LIMIT THEOREMS FOR A ONE-DIMENSIONAL SYSTEM WITH RANDOM SWITCHINGS

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Tobias Hurth

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Approved by:

Professor Yuri Bakhtin,
Committee Chair
School of Mathematics
Georgia Institute of Technology

Professor Yuri Bakhtin, Advisor
School of Mathematics
Georgia Institute of Technology

Professor Yuri Bakhtin
School of Mathematics
Georgia Institute of Technology

Professor Leonid Bunimovich
School of Mathematics
Georgia Institute of Technology

Professor Vladimir Koltchinskii
School of Mathematics
Georgia Institute of Technology

Date Approved: 2 November 2010

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SUMMARY

In the present article, we will discuss the randomly driven motion of a walker within the interval $]0;1[$. We imagine that at a fixed starting time $t_0 \in \mathbb{R}$, our walker is located at the starting position $\xi \in]0;1[$. Moreover, the point 0 shall serve as an attractor at time t_0, meaning that the walker, enticed by an attracting force, starts moving towards this point. As the walker approaches 0, its speed shall decrease at a rate preventing it from ever reaching the attractor. One simple way of modeling such a rapidly decreasing velocity works as follows: We introduce a function d which delivers the Euclidean distance between the walker and the attractor as a function of time. For instance, in the context of our initial conditions, d_{t_0} is given by the starting position ξ. At future times $t > t_0$, we postulate that $d_t = d_{t_0} \cdot \exp(t_0 - t)$. The exponential term starring in this formula indicates the exponentially slow rate at which the distance to the attractor decreases. If we were contented with the mechanism established up to this point, our object of scrutiny would be a movement whose future evolvement is entirely determined by its starting conditions. However, we increase the degree of complexity by allowing for alternating attraction points. Both endpoints 0 and 1 of the open interval $]0;1[$ are declared potential attractors, with the attractor role relentlessly switching between them. So, while 0 was attractor at time t_0, the point 1 starts attracting the walker at some time $t_1 > t_0$, ending the attraction regime of 0. Yet, at time $t_2 > t_1$, 0 and 1 might reverse their roles once more, leaving 0 as the new attracting point until the next change occurs. With respect to the properties of our walk, it is crucial to define the law governing the switch of attractors over time, and we shall require these changes to take place randomly, with the times between two subsequent switches subject to an exponential distribution. The intensity of this

exponential distribution shall be a positive constant independent of time. After each change of attracting point, the distance function d is defined analogously to the initial case, where the initial time is replaced with the time of the previous switch.

Some of the postulates stated in this article can be relaxed without significantly changing the problem at hands. We could, for example, permit a starting point outside the interval, with the certainty that the walker will eventually enter $]0;1[$ and stay there for good. The exponential function could also yield to some function with similar decay features, but this would deprive us from exploiting the functional identity $\exp(s+t) = \exp(s) \cdot \exp(t)$. One should also note that, instead of considering a movement that started at a fixed point of time t_0 and is therefore lacking any prior history, we may assume that the movement has always existed and has neither beginning nor ending.

CHAPTER I

ONE FORCE-ONE SOLUTION PRINCIPLE

In the first two sections of this chapter, we prove a couple of easy auxiliary statements in the spirit of forward and pullback attraction results, examples of which can be found in [8]. The distance function we introduce right at the beginning can be interpreted as the distance between two walkers governed by the aforementioned mechanism at time t, provided that at time $s < t$, one of them was in ξ and the other in η.

1.1 Forward attraction

For $s < t \in \mathbb{R}$ and $\xi, \eta \in]0;1[$, we define

$$d_s(\xi,\eta)(t) := |\xi - \eta| \cdot \exp(s-t).$$

Theorem 1 *For each $t_0 \in \mathbb{R}, \xi, \eta \in]0;1[$ and for every $\epsilon > 0$, there exists a $T > t_0$ such that $d_{t_0}(\xi,\eta)(t) < \epsilon$ for $t \geq T$.*

Proof. The desired result follows immediately from the fact that

$$\lim_{t\to\infty} d_{t_0}(\xi,\eta)(t) = |\xi-\eta| \cdot \lim_{t\to\infty} \exp(t_0 - t) = 0.$$

■

1.2 Pullback attraction

Theorem 2 *For each $t_0 \in \mathbb{R}, \xi, \eta \in]0;1[$ and $\epsilon > 0$, we can find a $\tilde{t} < t_0$ such that $d_t(\xi,\eta)(t_0) < \epsilon$ whenever $t \leq \tilde{t}$.*

Proof. Since $d_t(\xi,\eta)(t_0) = |\xi-\eta|\cdot\exp(t-t_0)$, we have $\lim_{t\to-\infty} d_t(\xi,\eta)(t_0) = 0$. ■

In fact, we can prove an even stronger result to which we will recur in the sequel:

Theorem 3 *Given $t_0 \in \mathbb{R}$ and $\epsilon > 0$, there is a $\tilde{t} < t$ so that for any $\xi, \eta \in]0;1[$ and for any $t \le \tilde{t}$, we have $d_t(\xi,\eta)(t_0) < \epsilon$.*

Proof. This is obvious, since $d_t(\xi,\eta)(t_0) = |\xi-\eta|\cdot\exp(t-t_0)$ and $\lim_{t\to-\infty}|\xi-\eta|\cdot\exp(t-t_0) = 0$. ■

1.3 Main existence and uniqueness result

Let Σ denote the set of all real-valued two-sided sequences $(a_k)_{k\in\mathbb{Z}}$ that increase monotonely and satisfy $\lim_{k\to-\infty} a_k = -\infty$ and $\lim_{k\to\infty} a_k = \infty$. If a is such a sequence in Σ, we can assign it a history of switchings between 0 and 1 as follows:

At time $t \in \mathbb{R}$, let 0 be the attracting point if $t \in [a_k; a_{k+1}[$ for an even number k, and let 1 be the attracting point if $t \in [a_k; a_{k+1}[$ for an odd k.

Owing to the properties of a, this provides a well-defined switching environment. At present, we assume a to be a deterministic sequence, but in due course, we will conceive a as a stochastic process in discrete time and we will require that its increments $(a_{k+1} - a_k)$ be exponentially distributed with a uniform intensity parameter λ.

Let $a \in \Sigma$ determine a fixed switching environment. In order to formulate our existence and uniqueness result, we need to define a flow function F depending on this environment which acts on elements of $\mathbb{R}\times]0;1[\times\mathbb{R}$. The dependence on the environment will be suppressed in our notation, so we write $F(s,x;t)$ to denote that F is applied to a point (s,x) (where $s \in \mathbb{R}$ represents a point of time and $x \in]0;1[$ is commonly interpreted as the location of our path at time s) and to a future time

$t \geq s$. The flow function F shall give the place of our walker at time t, that is the mechanism of the walk we consider will be encoded in the definition of F.

For $s \leq t \in \mathbb{R}$ and for $\xi \in]0;1[$ we define $F(s,\xi;t)$ by the following scheme:

Let $k_s := \sup\{j \in \mathbb{Z} : \quad a_j \leq s\}$.

If $t \in [s; a_{k_s+1}[$, set

$$F(s,\xi;t) := \begin{cases} \xi \cdot \exp(s-t) & \text{if } k_s \text{ is even} \\ 1-(1-\xi)\cdot\exp(s-t) & \text{if } k_s \text{ is odd.} \end{cases}$$

Assume we have defined $F(s,\xi;t)$ for $t \in [s; a_{k_s+n}[$, where n is some positive integer. Now, we define $F(s,\xi;t)$ for $t \in [a_{k_s+n}; a_{k_s+n+1}[$ inductively by setting

$$F(s,\xi;t) := \begin{cases} F(s,\xi;a_{k_s+n})\cdot\exp(a_{k_s+n}-t) & \text{if } (k_s+n) \text{ is even} \\ 1-(1-F(s,\xi,a_{k_s+n}))\cdot\exp(a_{k_s+n}-t) & \text{if } (k_s+n) \text{ is odd.} \end{cases}$$

To complete the definition of F, we have to define $(F(s,\xi;a_{k_s+n}))_{n\geq 1}$ inductively. We set

$$F(s,\xi;a_{k_s+1}) := \begin{cases} \xi\cdot\exp(s-a_{k_s+1}) & \text{if } k_s \text{ is even} \\ 1-(1-\xi)\cdot\exp(s-a_{k_s+1}) & \text{if } k_s \text{ is odd} \end{cases}$$

and

$$F(s,\xi;a_{k_s+n+1}) := \begin{cases} F(s,\xi;a_{k_s+n})\cdot\exp(a_{k_s+n}-a_{k_s+n+1}) & \text{if } (k_s+n) \text{ is even} \\ 1-(1-F(s,\xi;a_{k_s+n}))\cdot\exp(a_{k_s+n}-a_{k_s+n+1}) & \text{if } (k_s+n) \text{ is odd.} \end{cases}$$

Why does this definition of F constitute a sensible description of the mechanism we study? First assume that $t \in [s; a_{k_s+1}[$. If k_s is even, 0 acts as an attracting point during the time interval $[a_{k_s}; a_{k_s+1}[$ which encompasses both s and t. Assume that the path is in some location $\chi \in]0;1[$ at time a_{k_s}. Then, by virtue of the underlying mechanism, it is in $\chi\cdot\exp(a_{k_s}-s)$ at time s. But when considering $F(s,\xi;t)$, we make the implicit assumption that ξ is the location of the path at time s, yielding

$$\xi = \chi\cdot\exp(a_{k_s}-s) \quad \Leftrightarrow \quad \chi = \xi\cdot\exp(s-a_{k_s}).$$

Hence, the process is in $\xi \cdot \exp(s - a_{k_s})$ at time s and in

$$\xi \cdot \exp(s - a_{k_s}) \cdot \exp(a_{k_s} - t) = \xi \cdot \exp(s - t) = F(s, \xi; t)$$

at time t. If k_s is odd, 1 is the attracting point and the process is located in $1 - (1 - \chi) \cdot \exp(a_{k_s} - s)$ at time s, provided that it was in χ at time a_{k_s}. Thus,

$$\xi = 1 - (1 - \chi) \cdot \exp(a_{k_s} - s) \quad \Leftrightarrow \quad \chi = 1 - (1 - \xi) \cdot \exp(s - a_{k_s}).$$

Then, at time t, the process has attained

$$1 - (1 - \chi) \cdot \exp(a_{k_s} - t) = 1 - (1 - \xi) \cdot \exp(s - t) = F(s, \xi; t).$$

A similar argument shows that the definition of $F(s, \xi; t)$ for $t \in [a_{k_s+n}; a_{k_s+n+1}[$ is also consistent with the heuristics of our motion.

Now, we possess the tools to state our main existence and uniqueness theorem.

Theorem 4 *One force - one solution principle*
Given a fixed switching environment E, there exists a unique path $X = (X_t)_{t \in \mathbb{R}}$ such that for every $s < t \in \mathbb{R}$, we have $X_t = F(s, X_s; t)$.

Another instance of the one force-one solution principle, related to a partial differential equation with random boundary conditions, is presented in [1].

Before we embark on the proof of this theorem, we mention two lemmas whose rather technical proofs are given in the appendix. The tediousness of these proofs stems from the necessity to treat many different cases. Despite this fact, we can easily see why they should be true by appealing to the underlying driving mechanism: Lemma 1 roughly states that if two walkers are at the same location at present, they will continue to stick together in the future. Lemma 2 is an immediate consequence of the heuristic definitions we provided for F and d.

Lemma 1 *If $r < s < t$ and $\xi \in]0; 1[$, we have*

$$F(r, \xi; t) = F(s, F(r, \xi; s); t).$$

Lemma 2 *For $s < t$ and $\xi, \eta \in]0;1[$, we have*

$$|F(s,\xi;t) - F(s,\eta;t)| = d_s(\xi;\eta)(t).$$

Having stated these lemmas, we can prove our main theorem.

Proof. The proof consists of three parts. First, we define a path X, then we show that X matches the description in the theorem and finally we establish uniqueness.

Let a be the sequence determining our switching environment, $\xi \in]0;1[$ and $t_0 \in \mathbb{R}$. Consider the sequence $(F(a_{-k}, \xi; t_0))_{k\geq 1}$. This is a Cauchy sequence:

Let $\epsilon > 0$. According to theorem 3, there is a $\tilde{t} < t_0$ so that $d_t(\xi,\eta)(t_0) < \epsilon$ whenever $t \leq \tilde{t}$ and for arbitrarily selected $\xi, \eta \in]0;1[$. Then, we can find a $K \in \mathbb{N}$ with $a_{-k} < \tilde{t}$ for all $k \geq K$. Due to lemmas 1 and 2, we have for $l > k \geq K$ that

$$\begin{aligned}
&|F(a_{-k},\xi;t_0) - F(a_{-l},\xi;t_0)| \\
=&|F(a_{-k},\xi;t_0) - F(a_{-k}, F(a_{-l},\xi;a_{-k});t_0)| \\
=&d_{a_{-k}}(\xi, F(a_{-l},\xi;a_{-k}))(t_0) < \epsilon
\end{aligned}$$

as $a_{-k} < \tilde{t}$.

This proves that $(F(a_{-k},\xi;t_0))_{k\geq 1}$ is indeed a Cauchy sequence of real numbers, hence convergent. Set X_{t_0} to be the limit of this sequence as k approaches infinity.

Now, we should verify that for $s < t \in \mathbb{R}$, we have $X_t = F(s, X_s; t)$. If we knew that F was continuous in its second argument, we would have

$$\begin{aligned}
F(s, X_s; t) &= F(s, \lim_{k\to\infty} F(a_{-k},\xi;s);t) \\
&= \lim_{k\to\infty} F(s, F(a_{-k},\xi;s);t) \\
&= \lim_{k\to\infty} F(a_{-k},\xi;t) = X_t.
\end{aligned}$$

Note that we evoked lemma 1 in deriving the penultimate equality. So, it remains to show that F is continuous in its second component. But this follows from lemma 2:

$$|F(s,\xi;t) - F(s,\eta;t)| = d_s(\xi,\eta)(t) = |\xi - \eta| \cdot \exp(s-t).$$

For the sake of establishing uniqueness of X, let $(Y_t)_{t\in\mathbb{R}}$ be a path different from X that also satisfies $Y_t = F(s, Y_s; t)$ for $s < t \in \mathbb{R}$. Without loss of generality, we may assume that $Y_t > X_t$ for some $t \in \mathbb{R}$, so that $\epsilon := Y_t - X_t > 0$. By theorem 3, we find a $\tilde{t} < t$ such that $d_s(\xi, \eta)(t) < \frac{\epsilon}{2}$ for every $\xi, \eta \in]0; 1[$ and $s \leq \tilde{t}$. So, in particular,

$$|X_t - Y_t| = |F(s, X_s; t) - F(s, Y_s; t)| = d_s(X_s, Y_s)(t) < \frac{\epsilon}{2},$$

which contradicts our assumption on X_t and Y_t.

This establishes the theorem in its entirety. ■

1.4 Forward and pullback attraction for the unique solution

For our uniquely determined path X satisfying $X_t = F(s, X_s; t)$, we can derive forward and pullback attraction results that resemble closely the ones given in the first section of our discussion. Throughout this paragraph, X shall always denote this unique solution.

1.4.1 Forward attraction

Theorem 5 *Let $t_0 \in \mathbb{R}, \xi \in]0; 1[$ and define*

$$Y_t := F(t_0, \xi; t) \quad \forall t \geq t_0.$$

Given $\epsilon > 0$, there is a $T > t_0$ such that

$$|X_t - Y_t| < \epsilon \quad \forall t \geq T.$$

Proof. For $t_0 \in \mathbb{R}, \xi, X_{t_0} \in]0; 1[$ and $\epsilon > 0$, theorem 1 guarantees existence of a time $T > t_0$ such that

$$d_{t_0}(\xi, X_{t_0})(t) < \epsilon \quad \forall t \geq T.$$

Further, lemma 2 implies

$$\epsilon > d_{t_0}(\xi, X_{t_0})(t) = |F(t_0, \xi; t) - F(t_0, X_{t_0}; t)| = |Y_t - X_t|.$$

■

1.4.2 Pullback attraction

Theorem 6 *If* $\xi \in]0;1[, \epsilon > 0$ *and* $t_0 \in \mathbb{R}$, *there is a* $\tilde{t} < t_0$ *such that for any* $t \leq \tilde{t}$, *we have* $|Y_{t_0} - X_{t_0}| < \epsilon$, *where* $Y_{t_0} := F(t, \xi; t_0)$.

Proof. For $t_0 \in \mathbb{R}$ and $\epsilon > 0$, there exists a $\tilde{t} < t_0$ such that for any $\eta \in]0;1[$ and for any $t \leq \tilde{t}$, we have

$$d_t(\xi, \eta)(t_0) < \epsilon,$$

as can be easily deduced from theorem 3. Then, if $t \leq \tilde{t}$, replace η with X_t to obtain

$$\epsilon > d_t(\xi, X_t)(t_0) = |F(t, \xi; t_0) - F(t, X_t; t_0)| = |Y_{t_0} - X_{t_0}|.$$

■

CHAPTER II

DERIVATION OF KOLMOGOROV FORWARD EQUATIONS

So far, we have been working with a deterministic flow function F that was essentially determined by a two-sided sequence $(a_k)_{k\in\mathbb{Z}}$. From now on, we will assume that our motion starts at time 0 in a randomly selected starting point $\xi \in]0;1[$ and is governed by a sequence of exponentially distributed switching times. The starting point is chosen at random in order to ensure that the random variable encoding the position of the walker at a certain time t has a density function. A convenient side-effect of this setting is that we may think of the random walk as an "‘eternal"’ motion without actual initiation. Under these circumstances, time $t = 0$ can be interpreted as the point of time at which we started our observation of the random walk.

On a probability space $(\Omega, \mathscr{F}, P)$, let $(T_k)_{k\geq 1}$ be a sequence of independent, exponentially distributed random variables with intensity $\lambda > 0$, and let ξ be a random variable, independent of $(T_k)_{k\geq 1}$, which maps to $(]0;1[;\mathscr{B}(]0;1[))$ and has a continuously differentiable probability density function p with continuous extension to $[0;1]$. Given an $\omega \in \Omega$, we set $a_j(\omega) := \sum_{k=1}^{j} T_k(\omega)$ for every $j \in \mathbb{N}$. With this setting, $(a_j(\omega))_{j\geq 1}$ is a monotone increasing sequence that diverges to $+\infty$ for P-almost every ω in Ω. We denote the subset of Ω on which $(a_j)_{j\geq 1}$ diverges by Ω_0. We may now define a stochastic process $(X_t)_{t\geq 0}$ in continuous time via

$$X_0 \equiv \xi$$

$$X_t \equiv F(0, \xi; t) \quad \text{for} \quad t > 0$$

where $F(0, \xi; t)$ depends on both the random starting point ξ and the sequence of random variables $(a_j)_{j\geq 1}$ and is therefore itself a random variable. In defining $X_t(\omega)$, we require that ω be contained in Ω_0, as otherwise, $F(0, \xi; t)$ might not be defined at ω. The process $(X_t)_{t\geq 0}$ gives rise to a random walk whose essence can be captured in the following description:

At time $t = 0$, the walker begins his motion in $\xi \in]0; 1[$ and is attracted by 0. As the walker approaches 0 linearly, his velocity decreases exponentially, preventing him from ever reaching the current attracting point. After a random, exponentially distributed, span of time, the attractor switches from 0 to 1, prompting the walker to change direction and to move towards 1 instead of 0. Due to the exponential decrease in velocity, the walker's quest to reach 1 is again doomed and he will invariably change course once the exponentially-λ distributed attracting time has elapsed. It is then reasonable to conjecture that the intensity λ of the switchings exerts a tremendous influence on our random walk. In the extremal cases of a very high and a very low intensity, we expect a localization of the walk in certain areas of $]0; 1[$. This question will receive rigorous treatment in section 7.

