - v_0$ in part (ii) of Lemma 3.3.1 can be extended to a 5-coloring of G_0. This shows that $1 \leq k \leq m \leq 2$. Possibly $\{z_1, z_2\} \cap \{u_1, \ldots, u_k\} \neq \emptyset$.

We now 5-color G_0 minus the interior of the walk $W = v_2 z_1 z_2 v_1 u_1 \cdots u_k v_2$. By Lemma 3.2.1, the interior of W either contains just one vertex (and this must be v_0) adjacent to all five colors, or two vertices v_0, v_0' and edges $v_0 v_2, v_0 z_1, v_0 z_2, v_0 v_1, v_0 v_0', v_0' v_1$, $v_0' u_1, v_0' u_2, v_0' v_2$. Suppose first that the interior of W contains both v_0 and v_0'. Since $G_{v_1 v_2}$ is $C_3 + C_5$ or $K_2 + h_7$ with an additional vertex v_0', we can find a vertex, r_1 with degree five in G_0 such that $N(r_1)$ has at least one triangle. In addition if $N(v_0)$ has at least one triangle, then r_1 can be chosen such that $N(r_1)$ has at least two triangles. So $N(v_0)$ has at least two triangles by property (iii) of an optimal pair. By Euler's formula, G_0 can not contain both edges $v_1 z_1, v_2 z_2$. So $k = 2$ and $\{z_1, z_2\} \cap \{u_1, u_2\} \neq \emptyset$. It cannot be that $\{z_1, z_2\} = \{u_1, u_2\}$ because then $H'_{v_1 v_2}$ would have two double edges, again contradicting Euler's formula. So we may assume that $u_1 = z_1$ and $u_2 \neq z_2$. By Lemma 3.2.1, z_2 and u_2 have the same color in any 5-coloring of $G_0 - \{v_0, v_0'\}$. So $G_1 = (G_0 - \{v_0, v_0'\}) \cup \{z_2 u_2\}$ is not 5-colorable. Hence G_1 contains a subgraph, G_1' which is one of the graphs on our list, by the minimality of G_0. If $z_2 u_2$ is contained in a facial cycle $z_2u_2qz_2$ of G_1', then either $qz_2v_1u_1u_2q$ or $qz_2z_1v_2u_2q$ is a contractible 5-cycle with more than one vertex in its interior, contradicting Lemma 3.2.1. So z_2u_2 is not in a facial 3-cycle of G_1'. Thus $G_1' = K_6$. This K_6 contains z_2 and u_2 but none of v_1, v_2 because $H'_{v_1v_2}$ has only one double edge; v_1 or v_2 has degree at most six in G_0, since otherwise v_1, v_2, v_0, v_0' are inside a walk of length six. So $G_1' = K_6$ can be obtained from $H_{v_1v_2}$ by first deleting a vertex of degree at most six (and some more vertices) and then adding an edge. This is impossible because $H_{v_1v_2} = C_3 + C_5$ or $K_2 + H_7$.

We now return to the case when W has at most one vertex, namely v_0, in its interior. If $k = 1$, then there is at most one vertex distinct from u_1, say u_0 that is adjacent to both v_1 and v_2, by Euler's formula. If $k = 2$, we can interchange between $\{z_1, z_2\}$ and $\{u_1, u_2\}$. In any case we can assume that $N(v_0)$ has vertex set $\{x, y, z_1, z_2, u_1\}$ and can apply Lemma thomlem5.2 as well as an analogous lemma on $K_2 + H_7$ to 5-color G_0.

□

Lemma 3.3.3. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, let J be a subgraph of $G_{v_1v_2}$ isomorphic to L_1, L_2, L_5 or L_6, and let R_1, R_2 be the hinges of J. If R_1 and R_2 share a vertex $u \neq z_0$ and at least one of them has length three, then the other one has length five and there exists an index $i \in \{1, 2\}$ such that $\hat{R}_1 \cup \hat{R}_2 \backslash \{v_0, v_i\}$ is a cycle in G_0 that bounds an open disk containing v_0 and v_i.*

Proof. By the symmetry we may assume that R_2 has length three. Thus u is adjacent to z_0 in J. Since R_1 is an induced cycle, the cycles R_1, R_2 share the edge z_0u. Thus $\hat{R}_1$, $\hat{R}_2$ share the edge v_iu for some $i \in \{1, 2\}$, and the second conclusion follows. By Lemma 3.2.1 the cycle $\hat{R}_1 \cup \hat{R}_2 \backslash \{v_0, v_i\}$ has length at least six, and hence R_1 has length five, as desired. □

We denote by K_5^- the graph obtained from K_5 by deleting an edge, and by $K_5 - P_3$

the graph obtained from K_5 by deleting two adjacent edges.

Lemma 3.3.4. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1 v_2}$ isomorphic to L_1, L_2, L_5 or L_6. Then there exists a vertex $s \in V(G_0) - \{v_0\}$ of degree five such that*

(i) *$N_{G_0}(s)$ has a subgraph isomorphic to $K_5 - P_3$, and*

(ii) *if both hinges of J have length five, then $N_{G_0}(s)$ has a subgraph isomorphic to K_5^-.*

Proof. We begin by proving the first assertion. Assume that the notation is as in the paragraph prior to Lemma 3.3.2, and suppose first that $J = L_5$. Let the vertices of J be numbered as in Figure 12. It follows from Lemma 3.2.9 that the indices of a_i and b_j can be renumbered so that the faces of J around c_1 are $a_1c_1a_2, a_2c_1a_3$, $a_3c_1a_4, a_4c_1b_3b_5c_2$, $b_3c_1b_2, b_2c_1b_1$, $b_1c_1a_1c_2b_4$, in order. Recall that z_0 is the vertex of J that results from the identification of v_1 and v_2. If $z_0 \neq c_1$, then one of the vertices a_2, a_3, b_2 is not incident with $\hat{R}_1$ or $\hat{R}_2$, and hence has the same neighbors in J and in G_0. It follows that such a vertex satisfies the conclusion of the lemma, as desired. We will use the same argument again later, whereby we will simply say that a certain vertex satisfies the conclusion of the lemma and assume that this means condition (i) in this part of the proof.

Thus we may assume that $z_0 = c_1$, and since we may assume that no vertex satisfies the conclusion of the lemma, we deduce that one of R_1 and R_2 is the face $a_2c_1a_3$ and the other is $b_1c_1b_2$ or $b_2c_1b_3$. Thus we may assume that R_1 is $a_2c_1a_3$ and R_2 is $b_1c_1b_2$. We may assume, by swapping v_1 and v_2, that the neighbors of v_1 in $\hat{J}$ are a_1, a_2, v_0, b_1 and that the neighbors of v_2 are a_3, a_4, b_3, b_2, v_0. Hence the face $\hat{R}$ is $v_1a_2a_3v_2b_2b_1$. Now v_1 is not adjacent to a_3 in G_0, for otherwise a_2 satisfies the conclusion of the lemma. We shall abbreviate this argument by $a_2 \Rightarrow v_1 \not\sim a_3$. Similarly, we have $b_5 \Rightarrow b_3 \not\sim c_2$ and $b_3 \Rightarrow v_2 \not\sim b_5$. We shall define a 5-coloring c of

$\hat{J} \setminus v_0$. Let $c(a_1) = c(v_2) = c(b_5) = 1, c(a_2) = c(b_1) = 2, c(a_3) = c(v_1) = 3, c(a_4) = 4$, and $c(c_2) = c(b_3) = 5$. Assume first that b_4 is adjacent to a_1. Then b_2 is not adjacent to v_1, for otherwise b_1 satisfies the conclusion of the lemma. Furthermore, there is no vertex of G in the face of $\hat{J}$ bounded by $b_1v_1a_1c_2b_4$. In that case we let $c(b_4) = 4$ and $c(b_2) = 3$. If b_4 is not adjacent to a_1, then we let $c(b_4) = 3$ and $c(b_2) = 4$. In either case it follows from Lemma 3.2.1 and the fact that v_0 is adjacent to v_1 and v_2 that c extends to a 5-coloring of G_0, a contradiction. This completes the case $J = L_5$.

If $J = L_6$ we proceed analogously. By Lemma 3.2.9 we may assume that the faces around c_1 are $a_1c_1a_2$, $a_2c_1a_3$, $a_3c_1a_4$, $a_4c_1b_4b_5c_2$, $b_4c_1b_3$ $b_3c_1b_2$, $b_2c_1b_1$ and $b_1c_1a_1c_2b_5$. If $z_0 \neq c_1$, or if one of R_1, R_2 is not $a_1c_1a_2$ or $b_2c_1b_3$, then one of a_2, a_3, b_2, b_3 satisfies the conclusion of the lemma. Thus we may assume that R_1 is $a_2c_1a_3$ and R_2 is $b_2c_1b_3$. We may also assume, by swapping v_1 and v_2 that the neighbors of v_1 in $\hat{J}$ are a_1, a_2, v_0, b_1 and b_2 and the neighbors of v_2 in $\hat{J}$ are a_3, a_4, b_3, b_4, and v_0. Now $a_1 \Rightarrow v_1 \not\sim y$, $b_4 \Rightarrow v_2 \not\sim b_5$, $a_3 \Rightarrow a_2 \not\sim v_2$, and $b_2 \Rightarrow b_3 \not\sim v_1$. With these constraints in mind and recalling that v_0 is adjacent to v_1 and v_2, consider the following coloring: $c(a_4) = c(b_1) = 1, c(a_1) = c(b_2) = 2, c(b_3) = c(v_1) = c(c_2) = 3, c(a_3) = c(b_4) = 4$ and $c(b_5) = c(a_2) = c(v_2) = 5$. It follows from Lemma 3.2.1 that c extends to a 5-coloring of G_0, a contradiction. This completes the case $J = L_6$.

We now consider the case $J = L_1$. By Lemma 3.2.8 exactly one face of J, say F, is bounded by a cycle of length five, and the remaining faces are bounded by triangles. Furthermore, we may assume, by swapping b_1, b_2, and by permuting a_1, a_2, a_3 that the faces around c_1 in order are $F, b_2c_1b_1, b_1c_1c_0, c_0c_1a_1, a_3c_1a_1, a_2c_1a_3$. By swapping b_3, b_4 we may assume that the faces around c_2 are $F, b_3c_2b_4, b_4c_2c_0, c_0c_2a_\alpha, a_\beta c_2a_\alpha, a_\gamma c_2a_\beta$ for some distinct indices $\alpha, \beta, \gamma \in \{1, 2, 3\}$. Thus the face F is bounded by the cycle $c_1a_2c_2b_3b_2$, and hence $\gamma = 2$. Since $a_1c_0c_1, c_1c_0b_1, b_4c_0c_2$ and $c_2c_0a_\alpha$ are faces of J we deduce that the faces around c_0 in order are $a_1c_0c_1, c_1c_0b_1, b_1c_0b_i, b_ic_0b_j, b_jc_0b_4, b_4c_0c_2$, $c_2c_0a_\alpha, a_\alpha c_0a_\delta, a_\delta c_0a_1$ for some integers i, j, δ with $\{i, j\} = \{2, 3\}$ and $\delta \in \{2, 3\} - \{\alpha\}$.

Since $\gamma = 2$ we have $\alpha \neq 2$, and hence $\alpha = 3$ and $\delta = 2$.

Now if $z_0 \neq c_0$, then one of the vertices $a_1, a_2, a_3, b_1, b_2, b_3, b_4$ satisfies the conclusion of the lemma, and hence we may assume that $z_0 = c_0$. Furthermore, it is not hard to see that one of the above vertices satisfies the conclusion of the lemma unless one of R_1, R_2 is $a_1c_0a_2$ or $a_2c_0a_3$ and the other is one of $b_1c_0b_i$, $b_ic_0b_j$, $b_jc_0b_4$. Thus by symmetry we may assume that R_1 is $a_1c_0a_2$ and that R_2 is one of $b_1c_0b_i$, $b_ic_0b_j$, $b_jc_0b_4$.

We may assume that in $\hat{J}$ the vertex v_1 is adjacent to c_1 and v_2 is adjacent to c_2. We see that $a_3 \Rightarrow c_1 \not\sim c_2$ and $a_3 \Rightarrow a_1 \not\sim v_2$. Furthermore, if R_2 is the face $b_1c_0b_i$, then $b_4 \Rightarrow b_1 \not\sim v_2$, and if R_2 is the face $b_1c_0b_i$, then $b_1 \Rightarrow v_1 \not\sim b_4$. Let c be the coloring of $\hat{J}\backslash v_0$ defined by $c(b_1) = c(v_2) = 1$, $c(b_i) = c(a_1) = 2$, $c(b_j) = c(v_1) = c(a_3) = 3$, $c(b_4) = c(a_2) = 4$, and $c(x) = c(y) = 5$, and let c' be obtained from c by changing the colors of the vertices v_1, v_2, a_2 to $4, 2, 1$, respectively. It follows from Lemma 3.2.1 by examining the three cases for R_2 separately that one of c, c' extends to a 5-coloring of G, a contradiction. This completes the case $G = L_1$.

Finally, let $J = L_2$. We proceed similarly as above, using Lemma 3.2.8. Let F be the unique face of J of size five. By renumbering a_1, a_2, a_3 and b_1, b_2, b_3 we may assume that the faces around c_1 are $F, b_3c_1b_2, b_2c_1b_1, b_1c_1c_0, c_0c_1a_1, a_1c_1a_3, a_3c_1a_2$. Then the faces around c_2 are $F, b_4c_2c_0, c_0c_2a_\alpha, a_\alpha c_2a_\beta, a_\beta c_2a_\gamma$ for some distinct integers $\alpha, \beta, \gamma \in \{1, 2, 3\}$. It follows that $\gamma = 2$ and that F is bounded by $c_1b_3b_4c_2a_2$. Since $b_1c_1c_0, c_0c_1a_1, b_4c_2c_0, c_0c_2a_\alpha$ are faces of J we deduce that $\alpha \neq 1$ (and hence $\alpha = 3$ and $\beta = 1$) and that the cyclic order of the neighbors of c_2 around c_2 is $c_1b_1b_ib_jb_4c_2a_3a_2a_1$ for some distinct integers $i, j \in \{2, 3\}$. (Recall that all faces incident with c_0 are triangles.) Since b_4 is adjacent to b_3 in the boundary of F we deduce that $i = 3$ and $j = 2$.

Similarly as above, it is easy to see that some a_i or b_j satisfies the conclusion of the lemma, unless $z_0 \in \{c_0, c_1\}$. Suppose first that $z_0 = c_1$. We may assume that R_1 is $b_1b_2c_1$ and R_2 is $a_1a_3c_1$, for otherwise some vertex satisfies the conclusion of the lemma. We may assume that v_1 is adjacent to a_2, a_3, b_2, b_3. We have $a_2 \Rightarrow v_1 \not\sim c_2$, $a_1 \Rightarrow a_3 \not\sim v_2$ and $b_2 \Rightarrow v_1 \not\sim b_1$. Let $c(a_2) = c(b_2) = 1, c(a_3) = c(b_4) = c(v_2) = 2, c(a_1) = c(b_3) = 3, c(v_1) = c(b_1) = c(c_2) = 4$, and $c(c_0) = 5$. It follows from Lemma 3.2.1 that c extends to a 5-coloring of G_0, a contradiction. Thus we may assume that $z_0 = c_0$. Similarly as above we may assume that R_1 is $b_1b_3c_0$ or $b_3b_2c_0$ and that R_2 is $a_1a_2c_0$ or $a_2a_3c_0$. We may assume that v_1 is adjacent to a_1 and b_1. If R_2 is $a_1a_2c_0$, then we have $a_3 \Rightarrow c_1 \not\sim c_2$ and $a_3 \Rightarrow a_1 \not\sim v_2$. If R_2 is $a_2a_3c_0$, then $a_1 \Rightarrow c_1 \not\sim c_2$ and $a_1 \Rightarrow a_3 \not\sim v_2$. If R_1 is $b_1b_3c_0$, then $b_2 \Rightarrow b_1 \not\sim v_2$. Let $c(a_1) = c(b_1) = c(v_2) = 1, c(b_3) = 2, c(a_2) = c(b_2) = 3, c(a_3) = c(b_4) = c(v_1) = 4$ and $c(c_1) = c(c_2) = 5$. It follows from Lemma 3.2.1 that c extends to a 5-coloring of G_0, a contradiction. □

We now prove the second assertion, using similar techniques as the previous case. We begin by showing that z_0 is not a vertex of degree five, and then proceed by handling each of L_1, L_2, L_5 or L_6 individually. First suppose that $z_0 = c_2$ in L_2 or L_6. In this case, one of b_1, b_2, b_3 is a vertex that satisfies the condition of the lemma (by which, in this and in subsequent cases we mean condition (ii)). Now, suppose that z_0 is a vertex of degree five in L_1, L_2 not already handled above. In the original graph, by criticality, z_0's five neighbors are colored 1, 2, 3, 4, 5, respectively. Further, the cycle containing z_0, call it C_1, must be a cycle of length seven. Cycle C_1 can be no larger as z_0 is degree five and L_1, L_2 consists of a single 5-cycle. Further, if C_1 has length less than seven, then after the split operation there are three internal vertices inside C_1, not all mutually adjacent, a contradiction. So there are no chords inside the 5-cycle contained within the 7-cycle.

Suppose that $C_1 = d_1d_2d_3d_4d_5f_1f_2d_1$. Here $c(d_i) = i$. Let $c(f_1) = 1, c(f_2) = 5$. We must now check cases (i) - (vi) of Lemma 3.2.1 to ensure that every coloring of the rest of the 6-cycle extends inside it. To do this, first notice that cases (iv)-(vi) do not hold because the vertices colored the same around the 7-cycle are not in the locations stated in the lemma. In case (i) for an internal vertex to be adjacent to five vertices on C_1, the extra added edges creates a situation where there are two vertices inside a 5-cycle, a contradiction. Similarly in case (ii), where two adjacent vertices must be adjacent to four vertices on C_1, this leaves all triangles and a 4-cycle inside C_1, but there is an additional internal vertex (as the split creates at least three internal vertices), a contradiction. In case (iii), we must have three internal vertices to be adjacent to each other and to three vertices on C_1. But this is impossible as $v_1 \not\sim v_2$ and so there must be an internal vertex in either R_1, R_2, but this internal vertex can only see two vertices on C_1. This completes the case when z_0 is a vertex of degree five in L_1 or L_2.

Now suppose there is a vertex of degree 5 in L_5, L_6 not previously considered above. Notice that if $z_0 = b_i$, then one of a_2, a_3 satisfies the condition of the lemma. So we may assume that $z_0 = a_i$ for some i. We wish to apply the same argument as the previous paragraph, but now we may not be able to color vertices f_1, f_2 in C_1 arbitrarily because they may be part of the other 5-cycle in L_5 or L_6. In this case without loss of generality, the vertices of C_1, are, in order $c_2a_4a_3a_2c_1b_1b_5c_2$. Let $c(c_1) = 1, c(c_2) = 5, c(a_2) = 2, c(a_3) = 3, c(a_4) = 4$. Let C_2 be the other 5-cycle, defined by $c_1a_3c_2b_ib_jc_1$. The vertices b_i, b_j differ when considering L_5 and L_6. Suppose first that we are considering L_5. Then $b_i = b_4$ and $b_j = \{b_1, b_2, b_3\}$. Suppose there is no chord in C_2. Then let $c(b_i) = 3$, let $c(b_5) = 1, c(b_1) = 5$, and color the rest of the b_i's properly. Then the argument for coloring inside C_1 is identical to the case for L_1 and L_2 above. If instead there were chords in C_2, the only chord that would force us to color C_1 differently is if there was an edge between b_1 and c_2. In this case, let $c(b_1) = 2$ and let $c(b_5) = 1$. Color the rest of the vertices on C_1 as before. Notice that the arguments for cases (i) - (iii) above did not depend on the specific coloring of C_1, so these arguments also hold here. Also observe that cases (v) and (vi) do not hold because in our coloring the pairs of vertices colored the same are consecutive.

In case (iv) the only time the pairs of vertices colored the same are consecutive is not in the same configuration as our coloring of C_1. This completes the case when z_0 is a vertex of degree 5 in L_5.

Suppose instead that z_0 is a vertex of degree five in L_6. Now, vertex b_5 is present in both C_1 and C_2. So C_2 is the 5-cycle, defined by $c_1a_3c_2b_5b_jc_1$, where $j = \{2, 3\}$. Suppose there are no chords in C_2. Then we may color the vertices of C_1 and C_2 as in the previous paragraphs. So suppose that C_2 has a chord. In particular, the only chord that affects the coloring of C_1 is edge c_1b_5. In this case, let $c(b_5) = 4, c(b_1) = 5, c(b_2) = 2$. The same arguments concerning cases (i) - (vi) of Lemma 3.2.1 now apply to this situation. This concludes the case when z_0 is a vertex of degree 5 in L_6.

So we may now assume that z_0 has degree at least six. Further, we may assume that for L_6, $z_0 = c_1$ and we first consider this case. Notice that one hinge must use vertices of the form a_i and the other must use vertices of the form b_i. Otherwise, one of vertices b_2, b_3, a_2, a_3 satisfy the conditions of the lemma. Suppose that c_1 is split into vertices v_1, v_2 such that without loss of generality, vertex v_1 is adjacent to at least one vertex of the form b_i. Let $c(c_2) = 2$ and $c(b_5) = 3$. If there is an edge between b_5 and v_1, then $c(v_1) = 1$ and $c(a_1) = 3$. Otherwise let $c(v_1) = 3$ and $c(a_1) = 1$. If there is an edge between v_2 and c_2, then $c(v_2) = 5$ and $c(b_4) = 2$. Otherwise let $c(v_2) = 2$ and $c(b_4) = 1$. Let b_α be the vertex of the form b_i of highest index not adjacent to v_2 via the split, and let a_α be the vertex of lowest index not adjacent to v_1 via the split. If there is no edge in R_1 between v_2 and b_α, let $c(b_\alpha) = c(v_2)$. Otherwise, color b_α properly. Similarly if there is no edge in R_2 between v_1 and a_β, let $c(a_\beta) = c(v_1)$. Otherwise, color a_β properly. Notice that both edges are not present else v_0 is contained in a 4-cycle. If one such edge is present, notice that v_0 is contained in a 5-cycle but two vertices are colored the same. The rest of the vertices of the form a_i and b_i may now be properly colored, and so the graph can be properly colored.

We now consider the case of L_5.

First suppose that $z_0 = c_2$, the vertex of degree six. Notice that if the split uses two triangles containing vertices of the form a_i, but not b_i, then $a_1a_2a_3a_4v_2a_1$ is a 5-cycle that contains two internal vertices, namely v_0 and v_1 from the split construction, a contradiction. So it follows, up to symmetry, that the split is between $b_4c_2b_5$ and $a_1c_2a_2$ or $b_4c_2b_5$ and $a_2c_2a_3$. Also, assume without loss of generality that the 5-cycles in this graph are $a_1c_1b_1b_4v_1$ and $a_4c_1b_2b_5v_2$. First suppose that the split is between $b_4c_2b_5$ and $c_2a_1a_2$. Let $c(a_i) = i$ for $i = \{1, 2, 3, 4\}$. Let $c(c_1) = 5$. Let $c(v_1) = 2, c(v_2) = 1$. Here v_1 is adjacent to a_1 and b_4, and v_2 is adjacent to a_2, a_3, a_4, b_5. Edge a_1v_2 is not present else a_3 is a vertex that satisfies the conditions of the lemma. Edge b_1c_5 is not present else either vertex b_2, b_3 is one we desire. Also, b_4a_1 is not present else there are two vertices in 5-cycle $b_4a_1a_2v_2b_5$. Let $c(b_1) = 2, c(b_2) = 3, c(b_3) = 4, c(b_4) = 1, c(b_5) = 5$. If edge v_1b_1 or v_1a_2 is present color v_2 color 3. If b_4v_2 is present, let $c(b_4) = 4, c(b_3) = 1$. These colorings do not allow for cases (i) - (iii) in 6-cycle $a_1a_2v_2b_5b_4v_1$, and so we are finished with this case. Now suppose the split is between $b_4c_2b_5$ and $a_2c_2a_3$. Suppose that $c(a_1) = 1, c(a_2) = 2, c(a_3) = 3, c(a_4) = 4$, $c(c_1) = 5, c(b_1) = 2, c(b_2) = 3, c(b_3) = 4, c(b_4) = 1, c(b_5) = 5, c(v_1) = 3, c(v_2) = 2$. Now, if a_2v_2 is present let $c(v_2) = 1$. If $a3v_1$ is present let $c(v_1) = 4$. Notice that edges b_4a_1 and b_5c_1 can not occur as in the argument above. So all we must do is ensure that conditions (i)-(iii) of Lemma 3.2.1 do not hold in the 6-cycle $a_2a_3v_2b_5b_4v_1$. If neither a_3v_1 nor b_4a_1 were present, then none of these conditions hold. The same is true if edge a_2v_1 was present. However, if edge a_3v_1 was present then let $c(v_2) = 1$, and then none of the conditions of Lemma 3.2.1. This completes the proof that $z_0 \neq c_2$ in L_5.

