Project No. E-25-B01 (R5886-OAO)  
Project Director: D. L. McDowell  
Sponsor: Martin Marietta Energy Systems

Type Agreement: Project Authorization X-19 under BOA 19B-07802C (OCA File #46)

Award Period: From 1/1/85 To 12/30/85 (Performance) 2/30/85 (Reports)

Sponsor Amount:
Estimated: $43,242  
Funded: $43,242

Cost Sharing Amount: $  
Cost Sharing No:

Title: Anisotropy of Creep-Fatigue Deformation and Damage under Non-Proportional Loading

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Defense Priority Rating: N/A  
Military Security Classification: N/A  
(or) Company/Industrial Proprietary: N/A

RESTRICTIONS
See Attached Supplemental Information Sheet for Additional Requirements.

Travel: Foreign travel must have prior approval — Contact OCA in each case. Domestic travel requires sponsor approval where total will exceed greater of $500 or 125% of approved proposal budget category.

Equipment: Title vests with Sponsor

COMMENTS:
This project administered under terms of Martin Marietta Basic Agreement No. 19B-07802C (OCA BOA #46)
Date: 6/30/86

Project No. E-25-B01

Includes Subproject No.(s) N/A

Project Director(s) D. L. McDowell

Sponsor Martin Marietta Energy Systems

Title Anisotropy of Creep-Fatigue Deformation and Damage under Non-proportional Loading

Effective Completion Date: 2/28/86

Grant/Contract Closeout Actions Remaining:

- [x] Final Invoice or Final Fiscal Report
- [x] Govt. Property Inventory & Related Certificate

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PROGRESS REPORT

ANISOTROPY OF CREEP-FATIGUE DEFORMATION AND DAMAGE
UNDER NONPROPORTIONAL LOADING

for the period
January 1, 1985 through February 28, 1985

D. L. McDowell
School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332
Project No. E25-B01

Attention:
R. L. Huddleston
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February 1985
OVERVIEW OF YEAR 1 OBJECTIVES

The objective of the first year of this project is to develop a tensor-based, theoretical model for damage accumulation and failure due to high-temperature creep in type 304 stainless steel. A series of creep tests is currently being performed at ORNL in which the orientation of the principal stress axes are changed prior to the onset of tertiary creep. From these tests, records of the components of the creep strain tensor and metallographic examination of the failed specimens at Georgia Tech will be explored to help validate/guide the analytical model formulation.

OUTLINE OF TASKS

The following tasks have been identified to meet program objectives. There are two parallel sets of tasks pertaining to analytical formulation and physical measurement of creep damage, respectively.

Task Set A

Task 1: Review of existing models for

a. Continuum creep damage
b. Anisotropic representations of creep damage
c. Coupled creep deformation-damage equations
   i. Decomposition of time-independent plastic and creep strains
   ii. Unified theories for which this decomposition is not made

Review of literature pertaining to:

d. Relationship of physical creep damage (cavitation, coalescence, grain boundary sliding) to primary loading directions for nonproportional histories.
e. Physical damage processes in type 304 stainless steel that must be reflected by anisotropic damage theory.

Duration: 1/1/85 through 3/30/85

Task 2: Damage Model Development

a. Development of the damage tensor $\mathbf{D}$ (appropriate rank, symmetry conditions, relationship to physical damage). If existing forms for $\mathbf{D}$ are inappropriate, more general or pertinent forms will be developed.

b. Development of the evolution equations for growth of damage $\mathbf{D}$. Equations must depend on current state of loading and material in addition to history of loading.

Duration: 4/1/85 through 6/30/85

Task 3: Development of Couplings Between Damage and Deformation Models

a. Investigation of creep strain rate equations appropriate for nonproportional loading.

b. Introduction of damage $\mathbf{D}$ into creep strain rate equation.

Duration: 7/1/85 through 8/15/85

Task 4: Demonstration of capability of model to correlate experimental
data obtained at ORNL. Guidance for further model development.

Duration: 8/16/85 through 9/30/85

Task Set B

Physical Damage Measurement

Task 1: Development of appropriate quantitative measurement techniques for creep damage in type 304 Stainless steel

Duration: 1/1/85 through 4/30/85

Task 2: Quantitative measurement of magnitude and direction of creep damage (interface with analytical development).

Duration: 1/1/85 through 9/30/85

PROGRESS TO DATE ON TASK SETS A AND B

Task Set A

A Ph.D. student in Mechanical Engineering has been hired and is currently performing the interpretative literature review stated in Task 1. The student is meeting twice a week with Dr. McDowell. The goal of this review is to objectively evaluate existing theory and to shed light on necessary modifications or development of new theory.
Task Set B

A student in Metallurgy/Chemical Engineering has been identified to perform this task set under the supervision of Dr. E. E. Underwood. Dr. Underwood is an expert in quantitative metallography, and has established an impressive laboratory with computer-assisted imaging equipment. Hence, quantitative measures of the directionality of creep damage will be possible. It should be noted that the portion of the student's stipend not supported by this contract will be provided by the Metallurgy Program directed by Dr. S. D. Antolovich.

Currently, the principal investigator is meeting with Dr. Underwood to clarify the measurements required and task schedule.

WORK TO BE PERFORMED PRIOR TO NEXT PROGRESS REPORT (4/30/85)

The review of existing models and pertinent literature (Task Set A, Task 1) should be completed by the next progress report. Also, progress will be made on the development of appropriate quantitative measurement techniques for creep damage in type 304 stainless steel (Task Set B, Task 1). The next progress report will contain more details on these initial tasks.
Martin Marietta Energy Systems

PROGRESS REPORT

ANISOTROPY OF CREEP-FATIGUE DEFORMATION AND DAMAGE
UNDER NONPROPORTIONAL LOADING

for the period
March 1, 1985 through April 30, 1985

D. L. McDowell
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Project No. E25-B01

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April 1985
OVERVIEW OF PROGRESS FOR PERIOD MARCH 1, 1985 TO APRIL 30, 1985

Referring to the outline of tasks as defined in the first progress report (see appendix), task 1 of task set A has been completed on schedule. This task involved the review of existing continuum creep damage models with an emphasis on anisotropic damage. The relationship of these models to physical creep damage processes and materials science approaches for cavity nucleation and growth has also been studied. Physical damage processes in type 304 stainless steel have also been reviewed for initial guidance in the formulation. Finally, an initial framework, based on critical review and pertinent extensions, is suggested for general anisotropic damage growth under nonproportional loading.

The literature review covers all of the major work known to the author on anisotropic continuum creep damage approaches. Emphasis in criticism of the various approaches is placed on sufficient generality, capability to model physical creep damage in a mechanically consistent way, physically reasonable specification of the dependence of creep strain rate on damage, ease of specialization and determination of material constants for specific materials, and practicality of engineering implementation and understanding. This rather extensive review and the resulting suggested initial framework for the damage analysis will be reported in the May 1985 semi-annual progress report.

R. K. Payne, a student in Metallurgy, has been assigned the responsibility of task set B (see appendix). Mr. Payne is co-advised by Drs. McDowell and Antolovich. He is currently engaged in task 1 of task set B, the development of appropriate quantitative measurement techniques for creep damage in type 304 stainless steel. The goal is to quantitatively determine the damage tensor in a statistically meaningful sense for the specimens currently being tested at ORNL. This damage tensor will be compared with that
obtained from theoretical calculations in the tertiary stage with respect to magnitude and direction. This comparison will be rather unique and should lead to specification of further tests and refinements. Measurement of the damage tensor may involve writing software necessary to implement interactive, computer-aided quantification of void or fissure density and orientation of damaged grain boundaries. To this end, the quantitative metallography lab, directed by Dr. E. E. Underwood, should be of assistance.

The complete critical literature review with suggested initial damage formulation and details on the quantitative damage measurement task will appear in the semi-annual progress report to be presented May 15, 1985. At that time, the principal investigator would like to meet with the sponsor technical contact, R. L. Huddleston, at ORNL to discuss progress on the experimental aspects of the program. Work is now underway on implementation of anisotropic damage and deformation equations in the axial-torsional subspace. Possibly, enough information is available in the literature to venture predictions of the creep rate, damage growth, and rupture time for the current set of tests on type 304 stainless steel at 593°C.

APPENDIX

Outline of Tasks

The following tasks have been identified to meet program objectives. There are two parallel sets of tasks pertaining to analytical formulation and physical measurement of creep damage, respectively.

Task Set A

Task 1: Review of existing models for
   a. Continuum creep damage
   b. Anisotropic representations of creep damage
   c. Coupled creep deformation-damage equations
i. Decomposition of time-independent plastic and creep strains
ii. Unified theories for which this decomposition is not made

Review of literature pertaining to:
d. Relationship of physical creep damage (cavitation, coalescence, grain boundary sliding) to primary loading directions for nonproportional histories.
e. Physical damage processes in type 304 stainless steel that must be reflected by anisotropic damage theory.

Duration: 1/1/85 through 3/30/85

Task 2: Damage Model Development
a. Development of the damage tensor D (appropriate rank, symmetry conditions, relationship to physical damage). If existing forms for D are inappropriate, more general or pertinent forms will be developed.
b. Development of the evolution equations for growth of damage D. Equations must depend on current state of loading and material in addition to history of loading.

Duration: 4/1/85 through 6/30/85

Task 3: Development of Couplings Between Damage and Deformation Models
a. Investigation of creep strain rate equations appropriate for nonproportional loading.
b. Introduction of damage $D$ into creep strain rate equation.

Duration: 7/1/85 through 8/15/85

Task 4: Demonstration of capability of model to correlate experimental data obtained at ORNL. Guidance for further model development.

Duration: 8/16/85 through 9/30/85

Task Set B

Physical Damage Measurement

Task 1: Development of appropriate quantitative measurement techniques for creep damage in type 304 stainless steel.

Duration: 1/1/85 through 4/30/85

Task 2: Quantitative measurement of magnitude and direction of creep damage (interface with analytical development).

Duration: 1/1/85 through 9/30/85
Martin Marietta Energy Systems

Progress Report

ANISOTROPY OF CREEP-FATIGUE DEFORMATION
AND DAMAGE UNDER NONPROPORTIONAL LOADING

for the period

May 1, 1985 through June 30, 1985

D. L. McDowell
School of Mechanical Engineering
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Project No. E25-B01

Attention

R. L. Huddleston
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P.O. Box M
Oak Ridge, Tennessee 37831
I. DEVELOPMENT OF ANISOTROPIC CREEP DAMAGE APPROACH

A preliminary form of the Kachanov-Rabotnov creep damage equations have been derived for nonproportional creep damage. The net stress tensor formulation of Murakami and Ohno [1-2] is proposed for use in the creep damage rate equation. The magnitude of the tensorial damage rate depends on the isochronous stress proposed by Huddleston [3]. A single material constant is used to apportion the relative contribution of the isotropic and anisotropic components of the symmetric, second rank damage tensor. The rupture criterion can assume several possible forms, depending on experimental results. Possible forms include:

a. damage $\rightarrow 1$ in direction of maximum principal stress.

b. scalar equivalent damage $\rightarrow 1$

c. rupture is indicated by some product of maximum principal stress and damage in that direction, modified by the magnitude of the first invariant of damage [4].

The constitutive equation for creep strain rate is not modified by any of the damage tensor components taken separately, but rather by the first invariant of damage. This formulation is in accord with experiments on perforated specimens [2] and the fact that tertiary creep is not observed at all rupture stress or damage levels.

Preliminary forms of the proposed anisotropic creep damage and deformation equations have already been reported during this performance interval in the May 15 Semi Annual Progress Report. Refer to that report for specific forms and detailed discussion.

II. PROGRESS IN IMPLEMENTATION OF DAMAGE EQUATIONS

A computer program has been written which implements the proposed anisotropic damage rate equation. Three different rupture criteria are
available, as mentioned earlier. The most suitable choice of rupture
criterion will depend on results of metallographic examination (i.e. extent of
anisotropy) in addition to observed rupture times.

The next task is to debug the program, select appropriate constants based
on the existing proportional loading test database for type 304 stainless
steel at 593°C, and integrate the damage rate equations for the experimental
histories to be run. Then the creep strain rate equations will be integrated
in coupled fashion with the damage rate equation; the change in cross-
sectional area due to large strains can then be estimated and true stress can
be used in the damage rate equation rather than engineering stress.

III. PROGRESS IN METALLOGRAPHIC EVALUATION

Metallographic analysis will commence in July 1985. As reported in the
Semi Annual Progress Report in May, the area fraction of grain boundary
cavitation and/or wedge cracking will be determined quantitatively. The data
acquisition will be largely automated, and the database generated will enable
calculation of the damage components with damage defined as the area fraction
of cracking or voids on grain boundaries associated with the outer product of
the unit normal vectors to the grain boundaries.

Mr. Payne is currently reviewing damage mechanisms reported for type 304
stainless steel reported in the literature.

SUMMARY OF PROGRESS ON TASK SETS A AND B

Task Set A, dealing with the analytical development of appropriate
anisotropic damage models, requires that damage model development be completed
by June 30, 1985. This task has essentially been completed, with the initial
damage formulation reported in the Semi Annual Progress Report (refer to
progress reports for January - April for task set definitions).
Progress on Task Set B, physical damage measurement, has to date satisfied part of Task 1. The computer programs for quantification of creep damage data need to be written. Task 1 should be completed in the next two-month performance period; progress should also be made on Task 2 of Task Set B, i.e. quantitative measurement of damage in ruptured specimens.
REFERENCES


ANISOTROPY OF CREEP-FATIGUE DEFORMATION AND DAMAGE UNDER NONPROPORTIONAL LOADING

for the period
July 1, 1985 through August 31, 1985

D. L. McDowell
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Attention:
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August 1985
Summary of Work

Tasks 1 and 2 of task set A in the Appendix have been completed. Currently, task 3 of task set A is underway; upon completion of task 3, task 4 can be performed. Task 4 deals with demonstration of the capability of the model to correlate experimental data obtained at ORNL.

Task 1 of task set B has been completed. Appropriate quantitative measurement techniques have been developed. Task 2 of task set B is underway currently. To date, the single biaxial specimen received from ORNL has been examined. Also, uniaxial specimens are currently being examined to determine the extent of creep damage transverse to the loading direction and whether the damage tensor is collinear with the loading axis.
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Outline of Tasks

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a. Continuum creep damage

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Review of literature pertaining to:

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e. Physical damage processes in type 304 stainless steel that must be reflected by anisotropic damage theory.

Duration: 1/1/85 through 3/30/85
Task 2: Damage Model Development

a. Development of the damage tensor D (appropriate rank, symmetry conditions, relationship to physical damage). If existing forms for D are inappropriate, more general or pertinent forms will be developed.

b. Development of the evolutions equations for growth of damage D. Equations must depend on current state of loading and material in addition to history of loading.

Duration: 4/1/85 through 6/30/85

Task 3: Development of Coupling Between Damage and Deformation Models

a. Investigation of creep strain rate equations appropriate for nonproportional loading.

b. Introduction of damage into creep strain rate equation.

Duration: 7/1/85 through 8/15/85

Task 4: Demonstration of capability of model to correlate experimental data obtained at ORNL. Guidance for further model development.

Duration: 8/16/85 through 9/30/85
Task Set B

Physical Damage Measurement

Task 1: Development of appropriate quantitative measurement techniques for creep damage in type 304 stainless steel.

Duration: 1/1/85 through 4/30/85

Task 2: Quantitative measurement of magnitude and direction of creep damage (interface with analytical development).

Duration: 1/1/85 through 9/30/85
PROGRESS REPORT:
ANISOTROPY OF CREEP-FATIGUE DEFORMATION AND DAMAGE UNDER
NONPROPORTIONAL LOADING

Prepared for MARTIN MARIETTA ENERGY SYSTEMS
Attn: R.L. Huddleston
Oak Ridge, TN 37831

Report for the period September 1, 1985 through October 31, 1985

October 1985

D. L. McDowell
Assistant Professor
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Summary

Significant progress has been made on the development of a rational tensorial creep damage formulation for nonproportional multiaxial loading. The theory can best be described as a continuous phenomenological representation of the direction and magnitude of the creep damage, including cavitation and wedge cracking. It is similar to continuum damage theories of Kachanov [1] and Rabotnov [2] based on net section stress and the more recent extensions due to Chaboche [3-4], Leckie [5-7], and Murakami and Ohno [8-9]. Unlike these theories, with the exception of Murakami and Ohno's, the creep strain rate tensor is assumed to depend not only on the net stress tensor, but also on the first invariant of damage. Also, the rupture criterion allows for the contribution of the various tensorial damage components, following the work of Leckie and Onat [10].

Integration of the model for proposed experimental histories to be run at ORNL has been completed. The first set of model predictions have been based on the use of Second Piola-Kirchoff (engineering) stress and strain definitions to assess the damage evolution and rupture times in a consistent manner without the complicating consideration of finite deformation which exists in the experiments. The objective here is to show how the theory can parametrically reflect evolving physical creep damage processes during complex nonproportional loading. Consideration of finite strains will be a subject of future work and should not substantially affect applicability of the damage framework.
**Discussion of Anisotropic Damage Formulation**

The motivation for the suggested anisotropic damage formulation was presented in earlier work [11]. Here, the full equations will again be presented and appropriate simplifications will be introduced for purposes of demonstration and correlation with experiments completed to date at ORNL.

The proposed general framework for the isothermal coupled damage and creep strain rate equations is:

**Damage Rate Equation:**

\[
\dot{D} = B [\sigma^*(S)]^k \left[ \eta \frac{\phi^3}{(1-\eta)} + (1-\eta) \sum_{j=1}^{3} \gamma^{(j)} \otimes \gamma^{(j)} M(\gamma^{(j)}, \phi) \right]
\]

(1)

where \( \eta, B, k, \) and \( l \) are material constants, and

\[
\gamma^{(j)} \otimes \gamma^{(j)} M(\gamma^{(j)}, \phi)
\]

or

\[
M(\gamma^{(j)}, \phi) = [1 - \gamma^{(j)} \otimes \gamma^{(j)}]^{1/2} M_j^*(\sigma_j)
\]

where \( M_j^*(\sigma_j) \) admits anisotropic contribution of non-maximal principal net stresses, and \( \sigma^*(S) \) is the isochronous stress given by

\[
\sigma^*(S) = \frac{3}{2} H^2 \left( \frac{S \sigma}{3 H} \right)^2 \exp\left(\frac{S H}{S} - 1\right)
\]

(2)
due to Huddleston [12], where

\[ S_{eq} = ((3/2) \sum' \sum')^{1/2} \]

\[ H_1 = S_1 - S_{kk}/3 \]

\[ \sum' = S - (1/3)S_{kk}I \]

\[ S_s = (S_1^2 + S_2^2 + S_3^2)^{1/2} \]

\[ S = \text{net stress tensor} = (1/2)(\sigma \cdot \phi + \phi \cdot \sigma) \]

\[ \phi = (I - P)^{-1} \]

\[ D = 2^{nd} \text{ order damage tensor} \]

\[ v_j = \text{unit vector in principal net stress direction, } S_j \]

and a and b are material constants. \( \sigma \) is the Cauchy stress.

**Rupture Criterion:**

Possible rupture criteria include

\[ D_j = D_{cr} \text{ for } j = 1, 2, \text{ or } 3 \] (3)

where \( D_j \) are the principal values of \( D \), and \( D_{cr} \) is a critical damage level, or

\[ R(\sigma_{\max}, D_{kk}, n_i D_{ij} n_j^*) = 0 \] (4)

where \( \sigma_{\max} = n_i^* \sigma_{ij} n_j^* \) is the component of stress in the maximum principal stress direction, \( n_j^* \). Here, a critical combination of maximum principal stress, total void or cavity fraction, and void density on grain boundaries normal to the maximum principal stress dictates rupture. Leckie has also proposed the criterion
\[
\text{max}_{\overline{n}} (\overline{n}^i \overline{D}^i_{ij} \overline{n}^j \cdot \overline{n}^k \overline{\sigma}^k_{lj} \overline{n}^l) = \text{constant}
\]

where \(\overline{n}\) is an arbitrary unit vector.

**Coupled Creep Strain Rate Equation:**

Suggested coupling with a rate-dependent unified creep plasticity theory is given by:

\[
\dot{\varepsilon}^n = f(\|\overline{S}\| - \alpha \|/k)(\overline{S}' - \alpha) \tag{6}
\]

\[
\dot{\alpha} = h_\alpha (\varepsilon^n : \overline{\nu}) \overline{\nu} - r_\alpha \\dot{\alpha} \tag{7}
\]

\[
\dot{\nu} = h_\nu \varepsilon^n - r_\nu \tag{8}
\]

where \(\overline{\nu}\) is a selected directional index, and

\[
\overline{S} = f_1 (D_{kk}) \overline{\sigma} + f_2 (D_{kk}) \overline{S} \tag{9}
\]

\[
\overline{\alpha} = f_1 (D_{kk}) \overline{\alpha} + f_2 (D_{kk}) \overline{\alpha} \tag{10}
\]

\[
\overline{\nu} = (1/2)(\alpha \overline{\phi} + \dot{\phi} \overline{\alpha}) \tag{11}
\]

\[
\overline{S}' = \overline{S} - (1/3) \overline{S}_{kk} \overline{I} \tag{12}
\]

Here, \(h_\alpha, h_\nu, r_\alpha, r_\nu\) are hardening and recovery functions, respectively. It is important to note that the strain rate is assumed to depend on the first invariant of damage in keeping with the findings of Leckie et al. [6] and Murakami et al. [9].
Application to Type 304 Stainless Steel at 593°C

The proposed theory in equations (1)-(12) is quite general in applicability. There exists various levels of complexity or sophistication at which one can choose to apply this theory. Compatible with the goals of the first year of this program, the equations were simplified to a form involving engineering stress in the isochronous stress equation, a simple multiaxial creep strain rate equation, and a simple rupture criterion. Lack of a sufficiently exhaustive base of multiaxial rupture tests warrants this level of sophistication. Actually, the limiting assumptions made in this section are also those made by other investigators of the creep continuum damage approach and should not be viewed as unusually restrictive. Future research in this program should increase the model accuracy and sophistication.

Briefly, the pertinent equations are:

\[
\dot{P} = B[\sigma^*(\sigma^{pk})]^k(\dot{\phi}; \dot{\phi})^{1/2}[\eta I + (1-\eta) \sum_{k=1}^{3} \gamma^{(k)} \otimes \gamma^{(k)} M^{(k)}}] \tag{13}
\]

where

\[
M^{(k)} = \gamma^{(k)} \cdot \frac{\sigma^{pk}}{\sigma_1} \cdot \gamma^{(k)}
\]

and \(<F> = F \text{ if } F \geq 0; <F> = 0 \text{ if } F < 0. \sigma^{pk} \text{ is the } 2^{nd} \text{ Piola-Kirchoff stress tensor. For purposes of brevity, it will be understood that } \sigma \text{ will be taken to represent } \sigma^{pk} \text{ in all that follows.}

The isochronous stress is defined by
\[ \sigma^* = \frac{3}{2} S \left( \frac{2 \bar{\sigma}}{3 S_1} \right)^3 \exp\left[ b \left( \frac{J}{S_s} - 1 \right) \right] \quad (14) \]

where

\[ J_1 = \sigma_1 + \sigma_2 + \sigma_3 \]
\[ \bar{\sigma} = \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2} / 2 \]
\[ S_1 = \sigma_1 - J_1 / 3 \]
\[ S_s = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} \]

and \( B, k, l, a, \) and \( b \) are material constants. Note that only positive principal stresses are permitted to contribute to damage evolution.

Also,

\[ \psi(k) = \text{unit vector in } k^{th} \text{ principal stress direction} \]
\[ \phi = (I - D)^{-1} \]
\[ \eta = \text{isotropic damage weighting factor} \quad (0 \leq \eta \leq 1) \]

A simple form of the creep strain rate equation was used to obtain an estimate of the evolution of in-plane creep strain components in the axial-torsional tests. State variables \( z \) and \( x \) (e.g. backstress and dragstress) were not included due to lack of characterization of the hardening and recovery functions; the initial scope of this investigation does not require their inclusion. Prediction of creep strain accumulation for alternating nonproportional loading would, in general, require their inclusion to reflect deformation-induced anisotropy effects.

Also, compatible with the goals of the initial analysis,
predictions and results are compared on the basis of nominal or engineering strain. The further assumption is made that creep strain rate is coupled to damage through the first invariant of damage only, i.e.

\[ \dot{\varepsilon}_n = -\frac{3}{2} \left( \frac{\sigma}{A} \right) \frac{n \sigma}{\dot{\varepsilon}} \left[ 1 + (cD_{kk})^m \right] \]  

where \( c \) and \( m \) are constants determined by fitting the secondary and tertiary creep regimes of a uniaxial creep test at fixed stress.
Determination of Material Constants

All material constants were determined from uniaxial creep tests found in the literature and by examination of ruptured uniaxial specimens. Constants \( a \) and \( b \) were given by Huddleston [12] for type 304 stainless steel at 593\(^0\)C as

\[
a = 1.086 \quad b = 0.289
\]

The exponent \( n \) was determined from a two level stress step test shown in Figure 1 [13]. By comparing the secondary creep rates (based on engineering strain) for the two nominal stress levels,

\[
n = \frac{\log(e_{ss_1}/e_{ss_2})}{\log(\sigma_1/\sigma_2)} \approx 12.5
\]  

Therefore,

\[
A = \frac{\sigma_2}{(e_{ss_2})^{1/n}} = 47.66
\]

where units of stress are ksi and strain rate in hr\(^{-1}\).

Integrating equation (13) for a uniaxial constant load creep test with axial stress \( \sigma_{11} \) leads to a rupture time \( t_R \) of

\[
t_R = \frac{1}{B(\sigma)} \int_0^D \left[ \left( 1-D_{11} \right)^2 + 2 \left( \frac{1}{1-\eta D_{11}} \right)^2 \right]^{-1/2} dD_{11}
\]

where \( D_{11} \) is the damage component in the axial direction.

If \( D_{11} = \) critical value = \( D_c \) at rupture, then rupture time
depends explicitly on stress level and equation (18) can be expressed as

\[ \log t_R = A_1 - k \log \sigma_{11} \]  

where \( A_1 \) is a constant. From Huddleston's data for uniaxial tests \[16\], \( k = 8.5551 \).

From analysis of the damaged microstructure of two uniaxial creep specimens (to be discussed later), an average value of \( \eta \) was found to be 0.61, based on wedge cracking. Hence, a value of \( \eta = 0.61 \) was selected as representative of the degree of anisotropy of damage since wedge cracking is the dominant failure mode at the temperature and stress levels of this study. Precise determination of the nonlinearity of damage evolution, reflected by the exponent \( \eta \), requires the interruption of tests at various points in the creep history along with sectioning and examination. Since interrupted test specimens are not available at this time, it was thought that values of \( B \) and \( \eta \) would be selected in the range typical of other stainless steels reported in the literature \[3-4\]. Hence,

\[ B = 1.539 \times 10^{17} \]
\[ \eta = 4.80 \]

As will be discussed shortly, the values for \( B \) and \( \eta \) were adjusted in an iteration process which involved fitting the tertiary portion of a uniaxial engineering creep strain curve.

In order to determine the coupling of strain rate with damage, a uniaxial creep curve shown in Figure 2 \[13\] was digitized and a computer...
program was written to integrate the coupled equations (15) and (18) for the uniaxial case, i.e.

\[ D_{11} = B (\sigma^*)^k \left[ \left( \frac{1}{1-D_{11}} \right)^2 + 2 \left( \frac{1}{1-\eta D_{11}} \right)^2 \right]^{1/2} \]  

(20)

\[ e_{11}^n = \left( \frac{\sigma_{11}}{A} \right)^n \left[ 1 + (cD_{kk})^m \right] \]  

(21)

A flowchart of the computer program used to fit the uniaxial creep curve appears in Figure 3. In the first iterative loop, constants \( B \) and \( \eta \) are adjusted until the value of the axial damage component \( D_{11} \) at rupture is approximately unity. Then, the values of \( c \) and \( m \) in equation (21) are determined iteratively by plotting the resulting predicted tertiary creep response as an overlay on the digitized experimental data. The values of \( B \) and \( \eta \) can be adjusted slightly to best match the tertiary region, with the stipulation that the axial damage component at rupture should essentially be unity. It is important to realize that the area fraction of damaged grain boundaries at rupture perpendicular to the loading axis in a uniaxial creep test is usually significantly less than one [14], but the current formulation does not require that the area fraction be identically equal to the damage parameter. It is required, however, that the predicted damage components are of the same ratio as those experimentally determined. For the purpose of comparing tests conducted at the same isochronous stress level, specifying that \( D_{11} = 1 \) at rupture is sufficient. For variable stress histories, though, one would like to employ the full
capability of the model by relating area fractions identically to components of damage.

A plot of the predicted versus experimental creep curve for the values of the "best-fit" parameters is shown in Figure 4. Note that $c = 1.35$ and $m = 2.25$ provided the most accurate fit of the creep response for the assumed values of $B$ and $l$. It must again be emphasized that a more rigorous determination of $B$ and $l$ would require interrupted testing; however, the stress-independent rupture criterion and damage-independent isochronous stress of this initial formulation should provide rupture times consistent with the isochronous stress concept regardless of the $B$ and $l$ values for constant load, proportionally loaded creep tests. Accurate values of $B$ and $l$ are particularly important for step stress tests or for nonproportional loading. Interrupted testing will be pursued in years 2-3 of this work.

The rupture criterion employed in this demonstration was

$$v^{(1)}_\sim \cdot D \cdot v^{(1)}_\sim = 1$$

(22)

where $i = 1, 2, 3$. This is equivalent to the criterion stated in equation (3) with $D_{cr} = 1$. Though the area fraction of grain boundaries in the continuum sense in any direction is generally not unity at rupture, the high nonlinearity of terms involving damage in the damage rate equation results in very little difference in predicted rupture time for critical maximum principal damage values between about 0.5 and 1.
In order to determine the appropriate value of \( \eta \) for type 304 stainless steel at 593°C and to compare analytical predictions with observed physical damage, biaxial creep specimens were sectioned, polished, and etched. Then, micrographs were taken at various locations to obtain a sample distribution rather than a single micrograph. Since the rupture surface is viewed primarily as a fracture phenomenon resulting from linkage of voids and wedge cracks into an unstably propagating crack, all samples were taken from points away from the rupture crack but still in the zone of uniform temperature and deformation. A discolored region was observed in the middle third of the 2.43 inch gage section, evidently associated with localization of deformation; all micrographs were taken in this region.

Since both wedge cracking and cavitation contribute to grain boundary damage, it was initially desired to include both in constructing the damage tensor from micrographs. Due to the different nature of each type of damage, however, the wedge cracking and cavitation components of the damage tensor were computed separately, i.e.

\[
\tilde{D}_{\text{exp}} = \tilde{D}_w + \tilde{D}_c
\]  

(23)

where \( \tilde{D}_{\text{exp}} \) represents the total damage tensor measured experimentally. In general, the cavitation damage is more difficult to quantify than the wedge cracking. Lack of resolution in the micrographs, even at 1000X,
made it difficult to assign an area fraction value quantitatively to a cavitated grain boundary. Furthermore, due to the relatively large deformations encountered at rupture in type 304 stainless steel, cavities are smeared due to grain boundary sliding and elongation. This elongation of grains in the primary stretch direction also created a preferred orientation for grain boundary segments which skewed the calculation of the cavitation damage tensor from micrographs, since cavitation was observed almost uniformly on all grain boundaries. In contrast, wedge cracking was much more readily quantifiable.