Apart from the resulting motion of the random walker, we also intend to keep track of the switching environment. To this end, we introduce another stochastic process $(A_t)_{t\geq 0}$ by

$$\begin{aligned} A_0 &\equiv 0 \\ A_t &\equiv \frac{1}{2} \cdot (1 - (-1)^{k_t}), \end{aligned}$$

where k_t is defined as the supremum over the set of positive integers j for which $a_j \leq t$. Once more, we emphasize that randomness comes into play because of the sequence of random variables $(a_j)_{j\geq 1}$. At any given time $t \geq 0$, the process A_t yields the current attracting point.

Our goal in this section is to derive a system of partial differential equations, the so-called Kolmogorov forward equations, that will eventually allow us to provide explicit formulas for the invariant densities of our system. Especially in mathematical physics, these equations are also referred to as Fokker-Planck equations. For an introduction to Kolmogorov's forward and backward equations, the reader may confer the 11th chapter of [7] on diffusions.

In a first step, we fix a time $t > 0$ at which we want to study our process. At time t, either 0 or 1 acts as attractor, and we introduce two probability measures $P_0(.) := P(.|A_t = 0)$ and $P_1(.) := P(.|A_t = 1)$ by conditioning P on these potential realizations of A_t. We claim that we can assign probability density functions $p_0(., t)$ and $p_1(., t)$ to the random variable X_t, such that

$$P_0(X_t \in B) = \int_B p_0(x, t)dx$$

and

$$P_1(X_t \in B) = \int_B p_1(x, t)dx$$

for any $B \in \mathscr{B}(]0; 1[)$. This result will be a consequence of the following lemma:

Lemma 3 *For $i \in \{0; 1\}$, the P_i-distribution of X_t is absolutely continuous with respect to the Lebesgue measure on* $]0; 1[$.

Proof. We prove the lemma for $i = 0$. Given a set $E \subseteq]0; 1[$ of Lebesgue measure zero, we have to show that $P_0(X_t \in E) = 0$. For any $n \in \mathbb{N}_0$, we set

$$C_n := \{\omega \in \Omega : \quad k_t(\omega) = n\}.$$

From its definition, we deduce that the stochastic process $(k_j)_{j \geq 0}$ is a Poisson process. Accordingly, the random variable k_t is Poisson-distributed with intensity parameter $\lambda \cdot t$. Thus, we have

$$P(C_n) = \exp(-\lambda \cdot t) \cdot \frac{(\lambda \cdot t)^n}{n!}$$

for any $n \in \mathbb{N}_0$. Since all of the events C_n have positive probability, we may compute the probabilities of certain events conditioned on C_n. And as the sets C_n are pairwise disjoint, countable additivity of P implies

$$\begin{aligned} P(X_t \in E) &= P(X_t \in E; \bigcup_{n=0}^{\infty} C_n) = \sum_{n=0}^{\infty} P(X_t \in E; C_n) \\ &= \sum_{n=0}^{\infty} P(X_t \in E | C_n) \cdot P(C_n). \end{aligned}$$

We show that $P(X_t \in E|C_n) = 0$ for any $n \in \mathbb{N}_0$, whence it follows that $P(X_t \in E) = 0$.

If we condition on the event C_n, we assume that there have been exactly n switches during the time interval $]0;t]$. The position of the walker at time t, denoted by X_t, then depends on the random starting point ξ, a random variable with values in $]0;1[$ and density function p, and on the exponentially distributed stopping times $T_1, ..., T_n$. Note also that the random variables $\xi, T_1, ..., T_n$ are independent. If we define a simplex Δ in $\mathbb{R}^n$ by

$$\Delta := \{(t_1, ..., t_n) \in \mathbb{R}^n : \quad t_1, ..., t_n \geq 0; \sum_{i=1}^{n} t_i \leq t\},$$

the random vector $(\xi, T_1, ..., T_n)$ maps from Ω to $]0;1[\times\Delta$ with $P(.|C_n)$-probability 1. We examine whether the distribution of $(\xi, T_1, ..., T_n)$ under the probability measure $P(.|C_n)$ has a probability density function. Let A and B be elements of the Borel-σ-algebras $\mathscr{B}(]0;1[)$ and $\mathscr{B}(\Delta)$, respectively. Then, the product set $I := A \times B$ is contained in $]0;1[\times\Delta$ and we have

$$\begin{aligned} P((\xi, T_1, ..., T_n) \in I | C_n) &= \frac{P(\xi \in A; (T_1, ..., T_n) \in B; C_n)}{P(C_n)} \\ &= \frac{P(\xi \in A; (T_1, ..., T_n) \in B; \sum_{i=1}^{n} T_i \leq t < \sum_{i=1}^{n+1} T_i)}{P(C_n)}. \end{aligned}$$

Since $B \subseteq \Delta$, the event $(T_1, ..., T_n) \in B$ automatically implies that $\sum_{i=1}^{n} T_i \leq t$. Hence, the numerator of the above ratio becomes

$$\begin{aligned} &P(\xi \in A; (T_1, ..., T_n) \in B; t < \sum_{i=1}^{n+1} T_i) \\ =&P(\xi \in A) \cdot P((T_1, ..., T_n) \in B; T_{n+1} > t - \sum_{i=1}^{n} T_i). \end{aligned}$$

As $T_1, ..., T_{n+1}$ are independent exponentially distributed random variables, the random vector $(T_1, ..., T_{n+1})$ has the joint density function

$$\psi(x_1, ..., x_{n+1}) = \lambda^{n+1} \cdot \exp(-\lambda \cdot \sum_{i=1}^{n+1} x_i) \quad \text{for } x_1, ..., x_{n+1} \geq 0.$$

Thus,

$$\begin{aligned}
&P((T_1, ..., T_n) \in B; T_{n+1} > t - \sum_{i=1}^{n} T_i) \\
=&P((T_1, ..., T_{n+1}) \in \{(x_1, ..., x_{n+1}) \in [0; \infty[^{n+1}: \quad (x_1, ..., x_n) \in B, x_{n+1} > t - \sum_{i=1}^{n} x_i\}) \\
=&\int_{[0;\infty[^{n+1}} \mathbb{1}_{\{(x_1,...,x_n) \in B, x_{n+1} > t - \sum_{i=1}^{n} x_i\}} \cdot \psi(x_1, ..., x_{n+1}) d(x_1, ..., x_{n+1}).
\end{aligned}$$

By Tonelli's theorem (cf. for instance [6] or [4]), this integral can be decomposed into

$$\int_0^\infty ... \int_0^\infty \lambda^n \cdot \int_0^\infty \mathbb{1}_{\{(x_1,...,x_n) \in B, x_{n+1} > t - \sum_{i=1}^{n} x_i\}} \cdot \lambda \cdot \exp(-\lambda \cdot \sum_{i=1}^{n+1} x_i) dx_{n+1} dx_n ... dx_1. \quad (1)$$

The substitution $x_{n+1} \to y := \sum_{i=1}^{n+1} x_i$ then yields

$$\begin{aligned}
&\int_0^\infty \mathbb{1}_{\{(x_1,...,x_n) \in B, x_{n+1} > t - \sum_{i=1}^{n} x_i\}} \cdot \lambda \cdot \exp(-\lambda \cdot \sum_{i=1}^{n+1} x_i) dx_{n+1} \\
=&\int_{-\sum_{i=1}^{n} x_i}^\infty \mathbb{1}_{\{(x_1,...,x_n) \in B, y > t\}} \cdot \lambda \cdot \exp(-\lambda \cdot y) dy \\
=&\mathbb{1}_{\{(x_1,...,x_n) \in B\}} \cdot \int_t^\infty \lambda \cdot \exp(-\lambda y) dy \\
=&\mathbb{1}_{\{(x_1,...,x_n) \in B\}} \cdot \exp(-\lambda t).
\end{aligned}$$

Accordingly, the multiple integral in (1) can be rewritten as

$$\begin{aligned}
&\exp(-\lambda t) \cdot \lambda^n \cdot \int_0^\infty ... \int_0^\infty \mathbb{1}_{\{(x_1,...,x_n) \in B\}} dx_n ... dx_1 \\
=&\exp(-\lambda t) \cdot \lambda^n \cdot \int_{[0;\infty[^n} \mathbb{1}_{\{(x_1,...,x_n) \in B\}} d(x_1, ..., x_n) \\
=&\exp(-\lambda t) \cdot \lambda^n \cdot |B|.
\end{aligned}$$

The number of switches in $[0; t]$ being Poisson-distributed with intensity parameter $\lambda \cdot t$, we also have

$$P(C_n) = \exp(-\lambda t) \cdot \frac{(\lambda t)^n}{n!},$$

implying that

$$\begin{aligned}
P((\xi, T_1, ..., T_n) \in I | C_n) &= P(\xi \in A) \cdot |B| \cdot \frac{n!}{t^n} \\
&= \int_A p(x) dx \cdot \int_B \frac{n!}{t^n} dy \\
&= \int_I p(x) \cdot \frac{n!}{t^n} d(x, y).
\end{aligned}$$

Note that $\frac{t^n}{n!}$ gives the volume of an n-dimensional simplex whose edges have length t, that is the volume of our set Δ. The mapping $x \mapsto \frac{n!}{t^n}$ is therefore the density function of the uniform distribution on Δ.

For arbitrary, not necessarily rectangular sets S in the product-σ-algebra $\mathscr{B}(]0;1[) \otimes \mathscr{B}(\Delta)$, we consider the family of sets

$$\mathscr{S} := \{S \in \mathscr{B}(]0;1[) \otimes \mathscr{B}(\Delta) : \quad P((\xi, T_1, ..., T_n) \in S | C_n) = \int_S p(x) \cdot \frac{n!}{t^n} d(x, y)\}$$

and show that it is a σ-algebra. Since $]0;1[\times \Delta$ is a rectangular set, it is contained in $\mathscr{S}$. And if $R, S \in \mathscr{S}$ with $R \subseteq S$, we have

$$\begin{aligned}
P((\xi, T_1, ..., T_n) \in S \setminus R | C_n) &= P(\{(\xi, T_1, ..., T_n) \in S\} \setminus \{(\xi, T_1, ..., T_n) \in R\} | C_n) \\
&= P((\xi, T_1, ..., T_n) \in S | C_n) - P((\xi, T_1, ..., T_n) \in R | C_n) \\
&= \int_S p(x) \cdot \frac{n!}{t^n} d(x, y) - \int_R p(x) \cdot \frac{n!}{t^n} d(x, y) \\
&= \int_{S \setminus R} p(x) \cdot \frac{n!}{t^n} d(x, y)
\end{aligned}$$

so that $S \setminus R \in \mathscr{S}$. Finally, if $(S_k)_{k \geq 1}$ is a countable family of disjoint sets from $\mathscr{S}$, we obtain

$$\begin{aligned}
P((\xi,T_1,...,T_n)\in\bigcup_{k=1}^{\infty}S_k|C_n) &= P(\bigcup_{k=1}^{\infty}\{(\xi,T_1,...,T_n)\in S_k\}|C_n)\\
&=\sum_{k=1}^{\infty}P((\xi,T_1,...,T_n)\in S_k|C_n)\\
&=\sum_{k=1}^{\infty}\int_{S_k}p(x)\cdot\frac{n!}{t^n}d(x,y)\\
&=\lim_{k\to\infty}\int_{]0;1[\times\Delta}\mathbb{1}_{\bigcup_{l=1}^{k}S_l}(x,y)\cdot p(x)\cdot\frac{n!}{t^n}d(x,y)\\
&=\int_{\bigcup_{k=1}^{\infty}S_k}p(x)\cdot\frac{n!}{t^n}d(x,y)
\end{aligned}$$

by monotone convergence. Hence, we have verified that $\mathscr{S}$ is a Dynkin system. As the family of rectangular sets $A\times B$, a π-system, is contained in $\mathscr{S}$, Dynkin's π-λ-theorem, to be found, among other sources, in [2], implies that the σ-algebra generated by the rectangular sets, namely $\mathscr{B}(]0;1[)\otimes\mathscr{B}(\Delta)$, is a subset of $\mathscr{S}$. The converse set inclusion being trivial, we see that

$$P((\xi,T_1,...,T_n)\in S|C_n)=\int_S p(x)\cdot\frac{n!}{t^n}d(x,y)\quad\forall S\in\mathscr{B}(]0;1[)\otimes\mathscr{B}(\Delta)$$

. This establishes that $p\cdot\frac{n!}{t^n}$ is the probability density function of the random vector $(\xi,T_1,...,T_n)$ with respect to the measure $P(.|C_n)$.

In a subsequent step, we define a function

$$f:\quad]0;1[\times\Delta\to]0;1[,\quad (x,t_1,...,t_n)\mapsto F(0,x;t)$$

where $a_j:=\sum_{k=1}^{j}t_k$ for $1\le j\le n$. Our objective is to prove that f is continuously differentiable in each of its $(n+1)$ components, for

$$X_t(\omega)=f(\xi(\omega),T_1(\omega),...,T_n(\omega)),$$

provided that $\omega\in C_n$. As far as the x-component is concerned, we evoke lemma (2) to get

$$
\begin{aligned}
&\frac{1}{h} \cdot (f(x+h, t_1, ..., t_n) - f(x, t_1, ..., t_n)) = \frac{1}{h} \cdot (F(0, x+h; t) - F(0, x; t)) \\
=&\frac{1}{h} \cdot d_0(x; x+h)(t) = \frac{1}{h} \cdot h \cdot e^{-t} = e^{-t}.
\end{aligned}
$$

Accordingly, f is differentiable in its first component with the constant derivative $x \mapsto e^{-t}$. In order to prove differentiability for the remaining components $t_1, ..., t_n$, we first assume that n is an even integer. With $|h|$ sufficiently small, we show inductively that for any even $k \in \{1, ..., n\}$, we have

$$
\begin{aligned}
&\frac{1}{h} \cdot (f(x, t_1, ..., t_l + h, ..., t_n) - f(x, t_1, ..., t_l, ..., t_n)) \\
=&\frac{e^h - 1}{h} \cdot (\sum_{j=0}^{n-k-1} (-1)^j \cdot \exp(a_{n-j} - t)) \\
&+ \frac{1}{h} \cdot (F(0, x; a_k + h) \cdot \exp(a_k + h - t) - F(0, x; a_k) \cdot \exp(a_k - t)),
\end{aligned}
$$

provided that $l \leq k$. If k is an odd integer in $\{1, ..., n\}$, we claim that

$$
\begin{aligned}
&\frac{1}{h} \cdot (f(x, t_1, ..., t_l + h, ..., t_n) - f(x, t_1, ..., t_l, ..., t_n)) \\
=&\frac{e^h - 1}{h} \cdot (\sum_{j=0}^{n-k} (-1)^j \cdot \exp(a_{n-j} - t)) \\
&+ \frac{1}{h} \cdot (F(0, x; a_k + h) \cdot \exp(a_k + h - t) - F(0, x; a_k) \cdot \exp(a_k - t))
\end{aligned}
$$

whenever $l \leq k$. First, let us consider the case $k = n$. For $l \leq n$, we have

$$
\begin{aligned}
&\frac{1}{h} \cdot (f(x, t_1, ..., t_l + h, ..., t_n) - f(x, t_1, ..., t_l, ..., t_n)) \\
=&\frac{1}{h} \cdot (F(0, x; a_n + h) \cdot \exp(a_n + h - t) - F(0, x; a_n) \cdot \exp(a_n - t)).
\end{aligned}
$$

Now, in the first induction step, let us assume that k is an even number in $\{1, ..., n\}$ for which the statement holds. Then, if $l \leq (k-1)$, the integer l is in particular less than k, and, according to our induction assumption, we have

$$
\begin{aligned}
&\frac{1}{h}\cdot(f(x,t_1,...,t_l+h,...,t_n)-f(x,t_1,...,t_l,...,t_n))\\
=&\frac{e^h-1}{h}\cdot(\sum_{j=0}^{n-k-1}(-1)^j\cdot\exp(a_{n-j}-t))\\
&+\frac{1}{h}\cdot(F(0,x;a_k+h)\cdot\exp(a_k+h-t)-F(0,x;a_k)\cdot\exp(a_k-t))\\
=&\frac{e^h-1}{h}\cdot(\sum_{j=0}^{n-k-1}(-1)^j\cdot\exp(a_{n-j}-t))\\
&+\frac{1}{h}\cdot((1-(1-F(0,x;a_{k-1}+h))\cdot\exp(a_{k-1}-a_k))\cdot\exp(a_k+h-t)\\
&-(1-(1-F(0,x;a_{k-1}))\cdot\exp(a_{k-1}-a_k))\cdot\exp(a_k-t))\\
=&\frac{e^h-1}{h}\cdot(\sum_{j=0}^{n-k-1}(-1)^j\cdot\exp(a_{n-j}-t))\\
&+\frac{1}{h}\cdot(\exp(a_k+h-t)-\exp(a_k-t)\\
&+(1-F(0,x;a_{k-1}))\cdot\exp(a_{k-1}-t)-(1-F(0,x;a_{k-1}+h))\cdot\exp(a_{k-1}+h-t))\\
=&\frac{e^h-1}{h}\cdot(\sum_{j=0}^{n-k}(-1)^j\cdot\exp(a_{n-j}-t))\\
&+\frac{1}{h}\cdot(\exp(a_{k-1}-t)-\exp(a_{k-1}+h-t)\\
&+F(0,x;a_{k-1}+h)\cdot\exp(a_{k-1}+h-t)-F(0,x;a_{k-1})\cdot\exp(a_{k-1}-t))\\
=&\frac{e^h-1}{h}\cdot(\sum_{j=0}^{n-k+1}(-1)^j\cdot\exp(a_{n-j}-t))\\
&+\frac{1}{h}\cdot(F(0,x;a_{k-1}+h)\cdot\exp(a_{k-1}+h-t)-F(0,x;a_{k-1})\cdot\exp(a_{k-1}-t)).
\end{aligned}
$$

And if the statement holds true for an odd number $k\in\{1,...,n\}$ and l is less than or equal to the even number $(k-1)$, we have

$$
\begin{aligned}
&\frac{1}{h}\cdot(f(x,t_1,...,t_l+h,...,t_n)-f(x,t_1,...,t_l,...,t_n))\\
=&\frac{e^h-1}{h}\cdot(\sum_{j=0}^{n-k}(-1)^j\cdot\exp(a_{n-j}-t))\\
&+\frac{1}{h}\cdot(F(0,x;a_k+h)\cdot\exp(a_k+h-t)-F(0,x;a_k)\cdot\exp(a_k-t))\\
=&\frac{e^h-1}{h}\cdot(\sum_{j=0}^{n-k}(-1)^j\cdot\exp(a_{n-j}-t))\\
&+\frac{1}{h}\cdot(F(0,x;a_{k-1}+h)\cdot\exp(a_{k-1}+h-t)-F(0,x;a_{k-1})\cdot\exp(a_{k-1}-t)).
\end{aligned}
$$

For an even integer k, the previous result implies the following identity:

$$
\begin{aligned}
&\frac{1}{h}\cdot(f(x,t_1,...,t_k+h,...,t_n)-f(x,t_1,...,t_k,...,t_n))\\
=&\frac{e^h-1}{h}\cdot(\sum_{j=0}^{n-k-1}(-1)^j\cdot\exp(a_{n-j}-t))\\
&+\frac{1}{h}\cdot(F(0,x;a_k+h)\cdot\exp(a_k+h-t)-F(0,x;a_k)\cdot\exp(a_k-t))\\
=&\frac{e^h-1}{h}\cdot(\sum_{j=0}^{n-k-1}(-1)^j\cdot\exp(a_{n-j}-t))\\
&+\frac{1}{h}\cdot((1-(1-F(0,x;a_{k-1}))\cdot\exp(a_{k-1}-a_k-h))\cdot\exp(a_k+h-t)\\
&-(1-(1-F(0,x;a_{k-1}))\cdot\exp(a_{k-1}-a_k))\cdot\exp(a_k-t))\\
=&\frac{e^h-1}{h}\cdot(\sum_{j=0}^{n-k-1}(-1)^j\cdot\exp(a_{n-j}-t))\\
&+\frac{1}{h}\cdot(\exp(a_k+h-t)-\exp(a_k-t))\\
=&\frac{e^h-1}{h}\cdot(\sum_{j=0}^{n-k}(-1)^j\cdot\exp(a_{n-j}-t)),
\end{aligned}
$$

so that

$$\partial_{t_k}f(x,t_1,...,t_k,...,t_n)=\sum_{j=0}^{n-k}(-1)^j\cdot\exp(a_{n-j}-t).$$

A calculation in the spirit of the preceeding ones shows that this equality is also valid for odd integers. This establishes f as a continuously differentiable scalar field.