Now suppose that $z_0 = c_1$ in L_5. As before, notice that if the split uses two triangles containing vertices of the form a_i, but not b_i, then $a_1a_2a_3a_4v_2a_1$ is a 5-cycle that contains two internal vertices, namely v_0 and v_1 from the split construction, a contradiction. Also, assume without loss of generality that the 5-cycles in this graph are $a_1c_1b_1b_4v_1$ and $a_4c_1b_2b_5v_2$. Up to symmetry there are two situations. First suppose that we split between triangles $b_1c_1b_3$ and $a_1c_1a_2$. Now, suppose that edge v_2b_1 is not present. Then let $c(a_1) = 1$, $c(a_2) = 2$, $c(a_3) = 3$, $c(a_4) = 4$, $c(c_2) = 5$, $c(b_1) = 1$, $c(b_2) = 2$, $c(b_3) = 5$,$c(b_4) = 3$, $c(b_5) = 1$, $c(v_1) = 3$, $c(v_2) = 1$. Here v_1 is adjacent to b_1, a_1 and v_2 is adjacent to a_2, a_3, a_4, b_2, b_3. Edge a_1b_1 is not present else there are two vertices inside 5-cycle $a_1b_1b_3v_2a_2a_1$. Also, edge a_1v_2 is not present else one of a_2, a_3 is a vertex that satisfies the conditions of this lemma. If edge b_4a_4 is present the given coloring is a proper 5-coloring. If this edge is not present, then there may be a vertex in 5-cycle $a_4c_2b_4b_2v_2a_4$, and if this is the case, then let $c(b_5) = 3, c(v_1) = 4, c(b_4) = 4$. Thus, we may assume that v_2b_1 is present. Then edge v_2b_4 is not present else either b_2, b_3 satisfies the condition of the lemma. Again, edge a_1b_1 is not present else there are two vertices inside 5-cycle $a_1b_1b_3v_2a_2a_1$. Also, edge a_1v_2 is not present else an internal vertex is inside a 4-cycle. Now, let $c(a_1) = 1$, $c(a_2) = 2$, $c(a_3) = 3$, $c(a_4) = 4$, $c(c_2) = 5$, $c(b_1) = 4$, $c(b_2) = 3$, $c(b_3) = 5$,$c(b_4) = 1$, $c(b_5) = 2$, $c(v_1) = 3$, $c(v_2) = 1$. This coloring gives a proper 5-coloring if there is a vertex of degree 5 in cycle $a_1c_1b_1b_4v_2a_1$. In this case, let $c(b_1) = 3, c(b_2) = 2$. This finishes the case of the first split for c_1 in L_5.

Now, suppose that we split between triangles $b_1c_1b_3$ and $a_2c_1a_3$. Suppose that edge a_2v_2 was present. Then let $c(a_1) = 1$, $c(a_2) = 2$, $c(a_3) = 3$, $c(a_4) = 4$, $c(c_2) = 5$, $c(b_1) = 1$, $c(b_2) = 2$, $c(b_3) = 5$,$c(b_4) = 4$, $c(b_5) = 3$, $c(v_1) = 4$, $c(v_2) = 1$. Notice that edge a_1b_1 is not present as the 5-cycle $a_1b_1b_3v_2a_2a_1$ contains two internal vertices. This coloring is a proper 5-coloring if edge a_4b_4 does not exist. If a_4b_4 is present, then let $c(b_4) = 3, c(b_5) = 4$. So we may suppose a_2v_2 is not present. Similarly, suppose edge b_1v_2 is not present. Then let $c(a_1) = 1$, $c(a_2) = 2$, $c(a_3) = 3$, $c(a_4) = 4$, $c(c_2) = 5$, $c(b_1) = 1$, $c(b_2) = 3$, $c(b_3) = 5$,$c(b_4) = 4$, $c(b_5) = 2$, $c(v_1) = 4$, $c(v_2) = 2$. Again, a_1b_1 is not present. This is a valid coloring unless edge b_4v_4 is present. If so, then let $c(b_4) = 5$, $c(b_3) = 4$. This gives a proper 5-coloring. Thus we may assume edges a_2v_2 and b_1v_2 are not present.

Now we will condition on whether chords b_5v_1 and a_4b_4 are present. In this series of four colorings, let $c(a_1) = 1$, $c(a_2) = 2$, $c(a_3) = 3$, $c(a_4) = 4$, $c(c_2) = 5$, $c(v_2) = 2$. Now suppose that neither chord is present. Then let $c(b_1) = 2$, $c(b_2) = 1$, $c(b_3) = 5$,$c(b_4) = 4$, $c(b_5) = 3$, $c(v_1) = 3$. If chord a_4b_4 is present but b_5v_1 is not, then let $c(b_1) = 2$, $c(b_2) = 5$, $c(b_3) = 1$,$c(b_4) = 3$, $c(b_5) = 4$, $c(v_1) = 4$. If chord b_5v_1 is present but a_4b_4 is not, then let $c(b_1) = 2$, $c(b_2) = 1$, $c(b_3) = 3$,$c(b_4) = 4$, $c(b_5) = 3$, $c(v_1) = 4$. If both chords b_5v_1, a_4b_4 are present, then let $c(b_1) = 2$, $c(b_2) = 1$, $c(b_3) = 5$,$c(b_4) = 3$, $c(b_5) = 4$, $c(v_1) = 3$. Notice that in the 6-cycle $a_2a_3v_2b_3b_1v_1a_2$, there are three vertices colored the same. Thus, the coloring extends inside this 6-cycle and each of these colorings are valid. This completes the case when z_0 is a vertex in L_5.

Suppose that z_0 is a vertex in L_2. Consider the case when $z_0 = c_1$ of L_2. Consider the following cycle of vertices around c_1: $b_1b_2b_3b_4c_2a_3a_2a_1c_0b_1$. Notice that the split must use one triangle that involves vertices of the form b_i and one triangle that involves vertices of the form a_i, else either a_2 or b_2 is a vertex that satisfies the conditions of the lemma. Further, we know that the 5-cycle that includes c_1 in L_2 is $b_3b_4c_2a_3c_1b_3$. Also note that around c_1, the two triangles that are split must be at least three edges apart. In general let v_1 be the vertex of the split that is adjacent to c_0. Up to symmetry, this means we must split along the following pairs of triangles: $(b_2c_1b_3, a_2c_1a_3), (b_2c_1b_3, a_1c_1a_2), (b_2c_1b_3, c_0c_1a_1), (b_1c_1b_2, a_1c_1a_2)$.

First suppose that we split between triangles $b_2c_1b_3$ and $a_2c_1a_3$. In this case notice that $b_3 \not\sim a_3$ else two vertices are contained in the 5-cycle $b_2b_32a_3a_2v_1b_2$. Also, $c_1 \not\sim a_3$ and $b_3 \not\sim c_1$ else a_2 or b_2, respectively would be a vertex that satisfies the conditions of the lemma. Now let $c(b_1) = 1, c(b_2) = 2, c(b_3) = 3$, $c(b_4) = 4, c(c_0) = 5, c(v_1) = 3$, $c(c_2) = 1, c(a_1) = 2, c(a_2) = 4$, $c(a_3) = 3$. Notice that three vertices are colored the same in the 7-cycle $b_2b_3b_4c_0a_3a_2v_1b_2$. This eliminates cases (iv) - (vi) of Lemma 3.2.1. Further, condition (i) does not hold since the 7-cycle contains only four colors. Condition (iii) does not hold since two colors are only used once, and condition (ii) does not hold because there does not exist a pair of internal vertices that can each see four different colors.

Now suppose the split was between triangles $b_2c_1b_3$ and $a_1c_1a_2$. If there is an edge between c_1a_2 then we argue as in the case above. Further, $c_1 \not\sim b_3$ else b_2 satisfies the conditions of the lemma. So let $c(b_1) = 1, c(b_2) = 2, c(b_3) = 3,\ c(b_4) = 4, c(c_0) = 5, c(v_1) = 3,\ c(a_1) = 4, c(a_2) = 3, c(a_3) = 1,\ c(v_2) = 5, c(c_2) = 2$. This coloring is a proper 5-coloring unless there is a vertex adjacent to all five vertices of the 5-cycle $a_3c_2b_4b_3v_2a_3$. In this case, let $c(c_2) = 2$ or 4, depending upon whether there edge b_2v_2 or a_1v_2 is present.

Next, suppose the split was between triangles $b_2c_1b_3$ and $c_0c_1a_1$. In this case edge c_0v_2 is not present else a_2 satisfies the conditions of the lemma. Also, $b_3 \not\sim v_1$ else vertex b_2 satisfies the conditions of the lemma. Then let $c(b_1) = 1, c(b_2) = 2, c(b_3) = 3$, $c(b_4) = 4, c(c_0) = 5, c(v_1) = 3,\ c(a_1) = 4, c(a_2) = 3, c(a_3) = 1,\ c(v_2) = 5, c(c_2) = 2$. This works unless there is a vertex adjacent to all five vertices of the 5-cycle $a_3c_2b_4b_3v_2a_3$. In this case, let $c(a_1) = 1$ and $c(a_3) = 4$.

Finally suppose the split was between triangles $b_1c_1b_2$ and $a_1c_2a_2$. Notice that edges a_2v_1 and b_2v_1 are not present else we can use arguments from the last three cases to satisfy the lemma. Now, let $c(b_1) = 1, c(b_2) = 2, c(b_3) = 3,\ c(b_4) = 4, c(c_0) = 5, c(v_1) = 2,\ c(a_1) = 3, c(a_2) = 2, c(a_3) = 4,\ c(v_2) = 5, c(c_2) = 1$. This coloring holds unless there is a chord between b_4 and a_3. If this is the case, let $c(c_2) = 3, c(a_3) = 1, c(a_1) = 4$. Notice that this coloring holds unless there is a chord between b_3 and c_2. But both these chords may not be present simultaneously, so this completes the proof of this split and the proof when $z_0 = c_1$ in L_2.

Now suppose that $z_0 = c_0$ in L_2. In this case, observe that the neighborhood around c_0 in L_2 is entirely triangles, but there is a 5-cycle that is defined by $c_1a_2c_2b_4b_2c_1$. Notice that triangle $c_0c_2b_4$ can not be used for any split else either a_2 or b_2 satisfies the conditions of the lemma. Further, one triangle must contain a vertex of the form a_i and another triangle must contain a vertex of the form b_i, else either a_2 or b_2 is a vertex that satisfies the conditions of the lemma. Also note that around c_0, the two triangles that are split must be at least three edges apart around the neighborhood of c_0. Up to symmetry, this gives seven pairs of triangles which must be analyzed, $(a_3c_0c_2, b_2c_0b_3)$, $(a_1c_0a_2, b_3c_0b_4)$, $(c_1c_0a_1, b_3c_0b_4)$, $(a_2c_0a_3, b_2c_0b_3)$, $(a_1c_0a_2, b_2c_0b_3)$, $(c_1c_0a_1, b_2c_0b_3)$, $(c_1c_0a_1, b_2c_0b_3)$, $(a_1c_0a_2, b_1c_0b_2)$. Without loss of generality, let v_1 be the vertex of the split adjacent to c_1. In all cases, assume that the vertices around c_0 are, in order, $c_1b_1b_2b_3b_4c_2a_3a_2a_1c_1$. Further, suppose that the 5-cycle in L_2 is defined by, $b_2b_4c_2a_2c_1b_2$. In all the colorings when $z_0 = c_0$, let $c(b_1) = 1, c(b_2) = 2$, $c(b_3) = 3, c(b_4) = 4$, $c(c_1) = 5$.

First consider the case when triangles $a_3c_0c_2$ and $b_2c_0b_3$ are split. Notice that $v_1 \not\sim c_2$ else there are two vertices in the 5-cycle $v_1c_2b_4b_3b_2$. Also $v_1 \not\sim b_3$, else there are two vertices in the 5-cycle $c_1b_3b_4c_2a_3c_1$. With this in mind, let $c(v_1) = 3, c(v_2) = 2$, $c(c_2) = 3, c(a_1) = 4$, $c(a_2) = 1$, $c(a_3) = 2$. This coloring holds unless there is vertex inside the 5-cycle in L_2. In this case, let $c(a_1) = 1, c(a_2) = 4$. Observe that this coloring does not allow for conditions (i)-(iii) of Lemma 3.2.1 to hold around the 6-cycle $b_2b_3v_2c_2a_3v_1b_2$. This completes the case, call it Case 1.

Now suppose that triangles $a_1c_0a_2$ and $b_3c_0b_4$ are split. First suppose that $v_2 \not\sim a_1$. Now, $v_1 \not\sim b_4$, else b_2 is a vertex that satisfies the conditions of the lemma. Vertex $v_1 \not\sim a_2$ as this is symmetric to Case 1 of this series of arguments. Let $c(v_1) = 4, c(v_2) = 1$, $c(c_2) = 3, c(a_1) = 1$, $c(a_2) = 4$, $c(a_3) = 2$. This coloring admits a proper 5-coloring unless edge a_2b_4 exists. In this case let $c(c_2) = 5, c(a_2) = 3$. This coloring holds unless edge c_1c_2 exists, but both c_1c_2 and a_2b_4 can not be present simultaneously. Now, suppose that $v_2 \sim a_1$. Then let $c(v_1) = 4, c(v_2) = 1$, $c(c_2) = 5, c(a_1) = 2$, $c(a_2) = 3$, $c(a_3) = 4$. Notice that if c_1c_2 exists then a_3 is a vertex we desire, so this coloring admits a proper 5-coloring. Call this Case 2.

Next, suppose that triangles $c_1c_0a_1$ and $b_3c_0b_4$ are split together. Then notice that $c_1 \not\sim v_2$ and $c_1 \not\sim b_4$ else a_2 is a vertex that satisfies the conditions of the lemma. Further $v_1 \not\sim a_1$ as this would give a situation equivalent to Case 2. Let $c(v_1) = 4, c(v_2) = 5$, $c(c_2) = 1, c(a_1) = 4$, $c(a_2) = 3$, $c(a_3) = 2$. This coloring admits a proper 5-coloring unless there is a vertex in 5-cycle $b_2b_4c_2a_2c_1b_2$. In this case, let $c(a_2) = 2$ and $c(a_3) = 3$. Call this Case 3.

Suppose that triangles $a_2c_0a_3$ and $b_2c_0b_3$ are split. Notice that $v_1 \not\sim a_3$, else we are in Case 1. Further $v_1 \not\sim b_3$ else we are in a situation that is symmetric to Case 3. So let $c(v_1) = 3, c(v_2) = 5$, $c(c_2) = 1, c(a_1) = 2$, $c(a_2) = 4$, $c(a_3) = 3$. This coloring admits a proper 5-coloring unless edge a_2b_4 is present in 5-cycle $b_2b_4c_2a_2c_1b_2$. In this case, let $c(a_2) = 1$ and $c(c_2) = 2$. Notice that edges b_2c_2 and a_2b_4 can not exist simultaneously. This allows us to complete a proper 5-coloring. Call this Case 4.

Suppose now that triangles $a_1c_0a_2$ and $b_2c_0b_3$ are split. In this case, notice that $v_1 \not\sim a_2$ else we are in the situation defined by Case 4, and $v_1 \not\sim b_3$ else we are in Case 2. Now, let $c(v_1) = 3, c(v_2) = 5$, $c(c_2) = 2, c(a_1) = 1$, $c(a_2) = 3$, $c(a_3) = 4$. This coloring admits a proper 5-coloring unless edge b_2c_2 is present in 5-cycle $b_2b_4c_2a_2c_1b_2$. In this case, let $c(a_1) = 2$ and $c(c_2) = 1$. This allows us to complete a proper 5-coloring. Call this Case 5.

Next, suppose that triangles $c_1c_0a_1$ and $b_2c_0b_3$ are split. In this case, notice that $c_1 \not\sim v_2$ else there are two vertices in 5-cycle $c_1v_2b_3b_2b_1c_1$ and $b_2 \not\sim v_2$ else there are two vertices in 5-cycle $c_1a_1v_2b_2b_1c_1$. Further, notice that if there is a chord between c_1c_2 in 5-cycle $b_2b_4c_2a_2c_1b_2$, then a_2 is a vertex that satisfies the conditions of the lemma. Now, let $c(v_1) = 4, c(v_2) = 2$, $c(c_2) = 5, c(a_1) = 3$, $c(a_2) = 1$, $c(a_3) = 4$. This coloring allows us to complete a 5-coloring. Call this Case 6.

Finally, suppose that triangles $a_1c_0a_2$ and $b_1c_0a_2$ were split. Then notice that $a_1 \not simv_2$ else there are two vertices in 5-cycle $a_1v_2b_2b_1c_1a_1$, $b_1 \not\sim v_2$ else there are two vertices in 5-cycle $b_1v_2a_2a_1c_1b_1$. Further, $v_1 \not\sim b_2$ and $v_1 \not\sim a_2$ else we are in Case 5. Let $c(v_1) = 2, c(v_2) = 1$, $c(c_2) = 5, c(a_1) = 1$, $c(a_2) = 3$, $c(a_3) = 2$. This coloring admits a proper 5-coloring unless there is an edge between c_1 and c_2 in the 5-cycle $b_2b_4c_2a_2c_1b_2$. In this case let $c(c_2) = 1, c(a_1) = 4$. This completes the final case when $z_0 = c_0$ in L_2. This completes the proof for L_2.

Suppose that z_0 is a vertex in L_1. We have already considered vertices of degree five. This leaves c_0, c_1, c_2. In this case, observe that c_1 and c_2 are identical up to symmetry so without loss of generality, consider the case when $z_0 = c_1$. The vertices forming an 8-cycle around c_1 are $c_0b_1b_2b_3c_2a_3a_2a_1c_0$. Notice that this includes the neighbors of c_1 and the 5-cycle that contains c_1, $c_1b_2b_3c_2a_3c_1$. Note that around c_1, the two triangles that are split must be at least three edges apart around the neighborhood of c_0. Up to symmetry, this gives three pairs of triangles which must be analyzed, namely $(a_1c_1a_2, b_1c_1b_2)$, $(a_2c_1a_3, b_1c_1b_2)$, and $(a_2c_1a_3, c_0c_1b_1)$. In all the colorings when $z_0 = c_1$, let $c(b_1) = 1, c(b_2) = 2$, $c(b_3) = 3, c(b_4) = 4$, $c(c_1) = 5$.

Now, first suppose we are in the case when triangles $a_1c_1a_2$ and $b_1c_1b_2$ are split. Notice that $b_1 \not\sim v_2$ else there are two vertices contained in 5-cycle $b_1v_2a_2a_1c_0b_1$. Also $a_1 \not\sim v_2$ else there are two vertices contained in 5-cycle $a_1v_2b_2b)1c_0a_1$. Let $c(v_2) = 5$, $c(c_2) = 1$, $c(a_1) = 3$, $c(a_2) = 2$, $c(a_3) = 4$. This coloring, unless there is a vertex of degree 5 in 5-cycle $b_2b_3c_2a_3v_2$ gives a valid 5-coloring as the 6-cycle $b_1b_2v_2a_2a_1c_0b_1$ does not satisfy conditions (i)-(iii) in Lemma 3.2.1. If such a vertex of degree five exists, then let $c(a_2) = 4, c(a_3) = 2$. This completes Case 1 of the situation when $z_0 = c_1$ of L_1.

Suppose that triangles $a_2c_1a_3$ and $b_1c_1b_2$ are split. If both v_1b_2 and v_1a_3 were present then two vertices are inside the 5-cycle $v_1b_2b_3c_2a_3v_1$. Then suppose that neither edge v_1b_2 nor v_1a_3 were present. In this case, the coloring where $c(v_1) = 2$, $c(v_2) = 4$, $c(c_2) = 1$, $c(a_1) = 4$, $c(a_2) = 3$, $c(a_3) = 2$ admits a valid 5-coloring. Notice that edge b_3a_3 is not present else there would be two vertices inside the 5-cycle $b_2a_3a_2v_1b_1b_2$. Suppose edge v_1b_2 was present but v_1a_3 was not. Let $c(a_2) = 2$,$c(v_1) = 3$, $c(a_3) = 3$, $c(c_2) = 2$. Notice that if one of edges b_3a_3 or b_2c_2 was present, two vertices would again be inside a 5-cycle. So this coloring admits a valid 5-coloring. finally suppose that v_1a_3 was present. Then let $c(a_2) = 1$,$c(v_1) = 2$, $c(a_3) = 3$, $c(c_2) = 2$. For the same reasons as above in this paragraph, this coloring also admits a valid 5-coloring. This completes Case 2 of the situation when $z_0 = c_1$ of L_1.

Finally, suppose that triangles $a_2c_1a_3$ and $c_0c_1b_1$ are part of the split. In this case $c_0 \not\sim v_2$, else there are two vertices in the 5-cycle $c_0v_2a_3a_2a_1c_0$. Also, $v_1 \not\sim b_1$ else we are in Case 2 of this analysis. Vertex $b_2 \not\sim a_3$ as there are three non-mutually adjacent vertices in cycle $b_2a_3a_2a_1c_0b_1b_2$. Also $v_1 \not\sim a_3$, else a_2 is a vertex that satisfies the conditions of the lemma. With this in mind, let $c(a_1) = 2$, $c(a_2) = 3$, $c(a_3) = 4$, $c(c_2) = 1$, $c(v_2) = 1$. Notice that the 6-cycle $c_0a_1a_2a_3v_2b_1c_0$ does not satisfy conditions (i)-(iii) in Lemma 3.2.1. So this gives a valid coloring unless there is a vertex of degree five in the 5-cycle $c_1a_3c_2b_3b_2c_1$. In this case, let $c(a_2) = 3, c(a_3) = 4$. This completes the proof when $z_0 = c_1$ of L_1.

The final vertex to check is when $z_0 = c_0$ of L_1. In this case, observe that the neighborhood around c_0 in L_1 is entirely triangles, but there is a 5-cycle that is defined by $c_1a_2c_2b_3b_2c_1$. Further, one triangle must contain a vertex of the form a_i else a_2 is a vertex that satisfies the conditions of the lemma. Also note that around c_0, the two triangles that are split must be at least three edges apart around the neighborhood of c_0. Up to symmetry, this gives eight pairs of triangles which must be analyzed, $(c_1c_0a_1, b_3c_0b_2)$, $(c_1c_0a_1, b_2c_0b_4)$,$(c_1c_0a_1, b_4c_0c_2)$,$(c_1c_0a_1, c_2c_0c_3)$, $(a_1c_0a_2, b_1c_0b_3)$, $(a_1c_0a_2, b_3c_0b_2)$, $(a_1c_0a_2, b_2c_0b_4)$ and $(a_1c_0a_2, b_4c_0c_2)$. Without loss of generality, let v_1 be the vertex of the split adjacent to c_1. In all cases, assume that the vertices around c_0 are, in order, $c_1b_1b_3b_2b_4c_2a_3a_2a_1c_1$. In all the colorings when $z_0 = c_0$, let $c(b_1) = 1, c(b_2) = 2$, $c(b_3) = 3, c(b_4) = 4$.

First suppose that c_0 is split between triangles $a_1c_0a_2$ and $b_4c_0c_2$. Note that $c_2 \not\sim v_1$ as there would be two vertices in 5-cycle $a_1v_1c_2a_3a_2a_1$. Also, $v_1 \not\sim a_2$, else two vertices are in 5-cycle $v_1a_2a_3c_2b_4$ In this case, let $c(c_1) = 3$, $c(v_1) = 5$, $c(a_1) = 2$, $c(a_2) = 5$,

$c(a_3) = 4$, $c(c_2) = 1$, $c(v_2) = 3$. Notice that if edge $c_1 \sim b_3$ then b_2 is a vertex that satisfies the conditions of the lemma. This coloring admits a proper 5-coloring. Call this Case 1.

Now suppose that c_0 is split between triangles $a_1c_0a_2$ and $b_2c_0b_4$. Notice $v_1 \not\sim b_4$ else we are in Case 1. Now, let $c(c_1) = 5$, $c(v_1) = 4$, $c(a_1) = 2$, $c(a_2) = 4$, $c(a_3) = 3$, $c(c_2) = 1$, $c(v_2) = 5$. This coloring holds if there is not a vertex of degree 5 inside the cycle $c_1b_2b_3c_2a_2c_1$ or edge v_1a_2 is not present. if v_1a_2 is present, let $c(a_2) = 3, c(a_3) = 4$. This coloring holds unless there edge a_2b_3 is present. Then let $c(c_2) = 5$, $c(a_2) = 1$. This completes Case 2.

Next, suppose that c_0 is split between triangles $a_1c_0a_2$ and $b_1c_0b_3$. Observe that $b_1 \not\sim v_2$, else 5-cycle $b_1v_2a_2a_1c_1b_1$ contains two vertices. Similarly, $a_1 \not\sim v_2$ else 5-cycle $a_1v_2b_3b_1c_1a_1$ contains two vertices. Now, let $c(c_1) = 5$, $c(a_1) = 2$, $c(a_2) = 4$, $c(a_3) = 3$, $c(c_2) = 1$, $c(v_2) = 5$. Notice that $b_1b_3v_2a_2a_1c_1b_1$ form a 6-cycle that satisfies conditions (i)-(iii) in Lemma 3.2.1. This coloring holds unless there is a vertex adjacent to all five vertices of cycle $a_2c_1b_2b_3c_2a_2$. In this case let $c(a_1) = 4, c(a_2) = 2$. This completes Case 3.

Now, suppose that c_0 is split between triangles $a_1c_0a_2$ and $b_3c_0b_2$. Notice $v_1 \not\sim b_2$ else we are in Case 2, and $v_2 \not\sim b_3$ else we are in Case 3. First suppose that edge a_1v_2 does not exist. Then let $c(c_1) = 4$, $c(a_1) = 3$, $c(a_2) = 5$, $c(a_3) = 2$, $c(c_2) = 1$, $c(v_2) = 3$, $c(v_1) = 2$. This holds unless there is a vertex adjacent to all vertices of the 5-cycle $a_2c_1b_2b_3c_2a_2$. In this case, let $c(c_2) = 2, c(a_3) = 1$. Finally, if a_1v_2 was present, let $c(a_1) = 1, c(a_2) = 4$,$c(a_3) = 2, c(c_2) = 5$. In this situation, notice that $c_1 \not\sim c_2$ as then a_2 is a vertex that satisfies the conditions of the lemma. This completes Case 4.

Suppose next that c_0 is split between triangles $c_1c_0a_1$ and $a_3c_0c_2$. Note that $a_1 \not\sim v_1$ else $a_1v_1c_2a_3a_2a_1$ contains two vertices. Also, $c_1 \not\sim v_2$ as then 5-cycle $c_1v_2a_3c_2v_1c_1$ also contains two vertices. Similarly $a_3 \not\sim v_1$ and $c_2 \not\sim v_2$, else two vertices are in 5-cycles $a_3v_1c_1a_1a_2a_3$, and $c_2a_2a_1c_1v_1c_2$, respectively. Now, let $c(c_1) = 4$, $c(a_1) = 2$, $c(a_2) = 5$,

$c(a_3) = 3$, $c(c_2) = 1$, $c(v_2) = 4$, $c(v_1) = 5$. This coloring holds unless there is a vertex of degree five in $c_2b_3b_2c_1a_2c_2$. In this case let $c(a_2) = 2, c(a_1) = 5$. This gives a proper 5-coloring of the graph. Call this Case 5.