It is desirable to obtain a representative sample of grains for quantitative damage measurement to ensure reliable results. To include a sufficient sample size of grain boundaries, a magnification of 200X was used for determining the wedge crack damage tensor. A magnification of 1000X was used to determine the cavitation damage tensor to improve the resolution of grain boundaries. As previously mentioned, determination of the cavitation damage tensor was fraught with problems; perhaps the most serious reservation is that the cavitation observed or measured in the ruptured specimen is not representative of the evolution of cavitation and eventual linkage of cavities due to the smearing effects. It is our current thinking that quantification of the wedge cracking will most successfully describe the rupture state and its link to damage history.

Sections were taken at two locations each at the inside and outside specimen diameters. At each section, five different locations were photographed to provide a suitable sample lot. For the uniaxial specimens, the damage was evaluated at the specimen centerline at five
locations. Typical micrographs for the wedge crack and cavity distributions appear in Figure 5 for the single biaxial specimen evaluated to date. It is interesting to note that a radial gradient of damage is clearly observed in micrographs of the entire specimen wall with a heavier distribution of wedge cracking at the specimen outer diameter. This effect is possibly due in part to the higher shear strain accumulated at the outer diameter than at the inner. Figure 6 illustrates the radial damage gradient.

Computer programs were written for a Weiss Videoplan (available in Dr. Underwood's quantitative metallography laboratory) to allow the user to move the cursor along grain boundaries and to mark wedge cracks or cavitated segments. The results were immediately digitized on floppy disk. Post-processing programs were written to convert the raw data in the wedge crack or cavity files to print to any output file the grain boundary segment length, fraction wedge cracked or cavitated, and the normal vector to the segment. All grain boundary segments in a given micrograph were digitized, regardless of whether or not any damage was present. Hence, total grain boundary length for each micrograph is available.

The processed data files were stored on floppy disks formatted with MSDOS. BASIC computer programs were written to perform the computation of wedge crack and cavitation damage tensors on an IBM PC based on the numerical integration implied by the equations

$$D_w = \frac{1}{L_T} \sum_{k=1}^{N} n^{(k)} \otimes n^{(k)} \Delta S_{gw}^{(k)}$$ (24)
where $L_T$ is the total grain boundary length in the micrograph, $\mathbf{n}^{(k)}$ is the unit normal vector to the $k$th grain boundary segment, and $\Delta S_{gw}^{(k)}$ and $\Delta S_{gc}^{(k)}$ are, respectively, the length of the wedge crack or cavitated segment associated with the $k$th grain boundary segment.

Copies of the two computer programs written to compute the damage tensors appear in the APPENDIX.
Prediction of Damage Evolution and Comparisons with Data

In the present continuum damage approach, no distinction is made between cavitation damage or wedge crack damage. The damage tensor simply reflects area fraction of both types of damage. A computer program was written to integrate the coupled equations (20)-(21). The numerical integration technique used was a Runge-Kutta with fixed time step size. This method was found to result in very efficient, accurate integration of unified creep-plasticity theory in an earlier study [15] for nonproportional cyclic loading. A flowchart of the computer program appears in Figure 7.

Since the damage evolution depends on \( \eta \), it was necessary to establish a value of \( \eta \) from micrographs of uniaxial creep tests. Two specimens were provided by ORNL for this purpose. The value of \( \eta \) was based on quantification of wedge type damage only; this should not be too restrictive since this type of damage most likely dominates the rupture process in the stress and temperature regimes of the tests. Referring to equation (20), it is clear that the transverse damage \( D_{22} \) is related to the axial damage \( D_{11} \) by \( D_{22} = \eta D_{11} \). Hence, \( \eta \) is the ratio of the transverse to axial components of damage in a uniaxial creep test. A value of \( \eta = 0.31 \) was determined as the average value of \( \eta \) computed from ten micrographs, five from each specimen. It should be noted that this value of \( \eta \) indicates that the isotropic component of the damage tensor is "larger" than the anisotropic component. Hence, we would not expect a change in loading direction at the same isochronous stress level to result in a factor of two difference in rupture life as
in pure copper.

The computer program was written to allow any nominal stress history to be input. Output includes rupture time and plots of damage components $D_{11}$ and $D_{12}$ in addition to engineering strain components $e_{11}$ and $e_{12}$.

Since only the first biaxial specimen (GT-1) was evaluated at the time of this writing, the comparison of predicted and experimental rupture times and engineering creep strain components at rupture are:

<table>
<thead>
<tr>
<th>Predicted</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{11} = 0.77$</td>
<td>$e_{11} = 0.19$</td>
</tr>
<tr>
<td>$e_{12} = 0.38$</td>
<td>$e_{12} = 0.11$</td>
</tr>
<tr>
<td>$t_R = 1030$ hrs</td>
<td>$t_R = 892$ hrs</td>
</tr>
</tbody>
</table>

The evolution of damage components and creep strain components through the history is shown in Figures 8-9. The results of the wedge crack damage tensor components for the predicted and experimental cases at rupture are:

<table>
<thead>
<tr>
<th>Predicted</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{11} = 0.89$</td>
<td>$D_{11} = 0.31$</td>
</tr>
<tr>
<td>$D_{12} = 0.12$</td>
<td>$D_{12} = 0.037$</td>
</tr>
<tr>
<td>$\theta = 17^\circ$</td>
<td>$\theta = 23^\circ$</td>
</tr>
</tbody>
</table>

where $\theta$ is the angle, measured counter-clockwise at the specimen outer surface, from the longitudinal specimen axis to the maximum principal damage direction.
It should be noted that the experimental values of $e_{11}$ and $e_{12}$ were computed using the equations

$$e_{11} = \frac{\delta L}{L_o}$$  \hspace{1cm} (26)

$$e_{12} = \frac{1}{2} \bar{r} \delta \psi / L_o$$  \hspace{1cm} (27)

where $L_o$ is the gage length, $\bar{r}$ is the average radius, and $\delta \psi$ is the relative rotation of the gage section in radians. Obviously, engineering strain is not indicative of the true strains occurring at such large deformations. It is not entirely unreasonable, though, to compare the engineering strains predicted by the model with those obtained experimentally. In particular, the ratio of creep strain predicted in the tertiary regime to that which would be computed by using only a secondary creep model is of much interest. These ratios are

\begin{align*}
\text{Predicted} & \hspace{1cm} \text{Experimental} \\
\frac{e_{11}}{e_{11}^{ss}} &= 1.45 & \frac{e_{11}}{e_{11}^{ss}} &= 1.34 \\
\frac{e_{12}}{e_{12}^{ss}} &= 1.39 & \frac{e_{12}}{e_{12}^{ss}} &= 1.36
\end{align*}

where the $ss$ superscript denotes secondary creep strain projected to failure. Instantaneous plastic strain and primary creep strain are neglected in the above ratios. Note that the predicted and experimental ratios are relatively consistent, indicating that the model provides a good prediction of the tertiary stage. The models of Chaboche and Leckie, which employ equivalence of the damaged and undamaged material in
the sense of elastic strain, would predict the above ratios to be infinite at rupture in contrast. The reason for the rather large discrepancy between predicted and measured magnitudes of creep strain components will be discussed shortly.

Another very revealing ratio is that of the wedge crack damage components at rupture:

<table>
<thead>
<tr>
<th>Predicted</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{11}/D_{12} = 7.4$</td>
<td>$D_{11}/D_{12} = 8.4$</td>
</tr>
</tbody>
</table>

Note that the ratios are comparable. Also, note that the predicted and experimental values of $\theta$ are relatively close, with differences probably attributable to the rotation of material elements occurring for this finite deformation case. It is quite interesting to note that the angle $\theta$ determined for wedge cracking in the uniaxial case is only $-8.3^\circ$, indicating that the maximum principal damage value is more nearly collinear with the loading axis than for the proportionally loaded biaxial specimen GT-1. Since only two uniaxial specimens were sectioned and examined, it is likely that the angle of $8.3^\circ$ is not of high confidence level. Accuracy of the angle is more highly dependent on sample size than is accuracy of $\eta$ since orientation is particularly important for calculation of the shear component of damage. More samples will be digitized and evaluated from the uniaxial specimens to gain a higher confidence level in the principal damage orientation; at this point, it appears that the uniaxial principal damage orientation is significantly different from the biaxial case with alignment with the principal stress axes observed in each case.
Discussion

Several remarks are in order regarding the analysis and comparison of predicted and experimental results.

It is interesting to compare the value of $\eta$ determined from biaxial test GT-1 with that obtained from the uniaxial tests. The ratio of the principal wedge cracking values (computed from the average of 20 digitized micrographs) of specimen GT-1 is $\eta = D_2/D_1 = 0.69$ which is approximately equal to the value of $\eta = 0.61$ found from analysis of the uniaxial specimens. This result and the agreement of 0 between theory and experiment tends to confirm the fundamental structure of the analytical formulation.

It should also be noted that the critical value of damage defining rupture, $D_{cr}$ in equation (3), could be defined as the axial component of wedge cracking observed in a uniaxial test. In this case, the predicted magnitudes of damage components at rupture in the biaxial case would more nearly reflect the experimental values. Further pursuit of this particular point is not seen as essential in the current study since the two uniaxial specimens used to determine $\eta$ were from the same heat, but were tested at a higher temperature ($663^\circ$C) and lower isochronous stress level (11 ksi) than the biaxial specimen.

The largest discrepancy was observed between the predicted and experimental engineering creep strains and strain rates. Referring to Figure 10 [13], it is clear that there are significant differences between annealed and annealed and aged uniaxial specimens in terms of minimum creep rates at the same stress levels. Figures 1 and 2 show the
creep behavior of annealed and aged specimens. Since $A$ and $n$ in equation (21) were determined for the annealed and aged creep response, it is quite likely that the computed value of $n$ exceeds the actual value for the annealed biaxial specimens. Again referring to Figure 10, a factor of approximately four difference is observed at 23 ksi between annealed and annealed and aged specimens. This is also the factor which exists between predicted and experimental creep strain rates for biaxial specimen GT-1. Therefore, the secondary creep equation in Figure 10 will be applied in future biaxial correlations at $593^\circ$C, i.e.

$$
\dot{\varepsilon}_{11}^n = \left( \frac{\sigma_{11}}{60.0} \right)^{10.74} \tag{28}
$$

where the unit of strain rate is hr$^{-1}$ and stress is in ksi.

Future model development is required in several areas. First, multiple load level experiments must be conducted to determine if exponent $l$ is a function of stress level, or if a stress dependent rupture criterion can model sequence effects. Interrupted tests must be run to more accurately determine constants $B$ and $l$ and to afford the possibility to quantify cavitation damage in relatively early stages in life. By examining all specimens at the highly deformed rupture state, there is the possibility that the observed orientation of cracking is due primarily to grain boundary sliding and accommodation of the deformation rather than the history of the applied principal stress orientation. Examination of nonproportionally loaded specimens should help clarify this issue.
REFERENCES


Fig. 1 Two level stress step test from 23.0 ksi to 21.0 ksi at 593°C [13].
Fig. 2  Constant load uniaxial creep curve at 593°C and 23.0 ksi [13].
Flowchart for fitting uniaxial creep curve

Begin

Input experimental data

Plot experimental creep curve on monitor

Input n
Input B, k, l

Compute $D_{11}, e_{11}$

Plot creep strain vs. time

Is $D_{11} = D_{cr}$?

yes

no

Adjust B, l

Adjust c, m

Is curve fit?

no

yes

Plot creep strain versus time

Compute $D_{11}, e_{11}$

Input c, m

Stop

Fig. 3 Flowchart of computer program used to iteratively fit the uniaxial creep curve.
damage(49) = .9850499

\[ \begin{align*}
\text{change} & = ? \text{ ?} \\
b & = 1.539 \times 10^{-17} \\
l & = 4.8 \\
\alpha & = .61 \\
c & = 1.35 \\
M & = 2.25 \\
.d & = 440855
\end{align*} \]

**Fig. 4** Overlay of experimental creep curve and output of model.
Fig. 5  Typical micrographs for determination of wedge crack (top) and cavitation (bottom) damage tensors.
Fig. 6  Specimen transverse cross-section, illustrating radial damage gradient.
Read Data
a, b, B, k, l,
c, m, n, σ_{ij}|t=0, n

Write
initial values

Compute
σ_{ij}, σ^*, \dot{\phi}, \dot{\psi}(i)
M(i), S_1, J_1, S_s

Compute
D_{ij}(t + \Delta t)
e_{ij}(t + \Delta t)
viā Runge Kutta

Write
time, D_{ij}, e_{ij}

Is
\max_j \sigma_{ij}^* > 1
?

no

yes

Stop

Fig. 7 Flowchart of computer program used to integrate coupled deformation-damage equations for the biaxial case in this study.
Fig. 8 Predicted damage and engineering creep strain component evolution for specimen GT-1. The applied nominal stresses are $\sigma_{11} = 22.73$ ksi and $\sigma_{12} = 7.577$ ksi.
Fig. 9 Experimental curves for axial displacement of crosshead versus time (top) and relative angular rotation (bottom) for specimen GT-1. The applied nominal stresses are $\sigma_{11} = 22.73$ ksi and $\sigma_{12} = 7.58$ ksi, where the shear stress is computed as uniform across the wall thickness.

- 32 -
CORRELATION FOR ANNEALED 48-mm (1 7/8-in.) BAR FOR CONSTANT LOAD TESTING:

\[ \dot{\epsilon} \text{ (\%h)} = [\sigma \text{ (MPa)/269.8}]^{0.14} \]

B1-4 STEP LOAD TESTS, CREEP RATE AT FIRST LOAD LEVEL (158.5 MPa (23 ksi)); AVERAGE WITH 6 DATA POINTS FALLING IN RANGE SHOWN

A1-3 BASELINE CONSTANT LOAD TESTS AT 144.8 MPa (21 ksi); AVERAGE WITH 6 DATA POINTS FALLING IN RANGE SHOWN

A1-3 BASELINE CONSTANT LOAD TESTS AT 144.8 MPa (21 ksi); CREEP RATE AT SECOND LOAD LEVEL [144.8 MPa (21 ksi)]; AVERAGE WITH 6 DATA POINTS FALLING IN RANGE SHOWN

Fig. 10 Comparison of minimum creep rates at 593°C for several types of uniaxial tests [13].
APPENDIX

LISTING OF COMPUTER PROGRAMS

I. Wedge crack damage tensor analysis program.
II. Cavitation damage tensor analysis program.
III. Coupled deformation-damage integration program.
this program computes the 2nd order wedge crack damage tensor.
input file is generated by program wedge.for.

PRINT "input name of wedge crack data file:"; INPUT A$

'***** definitions *****
gblen = total grain boundary length
bl = segment boundary length
bn1 = x dir. normal vector comp. to segment boundary
bn2 = y dir. normal vector comp. to segment boundary

OPEN A# FOR INPUT AS #1
GBLEN=0 : ' initialize grain boundary length = 0
INPUT #1,BL
PRINT BL
IF BL=0 THEN 72
GBLEN=GBLEN+BL
INPUT BN1,BN2
PRINT BN1,BN2
TNORM=SOR(BN1^2+BN2^2):BN1=BN1/TNORM:BN2=BN2/TNORM
GOTO 50
PRINT "grain boundary length =";GBLEN
PRINT "press any key to continue:";INPUT H$
INPUT #1,BL:PRINT BL
D11=0!: D22=0! : D12=0!

'input wedge crack information
'and compute wedge crack damage tensor.

wlen = wedge crack length
wn1= x dir. normal vector comp. to wedge crack
wn2= y dir. normal vector comp. to wedge crack
dwil = damage component 11
dw22 = damage component 22
dw12 = damage component 12

INPUT #1,WLEN
INPUT #1,WN1,WN2
PRINT WN1,WN2
5 TNORM=SOR(WN1^2+WN2^2):WN1=WN1/TNORM:WN2=WN2/TNORM
T11=WN1^2 : T22=WN2^2 : T12=WN1*WN2
T11=T11*WLEN : T22=T22*WLEN : T12=T12*WLEN
D11=D11+T11 : D22=D22+T22 : D12=D12+T12
DW11=(1!/GBLEN)*D11 : DW22=(1!/GBLEN)*D22 : DW12=(1!/GBLEN)*D12
PRINT DW11,DW22,DW12
7 IF WLEN <> -9999 THEN 90
D1=(DW11+DW22)/2!+SOR((DW11-DW22)^2/4+DW12^2/4)
D3=(DW11+DW22)/2!-SOR((DW11-DW22)^2/4+DW12^2/4)
PRINT "d1=",D1,"d3=",D3
5 6 7
CLOSE #1
END
this program computes the 2nd order cavitation damage tensor.
input file is generated by program cavity.for.
PRINT "input name of cavitation data file:": INPUT A$

'***** definitions *****
'gblen = total grain boundary length
'bl = cavitated segment boundary length
'bn1 = x dir. normal vector comp. to segment boundary
'bn2 = y dir. normal vector comp. to segment boundary
'fca = fraction of segment boundary cavitated.
nseg = total number of cavitated segments

GBLEN=0
OPEN A$ FOR INPUT AS #1
D11=0 : D12=0 : D22=0!
INPUT #1,NSEG
   input cavity information and compute
cavity damage tensor.
FOR I=1 TO NSEG
INPUT #1, BL
GBLEN=GBLEN+BL
INPUT #1, BN1, BN2
TNORM=SOR(BN1^2+BN2^2) : BN1=BN1/TNORM : BN2=BN2/TNORM
INPUT #1, FCA
D11=D11+FCA*BL*BN1^2 : D22=D22+FCA*BL*BN2^2 : D12=D12+FCA*BL*BN1*BN2
DC11=(1/GBLEN)*D11 : DC22=(1/GBLEN)*D22 : DC12=(1/GBLEN)*D12
PRINT DC11, DC22, DC12
   dc11 = damage component 11
   dc22 = damage component 22
   dc12 = damage component 12
NEXT I
END
PROGRAM START(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
REAL KAPPA,LAMBDA,M,NU,INEDAM
COMMON /HOKY1/ALPHA,BETA,B,KAPPA,LAMBDA,ETA,SIGMA(3,3)
COMMON /HOKY2/DAMAGE(3,3),DAMRAT(3,3),NU(3,3,3),SSTAR
COMMON /HOKY3/INEDAM(3,3),FI(3,3)
COMMON /HOKY4/SEI(3),SP(3),SIGBAR,SIFRIM(3,3)
COMMON /HOKY5/EIVEC(3,3),TEN(3,3),VEC(3),M(3)
COMMON /HOKY6/ERROR,MITER
COMMON /HOKY7/STRAIN(3,3),STRATE(3,3),DKK
COMMON /HOKY8/ACONST,CTIMES,POWERM,POWERN
DATA ALPHA,BETA/1.0859,0.2893/
DATA B,KAPPA,LAMBDA/1.44021E-17,8.5551,4.8/
DATA ACONST,CTIMES,POWERM,POWERN/47.66,1.49,2.9,12.5/
SIGMA(1,1)=0.0
SIGMA(1,2)=19.45
SIGMA(1,3)=0.0
SIGMA(2,1)=19.45
SIGMA(2,2)=0.0
SIGMA(2,3)=0.0
SIGMA(3,1)=0.0
SIGMA(3,2)=0.0
SIGMA(3,3)=0.0
SIGM=(SIGMA(1,1)+SIGMA(2,2)+SIGMA(3,3))/3.
DO 243 IU=1,3
DO 243 IL=1,3
SIFRIM(IU,IL)=SIGMA(IU,IL)-SIGM
IF(IU.NE.IL) SIFRIM(IU,IL)=SIGMA(IU,IL)
243 CONTINUE
ETA=.65
INTEVAL=49
ERROR=1.E-5
MITER=20
DELTIME=10
C HEADING-INPUT-DATA
WRITE(6,30)
30 FORMAT(10X,"SIGMA")
   DO 50 I=1,3
50   WRITE(6,100) (SIGMA(I,J),J=1,3)
100  FORMAT(3(2X,E15.7,5X))
WRITE(6,110) ALPHA,BETA
WRITE(6,120) B,KAPPA,LAMBDA
120  FORMAT(5X,E15.7,5X,F14.7,5X,F14.7)
WRITE(6,205) ETA
205  FORMAT(2X," ETA = ",F5.2)
C INITIALIZ E
666  DO 10 I=1,3
10   DO 10 J=1,3
10   DAMAGE(I,J)=0.0
   CALL STPNU
C C
   I=1
   TIME=(I-1)*DELTIME
467  WRITE(6,200) TIME
200  FORMAT(5X,"TIME = ",F10.5)
   DO 40 LLLL=1,3
40   WRITE(6,210) (DAMAGE(LLLL,JJ),JJ=1,3)
      - 37 -
DO 42 IN=1,3
   WRITE(6,210) (STRAIN(IN,IM),IM=1,3)
210   FORMAT(3(3X,E15.7))
IF(MOD(I-1,INTERVAL).EQ.0) GO TO 700
750   DKK=DAMAGE(1,1)+DAMAGE(2,2)+DAMAGE(3,3)
   CALL RUNGE(1,DELTIME)
   CALL BIJAIJ(EIVEC,DAMAGE,VALUE)
   CALL RUNGE2(1,DELTIME,1)
   IF(VALUE.GE.1.) GO TO 999.
   I=I+1
   TIME=(I-1)*DELTIME
   GO TO 467
700   SIGMA(1,2)=-SIGMA(1,2)
   SIGMA(2,1)=-SIGMA(2,1)
   SIGM=(SIGMA(1,1)+SIGMA(2,2)+SIGMA(3,3))/3.
   DO 710 IU=1,3
   DO 710 IL=1,3
   SIFRIM(IU,IL)=SIGMA(IU,IL)-SIGM
   IF(IU.NE.IL) SIFRIM(IU,IL)=SIGMA(IU,IL)
   CONTINUE
710   CALL STPNU
   GO TO 750
999   CONTINUE
333   CONTINUE
STOP
END
C
SUBROUTINE RUNGE-KUTTA
SUBROUTINE RUNGE(IN,H)
REAL KAPPA,LAMBDA,NU,INEDAM,M
DIMENSION ERRY(3,3)
DIMENSION A(4),B(4),C(4),D(4),CDX(4)
COMMON /HOKY1/ALPHA,BETA,BI,KAPPA,LAMBDA,ETA,SIGMA(3,3)
COMMON /HOKY2/Y(3,3),YPRIME(3,3),NU(3,3,3),SSTAR
COMMON /HOKY3/INEDAM(3,3),FI(3,3)
COMMON /HOKY4/SEI(3),SP(3),SIGBAR,SIFRIM(3,3)
COMMON /HOKY5/EIVEC(3,3),TEN(3,3),VEC(3),M(3)
COMMON /HOKY6/ERROR,MITER
COMMON /HOKY7/STRAIN(3,3),STRATE(3,3),DKK
COMMON /HOKY8/ACONST,CTIMES,POWERM,POWERN
IF(IN.NE.1) GO TO 2100
IN=0
CDX(1)=0.0
A(1)=0.5
B(1)=2.0
C(1)=1.5
D(1)=0.5
CDX(2)=0.5
A(2)=1.-SQRT(0.5)
B(2)=1.0
C(2)=3*(1.-SQRT(0.5))
D(2)=1.-SQRT(0.5)
CDX(3)=0.0
A(3)=1.+SQRT(0.5)
B(3)=1.0
C(3)=3*(1.+SQRT(0.5))
D(3)=1.+SQRT(0.5)
CDX(4)=0.5
A(4)=1./6
B(4)=2.0
C(4)=0.5
D(4)=0.5
DO 2000 I=1,3
DO 2000 II=1,3
2000 CONTINUE
EROY(I,II)=0.0
2100 CONTINUE
DO 2300 I=1,4
CALL DDOT2
   DO 2200 J=1,3
   DO 2200 JJ=1,3
      ERRY(J,JJ)=Y(J,JJ)+A(I)*ERRY(J,JJ)
      Y(J,JJ)=Y(J,JJ)+A(I)*ERRY(J,JJ)
      ERRY(J,JJ)=ERRY(J,JJ)+C(I)*ERRY(J,JJ)+D(I)*H*YPRIME(J,JJ)
   2200 CONTINUE
2300 CONTINUE
RETURN
END

SUBROUTINE RUNGE2(IN,H,II)
REAL KAPPA,LAMBDA,NU,INEDAM
DIMENSION ERRY(3,3)
DIMENSION A(4),B(4),C(4),D(4),CDX(4)
COMMON /HOKY1/ALPHA,BETA,BI,KAPPA,LAMBDA,ETA,SIGMA(3,3)
COMMON /HOKY2/DAMAGE(3,3),DAMRATE(3,3),NU(3,3,3),SSTAR
COMMON /HOKY3/INEDAM(3,3),FI(3,3)
COMMON /HOKY4/SEI(3,3),SP(3),SIGBAR,SIFRIM(3,3)
COMMON /HOKY5/EIVEC(3,3),TEN(3,3),VEC(3),M(3)
COMMON /HOKY6/ERROR,MITER
COMMON /HOKY7/Y(3,3),YPRIME(3,3),DKK
COMMON /HOKY8/ACONST,CTIMES,POWERM,POWERN
CDX(1)=0.0
A(1)=0.5
B(1)=2.0
C(1)=1.5
D(1)=0.5
CDX(2)=0.5
A(2)=1.0-SQRT(0.5)
B(2)=1.0
C(2)=3*(1.-SQRT(0.5))
D(2)=1.-SQRT(0.5)
CDX(3)=0.0
A(3)=1.+SQRT(0.5)
B(3)=1.0
C(3)=3*(1.+SQRT(0.5))
D(3)=1.+SQRT(0.5)
CDX(4)=0.5
A(4)=1./6
B(4)=2.0
C(4)=0.5
D(4)=0.5
DO 2000 I=1,3
DO 2000 II=1,3
2000 CONTINUE
DO 2300 I=1,4
CALL EDOT2(IJ)
   DO 2200 J=1,3
   DO 2200 JJ=1,3
      ERRY(J,JJ)=Y(J,JJ)+A(I)*ERRY(J,JJ)
      Y(J,JJ)=Y(J,JJ)+A(I)*ERRY(J,JJ)
      ERRY(J,JJ)=ERRY(J,JJ)+C(I)*ERRY(J,JJ)+D(I)*H*YPRIME(J,JJ)
   2200 CONTINUE
2300 CONTINUE
RETURN
END
SUBROUTINE STRAIN-RATE
SUBROUTINE EDOT2(IJ)
REAL KAPPA, LAMBDA, M, NU, INEDAM
COMMON /HOKY1/ALPHA, BETA, B, KAPPA, LAMBDA, ETA, SIGMA(3,3)
COMMON /HOKY2/DAMAGE(3,3), DAMRATE(3,3), NU(3,3,3), SSTAR
COMMON /HOKY3/INEDAM(3,3), FI(3,3)
COMMON /HOKY4/SEI(3), SP(3), SIGBAR, SIFRIM(3,3)
COMMON /HOKY5/EIVEC(3,3), TEN(3,3), VEC(3), M(3)
COMMON /HOKY6/ERROR, MITER
COMMON /HOKY7/STRAIN(3,3), STRATE(3,3), DKK
COMMON /HOKY8/ACONST, CTIMES, POWERM, POWERN
DUM1=(SIGBAR/ACONST)**POWERN
DUM2=(1.+(CTIMES*DKK)**POWERM)
DO 10 I=1,3
DO 10 J=1,3
STRATE(I,J)=1.5*DUM1*DUM2*(SIFRIM(I,J)/SIGBAR)
10 CONTINUE
RETURN
END

SUBROUTINE FIFI(A,B,C)
DIMENSION A(3,3)

SUBROUTINE SUBSTITION
SUBROUTINE SUBN(A,B,N)
DIMENSION A(3,3), B(3,3)

SUBROUTINE MULTIPLICATION
SUBROUTINE MPLY(A,B,C,N)
DIMENSION A(3,3), B(3,3), C(3,3)
DO 10 K=1,N
   C(I,J)=C(I,J)+A(I,K)*B(K,J)
RETURN
END

SUBROUTINE IDENTITY - B
SUBROUTINE INEGAB(B,IMB)
REAL IMB(3,3)
DIMENSION B(3,3)

C VARIABLES
C B DAMAGE TENSOR (3,3)
C IMB I-B (3,3)

DO 10 I=1,3
DO 10 J=1,3
IF(J.EQ.I) THEN
   IMB(I,J)=1.-B(I,J)
ELSE
   IMB(I,J)=-B(I,J)
ENDIF
10 CONTINUE
RETURN
END

SUBROUTINE INVERSE
SUBROUTINE INVS(H,HINVS,N)
DIMENSION H(3,3),HINVS(3,3),A(3,3),B(3,3)

C VARIABLES
C H ORIGINAL MATRIX (N,N)
C HINVS INVERSED MATRIX (N,N)
C N DIMENSION
C REQUIRED SUBROUTINES
C 1) SUBN
C 2) MPLY
C CHECK IDENTITY MATRIX
IF(N.EQ.3) GO TO 300
   IF(H(1,1).NE.H(2,2)) GO TO 450
   IF(H(1,2).NE.H(2,1)) GO TO 450
   IF(H(1,2).NE.0.0 ) GO TO 450
   DO 210 I=1,N
   DO 210 J=1,N
      210 HINVS(I,J)=H(I,J)/(H(1,1)*H(2,2))
   GO TO 999
300 IF(H(1,1).NE.H(2,2)) GO TO 450
   IF(H(1,1).NE.H(3,3)) GO TO 450
   IF(H(1,2).NE.H(1,3)) GO TO 450
   IF(H(2,1).NE.H(2,3)) GO TO 450
   IF(H(3,1).NE.H(3,2)) GO TO 450
   IF(H(1,2).NE.H(2,1)) GO TO 450
   IF(H(1,2).NE.H(3,1)) GO TO 450
   IF(H(1,2).NE.0.0 ) GO TO 450
   DO 310 I=1,N
   DO 310 J=1,N
      310 HINVS(I,J)=H(I,J)/(H(1,1)*H(2,2)*H(3,3))
   GO TO 999
450 CALL SUBN(H,A,N)
NM1=N-1
   DO 10 I=1,NM1
      SUM=0.
      DO 11 K=1,N
         SUM=SUM+A(K,K)
      11 SUM=SUM/I
      DO 12 J=1,N

A(J,J) = A(J,J) - SUM
IF(I.EQ.NM1) CALL SUBN(A,HINVS,N)
CALL MPLY(H,A,B,N)
CALL SUBN(B,A,N)
DO 13 I=1,N
DO 13 J=1,N
HINVS(I,J) = HINVS(I,J) / A(1,1)
13 RETURN
CONTINUE
RETURN
999 RETURN
END

SUBROUTINE EIGEN - VALUE
SUBROUTINE EIGEN(H,EIGENS,ERROR,MITER,N)
DIMENSION B(4), C(5), DBDA(4), H(3,3), EIGENS(3)
DATA B(1), DBDA(1) /1.0, 0.0/

VARIABLES
H ORIGINAL MATRIX (N,N)
EIGENS EIGENVALUES OF ORIGINAL MATRIX (N)
ERROR MAXIMUM ERROR-RANGE OF EIGENVALUES
MITER MAXIMUM ITERATION COUNTS
N DIMENSION