Next, we define a mapping

$$\tilde{f}: \quad]0;1[\times\Delta \to \tilde{f}(]0;1[\times\Delta), \quad (x,t_1,...,t_n) \mapsto (f(x,t_1,...,t_n),t_1,...,t_n).$$

This mapping is bijective, for if

$$\tilde{f}(x,t_1,...,t_n) = \tilde{f}(y,s_1,...,s_n)$$

for some $x,y \in]0;1[$ and $(t_1,...,t_n),(s_1,...,s_n) \in \Delta$, we have

$$(f(x,t_1,...,t_n),t_1,...,t_n) = (f(y,s_1,...,s_n),s_1,...,s_n)$$

and equality holds componentwise. Thus, $t_k = s_k$ for $1 \le k \le n$ and $f(x,t_1,...,t_n) = f(y,t_1,...,t_n)$. By lemma (2),

$$0 = |f(x,t_1,...,t_n) - f(y,t_1,...,t_n)| = |F(0,x;t) - F(0,y;t)| = d_0(x;y)(t) = |x-y| \cdot e^{-t},$$

so that $x = y$. Its Jacobi matrix reads

$$J_{\tilde{f}}(x,t_1,...,t_n) = \begin{pmatrix} e^{-t} & \sum_{j=0}^{n-1}(-1)^j \cdot \exp(\sum_{l=1}^{n-j} t_l - t) & \dots & \dots & \exp(\sum_{l=1}^{n} t_l - t) \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

and its lower left triangle consists entirely of zeros. Therefore, the Jacobi determinant equals

$$\det J_{\tilde{f}}(x,t_1,...,t_n) = e^{-t} \neq 0$$

for any point $(x,t_1,...,t_n)$ in $]0;1[\times\Delta$.

According to the transformation formula for densities, the distribution of the random vector $\tilde{f}(\xi,T_1,...,T_n)$, taken with respect to $P(.|C_n)$, then has a probability density function φ. If E is now the arbitrary set of Lebesgue measure 0 introduced at the very start of this proof, we obtain

$$
\begin{aligned}
P(X_t \in E|C_n) &= P(f(\xi, T_1, ..., T_n) \in E|C_n) \\
&= P((f(\xi, T_1, ..., T_n), T_1, ..., T_n) \in E \times [0; \infty[^n |C_n) \\
&= P(\tilde{f}(\xi, T_1, ..., T_n) \in E \times [0; \infty[^n |C_n) \\
&= \int_E \int_0^\infty ... \int_0^\infty \varphi(x, t_1, ..., t_n) dt_n ... dt_1 dx \\
&= 0.
\end{aligned}
$$

This completes our proof. ■

Given a Borel-set $B \subseteq]0; 1[$, we then have

$$
P(X_t \in B; A_t = i) = P_i(X_t \in B) \cdot P(A_t = i) = \int_B p_i(x, t) \cdot P(A_t = i) dx
$$

for $i = 0, 1$. When we set $\rho_i(x, t) := p_i(x, t) \cdot P(A_t = i)$, the previous equality becomes

$$
P(X_t \in B; A_t = i) = \int_B \rho_i(x, t) dx.
$$

The functions ρ_i are density functions which do not integrate to 1, due to the factor $P(A_t = i)$. However, once we set $\rho(x, t) := \rho_0(x, t) + \rho_1(x, t)$, we obtain the probability density function $\rho(., t)$ of the random variable X_t, for

$$
\begin{aligned}
P(X_t \in B) &= P(X_t \in B; A_t = 0) + P(X_t \in B; A_t = 1) \\
&= \int_B (\rho_0(x, t) + \rho_1(x, t)) dx = \int_B \rho(x, t) dx.
\end{aligned}
$$

Let $N_{r,s} := k_s - k_r$ denote the number of switches within the time-interval $]r; s]$. With this convention, we have

$$
\begin{aligned}
P(X_t \in B; A_t = i) =& P(X_t \in B; A_t = i; N_{t,t+h} = 0) + P(X_t \in B; A_t = i; N_{t,t+h} = 1) \\
&+ P(X_t \in B; A_t = i; N_{t,t+h} > 1)
\end{aligned}
$$

for a small $h > 0$. The stochastic process $(k_s)_{s\geq 0}$ being a Poisson process, the increment $N_{t,t+h} = k_{t+h} - k_t$ is independent of the random variables X_t and A_t as these depend on ξ and the history of the Poisson process $(k_s)_{s\geq 0}$ up to time t. Therefore, we can represent the right-hand side of the prior equation as

$$\begin{aligned}&P(X_t \in B; A_t = i) \cdot P(N_{t,t+h} = 0) + P(X_t \in B; A_t = i) \cdot P(N_{t,t+h} = 1)\\ &+ P(X_t \in B; A_t = i) \cdot P(N_{t,t+h} > 1),\end{aligned}$$

and we now have a closer look at the distribution of $N_{t,t+h}$. The random variable $N_{t,t+h} = k_{t+h} - k_t$ is Poisson-distributed with intensity parameter $\lambda \cdot h$. This observation yields the formulas

$$\begin{aligned}P(N_{t,t+h} = 0) &= \exp(-\lambda \cdot h)\\ P(N_{t,t+h} = 1) &= \exp(-\lambda \cdot h) \cdot \lambda h\\ P(N_{t,t+h} > 1) &= 1 - \exp(-\lambda \cdot h) \cdot (1 + \lambda h).\end{aligned}$$

As

$$\frac{1}{h} \cdot (\exp(-\lambda \cdot h) + \lambda h - 1) = \lambda - \lambda \cdot \frac{\exp(-\lambda \cdot h) - 1}{-\lambda \cdot h}$$

and

$$\lim_{h \searrow 0} \frac{\exp(-\lambda \cdot h) - 1}{-\lambda \cdot h} = 1,$$

we gather that

$$P(N_{t,t+h} = 0) = 1 - \lambda h + o(h)$$

as h decreases to 0. In addition,

$$\lim_{h \searrow 0} \frac{1}{h} \cdot (\exp(-\lambda h) \cdot \lambda h - \lambda h) = \lambda \cdot \lim_{h \searrow 0} (\exp(-\lambda h) - 1) = 0,$$

whence we infer that

$$P(N_{t,t+h} = 1) = \lambda h + o(h).$$

Accordingly,

$$P(N_{t,t+h} > 1) = o(h)$$

as h tends to 0.

Now, we fix two numbers $a < b$ in the interval $]0;1[$ and consider the density function ρ_0. The total ρ_0-mass contained in $]a;b[$ at time t is then given by $\int_a^b \rho_0(x,t)dx$, and by calculating its time-derivative $\partial_t \int_a^b \rho_0(x,t)dx$, we get the change of total ρ_0-mass at time t. Clearly,

$$\partial_t \int_a^b \rho_0(x,t)dx = \lim_{h\searrow 0} \frac{1}{h} \cdot (\int_a^b \rho_0(x,t+h)dx - \int_a^b \rho_0(x,t)dx)$$

and we call $(\int_a^b \rho_0(x,t+h)dx - \int_a^b \rho_0(x,t)dx)$ the net density flux during $]t;t+h[$ taken with respect to $]a;b[$.

This net density flux equals

$$\begin{aligned}
&P(X_{t+h} \in]a;b[; A_{t+h} = 0) - P(X_t \in]a;b[; A_t = 0) \\
=&(P(X_{t+h} \in]a;b[; A_{t+h} = 0|N_{t,t+h} = 0) - P(X_t \in]a;b[; A_t = 0)) \cdot P(N_{t,t+h} = 0) \\
&+ (P(X_{t+h} \in]a;b[; A_{t+h} = 0|N_{t,t+h} = 1) - P(X_t \in]a;b[; A_t = 0)) \cdot P(N_{t,t+h} = 1) \\
&+ (P(X_{t+h} \in]a;b[; A_{t+h} = 0|N_{t,t+h} > 1) - P(X_t \in]a;b[; A_t = 0)) \cdot P(N_{t,t+h} > 1).
\end{aligned}$$

Let us first consider the case $N_{t,t+h} = 0$, that is we do not witness a switch during the time interval $]t;t+h]$. Under this assumption, the statements $A_t = 0$ and $A_{t+h} = 0$ are equivalent, yielding

$$P(X_{t+h} \in]a;b[; A_{t+h} = 0|N_{t,t+h} = 0) = P(X_{t+h} \in]a;b[; A_t = 0|N_{t,t+h} = 0).$$

If we also condition on the event $A_t = 0$, we get

$$\begin{aligned}
&P(X_{t+h} \in]a;b[; A_t = 0|N_{t,t+h} = 0) \\
=&P(X_{t+h} \in]a;b[|A_t = 0; N_{t,t+h} = 0) \cdot P(A_t = 0|N_{t,t+h} = 0).
\end{aligned}$$

And

$$X_{t+h}(\omega) = X_t(\omega) \cdot \exp(-h),$$

provided that $A_t(\omega) = 0$ and $N_{t,t+h}(\omega) = 0$. Hence,

$$P(X_{t+h} \in]a;b[|A_t = 0; N_{t,t+h} = 0) = P(X_t \in]e^h a; e^h b[|A_t = 0; N_{t,t+h} = 0)$$

As a result, we have

$$
\begin{aligned}
&P(X_{t+h} \in]a;b[; A_t = 0 | N_{t,t+h} = 0) \\
=&P(X_t \in]e^h \cdot a; e^h \cdot b[| A_t = 0; N_{t,t+h} = 0) \cdot P(A_t = 0 | N_{t,t+h} = 0) \\
=&P(X_t \in]e^h \cdot a; e^h \cdot b[; A_t = 0 | N_{t,t+h} = 0).
\end{aligned}
$$

As before, the event $N_{t,t+h} = 0$ does not depend on the situation at time t. Consequently,

$$
P(X_t \in]e^h \cdot a; e^h \cdot b[; A_t = 0 | N_{t,t+h} = 0) = P(X_t \in]e^h \cdot a; e^h \cdot b[; A_t = 0),
$$

and we eventually obtain

$$
\begin{aligned}
&P(X_{t+h} \in]a;b[; A_{t+h} = 0 | N_{t,t+h} = 0) - P(X_t \in]a;b[; A_t = 0) \\
=&P(X_t \in]e^h \cdot a; e^h \cdot b[; A_t = 0) - P(X_t \in]a;b[; A_t = 0) \\
=&\int_{e^h \cdot a}^{e^h \cdot b} \rho_0(x,t)dx - \int_a^b \rho_0(x,t)dx \\
=&\int_b^{e^h \cdot b} \rho_0(x,t)dx - \int_a^{e^h \cdot a} \rho_0(x,t)dx.
\end{aligned}
$$

In the previous line, the term $\int_b^{e^h \cdot b} \rho_0(x,t)dx$ can be interpreted as the total mass influx into the interval $]a;b[$ that occurred between times t and $(t+h)$, assuming that no switch took place in $]t;t+h[$. By subtracting the total mass outflux $\int_a^{e^h \cdot a} \rho_0(x,t)dx$, we obtain the net flux, always bearing in mind that we have conditioned on the event $N_{t,t+h} = 0$. We assume that $\rho_0(.,t)$ is sufficiently regular, more precisely that it is continuously differentiable in $[a; e^{h_0} \cdot b]$ for a sufficiently small $h_0 > 0$. The derivative $\partial_x \rho_0$ is then a continuous function in $[a; e^{h_0} \cdot b]$ and assumes its maximum $\mu := \max_{\eta \in [a; e^{h_0} \cdot b]} \partial_x \rho_0(\eta, t)$. For $h \leq h_0$, Taylor's theorem implies the existence of $\eta_h \in]b; e^h \cdot b[$, such that

$$
\begin{aligned}
&\frac{1}{h} \cdot \int_b^{e^h \cdot b} (\rho_0(x,t) - \rho_0(b,t))dx = \frac{1}{h} \cdot \int_b^{e^h \cdot b} \partial_x \rho_0(\eta_h, t) \cdot (x-b)dx \\
=&\frac{1}{h} \cdot \partial_x \rho_0(\eta_h, t) \cdot [\frac{1}{2}x^2 - bx]_{x=b}^{x=e^h \cdot b} \leq \frac{1}{h} \cdot \mu \cdot \frac{b^2}{2} \cdot (e^h - 1)^2.
\end{aligned}
$$

Since $\lim_{h\searrow 0}\frac{e^h-1}{h}=1$ and $\lim_{h\searrow 0}(e^h-1)=0$, the term

$$\frac{1}{h}\cdot\int_b^{e^h\cdot b}(\rho_0(x,t)-\rho_0(b,t))dx$$

converges to 0 as h approaches 0. This gives us

$$\int_b^{e^h\cdot b}\rho_0(x,t)dx=\int_b^{e^h\cdot b}\rho_0(b,t)dx+o(h)=b\cdot(e^h-1)\cdot\rho_0(b,t)+o(h).$$

As (e^h-1) and h are asymptotically equivalent as $h\searrow 0$, our final estimate on the influx term reads

$$\int_b^{e^h\cdot b}\rho_0(x,t)dx=b\cdot h\cdot(1+o(1))\cdot\rho_0(b,t)+o(h). \tag{2}$$

For the integral describing the mass outflux, the (justified) replacement of b by a reveals that

$$\int_a^{e^h\cdot a}\rho_0(x,t)dx=a\cdot h\cdot(1+o(1))\cdot\rho_0(a,t)+o(h). \tag{3}$$

Before we calculate the net flux and rescale it appropriately, we cast the above identities into new forms involving a drift function v_0. The purpose of this aside is to pave the way for generalizations of the current problem. These will arguably bring about more complicated terms, but might still be accessible with the aid of the presently developed tools.

If ω satisfies the condition $N_{t,t+h}(\omega)=0$, we obtain

$$X_{t+h}(\omega)=X_t(\omega)\cdot e^{-h}=:\Phi_0(X_t(\omega),h).$$

Taking the partial derivative of Φ_0 with respect to time then yields the formula

$$\partial_h\Phi_0(X_t(\omega),h)=-X_t(\omega)\cdot e^{-h}=-\Phi_0(X_t(\omega),h),$$

which can be rewritten as

$$\partial_h\Phi_0(X_t(\omega),h)=v_0(\Phi_0(X_t(\omega),h))$$

by introducing the drift function $v_0(y):=-y$.

And with this drift function, equations (2) and (3) become

$$\int_b^{e^h \cdot b} \rho_0(x,t)dx = |v_0(b)| \cdot h \cdot (1+o(1)) \cdot \rho_0(b,t) + o(h),$$
$$\int_a^{e^h \cdot a} \rho_0(x,t)dx = |v_0(a)| \cdot h \cdot (1+o(1)) \cdot \rho_0(a,t) + o(h).$$

Thus, the net flux in case of no switch is accounted for by the term

$$\begin{aligned}
&(P(X_{t+h} \in]a;b[; A_{t+h} = 0 | N_{t,t+h} = 0) - P(X_t \in]a;b[; A_t = 0)) \cdot P(N_{t,t+h} = 0) \\
=&(|v_0(b)| \cdot \rho_0(b,t) - |v_0(a)| \cdot \rho_0(a,t) + o(h)) \cdot h \cdot (1+o(1)) \cdot (1 - \lambda h + o(h)) \\
=&(|v_0(b)| \cdot \rho_0(b,t) - |v_0(a)| \cdot \rho_0(a,t)) \cdot h + o(h).
\end{aligned}$$

For the net flux in case of multiple switches, we immediately obtain

$$\begin{aligned}
&|(P(X_{t+h} \in]a;b[; A_{t+h} = 0 | N_{t,t+h} > 1) - P(X_t \in]a;b[; A_t = 0)) \cdot P(N_{t,t+h} > 1)| \\
\leq &2 \cdot o(h) = o(h).
\end{aligned}$$

It remains to discuss the case of exactly one switch in $]t; t+h]$.