Now, suppose that c_0 is split between triangles $c_1c_0a_1$ and $b_4c_0c_2$. Notice that $v_1 \not\sim c_2$ else we are in Case 5. Similarly, $a_1 \not\sim v_1$ else we are in Case 1. Let $c(c_1) = 4$, $c(a_1) = 2$, $c(a_2) = 5$, $c(a_3) = 3$, $c(c_2) = 1$, $c(v_2) = 4$, $c(v_1) = 5$. This admits a proper 5-coloring unless there is a vertex of degree five in $c_2b_3b_2c_1a_2c_2$. In this case let $c(a_2) = 3, c(a_3) = 5$. This completes Case 6.

Suppose that c_0 is split between triangles $c_1c_0a_1$ and $b_3c_0b_2$. Then it follows that $c_1 \not\sim v_2$ else $c_1v_2b_2b_3b_1c_1$ is a 5-cycle that contains two vertices. Similarly, $a_1 \not simv_1$, $b_3 \not\sim v_2$, else 5-cycles $a_1v_1b_3b_2v_2a_1$ and $b_3v_2a_1c_1b_1$, respectively, contain two vertices. Let $c(c_1) = 5$, $c(a_1) = 4$, $c(a_2) = 1$, $c(a_3) = 2$, $c(c_2) = 5$, $c(v_2) = 3$, $c(v_1) = 4$. Notice that $c_1 \not\sim c_2$ else a_2 is a vertex that satisfies the conditions of the lemma. The coloring given admits a proper 5-coloring. This completes Case 7.

Finally, suppose that c_0 is split between $b_2c_0b_4$ and $c_1c_0a_1$. Then notice that $v_1 \not\sim b_4$ else we are in Case 6, $v_2 \not\sim b_2$ else we are in Case 7 and $v_1 \not\sim a_1$ else we are in Case 2. Now, let $c(c_1) = 5$, $c(a_1) = 1$, $c(a_2) = 4$, $c(a_3) = 2$, $c(c_2) = 5$, $c(v_2) = 3$, $c(v_1) = 4$. Notice this coloring admits a proper 5-coloring unless there is an edge between c_1 and c_2. However, in this case, if $c_1 \sim c_2$, then a_2 is a vertex that satisfies the conditions of the lemma. This completes the case when $z_0 = c_0$, the case of L_1 and the proof.

□

Lemma 3.3.5. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1v_2}$. Then J is not isomorphic to L_1, L_2, L_5 or L_6.*

Proof. Let G, v_0, v_1, v_2 and J be as stated, and suppose for a contradiction that J is isomorphic to L_1, L_2, L_5 or L_6. Let R_1, R_2 be the hinges of J, and let $\hat{J}$, $\hat{R}_1$ and

$\hat{R}_1$ be as prior to Lemma 3.3.2. From Lemma 3.3.4 and conditions (ii)–(iv) in the definition of an optimal pair we deduce that

(1) $N_{G_0}(v_0)$ has a subgraph isomorphic to $K_5 - P_3$,

and

(2) if both R_1 and R_2 have length five, then v_1, v_2 is the only non-adjacent pair of vertices in $N_{G_0}(v_0)$.

Let v_3, v_4, v_5 be the remaining neighbors of v_0 in G_0. If at least two of them belong to the interior of $\hat{R}_1$ or $\hat{R}_2$, then they belong to the interior of the same face, say R_1, by (1). But then $\hat{R}_1$ is bounded by a cycle of length seven, and that again contradicts (1) by inspecting the outcomes of Lemma 3.2.1. Thus at most one of v_3, v_4, v_5 belongs to the interior of $\hat{R}_1$ or $\hat{R}_2$.

From the symmetry we may assume that the edges v_0v_4 and v_0v_5 belong to the face $\hat{R}_1$. We may also assume that v_5 belongs to the boundary of $\hat{R}_1$, and that if v_4 does not belong to the boundary of $\hat{R}_1$, then the edge v_0v_3 belongs to $\hat{R}_2$. We claim that v_4 belongs to the boundary of $\hat{R}_1$. To prove this suppose to the contradiction that v_4 belongs to the interior of $\hat{R}_1$. Then one of the edges v_1v_4, v_2v_4 does not belong to G_0, and so we may assume v_2v_4 does not. By (1) v_1, v_2 and v_2, v_4 are the only non-adjacent pairs of vertices in $N_{G_0}(v_0)$, and by (2) at least one of R_1, R_2 has length three. It follows that v_3 belongs to the boundary of $\hat{R}_1$, and the choice of v_4, v_5 implies that the edge v_0v_3 lies in the face $\hat{R}_2$. Thus v_3 belongs to the boundary of $\hat{R}_2$. By Lemma 3.3.3 there exists an index $i \in \{1, 2\}$ such that the cycle $R_1 \cup R_2 \backslash \{v_0, v_i\}$ bounds a disk containing v_0, v_i in its interior. By shortcutting this cycle through v_0 we obtain a cycle of G_0 of length at most four bounding a disk that contains the vertex v_i in its interior, contrary to Lemma 3.2.1. This proves our claim that v_4 belongs to the boundary of $\hat{R}_1$. We may assume that v_0, v_1, v_4, v_5, v_2 occur on the boundary of $\hat{R}_1$ in the order listed.

Let $e \in E(G_0)$ have ends either v_1, v_5, or v_2, v_4. Then $e \notin E(\hat{J})$, because the boundary of $\hat{R}_1$ is an induced cycle of $\hat{J}$. Moreover, e does not belong to the face $\hat{R}_1$, because the edges v_0v_4, v_0v_5 belong to that face. Thus e belongs to $\hat{R}_2$ or a face of $\hat{J}$ of length five. We claim that e does not belong to $\hat{R}_2$. To prove the claim suppose to the contrary that it does, and from the symmetry we may assume that $e = v_2v_4$. We now argue that not both R_1, R_2 are pentagons. Indeed, otherwise v_1 is adjacent to v_5 by (2), and the edge v_1v_5 belongs to $\hat{R}_2$, because there is no other face of length at least five to contain it. In particular, v_4, v_5 belong to the boundary of $\hat{R}_2$, and because the edges v_1v_5, v_2v_4 do not cross inside $\hat{R}_2$, the vertices v_1, v_0, v_2, v_4, v_5 occur on the boundary of $\hat{R}_2$ in the order listed. It now follows by inspecting the 5-cycles of L_5 and L_6 that this is impossible. Thus not both R_1, R_2 are pentagons. By Lemma 3.3.3 the cycle $\hat{R}_1 \cup \hat{R}_2 \backslash \{v_0, v_1\}$ bounds a disk with v_0, v_1 in its interior. By shortcutting this cycle using the chord v_2v_4 we obtain a cycle in G_0 of length at most five bounding a disk with at least two vertices in its interior, contrary to Lemma 3.2.1. This proves our claim that v_1v_5 and v_2v_4 do not lie in the face $\hat{R}_2$.

By (1) and the symmetry we may assume that $v_2v_4 \in E(G_0)$, and hence the edge v_2v_4 belongs to a face $\hat{F}$ of $\hat{J}$ such that $\hat{F} \neq \hat{R}_1, \hat{R}_2$. Let F be the corresponding face of J. Since F is bounded by an induced cycle, we deduce that v_4 is not adjacent to z_0 in J. Consequently, R_1 has length five. Thus R_1 and F have length five, and all other faces of J, including R_2, are triangles. In particular, $J = L_5$ or $J = L_6$, and v_1, v_5 are not adjacent in G_0 (because no face of $\hat{J}$ can contain the edge v_1v_5). By (1) v_1, v_2 and v_1, v_5 are the only non-adjacent pairs of vertices in $N_{G_0}(v_0)$. Condition (1) also implies that v_3 belongs to the boundary of $\hat{R}_2$. Using that and the fact that v_3 is adjacent to v_1 and v_2 in G_0, it now follows that there exists a vertex of $G_0 \backslash v_0$ whose neighborhood in G_0 has a subgraph isomorphic to $K_5 - P_3$. Finding such a vertex requires a case analysis reminiscent of but simpler than the proof of Lemma 3.3.4. We omit further details. The existence of such a vertex contradicts the fact that (G_0, v_0)

is an optimal pair. □

Lemma 3.3.6. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1v_2}$. Then J is not isomorphic to L_3 or L_4.*

Proof. Let G_0, v_0, v_1, v_2 and J be as stated, and suppose for a contradiction that J is isomorphic to L_3 or L_4. Let R_1, R_2 be the hinges of J, and let $\hat{J}, \hat{R}_1, \hat{R}_2$ be as prior to Lemma 3.3.2. Since by Euler's formula J triangulates the Klein bottle, we deduce that the faces $\hat{R}_1, \hat{R}_2$ have size five, and every other face of $\hat{J}$ is a triangle. Let the boundaries of $\hat{R}_1$ and $\hat{R}_2$ be $v_1v_0v_2a_1b_1$ and $v_1v_0v_2cb_l$, respectively. Let the neighbors of v_1 in $\hat{J}$ in cyclic order be $v_0, b_1, b_2, \ldots, b_l$, and let the neighbors of v_2 in $\hat{J}$ be $v_0, a_1, a_2, \ldots, a_k, c$. Then $\deg_J(z_0) = k + l + 1$. Since J has no parallel edges the vertices $a_1, a_2, \ldots, a_k, c, b_l, b_{l-1}, \ldots, b_1$ are distinct, and since J is a triangulation they form a cycle, say C, in the order listed. Since v_1 is not adjacent to v_2 in G_0, Lemma 3.2.1 implies that $|V(C)| \geq 7$.

Let us assume that $|V(C)| = 7$. Then z_0 has degree seven, and hence $J = L_4$, because L_3 has no vertices of degree seven. Let X be the set of neighbors of z_0 in J. By inspecting the graph obtained from L_4 by deleting a vertex of degree seven, we find that for every $x \in X$, there exists a 5-coloring of $J \setminus z_0$ such that no vertex of $X - \{x\}$ has the same color as x. But this contradicts Lemma 3.2.1 applied to the subgraph of G_0 consisting of all vertices and edges drawn in the closed disk bounded by C, because $X = V(C)$. This completes the case when $|V(C)| = 7$.

Since L_3 and L_4 have no vertices of degree eight, it follows that $|V(C)| = 9$, and hence z_0 is the unique vertex of J of degree nine. From the symmetry between v_1 and v_2, we may assume that $\deg_{\hat{J}}(v_1) \leq 5$; in other words $l \leq 4$. The graph J is 6-critical. Since z_0 is adjacent to every other vertex of J, we deduce that $J \setminus z_0 \setminus x$ is 4-colorable for every vertex $x \in V(J) - \{z_0\}$, and hence

(1) *for every vertex $x \in V(J) - \{z_0\}$, the graph $J \setminus z_0$ has a 5-coloring such that x*

is the only vertex colored 5.

From Lemma 3.2.1 applied to the boundary of the face $\hat{R}$ of $\hat{J} \setminus v_0$, we deduce that one of $\hat{R}_1, \hat{R}_2$ contains no vertex of G_0 in its interior, and the other contains at most one. Since v_0 has degree five, we may assume from the symmetry between $\hat{R}_1$ and $\hat{R}_2$ that v_0 is adjacent to a_1 and b_1 (and hence $\hat{R}_1$ includes no vertices of G_0 in its interior). We claim that $l = 4$ and v_1 is adjacent to c. To prove the claim suppose to the contrary that either $l \leq 3$ or v_1 is not adjacent to c. Then $\deg_{\hat{J}}(v_1) \leq 5$. By (1) there exists a coloring of $J \setminus z_0 = \hat{J} \setminus \{v_0, v_1, v_2\}$ such that b_1 is the only vertex colored 5. We give v_2 the color 5, then we color v_1, then we color the unique vertex in the interior of $\hat{R}_2$ if there is one, and finally color v_0. The last three steps are possible, because each vertex being colored sees at most four distinct colors. Thus we obtain a 5-coloring of G_0, a contradiction. This proves our claim that $l = 4$ and v_1 is adjacent to c. It follows that $k = 4$ and $V(G_0) = \{v_0, v_1, v_2, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c\}$. We have $\deg_{G_0}(v_1) = \deg_{G_0}(v_2) = 6$, and since $\deg_J(c) \leq \deg_{G_0}(c) - 2$, we deduce that $\deg_{G_0}(c) \geq 7$. Thus we have shown that

(2) *if* x_1, x_2, x_3, x_4, x_5 *are the neighbors of* v_0 *in* G_0 *listed in their cyclic order around* v_0*, the vertex* x_1 *is not adjacent to* x_3 *in* G_0 *and* G_{x_1,x_3} *has a subgraph isomorphic to* L_3 *or* L_4*, then* $\deg_{G_0}(x_1) = \deg_{G_0}(x_3) = 6$ *and* $\deg_{G_0}(x_2) \geq 7$.

It also follows that v_1 is not adjacent to a_1 in G_0 and that v_2 is not adjacent to b_1 in G_0. Not both $G_{v_1 a_1}$ and $G_{v_2 b_1}$ have a subgraph isomorphic to L_3 or L_4 by (2), and so from the symmetry we may assume that $G_{v_1 a_1}$ does not. By the optimality of (G_0, v_0) and Lemmas 3.3.2 and 3.3.5, it follows that $G_{v_1 a_1}$ has a subgraph isomorphic to K_6. Thus $G \backslash \{v_0, v_1, v_2\}$ has a subgraph K isomorphic to K_5. If $v_2 \notin V(K)$, then $V(K) \cup \{z_0\}$ induces a K_6 subgraph in J, a contradiction. Thus $v_2 \in V(K)$, and hence $V(K) = \{v_2, a_2, a_3, a_4, c\}$. Let $i \in \{3, 4\}$. If a_1 is not adjacent to a_i in G_0, then we 5-color G_0 as follows. By (1) there is a 5-coloring of $G_0 \setminus \{v_0, v_1, v_2\}$ such that a_1 and a_i are the only two vertices colored 5. We give v_1 color 5, then color v_2 and finally v_0. Similarly as before, this gives a valid 5-coloring of G_0 a contradiction. Thus, a_1 is adjacent to a_3 and a_4 and hence a_1 is not adjacent to c, for otherwise $\{a_1, a_2, a_3, a_4, v_2, c\}$ induces a K_6 subgraph in G_0.

Since $\deg_{G_0}(v_2) = 6$, it follows from (2) that G_{ca_1} has no subgraph isomorphic to L_3 or L_4. By the optimality of (G_0, v_0) and Lemmas 3.3.2 and 3.3.5 it follows that G_{ca_1} has a subgraph isomorphic to K_6. By an analogous argument as above we deduce that $\{v_1, b_1, b_2, b_3, b_4\}$ is the vertex-set of a K_5 subgraph of G_0. The existence of the two K_5 subgraphs implies that $a_2, a_3, a_4, b_2, b_3, b_4$ have K_4 subgraphs in their neighborhoods, and the optimality of (G_0, v_0) implies that $a_2, a_3, a_4, b_1, b_2, b_3$ all have degree at least six in G_0, and hence in J. Thus a_1, b_1, c are the only vertices of J of degree five. Thus, $J = L_3$ and a_1, b_1, c are pairwise adjacent, a contradiction, because we have shown earlier that a_1 is not adjacent to c. □

The results of this section may be summarized as follows.

Lemma 3.3.7. *Let (G_0, v_0) be an optimal pair, and let v_1, v_2 be an identifiable pair. Then $G_{v_1v_2}$ has a subgraph isomorphic to K_6.*

Proof. Every 5-coloring of $G_{v_1v_2}$ may be extended to a 5-coloring of G_0, and hence $G_{v_1v_2}$ is not 5-colorable. By the choice of G_0 the graph $G_{v_1v_2}$ has a subgraph isomorphic to one of the graphs listed in Theorem 3.1.2. By Lemmas 3.3.2, 3.3.5 and 3.3.6 that subgraph is K_6, as desired. □

3.4 Using K_6

Lemma 3.4.1. *Let (G_0, v_0) be an optimal pair. Then G_0 has at least* 10 *vertices, and if it has exactly* 10*, then it has a vertex of degree nine.*

Proof. First, suppose that $|V(G_0)| \leq 10$. We now employ a result of Gallai [16], which states that if G is a k-critical graph with at most $2k-2$ vertices, then G is of the form $G = G_1 + G_2$, where G_i is k_i-critical for $i = 1, 2$, and $k_1 + k_2 = k$. By this result, it follows that G_0 is of the form $H_1 + H_2$, where H_i is k_i-vertex-critical, $k_1 \leq k_2$, and $k_1 + k_2 = 6$. If $k_1 = k_2 = 3$, then we obtain either K_6 or $C_3 + C_5$ for G_0, a contradiction. So $k_1 \leq 2$ and therefore G_0 has a vertex adjacent to all other vertices. Now, suppose for purposes of contradiction that $|V(G_0)| \leq 9$. Let $k_1 = 1$. Then $|V(H_2)| \leq 8$ and so H_2 is of the form $H_2' + H_2''$, where $H_2' = K_2$ or K_1. So $k_1 = 2$ and that H_2 is 4-vertex-critical. By results of [16] and [43], the only 4-vertex-critical graphs with at most seven vertices are $K_4, K_1 + C_5, H_7$ and M_7, where M_7 is obtained from a 6-cycle, $x_1x_2 \cdots x_6x_1$ by adding an additional vertex v and edges $x_1x_3, x_3x_5, x_5x_1, vx_2, vx_4, vx_6$. However, $G_0 \not\supseteq K_2 + (K_1 + C_5) = K_3 + C_5$ and $G_0 \not\supseteq K_2 + H_7$. This implies that $G_0 \supseteq K_2 + M_7$. The graph G_0 has nine vertices and at least 27 edges, so $G_0 = K_2 + N_7$. However, $N(v)$ is not a wheel neighborhood, a contradiction. This proves the lemma. □

Lemma 3.4.2. *Let (G_0, v_0) be an optimal pair. Then there are at least two identifiable pairs.*

Proof. Since G_0 has no subgraph isomorphic to K_6 there is at least one identifiable pair. Suppose for a contradiction that v_1, v_2 is the only identifiable pair. Thus the subgraph of G_0 induced by v_0 and its neighbors is isomorphic to K_6 with one edge deleted. By Lemma 3.3.7 the graph $G_0 \backslash \{v_0, v_1, v_2\}$ has a subgraph K isomorphic to K_5, and every vertex of K is adjacent to v_1 or v_2. Let t be the number of neighbors of v_0 in $V(K)$. If $t = 0$, then G_0 has a subgraph isomorphic to L_5 or L_6; if $t = 1$, then G_0 has a subgraph isomorphic to L_1 or L_2; if $t = 2$, then G_0 has a subgraph isomorphic to $K_2 + H_7$; and if $t = 3$, then G_0 has a subgraph isomorphic to $C_3 + C_5$. □

Lemma 3.4.3. *Let (G_0, v_0) be an optimal pair. Then v_0 has a wheel neighborhood.*

Proof. Let us say that a vertex $v \in V(G_0)$ is a *fan* if its neighbors form a cycle in the order determined by the embedding of G_0. We remark that if v is a fan, then the embedding of G_0 can be modified so that v will have a wheel neighborhood. Thus it suffices to show that v_0 is a fan. Suppose for a contradiction that there exist non-adjacent vertices $a, b \in N(v_0)$ that are consecutive in the cyclic order of the neighbors of v_0. By condition (iv) in the definition of an optimal pair, the graph $G' = G_0 + ab$ has a subgraph M isomorphic to one of the graphs from Theorem 3.1.2. Assume, for a contradiction, that $v_0 \notin V(M)$. By optimality condition (i), $G_0 \backslash v_0$ has a 5-coloring c. Since c is not a 5-coloring of M it follows that $c(a) = c(b)$. But then c can be extended to a 5-coloring of G_0, a contradiction. Thus $v_0 \in V(M)$. Since $\deg(v_0) = 5$, we get that $N_{G_0}(v_0) \subseteq V(M)$. Further note that a, b are adjacent in M, because M is not a subgraph of G_0.

First, assume $M = K_6$. Then $V(M) = \{v_0\} \cup N(v_0)$. This implies that there is at most one identifiable pair, contrary to Lemma 3.4.2. Second, assume $M = L_3$ or L_4. As each is a triangulation, Lemma 3.2.1 implies that $G_0 = M \backslash ab$. But M is 6-critical, so G_0 has a 5-coloring, a contradiction.

Third, assume that $M = C_3 + C_5$ or $K_2 + H_7$. Because M is one edge short of being a triangulation, there is a unique face in M of length four. As $ab \in E(M)$, the embedding of $M \backslash ab$ has at most two faces of size strictly bigger than three, and if it has two, then they both have size four. Since G_0 has at least 10 vertices by Lemma 3.4.1, Lemma 3.2.1 implies that $M \backslash ab$ has a face f of size five whose interior includes a vertex of degree five. However, f is the only face of $M \backslash ab$ of size at least four, and hence it also includes the edge ab, but that is impossible. This completes the case when $M = C_3 + C_5$ or $K_2 + H_7$.

Fourth, suppose that M is either L_5 or L_6, and let the notation be as in the proof of Lemma 3.3.4. In particular, every face incident with a_2 or b_2 is a triangle. At least one of a_2, b_2, say s, is not equal to v_0 and does not include both a, b in its neighborhood. But then the neighborhoods of s in G and in M are the same, and hence s satisfies conditions (ii)-(iv) in the definition of an optimal pair by Lemma 3.4.2. But s is a fan in M, and hence has a wheel neighborhood in some embedding of G_0, contrary to condition (vi) in the definition of optimal pair.

If $M = L_1$, then we apply the argument of the previous paragraph to the vertices a_1, b_1, b_4, using the notation of Lemma 3.3.4. Finally, suppose that M is L_2, and let the notation be again as in the proof of Lemma 3.3.4. Every face incident with one of the vertices a_3, b_2 is a triangle, and at least one of those vertices, say s, has the property that $s \neq v_0$ and if the neighborhood of s includes both a and b, then a, b are not consecutive in the cyclic ordering around s and $\{a, b\} \cap \{x, y\} \neq \emptyset$ for every pair of distinct non-adjacent vertices $x, y \in N_M(v_0)$. Since s is a fan in M its choice implies that it is a fan in G_0, and hence has a wheel neighborhood in some embedding of G_0, contrary to condition (vi) in the definition of optimal pair. Furthermore, in G_0 there are at most two pairs of non-adjacent vertices in the neighborhood of s, and if there are two, then they are not disjoint. Thus s satisfies conditions (ii)-(iv) in the definition of an optimal pair by Lemma 3.4.2, contrary to condition (vi) in the definition of an optimal pair. □

A drawing of a graph G in a surface is 2-*cell* if every face of G is homeomorphic to an open disk.

Lemma 3.4.4. *Let (G_0, v_0) be an optimal pair, and let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1 v_2}$ isomorphic to K_6. Then the drawing of J in the Klein bottle is 2-cell.*

Proof. Let $v_0, R_1, R_2, \hat{R}_1, \hat{R}_2$ be as before, and suppose for a contradiction that the drawing of J is not 2-cell. Since K_6 has a unique drawing in the projective plane [20, page 364], it follows that every face of J is bounded by a triangle, and exactly one face, say F, is homeomorphic to the Möbius strip. If F is not R_1 or R_2, then the boundary of F is a separating triangle of G_0, a contradiction, because no 6-critical graph has a separating triangle. Thus we may assume that $F = R_2$.

Since both R_1 and R_2 are triangles, and they share at least one vertex, there exists a vertex $s \in V(J)$ not incident with R_1 or R_2. Thus in $\hat{J}$ all the faces incident with s are triangles, and hence $\deg_{G_0}(s) = \deg_J(s) = 5$ by Lemma 3.2.1. Furthermore, if R_1 and R_2 share an edge, then $N_{G_0}(s)$ has a subgraph isomorphic to K_5^-, the complete graph on five vertices with one edge deleted. This implies, by the optimality of (G_0, v_0), that $N_{G_0}(v_0)$ has a subgraph isomorphic to K_5^-, contradicting Lemma 3.4.2.

So we may assume that R_1 and R_2 have no common edge. Let the facial walk incident with $\hat{R}_1$ be $v_0, v_1, z_1, z_2, v_2, v_0$, and the facial walk incident with $\hat{R}_2$ be $v_0, v_1, z_3, z_4, v_2, v_0$. Notice, from the embedding of J, that the z_i are distinct. Also notice that s is the lone vertex in $\hat{J}$ not incident with either $\hat{R}_1$ or $\hat{R}_2$, and $N_{G_0}(s)$ includes no two disjoint pairs of non-adjacent vertices. This implies, by the optimality of (G_0, v_0), that $N_{G_0}(v_0)$ includes no two disjoint pairs of non-adjacent vertices. We shall refer to this as the DP property.

Let $N(v_0) = \{v_1, v_2, v_3, v_4, v_5\}$. Assume that some neighbor of v_0, say v_3, belongs to $\hat{R}_1$. By Lemma 3.2.1, v_3 is adjacent to all vertices incident with $\hat{R}_1$. Thus v_4 and v_5 belong to the closure of R_2. In either case, v_3 and v_4 are not adjacent in G_0. Since v_1 and v_2 are also not adjacent, this contradicts the DP property.

Since v_1 is not adjacent to v_2 in G_0 it follows from Lemma 3.4.3 that at least one of v_3, v_4, v_5 belongs to the closure of $\hat{R}_1$. Thus there remain two cases, depending on whether one or two of those vertices are incident with $\hat{R}_1$. If it is two vertices, then we may assume without loss of generality that $v_3 = z_1$ and $v_4 = z_2$. As z_1 and z_2 are not incident to $\hat{R}_2$, v_3, v_2 and v_4, v_1 are not adjacent in G_0, contrary to the DP property. Thus we may assume that $v_3 = z_1$ and v_4 and v_5 belong to the closure of $\hat{R}_2$. By the DP property v_3, v_4 and v_3, v_5 are adjacent in G_0. Thus, without loss of generality, $v_4 = z_3$ and $v_5 = z_4$. Furthermore, it follows from the DP property that either v_1, v_5 or v_2, v_4 are adjacent in G_0. Thus the subgraph L of $\hat{J}$ consisting of the vertices v_0, v_1, v_2, v_4, v_5 and edges between them that belong to the closure of $\hat{R}_2$ has five vertices and at least eight edges. We can regard L as drawn in the Möbius band with the cycle $v_1v_0v_2v_5v_4$ forming the boundary of the Möbius band. As such the graph L has at least three faces. Since the sum of the lengths of the faces is at least 11, at most one of them has length at least five. That face of L includes at most one vertex of G_0 by Lemma 3.2.1, and the other faces of L include none. Thus G_0 has at most nine vertices, contrary to Lemma 3.4.1. □

Lemma 3.4.5. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1v_2}$ isomorphic to K_6. Then some face of J has length six.*

Proof. Let $\tilde{J}$ denote the graph consisting of $\hat{J}$ and edges of G_0 not in $\hat{J}$ from v_1 or v_2 to the boundary of $\hat{R}_1$ or $\hat{R}_2$ that are drawn inside $\hat{R}_1$ or $\hat{R}_2$. Let $\tilde{R}_1$ be the face in $\tilde{J}$ that contains v_0 and is contained in R_1, and similarly for $\tilde{R}_2$. We assume for a contradiction that no face of J has length six. By Lemma 3.4.4 the embedding of J is 2-cell, and so, by Euler's formula, all faces of J are bounded by triangles, except for either three faces of length four, or one face of length four and one face of length five. Each face of $\tilde{J}$ other than $\tilde{R}_1$ and $\tilde{R}_2$ will be called *special* if it has length at least four. Thus there are at most three special faces, and if there are exactly three, then they have length exactly four.