REQUIRED SUBROUTINES
1) COEFF

DUM01 = H(1,1) * (H(2,3)**2 - H(2,2) * H(3,3))
DUM02 = H(1,2) * (H(1,2) * H(3,3) - H(1,3) * H(2,3))
DUM03 = H(1,3) * (H(1,3) * H(2,2) - H(1,2) * H(2,3))
DUM0 = DUM01 + DUM02 + DUM03
DUM11 = H(1,1) * H(2,2) + H(2,2) * H(3,3) + H(3,3) * H(1,1)
DUM12 = H(1,2)**2 + H(2,3)**2 + H(3,1)**2
DUM1 = DUM11 - DUM12
DUM2 = -(H(1,1) + H(2,2) + H(3,3))

CONDITIONS
IF(ABS(DUM0) .LT. 1.E-40) THEN
IF(ABS(DUM1) .LT. 1.E-40) THEN
EIGENS(1) = 0.0
EIGENS(2) = 0.0
EIGENS(3) = - DUM2
ELSE
EIGENS(1) = 0.0
EIGENS(2) = (-DUM2 - SQRT(DUM2**2 - 4.*DUM1))/2.
EIGENS(3) = (-DUM2 + SQRT(DUM2**2 - 4.*DUM1))/2.
ENDIF
ELSE
C(1) = 0.
C(2) = 1.
C(3) = C(3) / C(2)
C(4) = C(4) / C(2)
C(5) = C(5) / C(2)
I = N + 1
A = 0.
IM1 = I - 1
DO 40 LI = 1, MITER
   DO 41 J = 2, I
      B(J) = C(J+1) - A*B(J-1)
   41 DBDA(J) = - A*DBDA(J-1) - B(J-1)
   DA = - B(I) / DBDA(I)
   A = A + DA
   IF(ABS(DA) - ERROR) 11, 11, 40
40 CONTINUE
11 DO 42 J1 = 1, IM1
42       C(J1+1)=B(J1)
        EIGENS(I-1)=-1./A
        IF(I.EQ.2) GO TO 12
        I=I-1
        GO TO 10
12       CONTINUE
        ENDIF
        RETURN
        END

C SUBROUTINE SORTING
SUBROUTINE SORT(A,B,N)
DIMENSION A(3),B(3)
C VARIABLES
C A ORIGINAL SERIES OF VALUES (N)
C B ASCENDINGLY SORTED SERIES OF VALUES (N)
DO 10 I=1,N-1
DO 10 J=I+1,N
IF(A(I).LE.A(J)) THEN
    BIG=A(J)
    SMALL=A(I)
    A(I)=BIG
    A(J)=SMALL
ELSE
ENDIF
10 CONTINUE
DO 20 K=1,N
20 B(K)=A(K)
RETURN
END

C SUBROUTINE EIGENVECTOR
SUBROUTINE EIVECTR(A,B,EI,N)
DIMENSION A(3,3),B(3,3),EI(3),C(3,3),X(3)
C VARIABLES
C A ORIGINAL MATRIX (N,N)
C B EIGENVECTORS N X (N)
C EI EIGENVALUES (N)
C N DIMENSION
C REQUIRED SUBROUTINES
C 1) HOMO
DO 20 K=1,N
DO 10 I=1,N
DO 10 J=1,N
IF(J.EQ.I) THEN
    C(I,J)=A(I,J)-EI(K)
ELSE
    C(I,J)=A(I,J)
ENDIF
10 CONTINUE
IF(C(2,2).EQ.0.0) GO TO 55
IF((C(1,3).EQ.0.0).AND.(C(2,3).EQ.0.0)) THEN
    X(3)=0.0
    X(1)=1.
    X(2)=-C(1,1)/C(1,2)
ELSE
    CALL HOMO(C,X,N)
ENDIF
GO TO 77
55 CALL HOMO(C,X,N)
77 XX=0.
    DO 15 JJ=1,N
    - 43 -
XX=XX+X(JJ)*X(JJ)
XX=SQRT(XX)
DO 30 KK=1,N
   B(K,KK)=X(KK)/XX
30 CONTINUE
RETURN
END

SUBROUTINE HOMOGENEOUS
SUBROUTINE HOMO(A,X,N)
DIMENSION A(3,3),B(3,3),BINVS(3,3),X(3),Y(3)

VARIABLES

A COEFFICIENT MATRIX (N,N)
X SOLUTION VECTOR (N)
N DIMENSION

REQUIRED SUBROUTINES

1) INVS
2) MPLY
3) SUBN

X(N)=1.
NM1=N-1
DO 40 I=1,NM1
   Y(I)=-A(I,N)
40   B(I,J)=A(I,J)
   IF(NM1.EQ.1) BINVS(1,1)=1./B(1,1)
IF(NM1.EQ.1) GO TO 10
CALL INVS(B,BINVS,NM1)
10   DO 41 I=1,NM1
   X(I)=0.
41   X(I)=X(I)+BINVS(I,J)*Y(J)
RETURN
END

SUBROUTINE OUT-DOT(VECTOR)
SUBROUTINE OUTVEC(A,B)
DIMENSION A(3),B(3,3)

VARIABLES

A VECTOR TO MAKE TENSOR (3)
B TENSOR (3,3)

DO 10 I=1,3
   DO 10 J=1,3
10   B(I,J)=A(I)*A(J)
RETURN
END

SUBROUTINE STRESS-STAR
SUBROUTINE STPNU
REAL INEDAM,NU,M,KAPPA,LAMBDA
COMMON /HOKY1/ALPHA,BETA,B,KAPPA,LAMBDA,ETA,SIGMA(3,3)
COMMON /HOKY2/DAMAGE(3,3),DAMRATE(3,3),NU(3,3,3),SSTAR
COMMON /HOKY3/INEDAM(3,3),FI(3,3)
COMMON /HOKY4/SEI(3),SP(3),SIGBAR,SIFRIM(3,3)
COMMON /HOKY5/EIVEC(3,3),TEN(3,3),VEC(3),M(3)
COMMON /HOKY6/ERROR,MITER
COMMON /HOKY7/STRAIN(3,3),STRATE(3,3),DKK
COMMON /HOKY8/ACONST,CTIMES,POWERM,POWERN
CALL EIGEN(SIGMA,SEI,ERROR,MITER,3)
CALL SORT(SEI,SP,3)
J1=SIGMA(1,1)+SIGMA(2,2)+SIGMA(3,3)
S1=SP(1)-J1/3.
T1=(SP(1)-SP(2))**2
\[ T_2 = (SP(2) - SP(3))^2 \]
\[ T_3 = (SP(3) - SP(1))^2 \]
\[ \text{SIGBAR} = \sqrt{(T_1 + T_2 + T_3)/2} \]
\[ SS = \sqrt{SP(1)^2 + SP(2)^2 + SP(3)^2} \]
\[ DUM1 = ((2/3)*(\text{SIGBAR}/S1))^{\text{ALPHA}} \]
\[ DUM2 = BETA*(J1/SS - 1) \]
\[ SSTAR = 1.5*S1*DUM1*EXP(DUM2) \]

!!! FIND NU TENSOR !!!
CALL EIVECTR(SIGMA,EIVEC,SP,3)
DO 10 I=1,3
DO 20 J=1,3
20 VEC(J)=EIVEC(I,J)
CALL OUTVEC(VEC,TEN)
DO 10 K=1,3
DO 10 L=1,3
10 NU(I,K,L)=TEN(K,L)
RETURN
END

SUBROUTINE DAMAGE-RATE(1)
SUBROUTINE DDOT2
REAL LAMBDA,NU,KAPPA,INEDAM,M
COMMON /HOKY1/ALPHA,BETA,B,KAPPA,LAMBDA,ETA,SIGMA(3,3)
COMMON /HOKY2/DAMAGE(3,3),DAMRATE(3,3),NU(3,3,3),SSTAR
COMMON /HOKY3/INEDAM(3,3),FI(3,3)
COMMON /HOKY4/SEI(3),SP(3),SIGBAR,SIFRIM(3,3)
COMMON /HOKY5/EIVEC(3,3),TEN(3,3),VEC(3),M(3)
COMMON /HOKY6/ERROR,MITER
COMMON /HOKY7/STRAIN(3,3),STRATE(3,3),DKK
COMMON /HOKY8/ACONST,CTIMES,POWERM,POWERN
CALL INEGAB(DAMAGE,INEDAM)
CALL INVS(INEDAM,FI,3)
CALL FIFI(FI,FISO,LAMBDA)

DUM1=B*((SSTAR)**KAPPA)*FISQ
CALL HEAVY(SIGMA,SP(1),M,EIVEC)
DO 10 I=1,3
DO 10 J=1,3
SUM2=0.0
DO 5 K=1,3
5 SUM2=SUM2+NU(K,I,J)*M(K)
IF(J.EQ.1) THEN
   DAMRATE(I,J)=DUM1*(ETA+(1.-ETA)*SUM2)
ELSE
   DAMRATE(I,J)=DUM1*(1.-ETA)*SUM2
ENDIF
CONTINUE
RETURN
END

SUBROUTINE BIJAIJ(A,B,P)
DIMENSION A(3,3),B(3,3)
P=0.0
DO 10 J=1,3
DO 10 K=1,3
10 P = P + B(J,K)*A(1,J)*A(1,K)
RETURN
END

SUBROUTINE HEAVYSIDE-FUNCTION
SUBROUTINE HEAVY(SIGMA,STRESS,M,EIVEC)
REAL M
DIMENSION SIGMA(3,3),EVEC(3,3),M(3)
DO 10 I=1,3
   M(I)=0.0
   DO 5 J=1,3
      DO 5 K=1,3
         M(I)=M(I)+EVEC(I,J)*SIGMA(J,K)*EVEC(I,K)
      M(I)=M(I)/STRESS
      IF(M(I).GE.0.) THEN
         M(I)=M(I)
      ELSE
         M(I)=0.
      ENDIF
   CONTINUE
10 CONTINUE
RETURN
END
Martin Marietta Energy Systems

PROGRESS REPORT

ANISOTROPY OF CREEP-FATIGUE DEFORMATION AND DAMAGE
UNDER NONPROPORTIONAL LOADING

for the period
November 1, 1985 through December 31, 1985

D. L. McDowell
School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332
Project No. E25-B01

Attention:
R. L. Huddleston
Martin Marietta Energy Systems
P.O. Box M
Oak Ridge, TN 37831

December 1985
Summary of Work

Tasks 1-3 of task set A in the Appendix have been completed. Currently, task 4 of task set A is underway, pending execution of task 2 of task set B. Good correlation has been obtained for the proportional biaxial histories. A metallurgy graduate student is in the process of measuring damage in the other biaxial specimens you sent.

As mentioned in the last progress report, we are now using the constants appropriate for annealed type 304 stainless steel in our predictions and are obtaining much more realistic estimates of axial and shear creep strain at rupture.

It is anticipated that we may possibly have to develop a more sophisticated repository for memory of prior creep damage to accurately model the nonproportional tests. After evaluating the current formulation on the basis of all tests proposed in the first year, it would be a good idea to formulate a series of tests in the second year which would involve a greater rotation of the maximum principal stress direction and hence provide greater "resolution" for our evaluation.
APPENDIX

Outline of Tasks

The following tasks have been identified to meet program objectives. There are two parallel sets of tasks pertaining to analytical formulation and physical measurement of creep damage, respectively.

Task Set A

Task 1: Review of existing models for

a. Continuum creep damage

b. Anisotropic representations of creep damage

c. Coupled creep deformation-damage equations

i. Decomposition of time-independent plastic and creep strains

ii. Unified theories for which this decomposition is not made

Review of literature pertaining to:

d. Relationship of physical creep damage (cavitation, coalescence, grain boundary sliding) to primary loading directions for nonproportional histories.

e. Physical damage processes in type 304 stainless steel that must be reflected by anisotropic damage theory.

Duration: 1/1/85 through 3/30/85
Task 2: Damage Model Development

a. Development of the damage tensor $D$ (appropriate rank, symmetry conditions, relationship to physical damage). If existing forms for $D$ are inappropriate, more general or pertinent forms will be developed.

b. Development of the evolutions equations for growth of damage $D$. Equations must depend on current state of loading and material in addition to history of loading.

Duration: 4/1/85 through 6/30/85

Task 3: Development of Coupling Between Damage and Deformation Models

a. Investigation of creep strain rate equations appropriate for nonproportional loading.

b. Introduction of damage into creep strain rate equation.

Duration: 7/1/85 through 8/15/85

Task 4: Demonstration of capability of model to correlate experimental data obtained at ORNL. Guidance for further model development.

Duration: 8/16/85 through 9/30/85
Task Set B

Physical Damage Measurement

Task 1: Development of appropriate quantitative measurement techniques for creep damage in type 304 stainless steel.

Duration: 1/1/85 through 4/30/85

Task 2: Quantitative measurement of magnitude and direction of creep damage (interface with analytical development).

Duration: 1/1/85 through 9/30/85
MARTIN MARIETTA ENERGY SYSTEMS

PROGRESS REPORT

ANISOTROPY OF CREEP-FATIGUE DEFORMATION AND DAMAGE UNDER NONPROPORTIONAL LOADING

for the period
January 1, 1986 through February 28, 1986

D.L. McDowell
George W. Woodruff School of Mechanical Engineering
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Project No. E25-B01

Attn:
R.L. Huddleston
Martin Marietta Energy Systems
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Oak Ridge, TN 37831
Summary of Work

Metallurgical evaluation and quantification of damage of specimens GT-2 through GT-4 has been completed. Currently, we are comparing the results with those predicted by the damage model.

A second-year test matrix has been discussed. The tests include tension-compression as well as a larger angle of principal stress rotation.

Further concepts are being developed to enhance the damage model. We are seeking ways to embed the model in a thermodynamically consistent damage potential. Also, we are considering use of higher order tensors to more accurately represent damage distribution.

All tasks set forth in the first year of the contract have essentially been completed.
APPENDIX

Outline of Tasks

The following tasks have been identified to meet program objectives. There are two parallel sets of tasks pertaining to analytical formulation and physical measurement of creep damage, respectively.

**Task Set A**

Task 1: Review of existing models for

a. Continuum creep damage

b. Anisotropic representations of creep damage

c. Coupled creep deformation-damage equations

i. Decomposition of time-independent plastic and creep strains

ii. Unified theories for which this decomposition is not made

Review of literature pertaining to:

d. Relationship of physical creep damage (cavitation, coalescence, grain boundary sliding) to primary loading directions for nonproportional histories.

e. Physical damage processes in type 304 stainless steel that must be reflected by anisotropic damage theory.

Duration: 1/1/85 through 3/30/85
Task 2: Damage Model Development
   a. Development of the damage tensor $D$ (appropriate rank, symmetry conditions, relationship to physical damage). If existing forms for $D$ are inappropriate, more general or pertinent forms will be developed.
   b. Development of the evolutions equations for growth of damage $D$. Equations must depend on current state of loading and material in addition to history of loading.

Duration: 4/1/85 through 6/30/85

Task 3: Development of Coupling Between Damage and Deformation Models
   a. Investigation of creep strain rate equations appropriate for nonproportional loading.
   b. Introduction of damage into creep strain rate equation.

Duration: 7/1/85 through 8/15/85

Task 4: Demonstration of capability of model to correlate experimental data obtained at ORNL. Guidance for further model development.

Duration: 8/16/85 through 9/30/85
Task Set B

Physical Damage Measurement

Task 1: Development of appropriate quantitative measurement techniques for creep damage in type 304 stainless steel.

Duration: 1/1/85 through 4/30/85

Task 2: Quantitative measurement of magnitude and direction of creep damage (interface with analytical development).

Duration: 1/1/85 through 9/30/85
ANISOTROPY OF CREEP-FATIGUE DEFORMATION AND DAMAGE UNDER NONPROPORTIONAL LOADING

By
D. L. McDowell
School of Mechanical Engineering

Prepared for
MARTIN MARIETTA ENERGY SYSTEMS
Attn:  R. L. Huddleston
Oak Ridge, TN  37831

Report for the period January 1, 1985 through May 15, 1985

May 1985

GEORGIA INSTITUTE OF TECHNOLOGY
A UNIT OF THE UNIVERSITY SYSTEM OF GEORGIA
SCHOOL OF MECHANICAL ENGINEERING
ATLANTA, GEORGIA 30332
Martin Marietta Energy Systems

Semi Annual Progress Report

ANISOTROPY OF CREEP-FATIGUE DEFORMATION AND DAMAGE UNDER NONPROPORTIONAL LOADING

for the period

January 1, 1985 through May 15, 1985

D. L. McDowell
School of Mechanical Engineering
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Project No. E25-B01

Attention:

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May 1985
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BASES OF CONTINUUM DAMAGE MECHANICS

The notion of applying continuum mechanics principles to "track" creep damage was first proposed by Kachanov [1-2]. Since creep damage occurs in a number of grains and grain boundaries, it can be treated in a bulk-averaged sense. Creep damage mechanisms are relatively well-understood; voids nucleate, grow and coalesce driven by diffusion processes, viscous creep of surrounding matrix material, or coupled processes [3].

Continuum damage mechanics seeks to reflect the growth of cavities and the mechanical behavior of damaged material by representing physical creep damage (cavitation) by internal mechanical variables. These mechanical variables, then, modify the creep strain rate equations to produce transition from secondary to tertiary creep (void coalescence). The advantage to such an approach is that damage is treated as a path-dependent variable coupled with nonlinear stress analysis.

Isotropic Damage Formulations

If physical creep damage is represented by a scalar, then no directional dependence or variation of damage is assumed. Nonproportional creep tests [4-9] have shown that creep damage does have a directional character; changing the maximum principal stress orientation may result in different subsequent creep damage and strain rates than would be predicted by using a scalar damage variable. We will begin discussion of continuum damage mechanics with a review of isotropic damage models, since much of the rationale for extension to anisotropic damage is derived from these models.

(i) Kachanov's model:

Kachanov [1-2] defined the evolution of a "continuity" variable $\psi$ at a point $X$ by
with initial condition \( \psi = 1 \) in the undamaged state. Here, \( B \) and \( \nu \) are material constants and \( \sigma_{\text{max}} \) is the maximum tensile stress at point \( X \) in the steady creep field in the perpendicular direction of creep crack growth. Fracture occurs when \( \psi = 0 \).

Kachanov did not discuss the relationship of \( \psi \) to physical damage, nor prediction of strain at rupture. In essence, the creep strain rate was assumed unaffected by the presence of damage.

(ii) Rabotnov's model:

Rabotnov [10-11] generalized Kachanov's model to predict rupture strain in addition to rupture time. He defined a damage parameter \( \omega \) by

\[
\omega = 1 - \psi
\]

which evolves according to

\[
\dot{\omega} = G(\sigma, \omega)
\]

for uniaxial loading. Rabotnov also allowed the uniaxial creep strain rate to depend on \( \omega \), i.e.

\[
\dot{\varepsilon}^C = F(\sigma, \omega)
\]

where \( \omega = 0 \) represents the undamaged state, and \( \omega = 1 \) the ruptured state. The particular forms of these equations were written as
\[ \dot{\omega} = B \sigma^\nu/(1 - \omega)^\mu \]  
(5)

\[ \dot{\varepsilon}^c = A \sigma^n/(1 - \omega)^m \]  
(6)

where \( A, B, m, n, \mu \) and \( \nu \) are material constants, dependent on temperature.

It should be noted that the term \( 1 - \omega \) is often interpreted in the literature as the reduction in area due to the presence of voids or cracks in the material, i.e.

\[ (1 - \omega) = \frac{A_a - A_r}{A_a} \]  
(7)

where \( A_r \) is the reference area (undamaged) and \( A_a \) is the reduced (due to damage) area. Rabotnov did not specifically make this interpretation. As a consequence of equation (7), the net stress may be expressed as

\[ S = \sigma/(1 - \omega) \]  
(8)

such that \( S \) reflects the increase in stress due to reduced load bearing area. Equation (6) can be identified as Norton's creep law if \( m = n \) written in terms of net stress. These equations can be easily integrated for conventional creep tests (constant \( \sigma \)) when the deformation at failure can be considered small.

(iii) Leckie-Hayhurst model:

The model of Rabotnov was derived for unaxial creep behavior. Leckie and Hayhurst [4-6,10,12] generalized the Rabotnov approach to multiaxial loading. The effect of state of stress (for proportional loading) on rupture is normally expressed in terms of isochronous equi-damage surfaces which Hayhurst and Leckie expressed as
\[ \sigma^*(\sigma_{ij}) = \alpha \sigma_1 + \beta \bar{\sigma} + (1 - \alpha - \beta)\sigma_{kk} \quad (9) \]

where \( \sigma_1 \) is the maximum principal stress, effective stress \( \bar{\sigma} = (3\sigma_{ij}^\prime \sigma_{ij}^\prime/2)^{1/2} \), \( \sigma_{ij}^\prime \) is the deviatoric stress tensor, and \( \sigma_{kk} \) is the first invariant of stress. Chaboche [13] suggests that net stress be used in computation of \( \sigma_1, \bar{\sigma} \) and \( \sigma_{kk} \) in equation (9) to reflect damage growth.

Leckie and Hayhurst's multiaxial generalization now takes the form

\[ \dot{\varepsilon}_{ij}^C = (3A/2)(\bar{\sigma}/(1 - \omega))^n(\sigma_{ij}^\prime/\bar{\sigma}) \quad (10) \]

\[ \dot{\omega} = B(\sigma^*)^\nu/(1 - \omega)^\mu \quad (11) \]

where \( \alpha, \beta, A, B, n, \nu \) and \( \mu \) are temperature-dependent material constants.

Two comments should be made regarding equations (10) and (11). First, for a creep test with fixed \( \sigma_{ij}^\prime \), the ratio of creep strain rate components approximately follow a deviatoric flow rule even in the presence of damage. This point was made by Leckie [5-6,10]. Damage growth acts to accelerate the creep strain rate through \( \bar{\sigma}/(1 - \omega) \) in equation (1). Secondly, use of \( \sigma^* \) in equation (11) allows for correlation of materials which are maximum principal stress dependent \( (\alpha + 1) \) or effective stress dependent \( (\beta + 1) \). Increased damage rate due to higher hydrostatic tensile stress is also included.

Though different forms of isochronous surfaces are permitted by variation of \( \alpha \) and \( \beta \), equations (9) - (11) can only apply to proportional loading \( (i.e. \sigma_{ij}^\prime/\bar{\sigma} \text{ fixed for all } i \text{ and } j) \), or for nonproportional loading for materials which damage isotropically. A material is defined to damage isotropically if rotations of the principal stresses (or deviatoric stresses)
would result in the same rupture time at the same isochronous stress level as for a unaxial test. Hence, for nonproportional loading of materials which damage anisotropically, equations (9) - (11) cannot be used since \( \omega \) is a scalar parameter. It should also be noted that equation (10) may not be applicable for combined creep-fatigue loading since no measures of work-hardening or backstress are included.

(iv) Chaboche's model:

Chaboche makes the interpretation of \((1 - \omega)\) as reduced area and defines the net stress as

\[
S = \sigma/(1 - D) \tag{12}
\]

where \( \omega \) is replaced by \( D \), the scalar damage parameter. He then proceeds to couple static plastic damage \( D_1 \), creep damage \( D_2 \), and fatigue damage \( D_3 \):

\[
dD_1 = f_1(\phi, \alpha, D_1, D_2, D_3, \ldots)d\sigma \tag{13}
\]

\[
dD_2 = f_2(\phi, \alpha, D_1, D_2, D_3, \ldots)dt \tag{14}
\]

\[
dD_3 = f_3(\phi, \alpha, D_1, D_2, D_3, \ldots)dN \tag{15}
\]

where \( \phi \) is the forcing variable, \( \alpha \) represents internal variables describing hardening state, and \( N \) represents cycles.

Chaboche and Lemaitre [14], assuming creep and fatigue damage to be additive, have defined creep-fatigue damage in the following way:

\[
dD_2 = f_2(\sigma, (D_2 + D_3))dt \tag{16}
\]
\[ dD_3 = f_3(\Delta \sigma, \sigma_m, (D_2 + D_3))dN \] (17)

where \( \Delta \sigma \) and \( \sigma_m \) are stress range and mean stress, respectively. Defining \( D = D_2 + D_3 \) gives

\[ dD = f_2(\sigma, D)dt + f_3(\Delta \sigma, \sigma_m, D)dN \] (18)

A Rabotnov-Kachanov approach was taken for both creep and fatigue damage,

\[ dD = C^r(1 - D)^{-k(\sigma)} dt + [1 - (1 - D)^{B+1}]^{a(\Delta \sigma)} \left[ \frac{\Delta \sigma}{M(\sigma_m)(1-D)} \right]^B dN \] (19)

where the first term is the creep damage increment. \( k(\sigma), a(\Delta \sigma), M(\sigma_m), r \) and \( B \) are temperature-dependent functions and material constants.

Note the stress dependence of the exponents in the damage rate equation (19). This allows for nonlinear damage accumulation rather than a unique relationship between time fraction and damage.

Chaboche also introduced isotropic and kinematic hardening variables in the uniaxial strain rate equation to model cyclic viscoplasticity in addition to conventional creep strain, i.e.

\[ \dot{\varepsilon}^{in} = \left\{ \frac{|S - X^e| - R}{K} \right\}^n \text{sign}(S - X^e) \] (20)

\[ \dot{X}^e = C^f(p)(a\dot{\varepsilon}^{in} - b|X^e|^{m}\text{sign}(X^e)) \] (21)

where \( R \) and \( X^e \) are isotropic and kinematic hardening variables, and \( n, K, C, a, b, \) and \( m \) are temperature-dependent coefficients. Also,
\[ f(p) = \varepsilon + (1 - \varepsilon) e^{-\beta p}, \quad p = \int_0^t |\varepsilon^{in}(\tau)| d\tau \quad (22) \]

where \( \beta \) and \( \varepsilon \) are temperature-dependent material constants. The inelastic strain rate is given by \( \varepsilon^{in} \). Note that net stress is used in equations (20) - (22) to introduce coupling of the inelastic strain rate with damage. In equation (20), \( \langle \varepsilon \rangle = YH(Y) \), \( H \) being the unit step function. Note that \( x^e \) represents a net backstress for consistency since net stress is used in equations (20) - (21).

Equations for viscoplastic strain rate with an internal variable structure as in equations (20) - (22) can describe effects of unloading and cyclic loading unlike the more restricted creep strain rate equations derived from a dissipative potential (e.g. equations (6) and (10)). These effects include nonlinear hardening, Bauschinger effects, cyclic hardening or softening, and recovery. Additional state variables may be introduced to account for microstructural aging effects or transformation due to temperature changes.

**ANISOTROPIC DAMAGE FORMULATION**

The Rabotnov-Kachanov scalar damage parameter \( \omega \) is useful for describing creep damage under uniaxial or proportional loading conditions. Use of an isochronous surface to correlate multiaxial creep rupture is in general valid only for proportional loading; furthermore, the isochronous surface is essentially a description of "initial" creep anisotropy rather than deformation- or damage-induced anisotropy. This distinction is important since it leads to the need to define a damage tensor. Anisotropic deformation of the isochronous surface is equivalent to tensorial damage for loading histories in which principal stress axes rotate.
One is guided in formulation of anisotropic damage by knowledge of dominant cavity growth mechanisms, relevant measures of driving stress, and the general framework of continuum damage mechanics.

Trampczynski, Hayhurst, and Leckie [4-5] conducted experiments on aluminum and copper. Copper cavitates much more easily than the aluminum or austenitic stainless steels [15]. They subjected thin-walled tubular specimens to an axial stress of 42.5 MPa and shear stress of 14.2 MPa for nearly the entire steady load lifetime, then reversed the shear stress to -14.2 MPa. For aluminum, the damage was well-distributed along grain boundaries, and the shear stress reversal resulted in no change of creep strain rates, damage rates, or rupture time. In contrast, damage in the form of cavity nucleation and growth occurred in copper on grain boundaries normal to the maximum principal stress. The shear stress reversal resulted in an increase in rupture time by a factor of approximately two for copper. A decrease in creep strain rates was attributed by the authors to deformation-induced anisotropy which is not accounted for in equation (10).

Hence, the anisotropy of damage exhibited by copper cannot be described by the generalized Rabotnov-Kachanov equations with scalar damage parameter $\omega$. The evolution of damage must be computed separately for each of the two planes normal to the maximum principal stress. For aluminum, though, they are satisfactory provided the inelastic strain is predominately viscous or creep strain during and after rotation of principal stress axes. It is interesting to note that the isochronous surfaces for copper and aluminum follow maximum principal stress and effective stress criterion, respectively. The experimental results for nonproportional loading would then imply that the isochronous locus for aluminum is underformed while that for copper is deformed, consistent with the earlier notion of damage-induced anisotropy of the isochronous surface.
At this point, it should be noted that more precise forms of isochronous surfaces have been introduced. Huddleston [16] introduced the form

\[ \sigma^*(\sigma_{ij}) = \frac{3}{2} H_1 \left( \frac{2}{3} \bar{\sigma} \right)^a \exp \left[ b \left( \frac{J_1}{S_s} - 1 \right) \right] \quad (23) \]

where

\[ H_1 = \sigma_1 - \frac{J_1}{3}, \quad J_1 = \sigma_{KK}, \]

\[ S_s = \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right)^{1/2}, \]

and \( a \) and \( b \) are temperature-dependent material constants. The results for type 304 stainless steel tested at \( 593^\circ \text{C} \) show much better correlation with equation (23) than with effective stress \( \bar{\sigma} \), maximum shear stress, or maximum principal stress. Hence, Huddleston's formulation would be a likely candidate for inclusion in a damage rate equation such as equation (11). Of course, the net stress can be used to compute all stress-related variables in equation (23).

There are several classifications of approaches which will be reviewed for generalization to anisotropic damage:

(i) a scalar damage parameter with a tensor multiplier to apportion damage effects among the various stress components,

(ii) definition of physical creep damage (e.g. cavity or fissure density) in terms of an appropriate rank tensor,

(iii) measure of deterioration of elastic behavior, and

(iv) creep damage potential functions.
(i) Tensor Multiplier Functions

In this approach, damage growth is governed by a scalar, but the growth of damage influences various stress-strain components differently. This representation of damage is implied by the procedure of Leckie et al. [5-6,10] which involved separate calculation of the damage on two non-interacting planes in copper. Defining the scalar damage parameter as \( \omega \), the rate of growth of the damage tensor \( \tilde{D} \) is given by

\[
\dot{\tilde{D}} = Q(\tilde{S}) \omega
\]  

(24)

where \( Q(\tilde{S}) \) is an operator of the same rank as \( \tilde{D} \) which defines the preferential orientation of cavity growth and crack formation. Note the dependence of the directionality of damage growth on the net stress tensor \( \tilde{S} \). This is due to the fact that the effective net stress exceeds the effective true stress, and rotation of the net stress tensor relative to the true stress tensor will occur as cavitation proceeds if the principal axes of true stress are rotated (i.e. nonproportional loading).

As stated by Hayhurst [17], it is likely that most metals suffer creep damage of a mixed cavitation/wedge cracking nature along grain boundaries; copper and aluminum offer two "bounds" of mostly cavitation damage and mostly grain boundary sliding-induced wedge cracking, respectively. From an analytical standpoint, this means that if \( Q(\tilde{S}) \) is defined as

\[
Q = \gamma \mathbb{I} + (1 - \gamma) \tilde{S}
\]

(25)

where \( \mathbb{I} \) is the identify tensor and \( \tilde{S} \) apportions damage anisotropy, then \( \gamma = 0 \) for copper and \( \gamma = 1 \) for aluminum, at least in the range of stresses tested by Leckie and associates. From consideration of deformation
mechanism maps [18], to be discussed in a later section, it is quite possible that both \( \gamma \) and \( \tau \) are net stress and temperature dependent.