If there has been exactly one switch of attractor within $]t; t+h]$, 0 is the attracting point at time $(t+h)$ if and only if 1 acts as an attractor at time t. For that reason,

$$\begin{aligned}
&P(X_{t+h} \in]a;b[; A_{t+h} = 0 | N_{t,t+h} = 1) = P(X_{t+h} \in]a;b[; A_t = 1 | N_{t,t+h} = 1) \\
=&P(X_{t+h} \in]a;b[| A_t = 1; N_{t,t+h} = 1) \cdot P(A_t = 1 | N_{t,t+h} = 1).
\end{aligned}$$

If $\omega \in \Omega$ is chosen in such a way that $A_t(\omega) = 1$ and $N_{t,t+h}(\omega) = 1$, there exists a uniquely determined time $s(\omega) \in]t; t+h]$ at which the point of attraction switches from 1 to 0. For such an ω, we have

$$\begin{aligned}
X_{t+h}(\omega) &= X_{s(\omega)}(\omega) \cdot \exp(s(\omega) - (t+h)) \\
&= (1 - (1 - X_t(\omega)) \cdot \exp(t - s(\omega))) \cdot \exp(s(\omega) - (t+h)) \\
&= \exp(s(\omega) - (t+h)) - (1 - X_t(\omega)) \cdot \exp(-h),
\end{aligned}$$

whence it follows that for those ω satisfying $A_t(\omega) = 1$ and $N_{t,t+h}(\omega) = 1$, the inequality chain $a < X_{t+h}(\omega) < b$ is equivalent to

$$e^h \cdot a + 1 - \exp(s(\omega) - t) < X_t(\omega) < e^h \cdot b + 1 - \exp(s(\omega) - t).$$

Accordingly,

$$\begin{aligned}
&P(X_{t+h} \in]a; b[; A_{t+h} = 0 | N_{t,t+h} = 1) \\
=&P(X_t \in]e^h a + 1 - \exp(s - t); e^h b + 1 - \exp(s - t)[| A_t = 1; N_{t,t+h} = 1) \cdot P(A_t = 1 | N_{t,t+h} = 1) \\
=&P(X_t \in]e^h \cdot a + 1 - \exp(s - t); e^h \cdot b + 1 - \exp(s - t)[; A_t = 1 | N_{t,t+h} = 1).
\end{aligned}$$

Then, as $0 < s(\omega) - t \leq h$, we get the inequalities

$$e^h \cdot (a - 1) + 1 \leq e^h \cdot a + 1 - \exp(s(\omega) - t) < e^h \cdot a$$

as well as

$$e^h \cdot (b - 1) + 1 \leq e^h \cdot b + 1 - \exp(s(\omega) - t) < e^h \cdot b.$$

From these inequalities, we deduce

$$\begin{aligned}
&P(X_t \in]e^h \cdot (a - 1) + 1; e^h \cdot b[; A_t = 1 | N_{t,t+h} = 1) &(4)\\
\geq &P(X_t \in]e^h a + 1 - \exp(s - t); e^h b + 1 - \exp(s - t)[; A_t = 1 | N_{t,t+h} = 1) &(5)\\
\geq &P(X_t \in]e^h \cdot a; e^h \cdot (b - 1) + 1[; A_t = 1 | N_{t,t+h} = 1). &(6)
\end{aligned}$$

The conditional probabilities (4) and (6) can be restated as

$$P(X_t \in]e^h \cdot (a - 1) + 1; e^h \cdot b[; A_t = 1)$$

and

$$P(X_t \in]e^h \cdot a; e^h \cdot (b - 1) + 1[; A_t = 1),$$

as both events are independent of the events on which we condition. Therefore, probability (4) equals

$$\int_{e^h \cdot (a-1)+1}^{e^h \cdot b} \rho_1(x, t) dx,$$

while expression (6) has the integral representation

$$\int_{e^h\cdot a}^{e^h\cdot(b-1)+1} \rho_1(x,t)dx.$$

Combining these results, we obtain

$$\int_{e^h\cdot a}^{e^h\cdot(b-1)+1} \rho_1(x,t)dx - \int_a^b \rho_0(x,t)dx \tag{7}$$

$$\leq P(X_{t+h} \in]a;b[; A_{t+h}=0|N_{t,t+h}=1) - P(X_t \in]a;b[; A_t=0) \tag{8}$$

$$\leq \int_{e^h\cdot(a-1)+1}^{e^h\cdot b} \rho_1(x,t)dx - \int_a^b \rho_0(x,t)dx. \tag{9}$$

Moreover, we have

$$\begin{aligned}
&\int_{e^h\cdot a}^{e^h\cdot(b-1)+1} \rho_1(x,t)dx - \int_a^b \rho_0(x,t)dx \\
=&\int_{e^h\cdot a}^{e^h\cdot(b-1)+1} \rho_1(x,t)dx - \int_a^b \rho_1(x,t)dx + \int_a^b \rho_1(x,t)dx - \int_a^b \rho_0(x,t)dx \\
=&\int_b^{e^h\cdot(b-1)+1} \rho_1(x,t)dx - \int_a^{e^h\cdot a} \rho_1(x,t)dx + \int_a^b (\rho_1(x,t)-\rho_0(x,t))dx
\end{aligned}$$

and, analogously,

$$\begin{aligned}
&\int_{e^h\cdot(a-1)+1}^{e^h\cdot b} \rho_1(x,t)dx - \int_a^b \rho_0(x,t)dx \\
=&\int_b^{e^h\cdot b} \rho_1(x,t)dx - \int_a^{e^h\cdot(a-1)+1} \rho_1(x,t)dx + \int_a^b (\rho_1(x,t)-\rho_0(x,t))dx,
\end{aligned}$$

so that (7) and (9) turn into

$$(\int_b^{e^h\cdot(b-1)+1} \rho_1(x,t)dx - \int_a^{e^h\cdot a} \rho_1(x,t)dx) + \int_a^b (\rho_1(x,t)-\rho_0(x,t))dx$$

and

$$(\int_b^{e^h\cdot b} \rho_1(x,t)dx - \int_a^{e^h\cdot(a-1)+1} \rho_1(x,t)dx) + \int_a^b (\rho_1(x,t)-\rho_0(x,t))dx.$$

Next, we exploit the fact that the density function $\rho_1(.,t)$ is non-negative in order to glean the estimates

$$\int_b^{e^h\cdot(b-1)+1} \rho_1(x,t)dx - \int_a^{e^h\cdot a} \rho_1(x,t)dx \geq -\int_a^{e^h\cdot a} \rho_1(x,t)dx$$

and

$$\int_b^{e^h \cdot b} \rho_1(x,t)dx - \int_a^{e^h\cdot(a-1)+1} \rho_1(x,t)dx \leq \int_b^{e^h\cdot b} \rho_1(x,t)dx.$$

Assuming that ρ_1 has the same regularity properties as ρ_0, we may extend identities (2) and (3) to ρ_1, yielding

$$\begin{aligned} \int_b^{e^h\cdot b} \rho_1(x,t)dx &= b\cdot\rho_1(b,t)\cdot h\cdot(1+o(1))+o(h), \\ -\int_a^{e^h\cdot a} \rho_1(x,t)dx &= -a\cdot\rho_1(a,t)\cdot h\cdot(1+o(1))+o(h). \end{aligned}$$

Finally, we have

$$\begin{aligned} &(-a\cdot\rho_1(a,t)\cdot h\cdot(1+o(1))+o(h)+\int_a^b(\rho_1(x,t)-\rho_0(x,t))dx)\cdot(\lambda h+o(h)) \\ \leq&(P(X_{t+h}\in]a;b[;A_{t+h}=0|N_{t,t+h}=1)-P(X_t\in]a;b[;A_t=0))\cdot P(N_{t,t+h}=1) \\ \leq&(b\cdot\rho_1(b,t)\cdot h\cdot(1+o(1))+o(h)+\int_a^b(\rho_1(x,t)-\rho_0(x,t))dx)\cdot(\lambda h+o(h)). \end{aligned}$$

This chain of inequalities eventually establishes that

$$\begin{aligned} &(P(X_{t+h}\in]a;b[;A_{t+h}=0|N_{t,t+h}=1)-P(X_t\in]a;b[;A_t=0))\cdot P(N_{t,t+h}=1) \\ =&\lambda h\cdot\int_a^b(\rho_1(x,t)-\rho_0(x,t))dx+o(h). \end{aligned}$$

With these results, the net density flux during $]t;t+h[$ becomes

$$\begin{aligned} &\int_a^b \rho_0(x,t+h)dx-\int_a^b\rho_0(x,t)dx \\ =&(|v_0(b)|\cdot\rho_0(b,t)-|v_0(a)|\cdot\rho_0(a,t))\cdot h+\lambda h\cdot\int_a^b(\rho_1(x,t)-\rho_0(x,t))dx+o(h), \end{aligned}$$

and

$$\begin{aligned} &\frac{d}{dt}\int_a^b\rho_0(x,t)dx \\ =&\lim_{h\searrow 0}\frac{1}{h}\cdot(o(h)+\lambda h\cdot\int_a^b(\rho_1(x,t)-\rho_0(x,t))dx-(\rho_0(b,t)\cdot v_0(b)-\rho_0(a,t)\cdot v_0(a))\cdot h) \\ =&\lambda\cdot\int_a^b(\rho_1(x,t)-\rho_0(x,t))dx-(\rho_0(b,t)\cdot v_0(b)-\rho_0(a,t)\cdot v_0(a)). \end{aligned}$$

Assuming that we may interchange the differentiation operator and the integral, this yields the identity

$$\int_a^b \partial_t \rho_0(x,t)dx = -\int_a^b \frac{d}{dx}(\rho_0(x,t) \cdot v_0(x))dx + \lambda \cdot \int_a^b (\rho_1(x,t) - \rho_0(x,t))dx.$$

As this holds true regardless of the $a < b \in]0;1[$, the integrands on both sides must coincide in almost every point. Applying the product rule for differentiation, we get

$$\partial_t \rho_0(x,t) = -(v_0'(x) \cdot \rho_0(x,t) + \partial_x \rho_0(x,t) \cdot v_0(x)) + \lambda \cdot (\rho_1(x,t) - \rho_0(x,t)).$$

If we insert $v_0(x) = -x$ and $v_0'(x) = -1$, this turns into

$$\partial_t \rho_0(x,t) = -(1+\lambda) \cdot \rho_0(x,t) + \lambda \cdot \rho_1(x,t) - x \cdot \partial_x \rho_0(x,t),$$

which is the Kolmogorov forward equation for the family of density functions $(\rho_0(.,t))_{t \geq 0}$.

Due to the symmetry inherent in the subject, the Kolmogorov forward equation for $(\rho_1(.,t))_{t \geq 0}$ reads

$$\partial_t \rho_1(x,t) = -(v_1'(x) \cdot \rho_1(x,t) + \partial_x \rho_1(x,t) \cdot v_1(x)) + \lambda \cdot \rho_0(x,t) - \lambda \cdot \rho_1(x,t),$$

where v_0 has been replaced with v_1, the drift function corresponding to the attracting point 1. If 1 has been the attractor throughout the entire interval $[0;t]$ and the path of our process X assigned to some $\omega \in \Omega$ started at $\xi(\omega)$, we have

$$X_t(\omega) = 1 - (1 - \xi(\omega)) \cdot \exp(-t) =: \Phi_1(\xi(\omega);t)$$

and Φ_1 satisfies the differential equation

$$\partial_t \Phi_1(\xi(\omega);t) = 1 - \Phi_1(\xi(\omega);t) =: v_1(\Phi_1(\xi(\omega);t)).$$

Hence, we set $v_1(y) := 1 - y$ to derive

$$\partial_t \rho_1(x,t) = (1-\lambda) \cdot \rho_1(x,t) + (x-1) \cdot \partial_x \rho_1(x,t) + \lambda \cdot \rho_0(x,t)$$

as our second Kolmogorov equation.

CHAPTER III

DERIVATION OF INVARIANT DENSITIES

In the previous section, we have argued that for any time $t > 0$, probabilities $(P(X_t \in B; A_t = i))_{i=0,1}$ can be expressed via density functions $\rho_i(.,t)$ according to

$$P(X_t \in B; A_t = i) = \int_B \rho_i(x,t)dx,$$

and we have derived a system of partial differential equations describing in how far the density families $(\rho_0(.,t))_{t\geq 0}$ and $(\rho_1(.,t))_{t\geq 0}$ relate to each other. In general, these densities are apparently time-dependent objects and evolve in line with the process X. In this chapter, we venture forward to identify densities that are to a certain degree stable in time, that is we require that the time derivatives of our families vanish at these densities. This naturally imposes the following agenda:

First, we set the left-hand sides of our Kolmogorov forward equations to be zero. This will leave us with a system of linear ordinary differential equations that permits an explicit solution. We will determine two particularly relevant solutions, subject to appropriate side-conditions. Depending on the parameter λ, these solutions are so-called invariant densities of our problem.

Setting $\partial_t \rho_0(x,t) = \partial_t \rho_1(x,t) = 0$, we transform the Kolmogorov forward equations into

$$\begin{aligned} 0 &= (1-\lambda)\cdot\rho_0(x,t) + x\cdot\partial_x\rho_0(x,t) + \lambda\rho_1(x,t), \\ 0 &= (1-\lambda)\cdot\rho_1(x,t) + (x-1)\cdot\partial_x\rho_1(x,t) + \lambda\rho_0(x,t). \end{aligned}$$

If $\rho_0(x,t)$ and $\rho_1(x,t)$ do not depend on time in the first place, their time derivatives certainly vanish. Under this assumption, we would have to solve the following system of linear ordinary differential equations:

$$0 = (1-\lambda)\cdot\rho_0(x) + x\cdot\rho_0'(x) + \lambda\cdot\rho_1(x),$$
$$0 = (1-\lambda)\cdot\rho_1(x) + (x-1)\cdot\rho_1'(x) + \lambda\cdot\rho_0(x).$$

This system is equivalent to

$$\begin{aligned}\rho_0'(x) &= \frac{1}{x}\cdot(\lambda-1)\cdot\rho_0(x) - \frac{1}{x}\cdot\lambda\cdot\rho_1(x),\\ \rho_1'(x) &= \frac{1}{x-1}\cdot(\lambda-1)\cdot\rho_1(x) - \frac{1}{x-1}\cdot\lambda\cdot\rho_0(x).\end{aligned}$$

Now, write $\rho_0(x) = \alpha_0(x)\cdot\beta_0(x)$ and $\rho_1(x) = \alpha_1(x)\cdot\beta_1(x)$ with functions $\alpha_0, \alpha_1, \beta_0, \beta_1$ yet to be determined. The product rule then provides the equations

$$\alpha_0'(x)\cdot\beta_0(x) + \alpha_0(x)\cdot\beta_0'(x) = \frac{\lambda-1}{x}\cdot\alpha_0(x)\beta_0(x) - \frac{\lambda}{x}\cdot\alpha_1(x)\beta_1(x) \tag{10}$$

and

$$\alpha_1'(x)\cdot\beta_1(x) + \alpha_1(x)\cdot\beta_1'(x) = -\frac{\lambda}{x-1}\cdot\alpha_0(x)\beta_0(x) + \frac{\lambda-1}{x-1}\cdot\alpha_1(x)\beta_1(x). \tag{11}$$

Next, let us suppose that α_0 and β_1 solve the ordinary differential equations

$$\begin{aligned}\alpha_0'(x) &= \frac{\lambda-1}{x}\cdot\alpha_0(x),\\ \beta_1'(x) &= \frac{\lambda-1}{x-1}\cdot\beta_1(x),\end{aligned}$$

respectively. For instance, we may pick $\alpha_0(x) := x^{\lambda-1}$ and $\beta_1(x) := (1-x)^{\lambda-1}$. Then, equations (10) and (11) become

$$\begin{aligned}x^{\lambda-1}\cdot\beta_0'(x) &= -\frac{\lambda}{x}\cdot(1-x)^{\lambda-1}\cdot\alpha_1(x),\\ (1-x)^{\lambda-1}\cdot\alpha_1'(x) &= -\frac{\lambda}{x-1}\cdot x^{\lambda-1}\cdot\beta_0(x),\end{aligned}$$

which is equivalent to

$$\begin{aligned}x^{\lambda}\cdot\beta_0'(x) &= -\lambda\cdot(1-x)^{\lambda-1}\cdot\alpha_1(x),\\ (1-x)^{\lambda}\cdot\alpha_1'(x) &= \lambda\cdot x^{\lambda-1}\cdot\beta_0(x).\end{aligned}$$

This system is obviously solved by $\alpha_1(x) := x^\lambda$ and $\beta_0(x) := (1-x)^\lambda$. Combining these results, we see that a solution of our original system of ordinary differential equations is given by

$$\rho_0(x) = C \cdot x^{\lambda-1} \cdot (1-x)^\lambda,$$
$$\rho_1(x) = C \cdot (1-x)^{\lambda-1} \cdot x^\lambda.$$

The constant factor C is to be chosen in accord with the condition

$$1 = \int_0^1 (\rho_0(x) + \rho_1(x))dx$$

that stems from the fact that $\rho := \rho_0 + \rho_1$ is a probability density function. Hence, C should satisfy

$$1 = C \cdot \int_0^1 x^{\lambda-1} \cdot (1-x)^\lambda dx + C \cdot \int_0^1 (1-x)^{\lambda-1} \cdot x^\lambda dx = C \cdot (\beta(\lambda;\lambda+1) + \beta(\lambda+1;\lambda)).$$

Here, β denotes Euler's beta-function. As the beta-function is symmetric, C equals $\frac{1}{2 \cdot \beta(\lambda;\lambda+1)}$, implying

$$\rho_0(x) = \frac{1}{2 \cdot \beta(\lambda;\lambda+1)} \cdot x^{\lambda-1} \cdot (1-x)^\lambda,$$
$$\rho_1(x) = \frac{1}{2 \cdot \beta(\lambda;\lambda+1)} \cdot (1-x)^{\lambda-1} \cdot x^\lambda.$$

These are invariant densities of our process.

Clearly, both ρ_0 and ρ_1 depend on the intensity $\lambda > 0$ of the switchings. The discussion in appendix C summarizes features of the density function ρ_0 for different λ and highlights the sometimes fundamental shifts at certain threshold values. An immediate conclusion from these definitions is that

$$\rho_1(1-x) = \rho_0(x),$$

that is ρ_1 can be obtained from ρ_0 by reflecting its graph with respect to the line running parallel to the y-axis and intersecting the x-axis at $\frac{1}{2}$. The substitution $x \mapsto 1-x$ then also yields

$$\int_0^1 \rho_0(x)dx = \int_0^1 \rho_1(x)dx = \frac{1}{2}.$$

CHAPTER IV

WEAK CONVERGENCE RESULTS FOR ρ_0

We recall from the previous section that the invariant density ρ_0 is defined as

$$\rho_0(x) := \frac{1}{2 \cdot \beta(\lambda; \lambda+1)} \cdot x^{\lambda-1} \cdot (1-x)^{\lambda}.$$

To simplify notation, we substitute $\zeta(\lambda)$ for the normalizing constant $\frac{1}{2 \cdot \beta(\lambda;\lambda+1)}$ and set $g^{(\lambda)}(x) := x^{\lambda-1} \cdot (1-x)^{\lambda}$. Up to this point, it has been tacitly understood that ρ_0 is not only a function of the variable x, but also depends on how we choose the intensity $\lambda > 0$. Since the present and the subsequent sections will be devoted to gaining convergence statements as λ approaches some limit, it seems advisable to break with this convention and write $\rho_0^{(\lambda)}$ in lieu of ρ_0, emphasizing the explicit λ on which ρ_0 depends.