Let us denote the vertices on the boundary of $\tilde{R}_1$ as v_1, v_0, v_2, $u_1, \ldots, u_k$ in order, and let the vertices on the boundary of $\tilde{R}_2$ be v_2, v_0, v_1, $z_1, \ldots, z_l$ in order. Note that $u_1, u_2, \ldots, u_k$ are pairwise distinct, and similarly for $z_1, z_2, \ldots, z_l$. A special face of length five may include a vertex of G_0 in its interior; such vertex will be called *special.* It follows that there is at most one special vertex. An edge of G_0 is called *special* if it has both ends in $\hat{J}\backslash v_0$, but does not belong to $\hat{J}$, and is not v_1z_1 or v_2z_1 if $l = 1$, and is not v_1u_1 or v_2u_1 if $k = 1$. It follows that every special edge is incident with v_1 or v_2. Furthermore, the multigraph obtained from G_0 by deleting all vertices in the faces $\tilde{R}_1$ and $\tilde{R}_2$ and contracting the edges v_0v_1 and v_0v_2 has J as a spanning subgraph, and each special edge belongs to a face of J of length at least four. It follows that there are at most three special edges. Furthermore, if there is a special vertex, then there is at most one special edge, and each increase of k or l above the value of two decreases the number of special edges by one.

Since R_1 and R_2 have length three, four, or five, we deduce that $k, l \in \{1, 2, 3, 4\}$. The graph $\hat{J} \backslash \{v_0, v_1, v_2\} = J \backslash z_0$ is isomorphic to K_5, and $u_1, u_2, \ldots, u_k$ are its distinct vertices; let $u_{k+1}, \ldots, u_5$ be the remaining vertices of this graph. It follows that if c is a 5-coloring of $\tilde{J}$ and $c(u_i) = c(z_j)$, then $u_i = z_j$. We will refer to this property as *injectivity*. From the symmetry we may assume that $k \geq l$. Since J has at most one face of length five, it follows that $l \leq 3$. We distinguish three cases depending on the value of l.

Case 1: $l = 1$

By Lemma 3.4.3 the vertex v_0 is adjacent to z_1. Also notice then that $v_1 z_1 v_2 u_1 u_2 \ldots u_k$ is a null-homotopic walk W of length at most seven. Since G_0 is 6-critical, the graph $G \backslash v_0$ has a 5-coloring, say c. By Lemma 3.2.1 applied to the subgraph L of G_0 drawn in the disk bounded by W and the coloring c, the graph L satisfies one of (i)–(vi) of that lemma. We discuss those cases separately.

Case (i): There are eight vertices in $\tilde{J}$ and none in the interior of $\tilde{R}_1$ and $\tilde{R}_2$, and at most one special vertex. Thus $|V(G_0)| \leq 9$, contradicting Lemma 3.4.1.

Case (ii): As before $|V(G_0)| \leq 9$, a contradiction, unless there exists a special vertex v_0'. This implies that $|\tilde{R}_1| = |\hat{R}_1| = 6$. Without loss of generality suppose v_0 is adjacent to u_3, v_1, z_1, v_2 and a vertex v_3 which is adjacent to v_0, v_2, u_1, u_2, u_3. Notice that v_0' must have degree five in G_0 and its neighborhood must contain a subgraph isomorphic to $K_5 - P_3$, since four of its neighbors are in $J \setminus z_0$ and thus form a clique. Meanwhile the neighborhood of v_0 is missing the edges $v_1 v_2$, $v_1 v_3$, and $v_2 u_3$. The last one does not belong to $\tilde{J}$, does not lie in $\tilde{R}_1$ (because we have already described the graph therein), and is not special, because all special edges have been accounted for.

Thus the pair (G_0, v_0') contradicts the optimality of (G_0, v_0).

Case (iii): The graph $L\backslash W$ consists of three pairwise adjacent vertices, and v_0 is one of them. Let v_3, v_4 be the remaining two. By Lemma 3.4.3 we may assume, using the symmetry that exchanges v_1, v_4, u_1, u_2 with v_2, v_3, u_k, u_{k-1}, that v_3 has neighbors v_0, v_2, u_1, u_2, v_4 and v_4 is adjacent to v_1, v_0, v_3, u_2 and either u_3 or u_4. In either case z_1 and u_2 are colored the same, and hence they are equal by injectivity. To be able to treat both cases simultaneously, we swap u_3 and u_4 if necessary; thus we may assume that v_4 is adjacent to u_3. We can do this, because we will no longer use the order of $u_1, u_2, \ldots, u_k$ for the duration of case (iii). The vertex v_1 is adjacent to u_2, u_3, u_4, u_5, for otherwise its color can be changed, in which case the coloring c could be extended to L, contrary to the fact that G_0 has no 5-coloring. Similarly, v_2 is adjacent to u_1, u_2, u_4, u_5. It follows that G_0 has a subgraph isomorphic to L_3, a contradiction. To describe the isomorphism, the vertices corresponding to the top row of vertices in Figure 7(c) in left-to-right order are u_1, u_4, u_5, u_3, the vertices corresponding to the middle row are $v_3, v_2, u_2 = z_1, v_1, v_4$, and the bottom vertex is v_0. This completes case (iii).

Cases (iv)-(vi): We have $k = 4$. Hence R_1 has length five, and therefore there is at most one special edge. Consequently, one of v_1, v_2 is not adjacent in $\tilde{J}$ to at least two vertices among u_1, u_2, u_3, u_4. Since every face of $\tilde{J}$ except $\tilde{R}_1$ and one other face of length four is bounded by a triangle this implies that in the coloring c, one of v_1, v_2 sees at most three colors. From the symmetry we may assume that v_2 has this property. Thus $c(v_2)$ may be changed to a different color.

By using this fact and examining the cases (iv)-(vi) of Lemma 3.2.1 we deduce that L is isomorphic to the graph of case (iv). Let the vertices of L be numbered as in Figure 9(iv). It further follows that $v_2 = x_4$ or $v_2 = x_5$, and so from the symmetry we may assume the former. Since z_1 has a unique neighbor in $L\backslash W$ we deduce that $z_1 = x_3$, $v_1 = x_2$, $u_4 = x_1$ and so on. Notice that x_8 has degree five in G_0 and that its neighborhood is isomorphic to $K_5 - P_3$. Meanwhile, the neighborhood of v_0 is certainly missing the edges v_1v_2 and v_1x_9. Now if $x_3 \neq x_5$ then x_3 is not adjacent to x_9 and $N(v_0)$ is missing at least three edges, a contradiction to the optimality of (G_0, v_0), given the existence of x_8. So $x_3 = x_5$, but then the edges x_3v_2, x_5v_2 are actually the same edge, because $\tilde{J}$ does not have parallel edges. It follows that v_2 has degree at most four in G_0, a contradiction.

Case 2: $l = 2$

By Lemma 3.4.3 either v_0 is adjacent to both z_1 and z_2, in which case we define $\overline{v}_0 := v_0$, or there exists a vertex $\overline{v}_0$ in $\tilde{R}_2$ adjacent to v_0, v_1, v_2, z_1, z_2. Let W denote the walk $v_1\overline{v}_0v_2u_1 \ldots u_k$ of length at most seven, and let X denote the set of vertices of G_0 drawn in the open disk bounded by W. We claim that $X \neq \emptyset$. This is clear if $\overline{v}_0 \neq v_0$, and so we may assume that $\overline{v}_0 = v_0$. But then $X = \emptyset$ implies $|V(G_0)| \leq 9$, contrary to Lemma 3.4.1. Thus $X \neq \emptyset$. Let $x \in X$ have the fewest number of neighbors on W. Since G_0 is 6-critical, the graph $G_0 \backslash x$ has a 5-coloring, say c. By Lemma 3.2.1 applied to the subgraph L of G_0 drawn in the disk bounded by W and the coloring c, the graph L and coloring c satisfy one of (i)–(vi) of that lemma.

Suppose first that L and c satisfy (i). Then $|X| = 1$ by the choice of x. As before $|V(G_0)| \leq 9$, contradicting Lemma 3.4.1, unless there is a special vertex. Hence $k \leq 3$. If $k = 3$, then R_1 has length four, and all special faces have been accounted for. In particular, $\tilde{J} = \hat{J}$. The fact that the coloring c cannot be extended to L implies that $\{c(z_1), c(z_2)\} \subseteq \{c(u_1), c(u_2), c(u_3)\}$, and hence $\{z_1, z_2\} \subseteq \{u_1, u_2, u_3\}$ by injectivity. Thus u_1 or u_3 is equal to one of z_1, z_2. Since there are no special edges, either u_1v_2 and z_2v_2, or u_kv_1 and z_1v_1 are the same edge, but then v_1 or v_2 has degree at most four, a contradiction. If $k = 2$ we reach the same conclusion, using the fact that in that case there is at most one special edge. It follows that L and c do not satisfy (i).

Next we dispose of the case $k \leq 3$. To that end assume that $k \leq 3$. Then W has length at most six. Thus L and c satisfy either (ii) or (iii) of Lemma 3.2.1, and so W has length exactly six and $k = 3$. In particular, R_1 has length four, and so there is either at most one special vertex, or at most two special edges, and not both. It follows that either $c(v_1)$ or $c(v_2)$ can be changed without affecting the colors of the other vertices of $G_0 \backslash X$. That implies that L and c satisfy (ii). Let v_3 be the unique neighbor of $\overline{v}_0$ in X, and let v_4 be the other vertex of X. From the symmetry we may assume that v_3 is adjacent to $\overline{v}_0, v_1, v_2, u_1, v_4$, and v_4 is adjacent to v_1, v_3, u_1, u_2, u_3. By considering the walk $u_1u_2u_3v_1z_1z_2v_2$ and the subgraph drawn in the disk it bounds, and by applying Lemma 3.2.1 to this graph and the coloring c we deduce that $|\{c(u_1), c(u_2), c(u_3)\} \cap \{c(z_1), c(z_2)\}| = 1$. That implies $|\{u_1, u_2, u_3\} \cap \{z_1, z_2\}| = 1$ by injectivity, and so we may assume that u_5 is not equal to z_1 or z_2. It follows that the neighborhood of u_5 has a subgraph isomorphic to $K_5 - P_3$. However, the neighborhood of $\overline{v}_0$ is missing v_1v_2 and at least one of the edges v_3z_1 and v_3z_2, contrary to the optimality of (G_0, v_0) if $v_0 = \overline{v}_0$. Similarly, the neighborhood of v_3 is missing v_1v_2 and $\overline{v}_0v_4$, a contradiction if $v_0 = v_3$. This completes the case $k \leq 3$.

Thus we may assume that $k = 4$. It follows that R_1 has length five, and hence there is at most one special edge. Let $i \in \{1, 2\}$. If v_i is adjacent to both z_1 and z_2, then one of the edges v_iz_1, v_iz_2 is special. It follows that in G_0, either v_1 is not adjacent to z_2, or v_2 is not adjacent to z_1. But z_2 is the only neighbor of v_1 in $G_0 \backslash X$ colored $c(z_2)$, because $G_0 \backslash (X \cup \{v_0, v_1, v_2\}$ is isomorphic to $J \backslash z_0$, which, in turn, is isomorphic to K_5. Thus there is a (proper) 5-coloring c_1 of $G_0 \backslash X$ obtained by changing the color of at most one of the vertices v_1, v_2 such that either $c_1(v_1) = c_1(z_2)$ or $c_1(v_2) = c_1(z_1)$. Now $c_1(\overline{v}_0)$ can be changed to another color, thus yielding a coloring c_2 of $G_0 \backslash X$.

If L and c satisfy one of the cases (iii)-(vi), then one of the colorings c_1, c_2 extends into L, a contradiction. Thus L and c satisfy (ii) of Lemma 3.2.1. Let $v_3 \in X$ be the unique vertex of X adjacent to $\overline{v}_0$, and let v_4 be the other vertex in X. If both v_3 and v_4 have degree five in G_0, then one of the colorings c_1, c_2 extends into L, a contradiction.

Thus one of v_3, v_4 has degree five, and the other has degree six. It follows that v_3 is adjacent to v_1, v_2, and either u_1 or u_4, and so from the symmetry we may assume it is adjacent to u_1. If $c_1(v_1) = c_1(u_1)$, then we can extend one of the colorings c_1, c_2 into L by first coloring v_4 and then v_3. Thus $c_1(v_1) \neq c_1(u_1)$. If v_4 is not adjacent to u_1, then we can extend c_1 or c_2 by giving v_4 the color $c_1(u_1)$, and then coloring v_3. Thus v_4 is adjacent to v_1. If v_4 has degree five, then its neighbors are u_1, u_2, u_3, u_4, v_3, and the neighbors of v_3 are $\overline{v}_0, v_1, v_2, u_1, u_4, v_4$. Let d a 5-coloring of $G_0 \backslash \overline{v}_0$. Since the coloring d cannot be extended to $\overline{v}_0$, it follows that the neighbors of $\overline{v}_0$ receive different colors. Now similarly as in the contruction of c_1 above, we can change either the color of v_1, or the color of v_2. The resulting coloring then extends to $\overline{v}_0$, a contradiction. This completes the case when v_4 has degree five, and hence v_4 has degree six. It follows that the neighbors of v_4 are $u_1, u_2, u_3, u_4, v_1, v_3$ and the neighbors of v_3 are $\overline{v}_0, v_1, v_2, u_1, v_4$. Let d_1 be a 5-coloring of the graph $G_0 \backslash \{\overline{v}_0, v_3\}$. Since the coloring d_1 does not extend into $\overline{v}_0, v_3$, we deduce that $\{d_1(z_1), d_1(z_2)\} = \{d_1(v_4), d_1(u_1)\}$. By injectivity this implies that $u_1 = z_1$ or $u_1 = z_2$. If $u_1 = z_2$, then one of the edges v_2u_1, v_2z_2 is special, because they cannot be the same edge, given that v_2 has degree at least five in G_0. Thus all special edges have been accounted for, and so z_1 is not adjacent to u_1. Thus $d_1(v_1)$ can be changed to $d_1(u_1)$, and the new coloring extends to all of G_0, a contradiction. Thus $u_1 = z_1$. It follows that G_0 is isomorphic to L_3. First of all, the vertex v_1 is not adjacent to both u_2 and u_3, for otherwise the vertices $v_1, v_4, u_1, u_2, u_3, u_4$ form a K_6 subgraph in G_0. If v_1 is adjacent to neither u_2 nor u_3, then v_2 is adjacent to these vertices, and an isomorphism between G_0 and L_3 is given by mapping the vertices in the top row in Figure 7(c), in left-to-right order, to u_4, u_2, u_3, u_5, the middle row to $v_1, v_4, u_1 = z_1, v_2, \overline{v}_0$ and the bottom vertex to v_3. If v_1 is adjacent to exactly one of u_2, u_3, then due to the symmetry in the forthcoming argument we may assume that v_1 is adjacent to u_3, and hence v_2 is adjacent to u_2. Then an isomorphism is given by mapping the top row to v_4, u_4, u_3, u_2, the middle row to $v_3, v_1, u_1 = z_1, u_5, v_2$, and mapping the bottom vertex to $\overline{v}_0$. This completes the case $l = 2$.

Case 3: $l = 3$

Lemma 3.4.3 implies that v_0 has at most one neighbor among $\{z_1, z_2, z_3, u_1, u_2, \ldots, u_k\}$, and such neighbor must be u_1, u_k, z_1, or z_3.

We claim that either v_0 is adjacent to z_1 or z_3, or $k = 3$ and v_0 is adjacent to u_1 or u_3. To prove this claim let us assume that v_0 has no neighbor among $\{z_1, z_2, z_3\}$. Let C be the cycle $v_1z_1z_2z_3v_2v_0$, and let X denote the set of vertices of G_0 drawn in the open disk bounded by C. We have $X \neq \emptyset$ by Lemma 3.4.3. Let c be a coloring of $G\backslash X$, and let L denote the subgraph of G_0 consisting of all vertices and edges drawn in the closed disk bounded by C. By Lemma 3.2.1 the graph L satisfies one of the conditions (i), (ii), (iii) of that lemma. The vertices z_1 and z_3 are adjacent, because the graph obtained from $\hat{J}$ by deleting v_0, v_1, v_2 and the vertices drawn in the faces $\tilde{R}_1$ or $\tilde{R}_2$ is isomorphic to K_5. We may also assume, by the symmetry between v_1 and v_2, that v_1 is adjacent to z_2. We claim that we may assume that the neighborhood of v_0 is a 5-cycle. This is clear if v_0 has no neighbor in $\{u_1, u_2, u_3, u_4\}$, and so we may assume that v_0 is adjacent to u_1. Then we may assume that $k = 4$, for otherwise the claim we are proving holds. Thus there is no special edge. By Lemma 3.4.3 there exists a vertex inside $\tilde{R}_1$ adjacent to v_0, v_1, u_1. Since there is no special edge the vertex v_1 is not adjacent to u_1, and u_1 is not adjacent to z_1, because v_2 has degree at least five in G_0. It follows that the neighborhood of v_0 is indeed a 5-cycle. If $|X| \geq 2$, then there exists a vertex in X whose neighborhood has a subgraph that is a 5-cycle plus at least one additional edge, namely z_1z_3 or v_1z_2. That contradicts the optimality of (G_0, v_0). Thus $|X| = 1$. Let x denote the unique element of X, and let us assume first that $k = 4$. Then there are no special edges, and so v_1 is not adjacent to z_3 and v_2 is not adjacent to z_1. Let C' denote the cycle $v_1xv_2u_1u_2u_3u_4$, and let X' be the set of vertices of G_0 drawn in the open disk bounded by C'. Then $G_0\backslash(X' \cup \{x\})$ has a 5-coloring c' such that $c'(v_1) = c'(z_3)$ and $c'(v_2) = c'(z_1)$. Then c' can be extended to x in at least two different ways. By Lemmas 3.2.1 and 3.4.3 the coloring c' can be extended to all of G_0, unless (up to symmetry betweeen v_1 and v_2) v_0 is adjacent to u_1, there exists a vertex y adjacent to u_1, u_2, u_3, u_4 and $c'(v_1) = c'(u_5)$. But v_1 is not adjacent to u_1 (because v_2 is and there are no special edges), and hence the color of v_1 can be changed to $c'(u_1)$, and the resulting coloring extends to all of G_0, a contradiction. This completes the case $k = 4$. Thus $k = 3$, and so there is at most one special edge. Let c'' be a 5-coloring of $G_0 \backslash X'$. It follows that the color of at least one of the vertices v_1, v_2 can be changed to a different color, without affecting the colors of the other vertices of $G \backslash X'$. It follows from Lemma 3.2.1 that $|X'| \leq 2$. That, in turn, implies that v_0 is adjacent to u_1 or u_3, and hence proves our claim from the beginning of this paragraph.

Thus we may assume that v_0 is adjacent to z_3. By Lemma 3.4.3 there exists a vertex v_3 adjacent to v_0, v_1, z_1, z_2, z_3 and a vertex v_4 in $\tilde{R}_1$ that is adjacent to v_0, v_1, v_2. The neighborhood of v_3 includes the edge z_1z_3, and so by the optimality of (G_0, v_0) the neighborhood of v_0 includes the edge v_4z_3. Thus $z_3 \in \{u_1, u_2, u_3, u_4\}$. Assume first that $k = 4$. Then there are no special edges, and hence $z_3 \neq u_4$. Next we deduce that $z_3 \neq u_1$, for otherwise v_2u_1 and v_2z_3 are the same edge, which implies (given that $z_3 = u_1$ is adjacent to v_4) that v_2 has degree at most three, a contradiction. Thus $z_3 \in \{u_2, u_3\}$. Let Y consist of v_0 and all vertices in $\tilde{R}_1$ or $\tilde{R}_2$. Since z_3 is adjacent to v_4 we deduce that $|Y| \leq 4$. Since there are no special edges, z_3 is not adjacent to v_1, and v_2 is not adjacent to u_4. Thus $G_0 \backslash Y$ has a coloring d such that $d(v_1) = d(z_3)$ and $d(v_2) = d(u_4)$. Since $z_3 \in \{u_2, u_3\}$ this coloring can be extended to the vertices drawn in $\tilde{R}_1$, and since $d(v_1) = d(z_3)$ it can be further extended to v_0 and v_3, a contradiction.

Thus $k = 3$. Let W denote the walk $v_1v_3z_3v_2u_1u_2u_3$, and let d' be a 5-coloring of $G_0 \backslash (Y - \{v_3\})$. We now apply Lemma 3.2.1 to the graph drawn in the closed disk bounded by W and coloring d', and note that either the color of each of v_1, v_2 can be changed to a different color, independently of each other and independently of the colors of other vertices, except possibly v_3, or the color of one of v_1, v_2 can be changed to two different values. In either case, one of the resulting colorings extends to G_0, a contradiction. □

Lemma 3.4.6. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1v_2}$ isomorphic to K_6. Then the drawing of J in the Klein bottle does not have a facial walk of length six.*

Proof. Suppose for a contradiction that there exists a subgraph J of $G_{v_1v_2}$ isomorphic to K_6 such that the drawing of J in the Klein bottle has a face F_0 bounded by a walk W of length six. Let the vertices of J be $z_1, z_2, \ldots, z_6$. Since K_7 cannot be embedded in the Klein bottle, it follows that W has a repeated vertex. If W has exactly one repeated vertex, then (since J is simple) we may assume that the vertices on W are $z_6, z_2, z_4, z_6, z_3, z_5$, in order. There exists a closed curve ϕ passing through z_6 and otherwise confined to F_0 such that there is an edge of J on either side of ϕ in a neighborhood of z_6. The curve ϕ cannot be separating, because $G_0 \backslash z_0$ is connected, and it cannot be 2-sided, because $G_0 \backslash z_0$ is not planar. It follows that ϕ is 1-sided. By Euler's formula every face of J other than F_0 is bounded by a triangle. It follows that the triangles $z_4z_5z_6$, $z_1z_6z_3$, and $z_1z_6z_2$ bound faces of J. Furthermore, either $z_3z_5z_2$ or $z_3z_5z_4$ is a face, but since J is simple we deduce that it is the former. It follows that $z_1z_3z_4$, $z_2z_3z_4$, $z_1z_2z_5$ and $z_1z_4z_5$ are faces of J, and those are all the faces of J. The drawing of J is depicted in Figure 13, where diagonally opposite vertices and edges are identified, and the asterisk indicates another cross-cap.

Similarly, if W has at least two repeated vertices, then it has exactly two, and we may assume that the vertices of W are $z_6z_5z_4z_6z_2z_4$. Similarly as in the previous paragraph, the embedding is now uniquely determined, and is depicted in Figure 14.

In either case let R_1 and R_2 be the hinges of J, and let F_{ijk} denote the facial

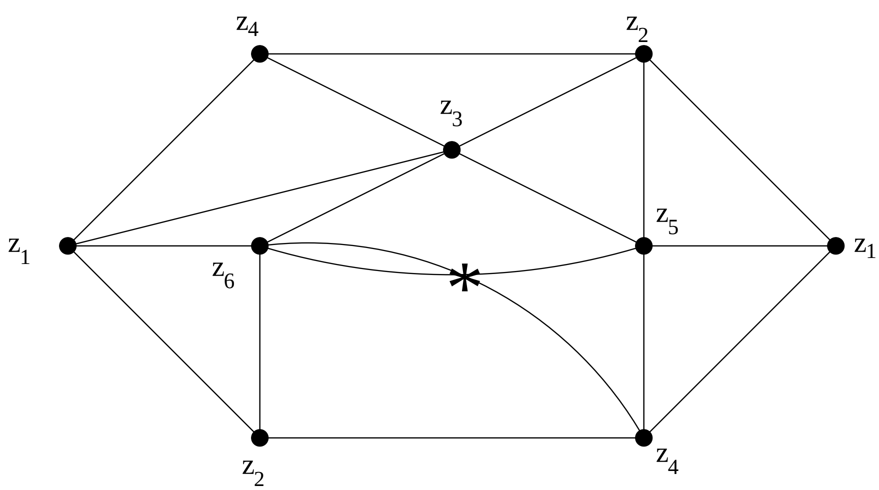


Figure 13: An embedding of K_6 with a facial walk on five vertices

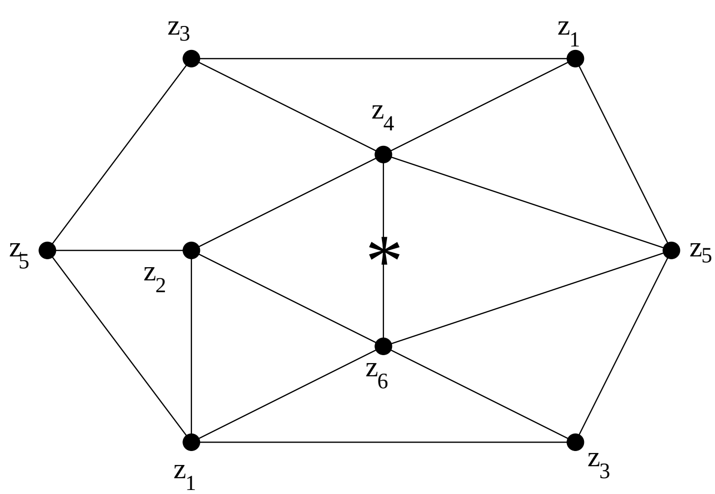


Figure 14: An embedding of K_6 with a facial walk on four vertices

triangle incident with z_i, z_j, z_k if it exists. We should note that specifying the hinges does not uniquely determine the graph $\hat{J}$, because the face F_0 has multiple incidences with some vertices. For instance, if W has five vertices, $z_0 = z_6$, and $R_1 = F_0$, then it is not clear whether the split occurs in the "angle" between the edges $z_3 z_6$ and $z_4 z_6$, or in the angle between $z_5 z_6$ and $z_2 z_6$. To overcome this ambiguity we will write $R_1 = F_{364}$ in the former case, and $R_1 = F_{265}$ in the latter case. Notice that this is just a notational device; there is no face bounded by $z_3 z_6 z_5$ or $z_2 z_6 z_4$. We proceed in a series of claims.