Studies which use the form of \( Q \) given in equation (25) usually assume damage is represented adequately by a second order tensor, which is only an approximation of physical damage for the general nonproportional loading case. Chaboche [13] extends the stress level dependence of the exponent of damage in the extended Rabotnov-Kachanov damage law to describe scalar damage evolution, i.e.

\[
\dot{\omega} = \left\langle \frac{\sigma^*(S)}{A} \right\rangle \Gamma / (1 - \omega)^k(\sigma^*)
\]

where we again note the use of the isochronous surface concept. Chaboche suggested the use of net stress to compute \( \sigma^* \) within the Macauley bracket in this equation.

Murakami and Ohno, using a second rank symmetric damage tensor, have suggested the general form [9,19]

\[
\mathbf{D} = n \mathbf{I} + \sum_{i} M^{(i)} \left[ \nu^{(i)} \otimes \nu^{(i)} \right] + \sum_{j} N^{(j)} : \left[ \nu_{D}^{(j)} \otimes \nu_{D}^{(j)} \right]
\]

(26)

where \( n \) and \( M^{(i)} \) are scalar functions of net stress, temperature and other internal variables, \( N^{(j)} \) is a fourth order tensor, and \( \nu^{(i)} \) and \( \nu_{D}^{(j)} \) are unit normal vectors in the positive principal value directions of the net stress tensor \( S \) and its deviator, \( S_D = S - (1/3)S_{kk}I \). The symbol \( \otimes \) stands for outer product. A particular form which seems to encompass the bounding behaviors of copper and aluminum is given by

\[
\mathbf{D} = B[\sigma^*(S)]^k(\eta I + (1-\eta)\nu^{(1)} \otimes \nu^{(1)}) (\mathbf{\phi} : \mathbf{\phi})^{\nu/2}
\]

(27)
where

$$\phi = (I - D)^{-1}.$$

Here, B, k, n and \( \lambda \) are material constants. Note the similarity of equation (27) with Chaboche's approach. The inverse dependence on damage is introduced through \( \phi \) in a consistent way in the Murakami-Ohno approach, without recourse to an additional scalar growth law. It should also be noted that \( \lambda \) in equation (27) could be made a function of net stress to correspond to Chaboche's modification to achieve a nonlinear damage versus time fraction relationship. Hayhurst and Leckie's generalization of the Rabotnov-Kachanov approach in equation (21) is equivalent to equation (27) if \( n = 1 \), i.e. isotropic damage is assumed.

Murakami and Ohno have also provided a rational definition of the net stress tensor consistent with area reduction due to physical damage. This will be discussed later in the section on anisotropic representation of physical damage. It should be emphasized that a second order damage tensor can only approximately reflect the effects of physical damage on the damage rate for general nonproportional loading. The computational simplicity of this approach is desirable, though, provided it is relatively accurate.

One item regarding equation (27) which appears to have been overlooked in the literature is the possibility that the rate of damage growth in a given direction is dependent upon the damage in that direction rather than the equivalent scalar damage, i.e.

$$\dot{D} = 8[\sigma^*(S)]^k(n\phi \cdot \phi)^{l/2} + (1 - n)\gamma^{(1)} \otimes \gamma^{(1)} \left[ \frac{1}{1 - \gamma^{(1)} \cdot \dot{D} \cdot \gamma^{(1)}} \right] \cdot \gamma^{(1)}$$ (28)
Of course, this effect could be studied via discrete rotations of maximum principal stress with metallographic examination of otherwise identical specimens at each state to quantify damage extent and direction. The criterion for rupture in this case could be $(\dot{\gamma}:\dot{\gamma})^{1/2} + \infty$ for $\eta + 1$ or $\gamma(1) \cdot \mathbf{D} \cdot \gamma(1) + 1$ for $\eta > 0$.

(ii) Tensorial Description of Physical Damage

The previous section dealt with heuristic descriptions of damage evolution which have evolved primarily out of extension of the Rabotnov-Kachanov continuum damage concepts to multiaxial nonproportional loading. When defining the damage tensor, through, it is necessary that the current state of damage be adequately reflected with regard to direction and sense. Creep damage in the continuum sense is almost universally considered to be quantitatively related to the area density of voids and fissures along grain boundaries in a global mean sense. That is, microscale effects in each grain are not considered. In a materials science approach, heterogeneity of damage from grain to grain can lead to void growth constraint; this would have to be treated as a nonlocal damage growth phenomenon in the continuum approach, i.e. damage in a neighboring grain influences local damage growth. These nonlocal effects are not usually addressed in continuum creep damage mechanics.

Leckie and Onat have proposed a generalized tensorial form for cavitation damage [20]. In their approach, we consider a material element large enough such that deformation and damage may be considered homogeneous within. Then, we consider a unit sphere with unit normal vector $\mathbf{n}$ at each point. The total volume of voids found in grain boundaries in the material element normal to $\mathbf{n}$ is denoted as
\[ V(n) \, dA(n) \]  

where \( V(n) \) is the density of the distribution of void volume. Since the physical damage state is invariant with respect to the sign of \( n \),

\[ V(n) = V(-n) \]

Various order damage tensors can be defined by the moments of the voids of grain boundaries, i.e.

\[ V_0 = \int_A V(n) \, dA(n) \]  

\[ V_i = \int_A V(n) \, n_i \, dA(n) \]  

\[ V_{ij} = \int_A V(n) \, n_i \, n_j \, dA(n) \]  

and so on, where \( A \) is over the unit sphere and \( n_i \) are the components of \( n \) in a fixed rectangular coordinate frame \( (i, j, \ldots = 1, 2, 3) \). These damage tensors transform in the usual way and result in invariance of damage with respect to rigid body rotation. From equation (30), all odd rank damage tensors must vanish. All even rank tensors are symmetric in all indices and
irreducible. We also note that \( V_{kk} = V_o \) where \( V_o \) is the total volume of voids per unit volume, or the isotropic damage tensor.

Leckie and Onat also define a series of tensors describing the density and direction of void nucleation sites, known to be particularly important in the early stages of creep. These tensors follow the same development as for \( V_o, V_{ij}, V_{ijkl} \), and are defined as \( N_o, N_{ij}, N_{ijkl}, \ldots \), with the totality of damage tensors defined by a tensor

\[
\mathbf{\mathcal{S}} = (V_o, V_{ij}, V_{ijkl}, \ldots; N_o, N_{ij}, N_{ijkl}, \ldots)
\]  

(34)

with damage evaluation given by

\[
\mathbf{\dot{s}} = g(s, \mathcal{S})
\]  

(35)

and a rupture criterion

\[
R(s, \mathcal{S}) < 0
\]  

(36)

The creep strain rate equation is assumed to be weakly dependent on \( \mathcal{S} \).

Obviously directionality of physical damage could be described with good quantitative accuracy by using fourth and higher order tensors. Yet such an approach is not necessarily economical nor practical. If use of a second order tensor can suitably approximate the directionality of damage, then it is indeed warranted. To this end, Leckie and Onat have suggested the use of second order symmetric tensors \( V \) and \( \mathcal{N} \). Again \( V_{kk} = V_o \). Applying the mean value theorem to equation (33), we get an estimate of \( V(\mathcal{S}) \) as
\[ V(n) = \frac{3}{4\pi} V_{ij} n_i n_j \]  

which also satisfies equation (31). Likewise,

\[ N(n) = \frac{3}{4\pi} N_{ij} n_i n_j \]  

The growth laws are then expressed as

\[ \dot{\varepsilon}^c = f(\overline{\sigma}, V_0) \sigma' \]  

\[ \dot{V} = g_V(g, V, N) \]  

\[ \dot{N} = g_N(g, V, N) \]  

with the rupture condition

\[ R(g, V) = 0 \]  

Here, \( \sigma' \) is the deviatoric stress tensor. A rupture criterion of the form

\[ R(\sigma_{\text{max}}, V_0, V_{ij} n_i^* n_j^*) = 0 \]  

was suggested, where \( \sigma_{\text{max}} = n_i^* \sigma_{ij} n_j^* \) is the component of stress in the direction of maximum principal stress, \( n^* \). Thus, a critical combination of maximum stress, total void volume, and void density on grain boundaries normal to the maximum stress dictates rupture. Later work by Leckie resulted in the proposed rupture condition.

- 16 -
\[
\max \frac{1}{n} \left( V_{ij} \bar{n}_i \bar{n}_j - \bar{n}_k \sigma_{k\ell} \bar{n}_\ell \right) = \text{constant}
\]

It is interesting to note that Leckie and Onat do not use net stress in the constitutive equations for damage and deformation. The effect of area reduction is evidently included via \( V_0 \) in the creep strain rate equations. Though both void nucleation and growth are included in this formulation, the specific forms for the growth equations are not yet well-developed.

Marakami and Ohno [9,19] have developed a rather complete formulation for creep damage, including finite deformation rotation invariance requirements. In their theory, damage is approximated by a second rank tensor which reflects the change in effective area of the Cauchy tetrahedron due to projected area of cavities on each face. Selecting a material volume element large in comparison to mean void or grain size, but small enough for stress and damage to be uniform, we define the damage tensor as

\[
\bar{D} = \frac{3}{S_g(V)} \sum_{k=1}^{N} \int_V \left[ n\,(k) \otimes n\,(k) + w\,(k) \left( I - n\,(k) \otimes n\,(k) \right) \right] dS_g(k) \tag{44}
\]

where \( dS_g(k) \) and \( n\,(k) \) denote the area of a grain boundary element occupied by the \( k \)-th cavity and the unit vector normal to \( dS_g(k) \), respectively, and \( S_g(V) \) is the total area of grain boundaries in \( V \). Murakami and Ohno do not distinguish between representation of void nucleation and growth. The definition of cavity can evidently include both wedge-type and \( r \)-type cavities as discussed by Raj et al. [15,22]. In equation (44), \( W(k) \) denotes the effect of the \( k \)-th cavity on the area reduction of planes whose normals are
perpendicular to \( n^{(k)} \); this effect is included since the voids are three-
dimensional and cannot be considered just as two-dimensional grain boundary

The principal values of the damage tensor are bounded by 0 and 1, i.e.

\[
0 < D_j < 1 \quad (j = 1, 2, 3)
\]

(45)

When \( D_j = 1 \) for any \( j \), creep rupture occurs on the \( j \)-th principal plane. For
undamaged material, \( D = 0 \). Note the similarity of \( D \) in equation (44) with \( \gamma \) in
equation (33). Equation (44) is integrated only over the portions of grain
boundaries which are cavitated in contrast to the entire unit sphere in
equation (33), but the two formulations are substantially equivalent apart
from the cavity width correction \( W^{(k)} \) in the Murakami-Ohno approach.

The effect of damage accumulation is to decrease the area over which the
force is carried on an infinitesimal element, with some directions
experiencing more area reduction than others due to preferential cavity
growth. Still, equilibrium must be satisfied; for an arbitrary plane in the
material, this leads to a definition of a net stress tensor which is not only
intensified, but also rotated with respect to the Cauchy stress tensor defined
for undamaged material. We define the symmetric part of the stress tensor
defined by

\[
g \cdot (I - D)^{-1} = g' \cdot \hat{g}
\]

(46)

as the net stress tensor \( \mathcal{S} \), i.e.

\[
\mathcal{S} = \frac{1}{2} (g \cdot (I - D)^{-1} + (I - D)^{-1} \cdot g)
\]

(47)
where $\sigma$ is Cauchy stress. $\mathcal{S}$ is essentially the stress acting on a fictitious undamaged material element which is equivalent to application of $\sigma$ on an element of damaged material. This approach is essentially a generalization of the Kachanov-Rabotnov scalar damage approach. Damage rate equations (27) or (28) could then be employed using $\mathcal{S}$ defined in equation (47), with appropriate specialization to combined isotropic and anisotropic damaging. The Jaumann derivative may be used for damage rate to ensure invariance of the damage rate equation with respect to rigid body rotations.

Again it is recognized that the creep strain rate is less affected by cavity growth than is damage rate or rupture, which are local phenomena. In fact, creep rupture may occur even when the creep strain rate is finite rather than infinite as implied by a Kachanov-Rabotnov scalar damage approach. Recalling the work of Leckie and associates, the creep strain rate is deviatoric with magnitude modified by $1/(1-\omega)$ as in equation (10). Murakami and Ohno point out that the net stress tensor for use in the creep strain rate equation is in general formed by a fourth rank tensor $\mathcal{E}$ operating on Cauchy stress, i.e.

$$\mathcal{S} = \frac{1}{2} (\mathcal{E} : \sigma + \sigma : \mathcal{E})$$

(48)

where $\mathcal{E} = \mathcal{E}(\phi)$. The damage and creep rate equations are then given by

$$\dot{\mathcal{S}} = G(\mathcal{S}, \phi, \kappa, T)$$

(49)

$$\dot{\mathcal{E}} = F(\mathcal{S}, \phi, \kappa, T)$$

(50)
where \( \kappa \) and \( T \) denote a matrix work-hardening parameter and absolute temperature, respectively. \( \dot{\varepsilon}_c^c \) may be interpreted here as the rate of deformation tensor for finite creep deformation. Note also that dependence on \( \dot{\eta} \) and \( \dot{\varphi} \) in equation (49) is equivalent to dependence on \( \dot{\varphi} \) and \( \dot{\psi} \), as is the pair \( \dot{\eta} \) and \( \dot{\varphi} \). In earlier work, Murakami and Ohno [9] had written equation (50) as

\[
\dot{\varepsilon}_c^c = \dot{\varphi}'(\dot{\eta}, \dot{\varphi}, \kappa, T) \tag{51}
\]

where

\[
\dot{\varphi} = (I - c\dot{\varphi})^{-1} \tag{52}
\]

and

\[
\dot{\eta} = \frac{1}{2}(\dot{\varphi} \cdot \dot{\varphi} + \dot{\varphi} \cdot \dot{\varphi}) \tag{53}
\]

where \( 0 < c < 1 \) allows variation from a deviatoric or other classical dissipative potential flow rule. The representation for creep strain rate given in equations (48) and (50) may be general, but is very difficult to implement. The approach in equations (51) - (53) is easier to implement, since \( c \) can be selected to fit a given data set, but use of \( \dot{\varphi} \) is a rather ad hoc procedure which is not well physically grounded. Fortunately, when the cavity volume fraction is on the order of a few percent or less [19], the effect of damage on creep deformation may be assumed to be isotropic. Hence, the creep strain rate equation may be simplified to

\[
\dot{\varepsilon}_c^c = (3/2)m A^1/m \kappa^{(m-1)/m} \bar{\varphi} (n-m)/m \bar{\eta}' \tag{54}
\]
where the net stress tensor modified by isotropic damage is given by

\[ \tilde{\Sigma} = \left( 1 + c D_{kk} \right) \Sigma \]  

(55)

where \( c (0 < c < 1) \) is a material constant. Furthermore,

\[ \tilde{\Sigma}_{eq} = \frac{3}{2} \tilde{\Sigma}^t \tilde{\Sigma}^{t'} \]  

(56)

\[ \tilde{\Sigma}^t = \tilde{\Sigma} - \frac{1}{3} \tilde{\Sigma}_{kk} \Sigma \]  

(57)

The evolution of matrix work-hardening parameter \( \kappa \) is given by

\[ \dot{\kappa} = mA_{m}^{1/m} \kappa^{(m-1)/m} \frac{\dot{\Sigma} - n}{m} \]  

(58)

In equations (54) - (58), \( A, m, n, \) and \( k \) are temperature-dependent material constants. Murakami and Ohno have demonstrated the applicability of the second rank damage tensor \( \tilde{D} \) for prediction of rupture time for copper subjected to nonproportional loading and for perforated specimens. Their prediction of creep strain under nonproportional reversal of stress using equation (54) is not suitably accurate, however; the introduction of backstress into equation (54) would allow for kinematic hardening (directional dislocation arrangement) and more accurate prediction for nonproportional loading and unloading sequences [22-24]. Hence, use of unified creep-plasticity theories might be warranted [25-30].

It should be noted that Betten [31], employing a tensor representation of area reduction due to damage, also showed that creep damage can be represented
by a second rank symmetric tensor. He derived the creep rate constitutive equations based on generalized tensor function theory. In contrast to Murakami and Ohno, Betten retains a non-symmetric net stress tensor. Furthermore, he introduces initial deformation-induced anisotropy effects into both the creep strain rate and damage rate equations. While it is obvious that initial anisotropy from a rolling or forming process would affect the creep strain rate magnitude and direction, the effect on magnitude and direction of damage rate is inconclusive. As a point of clarification, it should be noted that all previous growth equations in this paper have assumed initial isotropy.

While the work of Betten offers a very rigorous continuum mechanics approach to tensorial damage, its abstract nature and lack of simplification to practical creep constitutive equations limit its usefulness. The work of Murakami and Ohno is of similar nature yet oriented toward the engineering approach.

Baik and Raj [21] have proposed use of a second rank tensor to represent the development of three-dimensional creep damage for the case of wedge-cracking induced by grain boundary sliding for aluminum alloy and austenitic stainless steels subjected to creep-fatigue loading. They showed that wedge-cracking was likely to occur in the aluminum alloy and austenitic stainless steels under conditions of asymmetric load cycle shapes without tension hold, while hole cavitation ("r" type cavitation) was likely to occur under tension hold cycles. This points to the fact that the dominant mechanism of creep damage under creep-fatigue loading is not only sensitive to temperature and loading intensity, but the rate and sequence in which loads are applied. If wedge type creep damage is dominant, then Baik and Raj define damage growth in the principal frame by
for $i = 1, 2, 3$, where $\beta$ represents the ratio of the grain boundary sliding strain rate to the applied strain rate, $\alpha$ is a proportionality constant, and $\gamma$ is an adjustable parameter reflecting the extent to which damage produced in tension can be recovered in compression,

$$\frac{d(A_w)}{d\varepsilon} = \alpha \beta \left[ \gamma x_{ij} \frac{\sigma}{\sigma} \right]$$  \hfill (59)

and the measure of damage is defined by

$$A_w = \frac{c}{L}$$  \hfill (61)

where $c$ is the wedge crack length, and $L$ is the grain size. As in the earlier approaches, $0 < A_w < 1$, with $A_w = 0$ indicative of undamaged material, and $A_w = 1$ indicative of fracture. The tensor $x_{ij}$ allows correlation of the stress applied in direction $1$, $\sigma_1$, with development of damage in directions 2 and 3. Since wedge damage develops at triple junctions of grain boundaries, wedge cracks appear in normal directions along adjacent grain boundaries. It may be the case that when creep damage is dominated by wedge-cracking, the mean damage state (averaged over a number of grains) assumes a more isotropic nature due to random grain orientation than would "r" type cavitation, which is dependent on the maximum principal stress direction. Perhaps these considerations help explain the observed indifference to loading direction of creep strain and damage rates of Leckie and Hayhurst's fixed tension, alternating shear tests on aluminum. It would seem that materials which do
not cavitate easily may experience at least a component of triple junction wedge-cracking in creep-dominated situations with even occasional load reversals present.

Recalling the constructions of damage tensors reported earlier, it is clear that it is not necessary to distinguish between triple junction wedge cracks and well distributed cavities ("r" type) in a continuum damage approach. The three-dimensional character of damage can be described by a tensor with a suitable degree of anisotropy. It is significant, though, that previous continuum damage mechanics analyses have not substantially addressed the possibility of several dominant mechanisms, each with a different damage rate dependence on applied stress, over a range of temperatures and strain rates.

(iii) Measure of Deterioration of Elastic Behavior

It is very common in structural analysis to assume that the presence of damage is manifested by a decrease in stiffness. Usually, the concept of deterioration of stiffness is applied to a structural member rather than at each material point. Chaboche [13] introduced a form for the net stress tensor

$$\bar{S} = \bar{M}(D):\bar{\sigma}$$

where $\bar{M}$ is a fourth order tensor operating on stress. Equation (62) is applicable at each material point. Then $\bar{S}$ can be used in lieu of $\bar{\sigma}$ in the elastic-viscoplastic constitutive equations with damage-induced anisotropy being included in a natural way through flow rules based on scalar invariants of $\bar{S}$. Chaboche suggested that $\bar{M}$ could be measured from deterioration of the elastic response by using an equivalence in the elastic strain sense, i.e.
\[ \mathbf{\tilde{\sigma}} = \mathbf{\tilde{\Lambda}}(D) : \mathbf{\tilde{\varepsilon}} \]  
(63)

\[ \mathbf{\tilde{\varepsilon}} = \mathbf{\Lambda}(0) : \mathbf{\varepsilon} \]  
(64)

where \( \mathbf{\tilde{\varepsilon}} \) is the elastic strain tensor and \( \mathbf{\Lambda}(0) \) the fourth rank elasticity tensor for undamaged material. These equations lead to

\[ \mathbf{\tilde{M}}(D) = \mathbf{\Lambda}(0) : \mathbf{\tilde{\Lambda}}^{-1}(D) \]  
(65)

so that \( \mathbf{\tilde{M}}(D) \) could be determined if initial and subsequent elastic responses are known.

As mentioned by Chaboche, this description of damage is consistent with a general thermodynamical framework [13]. However, it is noted from earlier discussion that the dependence of creep deformation on creep damage is not direct as is the rupture condition since rupture may occur in a relatively brittle fashion at low void volume density. In other words the deformation of the matrix is not strongly affected by the presence of grain boundary damage up to tertiary creep, reflected through the applicability of a deviatoric flow rule for creep strain. For these reasons, as pointed out by Murakami and Ohno [19], it seems more appropriate to base the net stress tensor on reduction in projected area due to physical creep damage, and to use this net stress tensor in damage evolution equations but not necessarily in creep strain rate equations.

(iv) Creep Damage Potential Functions

The generalization of uniaxial concepts of creep deformation to the multiaxial case usually involves the assumption of a dissipative potential
function. The creep strain rate is then normal to this dissipative potential surface in stress space. It is tempting to generalize creep damage for the multiaxial case by assuming an appropriate creep damage potential. To this end, both phenomenological and coupled thermo-viscoplastic approaches have been offered.

From a phenomenological standpoint, Bodner and Lindholm [32] have suggested a damage growth law of the form

\[ \dot{D} = \alpha(W_s)\beta(\sigma_{kk})\gamma(D) \]  

(66)

where \( W_s \) is the stored energy of cold work, \( \sigma_{kk} \) is the first invariant of stress and \( D \) is a scalar damage parameter. The value of \( D \) at failure, unlike in the Kachanov-Rabotnov approach, is determined empirically. The mechanistic interpretation of this equation is that \( \alpha \) governs nucleation of voids and defects, \( \beta \) controls the rate of growth of voids under applied hydrostatic stress, and \( \gamma \) relates damage growth rate to current damage level. The use of the first invariant of stress as the sole stress parameter in equation (66) is incompatible with the concept of anisotropic creep damage. If \( \beta(\sigma_{kk})\gamma(D) \) were to be defined as a surface in stress space, the direction of the damage rate could be defined as the gradient of the damage surface if \( \beta \) depended on \( \sigma_{ij} \) instead of an invariant. The separation of void nucleation and growth processes in equation (66) is similar to the formulation of Leckie and Onat (equations (29)-(44)); the latter formulation, however, assigned directions to void nucleation growth processes in contrast to the Bodner-Lindholm approach. Essentially, then, equation (66) provides very limited capability for modeling nonproportional creep damage and is somewhat difficult to apply,
Krajcinovic [33] has proposed damage evolution laws based on the existence of a potential, suggesting that this approach is ultimately necessary for treatment of cyclic loads. In his approach, the Helmholtz free energy density is composed of elastic and viscous terms

$$\psi = \psi^{(e)}(\varepsilon^e, D, T) + \psi^{(c)}(a, p, D, T)$$

(67)

where $\psi^{(e)}$ and $\psi^{(c)}$ are elastic and creep potentials, $\varepsilon^e$ is the elastic strain, and $a$ and $p$ are kinematic and isotropic hardening variables which affect viscous response. In Krajcinovic's work, damage $D$ is a vector normal to a plane of cracks or crack-like defects. This definition of the damage tensor is not in agreement with the findings of Onat and Leckie, Murakami and Ohno, Betten, and others that physical damage must be represented by even rank tensors. Minimally, a second rank tensor (or scalar if damage is isotropic) is necessary. Use of a vector attributes a sense to the damage direction which incorrectly implies that the current state of damage is dependent on the sense of the direction; damage growth rate, not the current damage state, is dependent on sense of loading (tension or compression) and this is reflected through the damage evolution equation.

Since the rest of Krajcinovic's development does not hinge on this assertion of vectorial creep damage, we continue the discussion. From the Clausius-Duhem inequality,

$$\rho \dot{Q} = \varepsilon \cdot \dot{\varepsilon} - \rho \frac{\partial \psi}{\partial a} \dot{a} - \rho \frac{\partial \psi}{\partial p} \dot{p} - \frac{1}{T} q \cdot \gamma T > 0$$

(68)
where $\rho$ is mass density, $\dot{\varepsilon}^C$ is the creep strain rate, $q$ is the heat flux, and $T$ is temperature. Equation (68) states that the energy dissipation power density $\rho Q > 0$. The entropy term in the dissipation power density cancels with the term in the total time differential of $\psi$ resulting from dependence of $\psi$ on $T$. Since stress can be derived from $\psi$ by

$$\sigma = \rho \frac{\partial \psi}{\partial \varepsilon}$$

we can write

$$\dot{\sigma} = \rho \frac{\partial^2 \psi}{\partial \varepsilon \partial \varepsilon} \cdot \dot{\varepsilon}^C + \rho \frac{\partial^2 \psi}{\partial \varepsilon \partial \dot{\varepsilon}} \cdot \dot{T}$$

(70)

and the rates of other generalized forces $- \rho \frac{\partial \psi}{\partial \dot{\varepsilon}^D}, - \rho \frac{\partial \psi}{\partial \dot{\alpha}}, - \rho \frac{\partial \psi}{\partial \dot{\rho}}$ can be similarly found. Next, the rates of change of internal variables $\dot{\alpha}, \dot{\rho}, \dot{D}$ in addition to creep strain $\dot{\varepsilon}^C$ can be found by assuming the existence of a potential $\bar{\Omega}$ such that

$$\dot{j} = \frac{\partial \bar{\Omega}}{\partial \bar{x}}$$

(71)

where $\dot{j} = \{\dot{\varepsilon}^C, \dot{\alpha}, \dot{\rho}, \dot{D}, \frac{1}{T} \dot{T})^T$ and $\dot{x} = \{\dot{\varepsilon}^C, - \rho \frac{\partial \psi}{\partial \dot{\varepsilon}^C}, - \rho \frac{\partial \psi}{\partial \dot{\alpha}}, - \rho \frac{\partial \psi}{\partial \dot{\rho}}, - \rho \frac{\partial \psi}{\partial \dot{D}}, - \dot{T}\}^T$. If $\bar{\Omega}$ is quadratic in $\dot{x}$, then $\bar{\Omega}$ is equivalent to the energy dissipation power density $Q$. Krajcinovic suggests a flow potential of the form

$$\bar{\Omega} = G^C(\dot{x})H(G^C) + G^D(\dot{x})H(G^D)$$

(72)

when $H$ is the Heaviside function. For $G^C < 0$, no viscous deformation occurs, and for $G^D < 0$, no growth of damage occurs. Hence, the coupling of
damage with creep strain rate and growth of damage would emerge from equations (71) - (72). Possible forms for the viscous flow potential \( G^c \) and damage potential \( G^d \) are

\[
G^c = \left[ K \left( \sigma' + \rho \frac{\partial \psi}{\partial \sigma} \right) - \rho \frac{\partial \psi}{\partial p} \right] \quad \quad \quad \quad (73)
\]

\[
G^d = \left[ K_1 \left( R_1^2 + a D R_2^2 \right) - g_1(D) \right] \quad \quad \quad \quad (74)
\]

where \( K, K_1, a, m, \) and \( n \) are material parameters and \( \sigma' \) is deviatoric stress. \( R_1 \) and \( R_2 \) are the values of \(- \rho \frac{\partial \psi}{\partial \sigma}\) generalized forces in directions normal and tangential to the "plane of damage" in this formulation and therefore govern the normal and shear components of damage growth, respectively. The damage potential in equation (74) can reflect dominant shear-type damage accumulation, cleavage (normal) - type, or a combination.

The great advantage of using a flow potential is that damage is included as an internal variable. The damage and creep strain rate equations are coupled through the flow potential \( \Omega \), providing a rational treatment. The anisotropic character of damage is also defined through \( \Omega \). Rigorous application of the flow potential concept, though is difficult. It may be possible to simplify the flow potential or even to express some of the previously mentioned anisotropic theories in term of \( \Omega \) (e.g. Murakami-Ohno). Certainly, it seems necessary to minimally express \( \Omega \) in equations (68) - (74) as a second order tensor rather than a vector in future work.

As a final comment, it should be noted that considerable work has been done [e.g. 34-37] in characterizing uniaxial cumulative creep damage under varying stress levels. At higher stress levels, damage accumulation more
nearly approaches a linear time fraction accumulation rule than at lower stress levels. Bui-Quoc and associates [35] use a highly phenomenological, normalized damage curve approach to more accurately represent (fit) creep and creep-fatigue damage sequence effects. They suggest that the Chaboche-Lemaitre approach in equation (19) provides a damage parameter which is very small \( D \approx 0 \) even near the end of material life, and suggest that this is not representative of nonlinear damage accumulation processes. It appears to the current author that Chaboche et al. enforce this condition on damage to minimize the influence of damage on the creep strain rate equation until tertiary creep is reached. If the proper dependence of the creep strain rate on damage is prescribed, perhaps theoretical damage growth could more faithfully follow physical creep damage growth. It is strongly felt that the anisotropic damage tensor must accurately reflect physical damage to properly model loading sequence effects and creep-fatigue interaction.

SOME COMMENTS ON THE RELATIONSHIP OF MATERIAL SCIENCE AND CONTINUUM DAMAGE GROWTH LAWS

Naturally, description of creep damage has also been undertaken from the materials science or micromechanical viewpoint. Most of the work has dealt with the nucleation and growth of voids as a function of applied stress and temperature. The primary thrust of these studies has been to identify the regimes of stress and temperature in which voids grow by diffusion, power-law continuum plastic deformation, or by coupled mechanisms. Miller and Langdon [38] point out that three distinct void growth processes may occur:

(a) power-law creep of surrounding matrix at high stress levels,
(b) unconstrained diffusion growth at intermediate stress levels, and

(c) constrained diffusion growth at low stress levels.

Of course, coupling may exist between these mechanisms.

An essential feature of these different cavity growth mechanisms is that void growth rates will differ depending on the dominant mechanisms (highest growth rate). Svensson and Dunlop [39] have produced cavity growth mechanism maps which show the regimes of dominance of constrained diffusion growth, unconstrained diffusion growth, coupled power-law diffusional growth, power law growth, and non-equilibrium diffusional growth. These maps are very similar to the deformation mechanism maps introduced by Ashby and co-workers, but deal specifically with void growth mechanisms rather than creep deformation mechanisms. Other authors [3,15,21,38-44] have offered specific models for the nucleation of voids at grain boundary/slip band intersections, precipitate particles, or triple junctions, and the growth of these voids once nucleated.

Of particular interest to this investigation is the relationship of the cavity growth laws to the evaluation of the damage tensor in Kachanov-Rabotnov continuum damage theories. Obviously, the two must be somewhat compatible or we may extrapolate the continuum growth law into a regime where the damage evolution equation is of inappropriate form.