Since $\rho_0^{(\lambda)} \geq 0$ and $\int_0^1 \rho_0^{(\lambda)}(x)dx = \frac{1}{2}$ for any $\lambda > 0$, the density $2 \cdot \rho_0^{(\lambda)}$ constitutes a probability density function. This legitimizes the following theorem:

Theorem 7 *Define probability measures $\mu^{(\lambda)}$ by*

$$\mu^{(\lambda)}(A) := \int_A 2 \cdot \rho_0^{(\lambda)}(x)dx \quad \textit{for any} \quad A \in \mathscr{B}([0;1]).$$

Then, as λ decreases to 0, $\mu^{(\lambda)}$ converges weakly to the Dirac-delta measure at 0.

This statement receives a graphic motivation from the study of the plots of $\rho_0^{(\lambda)}$ for λ smaller than 1.

Proof. Let $F \in \mathscr{B}([0;1])$ be a closed set, and $(\lambda_n)_{n\geq 1}$ a sequence which decreases to zero. For any positive integer n, we have

$$\mu^{(\lambda_n)}([0;1]) = \int_0^1 2 \cdot \rho_0^{(\lambda_n)}(x)dx = 1$$

and

$$\delta_0([0;1]) = 1,$$

as 0 is contained in $[0;1]$. Therefore,

$$\lim_{n\to\infty} \mu^{(\lambda_n)}([0;1]) = \delta_0([0;1]).$$

Being a subset of $[0;1]$, F is bounded and has an infimum ι. And since F is closed, $\iota \in F$. If $\iota = 0$, 0 is an element of F and $\delta_0(F) = 1$. The sequence $(\mu^{(\lambda_n)}(F))_{n\geq 1}$ is bounded because $\mu^{(\lambda_n)}$ is a probability measure for any $n \in \mathbb{N}$. Thus, it has a greatest limit point $\limsup_{n\to\infty} \mu^{(\lambda_n)}(F)$. But since $\mu^{(\lambda_n)}(F) \leq 1$ for any $n \in \mathbb{N}$, the following inequality holds:

$$\limsup_{n\to\infty} \mu^{(\lambda_n)}(F) \leq \delta_0(F).$$

If $\iota > 0$, set $\epsilon := \frac{\iota}{2}$. Then 0 is not contained in F and $\delta_0(F) = 0$. Additionally, F is a subset of $]\frac{\iota}{2};1]$, from which we infer that

$$\mu^{(\lambda_n)}(F) \leq \mu^{(\lambda_n)}(]\frac{\iota}{2};1]) = \int_{\frac{\iota}{2}}^{1} 2\cdot\rho_0^{(\lambda_n)}(x)dx \leq \frac{\int_{\frac{\iota}{2}}^{1} 2\cdot\rho_0^{(\lambda_n)}(x)dx}{\int_0^{\frac{\iota}{2}} 2\cdot\rho_0^{(\lambda_n)}(x)dx},$$

owing to monotonicity of $\mu^{(\lambda_n)}$ and the fact that $\int_0^{\frac{\iota}{2}} 2\cdot\rho_0^{(\lambda_n)}(x)dx \leq 1$. If we can show that

$$\lim_{n\to\infty} \frac{\int_{\frac{\iota}{2}}^{1} 2\cdot\rho_0^{(\lambda_n)}(x)dx}{\int_0^{\frac{\iota}{2}} 2\cdot\rho_0^{(\lambda_n)}(x)dx} = 0,$$

it follows that

$$\lim_{n\to\infty} \mu^{(\lambda_n)}(F) = 0.$$

The Portemanteau theorem for weak convergence, as stated in [5], then yields the theorem.

In order to prove the remaining part, define

$$A_\epsilon(\lambda) := \int_0^{\epsilon} 2\cdot\rho_0^{(\lambda)}(x)dx,$$
$$B_\epsilon(\lambda) := \int_\epsilon^{1} 2\cdot\rho_0^{(\lambda)}(x)dx$$

for sufficiently small $\epsilon > 0$ and for $\lambda > 0$. We conjecture that $\lim_{\lambda \searrow 0} \frac{B_\epsilon(\lambda)}{A_\epsilon(\lambda)} = 0$ and consider the ratio $\frac{B_\epsilon(\lambda)}{A_\epsilon(\lambda)}$. For any positive integer n, we have

$$\frac{B_\epsilon(\lambda_n)}{A_\epsilon(\lambda_n)} = \frac{\int_\epsilon^1 2 \cdot \rho_0^{(\lambda_n)}(x)dx}{\int_0^\epsilon 2 \cdot \rho_0^{(\lambda_n)}(x)dx} = \frac{\int_\epsilon^1 x^{\lambda_n - 1} \cdot (1-x)^{\lambda_n} dx}{\int_0^\epsilon x^{\lambda_n - 1} \cdot (1-x)^{\lambda_n} dx} = \frac{\int_\epsilon^1 g^{(\lambda_n)}(x)dx}{\int_0^\epsilon g^{(\lambda_n)}(x)dx}.$$

We refer the reader to the brief discussion of ρ_0 in appendix C, where it is stated that for small λ, the function $\rho_0^{(\lambda)}$, and therefore also $g^{(\lambda)}$, is strictly monotone decreasing in $]0;1[$. In particular,

$$g^{(\lambda)}(x) \leq g^{(\lambda)}(\epsilon) \quad \forall x \in [\epsilon; 1[$$

and

$$\limsup_{n\to\infty} \int_\epsilon^1 g^{(\lambda_n)}(x)dx \leq \lim_{n\to\infty} \{(1-\epsilon)^{\lambda_n + 1} \cdot \epsilon^{\lambda_n - 1}\} = (1-\epsilon) \cdot \frac{1}{\epsilon}.$$

For $n \in \mathbb{N}$, we have

$$\begin{aligned} g^{(\lambda_{n+1})}(x) &= x^{\lambda_{n+1}-1} \cdot (1-x)^{\lambda_{n+1}} = \frac{1}{x} \cdot (x \cdot (1-x))^{\lambda_{n+1}} \\ &= \frac{1}{x} \cdot \exp(\lambda_{n+1} \cdot \ln(x \cdot (1-x))). \end{aligned}$$

As $x \cdot (1-x) < 1$, the term $\lambda_{n+1} \cdot \ln(x \cdot (1-x))$ is negative, implying

$$\lambda_{n+1} \cdot \ln(x \cdot (1-x)) \geq \lambda_n \cdot \ln(x \cdot (1-x)).$$

And since the exponential function is monotone increasing, we deduce that

$$g^{(\lambda_{n+1})}(x) \geq \frac{1}{x} \cdot \exp(\lambda_n \cdot \ln(x \cdot (1-x))) = \frac{1}{x} \cdot (x \cdot (1-x))^{\lambda_n} = g^{(\lambda_n)}(x).$$

As a result, the sequence $(g^{(\lambda_n)})_{n \geq 1}$ is monotone increasing. For any $x \in]0;1[$, we have

$$\lim_{n\to\infty} g^{(\lambda_n)}(x) = \lim_{n\to\infty} \frac{1}{x} \cdot \exp(\lambda_n \cdot \ln(x \cdot (1-x))) = \frac{1}{x} \cdot e^0 = \frac{1}{x}$$

by continuity of the exponential function. Since the sequence of non-negative functions $(g^{(\lambda_n)})_{n \geq 1}$ converges monotonically and at every point in $]0;1[$ to the function $x \mapsto \frac{1}{x}$, Levi's monotone convergence theorem applies:

$$\lim_{n\to\infty} \int_0^\epsilon g^{(\lambda_n)}(x)dx = \int_0^\epsilon \frac{dx}{x} = +\infty.$$

Hence,

$$\lim_{\lambda \searrow 0} \int_0^{\epsilon} g^{(\lambda)}(x)dx = \infty$$

and

$$\lim_{\lambda \searrow 0} \frac{\int_{\epsilon}^{1} g^{(\lambda)}(x)dx}{\int_0^{\epsilon} g^{(\lambda)}(x)dx} = 0$$

for any $\epsilon > 0$. This proves the outstanding claim. ■

On account of the symmetric relation between ρ_0 and ρ_1, we can glean an analogous statement for ρ_1 from what we have just proved: If we set $\nu^{(\lambda)}(B) := \int_B 2 \cdot \rho_1^{(\lambda)}(x)dx$, then the sequence $(\nu^{(\lambda_n)})_{n \geq 1}$ converges weakly to the delta measure at 1.

The next theorem is the complementary one to theorem (7), with the sequence $(\lambda_n)_{n \geq 1}$ diverging to $+\infty$. Its result is in the spirit of a law of large numbers for our random dynamicle system.

Theorem 8 *Let $(\lambda_n)_{n \geq 1}$ be a sequence of positive numbers that increases to $+\infty$, and define $\mu^{(\lambda)}$ for $\lambda > 0$ as above. Then, the sequence of measures $(\mu^{(\lambda_n)})_{n \geq 1}$ converges weakly to the Dirac-delta measure at the point $\frac{1}{2}$.*

Proof. Let $\epsilon > 0$ be a sufficiently small, fixed number. We purport that

$$\lim_{n \to \infty} \frac{\int_0^{\frac{1}{2}-\epsilon} 2 \cdot \rho_0^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^{1} 2 \cdot \rho_0^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} 2 \cdot \rho_0^{(\lambda_n)}(x)dx} = 0.$$

Given a positive integer n, we have

$$\frac{\int_0^{\frac{1}{2}-\epsilon} 2 \cdot \rho_0^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^{1} 2 \cdot \rho_0^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} 2 \cdot \rho_0^{(\lambda_n)}(x)dx} = \frac{\int_0^{\frac{1}{2}-\epsilon} g^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^{1} g^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} g^{(\lambda_n)}(x)dx}.$$

As $\lambda_n \to \infty$, we may assume without loss of generality that $\lambda_n > 2$. Under this assumption, $g^{(\lambda_n)}$ is strictly monotone increasing in $]0; \frac{\lambda_n - 1}{2\lambda_n - 1}[$ and strictly monotone decreasing in $]\frac{\lambda_n - 1}{2\lambda_n - 1}; 1[$ (confer also appendix C). Clearly,

$$\lim_{n \to \infty} \frac{\lambda_n - 1}{2\lambda_n - 1} = \frac{1}{2},$$

so we can choose n so large that $\frac{\lambda_n - 1}{2\lambda_n - 1} \in]\frac{1}{2} - \epsilon; \frac{1}{2} + \epsilon[$. Then,

$$\int_0^{\frac{1}{2}-\epsilon} g^{(\lambda_n)}(x)dx \leq (\tfrac{1}{2} - \epsilon) \cdot g^{(\lambda_n)}(\tfrac{1}{2} - \epsilon) = (\tfrac{1}{2} - \epsilon)^{\lambda_n} \cdot (\tfrac{1}{2} + \epsilon)^{\lambda_n}$$

and

$$\int_{\frac{1}{2}+\epsilon}^{1} g^{(\lambda_n)}(x)dx \leq (\tfrac{1}{2} - \epsilon)^{\lambda_n + 1} \cdot (\tfrac{1}{2} - \epsilon)^{\lambda_n - 1}.$$

Combining these two estimates, we obtain

$$\begin{aligned}\int_0^{\frac{1}{2}-\epsilon} g^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^{1} g^{(\lambda_n)}(x)dx \leq & (\tfrac{1}{2} - \epsilon)^{\lambda_n} \cdot (\tfrac{1}{2} + \epsilon)^{\lambda_n - 1} \cdot (\tfrac{1}{2} + \epsilon + \tfrac{1}{2} - \epsilon) \\ = & (\tfrac{1}{2} - \epsilon)^{\lambda_n} \cdot (\tfrac{1}{2} + \epsilon)^{\lambda_n - 1}.\end{aligned}$$

If n is chosen so large that $\frac{\lambda_n - 1}{2\lambda_n - 1} \in]\frac{1}{2} - \frac{\epsilon}{2}; \frac{1}{2} + \frac{\epsilon}{2}[$, we also have

$$\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} g^{(\lambda_n)}(x)dx \geq \int_{\frac{1}{2}-\frac{\epsilon}{2}}^{\frac{1}{2}+\frac{\epsilon}{2}} g^{(\lambda_n)}(x)dx \geq \epsilon \cdot \min\{g^{(\lambda_n)}(\tfrac{1}{2} - \tfrac{\epsilon}{2}); g^{(\lambda_n)}(\tfrac{1}{2} + \tfrac{\epsilon}{2})\}. \qquad (12)$$

Now, since

$$g^{(\lambda_n)}(\tfrac{1}{2} + \tfrac{\epsilon}{2}) = (\tfrac{1}{2} + \tfrac{\epsilon}{2})^{\lambda_n - 1} \cdot (\tfrac{1}{2} - \tfrac{\epsilon}{2})^{\lambda_n} = g^{(\lambda_n)}(\tfrac{1}{2} - \tfrac{\epsilon}{2}) \cdot \frac{\frac{1}{2} - \frac{\epsilon}{2}}{\frac{1}{2} + \frac{\epsilon}{2}} < g^{(\lambda_n)}(\tfrac{1}{2} - \tfrac{\epsilon}{2}),$$

the minimum in (12) equals $g^{(\lambda_n)}(\frac{1}{2} + \frac{\epsilon}{2})$. Hence,

$$\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} g^{(\lambda_n)}(x)dx \geq \epsilon \cdot g^{(\lambda_n)}(\tfrac{1}{2} + \tfrac{\epsilon}{2}),$$

providing the estimate

$$\begin{aligned}&\frac{\int_0^{\frac{1}{2}-\epsilon} g^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^{1} g^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} g^{(\lambda_n)}(x)dx} \\ \leq & (\tfrac{1}{2} - \epsilon)^{\lambda_n} \cdot (\tfrac{1}{2} + \epsilon)^{\lambda_n - 1} \cdot \frac{1}{\epsilon} \cdot \frac{1}{(\frac{1}{2} + \frac{\epsilon}{2})^{\lambda_n - 1} \cdot (\frac{1}{2} - \frac{\epsilon}{2})^{\lambda_n}} \\ = & \frac{1}{\epsilon} \cdot \Big(\frac{\frac{1}{2} - \epsilon}{\frac{1}{2} - \frac{\epsilon}{2}}\Big)^{\lambda_n} \cdot \Big(\frac{\frac{1}{2} + \epsilon}{\frac{1}{2} + \frac{\epsilon}{2}}\Big)^{\lambda_n - 1} \\ = & \frac{1}{\epsilon} \cdot \frac{\frac{1}{2} - \epsilon}{\frac{1}{2} - \frac{\epsilon}{2}} \cdot \Big(\frac{\frac{1}{4} - \epsilon^2}{\frac{1}{4} - \frac{\epsilon^2}{4}}\Big)^{\lambda_n - 1}.\end{aligned}$$

The term $\frac{1}{\epsilon} \cdot \frac{\frac{1}{2}-\epsilon}{\frac{1}{2}-\frac{\epsilon}{2}}$ is merely a constant factor. As $(\frac{1}{4}-\epsilon)$ is strictly less than $(\frac{1}{4}-\frac{\epsilon^2}{4})$, the entire expression converges to 0 as n goes to infinity. This establishes our initially stated assertion. The remaining arguments resemble closely to the ones invoked in the proof of theorem (7).

Let $F \in \mathscr{B}([0;1])$ be a closed set. As before, we have $\delta_{\frac{1}{2}}([0;1]) = 1$ and $\mu^{(\lambda_n)}([0;1]) = 1$ for any $n \in \mathbb{N}$, which implies

$$\lim_{n\to\infty} \mu^{(\lambda_n)}([0;1]) = \delta_{\frac{1}{2}}([0;1]).$$

If $\frac{1}{2} \in F$, the $\delta_{\frac{1}{2}}$-measure of F equals 1 and is therefore greater than or equal to the limes superior of $(\mu^{(\lambda_n)}(F))_{n\geq 1}$. Otherwise, there exists an $\epsilon > 0$ such that $F \cap]\frac{1}{2}-\epsilon; \frac{1}{2}+\epsilon[= \emptyset$; for if not, we could approximate $\frac{1}{2}$ by a sequence in F, and since F is closed, $\frac{1}{2}$ would necessarily be contained in F. Clearly, $\delta_{\frac{1}{2}}(F) = 0$. Now, we have

$$\mu^{(\lambda_n)}(F) \leq \mu^{(\lambda_n)}([0;\tfrac{1}{2}-\epsilon] \cup [\tfrac{1}{2}+\epsilon;1]) \tag{13}$$

$$= \int_0^{\frac{1}{2}-\epsilon} 2 \cdot \rho_0^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^1 2 \cdot \rho_0^{(\lambda_n)}(x)dx \tag{14}$$

$$= \frac{\int_0^{\frac{1}{2}-\epsilon} 2 \cdot \rho_0^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^1 2 \cdot \rho_0^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} 2 \cdot \rho_0^{(\lambda_n)}(x)dx} \cdot \int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} 2 \cdot \rho_0^{(\lambda_n)}(x)dx \tag{15}$$

$$\leq \frac{\int_0^{\frac{1}{2}-\epsilon} 2 \cdot \rho_0^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^1 2 \cdot \rho_0^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} 2 \cdot \rho_0^{(\lambda_n)}(x)dx}, \tag{16}$$

which tends to 0 as λ_n goes to infinity. We conclude that

$$\lim_{n\to\infty} \mu^{(\lambda_n)}(F) = 0 = \delta_{\frac{1}{2}}(F).$$

The Portemanteau theorem finally implies that $(\mu^{(\lambda_n)})_{n\geq 1}$ converges weakly to the delta measure at $\frac{1}{2}$. ■

CHAPTER V

CENTRAL LIMIT THEOREM

If Y is a random variable with probability density function $2 \cdot \rho_0^{(\lambda_n)}$, its distribution concentrates in a small environment of $\frac{1}{2}$, provided that λ is sufficiently large. By translating and rescaling Y, we can create a new random variable centered at 0 whose density function might, as λ grows to infinity, gradually assume the shape of a Gaussian density. We will show that this is indeed the case, and thereby derive a central limit theorem for our stochastic process X.