(1) *Not both R_1 and R_2 are bounded by triangles.*

To prove (1) suppose for a contradiction that R_1 and R_2 are both facial triangles. Let us recall that z_0 is the vertex of $G_{v_1 v_2}$ that results from the identification of v_1 and v_2. Suppose first that R_1 and R_2 are consecutive in the cyclic order around z_0. Then v_0 and one of v_1 or v_2 is in the interior of a 4-cycle in G_0, contrary to Lemma 3.2.1. Similarly, if the cyclic order around z_0 has R_1 followed by a facial triangle, followed by R_2, then there would be two vertices in the interior of a 5-cycle in G_0, contrary to Lemma 3.2.1. In addition, if the cyclic order has R_1, followed by two facial triangles, followed by R_2, then there are two vertices inside a 6-cycle. Hence, we are in either case (ii) or (iii) of Lemma 3.2.1. However, the boundary has five vertices that form a clique. So 5-color all but the interior of this 6-walk (using that G_0 is 5-critical); the boundary must have five colors, contrary to Lemma 3.2.1. We conclude that R_1 and R_2 must have F_0 in between them in the cyclic order around z_0, on both sides. In particular, W has five vertices.

Thus the only case remaining is that $z_0 = z_6$, where J is embedded with a facial 6-walk on five vertices. Suppose without loss of generality that $R_1 = F_{126}$ and $R_2 = F_{456}$, and that v_2 is adjacent to z_1, z_3 and z_4. Then the faces of the subgraph induced by $v_1, v_2, z_1, z_2, z_3, z_4, z_5$ are all triangles but perhaps for two six-cycles: $v_1, z_2, z_1, v_2, z_4, z_5$

and $v_1, z_5, z_3, v_2, z_4, z_2$. Since v_0 is adjacent to v_1 and v_2 it follows from Lemma 3.2.1 that the only vertex in G_0 in the interior of the first six-cycle is v_0. Hence there must be at least two vertices in the interior of the other six-cycle, else $|V(G_0)| \leq 9$, a contradiction. Thus we are in either case (ii) or (iii) of Lemma 3.2.1. Note that the disk bounded by the second cycle includes no chord. So v_1 is not adjacent to z_3. Now if v_1 is not adjacent to z_1, we color G_0 as follows. Let the color of z_i be i. Color v_1 with color 1. Then color v_0 and v_2, and extend the coloring to the interior of the second six-cycle by Lemma 3.2.1. Hence we may assume that v_1 is adjacent to z_1. But then v_0 is adjacent to z_1, z_4, z_5 while v_1 is not adjacent to z_4. Now v_1 may be colored either 3 or 4. One of these options extends to the interior of the second six-cycle after we color v_1, v_0, v_2 in that order. This proves claim (1).

In light of (1) we may assume that $R_1 = F_0$.

(2) *If R_2 is bounded by a triangle, then it is not consecutive with F_0 in the cyclic order around z_0 in J.*

To prove (2) suppose for a contradiction that R_2 is bounded by a triangle and that it is consecutive with F_0 in the cyclic order around z_0 in J. It follows that one of v_1, v_2 has degree two in $\hat{J}$, and so we may assume that it is v_1 and that its neighbors are v_0 and z_j. Consider the subgraph $\hat{J} \setminus \{v_0, v_1\}$. All of its faces are triangles but for a 7-walk. We 5-color this subgraph, which is isomorphic to K_6 minus an edge. Thus v_2 must receive the same color as z_j. Since this subgraph only has six vertices, the interior of the 7-walk must be as in case (v) or (vi) of Lemma 3.2.1, for otherwise there would be at most nine vertices in G_0, contrary to Lemma 3.4.1. Consider the edge $z_0 z_j$ in J, which must be on the boundary of F_0. Now the vertex or vertices not on the boundary of F_0 must be on the boundary of R_2, for otherwise the 7-walk would only have four colors and we could extend the 5-coloring to its interior, a contradiction. Since R_2 is a facial triangle this means that either z_0 or z_j is z_6 and that W has five vertices. However, then the color of z_0 and z_j appears three times on the boundary of the 7-walk. So the 5-coloring may also be extended, a contradiction. This proves (2).

By an *s-vertex* we mean a vertex $s \in V(G_0)$ of degree five such that $N_{G_0}(s)$ has a subgraph isomorphic to $K_5 - P_3$. If G_0 has an s-vertex, then the optimality of (G_0, v_0) implies that $N_{G_0}(v_0)$ does not include two disjoint pairs of non-adjacent vertices.

(3) *Let R_2 be bounded by a triangle; then $\hat{R}_2$ is bounded by a pentagon, say $v_0v_1r_1r_2v_2$. Assume further that G_0 has an s-vertex. Then either*

(a) *$\hat{R}_2$ includes a unique vertex v of G, and v is adjacent to v_0, r_1, r_2 and all neighbors of v_0 other than v, or*

(b) *v_0 is adjacent to r_1, r_2, and r_1, r_2 are adjacent to the neighbor of v_0 other than v_1, v_2, r_1, r_2, or*

(c) *v_0, v_1, v_2 are all adjacent to r_i for some $i \in \{1, 2\}$, and r_i is adjacent to the two neighbors of v_0 other than v_1, v_2, r_i.*

To prove (3) we first notice that $\hat{R}_2$ includes at most one vertex of G_0 by Lemma 3.2.1. If it includes exactly one vertex, then (a) holds by the existence of an s-vertex, and the optimality of (G_0, v_0). If $\hat{R}_2$ includes no vertex of G_0, then by Lemma 3.4.3 either v_0 is adjacent to both r_1 and r_2, or v_0, v_1, v_2 are all adjacent to r_i for some $i \in \{1, 2\}$. We deduce from the existence of an s-vertex and the optimality of (G_0, v_0) that either (b) or (c) holds. This proves (3).

(4) *The walk W has five vertices.*

To prove (4) we suppose for a contradiction that W has four vertices. Suppose first that $z_0 = z_2$. Then by (2) and the symmetry we may assume that $R_2 = F_{125}$. It follows that z_3 is an s-vertex, and so we may apply (3). But (a) does not hold, because in that case v_0 has four neighbors in $\hat{R}_1$ or on its boundary, and not all of them can be adjacent to the neighbor of v_0 in $\hat{R}_2$. If (b) holds, then v_0 is adjacent to z_1 and z_5, and v is adjacent to z_1, where v is the neighbor of v_0 other than v_1, v_2, z_1, z_5. Now $v \neq z_5$, because otherwise both $\hat{R}_1$ and $\hat{R}_2$ include an edge joining v_0 and z_5, contrary to the fact that G_0 is simple. Since v is adjacent to z_1 we deduce that $v = z_4$ or $v = z_6$. In either case Lemma 3.4.3 implies that v_1 or v_2 has degree at most four, a contradiction.

Thus we may assume that (c) holds, and so v_0, v_1, v_2 are all adjacent to z_1 or z_5. In the former case we can change notation so that $R_2 = F_{126}$, contrary to (2). Thus v_0, v_1, v_2 are all adjacent to z_5. Let v_1 be adjacent to z_3, z_4, z_5; then v_2 is adjacent to z_1, z_5, z_6. Let the vertices v_2, z_5, v_1, v_4, v_5 form the wheel neighborhood of v_0, in order. Since an s-vertex exists, the optimality of (G_0, v_0) implies that either v_1 is adjacent to v_5, or v_2 is adjacent to v_4, or both. We may assume from the symmetry that v_1 is adjacent to v_5. Since v_5 is adjacent to z_5 by (c), we deduce that $v_5 = z_4$ or $v_5 = z_6$, because $v_5 \neq z_5$ for the same reason as above. If $v_5 = z_6$, then v_2z_6 and v_2v_5 are the same edge, and it follows from Lemma 3.4.3 that v_2 has degree at most four. Thus $v_5 = z_4$. It follows that v_2 is adjacent to z_4, and hence the neighborhood of z_1 has a subgraph isomorphic to K_5^-, contrary to Lemma 3.4.2. This completes the case $z_0 = z_2$.

Thus by symmetry we may assume that $z_0 = z_4$. Again by symmetry we may assume that $R_1 = F_{246}$ and R_2 is either F_{134} or F_{145}. Assume first that $R_2 = F_{145}$. Let v_1 be adjacent to z_1, z_2, z_3. Then z_3 is an s-vertex, and so we may use (3). If (a) holds, and v is as in (a), then it is not possible for v to be adjacent to all neighbors of v_0 other than v, a contradiction. If (b) holds, then v_2 is not adjacent to z_1, and hence v_1 is adjacent to z_5, by the optimality of (G_0, v_0), because an s-vertex exists. Thus the neighborhood of z_3 in G_0 has a subgraph isomorphic to K_5^-, contrary to Lemma 3.4.2. Thus (c) holds. If v_0, v_1, v_2 are adjacent to z_5, then $N_{G_0}(z_3)$ has a subgraph isomorphic to K_5^-, contrary to Lemma 3.4.2. Hence v_0, v_1, v_2 are adjacent to z_1. By (c) the vertex z_1 is adjacent to v_4, v_5, the two neighbors of v_0 other than v_1, v_2, z_1. It follows that $\{v_4, v_5\} \subseteq \{z_2, z_5, z_6\}$. However, if v_0 is adjacent to z_2, then v_1 would be of degree at most four in G_0, a contradiction. Thus v_0 is adjacent to z_5 and z_6; hence v_1 is adjacent to z_5 by Lemma 3.4.3. Now the graph has eight vertices and perhaps one more inside the 5-cycle $v_1z_2z_6v_2z_5$. Hence G_0 has at most nine vertices, contrary to Lemma 3.4.1. This completes the case $R_2 = F_{145}$.

We may therefore assume that $R_2 = F_{246}$. From the symmetry we may assume that v_1 is adjacent to z_2 and z_3. If $\hat{R}_2$ includes an edge incident with v_1 or v_2, then Lemma 3.4.3 implies that v_0, v_1, v_2 are all adjacent to z_1 or z_3. Then we may change our notation so that either $R_2 = F_{145}$ or $R_2 = F_{234}$. In the former case we get a contradiction by the result of the previous paragraph, and in the latter case we get a contradiction by (2). Thus $\hat{R}_2$ includes no edge incident with v_1 or v_2. Hence either v_0 is adjacent to z_1 and z_3, or v_0 is adjacent to an internal vertex v_3 of degree five which is adjacent to z_1 and z_3. In either case there is a vertex of degree five in G_0 adjacent to v_1, z_3, z_1, and v_2. For this vertex, z_3, v_2 is an identifiable pair. Note that $G_{z_3v_2}$ is not 5-colorable. We 5-color the vertices z_1, z_2, $v_2 = z_3$, z_5, z_6 so that each gets a unique color. Then this coloring extends to $G_{z_3v_2}$, unless we are in case (ii) of Lemma 3.2.1 for the following walk on six vertices: $z_5, v_2 = z_3, z_6, z_2, v_2 = z_3, z_6$ in $G_{z_3v_2}[\{z_1, z_2, v_2 = z_3, z_4, z_5, z_6\}]$. This implies that there are two adjacent vertices w_1 and w_2 such that, in G_0, w_1 is adjacent to z_2, z_6, v_2, and z_5, while w_2 is adjacent to z_6, z_5, z_2, and one of v_2, z_3. But then the subgraph induced by the eight vertices: $z_1, z_2, z_3, z_5, z_6, v_2, w_1, w_2$ has all facial triangles except for perhaps one 5-cycle. Yet there can be at most one vertex in the interior of that 5-cycle. Thus G_0 has at most nine vertices, a contradiction. This proves (4).

(5) $z_0 \neq z_2, z_3$.

We may assume to a contradiction that $z_0 = z_2$ since the case where $z_0 = z_3$ is symmetric. By (2) $R_2 = F_{125}$ or F_{235}. Suppose first that some edge of G_0 is incident with v_1 or v_2 and lies inside $\hat{R}_2$. Then v_0, v_1, and v_2 are all adjacent to z_5, for otherwise we may change our notation so that $\hat{R}_2 = F_{126}$, contrary to (2). Let v_4 and v_5 be neighbors of v_0 such that the cyclic order around v_0 is v_1, z_5, v_2, v_5, v_4. Now notice that z_1 is degree five in G_0 and $N_{G_0}(z_1)$ has a subgraph isomorphic to $K_5 - P_3$. Since $N_{G_0}(z_0)$ is missing the edge v_1v_2, one of the edges v_1v_5, v_2v_4 must be present or z_1 would contradict the choice of v_0. This implies that v_1 and v_2 are both adjacent to v_j for some $j \in \{4, 5\}$. Thus the edges v_1v_j, v_2v_j must go to a repeated vertex on the boundary of R_1 or v_0 would be in a four-cycle in G_0, a contradiction. Thus $v_j = z_6$ and the edge v_2z_6 is already present. The edge v_1z_6 then implies that z_4 is degree five in G_0 and that $N_{G_0}(z_4)$ has a subgraph isomorphic to K_5^-, contrary to Lemma 3.4.2. Thus $\hat{R}_2$ includes no edge of G_0 incident with v_1 or v_2.

Now suppose that $R_2 = F_{125}$. We may assume that v_1 is adjacent to z_3, z_4, z_5. Then either the cyclic order around v_0 is v_1, z_5, z_1, v_2, and an unspecified vertex v_3, or v_0 is adjacent to a vertex v_3 of degree five with cyclic order: v_1, z_5, z_1, v_2, v_0. In either case, z_1v_1 is an identifiable pair for a vertex of degree five in G_0. Note that $G_{v_1z_1}$ is not 5-colorable. We 5-color the vertices $v_1 = z_1$, z_3, z_4, z_5, z_6 of $G_{v_1z_1}$ such that each gets a unique color. Since this coloring does not extend to $G_{v_1z_1}$ we deduce from Lemma 3.2.1 applied to the walk z_6, $v_1 = z_1$, z_4, z_6, z_3, z_5 on six vertices that case (i) of that lemma holds. That implies there exists a vertex w_1 in G_0 that is adjacent to v_1, z_4, z_6, z_3 and z_5. Let $H := G[\{z_1, z_3, z_4, z_5, z_6, v_1, w_1\}]$. The edge w_1z_6 may be embedded in two different ways. In one way of embedding the edge the graph H has all faces bounded by triangles, except for one bounded by a 4-cycle and one bounded by a 5-cycle. But then G_0 has at most eight vertices by Lemma 3.2.1, contrary to Lemma 3.4.1. It follows that the edge w_1z_6 is embedded in such a way that all faces of H are bounded by triangles, except for one face bounded by the walk $z_6z_1z_5v_1w_1z_5$ of length six. Since G_0 has at least ten vertices by Lemma 3.4.1, we must be in case

(iii) of Lemma 3.2.1 when applied to said walk. This can happen in two ways. In the first case there are pairwise adjacent vertices $a, b, c \in V(G_0)$ such that a is adjacent to z_1, z_5, z_6, the vertex b is adjacent to z_5, v_1, w_1 and c is adjacent to w_1, z_5, z_6. Now G_0 is isomorphic to L_4 by an isomorphism that maps z_3 and z_4 to the top two vertices in Figure 7(d) (in left-to-right order), z_6 and w_1 to the vertices in the second row, z_5 to the unique vertex of degree nine, and z_1, a, c, b, v_1 to the last row of vertices in that figure. In the second case there are pairwise adjacent vertices $a, b, c \in V(G_0)$ such that a is adjacent to z_1, z_5, v_1, the vertex b is adjacent to z_5, v_1, w_1 and c is adjacent to z_1, z_5, z_6. Now G_0 is isomorphic to L_3 by an isomorphism that maps the top row of vertices in Figure 7(c) to z_6, z_3, z_4, w_1 (again in left-to-right order), the middle row to c, z_1, z_5, v_1, b and the bottom vertex to a. Since either case leads to a contradiction, this completes the case $R_2 = F_{125}$.

It follows that $R_2 = F_{235}$. We may assume that v_2 is adjacent to z_1, z_5, z_6. Then either the cyclic order around v_0 is v_1, z_3, z_5, v_2, and an unspecified vertex v_3, or v_0 is adjacent to a vertex v_3 of degree five with cyclic order: v_1, z_3, z_5, v_2, v_0. Note that z_1 is degree five in G_0 and $N_{G_0}(z_1)$ has a subgraph isomorphic to $K_5 - P_3$. Thus in either case, v_2z_3 is an identifiable pair for a vertex of degree five in G_0, for otherwise $N_{G_0}(z_1)$ has a subgraph isomorphic to K_5^-, a contradiction. Note that $G_{v_2z_3}$ is not 5-colorable. We 5-color the vertices z_1, $v_2 = z_3$, z_4, z_5, z_6 of $G_{v_2z_3}$ such that each gets a unique color. Since this coloring does not extend to $G_{v_2z_3}$, we deduce that the 6-walk $z_6v_2 = z_3z_4z_6v_2 = z_3z_5$ satisfies (ii) of Lemma 3.2.1. Thus, in G_0, there exists two adjacent vertices w_1 and w_2 such that w_1 is adjacent to z_4, z_6, z_3, and z_5, while w_2 is adjacent to z_4, z_5, z_6 and v_2. But then w_1 is degree five in G_0 and $N_{G_0}(w_1)$ has a subgraph isomorphic to K_5^-, a contradiction. This proves (5).

(6) $z_0 \neq z_4, z_5$.

To prove (6) we may assume for a contradiction that $z_0 = z_4$ since the case where $z_0 = z_5$ is symmetric. Thus $R_2 = F_{134}$ or F_{145} by (2). Assume first that $R_2 = F_{145}$, and that $\hat{R}_2$ includes no edges incident with v_1 or v_2. Then either the cyclic order around v_0 is v_1, z_1, z_5, v_2, and an unspecified vertex v_3, or v_0 is adjacent to a vertex v_3 of degree five with cyclic order: v_1, z_1, z_5, v_2, v_0. If the edge v_1z_5 is present, then in the subgraph of G_0 induced by z_1, z_2, z_3, z_5, z_6 and v_2, there is only one face that is not bounded by a triangle or 4-cycle—the following walk on six vertices: z_5, $z_3z_6z_5z_1v_2$. Thus there are at most nine vertices in G_0 by Lemma 3.2.1, contrary to Lemma 3.4.1. Hence, in either case v_1z_5 is an identifiable pair for a vertex of degree five in G_0. Note that $G_{v_1z_5}$ is not 5-colorable. We 5-color the vertices z_1, z_2, z_3, $v_1 = z_5$, z_6 of $G_{v_1z_5}$ such that each gets a unique color. Since this 5-coloring does not extend to a 5-coloring of $G_{v_1z_5}$ we deduce that case (ii) of Lemma 3.2.1 holds for the following walk on six vertices: z_6, z_2, $v_1 = z_5$, z_6, z_3, $v_1 = z_5$. Thus, in G_0, there are two adjacent vertices w_1 and w_2 such that w_1 is adjacent to z_2, z_6, z_5, and z_3, while w_2 is adjacent to z_2, z_6, z_3 and v_1. But then w_1 is degree five in G_0 and $N_{G_0}(w_1)$ has a subgraph isomorphic to K_5^-, contrary to Lemma 3.4.2. This completes the case when $R_2 = F_{145}$ and $\hat{R}_2$ includes no edges incident with v_1 or v_2.

For the next case assume that $R_2 = F_{134}$, and again that $\hat{R}_2$ includes no edges incident with v_1 or v_2. Then either the cyclic order around v_0 is v_1, z_3, z_1, v_2, and an unspecified vertex v_3, or v_0 is adjacent to a vertex v_3 of degree five with cyclic order: v_1, z_3, z_1, v_2, v_0. Next we dispose of the case that v_2 is adjacent to z_3. In that case we consider the subgraph of G_0 induced by z_1, z_2, z_3, z_5, z_6 and v_2. There is only one face that is not bounded by a triangle or 4-cycle—the following walk on seven vertices: $z_5z_3v_2z_1z_3z_2z_6$. We 5-color the subgraph as follows: $c(z_i) = i$ for $i = 1, 2, 3, 5$, $c(z_6) = 4$, and $c(v_2) = 2$ and apply Lemma 3.2.1. By Lemma 3.4.1 cases (v) or (vi) of Lemma 3.2.1 hold. Since z_2 and v_2 have the same color and z_3 is a repeated vertex it follows from Lemma 3.2.1 that G_0 has four vertices a, b, c, d such that d is adjacent to z_2, z_3, z_5, z_6, the vertices a, b, c form a triangle and either a is adjacent to z_1, v_2, z_3, the vertex b is adjacent to z_1, z_2, z_3, and c is adjacent to z_2, z_3, d (case (v) of

Lemma 3.2.1), or a is adjacent to z_1, v_2, z_3, the vertex b is adjacent to v_2, z_3, d, and c is adjacent to z_2, z_3, d (case (vi) of Lemma 3.2.1). In the former case d is an s-vertex, and yet $v_0 = a$, c is not adjacent to z_1 and b is not adjacent to v_2, contrary to the optimality of (G_0, v_0). In the latter case G_0 is isomorphic to L_3 by a mapping that sends the top row of vertices in Figure 7(c) to z_1, z_6, z_5, z_2 (in left-to-right order), the middle row to a, v_2, z_3, d, c and the bottom vertex to b, a contradiction. Thus v_2 is not adjacent to z_3, and hence v_2z_3 is an identifiable pair for a vertex of degree five in G_0. Note that $G_{v_2z_3}$ is not 5-colorable. We 5-color the vertices z_1, z_2, $v_2 = z_3$, z_5, z_6 of $G_{v_2z_3}$ such that each gets a unique color. Since this coloring not extend to $G_{v_2z_3}$ we deduce that case (ii) of Lemma 3.2.1 holds for the following 6-walk: z_6, z_2, $z_3 = v_2$, z_6, $z_3 = v_2$, z_5. However this would imply that there are two internal vertices w_1 and w_2, both adjacent to z_2 and both adjacent to z_5. But then one of them is not adjacent to $z_3 = v_2$, a contradiction. This completes both cases when $\hat{R}_2$ includes no edges incident with v_1 or v_2.

Thus in the proof of (6) we have shown that $\hat{R}_2$ includes an edge incident with v_1 or v_2. Then v_0, v_1, v_2 are all adjacent to z_1, z_3 or z_5. However, if they are all adjacent to z_3, then we can change notation so that $R_2 = F_{234}$, contrary to (2), and if they are all adjacent to z_5, then we can change notation so that $R_2 = F_{456}$, again contrary to (2). Thus v_0, v_1, v_2 are all adjacent to z_1. We may assume that the notation is chosen so that v_1 is adjacent to z_2 and z_3 while v_2 is adjacent to z_5 and z_6. Let v_4 and v_5 be neighbors of v_0 numbered so that the cyclic order around v_0 is v_2, z_1, v_1, v_4, v_5.

Next we claim that v_1 is not adjacent to z_6. Suppose it were. The triangle $z_2v_1z_6$ is null-homotopic in G_0 by Lemma 3.2.1 applied to the 4-cycle $z_1z_5z_6v_1$. Now consider the subgraph induced by the vertices z_1, z_2, z_3, z_5, z_6, and v_1. All of its faces are triangles but for the 7-walk $z_1z_5z_6z_3z_5z_6v_1$. We 5-color these vertices as follows: $c(z_i) = i$ for $i = 1, 3, 5$, $c(z_6) = 4$, and $c(v_1) = 5$. Now we must be in case (v) or (vi) of Lemma 3.2.1, for otherwise $|V(G_0)| \leq 9$, contrary to Lemma 3.4.1. Yet, since the fifth color would appear three times on the boundary, we can extend this coloring to all of G_0, a contradiction. Thus v_1 is not adjacent to z_6.

Now we claim that $v_4, v_5 \notin \{z_1, z_2, \ldots, z_6\}$. To prove this claim we suppose the contrary. Then v_0 is adjacent to z_2, z_3, z_5 or z_6. If v_0 is adjacent to z_2, then v_1 has degree at most four in G_0. If v_0 is adjacent to z_6, then either v_2 is degree four in G_0, a contradiction, or v_1 is adjacent to z_6, a contrary to the previous paragraph. If $v_4 = z_3$, then the 5-cycle $v_1z_3z_6z_5z_1$ has the vertices v_0 and v_2 in its interior, contrary to Lemma 3.2.1. Let us assume that $v_5 = z_3$. Then v_2 is degree five and $N(v_2)$ is missing at most the edges v_0z_5 and v_0z_6. Yet these edges must not be present, for otherwise $N(v_2)$ has a subgraph isomorphic to K_5^-, contrary to Lemma 3.4.2. Hence $v_4 \notin \{z_1, z_2, \ldots, z_6\}$, but then it is not adjacent to z_1. Thus $N_{G_0}(v_0)$ includes two disjoint edges. However, $N_{G_0}(v_2)$ has a subgraph isomorphic to $K_5 - P_3$, contradicting the optimality of (G, v_0). Thus we may assume that v_0 is adjacent to z_5. This implies, by Lemma 3.4.3, that $v_4 = z_5$, because v_2 is already adjacent to z_5 and $v_5 = z_5$ would imply the existence of another edge from v_2 to z_5, not homotopic to the existing one. Then the subgraph of G_0 induced by z_1, z_2, z_3, z_5, z_6, and v_1 has only one face—a six-walk—that can have vertices in its interior. But then there are at most nine vertices in G_0 by Lemma 3.2.1, contrary to Lemma 3.4.1. This proves our claim that $v_4, v_5 \notin \{z_1, z_2, \ldots, z_6\}$.