Cocks and Ashby [3,43] have extensively studied the regimes of diffusion controlled and power-law controlled creep damage growth. They note that growth due to power-law deformation of surrounding matrix material becomes more important as the voids become larger, even if the growth was diffusion controlled when the cavities were small. They also have shown that consideration of both diffusion and power-law controlled damage growth is
necessary for accurate calculation of rupture times, particularly when variable loading histories are applied which impose diffusion controlled growth over part of the life and power-law growth over the rest. The linear time-fraction damage accumulation rule is not adequate when the mechanisms of cavity growth change due to temperature and/or stress changes. Edward and Ashby [45] have shown that coupled boundary diffusion/power-law growth is the mode of void growth over wide ranges of stress and temperature for structural metals.

Defining the area fraction of voids as

$$f_h = \frac{r_h^2}{12}$$  \hspace{1cm} (75)

where \( r_h \) is the average void radius and 21 is the average void spacing, then the growth of damage \( f_h \) is described by

$$\dot{f}_h = \frac{\varepsilon_0 \phi_0}{f_h^{1/2} \ln\left(\frac{z}{f_h}\right)} \left(\frac{\sigma_1}{\sigma_0}\right)$$  \hspace{1cm} (76)

for boundary diffusion alone and

$$\dot{f}_h = \varepsilon_0 \beta_0 \left[ \frac{1}{(1-f_h)^n} - (1-f_h) \right] \left(\frac{\sigma}{\sigma_0}\right)^n$$  \hspace{1cm} (77)

for power-law creep alone. In equation (76), \( \sigma_1 \) is the stress acting normal to the boundary on which the void(s) lies and \( \sigma_0 \) and \( \varepsilon_0 \) are normalizing constants in the uniaxial power-law creep equation

$$\dot{\varepsilon} = \varepsilon_0 \left(\frac{\sigma}{\sigma_0}\right)^n$$  \hspace{1cm} (78)

at a given temperature. Also,
where $D_B \delta$ is the boundary diffusivity, $\Omega$ the atomic volume, $k$ is Boltzmann's constant, and $T$ is absolute temperature. In equation (77),

$$\phi_0 = \frac{2\Omega D_B \delta}{kT} \frac{\sigma_0}{\epsilon_0}$$  \hspace{1cm} (79)$$

Comparison of the power-law creep equation (77) with the Rabotnov-Kachanov continuum damage theory in equation (5) with $\mu = \nu$ reveals a great deal of similarity. In fact the forms are identical apart from the second term $(1-f_h)$ in equation (77). When $f_h$ is large, the second term may be neglected and the two sets of equations match with $f_h$ being interpreted as the scalar damage parameter $\omega$. A very significant difference is that the continuum damage model gives a finite damage rate when the damage parameter is zero in contrast to the mechanistic approach, where the damage rate is zero for undamaged material. Cocks and Ashby point to this feature as being clearly indicative of the superiority of the mechanistic approach since non-existent holes do not grow. This is strictly true when damage is defined as void density; from the continuum viewpoint, though, a finite rate of damage when the damage parameter is zero could correspond to void nucleation processes or onset of creep damage.

For diffusion controlled growth of voids, as $f_h \rightarrow 1$,

$$f_h \approx \phi_0 \epsilon_0 \left( \frac{\sigma_1}{\sigma_0} \right) \left( \frac{1}{1-f_h} \right)$$  \hspace{1cm} (81)$$
which is identical to the Rabotnov-Kachanov relation when $\mu = \nu = 1$. As for power-law creep, for large $f_h$, the forms of the mechanistic and continuum models are the same, with only the damage power-law exponent differing between mechanisms. The differences between the mechanistic and continuum damage approaches are most pronounced when $f_h$ is small for both diffusion controlled and power-law creep. In fact, the boundary diffusion growth rate in equation (76) exhibits a decrease as $f_h$ increases for $f_h < .1$. Since Ashby et al. use $f_h = 0.25$ as a realistic failure condition, this behavior would be significant over much of life if voids grow by coupled boundary diffusion and power-law creep. Since Rabotnov-Kachanov type continuum damage laws imply that damage rate always increases with damage (for uniaxial creep loading), they cannot conform to boundary diffusion controlled void growth. Life predictions using the continuum damage law of Rabotnov-Kachanov can therefore be more conservative since power-law void growth is at a relatively high rate.

Cocks and Ashby [43] suggest the inclusion of $\phi_0$ and $(\sigma_1/\sigma_0)$ in the continuum damage approach to account for coupled boundary diffusion and power-law creep. Turning to the consideration of anisotropic damage, however, it seems likely that this refinement is not warranted in this initial study. Previous work on coupled diffusion and power-law creep have not involved rotation of the principal stress axes. The materials science studies do tell us, however, that consideration of the creep damage growth regime is very important in determining damage rates. If a transition from the boundary diffusion regime to power-law growth regime is encountered as voids grow and the net stress increases, rotation of the maximum principal stress may result in a significant under-prediction of rupture time using a power-law damage rate. The directionality of damage, though, is another issue; this directionality will depend on propensity to cavitate, grain boundary sliding,
etc. Grain boundary sliding, for example, results in a concentration of normal stress and an increase of hydrostatic stress in each grain due to constraint of surrounding grains. This effect can be suitably described through the dependence of isochronous stress on the first invariant of stress in the anisotropic damage growth law (c.f. equation (26) or (28)).

INCORPORATION OF DAMAGE IN CONSTITUTIVE EQUATIONS FOR DEFORMATION

To this point, the development of rational forms for tensorial damage have concentrated on description of physical damage and its evolution. In reality, the evolution of creep strain rate is not as highly dependent on the anisotropy of the current state of damage (prior to tertiary creep) as is the damage rate. This is due to the fact that creep strain rate in the primary and secondary regimes is primarily affected by matrix work-hardening and recovery processes. The cavitation and sliding processes which occur at grain boundaries do not greatly affect creep strain rate until the voids or grain boundary cracks coalesce. These comments pertain to power-law creep, grain boundary diffusion, surface diffusion, or coupled mechanisms. Hence, it is clear that

(a) the inelastic strain rate magnitude should be affected by a global or averaged measure of physical damage up to the tertiary stage,

(b) the inelastic strain rate direction should be modified by the directional distribution of physical damage, with this effect increasing in importance as damage approaches a critical level in one or more directions,
(c) the anisotropy of damage does not translate into an equivalent anisotropy of inelastic strain rate, since damage may accumulate as cavitation on grain boundaries oriented normal to the maximum principal stress (anisotropic damage) with no concurrent observation of anisotropic deformation [4-6,10,12],

(d) the magnitude of inelastic strain rate may actually be finite in a direction corresponding to a critical (rupture) damage value [5,9], and

(e) the inelastic strain rate is of a power-law form, dependent primarily on the effective stress [10,13,19,25-30].

It should be noted here that the term inelastic strain rate rather than creep strain rate is used in this section, following the unified or state variable theories of creep deformation. In these theories, no distinction is made between the time-independent plastic and time-dependent creep strains. One model structure can exhibit the essential characteristics of monotonic and cyclic, rate-dependent plasticity, and stress and temperature-dependent creep deformation or relaxation [25-30]. Furthermore, inclusion of backstress and drag stress in these unified theories allows for accurate description of inelastic strain rate direction under nonproportional loading [25-26,46] and matrix work-hardening, respectively.

A fairly general statement of the isothermal unified equations, including both backstress and drag stress, is:

\[
\dot{\varepsilon}^n = f \left( \frac{3J_2}{\kappa^2} \right) (\bar{\sigma} - \bar{\sigma}) \tag{82}
\]

\[
\dot{\bar{\sigma}} = h_\alpha \dot{\varepsilon}^n - d(\bar{\sigma}, T) \varepsilon^n \bar{\sigma} - r_\alpha(\bar{\sigma}, T) \bar{\sigma} \tag{83}
\]
\dot{\kappa} = n_\kappa \varepsilon^n - r_\kappa (\kappa, T)  \quad (84)

where \( h_a \) and \( h_\kappa \) are hardening functions, \( r_a \) and \( r_\kappa \) are static recovery functions, and \( d \) is a dynamic recovery function. \( a \) is backstress, \( \kappa \) is drag stress, \( J_{a2}' = \frac{1}{2} (\dot{a} - a) : (\dot{a} - a) \), and \( \varepsilon^n = (2/3 \varepsilon_n : \varepsilon_n)^{1/2} \). Terms involving temperature rate are dropped for the isothermal case but can be included if necessary. Common forms of the hardening and recovery functions will now be discussed:

\( h_a = \text{constant} \)  \quad (85)

for linear kinematic hardening or

\[ h_a = K_1 + K_2 \exp (-d_2 \zeta) \quad (86) \]

where \( K_1, K_2 \) and \( d_2 \) are material constants and \( \zeta = \dot{a}, \zeta = \varepsilon^n \), or \( \zeta = (\dot{a} - a) : a \). Experiments [26] have shown equation (86) to be most accurate. Also,

\[ h_\kappa = C_1 (\kappa^* - \kappa) \quad (87) \]

or

\[ h_\kappa = C_1 (\kappa^* - \kappa) + f_1 (\varepsilon^n, \varepsilon_n, a) \quad (88) \]

or

\[ h_\kappa = H(I \varepsilon) \left[ \left( \frac{3}{2} a : a \right)^{1/2} - b_1 (b_2 \kappa)^b \right] \quad (89) \]

where

\[ H(I \varepsilon) = [a_1 a_2 \exp \left( a_1 (\kappa - \kappa_0) \right)]^{-1} \quad (90) \]

Equations (89) - (90), formulated by Abrahamson, Cescotto, and Leckie [26,47] can accurately model cyclic hardening or softening. Of course, simple forms such as equation (87) may be sufficient for representation of matrix hardening during creep.
The static recovery functions $r_a$ and $r_\kappa$ are usually of the forms

$$r_a = C_2 (a : a)^{C_3}$$

$$r_\kappa = C_4 (\kappa - \kappa_0)^{C_5}$$

where $C_2$, $C_3$, $C_4$, $C_5$, and $\kappa_0$ are material constants.

Common forms of the modulus function $f(3J_2' / \kappa^2)$ are

$$[D_0 (3J_2'/\kappa^2)^n/J_2']^{1/2}$$

$$[D_0 \exp[-(\kappa^2/3J_2')]^n/J_2']^{1/2}$$

$$[D_0 \sinh(3J_2'/\kappa^2)^m]^{n/J_2'}^{1/2}$$

where $m$, $n$, and $D_0$ are constants [25].

Temperature dependence of an Arrhenius form may be included in the flow rule and recovery terms [25].

The structure of these equations have been derived from uniaxial experiments. Nonproportional loading, however, can introduce errors in the inelastic strain rate direction and work hardening rate $\dot{\kappa}$ [48]. Since the inelastic strain rate direction is governed by evolution of backstress, it is necessary to include nonproportionality effects in either the backstress evolution law or directly in the flow rule. The latter is the approach taken by Murakami and Ohno [22]; it seems more rational, though, to include this effect in the backstress evolution law, i.e.
\[ \dot{\sigma} = h_\alpha v \dot{\varepsilon}^n - d(\alpha,T) \dot{\varepsilon}^n \sigma - r_\alpha(\alpha,T) \sigma \]  

(96)

where unit vector \( v \) is based on a Mroz type hardening rule [23-24,48].

For the evolution of drag stress (isotropic hardening), it has been suggested [25-26] that \( h_\kappa \) be modified to account for additional hardening observed [23-24]. Possible forms for inclusion of this effect are

\[ h_\kappa = C(\psi(\phi)\kappa - \kappa) \]  

(97)

or

\[ h_\kappa = \frac{H(I_{\alpha\varepsilon})}{[\psi(\phi)]^{1/2}} \left[ \psi(\phi) \left( \frac{3}{2} a:a \right)^{1/2} - b_1(b_2\kappa)^3 \right] \]  

(98)

where \( \phi \) \((0 < \phi < 1)\) represents the additional hardening due to nonproportional loading. Proposed forms of \( \phi \) are

\[ \dot{\phi} = \mu(1 - \left| \frac{d}{dt} (\varepsilon_1 - \varepsilon_3) \right| - \phi)(\dot{\varepsilon}_n: \dot{\varepsilon}_n)^{1/2} \]  

(99)

due to McDowell [23-24] \((\varepsilon_1, \varepsilon_3\) are maximum and minimum principal strains), or

\[ \phi = \frac{\dot{\sigma}}{\| \dot{\sigma} \|} : \frac{\sigma}{\| \sigma \|} \]  

(100)

due to Bodner et al. [25]. It is felt that a history dependent measure of \( \phi \) such as equation (99) is more mechanistically desirable, and fits the data better for cyclic loading. An alternative definition for \( \phi \) might be the quotient of the inelastic strain path projection on the maximum inelastic strain direction with the total inelastic strain path length.

In the deformation is creep-dominated, the work of Oytana et al. [7] indicates that the hardening is primarily kinematic, i.e. backstress
evolution. With the backstress, primary, secondary and anelastic creep strains may be represented by the unified constitutive equations. Ohashi and associates [8], working with type 304 stainless steel, have shown that simple time or strain-hardening theory cannot predict transient softening observed in creep after stress reversals or nonproportional stress field rotations. Furthermore, the modified strain-hardening theory of ORNL is not suitable for nonproportional loading. Pure kinematic hardening theory gives excessive creep strain rate after large rotations of the principal stresses. They found that combined isotropic-kinematic theory (unified theory) incorrectly predicted a cycle-by-cycle decrease in amplitude of creep strain during nonproportional stress reversals. This suggests that the component of isotropic hardening should be weak compared to kinematic hardening in concurrence with the conclusions of Oytana et al. One troubling point in Ohashi's work is the use of a constant apportionment factor between isotropic and kinematic hardening (evidently 1/2), not included in most unified theories; this seems to artificially restrict the kinematic hardening. Regardless, it is clear that prediction of creep strain rate after relatively large, nonproportional stress reversals requires quite accurate constitutive laws more refined than of simple or modified strain-hardening type.

Cho and Findley [49-50] have shown the strong influence of aging at temperature on the subsequent creep deformation of type 304 stainless steel at 593°C (ORNL reference heat 9T2796). They include aging through power-law dependence of plastic, viscoelastic, and viscous strains on aging time. The same sort of manipulation could be accomplished by a power-law dependence of inelastic strain rate on aging time in the unified theories. These aging effects must be represented as Heaviside functions in rate-type constitutive laws and are therefore somewhat difficult to formulate [51]. In principle,
aging effects are not seen to be strongly related to the formulation of anisotropic damage and deformation constitutive laws and will not be considered at the present time. The nonproportional test program in this study does not investigate these effects, with the exception of aging which may occur continuously during the tests.

As mentioned earlier, experimental evidence suggests that the creep strain rate equations is relatively insensitive to damage anisotropy until tertiary creep. It follows that directional internal stresses and isotropic hardening in the matrix should develop with weak dependence on anisotropy of damage through the secondary creep stage. There seems to be two alternative methods to achieve this weak dependence. First the components of the damage tensor can remain small until the onset of tertiary creep. In this case, following a Kachanov-Rabotnov approach as in equation (6) (e.g. Chaboche),

\[ \dot{\varepsilon}^c = \mathcal{F} (\dot{\psi}, \kappa, \bar{\sigma}' - \bar{\sigma}, T) \]  

(101)

where \( \dot{\psi} = (I-D)^{-1} \) as in the Murakami-Ohno approach, and \( \bar{\sigma}' \) and \( \bar{\sigma} \) are the deviatoric modified net stress and backstress tensors. Such an expression would account for continuity of creep strain rate with respect to damage if damage is isotropic, or anisotropy of creep strain rate (reflected through net stress \( \bar{\sigma}' - \bar{\sigma} \)) if damage is anisotropic or mixed. These results are compatible with the experimental findings of Trampczynski and Hayhurst [4] for copper, aluminum, and Nimonic 80A.

The other method to include damage in the inelastic strain rate is to provide isotropic dependence on the first invariant of damage. This approach, stated in equations (54) - (58) for strain-hardening theory, seems adequate up to the tertiary creep stage since the cavity volume fraction is low for most
metais up to this stage, and anisotropy in the primary creep deformation subsequent to stress reversals can be accounted for through backstress in the unified theories [25-30]. Of course, the transition from secondary to tertiary creep is accompanied by an attendant increase in dependence of the inelastic strain rate on damage anisotropy.

Based on the current experimental evidence and state-of-the-art, a suggested initial form of the unified equations with inclusion of anisotropic damage is

\[
\dot{\varepsilon}^n = f \left\{ \frac{3\bar{J}_2}{\kappa^2} \right\} (\bar{s} - \bar{a}) 
\]

(102)

\[
\dot{\varepsilon}^n = h_a \varepsilon^n - d(\bar{a}, T) \varepsilon^n - r_a(\bar{a}, T) \bar{a} 
\]

(103)

\[
\dot{\varepsilon} = h_\kappa \varepsilon^n - r_\kappa(\kappa, T) 
\]

(104)

where

\[
\bar{s} = \gamma(D_{kk})(1 + C_k D_{kk}) \bar{a} + (1 - \gamma(D_{kk})) \bar{S} 
\]

(105)

and \( \bar{a} \) is similarly defined. The function \( \gamma(D_{kk}) \) introduces a smooth transition from isotropic dependence on damage \( \gamma(D_{kk}) = 1 \) to complete anisotropic dependence on net stress \( \bar{S} \) \( \gamma(D_{kk}) = 0 \) which is highly dependent on the tensorial character of damage. It is anticipated that \( \gamma(D_{kk}) \approx 1 \) based on previous results, at least if the cavity volume fraction at rupture is relatively low. Otherwise, results of Murakami [19] indicate that a power-law dependence on \( D_{kk} \) may be appropriate.

In equation (102), \( \bar{s}^i \) is the deviatoric modified net section stress, and

\[
\bar{s}^i = \frac{1}{2} \left( \bar{s}^i - \bar{a} \right) : (\bar{s}^i - \bar{a}) 
\]
SUMMARY OF INITIAL SUGGESTED ANISOTROPIC DAMAGE FORMULATION

At this point, it is desirable to propose a general framework for coupled anisotropic damage evolution and creep strain rate equations so that it is applicable to a number of metals. The specialization for type 304 stainless steel will be made on the basis of known creep damage mechanisms, results of previous work, and results of the current test program at ORNL in conjunction with this project. A proposed general framework for the isothermal coupled damage-deformation equations is:

\[
\dot{D} = B[\sigma^*(S)]^k[nJ(\phi;\phi)^{\phi/2} + (1 - n) \sum_{j=1}^{3} \gamma^{(j)} \otimes \gamma^{(j)} M(\gamma^{(j)}, \phi)]
\]

where \( n \), \( B \), \( k \), and \( \varepsilon \) are material constants, and

\[
M(\gamma^{(j)}, \phi) = (\phi;\phi)^{\phi/2} M^*(\sigma_j^{(j)})
\]

or

\[
M(\gamma^{(j)}, \phi) = \left[ \frac{1}{1 - \gamma^{(j)} \cdot \sigma^{(j)}} \right] ^{\phi/2} M^*(\sigma_j^{(j)})
\]

where \( M^*(\sigma_j^{(j)}) \) admits anisotropic contribution of non-maximal principal net stresses, and \( \sigma^*(S) \) is the isochronous stress given by

\[
\sigma^*(S) = \frac{3}{2} H_1 \left( \frac{2}{3} \frac{S_{eq}}{H_1} \right) a \exp \left[ b \left( \frac{S_{kk}}{S} - 1 \right) \right]
\]

where

\[
S_{eq} = ((3/2) S ; :S')^{1/2}
\]

\[
H_1 = S_1 - S_{kk}/3
\]

\[
S' = S - (1/3) S_{kk}
\]
\[ S_s = (S_1^2 + S_2^2 + S_3^2)^{1/2} \]

\[ \mathbf{\tilde{S}} \equiv \text{net stress tensor} = \frac{1}{2} (\mathbf{g} \cdot \mathbf{\hat{g}} + \mathbf{\hat{g}} \cdot \mathbf{\hat{g}}) \]

\[ \mathbf{\hat{g}} = (\mathbf{I} - \mathbf{D})^{-1} \]

\[ \mathbf{\tilde{D}} \equiv \text{2nd order damage tensor} \]

\[ \mathbf{v}^{(j)} \equiv \text{unit vector in principal net stress direction, } S_j \]

and \( a \) and \( b \) are material constants given by Huddleston [16].

The rupture criterion is given by

\[ D_j = 1, j = 1, 2, \text{ or } 3 \]

where \( D_j \) are the principal values of \( \mathbf{\tilde{D}} \).

Suggested coupling with an appropriate unified creep-plasticity theory is given by:

\[ \dot{\varepsilon}^n = D_0 \exp \left\{ -\frac{1}{2} \left\{ \frac{\kappa^2}{3\mathbf{1}} \right\} n \right\} \left[ \frac{\mathbf{\dot{v}}^n - \mathbf{\hat{a}}}{(\mathbf{\hat{v}}^n)^{1/2}} \right] \]

\[ \dot{\alpha}^n = h_\alpha (\varepsilon^n : \mathbf{\hat{v}}) \mathbf{\hat{v}} = r_\alpha \alpha^n \]

\[ \dot{\kappa}^n = h_\kappa \varepsilon^n \mathbf{\hat{k}} = r_\kappa \]

- 44 -
where the dynamic recovery term in backstress evolution has been neglected and

\[ \dot{\bar{\Sigma}} = \gamma(D_{kk}) (1 + C_1 D_{kk}) \bar{\sigma} + (1 - \gamma(D_{kk})) \dot{\bar{\sigma}} \]

\[ \dot{a} = \frac{1}{2} (a \cdot \dot{\sigma} + \dot{\sigma} \cdot a) \]

\[ \dot{\bar{\sigma}} = \frac{1}{2} (\bar{\sigma} \cdot \dot{\sigma} + \dot{\sigma} \cdot \bar{\sigma}) \]

\[ \dot{\Sigma} = (R/\kappa)(\dot{\Sigma} - \bar{\sigma}) - \dot{\sigma} \]

\[ \dot{\Sigma}' = \dot{\Sigma} - \frac{1}{3} \bar{\sigma}_{kk} \dot{\sigma} \]

\[ \dot{\bar{\Sigma}}' = \frac{1}{2} (\dot{\Sigma}' - \bar{\sigma}); (\dot{\Sigma}' - \bar{\sigma}) \]

\[ h_{\alpha} = \left\{ \begin{array}{ll}
K_1 + K_2 \exp (-d_2 \sqrt{\frac{\bar{\sigma}}{\bar{\sigma}}} : \bar{\sigma}) & \text{if } \dot{\bar{\sigma}} : \bar{\sigma} > 0 \\
1/d_1 & \text{if } \dot{\bar{\sigma}} : \bar{\sigma} < 0 \end{array} \right. \]

\[ h_{\kappa} = \frac{H(I_{\alpha \kappa})}{[\psi(\phi)]^{1/2}} \left[ \psi(\phi) \left( \frac{3}{2} \frac{\bar{\sigma}}{\bar{\sigma}} \right)^{1/2} - b_1(b_2 \kappa) b_3 \right] \]

\[ r_{\alpha} = C_2(\bar{a} : \bar{a}) C_3 \]

\[ H(I_{\alpha \kappa}) = [a_1 a_2 \exp (a_1(\kappa - \kappa_0))]^{-1} \]

\[ \phi \equiv \text{nonproportional hardening state variable} \]
\[ r_\kappa = \nu_4 (k - k_1) C_5 \]

In the above, \( R, D_0, n, C_1, K_1, K_2, d_2, d_1, C_2, C_3, C_4, C_5, a_1, a_2, b_1, b_2, b_3 \) are material constants and \( \gamma(D_{kk}) \) is a material function. This represents a "maximal" form of the equations, but not the most efficient.

It may be appropriate to introduce reduced forms of the hardening and recovery functions which are sufficient for creep dominated deformation. The above unified equations can predict transient nonproportional cyclic plasticity [46] in addition to creep [25-30]. Note the introduction of the Mroz type rule through the use of directional index \( \tilde{\gamma} \). Whatever form of the unified constitutive equations is used, the initially proposed coupling with damage would be the same.
REFERENCES


APPENDIX

Summary of Progress on Task Sets A and B

The first-year effort in this project is divided between two tasks, A and B. Task A involves the development of anisotropic damage constitutive equations, while task B regards the quantification of creep damage anisotropy for ruptured specimens loaded nonproportionally at ORNL (see Section II). The material is type 304 stainless steel; test temperature is 593°C. The specimens are being tested at an isochronous stress corresponding to a rupture time of approximately 1000 hours.

The main body of this paper reports progress made in Task Set A (see Section I) regarding review of existing models for continuum anisotropic damage. Therefore, Task 1 of Task Set A has been completed, and significant progress made on Task 2 of Task Set A.

(I) Task Sets A and B

Outline of Tasks:

The following tasks have been identified to meet program objectives. There are two parallel sets of tasks pertaining to analytical formulation and physical measurement of creep damage, respectively.

Task Set A

Task 1: Review of existing models for

a. Continuum creep damage

b. Anisotropic representations of creep damage

c. Coupled creep deformation-damage equations

i. Decomposition of time-independent plastic and creep strains

ii. Unified theories for which this decomposition is not made

Review of literature pertaining to:
d. Relationship of physical creep damage (cavitation, coalescence, grain boundary sliding) to primary loading directions for nonproportional histories.

e. Physical damage processes in type 304 stainless steel that must be reflected by anisotropic damage theory.

Duration: 1/1/85 through 3/30/85

Task 2: Damage Model Development

a. Development of the damage tensor $\tilde{D}$ (appropriate rank, symmetry conditions, relationship to physical damage). If existing forms for $\tilde{D}$ are inappropriate, more general or pertinent forms will be developed.

b. Development of the evolution equations for growth of damage $\tilde{D}$. Equations must depend on current state of loading and material in addition to history of loading.

Duration: 4/1/85 through 6/30/85

Task 3: Development of Couplings Between Damage and Deformation Models

a. Investigation of creep strain rate equations appropriate for nonproportional loading.

b. Introduction of damage $\tilde{D}$ into creep strain rate equation.

Duration: 7/1/85 through 8/15/85
Task 4: Demonstration of capability of model to correlate experimental data obtained at ORNL. Guidance for further model development.

Duration: 8/16/85 through 9/30/85

Task Set B

Physical Damage Measurement

Task 1: Development of appropriate quantitative measurement techniques for creep damage in type 304 stainless steel.

Duration: 1/1/85 through 4/30/85

Task 2: Quantitative measurement of magnitude and direction of creep damage (interface with analytical development).

Duration: 1/1/85 through 9/30/85

(II). Year One Test Matrix

Program Outline

Definitions:  \( \sigma = \) axial stress
\( \tau = \) shear stress
\( \varepsilon = \) axial strain
\( \gamma = \) engineering shear strain
Year 1

Material: type 304 stainless steel

Proportional Baseline tests: (593°C)

Test #1: $\sigma/3\tau = 1$; maximum principal stress at 16.8° to longitudinal axis, $\sigma$ positive.

Test #2: $\sigma/3\tau = -1$; maximum principal stress same magnitude as for test #1, but with different orientation, 33.7° from that of test #1.

Test #3: $\sigma/3\tau = 1$; for 1/2 life of test #1; reverse shear stress such that $\sigma/3\tau = -1$ for remaining life. Same values of $\sigma$ and $\tau$ as tests 1 and 2.

Test #4: replicate of test #3.

Test #5: $\sigma/3\tau = 1$; a number of alternating periods of equal duration with $\sigma/3\tau = -1$ and $\sigma/3\tau = 1$.

Test #6: replicate of test #5.

Test #7: $\tau/\sigma = \infty$; pure torsion with positive shear stress.

Test #8: $\sigma/\tau = 0$; pure torsion with positive shear stress for 1/2 life, switching to negative shear stress of same magnitude for remaining life.

Notes:
(a) All imposed states of stress lie on same isochronous surface.
(b) Tests 1 and 2 provide baseline damage state and lives.
(c) Tests 3 and 4 investigate anisotropy of damage on the basis of life and the distribution of void formation and growth.
(d) Tests 5 and 6 introduce "cycling" to assess directional creep sequence effects.
(e) Tests 7 and 8 compare damage at same equivalent stress state, but with different maximum positive principal stress level and a 90° rotation of the maximum principal stress direction.

(f) The complete total strain versus time history for each test will be recorded.

(g) Tests 1-6 are similar to those performed by Hayhurst et al. [4-6] on copper and aluminum.

As of May 15, 1985, ruptured test specimens have not been received from ORNL for metallurgical quantification. Hence, progress to date on Task Set B has been limited to identification of techniques of damage measurement and quantification. It is important to quantify creep damage in terms of a percentage for the triple point, or wedge type cracking, and the occurrence as a function of orientation for the case of microvoids.

Sections will be cut normal to the principal tensile stress directions at first, and photomicrographs used for measurement. To determine the triple point damage is a simple exercise, performed by sectioning normal to the principal tensile stress, laying down a test grid over selected micrographs, and counting the intersections of both the wedge cracks \((p_{\text{wc}})_{\text{tp}}\) and of all the triple points of the grain boundaries \((p_{\text{tp}})_{\text{tp}}\). The ratio of these gives the

\[
\% \left( \frac{(p_{\text{wc}})_{\text{tp}}}{(p_{\text{tp}})_{\text{tp}}} \right) = \left( \frac{(p_{\text{wc}})_{\text{tp}}}{(p_{\text{tp}})_{\text{tp}}} \right) \times 100
\]

percentage of cracking occurring for this section plane. Stereologically, the point fraction is equivalent to the volume fraction \((V_{\text{wc}})_{\text{tp}}\)

\[
(V_{\text{wc}})_{\text{tp}} = \left( \frac{(p_{\text{wc}})_{\text{tp}}}{(p_{\text{tp}})_{\text{tp}}} \right)
\]
Quantification of orientational dependence of void formation and growth will be accomplished by sectioning the tested specimens in orientations both normal and parallel to the principal tensile stress directions in the loading history. The normal section will be used for determining the number of voids per unit area of the section plane \((P_A)_{mv}\). The parallel or longitudinal section will be evaluated in the following manner. Photomicrographs will be taken from various longitudinal positions of the specimens with respect to the known orientation of the principal tensile stress. A set of parallel lines with an equivalent spacing will then be superimposed on the micrographs, and a point count of the number of intersections of voids per unit length of test line \((P_L)_{mv}\) will be reported. Initially, the orientation of the grid, corresponding to \(\theta = 0^\circ\), will be normal to the principal tensile stress, and subsequent rotation will be clockwise in increments of \(5^\circ\). This will provide data for generation of plots of \((P_L)_{mv}\) as a function of \(\theta\), to determine orientational dependence of void formation as it relates to loading histories.

Actual measurements of all variables will be performed using semi-automated equipment, whereby each wedge crack or void will be digitized and stored on a microcomputer. Software will then be used to read coordinates of the "defects", orient them in terms of the principal tensile stress(es), connect selected points to create ligaments, and assign normals to these ligaments. With the known orientation and defined normal of the ligaments, in addition to the occurrence in the cross section, relationships can be developed which provide some quantification of the damage in three dimensions. This will provide input for comparison with the model.