In the sequel, an appropriate function will always be a monotone increasing function $f: \mathbb{R}^+ \to \mathbb{R}$ which satisfies $\lim_{\lambda\to\infty} f(\lambda) = \infty$. For $\lambda > 0$, we consider the translated and rescaled random variable $(Y - \frac{1}{2}) \cdot f(\lambda)$ and set

$$\varphi : \quad]0;1[\to] - \tfrac{1}{2} \cdot f(\lambda); \tfrac{1}{2} \cdot f(\lambda)[, \quad x \mapsto (x - \tfrac{1}{2}) \cdot f(\lambda),$$

so that the modified random variable permits the representation $\varphi(Y)$. The function φ is bijective and differentiable, with differentiable inverse $\varphi^{-1}(y) = \frac{y}{f(\lambda)} + \frac{1}{2}$, hence a diffeomorphism. Its first derivative is given by $\varphi'(x) = f(\lambda)$. By the transformation theorem for probability density functions, $\varphi(Y)$ has the density function

$$q^{(\lambda)}(y) = \frac{1}{f(\lambda)} \cdot 2 \cdot \rho_0^{(\lambda)}(\tfrac{y}{f(\lambda)} + \tfrac{1}{2}).$$

We extend $q^{(\lambda)}$ to the entire real line by setting

$$q^{(\lambda)}(y) := \begin{cases} \frac{1}{f(\lambda)} \cdot 2 \cdot \rho_0^{(\lambda)}(\frac{y}{f(\lambda)} + \frac{1}{2}) & \text{if } y \in] - \frac{1}{2} \cdot f(\lambda); \frac{1}{2} \cdot f(\lambda)[\\ 0 & \text{otherwise.} \end{cases}$$

As before, let $(\lambda_n)_{n\geq 1}$ be a sequence of positive real numbers that increases to $+\infty$ and for which $f(\lambda_1) > 0$. For any $n \in \mathbb{N}$, let Y_n be a random variable with probability density function $q^{(\lambda_n)}$, and let Y_∞ be a standard Gaussian random variable. These are the prerequisites for the following theorem:

Theorem 9 *In the above situation, there exists an appropriate f such that the sequence $q^{(\lambda_n)}$ converges pointwise to the density function of Y_∞; that is for any $y \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} q^{(\lambda_n)}(y) = \frac{1}{\sqrt{2\pi}} \cdot \exp(-\frac{y^2}{2}).$$

Proof. For any positive integer n and any $y \in \mathbb{R}$, we have

$$\begin{aligned} g^{(\lambda_n)}(\tfrac{y}{f(\lambda_n)} + \tfrac{1}{2}) &= (\tfrac{y}{f(\lambda_n)} + \tfrac{1}{2})^{\lambda_n - 1} \cdot (\tfrac{1}{2} - \tfrac{y}{f(\lambda_n)})^{\lambda_n} \\ &= (\tfrac{1}{4} - \tfrac{y^2}{f(\lambda_n)^2})^{\lambda_n - 1} \cdot (\tfrac{1}{2} - \tfrac{y}{f(\lambda_n)}) \\ &= (\tfrac{1}{4})^{\lambda_n - 1} \cdot (1 - \tfrac{4y^2}{f(\lambda_n)^2})^{\lambda_n - 1} \cdot (\tfrac{1}{2} - \tfrac{y}{f(\lambda_n)}). \end{aligned}$$

When setting $f(x) := 2\sqrt{2} \cdot \sqrt{x-1}$, this term becomes

$$(\tfrac{1}{4})^{\lambda_n - 1} \cdot (1 - \tfrac{y^2}{2\cdot(\lambda_n - 1)})^{\lambda_n - 1} \cdot (\tfrac{1}{2} - \tfrac{y}{f(\lambda_n)}).$$

Let us denote the factor $(1 - \frac{y^2}{2\cdot(\lambda_n-1)})^{\lambda_n-1} \cdot (\frac{1}{2} - \frac{y}{f(\lambda_n)})$ by $h_n(y)$. We may then represent the density function $q^{(\lambda_n)}$ as

$$\begin{aligned} q^{(\lambda_n)}(y) &= \frac{1}{f(\lambda_n)} \cdot 2 \cdot \zeta(\lambda_n) \cdot (\tfrac{1}{4})^{\lambda_n - 1} \cdot h_n(y) \\ &=: \kappa(\lambda_n) \cdot h_n(y). \end{aligned}$$

Then,

$$\lim_{n\to\infty} h_n(y) = \exp(-\frac{y^2}{2}) \cdot \frac{1}{2}$$

and

$$1 = \int_{-\infty}^{\infty} q^{(\lambda_n)}(y)dy = \kappa(\lambda_n) \cdot \int_{-\infty}^{\infty} h_n(y)dy,$$

whence it follows that

$$\kappa(\lambda_n) = \frac{1}{\int_{-\infty}^{\infty} h_n(y)dy} \quad \forall n \in \mathbb{N}.$$

The task ahead is now to study the convergence properties of $(\int_{-\infty}^{\infty} h_n(y)dy)_{n\geq 1}$. We give ourselves a fixed $n \in \mathbb{N}$. If $y \in \mathbb{R}\backslash]-\frac{1}{2}\cdot f(\lambda_n); \frac{1}{2}\cdot f(\lambda_n)[$, we have $h_n(y) = 0$; and if $y \in]-\frac{1}{2}\cdot f(\lambda_n); \frac{1}{2}\cdot f(\lambda_n)[$, we have

$$h_n(y) = (1 - \tfrac{y^2}{2\cdot(\lambda_n - 1)})^{\lambda_n - 1}\cdot(\tfrac{1}{2} - \tfrac{y}{f(\lambda_n)}).$$

In the latter case, it holds also true that $\lambda_n - 1 > \frac{y^2}{2}$. Then, the sequence $((1 - \frac{y^2}{2\cdot(\lambda_k - 1)})^{\lambda_k - 1})_{k\geq n}$ increases monotonically to $\exp(-\frac{1}{2}y^2)$. Therefore,

$$|(1 - \tfrac{y^2}{2\cdot(\lambda_n - 1)})^{\lambda_n - 1}| \leq \exp(-\tfrac{1}{2}y^2)$$

and, since

$$|\frac{1}{2} - \frac{y}{f(\lambda_n)}| \leq \frac{1}{2} + \frac{|y|}{f(\lambda_n)} < 1,$$

we get the function $\exp(-\frac{1}{2}y^2)$ as an upper bound for $(|h_n(y)|)_{n\geq 1}$. As $\int_{-\infty}^{\infty}\exp(-\frac{1}{2}y^2)dy = \sqrt{2\pi}$, the dominated convergence theorem yields

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} h_n(y)dy = \int_{-\infty}^{\infty}\frac{1}{2}\cdot\exp(-\frac{1}{2}y^2)dy = \frac{1}{2}\cdot\sqrt{2\pi}.$$

Accordingly,

$$\lim_{n\to\infty} q^{(\lambda_n)}(y) = \frac{2}{\sqrt{2\pi}}\cdot\frac{1}{2}\cdot\exp(-\frac{1}{2}y^2) = \frac{1}{\sqrt{2\pi}}\cdot\exp(-\frac{1}{2}y^2),$$

which proves our stated convergence result. ■

CHAPTER VI

LARGE DEVIATIONS PRINCIPLE

Having established results in the spirit of the law of large numbers and the central limit theorem, we complete the canon of classical probabilistic results in this chapter by proving a large deviations principle for our process. An introduction to large deviation principles can be found both in [5] and [3].

definition 1 *Let* $I : \ ]0;1[\to [0;\infty[$ *be a lower semi-continuous (lsc) function, that is for any* $x_0 \in]0;1[$*, we have* $\liminf_{x\to x_0} I(x) \geq I(x_0)$*. Then,* I *is said to be an entropy function.*

We consider, as we did before, the family of probability measures $(\mu^{(\lambda)})_{\lambda>0}$. We say that $(\mu^{(\lambda)})_{\lambda>0}$ satisfies a large deviations principle with respect to the entropy function I if for any sequence $(\lambda_n)_{n\geq 1}$ of positive numbers that increases to ∞, there is a positive rate sequence $(r_n)_{n\geq 1}$, also increasing to ∞, such that for any non-empty open set $G \subseteq]0;1[$, we have

$$\liminf_{n\to\infty} \frac{1}{r_n} \cdot \ln(\mu^{(\lambda_n)}(G)) \geq -\inf_{x\in G} I(x) \tag{17}$$

and for any non-empty closed set $F \subseteq]0;1[$, we have

$$\limsup_{n\to\infty} \frac{1}{r_n} \cdot \ln(\mu^{(\lambda_n)}(F)) \leq -\inf_{x\in F} I(x). \tag{18}$$

The underlying topology is the one induced by the natural topology on $\mathbb{R}$, but taken with respect to $]0;1[$. For instance, our closed sets will be the ones that are relatively closed with respect to $]0;1[$ in $\mathbb{R}$.

Let us establish inequality (18) for our system, assuming that I is minimized at the point $\frac{1}{2}$ and that $I(\frac{1}{2}) = 0$. We distinguish between the cases that $\frac{1}{2}$ is contained in F and that $\frac{1}{2}$ is not contained in F. If $\frac{1}{2} \in F$, there is hardly anything to do. As $I(\frac{1}{2}) = \inf_{x\in[0;1]} I(x) = 0$, the right-hand side of inequality (18) becomes 0. And with $\mu^{(\lambda_n)}$ being probability measures, we have

$$(\mu^{(\lambda_n)}(F)) \leq 1$$

for any $n \in \mathbb{N}$. This yields the contended inequality, irrespective of how we choose $(r_n)_{n\geq 1}$.

The case of $\frac{1}{2} \notin F$ requires considerably greater efforts. Since F is closed and non-empty, there exists a point $x_0 \in F$ where the minimal distance between F and $\{\frac{1}{2}\}$ is assumed, meaning that

$$|x - \frac{1}{2}| \geq |x_0 - \frac{1}{2}|$$

for any $x \in F$. Let $\epsilon > 0$ denote this minimal distance, so $|x_0 - \frac{1}{2}| = \epsilon$. We would like to compare the set F with its closed superset $\tilde{F} :=]0; \frac{1}{2} - \epsilon] \cup [\frac{1}{2} + \epsilon; 1[$. If we assume that our entropy function I is symmetric with respect to the line running parallel to the y-axis and passing through $\frac{1}{2}$, and if we further impose that I be strictly monotone decreasing in $]0; \frac{1}{2}[$ and strictly monotone increasing in $]\frac{1}{2}; 1[$, we obtain

$$-\inf_{x\in F} I(x) = -I(\frac{1}{2} - \epsilon) = -\inf_{x\in\tilde{F}} I(x),$$

implying that the term on the right-hand side of (18) does not depend on which closed subset of $]0; \frac{1}{2} - \epsilon] \cup [\frac{1}{2} + \epsilon; 1[$ we designate as our F. As probability measures are monotone set functions, the term on the left-hand side of (18) is maximized over the class of all closed subsets of $\tilde{F}$ when pickig $F := \tilde{F}$. Hence, it suffices to prove the inequality for the closed set $F =]0; \frac{1}{2} - \epsilon] \cup [\frac{1}{2} + \epsilon; 1[$.

Given a positive integer n, we have

$$\mu^{(\lambda_n)}(F) = \int_0^{\frac{1}{2}-\epsilon} 2 \cdot \rho_0^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^{1} 2 \cdot \rho_0^{(\lambda_n)}(x)dx,$$

which, according to inequality (16), is less than or equal to

$$\frac{\int_0^{\frac{1}{2}-\epsilon} 2\cdot\rho_0^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^1 2\cdot\rho_0^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} 2\cdot\rho_0^{(\lambda_n)}(x)dx} = \frac{\int_0^{\frac{1}{2}-\epsilon} g^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^1 g^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} g^{(\lambda_n)}(x)dx}. \quad (19)$$

In the sequel, we will evoke many results that have been established 'along the way' in the proof of theorem 8. First, for sufficiently large n, we have the estimate

$$\int_0^{\frac{1}{2}-\epsilon} g^{(\lambda_n)}(x)dx + \int_{\frac{1}{2}+\epsilon}^1 g^{(\lambda_n)}(x)dx \le (\tfrac{1}{2}-\epsilon)^{\lambda_n}\cdot(\tfrac{1}{2}+\epsilon)^{\lambda_n-1}.$$

Now, we fix a $k \in \mathbb{N}$ and choose n_k so large that $\frac{\lambda_{n_k}-1}{2\lambda_{n_k}-1} \in]\frac{1}{2}-\frac{\epsilon}{k};\frac{1}{2}+\frac{\epsilon}{k}[$. Then, we have

$$\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} g^{(\lambda_{n_k})}(x)dx \ge \int_{\frac{1}{2}-\frac{\epsilon}{k}}^{\frac{1}{2}+\frac{\epsilon}{k}} g^{(\lambda_{n_k})}(x)dx \ge \frac{2\epsilon}{k}\cdot g^{(\lambda_{n_k})}(\tfrac{1}{2}+\tfrac{\epsilon}{k}),$$

so that the term (19) can be estimated against the upper bound

$$\tfrac{k}{2\epsilon}\cdot\tfrac{\frac{1}{2}-\epsilon}{\frac{1}{2}-\frac{\epsilon}{k}}\cdot(\tfrac{\frac{1}{4}-\epsilon^2}{\frac{1}{4}-\frac{\epsilon^2}{k^2}})^{\lambda_{n_k}-1}.$$

Since the logarithm is a monotone increasing function, this chain of inequalities is preserved when applying ln to each term, yielding

$$\ln(\mu^{(\lambda_{n_k})}(F)) \le \ln(\tfrac{k}{2\epsilon}\cdot\tfrac{\frac{1}{2}-\epsilon}{\frac{1}{2}-\frac{\epsilon}{k}}) + (\lambda_{n_k}-1)\cdot\ln(\tfrac{\frac{1}{4}-\epsilon^2}{\frac{1}{4}-\frac{\epsilon^2}{k^2}})$$

and, dividing both sides by λ_{n_k}, we get

$$\frac{1}{\lambda_{n_k}}\cdot\ln(\mu^{(\lambda_{n_k})}(F)) \le \frac{1}{\lambda_{n_k}}\cdot\ln(\tfrac{k}{2\epsilon}\cdot\tfrac{\frac{1}{2}-\epsilon}{\frac{1}{2}-\frac{\epsilon}{k}}) + \frac{\lambda_{n_k}-1}{\lambda_{n_k}}\cdot\ln(\tfrac{\frac{1}{4}-\epsilon^2}{\frac{1}{4}-\frac{\epsilon^2}{k^2}}).$$

This inequality also holds for any $n \ge n_k$. As

$$\lim_{n\to\infty}\tfrac{1}{\lambda_n}\cdot\ln(\tfrac{k}{2\epsilon}\cdot\tfrac{\frac{1}{2}-\epsilon}{\frac{1}{2}-\frac{\epsilon}{k}}) = 0$$

and as

$$\lim_{n\to\infty}\tfrac{\lambda_n-1}{\lambda_n}\cdot\ln(\tfrac{\frac{1}{4}-\epsilon^2}{\frac{1}{4}-\frac{\epsilon^2}{k^2}}) = \ln(\tfrac{\frac{1}{4}-\epsilon^2}{\frac{1}{4}-\frac{\epsilon^2}{k^2}}),$$

we finally obtain

$$\limsup_{n\to\infty}\frac{1}{\lambda_n}\cdot\ln(\mu^{(\lambda_n)}(F)) \le \ln(\tfrac{\frac{1}{4}-\epsilon^2}{\frac{1}{4}-\frac{\epsilon^2}{k^2}}).$$

Since the positive integer k was arbitrarily selected, we may have k tend to infinity on the right-hand side of the previous inequality. The final version of our large deviations inequality for closed sets F then reads

$$\limsup_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(\mu^{(\lambda_n)}(F)) \leq \ln(1-4\epsilon^2),$$

where our original sequence $(\lambda_n)_{n\geq 1}$ plays the role of a rate sequence. In the light of this estimate, let us define our entropy function as

$$I(x) := -\ln(4 \cdot x \cdot (1-x))$$

for any $x \in]0;1[$. We briefly verify that a thus defined I satisfies all the conditions we had initially imposed. First, we have

$$I(\tfrac{1}{2}) = -\ln(4 \cdot (\tfrac{1}{2})^2) = 0,$$

and since the expression $x \cdot (1-x)$ is maximized at $\frac{1}{2}$, it easily follows that I is indeed non-negative, with a global minimum at $\frac{1}{2}$. Clearly, our entropy function is also symmetric with respect to $\frac{1}{2}$. The first derivative of I can be calculated as

$$I'(x) = \frac{8x-4}{4x-4x^2},$$

whence we infer the asserted monotonicity statement. A plot of the function I can be found in appendix D.

The monotonicity property allows us to conclude that I, considered over the interval $]0;\frac{1}{2}-\epsilon]$, attains its minimum at the right boundary point $\frac{1}{2}-\epsilon$. Hence,

$$-\inf\{I(x) : \quad x \in]0;\tfrac{1}{2}-\epsilon] \cup [\tfrac{1}{2}+\epsilon;1[\} = \ln(4 \cdot (\tfrac{1}{2}-\epsilon) \cdot (\tfrac{1}{2}+\epsilon)) = \ln(1-4\epsilon^2),$$

and we have established inequality (18) in due thoroughness.

With the newly acquired representations of I and $(r_n)_{n\geq 1}$, the alleged inequality (17) can be restated as

$$\liminf_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(\mu^{(\lambda_n)}(G)) \geq \ln(1-4d^2)$$

for any open subset G of $]0;1[$. Here, d marks the distance between the sets G and $\{\frac{1}{2}\}$. We will establish this inequality in several steps.