Continuing with the proof of (6), we note that v_2 is not adjacent to v_4, for otherwise v_5 is of degree four in G_0, a contradiction. Similarly v_1 is not adjacent to v_5. Since z_1 is not adjacent to v_4 or v_5, the neighborhood of v_0 in G_0 is a cycle of length five. The vertex v_2 is not adjacent to z_2, for otherwise the 4-cycle $z_2v_2v_0v_1$ includes the vertices v_4 and v_5 in its interior, contrary to Lemma 3.2.1. Furthermore, the vertex v_4 is not adjacent to z_2, for otherwise the neighborhood of v_1 in G_0 has a subgraph isomorphic to a 5-cycle plus one edge, contrary to the optimality of (G_0, v_0). We now consider the graph $G_{v_2v_4}$. It has a subgraph H isomorphic to K_6, and the new vertex w of H

obtained by identifying v_2 and v_4 belongs to H. Let Δ denote the open disk bounded by the walk $z_1z_5z_6z_3z_5z_6z_2z_3$ of $G_{v_2v_4}$. Since w belongs to Δ, all vertices of H belong to the closure of Δ. However, $z_2 \notin V(H)$, because z_2 is not adjacent to v_2 or v_4 in G_0. Since v_1 is not adjacent to z_6 as shown two paragraphs ago, we deduce that not both z_6 and v_1 belong to H. That implies that $z_1 \notin V(H)$, because at most six neighbors of z_1 in $G_{v_2v_4}$ (including $z_2 \notin V(H)$) belong to the closure of Δ. If $v_1 \notin V(H)$, then no edge incident with one of the two occurrences of z_3 on the boundary of Δ belongs to H. Thus regardless of which of v_1, z_6 does not belong to H, there is a planar graph H' obtained from H by splitting at most two vertices, and a drawing of H' in the unit disk with vertices p, q, r, s drawn on the boundary in order such that H is obtained from H' by identifying p with r, and q with s. It follows that H can be made planar by deleting one vertex, contrary to the fact that it is isomorphic to K_6. This proves (6).

Since $R_1 = F_0$ it follows that $z_0 \neq z_1$. Thus $z_0 = z_6$ by (5) and (6).

(7) *We may assume that* $R_2 \neq F_{136}$ *and* $R_2 \neq F_{126}$.

To prove (7) we may assume for a contradiction by symmetry that $R_2 = F_{136}$. Then by (2) we have $R_1 = F_{264}$. We may assume that v_1 and v_2 are numbered so that v_1 is adjacent to z_1 and z_2. We may assume that $\hat{R}_2$ includes no edge incident with v_1 or v_2; for if it includes the edge v_2z_1, then we can change notation so that $R_2 = F_{126}$, contrary to (2), and if it includes the edge v_1z_3, then we can change notation and reduce to the case when $R_2 = F_0$, which is handled below. Then either the cyclic order around v_0 is v_1, z_1, z_3, v_2, and an unspecified vertex v_3, or v_0 is adjacent to a vertex v_3 of degree five with cyclic order: v_1, z_1, z_3, v_2, v_0. In either case, z_1, v_2 is an identifiable pair for a vertex of degree five in G_0. Note that $G_{v_2z_1}$ is not 5-colorable. We 5-color the vertices $z_1 = v_2, z_2, z_3, z_4, z_5$ of $G_{v_2z_1}$ such that each gets a unique color. Since this coloring does not extend to the rest of $G_{v_2z_1}$ we deduce that case (i) of Lemma 3.2.1 holds for the following 6-walk on five vertices: z_1v_2, z_2, z_4, z_1v_2, z_3, z_5. This implies that there exists a vertex w_1 in G_0 such that w_1 is adjacent to z_2, z_4, v_2, z_3 and z_5 in G_0. In the subgraph of G_0 induced by those six vertices and z_1, all the faces are triangles but for the face bounded by the cycle $z_1z_3v_2z_5w_1z_2$. Since G_0 must have at least ten vertices, we must be in case (iii) of Lemma 3.2.1. Now 5-color the subgraph induced by those six vertices and z_4 such that $c(z_i) = i$ for $i = 1, 2, 3, 5$, $c(w_1) = 1$, and $c(v_2) = 2$. The above-mentioned cycle is colored using four colors, and hence the 5-coloring may be extended to G_0, a contradiction. This proves (7).

In light of (7) we may assume that both R_1 and R_2 are equal to F_0. Thus we may assume that $R_1 = F_{264}$ and $R_2 = F_{365}$. We may assume that v_1 and v_2 are numbered so that v_1 is adjacent to z_1, z_2 and z_3. Let the remaining neighbors of v_0 be v_3, v_4, v_5 numbered so that the cyclic order around v_0 is v_1, v_3, v_2, v_5, v_4. This specifies the cyclic order uniquely up to reversal, and so we may assume by symmetry that the cyclic order around v_1 (of a subset of the neighbors of v_1) is $z_1, z_3, v_3, v_0, v_4, z_2$, where possibly $v_3 = z_3$ and $z_2 = v_4$.

(8) *The vertex v_1 is not adjacent to z_4 or z_5.*

To prove (8) we note that z_1 has degree five in G_0 and that its neighborhood has a subgraph isomorphic to $K_5 - P_3$. If v_1 was adjacent to z_4 or z_5, then the neighborhood of z_1 would have a subgraph isomorphic to K_5^-, contrary to Lemma 3.4.2 and the optimality of (G_0, v_0). This proves (8).

Since z_1 has degree five in G_0 and its neighborhood has a subgraph isomorphic to $K_5 - P_3$, we deduce from the optimality of (G_0, v_0) and Lemma 3.4.2 that the neighborhood of v_0 is isomorphic to $K_5 - P_3$. It follows that

(9) *the vertex v_3 is adjacent to v_4 or v_5*

and

(10) *either* v_1 *is adjacent to* v_5, *or* v_2 *is adjacent to* v_4, *and not both.*

(11) *The vertex* v_2 *is adjacent to* v_4.

To prove (11) suppose for a contradiction that v_2 and v_4 are not adjacent. We will consider $G_{v_2v_4}$ and its new vertex w formed by identifying v_2 and v_4. Let us note that all faces of the subgraph of $G_{v_2v_4}$ induced by z_1, z_2, z_3, z_4, z_5, v_1, w are bounded by triangles except for a face bounded by the 8-walk $W_1 = v_1wz_5z_3v_1wz_4z_2$. Let D_1 be the open disk bounded by W_1, let $W_0 = v_1v_4v_5v_2z_5z_3v_1v_3v_2z_4z_2$ be a corresponding walk in G_0, and let D_0 be the open disk bounded by W_0. By Lemma 3.3.7 the graph $G_{v_2v_4}$ has a subgraph H isomorphic to K_6. Since G has no K_6 subgraph it follows that $w \in V(H)$. If $z_1 \in V(H)$, then, since z_1 has degree five in G_0, all neighbors of z_1 belong to $V(H)$, contrary to (8). Thus all vertices of H belong to W_1 or D_1, and by Lemma 3.4.3 each vertex of $H\backslash w$ (when regarded as a vertex of G_0) belongs to W_0 or D_0. Assume for a moment that all but possibly one vertex of H belong to W_1. Then z_4 or z_5 belongs to $V(H)$, and so $v_1 \notin V(H)$ by (8). Thus exactly one vertex of H, say w_1, belongs to D_1 and $V(H) = \{w, w_1, z_2, z_3, z_4, z_5\}$. It follows that $v_4 \notin \{w_1, z_2, z_3, z_4, z_5\}$. Thus v_4 is not adjacent to z_3 in G_0, because the edge z_3v_4 would have to lie in D_0, where it would have to cross the path $z_4w_1z_5$. But w is adjacent to z_3 in H, and so v_2 is adjacent to z_3 in G_0. It follows that the the 4-cycle $v_1v_0v_2z_3$ is null-homotopic, for otherwise the edge v_2z_3 and path $z_2w_1z_5$ would cross in D_0. We deduce from Lemma 3.2.1 applied to the 4-cycle $v_1v_0v_2z_3$ that $v_3 = z_3$. But v_3 is adjacent to v_4 by (9), and yet z_3 is not adjacent to v_4, a contradiction. This completes the case when at most one vertex of H belongs to D.

Thus at least two vertices of H, say w_1 and w_2 belong to of D_1. Since W_1 has exactly two repeated vertices, the argument used at the end of the proof of (6) shows that w_1 and w_2 are the only two vertices of H in D_1. Also, it follows that w, v_1, the two repeated vertices of W_0, belong to H. Since v_1 is in H, (8) implies that $z_4, z_5 \notin V(H)$.

It follows that $z_2, z_3 \in V(H)$, and consequently $v_4 \notin \{z_2, z_3\}$. Thus each of w_1, w_2 is adjacent in G_0 to v_1, z_2, z_3 and to v_2 or v_4. It follows from considering the drawing of G_0 inside D_0 that one of w_1, w_2, say w_1, is adjacent to v_2 and the 4-cycle $v_1v_0v_2w_1$ is null-homotopic. By Lemma 3.2.1 applied to this 4-cycle we deduce that $w_1 = v_3$. Thus the edge v_3v_4 belongs to D_0. But $w_2 \neq v_4$, because v_4 is not a vertex of H, and yet the edge v_3v_4 intersects the path $z_3w_2z_2$ inside D_0, a contradiction. This proves (11).

(12) *The vertex* v_5 *is adjacent to* v_1.

We prove (12) similarly as the previous claim. Suppose for a contradiction that v_1 and v_5 are not adjacent, and consider $G_{v_1v_5}$ and its new vertex w. The subgraph of $G_{v_1v_5}$ induced by z_1, z_2, z_3, z_4, z_5, w, v_2 has all faces bounded by triangles except for one bounded by the 8-walk $W_1 = wv_2z_5z_3wv_2z_4z_2$. Let D_1 be the open disk bounded by W_1, and let W_0, D_0 be as in (11). Similarly as in the proof of (11) the graph $G_{v_1v_5}$ has a subgraph H isomorphic to K_6 with $w \in V(H)$. We claim that $z_4 \notin V(H)$. Indeed, if z_4 is in H, then it is adjacent to w in H; but z_4 is not adjacent in G_0 to v_1 by (8), and hence z_4 is adjacent to v_5 in G_0. Yet v_2 is adjacent to v_4 by (10). Since $v_4 \notin \{z_4, z_5\}$ by (8), the edges v_2v_4 and z_4v_5 must cross inside D_0, a contradiction. This proves our claim that $z_4 \notin V(H)$. It follows that $z_1 \notin V(H)$, because z_1 has degree five in $G_{v_1v_5}$, and z_4 is one of its neighbors.

If D_1 includes at most one vertex of H, then $w, v_2, z_2, z_3, z_5 \in V(H)$, and exactly one vertex of H, say w_1, belongs to D_1. Thus w_1 is adjacent to z_2 and z_5 in G_0, and that implies that the edges v_3v_4 and v_3v_5 do not lie in D_1. Therefore $v_3, v_4, v_5 \in \{z_2, z_3, z_4, z_5\}$, but that is impossible, given the existence of w_1. This completes the case that D_1 includes at most one vertex of H. Thus, similarly as in (11), it follows that D_1 includes exactly two vertices of H, say w_1 and w_2. Now $V(H)$ includes w, v_2 and exactly two of $\{z_2, z_3, z_5\}$. But it cannot include z_5 and z_3, because otherwise for some $j \in \{1, 2\}$ the paths $z_5 w_j v_2$ and $z_3 w_{3-j} v_2$ cross inside D_0. Thus $V(H)$ includes z_2 and z_i for some $i \in \{3, 5\}$. Choose $j \in \{1, 2\}$ such that $w_j \neq v_3$. Then the path $z_2 w_j z_i$ is not disjoint from the edges $v_3 v_4$, $v_3 v_5$ (because they cross inside D_0), and so it follows that $i = 3$ and $v_3 = z_3$. Since there is no crossing in D_0 and w_1 and w_2 are adjacent to z_2 and z_3, they are not both adjacent to v_5. Thus we may assume that w_1 is adjacent to v_1. This argument shows, in fact, that the cycle $v_1 v_0 v_2 w_1$ is null-homotopic, and so it follows from Lemma 3.2.1 that $v_3 = w_1$, a contradiction, because w_1 lies in D_1 and $v_3 = z_3$ does not. This proves (12).

Now claims (10), (11), and (12) are contradictory. This completes the proof of Theorem 3.4.6. □

Proof of Theorem 3.1.2. It follows by direct inspection that none of the graphs listed in Theorem 3.1.2 is 5-colorable. Conversely, let G_0 be a graph drawn in the Klein bottle that is not 5-colorable. We may assume, by taking a subgraph of G_0, that G_0 is 6-critical. Then G_0 has minimum degree at least five. By Lemma 3.2.3 the graph G_0 has a vertex of degree exactly five, and so we may select a vertex v_0 of G_0 such that (G_0, v_0) is an optimal pair. If there is no identifiable pair, then G_0 has a K_6 subgraph, as desired. Thus we may select an identifiable pair v_1, v_2. Let $G' := G_{v_1 v_2}$. By Lemma 3.3.7 the graph G' has a subgraph H isomorphic to K_6. By Lemma 3.4.4 the drawing of H is 2-cell, and by Lemma 3.4.5 some face of H has length six, contrary to Lemma 3.4.6. □

CHAPTER IV

AN ANALOGUE TO STEINBERG'S CONJECTURE FOR SURFACES

4.1 Introduction

In this chapter, we will investigate the structure of 3-colorable graphs on fixed surfaces that exclude certain structures. In particular, we will investigate graphs embedded on surface Σ that exclude contractible cycles of length four through ten and show that we can bound the size of $\mathcal{C}$-critical graphs (which we will define later) on any fixed surface. The main technique of our proof is discharging and Chapter 4.2 describes this discharging process. With this structure in hand, in Chapter 4.3, we will prove some inductive results that allow us to bound a particular invariant quantity. We state and prove the main theorem in Chapter 4.4.

4.2 The Discharging Process

We will employ a discharging technique similar to that used by Borodin et al [6]. Given a graph G, if it contains a reducible configuration then we will reduce the graph to a smaller graph and apply induction to a carefully selected statement. In particular the general argument is as follows: Give each vertex and each face an initial charge. If there are no reducible configurations, then the result follows by Euler's formula using a discharging argument.

Definition 4.2.1. Let $\mathcal{C}$ be the set of facial cycles $\{C_1, C_2, \ldots, C_k\}$ in a graph G drawn in a surface Σ. We say that G is *$\mathcal{C}$-critical* if $G \neq C_1 \cup C_2 \cup \cdots \cup C_k$ and for every subgraph G' of G that includes all the cycles in $\mathcal{C}$, some 3-coloring of $C_1 \cup C_2 \cup \cdots \cup C_k$ extends to a 3-coloring of G' but not to a 3-coloring of G.

Let G be a graph drawn in a surface Σ, and let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a set of facial cycles. If v is a cutvertex, define the initial charge of vertex v to be $d(v)$, where $d(v)$ is the degree of vertex v. If v is not a cutvertex, let the initial charge of v be $d(v) - 4$. For a face f in G, the *length* of f is defined to be the sum of the lengths of the walks that form the boundary of f. For all faces except the faces bounded by cycles in $\mathcal{C}$, let the initial charge be defined by $|f| - 4$. Define $|f|$ to be the length of face f. Give each face bounded by C_i charge $4|V(C_i)|/3$ - $2/3$. Denote the initial charge of a vertex or face to be $ch(v)$ and $ch(f)$, respectively. Let $ch'(v)$ and $ch'(f)$ be the final charge of the specified vertex or face after the discharging rules listed below are applied. The additional charge based upon the lengths of the precolored cycles contributes an additional charge of $\sum_{C_i \in \mathcal{C}}(\frac{|V(C_i)|}{3} + 10/3)$. The idea of our proof is that for each fixed surface, we can limit the number of graphs which have charge bounded by $-4g + \sum_i(\frac{|V(C_i)|}{3} + 10/3)$ and use this to show that our invariant is bounded by a function linear in $|\mathcal{C}|$. This implies that the number of $\mathcal{C}$-critical graphs have bounded size.

We say that a face is *internal* if it is not bounded by a cycle contained in $\mathcal{C}$. A vertex is *bad* if it has degree 3 and is incident to a 3-face. These rules are a slight modification to those used in Borodin [6] and are defined as follows:

1. Each face of length three receives charge of $\frac{1}{3}$ from each of its incident vertices.

2. Each internal non-triangular face f sends to each incident vertex v charge $\frac{2}{3}$ if v is degree 2 or v is bad. Also, f sends to vertex v charge $\frac{1}{3}$ if v is degree 3 and v is not bad, or v is degree 4 and incident with a 3-face not adjacent to f or is incident with two 3-faces both adjacent to f.

3. Each vertex $v \in C_i \in \mathcal{C}$ receives charge $\frac{4}{3}$ from face C_i if the degree of v is two and charge $\frac{2}{3}$ otherwise.

4. For any cutvertex v and every face f incident with v, if f is not bounded by a

triangle, then v sends $1/2$ to f; otherwise let e be the edge incident with f but not v, and let g be the other face incident with e. Then v sends $1/2$ to g.

Notice that total charge is preserved via these discharging rules. We show that after discharging every vertex has nonnegative charge. In addition we show that every 3-face has nonnegative charge and every other face f has charge at least $1/3(|f|-9)$. We now prove a series of lemmas that show conclusions of the discharging rules for graphs that adhere to the properties of graphs that are not reducible.

Let G be a graph drawn in a surface Σ, and let $\mathcal{C}$ be a set of facial cycles of G. A cycle C of G is *$\mathcal{C}$-admissible* if either $|V(C)| = 3$ or $|V(C)| \geq 11$, or both of the following conditions hold:

(i) if C bounds a closed disk Δ, then Δ includes a member of $\mathcal{C}$, and

(ii) if C separates Σ into surfaces Σ_1, Σ_2 with boundary C, then there exist $v_1, v_2 \in V(C)$ such that for $i = 1, 2$, the surface Σ_i does not include an edge of $G \setminus E(C)$ incident with v_i.

We define $\mathcal{S}_{10}(\Sigma)$ to be the set of all pairs $(G, \mathcal{C})$ as above, such that

(i) every vertex of G of degree at most two belongs to a cycle in $\mathcal{C}$,

(ii) every cycle in G is $\mathcal{C}$-admissible, and

(iii) if a cycle C in G bounds a closed disk that includes no member of $\mathcal{C}$, then either $|V(C)| = 3$ or $|V(C)| \geq 11$.

Lemma 4.2.2. *Let $G, \mathcal{C}, \Sigma$ be as above, let C be a $\mathcal{C}$-admissible cycle in G, and let H be a subgraph of G containing C and every cycle in $\mathcal{C}$. Then C is $\mathcal{C}$-admissible in H.*

Proof. This is apparent as properties (i) - (ii) in the definition of $\mathcal{C}$-admissible are preserved under the subgraph operation. □

Lemma 4.2.3. *Let Σ be a surface, let $(G, \mathcal{C}) \in \mathcal{S}_{10}(\Sigma)$, let G be $\mathcal{C}$-critical, and let C be a cycle forming a subset of the boundary of a face of a component of G. Then*

$|V(C)| \geq 11$.

Proof. By definition of $\mathcal{S}_{10}$, it follows that cycle C is $\mathcal{C}$-admissible. As a result, if $4 \leq |V(C)| < 11$, by condition (ii) of the definition of $\mathcal{C}$-admissible, there exist $v_1, v_2 \in V(C)$ such that for $i = 1, 2$, the surface Σ_i does not include an edge of $G \setminus E(C)$ incident with v_i. Since C is a subset of the boundary of a face, one of v_1, v_2 must have degree two. This contradicts the assumption that G is $\mathcal{C}$-critical. □

Lemma 4.2.4. *Let Σ be a surface and let $(G, \mathcal{C}) \in \mathcal{S}_{10}(\Sigma)$. If $v \in V(G)$, then $ch'(v) \geq 0$.*

Proof. Suppose $d(v) = 2$. Suppose first that v is not a cutvertex. Then $v \in C_i$. Then it receives 4/3 units of charge from C_i and 2/3 from its other incident face. Thus $ch'(v) = -2+4/3+2/3 = 0$. If v is a cutvertex, then $ch'(v) = 2+4/3+2/3-2(1/3) = 10/3 \geq 0$. Suppose $d(v) = 3$. First suppose that v is not a cutvertex. If $v \in C_i$, then v receives 2/3 from C_i and 2/3 from one (or possibly two) additional non-triangular face. In addition, it may lose charge 1/3 if it is part of a 3-face. Observe that v can be part of only one 3-face else G contains a 4-cycle. If $v \notin C_i$ then $ch'(v) = 3-4+(1/3 \cdot 3) = 0$. If instead v is a cutvertex, then the above rules apply, but v also sends $3(1/3) = 1$ unit of charge from rule 4. So $ch'(v) \geq 3 + 3(1/3) - 1 = 3 \geq 0$.

Now suppose that $d(v) = 4$. Suppose that v is not a cutvertex. First, if $v \in C_i$ then v receives 2/3 from C_i and distributes at most 1/3 units of charge to two incident 3-faces. Suppose that $v \notin C_i$. If v is not incident to a 3-face, then $ch(v) = ch'(v) = 0$. If v is incident to one 3-face, then v loses charge 1/3 by rule 1 and gains charge 1/3 by rule 2. Similarly, if v is incident to two 3-faces, then v loses 1/3 units of charge to each of the two incident triangles but gains 1/3 units of charge from both its other incident faces. If instead v is a cutvertex, then aside from rule 4, the charge v receives and gives is the same. So including rule 4, $ch'(v) \geq 4 - 4(1/3) = 8/3 \geq 0$.

If $d(v) \geq 5$, then if v is not a cutvertex then $ch'(v) = d(v) - 4 \geq 0$. If instead v is a cutvertex, then $ch'(v) = d(v) - 1/3(d(v)) \geq 0$. □

Definition 4.2.5. Given a graph G and a collection of cycles $\mathcal{C}$, a tetrad is a subgraph isomorphic to the graph in Figure 1 such that $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3$. In particular, where $xv_1v_2v_3v_4x'$ is contained in a boundary of a face, $t' \neq x$ and $t \neq x'$. Also, vertices v_1, v_2, v_3 and v_4 are not contained in $\mathcal{C}$.

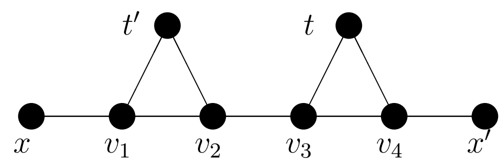


Figure 15: A tetrad.

We will show that if graph G contains a tetrad that does not use any vertices in $\mathcal{C}$, then G can be reduced to a smaller graph while preserving 3-colorability.

Proposition 4.2.6. *Let G be a graph with a tetrad T. Let G' be the multigraph obtained after deleting v_1, v_2, v_3, v_4 and identifying x with t. Call this multigraph G'. Every 3-coloring of G' extends to a 3-coloring of G.*

Proof. Let ϕ be a 3-coloring of G'. We will describe how to modify it to produce a 3-coloring of G. In particular, we must color vertices v_1, v_2, v_3 and v_4. The coloring ϕ gives rise to a 3-coloring ψ of $G \setminus \{v_1, v_2, v_3, v_4\}$ with $\psi(x) = \psi(t)$ and $\psi(v) = \phi(v)$ for every $v \in V(C)$. Since v_4 is adjacent to t and x', it has at least one color available, so we can extend ψ to v_4. Now, the only colored vertices v_3 is adjacent to are t and v_4, so it can be colored. If $\psi(v_3) = \psi(t')$, then let $\psi(v_2) = \psi(t)$ and color v_1 with its lone remaining color. If $\psi(v_3) \neq \psi(t')$ then let $\psi(v_1) = \psi(v_3)$. As a result, v_2 has one color available and thus can be colored. □

Define a tetrad to be *safe* to mean that none of x, t, t', x' is a cutvertex.

Lemma 4.2.7. *Let Σ be a surface and let $(G, \mathcal{C}) \in \mathcal{S}_{10}(\Sigma)$. Suppose G contains no safe tetrads. Let f be a face of G not in $\mathcal{C}$. Then if $|f| = 3$, $ch'(f) = 0$ and if $|f| > 3$, then $ch'(f) \geq \frac{1}{3}(|f| - 9)$.*

Proof. If $|f| = 3$, then $ch'(f) = -1 + 3 \cdot 1/3 = 0$. Now, suppose that $|f| \neq 3$. By our hypotheses, $|f| \geq 11$. Observe that if there are five consecutive bad vertices on f, then there exists either a safe tetrad or a cutvertex that sends charge $1/3$ into f. However, we are assuming there are no safe tetrads, so for every set of five consecutive vertices, either one of them must be not bad, or one of them must be a cutvertex that sends charge of $1/3$ into f. and hence only receives charge at most $1/3$ from f.

First suppose there are no cutvertices that send charge into f. Then since there are no safe tetrads, there must be at least three vertices that are not bad. Then $ch'(f) \geq |f| - 4 - \frac{2}{3}(|f| - 3) + 3\frac{1}{3} = \frac{1}{3}(|f| - 9)$. Suppose instead there was one cutvertex. Around this cutvertex, there are still two disjoint sets of five vertices, so there are at least two vertices that are not bad. Then $ch'(f) \geq |f| - 4 - \frac{2}{3}(|f| - 2) + 2\frac{1}{3} + \frac{1}{3} = \frac{1}{3}(|f| - 9)$. If there were two cutvertices, then regardless of how they are placed on a cycle of length at least 11, there still exists a set of five consecutive vertices on the cycle which does not contain a cutvertex. Hence, there is at least one vertex that is not bad. This gives, $ch'(f) \geq |f| - 4 - \frac{2}{3}(|f| - 1) + 1\frac{1}{3} + 2\frac{1}{3} = \frac{1}{3}(|f| - 9)$. Finally, if there are at least three cutvertices that contribute charge to f, then we obtain the lemma, even if all the other vertices in f were bad. This is because $ch'(f) \geq |f| - 4 - \frac{2}{3}(|f|) + 3\frac{1}{3} = \frac{1}{3}(|f| - 9)$. This completes the proof. □

Lemma 4.2.8. *Let Σ be a surface and let $(G, \mathcal{C}) \in \mathcal{S}_{10}(\Sigma)$. Let f be a face bounded by a cycle in $\mathcal{C}$. Then $ch'(f) \geq 0$.*

Proof. Observe that there is at least one vertex in each $C_i \in C$ with degree at least three. As a result, this vertex receives at most charge $2/3$ from f. So $ch'(C_i) \geq$

$4/3|V(C_i)| - 2/3 - 4/3(|V(C_i)| - 1) - 2/3 \geq 0$ as desired. □

Lemma 4.2.9. *Let Σ be a surface, let $(G, \mathcal{C}) \in \mathcal{S}_{10}(\Sigma)$ with Euler genus g, and let every face of G be homeomorphic to a disk. Then the sum of the charges of the vertices and faces in G is given by*

$$-4\chi(g) + \sum_{i=i}^{k} \left(10/3 + \frac{|V(C_i)|}{3}\right) + 4x,$$

where x is the number of cutvertices.