Previous experimental work on austenitic stainless steels (304 in particular) give preliminary notions regarding the anisotropy of damage and the structure of the coupled deformation-damage equations. Baik and Raj [A1]
report that cavitation is more difficult in the austenitic stainless steels, with a tendency toward wedge-cracking at high stress levels. For an isochronous stress level of 26 ksi (approximately 1000 hr. rupture time at 593°C) [A2] and a homologous temperature of 0.48, the deformation mechanism maps from Ashby [A3] for type 304 stainless steel (grain sizes of 50μm and 200μm) indicate that power-law creep is the dominant deformation mechanism. Since the boundary between power-law creep and diffusional flow is fairly well-defined for type 304 [A3], it would be expected that a power-law based damage growth rate would be appropriate, at least for proportional loading. The issue, however, may be complicated by rotation of the principal stresses, with a possible transition from power-law creep to diffusion-controlled creep or vice-versa on specific material planes. It seems appropriate to initially use a power-law creep damage rate equation even for the nonproportional case.

It is possible for 304 stainless steel that the anisotropy of damage will be dependent upon whether wedge-cracking or cavitation mechanisms are dominant. Previous work has shown that this material is quite well characterized by a deviatoric flow rule; hence, the proposed constitutive equations for deformation should be quite appropriate, even into the tertiary stage of creep.

**APPENDIX REFERENCES**


Martin Marietta Energy Systems

Final Report

ANISOTROPY OF CREEP-FATIGUE DEFORMATION AND DAMAGE UNDER NONPROPORTIONAL LOADING

for the period
January 1, 1985 through February 28, 1986

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May 1986
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BASES OF CONTINUUM DAMAGE MECHANICS

The notion of applying continuum mechanics principles to "track" creep damage was first proposed by Kachanov [1-2]. Since creep damage occurs in a number of grains and grain boundaries, it can be treated in a bulk-averaged sense. Creep damage mechanisms are relatively well-understood; voids nucleate, grow and coalesce driven by diffusion processes, viscous creep of surrounding matrix material, or coupled processes [3].

Continuum damage mechanics seeks to reflect the growth of cavities and the mechanical behavior of damaged material by representing physical creep damage (cavitation) by internal mechanical variables. These mechanical variables, then, modify the creep strain rate equations to produce transition from secondary to tertiary creep (void coalescence). The advantage to such an approach is that damage is treated as a path-dependent variable coupled with nonlinear stress analysis.

Isotropic Damage Formulations

If physical creep damage is represented by a scalar, then no directional dependence or variation of damage is assumed. Nonproportional creep tests [4-9] have shown that creep damage does have a directional character; changing the maximum principal stress orientation may result in different subsequent creep damage and strain rates than would be predicted by using a scalar damage variable. We will begin discussion of continuum damage mechanics with a review of isotropic damage models, since much of the rationale for extension to anisotropic damage is derived from these models.

(i) Kachanov's model:

Kachanov [1-2] defined the evolution of a "continuity" variable $\psi$ at a point $X$ by
\[
\dot{\psi} = - B \left( \frac{\sigma_{\text{max}}}{\psi} \right) ^\nu
\]  \hspace{1cm} (1)

with initial condition \( \psi = 1 \) in the undamaged state. Here, \( B \) and \( \nu \) are material constants and \( \sigma_{\text{max}} \) is the maximum tensile stress at point \( X \) in the steady creep field in the perpendicular direction of creep crack growth. Fracture occurs when \( \psi = 0 \).

Kachanov did not discuss the relationship of \( \psi \) to physical damage, nor prediction of strain at rupture. In essence, the creep strain rate was assumed unaffected by the presence of damage.

(ii) Rabotnov's model:

Rabotnov [10-11] generalized Kachanov's model to predict rupture strain in addition to rupture time. He defined a damage parameter \( \omega \) by

\[
\omega = 1 - \psi
\]  \hspace{1cm} (2)

which evolves according to

\[
\dot{\omega} = G(\sigma, \omega)
\]  \hspace{1cm} (3)

For uniaxial loading. Rabotnov also allowed the uniaxial creep strain rate to depend on \( \omega \), i.e.

\[
\dot{\varepsilon}^C = F(\sigma, \omega)
\]  \hspace{1cm} (4)

where \( \omega = 0 \) represents the undamaged state, and \( \omega = 1 \) the ruptured state. The particular forms of these equations were written as
\[ \dot{\omega} = B\sigma^v/(1 - \omega)^\mu \quad (5) \]

\[ \dot{\varepsilon}^c = A\sigma^n/(1 - \omega)^m \quad (6) \]

where \( A, B, m, n, \mu \) and \( v \) are material constants, dependent on temperature.

It should be noted that the term \((1 - \omega)\) is often interpreted in the literature as the reduction in area due to the presence of voids or cracks in the material, i.e.

\[ (1 - \omega) = \frac{A_a - A_r}{A_a} \quad (7) \]

where \( A_r \) is the reference area (undamaged) and \( A_a \) is the reduced (due to damage) area. Rabotnov did not specifically make this interpretation. As a consequence of equation (7), the net stress may be expressed as

\[ S = \sigma/(1 - \omega) \quad (8) \]

such that \( S \) reflects the increase in stress due to reduced load bearing area. Equation (6) can be identified as Norton's creep law if \( m = n \) written in terms of net stress. These equations can be easily integrated for conventional creep tests (constant \( \sigma \)) when the deformation at failure can be considered small.

(iii) **Leckie-Hayhurst model:**

The model of Rabotnov was derived for unaxial creep behavior. Leckie and Hayhurst [4-6,10,12] generalized the Rabotnov approach to multiaxial loading. The effect of state of stress (for proportional loading) on rupture is normally expressed in terms of isochronous equi-damage surfaces which Hayhurst and Leckie expressed as
\[
\sigma^*(\sigma_{ij}) = \alpha \sigma_1 + \beta \bar{\sigma} + (1 - \alpha - \beta) \sigma_{kk}
\]  
(9)

where \(\sigma_1\) is the maximum principal stress, effective stress \(\bar{\sigma} = (3\sigma_{ij} \sigma_{ij}/2)^{1/2}\), \(\sigma_{ij}\) is the deviatoric stress tensor, and \(\sigma_{kk}\) is the first invariant of stress. Chaboche [13] suggests that net stress be used in computation of \(\sigma_1\), \(\bar{\sigma}\) and \(\sigma_{kk}\) in equation (9) to reflect damage growth.

Leckie and Hayhurst's multiaxial generalization now takes the form

\[
\dot{\varepsilon}_{ij}^C = (3A/2)[\bar{\sigma}/(1 - \omega)]^n(\sigma_{ij}/\bar{\sigma})
\]

(10)

\[
\dot{\omega} = B(\sigma^*)^\nu/(1 - \omega)^\mu
\]

(11)

where \(\alpha, \beta, A, B, n, \nu\) and \(\mu\) are temperature-dependent material constants.

Two comments should be made regarding equations (10) and (11). First, for a creep test with fixed \(\sigma_{ij}\), the ratio of creep strain rate components approximately follow a deviatoric flow rule even in the presence of damage. This point was made by Leckie [5-6,10]. Damage growth acts to accelerate the creep strain rate through \(\bar{\sigma}/(1 - \omega)\) in equation (1). Secondly, use of \(\sigma^*\) in equation (11) allows for correlation of materials which are maximum principal stress dependent \((\alpha + 1)\) or effective stress dependent \((\beta + 1)\). Increased damage rate due to higher hydrostatic tensile stress is also included.

Though different forms of isochronous surfaces are permitted by variation of \(\alpha\) and \(\beta\), equations (9) - (11) can only apply to proportional loading \((i.e. \sigma_{ij}/\bar{\sigma} \text{ fixed for all } i \text{ and } j)\), or for nonproportional loading for materials which damage isotropically. A material is defined to damage isotropically if rotations of the principal stresses (or deviatoric stresses)
would result in the same rupture time at the same isochronous stress level as for a uniaxial test. Hence, for nonproportional loading of materials which damage anisotropically, equations (9) - (11) cannot be used since \( \omega \) is a scalar parameter. It should also be noted that equation (10) may not be applicable for combined creep-fatigue loading since no measures of work-hardening or backstress are included.

(iv) Chaboche's model:

Chaboche makes the interpretation of \( 1 - \omega \) as reduced area and defines the net stress as

\[
S = \sigma/(1 - D)
\]

(12)

where \( \omega \) is replaced by \( D \), the scalar damage parameter. He then proceeds to couple static plastic damage \( D_1 \), creep damage \( D_2 \), and fatigue damage \( D_3 \):

\[
dD_1 = f_1(\phi, \alpha, D_1, D_2, D_3, ...)d\sigma
\]

(13)

\[
dD_2 = f_2(\phi, \alpha, D_1, D_2, D_3, ...)dt
\]

(14)

\[
dD_3 = f_3(\phi, \alpha, D_1, D_2, D_3, ...)dN
\]

(15)

where \( \phi \) is the forcing variable, \( \alpha \) represents internal variables describing hardening state, and \( N \) represents cycles.

Chaboche and Lemaitre [14], assuming creep and fatigue damage to be additive, have defined creep-fatigue damage in the following way:

\[
dD_2 = f_2(\sigma, (D_2 + D_3))dt
\]

(16)
\[ dD_3 = f_3(\Delta \sigma, \sigma_m, (D_2 + D_3))dN \]  

(17)

where \( \Delta \sigma \) and \( \sigma_m \) are stress range and mean stress, respectively. Defining \( D = D_2 + D_3 \) gives

\[ dD = f_2(\sigma, D)dt + f_3(\Delta \sigma, \sigma_m, D)dN \]  

(18)

A Rabotnov-Kachanov approach was taken for both creep and fatigue damage,

\[ dD = C_{ur}(1 - 0)^{-k(\sigma)} dt + [1 - (1 - D)^{p+1}]^r(\Delta \sigma)[\frac{\Delta \sigma}{M(\sigma_m)(1-D)}] \beta dN \]  

(19)

where the first term is the creep damage increment. \( k(\sigma), \alpha(\Delta \sigma), M(\sigma_m), r \) and \( \beta \) are temperature-dependent functions and material constants.

Note the stress dependence of the exponents in the damage rate equation (19). This allows for nonlinear damage accumulation rather than a unique relationship between time fraction and damage.

Chaboche also introduced isotropic and kinematic hardening variables in the uniaxial strain rate equation to model cyclic viscoplasticity in addition to conventional creep strain, i.e.

\[ \dot{\varepsilon}^{in} = <\frac{|S - X^e| - R}{K}>^n \text{sign} (S - X^e) \]  

(20)

\[ X^e = C_f(p)(a \dot{\varepsilon}^{in} - X^e|\dot{\varepsilon}^{in}|) - bX^e|m| \text{sign}(X^e) \]  

(21)

where \( R \) and \( X^e \) are isotropic and kinematic hardening variables, and \( n, K, C, a, b, \) and \( m \) are temperature-dependent coefficients. Also,
\[ f(p) = \varepsilon + (1 - \varepsilon)e^{-\beta p}, \quad p = \int_0^t |\varepsilon^{\text{in}}(\tau)|d\tau \]  

(22)

where \( \beta \) and \( \varepsilon \) are temperature-dependent material constants. The inelastic strain rate is given by \( \varepsilon^{\text{in}} \). Note that net stress is used in equations (20) - (22) to introduce coupling of the inelastic strain rate with damage. In equation (20), \( \langle Y \rangle = YH(Y) \), \( H \) being the unit step function. Note that \( X^e \) represents a net backstress for consistency since net stress is used in equations (20) - (21).

Equations for viscoplastic strain rate with an internal variable structure as in equations (20) - (22) can describe effects of unloading and cyclic loading unlike the more restricted creep strain rate equations derived from a dissipative potential (e.g. equations (6) and (10)). These effects include nonlinear hardening, Bauschinger effects, cyclic hardening or softening, and recovery. Additional state variables may be introduced to account for microstructural aging effects or transformation due to temperature changes.

**ANISOTROPIC DAMAGE FORMULATION**

The Rabotnov-Kachanov scalar damage parameter \( \omega \) is useful for describing creep damage under uniaxial or proportional loading conditions. Use of an isochronous surface to correlate multiaxial creep rupture is in general valid only for proportional loading; furthermore, the isochronous surface is essentially a description of "initial" creep anisotropy rather than deformation- or damage-induced anisotropy. This distinction is important since it leads to the need to define a damage tensor. Anisotropic deformation of the isochronous surface is equivalent to tensorial damage for loading histories in which principal stress axes rotate.
One is guided in formulation of anisotropic damage by knowledge of dominant cavity growth mechanisms, relevant measures of driving stress, and the general framework of continuum damage mechanics.

Trampczynski, Hayhurst, and Leckie [4-5] conducted experiments on aluminum and copper. Copper cavitates much more easily than the aluminum or austenitic stainless steels [15]. They subjected thin-walled tubular specimens to an axial stress of 42.5 MPa and shear stress of 14.2 MPa for nearly the entire steady load lifetime, then reversed the shear stress to -14.2 MPa. For aluminum, the damage was well-distributed along grain boundaries, and the shear stress reversal resulted in no change of creep strain rates, damage rates, or rupture time. In contrast, damage in the form of cavity nucleation and growth occurred in copper on grain boundaries normal to the maximum principal stress. The shear stress reversal resulted in an increase in rupture time by a factor of approximately two for copper. A decrease in creep strain rates was attributed by the authors to deformation-induced anisotropy which is not accounted for in equation (10).

Hence, the anisotropy of damage exhibited by copper cannot be described by the generalized Rabotnov-Kachanov equations with scalar damage parameter $\omega$. The evolution of damage must be computed separately for each of the two planes normal to the maximum principal stress. For aluminum, though, they are satisfactory provided the inelastic strain is predominately viscous or creep strain during and after rotation of principal stress axes. It is interesting to note that the isochronous surfaces for copper and aluminum follow maximum principal stress and effective stress criterion, respectively. The experimental results for nonproportional loading would then imply that the isochronous locus for aluminum is underformed while that for copper is deformed, consistent with the earlier notion of damage-induced anisotropy of the isochronous surface.
At this point, it should be noted that more precise forms of isochronous surfaces have been introduced. Huddleston [16] introduced the form

$$\sigma^*(\sigma_{ij}) = \frac{3}{2} H_1 \left( \frac{2}{3} \frac{\sigma}{H_1} \right)^a \exp \left[ b \left( \frac{J_1}{S} - 1 \right) \right]$$

where

$$H_1 = \sigma_1 - J_1/3, \quad J_1 = \sigma_{KK},$$

$$S_s = (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)^{1/2},$$

and a and b are temperature-dependent material constants. The results for type 304 stainless steel tested at 593°C show much better correlation with equation (23) than with effective stress $\bar{\sigma}$, maximum shear stress, or maximum principal stress. Hence, Huddleston's formulation would be a likely candidate for inclusion in a damage rate equation such as equation (11). Of course, the net stress can be used to compute all stress-related variables in equation (23).

There are several classifications of approaches which will be reviewed for generalization to anisotropic damage:

(i) a scalar damage parameter with a tensor multiplier to apportion damage effects among the various stress components,

(ii) definition of physical creep damage (e.g. cavity or fissure density) in terms of an appropriate rank tensor,

(iii) measure of deterioration of elastic behavior, and

(iv) creep damage potential functions.
(1) Tensor Multiplier Functions

In this approach, damage growth is governed by a scalar, but the growth of damage influences various stress-strain components differently. This representation of damage is implied by the procedure of Leckie et al. [5-6,10] which involved separate calculation of the damage on two non-interacting planes in copper. Defining the scalar damage parameter as $\omega$, the rate of growth of the damage tensor $\mathbf{D}$ is given by

$$\dot{\mathbf{D}} = \mathbf{Q}(\mathbf{S}) \dot{\omega}$$  \hspace{1cm} (24)

where $\mathbf{Q}(\mathbf{S})$ is an operator of the same rank as $\mathbf{D}$ which defines the preferential orientation of cavity growth and crack formation. Note the dependence of the directionality of damage growth on the net stress tensor $\mathbf{S}$. This is due to the fact that the effective net stress exceeds the effective true stress, and rotation of the net stress tensor relative to the true stress tensor will occur as cavitation proceeds if the principal axes of true stress are rotated (i.e. nonproportional loading).

As stated by Hayhurst [17], it is likely that most metals suffer creep damage of a mixed cavitation/wedge cracking nature along grain boundaries; copper and aluminum offer two "bounds" of mostly cavitation damage and mostly grain boundary sliding-induced wedge cracking, respectively. From an analytical standpoint, this means that if $\mathbf{Q}(\mathbf{S})$ is defined as

$$\mathbf{Q} = \gamma \mathbf{I} + (1 - \gamma) \mathbf{\Sigma}$$  \hspace{1cm} (25)

where $\mathbf{I}$ is the identity tensor and $\mathbf{\Sigma}$ apportions damage anisotropy, then $\gamma = 0$ for copper and $\gamma = 1$ for aluminum, at least in the range of stresses tested by Leckie and associates. From consideration of deformation
mechanism maps [18], to be discussed in a later section, it is quite possible
that both $\gamma$ and $\tau$ are net stress and temperature dependent.

Studies which use the form of $Q$ given in equation (25) usually assume
damage is represented adequately by a second order tensor, which is only an
approximation of physical damage for the general nonproportional loading
case. Chaboche [13] extends the stress level dependence of the exponent of
damage in the extended Rabotnov-Kachanov damage law to describe scalar damage
evolution, i.e.

$$\dot{\omega} = <\frac{\sigma^*}{A}\gamma/(1-\omega)k(\sigma^*)$$

where we again note the use of the isochronous surface concept. Chaboche
suggested the use of net stress to compute $\sigma^*$ within the Macauley bracket in
this equation.

Murakami and Ohno, using a second rank symmetric damage tensor, have
suggested the general form [9,19]

$$\mathbf{D} = \Omega \mathbf{I} + \sum_i M^{(i)} [\mathbf{\nu}^{(i)} \otimes \mathbf{\nu}^{(i)}] + \sum_j N^{(j)} [\mathbf{\nu}^{(j)} \otimes \mathbf{\nu}^{(j)}]$$

(26)

where $\Omega$ and $M^{(i)}$ are scalar functions of net stress, temperature and other
internal variables, $N^{(j)}$ is a fourth order tensor, and $\mathbf{\nu}^{(i)}$ and $\mathbf{\nu}^{(j)}$ are unit
normal vectors in the positive principal value directions of the net stress
tensor $\mathbf{S}$ and its deviator, $S_D = S - (1/3)S_{kk} \mathbf{I}$. The symbol $\otimes$ stands for outer
product. A particular form which seems to encompass the bounding behaviors of
copper and aluminum is given by

$$\mathbf{D} = B[\sigma^*(S)]^k[\eta I + (1-\eta)\mathbf{\nu}^{(1)} \otimes \mathbf{\nu}^{(1)}] (\mathbf{a} : \mathbf{a})^{\nu/2}$$

(27)
where

\[ \hat{\varepsilon} = (I - D)^{-1} . \]

Here, B, k, n and \( \varepsilon \) are material constants. Note the similarity of equation (27) with Chaboche's approach. The inverse dependence on damage is introduced through \( \hat{\varepsilon} \) in a consistent way in the Murakami-Ohno approach, without recourse to an additional scalar growth law. It should also be noted that \( \varepsilon \) in equation (27) could be made a function of net stress to correspond to Chaboche's modification to achieve a nonlinear damage versus time fraction relationship. Hayhurst and Leckie's generalization of the Rabotnov-Kachanov approach in equation (11) is equivalent to equation (27) if \( n = 1 \), i.e. isotropic damage is assumed.

Murakami and Ohno have also provided a rational definition of the net stress tensor consistent with area reduction due to physical damage. This will be discussed later in the section on anisotropic representation of physical damage. It should be emphasized that a second order damage tensor can only approximately reflect the effects of physical damage on the damage rate for general nonproportional loading. The computational simplicity of this approach is desirable, though, provided it is relatively accurate.

One item regarding equation (27) which appears to have been overlooked in the literature is the possibility that the rate of damage growth in a given direction is dependent upon the damage in that direction rather than the equivalent scalar damage, i.e.

\[ \dot{D} = B[\sigma^*(S)]^k (1 - \eta) \gamma(1)^2 + (1 - \eta) \dot{\gamma}(1) \bullet \dot{\gamma}(1) \left[ \frac{1}{1 - \gamma(1) \cdot D \cdot \gamma(1)} \right]^2 \]  (28)
Of course, this effect could be studied via discrete rotations of maximum principal stress with metallographic examination of otherwise identical specimens at each state to quantify damage extent and direction. The criterion for rupture in this case could be $(\mathbf{\dot{\sigma}} : \mathbf{\dot{\sigma}})^{1/2} + \infty$ for $n > 1$

or $\gamma^{(1)} \cdot D \cdot \gamma^{(1)} + 1$ for $n = 0$.

(ii) Tensorial Description of Physical Damage

The previous section dealt with heuristic descriptions of damage evolution which have evolved primarily out of extension of the Rabotnov-Kachanov continuum damage concepts to multiaxial nonproportional loading. When defining the damage tensor, through, it is necessary that the current state of damage be adequately reflected with regard to direction and sense. Creep damage in the continuum sense is almost universally considered to be quantitatively related to the area density of voids and fissures along grain boundaries in a global mean sense. That is, microscale effects in each grain are not considered. In a materials science approach, heterogeneity of damage from grain to grain can lead to void growth constraint; this would have to be treated as a nonlocal damage growth phenomenon in the continuum approach, i.e. damage in a neighboring grain influences local damage growth. These nonlocal effects are not usually addressed in continuum creep damage mechanics.

Leckie and Onat have proposed a generalized tensorial form for cavitation damage [20]. In their approach, we consider a material element large enough such that deformation and damage may be considered homogeneous within. Then, we consider a unit sphere with unit normal vector $\mathbf{n}$ at each point. The total volume of voids found in grain boundaries in the material element normal to $\mathbf{n}$ is denoted as
where $V(\eta)$ is the density of the distribution of void volume. Since the physical damage state is invariant with respect to the sign of $\eta$,

$$V(\eta) = V(-\eta)$$

Various order damage tensors can be defined by the moments of the voids of grain boundaries, i.e.

$$V_0 = \int_A V(\eta)dA(\eta)$$

$$V_i = \int_A V(\eta) n_i dA(\eta)$$

$$V_{ij} = \int_A V(\eta) n_i n_j dA(\eta)$$

and so on, where $A$ is over the unit sphere and $n_i$ are the components of $\eta$ in a fixed rectangular coordinate frame ($i, j, \ldots = 1, 2, 3$). These damage tensors transform in the usual way and result in invariance of damage with respect to rigid body rotation. From equation (30), all odd rank damage tensors must vanish. All even rank tensors are symmetric in all indices and
irreducible. We also note that $V_{KK} = V_0$ where $V_0$ is the total volume of voids per unit volume, or the isotropic damage tensor.

Leckie and Onat also define a series of tensors describing the density and direction of void nucleation sites, known to be particularly important in the early stages of creep. These tensors follow the same development as for $V_0$, $V_{ij}$, $V_{ijkl}$, and are defined as $N_0$, $N_{ij}$, $N_{ijkl}$, ... , with the totality of damage tensors defined by a tensor

$$L = (V_0, V_{ij}, V_{ijkl}, ...; N_0, N_{ij}, N_{ijkl}, ...)$$

with damage evaluation given by

$$\dot{L} = g(\zeta, L)$$

and a rupture criterion

$$R(\zeta, L) < 0$$

The creep strain rate equation is assumed to be weakly dependent on $\zeta$.

Obviously directionality of physical damage could be described with good quantitative accuracy by using fourth and higher order tensors. Yet such an approach is not necessarily economical nor practical. If use of a second order tensor can suitably approximate the directionality of damage, then it is indeed warranted. To this end, Leckie and Onat have suggested the use of second order symmetric tensors $V$ and $\bar{N}$. Again $V_{KK} = V_0$. Applying the mean value theorem to equation (33), we get an estimate of $V(n)$ as
\[ V(n) = \frac{3}{4\pi} V_{ij} n_i n_j \]  

which also satisfies equation (31). Likewise,

\[ N(n) = \frac{3}{4\pi} N_{ij} n_i n_j \]  

The growth laws are then expressed as

\[ \dot{\varepsilon}_C = f(\sigma, V_o) \dot{\varepsilon}^' \]  

\[ \dot{V} = g_v(\dot{\varepsilon}, V, N) \]  

\[ \dot{N} = g_N(\dot{\varepsilon}, V, N) \]  

with the rupture condition

\[ R(\varepsilon, V) = 0 \]  

Here, \( \dot{\varepsilon}^' \) is the deviatoric stress tensor. A rupture criterion of the form

\[ R(\sigma_{\text{max}}, V_o, V_{ij} n_i^* n_j^*) = 0 \]  

was suggested, where \( \sigma_{\text{max}} = n_i^* \sigma_{ij} n_j^* \) is the component of stress in the direction of maximum principal stress, \( n^* \). Thus, a critical combination of maximum stress, total void volume, and void density on grain boundaries normal to the maximum stress dictates rupture. Later work by Leckie resulted in the proposed rupture condition.
It is interesting to note that Leckie and Onat do not use net stress in the constitutive equations for damage and deformation. The effect of area reduction is evidently included via $V_0$ in the creep strain rate equations. Though both void nucleation and growth are included in this formulation, the specific forms for the growth equations are not yet well-developed.

Marakami and Ohno [9,19] have developed a rather complete formulation for creep damage, including finite deformation rotation invariance requirements. In their theory, damage is approximated by a second rank tensor which reflects the change in effective area of the Cauchy tetrahedron due to projected area of cavities on each face. Selecting a material volume element large in comparison to mean void or grain size, but small enough for stress and damage to be uniform, we define the damage tensor as

$$D = \frac{3}{S_g(V)} \sum_{k=1}^{N} \int_V \left[ n^{(k)} \otimes n^{(k)} + W^{(k)} (I - n^{(k)} \otimes n^{(k)}) \right] dS_g^{(k)}$$  \hspace{1cm} (44)$$

where $dS_g^{(k)}$ and $n^{(k)}$ denote the area of a grain boundary element occupied by the k-th cavity and the unit vector normal to $dS_g^{(k)}$, respectively, and $S_g(V)$ is the total area of grain boundaries in $V$. Murakami and Ohno do not distinguish between representation of void nucleation and growth. The definition of cavity can evidently include both wedge-type and r-type cavities as discussed by Raj et al. [15,22]. In equation (44), $W^{(k)}$ denotes the effect of the k-th cavity on the area reduction of planes whose normals are
perpendicular to \( n^{(k)} \), this effect is included since the voids are three-
dimensional and cannot be considered just as two-dimensional grain boundary
cracks.

The principal values of the damage tensor are bounded by 0 and 1, i.e.

\[
0 < D_j < 1 \quad (j = 1, 2, 3) \tag{45}
\]

When \( D_j = 1 \) for any \( j \), creep rupture occurs on the \( j \)-th principal plane. For
undamaged material, \( D = 0 \). Note the similarity of \( D \) in equation (44) with \( V \)
equation (33). Equation (44) is integrated only over the portions of grain
boundaries which are cavitated in contrast to the entire unit sphere in
equation (33), but the two formulations are substantially equivalent apart
from the cavity width correction \( W^{(k)} \) in the Murakami-Ohno approach.

The effect of damage accumulation is to decrease the area over which the
force is carried on an infinitesimal element, with some directions
experiencing more area reduction than others due to preferential cavity
growth. Still, equilibrium must be satisfied; for an arbitrary plane in the
material, this leads to a definition of a net stress tensor which is not only
intensified, but also rotated with respect to the Cauchy stress tensor defined
for undamaged material. We define the symmetric part of the stress tensor
defined by

\[
\zeta \cdot (I - \mathcal{D})^{-1} = \zeta \cdot \hat{\mathcal{D}} \tag{46}
\]

as the net stress tensor \( \zeta \), i.e.

\[
\zeta = \frac{1}{2} (\zeta \cdot (I - \mathcal{D})^{-1} + (I - \mathcal{D})^{-1} \cdot \zeta) \tag{47}
\]
where $\tau$ is Cauchy stress. $S$ is essentially the stress acting on a fictitious undamaged material element which is equivalent to application of $\tau$ on an element of damaged material. This approach is essentially a generalization of the Kachanov-Rabotnov scalar damage approach. Damage rate equations (27) or (28) could then be employed using $S$ defined in equation (47), with appropriate specialization to combined isotropic and anisotropic damaging. The Jaumann derivative may be used for damage rate to ensure invariance of the damage rate equation with respect to rigid body rotations.

Again it is recognized that the creep strain rate is less affected by cavity growth than is damage rate or rupture, which are local phenomena. In fact, creep rupture may occur even when the creep strain rate is finite rather than infinite as implied by a Kachanov-Rabotnov scalar damage approach. Recalling the work of Leckie and associates, the creep strain rate is deviatoric with magnitude modified by $1/(1-\omega)$ as in equation (10). Murakami and Ohno point out that the net stress tensor for use in the creep strain rate equation is in general formed by a fourth rank tensor $\tilde{\xi}$ operating on Cauchy stress, i.e.

$$\tilde{\Sigma} = \frac{1}{2} (\xi : \tau + \tau : \xi)$$

(48)

where $\xi = \xi(\tau)$. The damage and creep rate equations are then given by

$$\dot{\xi} = G(S, \phi, \kappa, T)$$

(49)

$$\dot{\xi}^C = \xi(S, \phi, \kappa, T)$$

(50)
where $\kappa$ and $T$ denote a matrix work-hardening parameter and absolute temperature, respectively, $\dot{\varepsilon}^C$ may be interpreted here as the rate of deformation tensor for finite creep deformation. Note also that dependence on $S$ and $\phi$ in equation (49) is equivalent to dependence on $g$ and $D$, as is the pair $\bar{S}$ and $\bar{\phi}$. In earlier work, Murakami and Ohno [9] had written equation (50) as

$$
\dot{\varepsilon}^C = F'(\bar{\bar{\varepsilon}}, \bar{\varepsilon}, \kappa, T)
$$

where

$$
\bar{\varepsilon} = (I - cD)^{-1}
$$

and

$$
\bar{\bar{\varepsilon}} = \frac{1}{2} (g \cdot \bar{\varepsilon} + \bar{\varepsilon} \cdot g)
$$

where $c$ $(0 < c < 1)$ allows variation from a deviatoric or other classical dissipative potential flow rule. The representation for creep strain rate given in equations (48) and (50) may be general, but is very difficult to implement. The approach in equations (51) - (53) is easier to implement, since $c$ can be selected to fit a given data set, but use of $\bar{\varepsilon}$ is a rather ad hoc procedure which is not well physically grounded. Fortunately, when the cavity volume fraction is on the order of a few percent or less [19], the effect of damage on creep deformation may be assumed to be isotropic. Hence, the creep strain rate equation may be simplified to

$$
\dot{\varepsilon}^C = (3/2)m A^{1/m} \kappa^{(m-1)/m} S_{eq}^{(n-m)/m} \bar{\bar{S}}
$$

- 20 -
where the net stress tensor modified by isotropic damage is given by

\[ \bar{\Sigma} = (1 + c \cdot D_{kk}) \Sigma \]  

(55)

where \( c \) (0 < c < 1) is a material constant. Furthermore,

\[ \bar{\varepsilon}_{\text{eq}} = \left( \frac{3}{2} \bar{\Sigma} : \bar{\varepsilon}' \right)^{1/2} \]  

(56)

\[ \bar{\varepsilon}' = \bar{\Sigma} - \frac{1}{3} \bar{\varepsilon}_{kk} \bar{I} \]  

(57)

The evolution of matrix hardening parameter \( \kappa \) is given by

\[ \dot{\kappa} = mA^{1/m} \kappa^{(m-1)/m} \sigma^{n/m} \]  

(58)

In equations (54) - (58), \( A, m, n, \) and \( k \) are temperature-dependent material constants. Murakami and Ohno have demonstrated the applicability of the second rank damage tensor \( D \) for prediction of rupture time for copper subjected to nonproportional loading and for perforated specimens. Their prediction of creep strain under nonproportional reversal of stress using equation (54) is not suitably accurate, however; the introduction of backstress into equation (54) would allow for kinematic hardening (directional dislocation arrangement) and more accurate prediction for nonproportional loading and unloading sequences [22-24]. Hence, use of unified creep-plasticity theories might be warranted [25-30].