First, let us prove it for open intervals contained in $]\frac{1}{2};1[$, that is we scrutinize intervals of the form $]\frac{1}{2}+d;c[$ for positive numbers c,d with
$\frac{1}{2}+d<c$. The distance between the sets $\{\frac{1}{2}\}$ and $]\frac{1}{2}+d;c[$ is then given by d and the right-hand side of inequality (17) becomes $\ln(1-4d^2)$. On the left-hand side, we should consider the limes inferior of the sequence $(\frac{1}{\lambda_n}\cdot\ln(\int_{\frac{1}{2}+d}^{c}2\cdot\rho_0^{(\lambda_n)}(x)dx))_{n\geq 1}$. When fixing an $n\geq 1$, we have

$$\frac{\int_0^{\frac{1}{2}+d}2\cdot\rho_0^{(\lambda_n)}(x)dx+\int_c^1 2\cdot\rho_0^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}+d}^{c}2\cdot\rho_0^{(\lambda_n)}(x)dx}=\frac{\int_0^{\frac{1}{2}+d}g^{(\lambda_n)}(x)dx+\int_c^1 g^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}+d}^{c}g^{(\lambda_n)}(x)dx}.$$

The numerator of the above expression can be estimated as follows:

$$\begin{aligned}&\int_0^{\frac{1}{2}+d}g^{(\lambda_n)}(x)dx+\int_c^1 g^{(\lambda_n)}(x)dx\\ =&\int_0^{\frac{1}{2}-d}g^{(\lambda_n)}(x)dx+\int_{\frac{1}{2}-d}^{\frac{1}{2}+d}g^{(\lambda_n)}(x)dx+\int_c^1 g^{(\lambda_n)}(x)dx\\ \leq&(\tfrac{1}{2}-d)^{\lambda_n}\cdot(\tfrac{1}{2}+d)^{\lambda_n}+2d\cdot(\tfrac{\lambda_n-1}{2\lambda_n-1})^{\lambda_n-1}\cdot(\tfrac{\lambda_n}{2\lambda_n-1})^{\lambda_n}+(1-c)^{\lambda_n+1}\cdot c^{\lambda_n-1},\end{aligned}$$

where we have appealed to the fact that $g^{(\lambda_n)}$ attains its maximum at $\frac{\lambda_n-1}{2\lambda_n-1}$, while assuming that n was chosen so large that $\frac{\lambda_n-1}{2\lambda_n-1}\in]\frac{1}{2}-d;\frac{1}{2}+d[$. If, in addition, we fix a sufficiently large positive integer k, independent of n, the denominator admits the estimate

$$\int_{\frac{1}{2}+d}^{c}g^{(\lambda_n)}(x)dx\geq\tfrac{d}{k}\cdot(\tfrac{1}{2}+d\cdot\tfrac{k+1}{k})^{\lambda_n-1}\cdot(\tfrac{1}{2}-d\cdot\tfrac{k+1}{k})^{\lambda_n}.$$

Combining these two inequalities, we get

$$\begin{aligned}&\frac{\int_0^{\frac{1}{2}+d}g^{(\lambda_n)}(x)dx+\int_c^1 g^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}+d}^{c}g^{(\lambda_n)}(x)dx}\\ \leq&\tfrac{k}{d}\cdot(\tfrac{1}{2}-d\cdot\tfrac{k+1}{k})^{-1}\cdot(\tfrac{1}{4}-d^2\cdot\tfrac{(k+1)^2}{k^2})^{1-\lambda_n}\\ &\cdot((\tfrac{1}{4}-d^2)^{\lambda_n}+2d\cdot\tfrac{\lambda_n}{2\lambda_n-1}\cdot(\tfrac{\lambda_n^2-\lambda_n}{(2\lambda_n-1)^2})^{\lambda_n-1}+(1-c)^{\lambda_n+1}\cdot c^{\lambda_n-1})\\ =:&f(n,k).\end{aligned}$$

Bearing in mind that

$$(\int_0^{\frac{1}{2}+d} 2\cdot \rho_0^{(\lambda_n)}(x)dx + \int_c^1 2\cdot \rho_0^{(\lambda_n)}(x)dx) + \int_{\frac{1}{2}+d}^c 2\cdot \rho_0^{(\lambda_n)}(x)dx = 1,$$

solving this inequality for $\int_{\frac{1}{2}+d}^c 2\cdot \rho_0^{(\lambda_n)}(x)dx$ yields

$$\int_{\frac{1}{2}+d}^c 2\cdot \rho_0^{(\lambda_n)}(x)dx \geq \frac{1}{1+f(n,k)},$$

which is tantamount to saying that

$$\frac{1}{\lambda_n}\cdot \ln(\int_{\frac{1}{2}+d}^c 2\cdot \rho_0^{(\lambda_n)}(x)dx) \geq -\frac{1}{\lambda_n}\cdot \ln(1+f(n,k)).$$

This last term equals

$$\begin{aligned}
&-\frac{1}{\lambda_n}\cdot \ln(\tfrac{k}{d}\cdot(\tfrac{1}{2}-d\cdot\tfrac{k+1}{k})^{-1}\cdot(\tfrac{1}{4}-d^2\cdot\tfrac{(k+1)^2}{k^2})^{1-\lambda_n}\cdot((\tfrac{1}{4}-d^2)^{\lambda_n}\\
&+2d\cdot\tfrac{\lambda_n}{2\lambda_n-1}\cdot(\tfrac{\lambda_n^2-\lambda_n}{(2\lambda_n-1)^2})^{\lambda_n-1}+(1-c)^{\lambda_n+1}\cdot c^{\lambda_n-1}+\tfrac{d}{k}\cdot(\tfrac{1}{2}-d\cdot\tfrac{k+1}{k})\cdot(\tfrac{1}{4}-d^2\cdot\tfrac{(k+1)^2}{k^2})^{\lambda_n-1}))\\
=&-\frac{1}{\lambda_n}\cdot(\ln(\tfrac{k}{d})-\ln(\tfrac{1}{2}-d\cdot\tfrac{k+1}{k})\\
&+(1-\lambda_n)\cdot\ln(\tfrac{1}{4}-d^2\cdot\tfrac{(k+1)^2}{k^2})+\ln(c_1^{(n,k)}+c_2^{(n,k)}+c_3^{(n,k)}+c_4^{(n,k)})),
\end{aligned}$$

with

$$\begin{aligned}
c_1^{(n,k)} &:= (\tfrac{1}{4}-d^2)^{\lambda_n},\\
c_2^{(n,k)} &:= 2d\cdot\tfrac{\lambda_n}{2\lambda_n-1}\cdot(\tfrac{\lambda_n^2-\lambda_n}{(2\lambda_n-1)^2})^{\lambda_n-1},\\
c_3^{(n,k)} &:= (1-c)^2\cdot(c-c^2)^{\lambda_n-1},\\
c_4^{(n,k)} &:= \tfrac{d}{k}\cdot(\tfrac{1}{2}-d\cdot\tfrac{k+1}{k})\cdot(\tfrac{1}{4}-d^2\cdot\tfrac{(k+1)^2}{k^2})^{\lambda_n-1}.
\end{aligned}$$

Now, we have

$$\lim_{n\to\infty}(-\frac{1}{\lambda_n})\cdot(\ln(\tfrac{k}{d})-\ln(\tfrac{1}{2}-d\cdot\tfrac{k+1}{k}))=0$$

and

$$\lim_{n\to\infty}\frac{\lambda_n-1}{\lambda_n}\cdot\ln(\tfrac{1}{4}-d^2\cdot\tfrac{(k+1)^2}{k^2})=\ln(\tfrac{1}{4}-d^2\cdot\tfrac{(k+1)^2}{k^2}).$$

To deal with the term $(-\frac{1}{\lambda_n})\cdot\ln(c_1^{(n,k)}+c_2^{(n,k)}+c_3^{(n,k)}+c_4^{(n,k)})$, we will rely on the following lemma that is proved in appendix D and was inspired by lemma 23.9 in [5]:

Lemma 4 *Let N be a positive integer and let $(\lambda_n)_{n\geq 1}$ be a sequence of the type we have been working with throughout this section. For any $n \in \mathbb{N}$, let $c_1^{(n)}, ..., c_N^{(n)}$ be positive real numbers. Then, we have*

$$\limsup_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(\sum_{i=1}^{N} c_i^{(n)}) = \max_{i\in\{1,...,N\}} \limsup_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(c_i^{(n)}).$$

With this, we have

$$\begin{aligned}
&\liminf_{n\to\infty}(-\frac{1}{\lambda_n}) \cdot \ln(c_1^{(n,k)} + c_2^{(n,k)} + c_3^{(n,k)} + c_4^{(n,k)}) \\
=& -\limsup_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(c_1^{(n,k)} + c_2^{(n,k)} + c_3^{(n,k)} + c_4^{(n,k)}) \\
=& -\max\{\limsup_{n\to\infty} \tfrac{1}{\lambda_n} \cdot \ln(c_1^{(n,k)}); \limsup_{n\to\infty} \tfrac{1}{\lambda_n} \cdot \ln(c_2^{(n,k)}); \limsup_{n\to\infty} \tfrac{1}{\lambda_n} \cdot \ln(c_3^{(n,k)}); \limsup_{n\to\infty} \tfrac{1}{\lambda_n} \cdot \ln(c_4^{(n,k)})\} \\
=& -\max\{\ln(\tfrac{1}{4} - d^2); \ln(\tfrac{1}{4}); \ln(c - c^2); \ln(\tfrac{1}{4} - d^2 \cdot \tfrac{(k+1)^2}{k^2})\} \\
=& \ln(4).
\end{aligned}$$

Thus, we may infer that

$$\liminf_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(\int_{\frac{1}{2}+d}^{c} 2 \cdot \rho_0^{(\lambda_n)}(x)dx) \geq \ln(\tfrac{1}{4} - d^2 \cdot \tfrac{(k+1)^2}{k^2}) + \ln(4) = \ln(1 - 4d^2 \cdot \tfrac{(k+1)^2}{k^2}).$$

The prospected inequality is obtained when letting k go to infinity.

In a subsequent step, let us consider open intervals contained in $]0; \frac{1}{2}[$, that is intervals of the form $]c; \frac{1}{2} - d[$ for $c, d > 0$ and $c < \frac{1}{2} - d$. As before, the right-hand side of (17) reads $\ln(1 - 4d^2)$. Dealing with the left-hand side, we claim that

$$\int_{c}^{\frac{1}{2}-d} 2 \cdot \rho_0^{(\lambda_n)}(x)dx \geq \int_{\frac{1}{2}+d}^{1-c} 2 \cdot \rho_0^{(\lambda_n)}(x)dx$$

or, equivalently,

$$\frac{\int_{c}^{\frac{1}{2}-d} 2 \cdot \rho_0^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}+d}^{1-c} 2 \cdot \rho_0^{(\lambda_n)}(x)dx} \geq 1.$$

For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{\int_{c}^{\frac{1}{2}-d} 2 \cdot \rho_0^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}+d}^{1-c} 2 \cdot \rho_0^{(\lambda_n)}(x)dx} &= \frac{\int_{c}^{\frac{1}{2}-d} g^{(\lambda_n)}(x)dx}{\int_{\frac{1}{2}+d}^{1-c} g^{(\lambda_n)}(x)dx} \\
&= \frac{\int_{c}^{\frac{1}{2}-d} (x \cdot (1-x))^{\lambda_n - 1} \cdot (1-x)dx}{\int_{\frac{1}{2}+d}^{1-c} (x \cdot (1-x))^{\lambda_n - 1} \cdot (1-x)dx}.
\end{aligned}$$

By substituting y for $(1-x)$ in the denominator integral, this ratio becomes

$$\frac{\int_c^{\frac{1}{2}-d} (x \cdot (1-x))^{\lambda_n - 1} \cdot (1-x) dx}{\int_c^{\frac{1}{2}-d} (y \cdot (1-y))^{\lambda_n - 1} \cdot y dy}.$$

As the integrand in the denominator is majorized by the integrand in the numerator, the desired inequality holds. Therefore,

$$\begin{aligned}\liminf_{n\to\infty} \tfrac{1}{\lambda_n} \cdot \ln(\mu^{(\lambda_n)}(]c; \tfrac{1}{2} - d[)) &\geq \liminf_{n\to\infty} \tfrac{1}{\lambda_n} \cdot \ln(\mu^{(\lambda_n)}(]\tfrac{1}{2} + d; 1 - c[)) \\ &\geq \ln(1 - 4d^2).\end{aligned}$$

Finally, let us extend our result to a general open, non-empty set $G \subseteq]0;1[$. Then, every element of G is either contained in $]0; \frac{1}{2}[$ or in $]\frac{1}{2}; 1[$ or is the number $\frac{1}{2}$. If $x_0 \in G$ is a number smaller than $\frac{1}{2}$, the open set G encompasses an ϵ-environment of x_0, meaning that $]x_0 - \epsilon; x_0 + \epsilon[\subseteq G \cap]0; \frac{1}{2}[$ for a suitable $\epsilon > 0$. It is obviously save to assume that $x_0 + \epsilon < \frac{1}{2}$. We have just shown that

$$\liminf_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(\mu^{(\lambda_n)}(]x_0 - \epsilon; x_0 + \epsilon[)) \geq -I(x_0 + \epsilon),$$

so that

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(\mu^{(\lambda_n)}(G)) &\geq \liminf_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(\mu^{(\lambda_n)}(]x_0 - \epsilon; x_0 + \epsilon[)) \\ &\geq -I(x_0 + \epsilon) \\ &\geq -I(x_0).\end{aligned}$$

And if $x_0 > \frac{1}{2}$ is an element of G, we pick an $\epsilon > 0$ such that $x_0 - \epsilon > \frac{1}{2}$ and the interval $]x_0 - \epsilon; x_0 + \epsilon[$ is contained in G. Employing the same argument as above, we get

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(\mu^{(\lambda_n)}(G)) &\geq \liminf_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(\mu^{(\lambda_n)}(]x_0 - \epsilon; x_0 + \epsilon[)) \\ &\geq -I(x_0 - \epsilon) \\ &\geq -I(x_0).\end{aligned}$$

Since I is clearly a continuous function, the case $x_0 = \frac{1}{2}$ is implicitly taken care of and we obtain

$$\liminf_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(\mu^{(\lambda_n)}(G)) \geq - \inf_{x\in G} I(x).$$

This proves that our process satisfies indeed a large deviations principle.

APPENDIX A

PROOF OF LEMMA 1

We consider several distinct cases, depending on the switching environment and depending on how far the times r, s and t are apart from each other. As before, let $k_s := \sup\{j \in \mathbb{Z} : a_j \leq s\}$ and let $k_r := \sup\{j \in \mathbb{Z} : a_j \leq r\}$.

First assume that $t \in [s; a_{k_s+1}[$. If k_s is an even integer, we have

$$F(s, F(r, \xi; s); t) = F(r, \xi; s) \cdot \exp(s - t).$$

If $s \in [r; a_{k_r+1}[$, we have $k_r = k_s$, so k_r is even. Therefore,

$$F(r, \xi; s) = \xi \cdot \exp(r - s),$$

which implies

$$F(s, F(r, \xi; s); t) = \xi \cdot \exp(r - t).$$

Since $k_r = k_s$, the time t is contained in the interval $[r; a_{k_r+1}[$. This yields

$$F(r, \xi; t) = \xi \cdot \exp(r - t) = F(s, F(r, \xi; s); t).$$

Now, assume that $s \in [a_{k_r+n}; a_{k_r+n+1}[$ for some $n \in \mathbb{N}$. Then,

$$F(r, \xi; s) = F(r, \xi; a_{k_r+n}) \cdot \exp(a_{k_r+n} - s)$$

if $(k_r + n)$ is even. In this case, we get

$$F(s, F(r, \xi; s); t) = F(r, \xi; a_{k_r+n}) \cdot \exp(a_{k_r+n} - t).$$

We have $k_s = k_r + n$, so t is an element of $[a_{k_r+n}; a_{k_r+n+1}[$ and $(k_r + n)$ is even. From this consideration, we obtain

$$F(r, \xi; t) = F(r, \xi; a_{k_r+n}) \cdot \exp(a_{k_r+n} - t).$$

In a next step, assume that k_s is odd. Then,

$$F(s, F(r, \xi; s); t) = 1 - (1 - F(r, \xi; s)) \cdot \exp(s - t).$$

If $s \in [r; a_{k_r+1}[$, we have again equality between k_r and k_s, which entails that k_r is odd. Hence,

$$F(r, \xi; s) = 1 - (1 - \xi) \cdot \exp(r - s),$$

whence it follows that

$$F(s, F(r, \xi; s); t) = 1 - (1 - \xi) \cdot \exp(r - t).$$

Further, t is contained in $[r; a_{k_r+1}[$, so

$$F(r, \xi; t) = 1 - (1 - \xi) \cdot \exp(r - t).$$

And if $s \in [a_{k_r+n}; a_{k_r+n+1}[$ for some $n \geq 1$, we have $k_s = k_r + n$, so $(k_r + n)$ is odd. This yields

$$F(r, \xi; s) = 1 - (1 - F(r, \xi; a_{k_r+n})) \cdot \exp(a_{k_r+n} - s).$$

Consequently,

$$F(s, F(r, \xi; s); t) = 1 - (1 - F(r, \xi; a_{k_r+n})) \cdot \exp(a_{k_r+n} - t)$$

and $t \in [a_{k_r+n}; a_{k_r+n+1}[$, yielding

$$F(r, \xi; t) = 1 - (1 - F(r, \xi; a_{k_r+n})) \cdot \exp(a_{k_r+n} - t).$$

In a subsequent step, assume that $t \in [a_{k_s+1}; a_{k_s+2}[$. In the case of an even $(k_s + 1)$, the term $F(s, F(r, \xi; s); t)$ equals

$$F(s, F(r, \xi; s); a_{k_s+1}) \cdot \exp(a_{k_s+1} - t) = (1 - (1 - F(r, \xi; s)) \cdot \exp(s - a_{k_s+1})) \cdot \exp(a_{k_s+1} - t). \tag{20}$$

At this point, we differentiate between the cases $s \in [r; a_{k_r+1}[$ and $s \in [a_{k_r+m}; a_{k_r+m+1}[$. Under the assumption that s is contained in $[r; a_{k_r+1}[$, we have $k_r = k_s$ and k_r is odd. Thus,

$$F(r, \xi; s) = 1 - (1 - \xi) \cdot \exp(r - s).$$

Inserted in the term on the right-hand side of (20), this identity gives us

$$F(s, F(r, \xi; s); t) = (1 - (1 - \xi) \cdot \exp(r - a_{k_s+1})) \cdot \exp(a_{k_s+1} - t).$$

Moreover, $t \in [a_{k_r+1}; a_{k_r+2}[$, so

$$F(r, \xi; t) = F(r, \xi; a_{k_r+1}) \cdot \exp(a_{k_r+1} - t)$$

and

$$1 - (1 - \xi) \cdot \exp(r - a_{k_s+1}) = F(r, \xi; a_{k_s+1}) = F(r, \xi; a_{k_r+1}).$$

Assuming that $s \in [a_{k_r+m}; a_{k_r+m+1}[$ for some positive integer m, we notice that $k_r + m = k_s$ and that $(k_r + m)$ is odd. Therefore,

$$F(r, \xi; s) = 1 - (1 - F(r, \xi; a_{k_r+m})) \cdot \exp(a_{k_r+m} - s),$$

implying together with (20) that

$$\begin{aligned} F(s, F(r, \xi; s); t) &= (1 - (1 - F(r, \xi; a_{k_r+m})) \cdot \exp(a_{k_r+m} - s) \cdot \exp(s - a_{k_s+1})) \cdot \exp(a_{k_s+1} - t) \\ &= (1 - (1 - F(r, \xi; a_{k_r+m+1})) \cdot \exp(a_{k_r+m} - a_{k_r+m+1})) \cdot \exp(a_{k_r+m+1} - t). \end{aligned}$$

But we also have

$$\begin{aligned} F(r, \xi; t) &= F(r, \xi; a_{k_r+m+1}) \cdot \exp(a_{k_r+m+1} - t) \\ &= (1 - (1 - F(r, \xi; a_{k_r+m})) \cdot \exp(a_{k_r+m} - a_{k_r+m+1})) \cdot \exp(a_{k_r+m+1} - t). \end{aligned}$$

Next, if we suppose that $(k_s + 1)$ is odd, k_s is even and $F(s, F(r, \xi; s); t)$ equals

$$1-(1-F(s, F(r, \xi; s); a_{k_s+1}))\cdot\exp(a_{k_s+1}-t) = 1-(1-F(r, \xi; s)\cdot\exp(s-a_{k_s+1}))\cdot\exp(a_{k_s+1}-t). \tag{21}$$