Proof. Recall that we give each vertex charge $d(v) - 4$ and each face not in $\mathcal{C}$ charge $|f| - 4$. Each face in $\mathcal{C}$ is given charge $4/3|V(C)| - 2/3$. By Lemmas 4.2.4, 4.2.7, and 4.2.8, we know that every vertex has nonnegative charge after discharging. In addition, by Lemma 4.2.7, if $|f| > 3$, then the amount of charge face f contains after discharging is at least $\frac{1}{3}(|f| - 9)$. Notice that if there were no faces bounded by cycles in $\mathcal{C}$ or cutvertices, our discharging rules give

$$\sum_{v \in V}(d(v) - 4) + \sum_{f | f \notin \mathcal{C}} (|f| - 4) = -4\chi(g).$$

However, instead of giving face C_i charge $|V(C_i)| - 4$, we give it charge $4/3|V(C_i)| - 2/3$, which is $(10/3 + \frac{|V(C_i)|}{3})$ more than the uncolored faces. In addition, we give each cutvertex four more units of charge than vertices that are not cutvertices. As a result, the sum of the charges of the vertices and faces in G is given by $-4\chi(g) + \sum_{i=i}^{k} \left(10/3 + \frac{|V(C_i)|}{3}\right) + 4x$, completing the proof. □

Proposition 4.2.10. *Let Σ be the sphere, let $(G, \mathcal{C}) \in \mathcal{S}_{10}(\Sigma)$ where $\mathcal{C} = \{C\}$, suppose that G has no tetrads and that G is 2-connected. Then*

$$\sum_{f \in \mathcal{F}(G,\mathcal{C})} (|f| - 9) \leq |V(C)| - 14$$

Proof. By Lemma 4.2.9, the sum of the charges (before discharging) is $-8 + (10/3 + \frac{|V(C)|}{3})$, because every face is homeomorphic to a disk and there are no cutvertices. By our discharging rules, we know that the $ch'(v) \geq 0$, $ch'(t) = 0$, where t is a triangle, $ch'(C) \geq 0$ and $ch'(f) \geq \frac{1}{3}(|f| - 9)$ for all other faces f. As a result we have

$$\frac{1}{3} \sum_{f \in \mathcal{F}(G,\mathcal{C})} (|f| - 9) \leq \sum_{f \in \mathcal{F}(G,\mathcal{C})} ch'(f) \leq$$

$$\sum_{v} ch'(v) + \sum_{f} ch'(f) - ch'(C) \leq \sum_{f} ch(f) + \sum_{v} ch(v) = \frac{|V(C)|}{3} + \frac{10}{3} - 8.$$

Multiplying the left and right hand sides of the series of inequalities above by three gives

$$\sum_{f \in \mathcal{F}(G,\mathcal{C})} (|f| - 9) \leq |V(C)| - 14.$$

□

4.3 Inductive Lemmas

Let S be a cycle of graph G embedded in a surface Σ where S bounds an open disk Δ. Let $Int(S)$ be defined as the set of vertices lying inside Δ and let $Out(S)$ be defined as the set of vertices lying outside the closure of Δ. For a graph G, let $\mathcal{F}(G, \mathcal{C})$ be the set of faces $f \in G$ such that $|f| > 3, f \in G, f \neq \mathcal{C}$.

Lemma 4.3.1. *Let G be a plane graph with no cycles of length $4, 5, \ldots, 10$, excluding the induced cycle C that bounds the infinite face in G, so that $\mathcal{C} = \{C\}$. Suppose that G is $\mathcal{C}$-critical. Then*

$$\sum_{f \in \mathcal{F}(G,\mathcal{C})} (|f| - 9) \leq |V(C)| - 14.$$

Proof. We prove this result by induction on $|V(G)|$. The lemma is vacuously true if $|V(G)| \leq 3$, and we may assume that $|V(G)| \geq 4$ and that the lemma holds for all graphs on fewer than $|V(G)|$ vertices.

Now suppose there is a tetrad in G with its vertices labeled as in Figure 1. We will perform the operation used by Borodin et al [6] to remove the tetrad. In particular, delete v_1, v_2, v_3, v_4 and identify x with t as shown in Figure 1. Call this multigraph G'. By Proposition 4.2.6, every precoloring of C that extends to G' also extends to G.

We must ensure that after the tetrad operation is completed that $(G', \{C\}) \in \mathcal{S}_{10}(\mathbb{S}^2)$ and G' has no loops. In addition we must not identify two vertices of $\mathcal{C}$ nor can we create a chord of $\mathcal{C}$.

We first show that $(G', \{C\}) \in \mathcal{S}_{10}(\mathbb{S}^2)$. Suppose that after x is identified with t there is a cycle, B', such that B' has length between four and ten. Let B be the corresponding cycle of G. If B does not pass through t', then B' separates G, and so we may apply induction on the number of vertices of G to the graph $G - Out(B)$ and the graph contained inside cycle B. Since B separates G, then both $G - Out(B)$ and $G - Int(B)$ are nonempty so we may apply the lemma to each side side of B separately. However, it may be that B passes through vertex t'. In this case, in G this means there is a path, P_1, of length k from x to t' and a path, P_2 of length l from t' to t such that $k + l \leq 10$. So the length of P_1 or P_2 is at most five. Suppose, without loss of generality that the length of P_2 is at most five. In this case, $t'v_2v_3tP_2$ is a cycle of length between four and nine, a contradiction as $(G, \{C\}) \in \mathcal{S}_{10}(\mathbb{S}^2)$. Thus $(G', \{C\}) \in \mathcal{S}_{10}(\mathbb{S}^2)$.

Notice that G' is in fact a graph. Suppose there was a multiple edge in G'. This means a vertex, r, is adjacent to both x and t and so in G we have a 6-cycle, $xv_1v_2v_3trx$, a contradiction. If a loop exists in G', then x would be adjacent to t in G and would construct a 5-cycle, $xv_1v_2v_3tx$, a contradiction.

We now consider the case when there is a path P, consisting of vertices $u_1, \ldots u_k$, in order, of length at most five with both ends on C not forming a triangle. Let C, C_1, C_2 be the cycles of $C \cup P$. In order to apply induction on cycle C_i, it must be induced. So suppose that C_i was not induced. Notice that any chord of C_i must include at least one vertex of P, else C is not induced, a contradiction. If a chord does not use u_1 or u_k, then we can reroute P to make it shorter. So any chords are adjacent to u_1 and u_k only. Thus, if there exists a chord in C_1 or C_2, then the length of P is at most three.

First, suppose that the length of P is two. Then C is divided into $d \geq 2$ regions. Let $l_1, l_2, \ldots, l_d$ be the length of the outer cycle bounding each of the d, regions, respectively. If region i is facial, then it contributes $(l_i + 2) - 9$ to the sum, and if region i is not facial, then by induction, it contributes $(l_i+2)-14$. So a upper bound for this sum is to assume each of these regions are facial. This gives:

$$\sum_{f \in \mathcal{F}(G,C)} (|f| - 9) \leq \sum_{i=1}^{d} (l_i + 2) - 9 \leq |V(C)| - 14.$$

Now, suppose the length of P is three. Now, each vertex of P must have degree at least three, and so each internal vertex, v_1, v_2 of P is either part of at least two faces of length at least 11, and if v_i is adjacent to only two faces of length at least 11, then it is also adjacent to at least one triangle that has an edge on C. Otherwise, v_i is incident to at least three faces of length 11. First suppose that each vertex of P not on C is adjacent to two large faces and a triangle. Then,

$$\sum_{f \in \mathcal{F}(G,C)} (|f| - 9) \leq \sum_{i=1} d((l_i + 3) - 9) \leq \sum_{i=1}^{d} (l_i) + 2 - 14 \leq |V(C)| - 14.$$

If instead there are at least three faces of length 11 inside C, then $d \geq 3$ and we obtain

$$\sum_{f \in \mathcal{F}(G,C)} (|f| - 9) \leq \sum_{i=1}^{d} (l_i + 3) - 9 \leq |V(C)| - 14.$$

Thus, we may assume that C_1 and C_2 have no chords. First suppose that neither C_1 nor C_2 is facial. Then, let G_i be the subgraph of G consisting of vertices and edges drawn in the closed disk bounded by C_i. Then by induction

$$\sum_{f \in \mathcal{F}(G,C)} (|f| - 9) = \sum_{f \in \mathcal{F}(G_1,C)} (|f| - 9) + \sum_{f \in \mathcal{F}(G_2,C)} (|f| - 9)$$

$$\leq |V(C_1)| - 14 + |V(C_2)| - 14 = |V(C)| + 2|E(P)| - 28 \leq |V(C)| - 14.$$

This shows that if C_1 and C_2 are not facial, that there can be no path of length at most seven between C_1 and C_2. Notice that we can only apply induction when a cycle is not facial and is induced.

Now, suppose that path P has at least one internal vertex. It follows that at least one of C_1 and C_2 are not facial as internal vertices have degree at least three. Without loss of generality suppose that C_1 is not facial and C_2 is facial (as if both were not facial the result follows from the computation above). Now, our computation is

$$\begin{aligned}\sum_{f\in\mathcal{F}(G,C)}(|f|-9) &= \sum_{f\in\mathcal{F}(G_1,C)}(|f|-9)+|V(C_2)|-9\\ &\leq |V(C_1)|-14+|V(C_2)|-9=|V(C)|+2|V(P)-1|-23\\ &\leq |V(C)|-14.\end{aligned}$$

Notice this equation holds only if $|V(P)| \leq 5$, and this means that if there does not exist a path P of length at most four where vertices of P are only adjacent to vertices in the interior of one of C_1 and C_2.

Thus we can conclude that there is no path P of length at most four with both ends in C, and there is no path P with both ends in C of length at most five when both C_1 and C_2 are not facial cycles.

Now suppose that both vertices x and t from the tetrad reduction were elements of C. This contradicts the above claim as the path $xv_1v_2v_3t$ is a path of length four with both ends on C. Similarly, suppose that $x \in C$ and $t \notin C$ is adjacent to a vertex, v in C. After x is identified with t it may be that v and x have the same color. However, the path $xv_1v_2v_3tv$ is a path of length five with both ends on C. This also contradicts the above claim as this path partitions C into two non-facial cycles. Suppose instead that $x \notin C$ is adjacent to a vertex $v \in C$ and $t \in C$. After x is identified with t it may be that v and t have the same color. However, the path $vxv_1v_2v_3t$ is a path of length five with both ends on C. This contradicts the above claim as this path partitions C into two non-facial cycles.

Let G'' be a minimal subgraph of G' such that any 3-coloring of C that extends to G'' also extends to G' and hence to G. Then $G'' \neq C$ and G'' is C-critical. Observe that for every face f'' in G'' there is a corresponding cycle $D_{f''}$ in G, which may include several faces of G in its interior and therefore may not be facial. However if we let $D_{f''}$ act as C in the statement of this lemma, we can apply the induction hypothesis to $D_{f''}$ to give a bound for $\sum_{f\in\mathcal{F}(G,\mathcal{C}), f\subset D_{f''}}(|f|-9)$, where $f \subset D$ means that the face f is a subset of the open disk bounded by D. We can make an analogous estimate for each face in G''.

So for G'', $\sum_{f\in\mathcal{F}(G'',\mathcal{C})} |f|-9 \leq |V(C)|-14$ by the induction hypothesis. If there is nothing inside cycle D in G, the argument is the same as for G, which is described in the next paragraph. If D is not a facial cycle, then we will apply induction to the subgraph of G induced by D and $Int(D)$.

Now, we will reindex the sum so that each face in G'' is accounted for individually. For each one of these faces, call one such face D, we can apply induction to this face, noting that in the original graph G, the size of the face may have increased by at most four. So we know that $|D_{f''}| \leq |f''|+4$. So we have

$$\sum_{f\in\mathcal{F}(G,\mathcal{C})} |f|-9 \;=\; \sum_{f''\in\mathcal{F}(G'',\mathcal{C})}\left(\sum_{f\in\mathcal{F}(G,\mathcal{C}), f\subset D_{f''}} |f|-9\right) \quad (1)$$

$$\leq \sum_{f''\in\mathcal{F}(G'',\mathcal{C})} (|D_{f''}|-14) \leq \sum_{f''\in\mathcal{F}(G'',\mathcal{C})} (|f''|-9) \leq |C|-14, \quad (2)$$

where the second inequality of (2) follows from $|D_{f''}| \leq |f''|+4$, and the last inequality of (2) follows by induction applied to G'', as desired. Finally, suppose there are no tetrads in G. By Proposition 4.2.10, we obtain the desired result, because G is 2-connected, as is easily seen. □

A proposition that will be useful is a minor corollary of Lemma 4.3.1 that gives a stronger bound on the minimum size of the precolored cycle C.

Proposition 4.3.2. *Let G be a planar graph with a precolored induced outer cycle C. If G is C-critical, then $|V(C)| \geq 18$.*

Proof. We may assume that G consists of more than just the precolored cycle. In addition, by Lemma 4.3.1, we know that $\sum_{f\in\mathcal{F}(G,\mathcal{C})}(|f|-9) \leq |V(C)|-14$. There are at least three faces in G, as G consists more than C, and at least two of these faces is at least length 11. This implies that the sum on the left in the statement of Lemma 4.3.1 is at least four. As a result, $|V(C)| \geq 18$. □

In subsequent lemmas, there are complications that arise if G'' is not connected or has connectivity one after a tetrad reduction. One tool to assist this is Proposition 4.3.4, but first we need the following result.

Proposition 4.3.3. *Let $(G, \mathcal{C})$ be in $\mathcal{S}_{10}(\Sigma)$, where Σ is the sphere, and let $C = abP_1xyP_2a$ bounding the infinite face. We also assume that $a \not\sim x$, $b \not\sim y$. Here, P_1 and P_2 are paths, vertices a, x are precolored the same and vertices b, y are precolored the same but using a different color than the color of a and x. Then a precoloring of $\mathcal{C}$ with these properties extends to G.*

Proof. Suppose that the distance between vertices b and x along P_1 is equal to α and the distance between vertices y and a along P_2 is equal to β. If $\alpha > 7$, then draw another path, P_3, of length three, that connects b and x and lies in the face bounded by C. Notice that adding P_3 does not create a cycle of length less than 11 because the minimum distance between b and x is eight and P_3 is a path of length three, so any cycle that uses P_3 must involve vertices b and x and must have length at least 11. Similarly, if $\beta > 8$, then draw a path P_4 of length three that connects a and y which also lies in the face bounded by C. Again, adding P_4 does not create a new cycle of length less than eleven because any new cycle involves vertices a and x, the distance between a and x is at least nine and the length of P_4 is three. After these two possible constructions, the infinite face is bounded by a cycle of length at most 17. By Proposition 4.3.2, it follows that G is not $\mathcal{C}$-critical and so every precoloring of $G[a, b, x, y]$ extends to G. □

Proposition 4.3.4. *Suppose that* $(G, \mathcal{C}) \in \mathcal{S}_{10}(\Sigma)$ *where vertices* v_1 *and* v_2 *are pre-colored. The coloring extends unless* v_1 *and* v_2 *are colored the same and are adjacent.*

Proof. Suppose that there is an edge, e between v_1 and v_2, but that v_1 and v_2 are colored differently. Then we will consider instead the graph G' which is G after edge e was removed. If this graph extends, then so does G.

So now consider the case when v_1 and v_2 are not adjacent. Suppose that v_1 and v_2 are connected by a path P of length at most five. We will split open the internal vertices of this path to create a cycle D of size at most ten. Two of the vertices in D are already precolored, and now we will precolor the rest of D. We can then apply Proposition 4.3.3, where cycle D is the cycle in the statement of the proposition. Thus we may assume there is no path of length at most five between v_1 and v_2. If there is a tetrad, then we reduce G using our usual tetrad reduction and decrease the size of G.

If there is no tetrad, then we will again use a discharging argument with a modified discharging rule for the two precolored vertices and obtain a contradiction. The idea is that we can not use these precolored vertices in a tetrad reduction, and as a result these vertices can be additional bad vertices that require 2/3 units of charge from large faces incident to these vertices. As such, suppose that vertex v is precolored, has degree three and is part of a 3-face. Then give any face incident to v and of length at least eleven an additional 1/3 unit of charge. Since there are two precolored vertices, we may have added and extra 4/3 units of charge. By Lemma 4.2.9, the sum of the charges in G is at most -8 + 10/3 + 4/3 = -10/3. By our discharging rules, after the discharging process, the final charge of every vertex and face is nonnegative. Since charge is preserved during the discharging process, the sum of the charges must be nonnegative. This is a contradiction.

□

If there are more than two precolored cycles, it may be that there is more than one face whose length decreases by eight after a tetrad reduction. The following definitions and proposition is useful to better describe this situation.

Definition 4.3.5. Given a face f, define an *f-circle* as a set $X \subset f \cup \{v\}$ for some vertex v incident with f such that X is homeomorphic to a unit circle and null-homotopic. Let $w_2(f)$ be the maximum number of f-circles such that they are pairwise non-homotopic after we remove the faces bounded by cycles in $\mathcal{C}$ and pairwise share no points of f.

Proposition 4.3.6. *Suppose that G is a graph drawn in surface Σ that contains $k \geq 2$ disjoint closed disks $\Delta_1, \Delta_2, \ldots, \Delta_k$. Let $\tilde{\Sigma} = \Sigma - (\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k)$. Then the number of closed curves in $\tilde{\Sigma}$ that are pairwise not homotopic in $\tilde{\Sigma}$ and each is null homotopic in Σ is $2k-3$ if Σ is the sphere and $2k-1$ otherwise.*

Proof. We prove this via induction on the number of precolored cycles. First suppose that Σ is the sphere. Let f_n denote the maximum number of circles when there are n precolored cycles. Notice that $f_1 = 0$, $f_2 = 1$ and $f_3 = 3$.

We claim that $f_{n_1+n_2} = f_{n_1+1} + f_{n_2+1} - 1$. To see this notice that given a configuration of circles around a set of precolored cycles, we could cut open one circle to get two groups of precolored cycles, one containing $n_1 + 1$ precolored cycles and one with $n_2 + 1$ precolored cycles. However, we double-count the cycle that was cut open as it is part of the count in f_{n_1} and f_{n_2} and so this is why one is subtracted from the right hand side of the claim.

To prove the proposition, we apply induction. First notice that $f_3 = 3$ satisfies the $2k - 3$ bound. Now, suppose that $f_m \leq 2m + 3$ for $3 \leq m \leq k$. Let $k = m_1 + m_2$ So $f_{m_1+m_2} = 2(m_1 + 1) - 3 + 2(m_2 + 1) - 3 - 1 = 2(m_1 + m_2) - 3$, as desired.

If Σ is not the sphere, the argument is identical, except that the base cases are $f_1 = 1, f_2 = 3$ and $f_3 = 5$. □

Let Σ be a surface, and let $(G, \mathcal{C}) \in \mathcal{S}_{10}(\Sigma)$. By a *$\theta$-graph* in G, we mean a subgraph of G of the form $P_1 \cup P_2 \cup P_3$, where P_1, P_2, P_3 are three internally disjoint paths with the same ends. We say that a θ-graph H is *triseparating* if each face of H that is homeomorphic to a disk includes a member of $\mathcal{C}$.

Lemma 4.3.7. *Let Σ be a surface, let $(G, \mathcal{C}) \in \mathcal{S}_{10}(\Sigma)$, and let G have a tetrad with vertices labeled as in Figure 1. Let G_1 be obtained from $G \setminus \{v_1, v_2, v_3, v_4\}$ by identifying x and t and let G_2 be obtained from $G \setminus \{v_1, v_2, v_3, v_4\}$ by identifying x' and t'. If there is no triseparating θ-graph on at most 29 edges, then there is $i \in \{1, 2\}$ such that G_i is loopless and every cycle in G_i is $\mathcal{C}$-admissible.*

Proof. Suppose the lemma does not hold. Then in each of G_1 and G_2, there exists a cycle that violates looplessness or $\mathcal{C}$-admissibility. This Let P_1 be the path from x to t in G that will be identified to make a cycle, D_1, of length at most ten in G_1. Similarly, let P_2 be the path from t' to x' that will be identified to make a cycle, D_2, of length at most ten in G_2. Notice that P_1 and P_2 must share a vertex. Call this vertex v. Using P_1, P_2 and the tetrad, G is partitioned into four non-triangular regions. Call the region that includes the unbounded face R_4. Call the other three disks R_1, R_2, R_3, respectively such that x and t' are on the boundary of R_1, t' and t are on the boundary of R_2 and t and x' are on the boundary of R_3.

By hypothesis, there is no triseparating graph on 29 edges. In addition, there must be a cycle of $\mathcal{C}$ inside and outside D_1 and D_2. As a result, the two disks amongst R_1, R_2, R_3, R_4 that contain no member of $\mathcal{C}$ are either $\{R_1, R_3\}$ or $\{R_2, R_4\}$. The sum of the lengths of the boundaries of these disks are at most $10 + 10 + 3 + 5 = 28$. But each of D_1 and D_2 must be length at least 11, so both D_1 and D_2 have length at most 17. As a result, by Proposition 4.3.2, both these disks have no vertices in their interior. But then we can find vertices v_1, v_2 along those disks that satisfy assumption (ii) of the definition of $\mathcal{C}$ admissible for the other two R_i's. $\square$

Proposition 4.3.8. *Suppose that G is a graph drawn in surface Σ that contains $k \geq 2$ disjoint closed disks $\Delta_1, \Delta_2, \ldots, \Delta_k$. Let $\tilde{\Sigma} = \Sigma - (\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k)$. Then in a $\mathcal{C}$-critical graph, the number of closed curves in $\tilde{\Sigma}$ in each homotopy class that hit G exactly once is at most two.*

Proof. Suppose there are two closed curves C_1, C_2 in $\tilde{\Sigma}$ that are homotopic in $\tilde{\Sigma}$. Look at the graph H contained in the region between curves C_1, C_2. This is an annular region. Let $G' = G \setminus H$. Now, since G is $\mathcal{C}$-critical there exists a proper 3-coloring of G'. Let x and y be the vertices of G that hit curves C_1, C_2, respectively. By Proposition 4.3.4 it follows that x and y must be adjacent, else the 3-coloring of G' may be extended to H. Therefore if x and y were not adjacent or if there were more than two closed curves in this homotopy class, then G would not be $\mathcal{C}$-critical. $\square$

Definition 4.3.9. Let α, β, γ be constants whose value will be determined later. Let G be a graph embedded on a surface Σ. For each face f of G, if the boundary of f is connected, let

$$w_1(f) = |f| - 9 + \alpha(2 - \chi(f)).$$

If the boundary of f is disconnected with k components, let

$$w_1(f) = \alpha(2 - \chi(R_f)) + \beta(k) + \sum_{C_i \in R_f} |E(C_i)| - \gamma.$$

In either case, let

$$w(f) = w_1(f) + w_2(f).$$

Proposition 4.3.10. *Let Σ be a surface, let $(G, \mathcal{C}) \in \mathcal{S}_{10}(\Sigma)$. Suppose that every face is homeomorphic to a disk, the cycles in $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ are induced, $n \geq 2$, G is $\mathcal{C}$-critical and there are no safe tetrads in G. If $\alpha \geq 10\beta + 6, \beta \geq 66, 2\gamma \geq 3\beta + 50$, then*

$$\sum_{f \in \mathcal{F}(G,\mathcal{C})} w(f) \leq \alpha(2 - \chi(\Sigma)) + \beta|\mathcal{C}| + \left(\sum_{i=1}^{n} |V(C_i)|\right) - \gamma - 4.$$

Proof. Suppose there are no safe tetrads in G. By Lemma 4.2.9, the sum of the charges (before discharging) is $-4\chi(\Sigma)+\sum_{i=1}^{n}(10n/3+\frac{|V(C_i)|}{3})+4w_2(f)$. By our discharging rules, we know that $ch'(v)\geq 0, ch'(t)=0$, where t is a triangle, $ch'(C_i)\geq 0$ and $ch'(f)\geq \frac{1}{3}(|f|-9)$ for all other faces f. As a result, we have

$$\frac{1}{3}\sum_{f\in\mathcal{F}(G,\mathcal{C})} w(f) \leq \sum_{f\in\mathcal{F}(G,\mathcal{C})} ch'(f)+\frac{4}{3}w_2(f)\leq$$

$$\sum_{v} ch'(v)+\sum_{f} ch'(f)+\sum_{i=1}^{n} ch'(C_i)+\frac{4}{3}w_2(f)\leq \sum_{f} ch(f)+\sum_{v} ch(v)+\frac{4}{3}w_2(f)$$

$$\leq \left(\sum_{i=1}^{n}\frac{|V(C_i)|}{3}\right)+\frac{10n}{3}-4\chi(\Sigma)+4x+\frac{8}{3}n.$$

Here x denotes the number of cutvertices of G. Notice that $x\leq 4n$ by Proposition 4.3.7 and Proposition 4.3.8. Multiplying the left and right hand sides of the series of inequalities above by three and estimating for x gives

$$\sum_{f\in\mathcal{F}(G,\mathcal{C})} w(f)\leq \left(\sum_{i=1}^{n}(|V(C_i)|)\right)+10n+12(2-\chi(\Sigma))+48n+8n-24.$$

Since $n\geq 2, \alpha\geq 10\beta+6, \beta\geq 66, 2\gamma\geq 3\beta+50$, it follows that

$$\sum_{f\in\mathcal{F}(G,\mathcal{C})} w(f)\leq \alpha(2-\chi(\Sigma))+\beta|\mathcal{C}|+\left(\sum_{i=1}^{n}|V(C_i)|\right)-\gamma.$$

□

Lemma 4.3.11. *Let Σ be a surface, let $(G,\mathcal{C})\in\mathcal{S}_{10}(\Sigma)$, let $n\geq 2$ and suppose that G is $\mathcal{C}$-critical where $\mathcal{C}=\{C_1,C_2,C_3,\dots C_n\}$ and each C_i is induced. If α,β,γ satisfy $\alpha\geq 10\beta+6$, $\beta\geq 66$, $2\gamma\geq 3\beta+50$, then:*

$$\sum_{f\in\mathcal{F}(G,\mathcal{C})} w(f)\leq \alpha(2-\chi(\Sigma))+\beta|\mathcal{C}|+\left(\sum_{i=1}^{n}|V(C_i)|\right)-\gamma-4.$$

Proof. As before, we prove this result by induction on $|V(G)|$. The lemma is again vacuously true if $|V(G)|\leq 10$, and so we may assume that $|V(G)|\geq 11$ and that the lemma holds for all graphs on fewer than $|V(G)|$ vertices. First, recall from the computation in Proposition 4.3.6 that $\sum w_2(f) \leq 4|\mathcal{C}|$. Now, suppose that G has a non-null-homotopic closed curve that does not use any vertices of G. Then we can reduce the genus of the graph we are analyzing, so the result holds if $\alpha \geq 1$. Suppose instead that G has a non-null-homotopic closed curve, D that uses one vertex of G, call this vertex v. We will cut open along this curve, thus reducing genus by at least one and split vertex v into two vertices, v_1 and v_2. In addition, we will construct two new cycles of $\mathcal{C}$, call them D_1, D_2, which includes v_1 and v_2, respectively and are each length three. In addition we may have increased $w_2(f)$ by at most $4|\mathcal{C}|$. Construct a new graph H, embedded on the new surface obtained by Σ_1, and observe that $\chi(\Sigma_1) \geq \chi(\Sigma) + 1$. So this gives:

$$\begin{aligned}
\sum_{f \in \mathcal{F}(H, \{\mathcal{C} \cup \{D_1, D_2\})} w(f) &= \alpha(2 - (\chi(\Sigma) + 1)) + \beta(|\mathcal{C}| + 2) + 8|\mathcal{C}| \\
&+ \left(\sum_{i=1}^{n} |V(C_i)| + 6 \right) - \gamma - 4 \\
&\leq \alpha(2 - \chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{i=1}^{n} |V(C_i)| - \gamma - 4.
\end{aligned}$$

This holds because $\alpha \geq 10\beta + 6$.