It should be noted that Betten [31], employing a tensor representation of area reduction due to damage, also showed that creep damage can be represented
by a second rank symmetric tensor. He derived the creep rate constitutive equations based on generalized tensor function theory. In contrast to Murakami and Ohno, Betten retains a non-symmetric net stress tensor. Furthermore, he introduces initial deformation-induced anisotropy effects into both the creep strain rate and damage rate equations. While it is obvious that initial anisotropy from a rolling or forming process would affect the creep strain rate magnitude and direction, the effect on magnitude and direction of damage rate is inconclusive. As a point of clarification, it should be noted that all previous growth equations in this paper have assumed initial isotropy.

While the work of Betten offers a very rigorous continuum mechanics approach to tensorial damage, its abstract nature and lack of simplification to practical creep constitutive equations limit its usefulness. The work of Murakami and Ohno is of similar nature yet oriented toward the engineering approach.

Baik and Raj [21] have proposed use of a second rank tensor to represent the development of three-dimensional creep damage for the case of wedge-cracking induced by grain boundary sliding for aluminum alloy and austenitic stainless steels subjected to creep-fatigue loading. They showed that wedge-cracking was likely to occur in the aluminum alloy and austenitic stainless steels under conditions of asymmetric load cycle shapes without tension hold, while hole cavitation ("r" type cavitation) was likely to occur under tension hold cycles. This points to the fact that the dominant mechanism of creep damage under creep-fatigue loading is not only sensitive to temperature and loading intensity, but the rate and sequence in which loads are applied. If wedge type creep damage is dominant, then Baik and Raj define damage growth in the principal frame by
for $i = 1, 2, 3$, where $\beta$ represents the ratio of the grain boundary sliding strain rate to the applied strain rate, $\alpha$ is a proportionality constant, and $\gamma$ is an adjustable parameter reflecting the extent to which damage produced in tension can be recovered in compression,

$$d(A_w)_{il} = \alpha \beta [\gamma x_{il} \frac{\sigma_1}{\sigma}]$$

(59)

and the measure of damage is defined by

$$d\bar{\varepsilon} = (\frac{2}{3} d\varepsilon : d\varepsilon)^{1/2}$$

(60)

and the measure of damage is defined by

$$A_w = c/L$$

(61)

where $c$ is the wedge crack length, and $L$ is the grain size. As in the earlier approaches, $0 < A_w < 1$, with $A_w = 0$ indicative of undamaged material, and $A_w = 1$ indicative of fracture. The tensor $x_{il}$ allows correlation of the stress applied in direction 1, $\sigma_1$, with development of damage in directions 2 and 3. Since wedge damage develops at triple junctions of grain boundaries, wedge cracks appear in normal directions along adjacent grain boundaries. It may be the case that when creep damage is dominated by wedge-cracking, the mean damage state (averaged over a number of grains) assumes a more isotropic nature due to random grain orientation than would "r" type cavitation, which is dependent on the maximum principal stress direction. Perhaps these considerations help explain the observed indifference to loading direction of creep strain and damage rates of Leckie and Hayhurst's fixed tension, alternating shear tests on aluminum. It would seem that materials which do
not cavitate easily may experience at least a component of triple junction wedge-cracking in creep-dominated situations with even occasional load reversals present.

Recalling the constructions of damage tensors reported earlier, it is clear that it is not necessary to distinguish between triple junction wedge cracks and well distributed cavities ("r" type) in a continuum damage approach. The three-dimensional character of damage can be described by a tensor with a suitable degree of anisotropy. It is significant, though, that previous continuum damage mechanics analyses have not substantially addressed the possibility of several dominant mechanisms, each with a different damage rate dependence on applied stress, over a range of temperatures and strain rates.

(iii) Measure of Deterioration of Elastic Behavior

It is very common in structural analysis to assume that the presence of damage is manifested by a decrease in stiffness. Usually, the concept of deterioration of stiffness is applied to a structural member rather than at each material point. Chaboche [13] introduced a form for the net stress tensor

\[ \mathbf{S} = \mathbf{M}(D):\mathbf{a} \]  

(62)

where \( \mathbf{M} \) is a fourth order tensor operating on stress. Equation (62) is applicable at each material point. Then \( \mathbf{S} \) can be used in lieu of \( \mathbf{a} \) in the elastic-viscoplastic constitutive equations with damage-induced anisotropy being included in a natural way through flow rules based on scalar invariants of \( \mathbf{S} \). Chaboche suggested that \( \mathbf{M} \) could be measured from deterioration of the elastic response by using an equivalence in the elastic strain sense, i.e.
\[ \varepsilon = A(D) : \varepsilon^e \]  
\[ S = A(0) : \varepsilon^e \]  
(63)  
(64)

where \( \varepsilon^e \) is the elastic strain tensor and \( A(0) \) the fourth rank elasticity tensor for undamaged material. These equations lead to

\[ \mathcal{M}(D) = A(0) : A^{-1}(D) \]  
(65)

so that \( \mathcal{M}(D) \) could be determined if initial and subsequent elastic responses are known.

As mentioned by Chaboche, this description of damage is consistent with a general thermodynamical framework [13]. However, it is noted from earlier discussion that the dependence of creep deformation on creep damage is not direct as is the rupture condition since rupture may occur in a relatively brittle fashion at low void volume density. In other words the deformation of the matrix is not strongly affected by the presence of grain boundary damage up to tertiary creep, reflected through the applicability of a deviatoric flow rule for creep strain. For these reasons, as pointed out by Murakami and Ohno [19] it seems more appropriate to base the net stress tensor on reduction in projected area due to physical creep damage, and to use this net stress tensor in damage evolution equations but not necessarily in creep strain rate equations.

(iv) Creep Damage Potential Functions

The generalization of uniaxial concepts of creep deformation to the multiaxial case usually involves the assumption of a dissipative potential
function. The creep strain rate is then normal to this dissipative potential surface in stress space. It is tempting to generalize creep damage for the multiaxial case by assuming an appropriate creep damage potential. To this end, both phenomenological and coupled thermo-viscoplastic approaches have been offered.

From a phenomenological standpoint, Bodner and Lindholm [32] have suggested a damage growth law of the form

$$D = \alpha(W_s)\beta(\sigma_{kk})\gamma(D)$$

(66)

where $W_s$ is the stored energy of cold work, $\sigma_{kk}$ is the first invariant of stress and $D$ is a scalar damage parameter. The value of $D$ at failure, unlike in the Kachanov-Rabotnov approach, is determined empirically. The mechanistic interpretation of this equation is that $\alpha$ governs nucleation of voids and defects, $\beta$ controls the rate of growth of voids under applied hydrostatic stress, and $\gamma$ relates damage growth rate to current damage level. The use of the first invariant of stress as the sole stress parameter in equation (66) is incompatible with the concept of anisotropic creep damage. If $\beta(\sigma_{kk})\gamma(D)$ were to be defined as a surface in stress space, the direction of the damage rate could be defined as the gradient of the damage surface if $\beta$ depended on $\sigma_{ij}$ instead of an invariant. The separation of void nucleation and growth processes in equation (66) is similar to the formulation of Leckie and Onat (equations (29)-(44)); the latter formulation, however, assigned directions to void nucleation growth processes in contrast to the Bodner-Lindholm approach. Essentially, then, equation (66) provides very limited capability for modeling nonproportional creep damage and is somewhat difficult to apply.
Krajcinovic [33] has proposed damage evolution laws based on the existence of a potential, suggesting that this approach is ultimately necessary for treatment of cyclic loads. In his approach, the Helmholtz free energy density is composed of elastic and viscous terms

\[
\psi = \psi^{(e)}(\varepsilon^e, D, T) + \psi^{(c)}(\varphi, \rho, \varpi, T) \quad (67)
\]

where \(\psi^{(e)}\) and \(\psi^{(c)}\) are elastic and creep potentials, \(\varepsilon^e\) is the elastic strain, and \(\varphi\) and \(\rho\) are kinematic and isotropic hardening variables which affect viscous response. In Krajcinovic’s work, damage \(D\) is a vector normal to a plane of cracks or crack-like defects. This definition of the damage tensor is not in agreement with the findings of Onat and Leckie, Murakami and Ohno, Betten, and others that physical damage must be represented by even rank tensors. Minimally, a second rank tensor (or scalar if damage is isotropic) is necessary. Use of a vector attributes a sense to the damage direction which incorrectly implies that the current state of damage is dependent on the sense of the direction; damage growth rate, not the current damage state, is dependent on sense of loading (tension or compression) and this is reflected through the damage evolution equation.

Since the rest of Krajcinovic’s development does not hinge on this assertion of vectorial creep damage, we continue the discussion. From the Clausius-Duhem inequality,

\[
\rho \dot{Q} = g : \dot{\varepsilon} - \rho \frac{\partial \psi^{(c)}}{\partial \rho} \dot{\varphi} - \rho \frac{\partial \psi^{(c)}}{\partial \varphi} \dot{\varphi} - \rho \frac{\partial \psi^{(c)}}{\partial \rho} \dot{\rho} - \frac{1}{T} q \cdot \varpi \cdot \varpi^T \geq 0 \quad (68)
\]
where \( \rho \) is mass density, \( \dot{\varepsilon}_c \) is the creep strain rate, \( q \) is the heat flux, and \( T \) is temperature. Equation (68) states that the energy dissipation power density \( \rho \dot{Q} > 0 \). The entropy term in the dissipation power density cancels with the term in the total time differential of \( \dot{\psi} \) resulting from dependence of \( \dot{\psi} \) on \( T \). Since stress can be derived from \( \dot{\psi} \) by

\[
\sigma = \rho \frac{\partial \dot{\psi}}{\partial \dot{\varepsilon}_c} \tag{69}
\]

we can write

\[
\dot{\sigma} = \rho \frac{\partial^2 \dot{\psi}}{\partial \dot{\varepsilon}_c^2} \dot{\varepsilon}_c + \rho \frac{\partial^2 \dot{\psi}}{\partial \dot{\varepsilon}_c \partial \dot{D}} \dot{D} \tag{70}
\]

and the rates of other generalized forces \( -\rho \frac{\partial \dot{\psi}}{\partial \dot{D}} \), \( -\rho \frac{\partial \dot{\psi}}{\partial \dot{\alpha}} \), \( -\rho \frac{\partial \dot{\psi}}{\partial \dot{p}} \) can be similarly found. Next, the rates of change of internal variables \( \dot{\alpha}, \dot{\rho}, \dot{D} \) in addition to creep strain \( \dot{\varepsilon}_c \) can be found by assuming the existence of a potential \( \Omega \) such that

\[
\dot{j} = \frac{\partial \Omega}{\partial \dot{x}} \tag{71}
\]

where \( j = \{\dot{\varepsilon}_c, \dot{\alpha}, \dot{\rho}, \dot{D}, \frac{1}{T} \ln T \} \) and \( x = \{\dot{\varepsilon}_c, -\rho \frac{\partial \dot{\psi}}{\partial \dot{\alpha}}, -\rho \frac{\partial \dot{\psi}}{\partial \dot{p}}, -\rho \frac{\partial \dot{\psi}}{\partial \dot{D}}, -G\} \). If \( \Omega \) is quadratic in \( x \), then \( \Omega \) is equivalent to the energy dissipation power density \( \dot{Q} \). Krajcinovic suggests a flow potential of the form

\[
\Omega = G^{(c)}(x) H(G^{(c)}) + G^{(d)}(x) H(G^{(d)}) \tag{72}
\]

when \( H \) is the Heaviside function. For \( G^{(c)} < 0 \), no viscous deformation occurs, and for \( G^{(d)} < 0 \), no growth of damage occurs. Hence, the coupling of
damage with creep strain rate and growth of damage would emerge from equations (71) - (72). Possible forms for the viscous flow potential $G(c)$ and damage potential $G(d)$ are

$$G(c) = K \| \dot{\gamma} + \rho \frac{\partial \psi}{\partial \dot{\gamma}} \| - g(-\rho \frac{\partial \psi}{\partial \dot{\gamma}})^n$$  \hspace{1cm} (73)

$$G(d) = K_1 (R_1^2 + a D R_2^2) - g_1(D)^n$$  \hspace{1cm} (74)

where $K$, $K_1$, $a$, $m$, and $n$ are material parameters and $\dot{\gamma}$ is deviatoric stress. $R_1$ and $R_2$ are the values of $-\rho \frac{\partial \psi}{\partial \dot{\gamma}}$ generalized forces in directions normal and tangential to the "plane of damage" in this formulation and therefore govern the normal and shear components of damage growth, respectively. The damage potential in equation (74) can reflect dominant shear-type damage accumulation, cleavage (normal) - type, or a combination.

The great advantage of using a flow potential is that damage is included as an internal variable. The damage and creep strain rate equations are coupled through the flow potential $\psi$, providing a rational treatment. The anisotropic character of damage is also defined through $\psi$. Rigorous application of the flow potential concept, though is difficult. It may be possible to simplify the flow potential or even to express some of the previously mentioned anisotropic theories in term of $\psi$ (e.g. Murakami-Ohno). Certainly, it seems necessary to minimally express $D$ in equations (68) - (74) as a second order tensor rather than a vector in future work.

As a final comment, it should be noted that considerable work has been done [e.g. 34-37] in characterizing uniaxial cumulative creep damage under varying stress levels. At higher stress levels, damage accumulation more
nearly approaches a linear time fraction accumulation rule than at lower stress levels. Bui-Quoc and associates [35] use a highly phenomenological, normalized damage curve approach to more accurately represent (fit) creep and creep-fatigue damage sequence effects. They suggest that the Chaboche-Lemaitre approach in equation (19) provides a damage parameter which is very small \( D \approx 0 \) even near the end of material life, and suggest that this is not representative of nonlinear damage accumulation processes. It appears to the current author that Chaboche et al. enforce this condition on damage to minimize the influence of damage on the creep strain rate equation until tertiary creep is reached. If the proper dependence of the creep strain rate on damage is prescribed, perhaps theoretical damage growth could more faithfully follow physical creep damage growth. It is strongly felt that the anisotropic damage tensor must accurately reflect physical damage to properly model loading sequence effects and creep-fatigue interaction.

**SOME COMMENTS ON THE RELATIONSHIP OF MATERIAL SCIENCE AND CONTINUUM DAMAGE GROWTH LAWS**

Naturally, description of creep damage has also been undertaken from the materials science or micromechanical viewpoint. Most of the work has dealt with the nucleation and growth of voids as a function of applied stress and temperature. The primary thrust of these studies has been to identify the regimes of stress and temperature in which voids grow by diffusion, power-law continuum plastic deformation, or by coupled mechanisms. Miller and Langdon [38] point out that three distinct void growth processes may occur:

(a) power-law creep of surrounding matrix at high stress levels,
(b) unconstrained diffusion growth at intermediate stress levels, and
(c) constrained diffusion growth at low stress levels.
Of course, coupling may exist between these mechanisms.

An essential feature of these different cavity growth mechanisms is that void growth rates will differ depending on the dominant mechanisms (highest growth rate). Svensson and Dunlop [39] have produced cavity growth mechanism maps which show the regimes of dominance of constrained diffusion growth, unconstrained diffusion growth, coupled power-law diffusional growth, power law growth, and non-equilibrium diffusional growth. These maps are very similar to the deformation mechanism maps introduced by Ashby and co-workers, but deal specifically with void growth mechanisms rather than creep deformation mechanisms. Other authors [3,15,21,38-44] have offered specific models for the nucleation of voids at grain boundary/slip band intersections, precipitate particles, or triple junctions, and the growth of these voids once nucleated.

Of particular interest to this investigation is the relationship of the cavity growth laws to the evaluation of the damage tensor in Kachanov-Rabotnov continuum damage theories. Obviously, the two must be somewhat compatible or we may extrapolate the continuum growth law into a regime where the damage evolution equation is of inappropriate form.

Cocks and Ashby [3,43] have extensively studied the regimes of diffusion controlled and power-law controlled creep damage growth. They note that growth due to power-law deformation of surrounding matrix material becomes more important as the voids become larger, even if the growth was diffusion controlled when the cavities were small. They also have shown that consideration of both diffusion and power-law controlled damage growth is
necessary for accurate calculation of rupture times, particularly when variable loading histories are applied which impose diffusion controlled growth over part of the life and power-law growth over the rest. The linear time-fraction damage accumulation rule is not adequate when the mechanisms of cavity growth change due to temperature and/or stress changes. Edward and Ashby [45] have shown that coupled boundary diffusion/power-law growth is the mode of void growth over wide ranges of stress and temperature for structural metals.

Defining the area fraction of voids as

\[ f_h = \frac{r_h^2}{\eta^2} \]  \hspace{1cm} (75)

where \( r_h \) is the average void radius and \( \eta \) is the average void spacing, then the growth of damage \( f_h \) is described by

\[ \dot{f}_h = \frac{\varepsilon_0 \sigma_1}{f_h^{1/2} \ln(\frac{1}{f_h})} (\frac{\sigma_1}{\sigma_0}) \]  \hspace{1cm} (76)

for boundary diffusion alone and

\[ \dot{f}_h = \varepsilon_0 \beta_0 \left\{ \frac{1}{(1-f_h)^n} - (1-f_h) \right\} \left( \frac{\sigma}{\sigma_0} \right)^n \]  \hspace{1cm} (77)

for power-law creep alone. In equation (76), \( \sigma_1 \) is the stress acting normal to the boundary on which the void(s) lies and \( \sigma_0 \) and \( \varepsilon_0 \) are normalizing constants in the uniaxial power-law creep equation

\[ \dot{\varepsilon} = \varepsilon_0 \left( \frac{\sigma}{\sigma_0} \right)^n \]  \hspace{1cm} (78)

at a given temperature. Also,
where \( D_0 \delta \) is the boundary diffusivity, \( \Omega \) the atomic volume, \( k \) is Boltzmann's constant, and \( T \) is absolute temperature. In equation (77),

\[
\beta_0 = \sinh \left\{ -\frac{2}{3} \left( \frac{n - \frac{1}{2}}{n + \frac{1}{2}} \right) \right\}
\]  

(80)

Comparison of the power-law creep equation (77) with the Rabotnov-Kachanov continuum damage theory in equation (5) with \( u = \nu \) reveals a great deal of similarity. In fact the forms are identical apart from the second term \((1-f_h)\) in equation (77). When \( f_h \) is large, the second term may be neglected and the two sets of equations match with \( f_h \) being interpreted as the scalar damage parameter \( \omega \). A very significant difference is that the continuum damage model gives a finite damage rate when the damage parameter is zero in contrast to the mechanistic approach, where the damage rate is zero for undamaged material. Cocks and Ashby point to this feature as being clearly indicative of the superiority of the mechanistic approach since non-existent holes do not grow. This is strictly true when damage is defined as void density; from the continuum viewpoint, though, a finite rate of damage when the damage parameter is zero could correspond to void nucleation processes or onset of creep damage.

For diffusion controlled growth of voids, as \( f_h \rightarrow 1 \),

\[
\dot{f}_h = \frac{\sigma}{\sigma_0} \left( \frac{1}{1-f_h} \right)
\]  

(81)
which is identical to the Rabotnov-Kachanov relation when \( u = v = 1 \). As for power-law creep, for large \( f_h \), the forms of the mechanistic and continuum models are the same, with only the damage power-law exponent differing between mechanisms. The differences between the mechanistic and continuum damage approaches are most pronounced when \( f_h \) is small for both diffusion controlled and power-law creep. In fact, the boundary diffusion growth rate in equation (76) exhibits a decrease as \( f_h \) increases for \( f_h < .1 \). Since Ashby et al. use \( f_h = 0.25 \) as a realistic failure condition, this behavior would be significant over much of life if voids grow by coupled boundary diffusion and power-law creep. Since Rabotnov-Kachanov type continuum damage laws imply that damage rate always increases with damage (for uniaxial creep loading), they cannot conform to boundary diffusion controlled void growth. Life predictions using the continuum damage law of Rabotnov-Kachanov can therefore be more conservative since power-law void growth is at a relatively high rate.

Cocks and Ashby [43] suggest the inclusion of \( \phi_0 \) and \( (\sigma_1/\sigma_0) \) in the continuum damage approach to account for coupled boundary diffusion and power-law creep. Turning to the consideration of anisotropic damage, however, it seems likely that this refinement is not warranted in this initial study. Previous work on coupled diffusion and power-law creep have not involved rotation of the principal stress axes. The materials science studies do tell us, however, that consideration of the creep damage growth regime is very important in determining damage rates. If a transition from the boundary diffusion regime to power-law growth regime is encountered as voids grow and the net stress increases, rotation of the maximum principal stress may result in a significant under-prediction of rupture time using a power-law damage rate. The directionality of damage, though, is another issue; this directionality will depend on propensity to cavitate, grain boundary sliding,
etc. Grain boundary sliding, for example, results in a concentration of normal stress and an increase of hydrostatic stress in each grain due to constraint of surrounding grains. This effect can be suitably described through the dependence of isochronous stress on the first invariant of stress in the anisotropic damage growth law (c.f. equation (26) or (28)).

INCORPORATION OF DAMAGE IN CONSTITUTIVE EQUATIONS FOR DEFORMATION

To this point, the development of rational forms for tensorial damage have concentrated on description of physical damage and its evolution. In reality, the evolution of creep strain rate is not as highly dependent on the anisotropy of the current state of damage (prior to tertiary creep) as is the damage rate. This is due to the fact that creep strain rate in the primary and secondary regimes is primarily affected by matrix work-hardening and recovery processes. The cavitation and sliding processes which occur at grain boundaries do not greatly affect creep strain rate until the voids or grain boundary cracks coalesce. These comments pertain to power-law creep, grain boundary diffusion, surface diffusion, or coupled mechanisms. Hence, it is clear that

(a) the inelastic strain rate magnitude should be affected by a global or averaged measure of physical damage up to the tertiary stage,

(b) the inelastic strain rate direction should be modified by the directional distribution of physical damage, with this effect increasing in importance as damage approaches a critical level in one or more directions,
(c) the anisotropy of damage does not translate into an equivalent anisotropy of inelastic strain rate, since damage may accumulate as cavitation on grain boundaries oriented normal to the maximum principal stress (anisotropic damage) with no concurrent observation of anisotropic deformation [4-6,10,12],

(d) the magnitude of inelastic strain rate may actually be finite in a direction corresponding to a critical (rupture) damage value [5,9], and

(e) the inelastic strain rate is of a power-law form, dependent primarily on the effective stress [10,13,19,25-30].

It should be noted here that the term inelastic strain rate rather than creep strain rate is used in this section, following the unified or state variable theories of creep deformation. In these theories, no distinction is made between the time-independent plastic and time-dependent creep strains. One model structure can exhibit the essential characteristics of monotonic and cyclic, rate-dependent plasticity, and stress and temperature-dependent creep deformation or relaxation [25-30]. Furthermore, inclusion of backstress and drag stress in these unified theories allows for accurate description of inelastic strain rate direction under nonproportional loading [25-26,46] and matrix work-hardening, respectively.

A fairly general statement of the isothermal unified equations, including both backstress and drag stress, is:

\[ \dot{\varepsilon}_n = f \left( \frac{3J_2}{k^2} \right) (\dot{\sigma} - \dot{\sigma}) \]  \hspace{1cm} (82)

\[ \dot{\sigma} = h_\alpha \dot{\varepsilon}_n - d(\dot{\sigma}, T) \varepsilon - r_\alpha(\dot{\sigma}, T) \sigma \]  \hspace{1cm} (83)
\[ \dot{\kappa} = h_\kappa \dot{\varepsilon}^n - r_\kappa(\kappa,T) \quad (84) \]

where \( h_\alpha \) and \( h_\kappa \) are hardening functions, \( r_\alpha \) and \( r_\kappa \) are static recovery functions, and \( d \) is a dynamic recovery function. \( \varphi \) is backstress, \( \kappa \) is drag stress, \( J_2 = \frac{1}{2} (\sigma' - \varphi) : (\sigma' - \varphi) \), and \( \dot{\varepsilon}^n = (2/3 \dot{\varepsilon}^n : \dot{\varepsilon}^n)^{1/2} \). Terms involving temperature rate are dropped for the isothermal case but can be included if necessary. Common forms of the hardening and recovery functions will now be discussed:

\[ h_\alpha = \text{constant} \quad (85) \]

for linear kinematic hardening or

\[ h_\alpha = K_1 + K_2 \exp (-d_2 \zeta) \quad (86) \]

where \( K_1, K_2 \) and \( d_2 \) are material constants and \( \zeta = \alpha; \gamma, \zeta = \varepsilon^n \), or \( \zeta = (\sigma' - \varphi): \varphi \). Experiments [26] have shown equation (86) to be most accurate. Also,

\[ h_\kappa = C_1(\kappa^* - \kappa) \quad (87) \]

or

\[ h_\kappa = C_1(\kappa^* - \kappa) + f_1(\varepsilon^n, \dot{\varepsilon}^n, \varphi) \quad (88) \]

or

\[ h_\kappa = H(I_{\alpha \varepsilon}) \left[ \left( \frac{3}{2} \frac{1}{2} \varphi : \varphi \right)^{1/2} - b_1(b_2 \kappa)^{b_3} \right]. \quad (89) \]

where

\[ H(I_{\alpha \varepsilon}) = [a_1a_2 \exp (a_1(\kappa - \kappa_0))]^{-1} \quad (90) \]

Equations (89) - (90), formulated by Abrahamson, Cescotto, and Leckie [26,47] can accurately model cyclic hardening or softening. Of course, simple forms such as equation (87) may be sufficient for representation of matrix hardening during creep.
The static recovery functions $r_\alpha$ and $r_\kappa$ are usually of the forms

$$r_\alpha = C_2(\alpha:\alpha)^{C_3}$$  \hspace{1cm} (91)

$$r_\kappa = C_4(\kappa - \kappa_0)^{C_5}$$  \hspace{1cm} (92)

where $C_2$, $C_3$, $C_4$, $C_5$, and $\kappa_0$ are material constants.

Common forms of the modulus function $f(3J_2'/\kappa^2)$ are

$$[D_0 (3J_2'/\kappa^2)^n /J_2']^{1/2}$$  \hspace{1cm} (93)

$$[D_0 \exp[-(\kappa^2/3J_2')^n] /J_2']^{1/2}$$  \hspace{1cm} (94)

$$[D_0 [\sinh(3J_2'/\kappa^2)^n] /J_2']^{1/2}$$  \hspace{1cm} (95)

where $m$, $n$, and $D_0$ are constants [25].

Temperature dependence of an Arrhenius form may be included in the flow rule and recovery terms [25].

The structure of these equations have been derived from uniaxial experiments. Nonproportional loading, however, can introduce errors in the inelastic strain rate direction and work hardening rate $\dot{\kappa}$ [48]. Since the inelastic strain rate direction is governed by evolution of backstress, it is necessary to include nonproportionality effects in either the backstress evolution law or directly in the flow rule. The latter is the approach taken by Murakami and Ohno [22]; it seems more rational, though, to include this effect in the backstress evolution law, i.e.
\[ \dot{\alpha} = h_K \gamma \varepsilon^n - d(\alpha, T) \varepsilon^n \alpha - r(\alpha, T) \alpha \]  

(96)

where unit vector \( \gamma \) is based on a Mroz type hardening rule [23-24,48].

For the evolution of drag stress (isotropic hardening), it has been suggested [25-26] that \( h_K \) be modified to account for additional hardening observed [23-24]. Possible forms for inclusion of this effect are

\[ h_K = C_1(\psi(\phi) \kappa^* - \kappa) \]  

(97)

or

\[ h_K = \frac{H(1_\alpha)}{[\psi(\phi)]^{1/2}} \left[ \psi(\phi) \left( \frac{3}{2} \alpha : \alpha \right)^{1/2} - b_1(b_2 \kappa)^{b_3} \right] \]  

(98)

where \( \phi \) \((0 < \phi < 1)\) represents the additional hardening due to nonproportional loading. Proposed forms of \( \phi \) are

\[ \dot{\phi} = \mu \left(1 - \sqrt{\frac{d}{dt} \left( \frac{\varepsilon_1 - \varepsilon_3}{(\varepsilon_1 - \varepsilon_3)_1 - (\varepsilon_1 - \varepsilon_3)_3} \right) - \phi} \right) \left( \varepsilon^n : \varepsilon^n \right)^{1/2} \]  

(99)

due to McDowell [23-24] \((\varepsilon_1, \varepsilon_3)\) are maximum and minimum principal strains), or

\[ \phi = \frac{\dot{\alpha}}{\| \dot{\alpha} \|} \cdot \frac{\alpha}{\| \alpha \|} \]  

(100)

due to Bodner et al. [25]. It is felt that a history dependent measure of \( \phi \) such as equation (99) is more mechanistically desirable, and fits the data better for cyclic loading. An alternative definition for \( \phi \) might be the quotient of the inelastic strain path projection on the maximum inelastic strain direction with the total inelastic strain path length.

In the deformation is creep-dominated, the work of Oytana et al. [7] indicates that the hardening is primarily kinematic, i.e. backstress
With the backstress, primary, secondary and anelastic creep strains may be represented by the unified constitutive equations. Ohashi and associates [8], working with type 304 stainless steel, have shown that simple time or strain-hardening theory cannot predict transient softening observed in creep after stress reversals or nonproportional stress field rotations. Furthermore, the modified strain-hardening theory of ORNL is not suitable for nonproportional loading. Pure kinematic hardening theory gives excessive creep strain rate after large rotations of the principal stresses. They found that combined isotropic-kinematic theory (unified theory) incorrectly predicted a cycle-by-cycle decrease in amplitude of creep strain during nonproportional stress reversals. This suggests that the component of isotropic hardening should be weak compared to kinematic hardening in concurrence with the conclusions of Oytana et al. One troubling point in Ohashi's work is the use of a constant apportionment factor between isotropic and kinematic hardening (evidently 1/2), not included in most unified theories; this seems to artificially restrict the kinematic hardening. Regardless, it is clear that prediction of creep strain rate after relatively large, nonproportional stress reversals requires quite accurate constitutive laws more refined than of simple or modified strain-hardening type.

Cho and Findley [49-50] have shown the strong influence of aging at temperature on the subsequent creep deformation of type 304 stainless steel at 593°C (ORNL reference heat 9T2796). They include aging through power-law dependence of plastic, viscoelastic, and viscous strains on aging time. The same sort of manipulation could be accomplished by a power-law dependence of inelastic strain rate on aging time in the unified theories. These aging effects must be represented as Heaviside functions in rate-type constitutive laws and are therefore somewhat difficult to formulate [51]. In principle,
aging effects are not seen to be strongly related to the formulation of anisotropic damage and deformation constitutive laws and will not be considered at the present time. The nonproportional test program in this study does not investigate these effects, with the exception of aging which may occur continuously during the tests.