If $s \in [r; a_{k_r+1}[$, k_r is even and equals k_s. As a result

$$F(r, \xi; s) = \xi \cdot \exp(r - s)$$

and the right-hand side of equation (21) becomes

$$1 - (1 - \xi \cdot \exp(r - a_{k_r+1})) \cdot \exp(a_{k_r+1} - t).$$

Taking into account that $t \in [a_{k_r+1}; a_{k_r+2}[$, we readily obtain

$$\begin{aligned} F(r,\xi;t) &= 1-(1-F(r,\xi;a_{k_r+1}))\cdot\exp(a_{k_r+1}-t) \\ &= 1-(1-\xi\cdot\exp(r-a_{k_r+1}))\cdot\exp(a_{k_r+1}-t). \end{aligned}$$

And if $s \in [a_{k_r+m}; a_{k_r+m+1}[$ for some $m \geq 1$, the integer $k_r+m=k_s$ is even. Accordingly,

$$F(r,\xi;s) = F(r,\xi;a_{k_r+m})\cdot\exp(a_{k_r+m}-s),$$

and, citing (21) again, we receive

$$F(s,F(r,\xi;s);t) = 1-(1-F(r,\xi;a_{k_r+m})\cdot\exp(a_{k_r+m}-a_{k_r+m+1}))\cdot\exp(a_{k_r+m+1}-t).$$

On the other hand,

$$\begin{aligned} F(r,\xi;t) &= 1-(1-F(r,\xi;a_{k_r+m+1}))\cdot\exp(a_{k_r+m+1}-t) \\ &= 1-(1-F(r,\xi;a_{k_r+m})\cdot\exp(a_{k_r+m}-a_{k_r+m+1}))\cdot\exp(a_{k_r+m+1}-t). \end{aligned}$$

We have thus proved that our claim holds for any $r<s<t$ with $t \in [a_{k_s+1}; a_{k_s+2}[$. To state it differently, we have verified the special case $n=1$ within our broader objective of showing the claim for $t \in [a_{k_s+n}; a_{k_s+n+1}[$ for arbitrary $n \in \mathbb{N}$. If we succeed in this undertaking, we will have automatically established the lemma, bearing in mind that we have already dealt with the case $t \in [s; a_{k_s+1}[$. In the induction step, we will require that the statement be valid for any point t in $[s; a_{k_s+n+1}[$ and we will assume that $t \in [a_{k_s+n+1}; a_{k_s+n+2}[$. Let us first treat the case of an even (k_s+n+1). Then,

$$\begin{aligned} F(s,F(r,\xi;s);t) &= F(s,F(r,\xi;s);a_{k_s+n+1})\cdot\exp(a_{k_s+n+1}-t) \\ &= (1-(1-F(s,F(r,\xi;s);a_{k_s+n}))\cdot\exp(a_{k_s+n}-a_{k_s+n+1}))\cdot\exp(a_{k_s+n+1}-t). \end{aligned}$$

Since a_{k_s+n} lies in the interval $[a_{k_s+n}; a_{k_s+n+1}[$, this last term equals

$$\begin{aligned} &(1-(1-F(r,\xi;a_{k_s+n}))\cdot\exp(a_{k_s+n}-a_{k_s+n+1}))\cdot\exp(a_{k_s+n+1}-t) \\ =&F(r,\xi;a_{k_s+n+1})\cdot\exp(a_{k_s+n+1}-t) \\ =&F(r,\xi;t) \end{aligned}$$

by the induction hypothesis.

Now, assume that $(k_s + n + 1)$ is odd. In this case, we get, by a calculation similar to the one above, that

$$
\begin{aligned}
F(s, F(r, \xi; s); t) &= 1 - (1 - F(s, F(r, \xi; s); a_{k_s+n+1})) \cdot \exp(a_{k_s+n+1} - t) \\
&= 1 - (1 - F(s, F(r, \xi; s); a_{k_s+n}) \cdot \exp(a_{k_s+n} - a_{k_s+n+1})) \cdot \exp(a_{k_s+n+1} - t) \\
&= 1 - (1 - F(r, \xi; a_{k_s+n}) \cdot \exp(a_{k_s+n} - a_{k_s+n+1})) \cdot \exp(a_{k_s+n+1} - t) \\
&= 1 - (1 - F(r, \xi; a_{k_s+n+1})) \cdot \exp(a_{k_s+n+1} - t) \\
&= F(r, \xi; t).
\end{aligned}
$$

This completes our argument.

APPENDIX B

PROOF OF LEMMA 2

First, assume that $t \in [s; a_{k_s+1}[$ and that k_s is even. Then,

$$\begin{aligned}
&|F(s,\xi;t) - F(s,\eta;t)| \\
=&|\xi \cdot \exp(s-t) - \eta \cdot \exp(s-t)| \\
=&\exp(s-t)|\xi - \eta|,
\end{aligned}$$

and if k_s is odd, we have

$$\begin{aligned}
&|F(s,\xi;t) - F(s,\eta;t)| = |1 - (1-\xi) \cdot \exp(s-t) - 1 + (1-\eta) \cdot \exp(s-t)| \\
=&|\xi \cdot \exp(s-t) - \eta \cdot \exp(s-t)| = |\xi - \eta| \cdot \exp(s-t).
\end{aligned}$$

We are now concerned with the case $t \in [a_{k_s+n}; a_{k_s+n+1}[$ for $n \in \mathbb{N}$. If t is in $[a_{k_s+1}; a_{k_s+2}[$, let us start with discussing the case of an even (k_s+1). We have

$$\begin{aligned}
&|F(s,\xi;t) - F(s,\eta;t)| \\
=&|F(s,\xi;a_{k_s+1}) \cdot \exp(a_{k_s+1} - t) - F(s,\eta;a_{k_s+1}) \cdot \exp(a_{k_s+1} - t)| \\
=&\exp(a_{k_s+1} - t) \cdot |1 - (1-\xi) \cdot \exp(s - a_{k_s+1}) - 1 + (1-\eta) \cdot \exp(s - a_{k_s+1})| \\
=&\exp(a_{k_s+1} - t) \cdot \exp(s - a_{k_s+1}) \cdot |\xi - \eta| \\
=&\exp(s-t) \cdot |\xi - \eta|.
\end{aligned}$$

If $(k_s + 1)$ is odd, we get

$$\begin{aligned}
&|F(s,\xi;t) - F(s,\eta;t)| \\
=&|1 - (1 - F(s,\xi;a_{k_s+1})) \cdot \exp(a_{k_s+1} - t) - 1 + (1 - F(s,\eta;a_{k_s+1})) \cdot \exp(a_{k_s+1} - t)| \\
=&\exp(a_{k_s+1} - t) \cdot |F(s,\xi;a_{k_s+1}) - F(s,\eta;a_{k_s+1})| \\
=&\exp(a_{k_s+1} - t) \cdot |\xi \cdot \exp(s - a_{k_s+1}) - \eta \cdot \exp(s - a_{k_s+1})| \\
=&|\xi - \eta| \cdot \exp(s - t).
\end{aligned}$$

In the induction step, we assume that the lemma holds for all t contained in $[s; a_{k_s+n+1}[$. Given a t in $[a_{k_s+n+1}; a_{k_s+n+2}[$, we have for an even $(k_s + n + 1)$ that

$$\begin{aligned}
&|F(s,\xi;t) - F(s,\eta;t)| \\
=&|F(s,\xi;a_{k_s+n+1}) \cdot \exp(a_{k_s+n+1} - t) - F(s,\eta;a_{k_s+n+1}) \cdot \exp(a_{k_s+n+1} - t)| \\
=&\exp(a_{k_s+n+1} - t) \cdot |F(s,\xi;a_{k_s+n+1}) - F(s,\eta;a_{k_s+n+1})| \\
=&\exp(a_{k_s+n+1} - t) \cdot \exp(a_{k_s+n} - a_{k_s+n+1}) \cdot |F(s,\xi;a_{k_s+n}) - F(s,\eta;a_{k_s+n})| \\
=&\exp(a_{k_s+n} - t) \cdot \exp(s - a_{k_s+n}) \cdot |\xi - \eta| \\
=&|\xi - \eta| \cdot \exp(s - t).
\end{aligned}$$

In the remaining case of an odd (k_s+n+1), the proof proceeds in a very similar vein.

APPENDIX C

THE DENSITY FUNCTION ρ_0

For any $\lambda > 0$, the λ-dependent invariant density function ρ_0, introduced in the sixth chapter, is defined in the domain $]0;1[$ and is strictly positive. Other properties, such as its extremal and inflection points, do depend on the intensity parameter λ and are compiled below.

If λ lies in $]0;\frac{1}{2}[\cup]\frac{1}{2};1[$, ρ_0 does not have any extremal points and is strictly monotone decreasing in $]0;1[$. Moreover, its graph has an inflection point at $\frac{1-\lambda-\sqrt{\frac{\lambda}{2}}}{1-2\lambda}$, is convex in $]0;\frac{1-\lambda-\sqrt{\frac{\lambda}{2}}}{1-2\lambda}[$ and concave in $]\frac{1-\lambda-\sqrt{\frac{\lambda}{2}}}{1-2\lambda};1[$. With regard to asymptotics, we have

$$\lim_{x\searrow 0}\rho_0(x) = +\infty, \quad \lim_{x\nearrow 1}\rho_0(x) = 0, \quad \lim_{x\searrow 0}\rho_0'(x) = -\infty, \quad \lim_{x\nearrow 1}\rho_0'(x) = -\infty.$$

For $\lambda = \frac{1}{2}$, the density function ρ_0 is strictly monotone decreasing and devoid of extremal points. It has an inflection point at $\frac{3}{4}$, is convex in $]0;\frac{3}{4}[$ and concave in $]\frac{3}{4};1[$. Its asymptotics are exactly the same as in the case of $\lambda \in]0;\frac{1}{2}[\cup]\frac{1}{2};1[$.

If λ equals 1, $\rho_0(x) = 1 - x$ lacks any extremal points and is strictly monotone decreasing. It is an affine-linear function and therefore does not have any inflection points. Clearly, its asymptotics can be described by

$$\lim_{x\searrow 0}\rho_0(x) = 1, \quad \lim_{x\nearrow 1}\rho_0(x) = 0, \quad \lim_{x\searrow 0}\rho_0'(x) = -1, \quad \lim_{x\nearrow 1}\rho_0'(x) = -1.$$

If $\lambda \in]1;2[$, the function ρ_0 has a maximum at $\frac{\lambda-1}{2\lambda-1}$ and is strictly monotone increasing in $]0;\frac{\lambda-1}{2\lambda-1}[$ and strictly montone decreasing in $]\frac{\lambda-1}{2\lambda-1}[$. Its graph has an inflection point at $\frac{1-\lambda-\sqrt{\frac{\lambda}{2}}}{1-2\lambda}$, is concave in $]0;\frac{1-\lambda-\sqrt{\frac{\lambda}{2}}}{1-2\lambda}[$ and convex in $]\frac{1-\lambda-\sqrt{\frac{\lambda}{2}}}{1-2\lambda};1[$. In addition, we have

$$\lim_{x\searrow 0}\rho_0(x)=0,\quad \lim_{x\nearrow 1}\rho_0(x)=0,\quad \lim_{x\searrow 0}\rho_0'(x)=+\infty,\quad \lim_{x\nearrow 1}\rho_0'(x)=0.$$

If $\lambda=2$, ρ_0 attains its maximum at $\frac{1}{3}$, is strictly monotone increasing in $]0;\frac{1}{3}[$ and strictly monotone decreasing in $]\frac{1}{3};1[$. It has an inflection point at $\frac{2}{3}$, is concave in $]0;\frac{2}{3}[$ and convex in $]\frac{2}{3};1[$. As far as its asymptotics are concerned, we find that

$$\lim_{x\searrow 0}\rho_0(x)=0,\quad \lim_{x\nearrow 1}\rho_0(x)=0,\quad \lim_{x\searrow 0}\rho_0'(x)=6,\quad \lim_{x\nearrow 1}\rho_0'(x)=0.$$

Finally, for $\lambda>2$, ρ_0 has a maximum at $\frac{\lambda-1}{2\lambda-1}$, is strictly monotone increasing in $]0;\frac{\lambda-1}{2\lambda-1}[$ and strictly monotone decreasing in $]\frac{\lambda-1}{2\lambda-1};1[$. Besides, it has two inflection points, one at $\frac{-\sqrt{\frac{\lambda}{2}}-1+\lambda}{2\lambda-1}$, and the second one at $\frac{\sqrt{\frac{\lambda}{2}}-1+\lambda}{2\lambda-1}$. The graph is convex in $]0;\frac{-\sqrt{\frac{\lambda}{2}}-1+\lambda}{2\lambda-1}[\cup]\frac{\sqrt{\frac{\lambda}{2}}-1+\lambda}{2\lambda-1};1[$ and concave in $]\frac{-\sqrt{\frac{\lambda}{2}}-1+\lambda}{2\lambda-1};\frac{\sqrt{\frac{\lambda}{2}}-1+\lambda}{2\lambda-1}[$. Its asymptotics are

$$\lim_{x\searrow 0}\rho_0(x)=0,\quad \lim_{x\nearrow 1}\rho_0(x)=0,\quad \lim_{x\searrow 0}\rho_0'(x)=0,\quad \lim_{x\nearrow 1}\rho_0'(x)=0.$$

The plots in appendix E illustrate the behavior of ρ_0 for different λ.

APPENDIX D

PROOF OF LEMMA 4

From the obvious inequality chain

$$\max_{i\in\{1,\ldots,N\}} c_i^{(n)} \leq \sum_{i=1}^{N} c_i^{(n)} \leq N \cdot \max_{i\in\{1,\ldots,N\}} c_i^{(n)},$$

it follows that

$$\frac{1}{\lambda_n} \cdot \ln(\max_{i\in\{1,\ldots,N\}} c_i^{(n)}) \leq \frac{1}{\lambda_n} \cdot \ln(\sum_{i=1}^{N} c_i^{(n)}) \leq \frac{1}{\lambda_n} \cdot \ln(N) + \frac{1}{\lambda_n} \cdot \ln(\max_{i\in\{1,\ldots,N\}} c_i^{(n)})$$

for any $n \in \mathbb{N}$. Since the natural logarithm increases monotonically in its domain, the logarithm of the maximum of a finite set equals the maximum taken over the logarithms of the individual elements. Thus,

$$\max_{i\in\{1,\ldots,N\}} \frac{1}{\lambda_n} \cdot \ln(c_i^{(n)}) \leq \frac{1}{\lambda_n} \cdot \ln(\sum_{i=1}^{N} c_i^{(n)}) \leq \frac{1}{\lambda_n} \cdot \ln(N) + \max_{i\in\{1,\ldots,N\}} \frac{1}{\lambda_n} \cdot \ln(c_i^{(n)}).$$

As maximum and limes superior are interchangeable, this provides

$$\begin{aligned}
\max_{i\in\{1,\ldots,N\}} \limsup_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(c_i^{(n)}) &= \limsup_{n\to\infty} \max_{i\in\{1,\ldots,N\}} \frac{1}{\lambda_n} \cdot \ln(c_i^{(n)}) \\
&\leq \limsup_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(\sum_{i=1}^{N} c_i^{(n)}) \\
&\leq \lim_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(N) + \max_{i\in\{1,\ldots,N\}} \limsup_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(c_i^{(n)}) \\
&= 0 + \max_{i\in\{1,\ldots,N\}} \limsup_{n\to\infty} \frac{1}{\lambda_n} \cdot \ln(c_i^{(n)}).
\end{aligned}$$

This completes the proof.

APPENDIX E

TWO GRAPHS OF $\rho_0^{(\lambda)}$

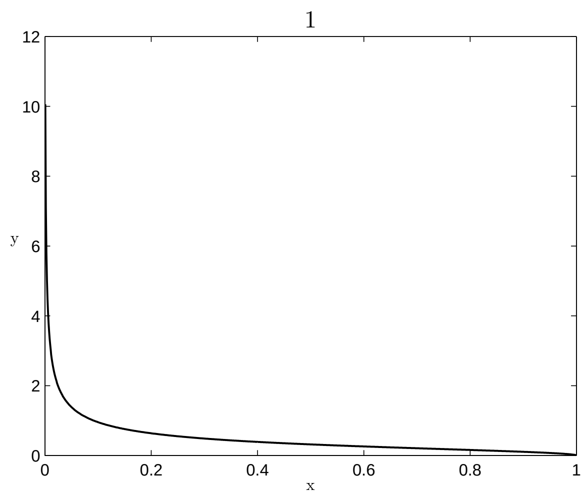

Figure 1: $\lambda = \frac{1}{2}$

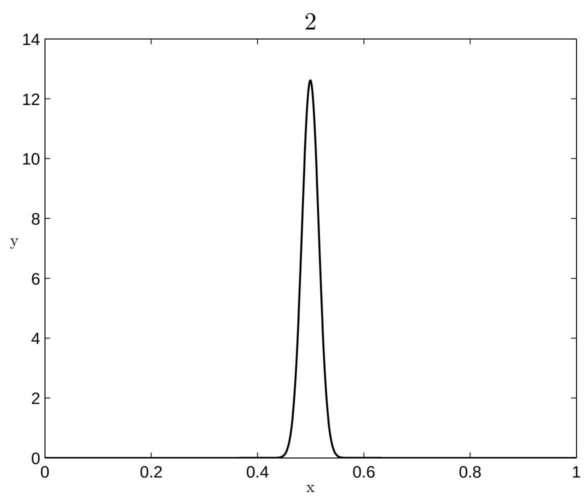

Figure 2: $\lambda = 500$

APPENDIX F

THE ENTROPY FUNCTION

The following plot showcases the graph of the entropy function I which was defined in the chapter on large deviations principle.

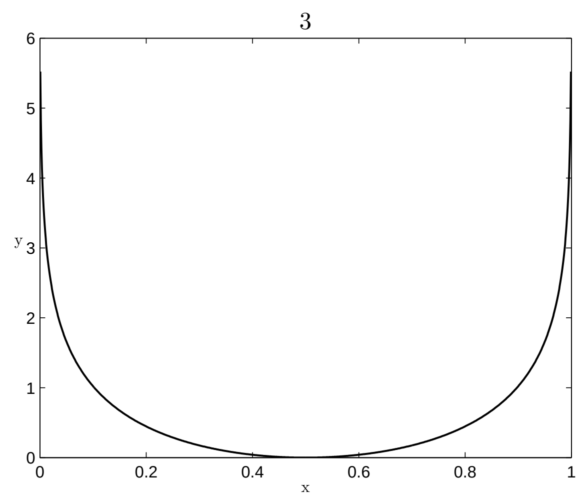

Figure 3: $I(x) = -\ln(4 \cdot x \cdot (1-x))$

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