Now, suppose that G is disconnected. Thus G has a face f_0 of disconnected boundary, and let $G'_1, G'_2, \ldots, G_l$ be subgraphs of G with union G, each incident with one boundary component of G. For $i = 1, 2, \ldots, l$ let f'_i be the face of G'_i containing the rest of G, and let G_i denote the embedded graph obtained from G'_i by capping off f'_i by a disk, resulting in a face f_i. By Lemma 4.2.3, it follows that G_i satisfies condition (iii) of the definition of $\mathcal{S}_{10}(\Sigma)$. Further, suppose that the graphs $G_1, \ldots, G_k$ contain at least two precolored cycles, and $G_{k+1}, \ldots G_l$ contain exactly one precolored cycle. Let $\mathcal{C}_i$ denote the collection of precolored cycles in graph G_i.

We now have by Lemma 4.3.1, and induction,

$$\begin{aligned}
\sum_{f\in\mathcal{F}(G,\mathcal{C})} w(f) &= \sum_{i=1}^{k}\sum_{f\in\mathcal{F}(G_i,\mathcal{C}_i)} w(f) + \sum_{i=k+1}^{l}\sum_{f\in\mathcal{F}(G_i,\mathcal{C}_i)} w(f) + w(f_0) - \sum_{i=1}^{l} w(f_i)\\
&\leq \sum_{i=1}^{k}\left(\alpha(2-\chi(\Sigma_i)) + \beta|\mathcal{C}_i| + \sum_{C\in\mathcal{C}_i} E(C) - \gamma - 4\right)\\
&+ \sum_{i=k+1}^{l}(\alpha(2-\chi(\Sigma_i)) + |E(C_i)| - 13) + \alpha(2-\chi(\Sigma_0))\\
&+ (l)\beta + \sum_{i=1} l|f_i| + 4w_2(f_0) - \gamma - \sum_{i=1}^{l} w(f_i) - 4\sum_{i=1}^{k} w_2(f_i)\\
&\leq \alpha(2-\chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{C_i\in\mathcal{C}} |E(C_i)| - \gamma - 4.
\end{aligned}$$

This completes the proof when G is disconnected.

Suppose there exists a non-contractible cycle D of length at most 14. We may assume that D is induced. Either it is two-sided and separating, two-sided and non-separating or one-sided. First, suppose that D was two-sided and non-separating. Cut D from surface Σ and construct a new graph H, embedded on Σ' where $\chi(\Sigma') \geq \chi(\Sigma) + 1$ with two copies of D, call them D_1 and D_2. Let $\mathcal{C}' = \mathcal{C} \cup \{D_1, D_2\}$. Then H is $\mathcal{C}'$-critical and

$$\begin{aligned}
\sum_{f\in\mathcal{F}(H,\{\mathcal{C}'\})} w(f) &= \alpha(2-(\chi(\Sigma)+1)) + \beta(|\mathcal{C}|+2) + \left(\sum_{i=1}^{n}|V(C_i)| + 28\right) - \gamma - 4\\
&\leq \alpha(2-\chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{i=1}^{n}|V(C_i)| - \gamma - 4,
\end{aligned}$$

because $\alpha \geq 2\beta + 28$.

Suppose instead that D was two-sided and separating. After cutting open along D, this produces two separate surfaces Σ_1 and Σ_2 and graphs H_1 and H_2 where H_1 is embedded on Σ_1 and H_2 is embedded on Σ_2. Graphs H_1, H_2 each contain a copy of D, called D_1, D_2, respectively. Here, $\chi(\Sigma_1) + \chi(\Sigma_2) = \chi(\Sigma) - 2$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the set of original precolored cycles on each surface. Let $\mathcal{C}_1' = \mathcal{C} \cup \{D_1\}$, and let

$\mathcal{C}_2' = \mathcal{C} \cup \{D_2\}$. Then H_i is $\mathcal{C}_i'$-critical for $i \in \{1, 2\}$ and

$$\begin{aligned}
\sum_{f\in\mathcal{F}(H_1,\{\mathcal{C}_1'\})} w(f) \quad &+ \sum_{f\in\mathcal{F}(H_2,\{\mathcal{C}_2'\})} w(f) \\
&= \alpha(2-\chi(\Sigma_1)) + \beta(|\mathcal{C}_1|+1) + \left(\sum_{i=1}^{n}|V(C_i)| + 14\right) - \gamma - 4 \\
&+ \alpha(2-\chi(\Sigma_2)) + \beta(|\mathcal{C}_2|+1) + \left(\sum_{i=1}^{n}|V(C_i)| + 14\right) - \gamma - 4 \\
&\leq \alpha(2-\chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{i=1}^{n}|V(C_i)| - \gamma - 4,
\end{aligned}$$

because $2\alpha \geq 2\beta + 28 - \gamma - 4$.

Finally, suppose that D was one-sided. Cutting open D produces a single cycle of length at most 28, and we will call this cycle D_1. This produces a graph H embedded on surface Σ_1 where $\chi(\Sigma_1) = \chi(\Sigma) + 1$. Let $\mathcal{C}_1' = \mathcal{C} \cup \{D_1\}$. Then H is $\mathcal{C}_1'$-critical, and we have

$$\begin{aligned}
\sum_{f\in\mathcal{F}(H,\{\mathcal{C},D_1\})} w(f) &= \alpha(2-\chi(\Sigma_1)) + \beta(|\mathcal{C}|+1) + \left(\sum_{i=1}^{n}|V(C_i)| + 10\right) - \gamma - 4 \\
&\leq \alpha(2-\chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{i=1}^{n}|V(C_i)| - \gamma - 4,
\end{aligned}$$

because $\alpha \geq \beta + 10$.

From these three results, we can assume there are no non-contractible cycles of length at most 14.

Next, suppose that there exists a triseparating θ-graph, H on at most 29 edges. Suppose that C_1, C_2, C_3 are the cycles of H and $\mathcal{C}_i$ is the set of precolored cycles in the face of C_i. Let H_i be the subgraph of H in face $i = \{1, 2, 3\}$. Further, suppose that H_1, H_2, H_3 are embedded on surfaces $\Sigma_1, \Sigma_2, \Sigma_3$ so that $\chi(\Sigma_1)+\chi(\Sigma_2)+\chi(\Sigma_3) \geq \chi(\Sigma) + 4$. Then

$$\begin{aligned}
\sum_{f\in\mathcal{F}(H,\{\mathcal{C}\cup C_1\cup C_2\cup C_3\})} w(f) &= \sum_{i=1}^{3}\left(\sum_{f\in\mathcal{F}(H,\{\mathcal{C}\cup C_i\})} w(f)\right)\\
&= \sum_{i=1}^{3}(\alpha(2-\chi(\Sigma_i))+\beta(|\mathcal{C}_i|+1)+|E(C_i)|)-\gamma-4)\\
&\leq \alpha(2-\chi(\Sigma))+\beta|\mathcal{C}|+\sum_{i=1}^{n}|V(C_i)|-\gamma-4,
\end{aligned}$$

because, $3\beta+2|E(H)|\leq 3\beta+58\leq 2(\gamma+4)$.

Suppose that there exists a path P of distance at most six between C_i and C_j. In this case, let P be a shortest path between cycles C_i and C_j with $k\leq 5$ vertices in this path not on C_i and C_j for $i\neq j$, $i,j\in\{1,2,\ldots,n\}$. Construct a new graph H that consists of a single precolored cycle containing C_i, C_j and two copies of P. Let x_1 be the vertex in P on C_i and let x_2 be the vertex in P on C_j. Let $p_1,\ldots,p_k$ be the vertices of P starting closest to C_1. The graph H, consists of a single precolored cycle C_i^* defined by $x_1p_1\cdots p_kx_2C_2x_2p_k\cdots p_1x_1C_1$. Edges incident to vertices on P, x_1 and x_2 are now incident to only one copy of P depending on their orientation in G. We can then apply induction on this lemma and the fact that $\beta\geq 66$ and $\sum 4w_2(f)\leq 16|\mathcal{C}|$ to get

$$\begin{aligned}
\sum_{f\in\mathcal{F}(G,\mathcal{C})} w(f) &\leq \alpha(2-\chi(\Sigma))+\beta(|\mathcal{C}|-1)+2|V(P)-1|\\
&+ \sum_{i=1}^{n}|V(C_i)|+\sum 4w_2(f)-\gamma-4\\
&\leq \alpha(2-\chi(\Sigma))+\beta|\mathcal{C}|+\sum_{i=1}^{n}|V(C_i)|-\gamma-4.
\end{aligned}$$

Again we must show that $(G',\mathcal{C})\in\mathcal{S}_{10}(\Sigma)$, has no loops or multiple edges, and that every coloring of $\mathcal{C}$ that extends to G' also extends to G. In addition we must not identify two vertices of $\mathcal{C}$ nor can we create a chord of $\mathcal{C}$. As a result of the above usage of the above argument, it follows that the two identified vertices of any tetrad must come from the same $C_i\in\mathcal{C}$ if they are in $\mathcal{C}$.

Suppose that there exists a path P of length at most six (where P has five vertices not on C_i) with both ends on the same $C_i \in \mathcal{C}$. Define D_1, D_2 to be the cycles in $C_i \cup P$ other than C_i.

First suppose that D bounds a disk that includes no member of $\mathcal{C}$. Let G_1 be the part of the graph contained in R including, cycle D and let G_2 be the remaining graph. If there are no cycles $C_j \in \mathcal{C}$ within D, then we have that

$$\begin{aligned}\sum_{f\in\mathcal{F}(G,\mathcal{C})} w(f) &= \sum_{f\in\mathcal{F}(G_1,\{C_i''\cup P\})} w(f) + \sum_{f\in\mathcal{F}(G_2,(\mathcal{C}\setminus C_i)\cup(C_i'\cup P))} w(f)\\ &\leq (|E(C_i'')| + |V(P)| - 1 - 14) + \alpha(2-\chi(\Sigma)) + \beta|\mathcal{C}|\\ &+ \sum_{j=1}^{i-1}|V(C_j)| + \sum_{j=i+1}^{n}|V(C_j)| + |E(C_i')| + |V(P)| - 1 - \gamma - 4\\ &\leq \alpha(2-\chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{i=1}^{n}|V(C_i)| - \gamma - 4.\end{aligned}$$

This holds because $|V(P) - 1| \leq 7$.

Now, suppose that D_1 and D_2 are surface separating. Then $\Sigma = \Sigma_1' \cup \Sigma_2' \cup f_i$, where f_i is the face bounded by C_i and for $j = \{1, 2\}$, Σ_j' is a surface with boundary D_i. Let Σ_j be obtained from Σ_j' by capping off D_i by a disk. By the result of the previous paragraph, we may assume each Σ_j either has positive genus or includes a member of $\mathcal{C}$. Suppose that the cycles in $\mathcal{C}$ contained in Σ are labeled $\mathcal{E}' = \{E_1, \ldots, E_k\}$. Suppose the cycles in $\mathcal{C}$ not in Σ are labeled $\mathcal{E}'' = \{F_1, \ldots, F_l\}$. In this case we have

$$
\begin{aligned}
\sum_{f\in\mathcal{F}(G,\mathcal{C})} w(f) &= \sum_{f\in\mathcal{F}(G_1,\{\mathcal{E}'\cup C_i''\cup P\})} w(f) + \sum_{f\in\mathcal{F}(G_2,(\mathcal{E}''\setminus C_i)\cup(C_i'\cup P))} w(f) \\
&\leq \beta(|\mathcal{E}'|+1) + \sum_{j=1}^{k}(|E(E_j)|) + |E(C_i')| + |V(P)| - 1 - \gamma - 4) \\
&+ \beta(|\mathcal{E}''|+1) + \sum_{j=1}^{l}|V(C_j)| + |E(C_i'')| + |V(P)| - 1 - \gamma - 4 \\
&+ \alpha(2-\chi(\Sigma)) \\
&= \alpha(2-\chi(\Sigma)) + \beta(|\mathcal{C}|+1) + \beta(|\mathcal{E}''|+1) \\
&+ \sum_{j=1}^{n}|V(C_j)| + 2(|V(P)|-1) - 2\gamma - 8 \\
&\leq \alpha(2-\chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{j=1}^{n}|V(C_j)| - \gamma - 4.
\end{aligned}
$$

This holds because $\gamma \geq \beta + 6$.

Finally, suppose that D does not separate Σ. Cut open along D, apply induction to the resulting graph and $(\mathcal{C} - \{C_i\} \cup \{D_1, D_2\})$. We now have

$$
\begin{aligned}
\sum_{f\in\mathcal{F}(G,\mathcal{C})} w(f) &= \sum_{f\in\mathcal{F}(G_1,\{\mathcal{E}'\cup C_i''\cup P\})} w(f) + \sum_{f\in\mathcal{F}(G_2,(\mathcal{E}''\setminus C_i)\cup(C_i'\cup P))} w(f) \\
&\leq \alpha(2-\chi(\Sigma_1)) + \beta(|\mathcal{E}'|+1) + \sum_{j=1}^{k}(|E(E_j)|) + |E(C_i')| + |V(P)| - 1 \\
&- \gamma - 4 + \alpha(2-\chi(\Sigma_2)) + \beta(|\mathcal{E}''|+1) \\
&+ \sum_{j=1}^{l}|V(C_j)| + |E(C_i'')| + |V(P)| - 1 - \gamma - 4 \\
&= \alpha(2-\chi(\Sigma)) + \beta(|\mathcal{C}|+1) + \beta(|\mathcal{E}''|+1) \\
&+ \sum_{j=1}^{n}|V(C_j)| + 2(|V(P)|-1) - 2\gamma - 8 \\
&\leq \alpha(2-\chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{j=1}^{n}|V(C_j)| - \gamma - 4.
\end{aligned}
$$

Again, this holds because $\gamma \geq \beta + 6$.

These arguments prove that there exists no P between two vertices of some C_i such that $|E(P)| \leq 5$.

Suppose there is no safe tetrad. At this point in the proof, we know that every face is homeomorphic to a disk. As a result, the lemma follows from Proposition 4.3.10. So suppose for the rest of the proof, there is a safe tetrad. Let T be a safe tetrad with vertices labeled as in Figure 4.2, and let G' be obtained from $G \setminus \{v_1, v_2, v_3, v_4\}$ by identifying x and t into a vertex w. By Lemma 4.3.7 and the fact that G has no triseparating subgraph on at most 29 edges, we may assume that G' is loopless and that every cycle in G' is $\mathcal{C}$-admissible. As before, we will perform the operation used by Borodin et al [6] to remove the tetrad as described in the proof of Lemma 4.3.1. Let the resulting multigraph be called G'.

Now suppose that both vertices x and t from the tetrad reduction were elements of some C_i or that one of x and t was on C_i and the other was adjacent to a vertex of C_i. Then either there is a path $xv_1v_2v_3t$ of length four with both ends on C_i or a path of length five $xv_1v_2v_3tv$ between two vertices of C_i, where vertex v is without loss of generality a vertex of C_i adjacent to t. Both of these paths contradict earlier claims.

By Proposition 4.2.6, any precoloring of $\mathcal{C}$ that extends to G' also extends to G.

Now, let G'' be a minimal subgraph of G' such that any 3-coloring of $\mathcal{C}$ that extends to G'' also extends to G' and hence to G. By Lemma 4.2.2, it follows that $(G'', \mathcal{C}) \in \mathcal{S}_{10}(\Sigma)$. Then G'' is $\mathcal{C}$-critical. This is because G'' is a subgraph of G and because any 3-coloring of $\mathcal{C}$ that extends to G'' also extends to G' and hence to G. So if every 3-coloring of $\mathcal{C}$ extends in G'', then it would extend in G, a contradiction. Notice that $G'' \neq C_1 \cup \cdots \cup C_k$ because this graph is always 3-colorable.

Let f'' be a face of G'' and let J denote its boundary. Then there is a canonical subgraph $D_{f''}$ of G that corresponds to J, defined as follows. If $w \notin V(J)$, then $D_{f''} = J$. We may therefore assume that $w \in V(J)$. Let E_1 be the set of edges of G'' incident with w that correspond to edges of G incident with x, and let E_2 be defined similarly with x replaced by t. If one of E_1, E_2 is disjoint from $E(J)$, then again $D_{f''} := J$; otherwise $D_{f''} := J \cup P$, where P is the path $xv_1v_2v_3t$. Let $\Delta_{f''}$ be the face of $D_{f''}$ that uniquely corresponds to f''.

We can apply the induction hypothesis to each $D_{f''}$ individually to give a bound for $\sum_{f \in \mathcal{F}(G,\mathcal{C}), f \subset \Delta''} w(f)$. We can make an analogous estimate for each face in G''.

So for G'', $\sum_{f \in \mathcal{F}(G'',\mathcal{C})} w(f) \leq \alpha(2-\chi(\Sigma)) + \beta(|\mathcal{C}|) + \sum_{j=1}^{n} |V(C_j)| - \gamma - 4$ by the induction hypothesis.

We claim that for every $f'' \in \mathcal{F}(G'', \mathcal{C})$,

$$\sum_{f \in \mathcal{F}(G,\mathcal{C}), f \subseteq \Delta_{f''}} w(f) \leq w(f'').$$

By induction, we have

$$\sum_{f \in \mathcal{F}(G,\mathcal{C}), f \subseteq \Delta_{f''}} w^*(f) \leq w_1(\Delta_{f''}) - 4,$$

where w^* is defined as follows. In order to apply induction, we need to split vertices so that the boundary components will become cycles; in the process the quantity $w_2(f)$ may decrease. Thus, $w^*(f) = w_1^*(f) + w_2^*(f)$, where $w_1^*(f) = w_1(f)$ and $w_2^*(f)$ is interpreted after the split. We have

$$w_1(\Delta_{f''}) \leq \begin{cases} w_1(f''), & \text{if } P \text{ is not incident with } \Delta_{f''}; \\ w_1(f'') + 4, & \text{if } P \text{ is incident with } \Delta_{f''} \text{ on one side only;} \\ w_1(f'') + 8, & \text{if } P \text{ is not incident with } \Delta_{f''} \text{ on both sides.} \end{cases}$$

and

$$\sum_{f \in \mathcal{F}(G,\mathcal{C}), f \subseteq \Delta_{f''}} w_2(f) \leq w_2(f'') + \sum_{f \in \mathcal{F}(G,\mathcal{C}), f \subseteq \Delta_{f''}} w_2^*(f),$$

where the inequality is strict if P is incident with $\Delta_{f''}$ on both sides, because T was chosen to be a safe tetrad. Thus letting $\epsilon = 4$ if P is incident with $\Delta_{f''}$ on both sides and $\epsilon = 0$ otherwise, we have

$$\begin{aligned}
\sum_{f\in\mathcal{F}(G,\mathcal{C}),f\subseteq\Delta_{f''}} w(f) &= \sum_{f\in\mathcal{F}(G,\mathcal{C}),f\subseteq\Delta_{f''}} w_1(f) + 4\sum_{f\in\mathcal{F}(G,\mathcal{C}),f\subseteq\Delta_{f''}} w_2(f) \\
&\leq 4\sum_{f\in\mathcal{F}(G,\mathcal{C}),f\subseteq\Delta_{f''}} w_2(f) + 4\sum_{f\in\mathcal{F}(G,\mathcal{C}),f\subseteq\Delta_{f''}} w^*(f) - \epsilon \\
&\leq 4w_2(f'') + w_1(\Delta_{f''}) - 4 - \epsilon \\
&\leq w_1(f'') + 4w_2(\Delta_{f''}) = w(f''),
\end{aligned}$$

as claimed.

This gives

$$\begin{aligned}
\sum_{f\in\mathcal{F}(G,\mathcal{C})} w(f) &= \sum_{f''\in\mathcal{F}(G'',\mathcal{C})}\ \sum_{f\in\mathcal{F}(G,\mathcal{C}),f\subseteq\Delta_{f''}} w(f) \\
&\leq \sum_{f''\in\mathcal{F}(G'',\mathcal{C})} w(f'') \leq \alpha(2-\chi(\Sigma)) + \beta|\mathcal{C}| + \sum_{j=1}^{n}|V(C_j)| - \gamma.
\end{aligned}$$

□

We can now prove the main theorem of this chapter.

Theorem 4.3.12. *Let Σ be a surface and let G be a 4-critical graph with no cycles of length four through ten. Then $|V(G)| \leq 2442g(\Sigma) + 37$.*

Proof. The theorem follows from Lemma 4.3.11 as soon as the constants for α, β, γ are determined. From the arguments above, the following inequalities must be satisfied

- $\beta \geq 66$
- $2\gamma \geq 3\beta + 50$
- $\alpha \geq 10\beta + 6$

The solution to these inequalities that minimizes each of α, β, γ is $\alpha = 666, \beta = 66, \gamma = 124$. By Lemma 4.3.11 we now have a bound for

$$\sum_{f\in\mathcal{F}(G,\mathcal{C})} w(f).$$

However, this lemma only holds if there are at least two cycles in $\mathcal{C}$. Let G be a 4-critical graph with no cycles of length four through ten. So, we will add two disjoint triangles, call them T_1, T_2 to G and let $\mathcal{C} = \{T_1, T_2\}$. We can then bound $w(f)$.

Let $g(\Sigma)$ be the Euler genus of surface Σ. Lemma 4.3.11 then states that

$$\sum_{f \in \mathcal{F}(G,\mathcal{C})} w(f) \leq 666g(\Sigma) + 132 + 6 - 128 = 666g(\Sigma) + 10.$$

We need to convert this result into a bound on the number of vertices of G. Let e be an edge of G that is not part of $\mathcal{C}$. Let f_1, f_2 be the faces (which may be the same) that edge e bounds. Define the *edge weight* of G to be the quantity $w(f_1)/|f_1| + w(f_2)/|f_2|$. Assume that $w(f_i) = 0$ if f_i is a face of length three. The sum of the edge weights of G is equivalent to $\sum_{f \in \mathcal{F}(G,\mathcal{C})} w(f)$. Since each edge in G is not part of two 3-cycles, else we obtain a cycle of length four, we may assume that at least one f_i has length at least 11 if f_i is connected, in which case $w(f_i) \geq 2$. If f_i is disconnected, then f_i has length at least three, and also $w(f_i) \geq 10$. Further, for every edge that f_i is longer than these minimum lengths, $w(f_i)$ increases by one. So a lower bound for the edge weight of an edge in this graph is an edge where f_1 is a triangle and f_2 is a connected face of length 11. In this case, the edge weight is $2/11$. As a result, we may conclude that there are at most

$$A = (666g(\Sigma) + 10)\left(\frac{11}{2}\right) = 3663g(\Sigma) + 55$$

edges in G. Now, every vertex in G has degree at least three. So there are at most $B = \lfloor \frac{2A}{3} \rfloor$ vertices in G. In our case, $B = 2442g(\Sigma) + 37$. Notice that B is a function that is linear in the genus.

□

Remark 4.3.13. A natural question to ask is whether there exist non-3-colorable graphs with the specifications of Theorem 4.3.12. It is well known that there exist graphs of arbitrarily large chromatic number and girth, so such graphs do exist. In fact, using the *Hajos construction* (or Hajos sum), we can also generate a sequence of such 4-critical graphs, such that each graph in the sequence embeds on a surface of successively higher genus. Suppose there exists a graph G that is 4-critical and embeds on a surface of genus g. Then take another copy of G, call it H. Let ab be an edge of G and uv be an edge of H. Contract vertices a and u into a single vertex, delete edges ab and uv and add the edge bv. If G is 4-critical, then this new graph is 4-critical and it embeds on a surface of genus $2g$. We could repeat this procedure by adding another copy of G to this new graph to create a 4-critical graph that embeds on a surface of genus $3g$. This shows that there exists a sequence of 4-critical graphs whose number of vertices increases linearly with genus.

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