As mentioned earlier, experimental evidence suggests that the creep strain rate equations is relatively insensitive to damage anisotropy until tertiary creep. It follows that directional internal stresses and isotropic hardening in the matrix should develop with weak dependence on anisotropy of damage through the secondary creep stage. There seems to be two alternative methods to achieve this weak dependence. First the components of the damage tensor can remain small until the onset of tertiary creep. In this case, following a Kachanov-Rabotnov approach as in equation (6) (e.g. Chaboche),

$$ C^c = F (\phi, \dot{\epsilon}, \kappa, S' - \bar{S}, T) $$

(101)

where \( \phi = (I-D)^{-1} \) as in the Murakami-Ohno approach, and \( S' \) and \( \bar{S} \) are the deviatoric modified net stress and backstress tensors. Such an expression would account for continuity of creep strain rate with respect to damage if damage is isotropic, or anisotropy of creep strain rate (reflected through net stress \( S' - \bar{S} \)) if damage is anisotropic or mixed. These results are compatible with the experimental findings of Trampczynski and Hayhurst [4] for copper, aluminum, and Nimonic 80A.

The other method to include damage in the inelastic strain rate is provide isotropic dependence on the first invariant of damage. This approach, stated in equations (54) - (58) for strain-hardening theory, seems adequate up to the tertiary creep stage since the cavity volume fraction is low for most
metals up to this stage, and anisotropy in the primary creep deformation subsequent to stress reversals can be accounted for through backstress in the unified theories [25-30]. Of course, the transition from secondary to tertiary creep is accompanied by an attendant increase in dependence of the inelastic strain rate on damage anisotropy.

Based on the current experimental evidence and state-of-the-art, a suggested initial form of the unified equations with inclusion of anisotropic damage is

$$
\ddot{\varepsilon}^n = r\left(\frac{3\ddot{J}_2}{\kappa^2}\right) (\ddot{\Sigma} - \ddot{a})
$$

(102)

$$
\ddot{a} = h_{\alpha} \dot{\varepsilon}^n - d(\ddot{a}, T) \ddot{\varepsilon}^n \ddot{a} - r_{\alpha}(\ddot{a}, T) \ddot{a}
$$

(103)

$$
\ddot{\kappa} = h_{\kappa} \dot{\varepsilon}^n - r_{\kappa}(\ddot{\kappa}, T)
$$

(104)

where

$$\ddot{\Sigma} = \gamma(D_{kk})(1 + C_{1}D_{kk})\ddot{g} + (1 - \gamma(D_{kk}))\ddot{S}
$$

(105)

and $\ddot{a}$ is similarly defined. The function $\gamma(D_{kk})$ introduces a smooth transition from isotropic dependence on damage ($\gamma(D_{kk}) = 1$) to complete anisotropic dependence on net stress $\ddot{S}$ ($\gamma(D_{kk}) = 0$) which is highly dependent on the tensorial character of damage. It is anticipated that $\gamma(D_{kk}) \approx 1$ based on previous results, at least if the cavity volume fraction at rupture is relatively low. Otherwise, results of Murakami [19] indicate that a power-law dependence on $D_{kk}$ may be appropriate.

In equation (102), $\ddot{J}_2$ is the deviatoric modified net section stress, and

$$\ddot{J}_2 = \frac{1}{2} (\ddot{\Sigma} - \ddot{a}) : (\ddot{\Sigma} - \ddot{a})$$
CORRELATION WITH EXPERIMENTS

The motivation for the suggested anisotropic damage formulation was presented earlier. Here, the full equations will again be presented and appropriate simplifications will be introduced for purposes of demonstration and correlation with experiments completed to date at ORNL.

The proposed general framework for the isothermal coupled damage and creep strain rate equations is:

Damage Rate Equation:

\[
\dot{D} = B \left[ \sigma^*(S) \right]^{k} \left[ \eta I \langle \phi : \phi \rangle^{\gamma/2} + (1 - \eta) \sum_{j=1}^{3} \nu(j) \otimes \nu(j) M(\nu^{(j)}, \phi) \right], \tag{106}
\]

where \( \eta, B, k, \) and \( \gamma \) are material constants, and

\[
M(\nu^{(j)}, \phi) = \langle \phi : \phi \rangle^{\gamma/2} M^*_j(\sigma_j)
\]

or

\[
M(\nu^{(j)}, \phi) = \left[ \frac{1}{1 - \nu^{(j)} \cdot D \cdot \nu^{(j)}} \right]^{\gamma} M^*_j(\sigma_j)
\]

where \( M^*_j(\sigma_j) \) admits anisotropic contribution of non-maximal principal net stresses, and \( \sigma^*(S) \) is the isochronous stress given by

\[
\sigma^*(S) = \frac{3}{2} H_1 \left( \frac{2}{3} \frac{S_{\text{eq}}}{H_1} \right)^a \exp \left[ b \left( \frac{S_{kk}}{S_s} - 1 \right) \right] \tag{107}
\]

due to Huddleston [16], where
\[ S_{eq} = ((3/2) S' : S')^{1/2} \]

\[ H_1 = S_1 - S_{kk}/3 \]

\[ S' = S - (1/3) S_{kk} I \]

\[ S_s = (S_1^2 + S_2^2 + S_3^2)^{1/2} \]

\[ S = \text{net stress tensor} = (1/2)(\sigma^\tau \phi + \phi^\tau \sigma) \]

\[ \phi = (I - D)^{-1} \]

\[ D = 2^{nd} \text{ order damage tensor} \]

\[ \nu(j) = \text{unit vector in principal net stress direction, } S_j \]

and a and b are material constants. \( \sigma \) is the Cauchy stress.

Rupture Criterion:

Possible rupture criteria include

\[ D_j = D_{cr} \text{ for } j = 1, 2, \text{ or } 3 \quad (108) \]

where \( D_j \) are the principal values of \( D \), and \( D_{cr} \) is a critical damage level, or

\[ R(\sigma_{max}, D_{kk}, n_i D_{ij} n_j^*) = 0 \quad (109) \]

where \( \sigma_{max} = n_i \sigma_{ij} n_j^* \) is the component of stress in the maximum principal stress direction, \( n^* \). Here, a critical combination of maximum principal
stress, total void or cavity fraction, and void density on grain boundaries normal to the maximum principal stress dictates rupture. Leckie has also proposed the criterion

\[
\max_{\bar{n}} \left( \bar{n}_i D_{ij} \bar{n}_j \cdot \bar{n}_k \bar{\sigma}_{kl} \bar{n}_l \right) = \text{constant}
\]  

(110)

where \( \bar{n} \) is an arbitrary unit vector.

Coupled Creep Strain Rate Equation:

Suggested coupling with a rate-dependent unified creep plasticity theory is given by:

\[
\dot{\bar{\varepsilon}}^\alpha = f \left( || \bar{S}^\alpha - \bar{a} || \kappa \right) \left( \bar{S} - \bar{a} \right)
\]

(111)

\[
\dot{\bar{a}} = h \bar{a} \left( \dot{\bar{\varepsilon}}^\alpha \cdot \bar{\nu} \right) - r \bar{a}
\]

(112)

\[
\dot{\bar{k}} = h \dot{\bar{\varepsilon}}^\alpha - r \bar{k}
\]

(113)

where \( \bar{\nu} \) is a selected directional index, and

\[
\bar{S} = f_1(D_{kk}) \bar{\sigma} + f_2(D_{kk}) \bar{S}
\]

(114)

\[
\bar{a} = f_1(D_{kk}) \bar{a} + f_2(D_{kk}) \bar{a}
\]

(115)

\[
\bar{a} = (1/2)(\bar{a} \cdot \bar{\phi} + \bar{\phi} \cdot \bar{a})
\]

(116)

\[
\bar{S}^\alpha = \bar{S} - (1/3) \bar{S}_{kk} \bar{I}
\]

(117)
Here, $h_\alpha$, $h_\kappa$ and $r_\alpha$, $r_\kappa$ are hardening and recovery functions, respectively. It is important to note that the strain rate is assumed to depend on the first invariant of damage in keeping with the findings of Leckie et al. [20] and Murakami et al. [9].

Application to Type 304 Stainless Steel at 593°C

The proposed theory in equations (106)-(117) is quite general in applicability. There exists various levels of complexity or sophistication at which one can choose to apply this theory. Compatible with the goals of the first year of this program, the equations were simplified to a form involving engineering stress in the isochronous stress equation, a simple multiaxial creep strain rate equation, and a simple rupture criterion. Lack of a sufficiently exhaustive base of multiaxial rupture tests warrants this level of sophistication. Actually, the limiting assumptions made in this section are also those made by other investigators of the creep continuum damage approach and should not be viewed as unusually restrictive. Future research in this program should increase the model accuracy and sophistication.

Briefly, the pertinent equations are:

$$\tilde{\sigma} = B[\sigma^\star(\tilde{\sigma}^{pk})]k(\phi:\phi)^{\frac{1}{2}}[\eta I + (1-\eta)\sum_{k=1}^{3} \nu(k) \otimes \nu(k)M(k)]$$

(118)

where

$$M(k) = \langle \nu(k) \cdot \frac{\sigma^{pk}}{\sigma_1} \cdot \nu(k) \rangle$$

and $\langle F \rangle = F$ if $F \geq 0$; $\langle F \rangle = 0$ if $F < 0$. $\sigma^{pk}$ is the 2nd Piola-Kirchoff stress tensor. For purposes of brevity, it will be understood that $\sigma$ will be taken to represent $\sigma^{pk}$ in all that follows.
The isochronous stress is defined by

\[
\sigma^* = \frac{3}{2} S_1 \left( \frac{2 \sigma \delta}{S_1} \right)^a \exp \left[ b \left( \frac{J_1}{S_s} - 1 \right) \right]
\]  

(119)

where

\[
J_1 = \sigma_1 + \sigma_2 + \sigma_3
\]

\[
\bar{\sigma} = \left[ \left( \sigma_1 - \sigma_2 \right)^2 + \left( \sigma_2 - \sigma_3 \right)^2 + \left( \sigma_1 - \sigma_3 \right)^2 \right]^{1/2}
\]

\[
S_1 = \sigma_1 - J_1/3
\]

\[
S_s = (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)^{1/2}
\]

and B, k, \( \lambda \), a and b are material constants. Note that only positive principal stresses are permitted to contribute to damage evolution. Also,

\[
u^{(k)} = \text{unit vector in } k^{\text{th}} \text{ principal stress direction}
\]

\[
\phi = (\bar{\Sigma} - \bar{\Sigma})^{-1}
\]

\[
\eta = \text{isotropic damage weighting factor } (0 \leq \eta \leq 1)
\]

A simple form of the creep strain rate equation was used to obtain an estimate of the evolution of in-plane creep strain components in the axial-torsional tests. State variables \( q \) and \( \kappa \) (e.g. backstress and dragstress) were not included due to lack of characterization of the hardening and recovery functions; the initial scope of this investigation does not require their inclusion. Prediction of creep strain accumulation for alternating nonproportional loading would, in general, require their inclusion to reflect deformation-induced anisotropy effects.
Also, compatible with the goals of the initial analysis, predictions and results are compared on the basis of nominal or engineering strain. The further assumption is made that creep strain rate is coupled to damage through the first invariant of damage only, i.e.

\[ \dot{e}^n = \frac{3}{2} \left( \frac{\sigma}{A} \right)^n \frac{\sigma'}{\sigma} \left[ 1 + (cD_{kk})^m \right] \]  

(120)

where \( c \) and \( m \) are constants determined by fitting the secondary and tertiary creep regimes of a uniaxial creep test at fixed stress.

**Determination of Material Constants**

All material constants were determined from uniaxial creep tests found in the literature and by examination of ruptured uniaxial specimens. Constants \( a \) and \( b \) were given by Huddleston [16] for type 304 stainless steel at 593°C as

\[ a = 1.086 \quad b = 0.289 \]

The exponent \( n \) was determined from uniaxial creep tests on annealed specimens as seen in Figure 1.

\[ n = 10.74 \]  

(121)

From the same tests,

\[ A = 60.0 \]  

(122)

where units of stress are ksi and strain rate in hr\(^{-1}\).

Integrating equations (118) for a uniaxial constant load creep test with axial stress \( \sigma_{11} \) leads to a rupture time \( t_R \) of
\[ t_R = \frac{1}{B(\sigma^*)^k} \int_0^{Dc} \left[ \frac{1}{1-D_{11}} \right]^2 + 2 \left[ \frac{1}{1-\eta D_{11}} \right]^{-\lambda/2} dD_{11} \]  \hspace{1cm} (123)

where \( D_{11} \) is the damage component in the axial direction.

If \( D_{11} = \) critical value = \( D_{Cr} \) at rupture, then rupture time depends explicitly on stress level and equation (123) can be expressed as

\[ \log t_R = A_1 - k \log \sigma_{11} \]  \hspace{1cm} (124)

where \( A_1 \) is a constant. From Huddleston's data for uniaxial tests [16], \( k = 8.5551 \).

From analysis of the damaged microstructure of two uniaxial creep specimens (to be discussed later), an average value of \( \eta = 0.61 \) was selected as representative of the degree of anisotropy of damage since wedge cracking is the dominant failure mode at the temperature and stress levels of this study. Precise determination of the nonlinearity of damage evolution, reflected by the exponent \( \lambda \), requires the interruption of tests at various points in the creep history along with sectioning and examination. Since interrupted test specimens are not available at this time, it was thought that values of \( B \) and \( \lambda \) should be selected in the range typical of other stainless steels reported in the literature [13-14]. Hence,

\[ B = 1.539 \times 10^{-17} \]

\[ \lambda = 4.80 \]
As will be discussed shortly, the values for \( B \) and \( \lambda \) were adjusted in an iteration process which involved fitting the tertiary portion of a uniaxial engineering creep strain curve.

In order to determine the coupling of strain rate with damage, a uniaxial creep curve shown in Figure 2 was digitized and a computer program was written to integrate the coupled equations (120) and (123) for the uniaxial case, i.e.

\[
\dot{\varepsilon}_{11} = B \left( \sigma^* \right)^\lambda \left[ \left( \frac{1}{1 - D_{11}} \right)^2 + 2 \left( \frac{1}{1 - \eta D_{11}} \right)^2 \right] ^{\lambda/2}
\]

\[
\dot{\varepsilon}_{11}^n = \left( \frac{\sigma_{11}}{A} \right)^n \left[ 1 + \left( c D_{kk} \right)^m \right]
\]

A flowchart of the computer program used to fit the uniaxial creep curve appears in Figure 3. In the first iterative loop, constants \( B \) and \( \lambda \) are adjusted until the value of the axial damage component \( D_{11} \) at rupture is approximately unity. Then, the values of \( c \) and \( m \) in equation (126) are determined iteratively by plotting the resulting predicted tertiary creep response as an overlay on the digitized experimental data. The values of \( B \) and \( \lambda \) can be adjusted slightly to best match the tertiary region, with the stipulation that the axial damage component at rupture should essentially be unity. It is important to realize that the area fraction of damaged grain boundaries at rupture perpendicular to the loading axis in a uniaxial creep test is usually significantly less than one [43], but the current formulation does not require that the area fraction be identically equal to the damage parameter. It is required, however, that the predicted damage components are of the same ratio as those experimentally determined. For the purpose of comparing tests conducted at the same isochronous stress level, specifying
that $D_{11} = 1$ at rupture is sufficient. For variable stress histories, though, one would like to employ the full capability of the model by relating area fractions identically to components of damage.

A plot of the predicted versus experimental creep curve for the values of the "best-fit" parameters is shown in Figure 4. Note that $c = 1.35$ and $m = 2.25$ provided the most accurate fit of the creep response for the assumed values of $B$ and $\mu$. It must again be emphasized that a more rigorous determination of $B$ and $\mu$ would require interrupted testing; however, the stress-independent rupture criterion and damage-independent isochronous stress of this initial formulation should provide rupture times consistent with the isochronous stress concept regardless of the $B$ and $\mu$ values for constant load, proportionally loaded creep tests. Accurate values of $B$ and $\mu$ are particularly important for step stress tests or for nonproportional loading.

The rupture criterion employed in this demonstration was

$$\nu(i) \cdot D \cdot \nu(i) = 1$$

(127)

where $i = 1, 2, 3$. This is equivalent to the criterion stated in equation (108) with $D_{cr} = 1$. Though the area fraction of grain boundaries in the continuum sense in any direction is generally not unity at rupture, the high nonlinearity of terms involving damage in the damage rate equation results in very little difference in predicted rupture time for critical maximum principal damage values between about 0.5 and 1.

Microstructural Damage Evaluation

In order to determine the appropriate value of $\eta$ for type 304 stainless steel at 593°C and to compare analytical predictions with observed physical damage, biaxial creep specimens were sectioned, polished, and etched. Then,
micrographs were taken at various locations to obtain a sample distribution rather than a single micrograph. Since the rupture surface is viewed primarily as a fracture phenomenon resulting from linkage of voids and wedge cracks into an unstably propagating crack, all samples were taken from points away from the rupture crack but still in the zone of uniform temperature and deformation. A discolored region was observed in the middle third of the 2.43 inch gage section, evidently associated with localization of deformation; all micrographs were taken in this region.

Since both wedge cracking and cavitation contribute to grain boundary damage, it was initially desired to include both in constructing the damage tensor from micrographs. Due to the different nature of each type of damage, however, the wedge cracking and cavitation components of the damage tensor were initially computed separately, i.e.

\[ D_{\text{exp}} = D_{\text{w}} + D_{\text{c}} \]  

(128)

where \( D_{\text{exp}} \) represents the total damage tensor measured experimentally. In general, the cavitation damage is more difficult to quantify than the wedge cracking. Lack of resolution in the micrographs, even at 1000X, made it difficult to assign an area fraction value quantitatively to a cavitated grain boundary. Also, grain boundary carbides were so prevalent that small cavities were indiscernable. Furthermore, due to the relatively large deformations encountered at rupture in type 304 stainless steel, cavities are smeared due to grain boundary sliding and elongation. This elongation of grains in the primary stretch direction also created a preferred orientation for grain boundary segments which skewed the calculation of the cavitation damage tensor from micrographs, since cavitation was observed almost uniformly on all grain boundaries. In contrast, wedge cracking was much more readily quantifiable.
The argument can be made, of course, that the damage tensor should only be computed based on grain boundary microcracks, since they represent coalesced voids in addition to triple point cracks.

It is desirable to obtain a representative sample of grains for quantitative damage measurement to ensure reliable results. To include a sufficient sample size of grain boundaries, a magnification of 200X was used for determining the wedge crack or microcrack damage tensor. A magnification of 1000X was used to determine the cavitation damage tensor to improve the resolution of grain boundaries. As previously mentioned, determination of the cavitation damage tensor was fraught with problems; perhaps the most serious reservation is that the cavitation observed or measured in the ruptured specimen is not representative of the evolution of cavitation and eventual linkage of cavities due to the smearing effects. It is our current thinking that quantification of the wedge cracking will most successfully describe the rupture state and its link to damage history, as mentioned above.

Sections were taken at two locations each at the inside and outside specimen diameters. At each section, five different locations were photographed to provide a suitable sample lot. For the uniaxial specimens, the damage was evaluated at the specimen centerline at five locations. Typical micrographs for the wedge crack and cavity distributions appear in Figure 5 for biaxial Specimen GT-1, a proportionally loaded specimen. It is interesting to note that a radial gradient of damage is clearly observed in micrographs of the entire specimen wall with a heavier distribution of wedge cracking at the specimen outer diameter. This effect is possibly due in part to the higher shear strain accumulated at the outer diameter than at the inner. Figure 6 illustrates the radial damage gradient.

Computer programs were written for a Weiss Videoplan (available in Dr. Underwood's quantitative metallography laboratory) to allow the user to
move the cursor along grain boundaries and to mark wedge cracks or cavitated segments. The results were immediately digitized on floppy disk. Post-processing programs were written to convert the raw data in the wedge crack or cavity files to print to any output file the grain boundary segment length, fraction wedge cracked or cavitated, and the normal vector to the segment. All grain boundary segments in a given micrograph were digitized, regardless of whether or not any damage was present. Hence, total grain boundary length for each micrograph is available.

The processed data files were stored on floppy disks formatted with MSDOS. BASIC computer programs were written to perform the computation of wedge crack and cavitation damage tensors on an IBM PC based on the numerical integration implied by the equations

\[ D_W = \frac{1}{L_T} \sum_{k=1}^{N} \vec{n}(k) \otimes \vec{n}(k) \Delta S_{gw}(k) \]  \hspace{1cm} (129)

\[ D_C = \frac{1}{L_T} \sum_{k=1}^{M} \vec{n}(k) \otimes \vec{n}(k) \Delta S_{gc}(k) \]  \hspace{1cm} (130)

where \( L_T \) is the total grain boundary length in the micrograph \( \vec{n}(k) \) is the unit normal vector to the \( k^{th} \) grain boundary segment, and \( \Delta S_{gw}(k) \) and \( \Delta S_{gc}(k) \) are, respectively, the length of the wedge crack or cavitated segment associated with the \( k^{th} \) grain boundary segment. Copies of the two computer programs written to compute the damage tensors appear in the APPENDIX.

**Prediction of Damage Evolution and Comparisons with Data**

In the present theoretical continuum damage approach, no distinction is made between cavitation damage or wedge crack damage. A computer program was
written to integrate the coupled equations (118)-(120). The numerical integration technique used was a Runge-Kutta with fixed time step size. This method was found to result in very efficient, accurate integration of unified creep-plasticity theory in an earlier study [46] for nonproportional cyclic loading. A flowchart of the computer program appears in Figure 7.

Since the damage evolution depends on $\eta$, it was necessary to establish a value of $\eta$ from micrographs of uniaxial creep tests. Two specimens were provided by ORNL for this purpose. The value of $\eta$ was based on quantification of wedge type damage only; this should not be too restrictive since this type of damage most likely dominates the rupture process in the stress and temperature regimes of the tests. Referring to equation (118) it is clear that the transverse damage $D_{22}$ is related to the axial damage $D_{11}$ by $D_{22} = \eta D_{11}$. Hence, $\eta$ is the ratio of the transverse to axial components of damage in a uniaxial creep test. A value of $\eta = 0.61$ was determined as the average value of $\eta$ computed from ten micrographs, five from each specimen. It should be noted that this value of $\eta$ indicates that the isotropic component of the damage tensor is "larger" than the anisotropic component. Hence, we would not expect a change in loading direction at the same isochronous stress level to result in a factor of two difference in rupture life as in pure copper.

The computer program was written to allow any nominal stress history to be input. Output includes rupture time and plots of damage components $D_{11}$ and $D_{12}$ in addition to engineering strain components $e_{11}$ and $e_{12}$.

It should be noted that the experimental values of $e_{11}$ and $e_{12}$ were computed using the equations

\[
e_{11} = \frac{\delta L}{L_0} \quad (131)
\]

\[
e_{12} = \frac{1}{2} \bar{\tau} \delta \psi / L_0 \quad (132)
\]
where $L_0$ is the gage length, $\bar{r}$ is the average radius, and $\delta\phi$ is the relative rotation of the gage section in radians. Obviously, engineering strain is not indicative of the true strains occurring at large deformations. It is not entirely unreasonable, though, to compare the engineering strains predicted by the model with those obtained experimentally; the difference between true strains and nominal strains in the experiments of this study are less than ten percent.

**Discussion of Results**

The biaxial loading histories of Specimens GT-1,2,3,4 appear in Figure 8. The applied axial and shear stresses plotted in these histories are nominal values, i.e. based on initial dimensions. The shear stress (moment) was applied at the bottom of each specimen in either a clockwise or counter-clockwise sense viewing down the specimen longitudinal axis. It should be noted that the isochronous stress for Specimens GT-1,2,3 is constant (based on nominal stresses) at $\sigma^* (\sigma^{pk}) = 25.54$ ksi. Hence, differences in rupture time should be attributable to nonproportionality of loading, neglecting the possibility of reversal of plastic strain. Specimens GT-1 and GT-2 were subjected to identical loading histories apart from the sign of shear stress. The loading history of Specimen GT-3 involved a reversal in sign of the shear stress at 456 hours.

Predicted and experimental results for the biaxial loading histories appear in Table 1. Note that a counter-clockwise applied moment corresponds to a positive shear stress in the $X_1,X_2$ coordinate system shown in Table 1. The predicted damage and creep strain components correspond to the time step immediately preceding satisfaction of the rupture criterion in equation (108) with $D_{cr} = 1$. It is observed in Table 1 that a rupture time of 1020 hours is predicted for Specimens GT-1 and GT-2; the actual values are 892 hours and
1173 hours, respectively. Good estimates of rupture time and accumulated creep strain components are achieved by the model, including the tertiary regime. Note that the actual creep strain components in Table 1 include only secondary and tertiary components as measured from plots of gage length extension and angle of twist versus time.

Plots of predicted damage and strain rate evolution appear in Figures 9-10 for Specimens GT-1 and GT-2. For reference, plots of relative angle of twist and gage length elongation versus time appear in Figures 11-12. Note the excellent agreement of the onset of tertiary creep and magnitude of creep strain at rupture between theory and experiment. Also note that the slope of the predicted creep strain rate is not infinite at rupture, which is confirmed by the experiments. The actual grain boundary microcrack damage at rupture is significantly less than unity for Specimens GT-1 and GT-2, with an average maximum principal damage of $D_1 = 0.28$. It is quite interesting to note that this value is very close to the rupture criterion used by Ashby et al. [43], i.e. $f_h = 0.25$ ($=D_1$). This result provides confirmation that the rupture event is a process of unstable linkage and propagation of grain boundary cracks after a critical level of grain boundary damage is reached. It is important to note that the rupture criterion used in these calculations, $D_j = D_{cr} = 1$, may be replaced by any other, more realistic, criterion such as $D_{cr} = 0.25$ or equation (109), provided that the coupling constants $c$ and $m$ in the creep strain rate equation are determined in conjunction with this criterion.

It is concluded that the second rank damage tensor approach of this study offers an improvement on prior isotropic damage models with regard to assigning magnitude and direction of creep damage for proportional loading. Note in Table 1 that the average orientation of the measured maximum principal damage values (among 42 digitized micrographs) were in relatively good agreement with the maximum principal stress directions. The relatively large standard
deviations for the principal values of damage and orientation, given in Table 2, are reflective of the heterogeneity of damage and the relatively large isotropic components. A highly anisotropically damaging material would obviously display less scatter.

The purpose of experiments GT-3 and GT-4 was to evaluate the capability of the second order tensor damage model to predict direction and magnitude of accumulated creep damage in addition to rupture time for nonproportional loading histories. As seen in Figure 13, Specimen GT-3 was subjected first to a counter-clockwise shear stress and an axial stress at the same isochronous stress level as Specimens GT-1 and GT-2; after 456 hours, however, the shear stress was reversed while maintaining the same isochronous stress level. A maximum principal stress rotation of 34° resulted from this shear stress reversal. Evolution of damage components and creep strain components are shown in Figure 14. The measured gage length extension and relative angle of twist versus time is shown in Figure 15 for comparison.

It is noted first that the predicted rupture time of 1030 hours listed in Table 1 is significantly lower than the observed 1398 hours. Also, the measured orientation of maximum principal damage is 13° from the predicted orientation. This result indicates that the actual rotation of the damage tensor was not as great as that predicted by the theory. Also, note that the magnitude of damage at rupture is less than in proportionally loaded Specimens GT-1 and GT-2. There are several likely reasons for these discrepancies.

Firstly, it is noted that there is a significantly higher creep shear strain observed experimentally than would be predicted using a simple deviatoric flow rule. This can be attributed to creep-plasticity interaction. In particular, memory of plastic deformation incurred in the initial loading direction was retained upon reversal of applied shear stress. Hence, power law creep does not occur radially after load reversal, but with respect to a
backstress imposed by the initial loading and slightly altered by plasticity during the shear stress reversal. Another consequence of this memory of the initial loading direction is that secondary creep rate is reduced (by ~10%) after reversal of shear stress. A simple creep strain rate equation without backstress cannot reflect this memory. Since grain boundary damage in these particular experiments is driven by matrix power law creep, the isochronous stress dependence of the coefficient in the damage rate equation implies a unique relationship between applied stress and time to rupture. It is felt that dependence on isochronous overstress $(\sigma - a)^*$ may be more appropriate for the damage rate equation under power law creep conditions. In this case $(\sigma - a)^*$ is based on the backstress from an appropriate unified creep-plasticity constitutive law (e.g. equations (111) - (113)) and is computed in identical fashion to $a^*$. This modification would more appropriately reflect the driving force for creep damage growth.

Secondly, regarding the measured damage distribution, it is noted that the maximum principal value of damage is nearly collinear with the specimen longitudinal axis. This is quite possibly an artifact of the assumed second order tensor distribution of damage. For proportional loading, a second order tensor distribution for the anisotropic component of damage appears appropriate as evidenced by the correlations achieved for Specimens GT-1 and GT-2. However, a second rank damage tensor simply rotates its three eigenvectors in response to change in principal stress orientations. This rotation amounts to an "averaging effect" on the damage distribution. As pointed out by the work of Leckie and Onat (equations (29)-(43)), it may be necessary to represent the anisotropic component of the damage distribution by higher, even-ordered tensors. For example, Specimen GT-3 may require a fourth order distribution to properly model the physical damage incurred by the two loading directions. The required distribution must be determined by further examination of the
wedge crack damage without the constraint of a second order damage tensor assumption. It is fully intended to pursue this determination.

Thirdly, the experimental result that the effective damage at rupture in nonproportionally loaded Specimen GT-3 is less than that for the proportionally loaded Specimens GT-1 and GT-2 is of concern. There is, of course, the question of statistical significance of this single result which tempers any mechanistic interpretation. However, this result possibly reflects influence of the first invariant of damage (or mean value of damage) on the rupture event since it is likely a higher fraction of the maximum value of damage in a nonproportional test than in a proportional test. This assertion can also be evaluated by considering the higher order damage distribution in the plane of the specimen. Such an evaluation will be pursued.

As seen in Figure 16, Specimen GT-4 was subjected to a relatively complex loading history. The experiment was originally intended to be a repeat of GT-3, but experimental difficulties related to an initial zero offset of torque of approximately 44% of the intended torque resulted in an initial overload which was later realized and corrected to produce an appropriate rupture time.

The problems involving creep-plasticity interaction and assumption of a second order anisotropic damage distribution are exacerbated by the nature of this history. Conclusions of model performance are somewhat difficult to draw regarding damage at rupture and creep strain at rupture due to the variable isochronous stress loading history and greater number (four) of principal stress orientations. It may be concluded, however, that the model refinements suggested by history GT-3 are also germane in this case. In particular, the inelastic strain rate equation should be modified to include backstress, the isochronous stress should be referenced to backstress, and the physical damage distribution should be studied to evaluate the necessity of a higher order
damage distribution. It is the author's judgment that these refinements will provide much more accurate modeling of both of histories GT-3 and GT-4.

There is a noteworthy aspect of GT-4 which deserves comment. The initial overload resulted in a model prediction of a rupture time of 440 hours, significantly shorter than the observed 1230 hours. This discrepancy is not attributable to the anisotropic form of the damage model, but to the dependence of rupture time on isochronous stress since the loading is proportional. The value of \( k \) has been properly determined from uniaxial rupture tests. However, the values of \( B \) and \( \lambda \) have been assumed and require further determination from experiments. To see that \( B \) and \( \lambda \) can influence rupture time, a function of stress, it is necessary only to express the integrated damage model at rupture, assuming \( D=\text{constant at rupture and uniaxial loading} \), as:

\[
\log t_R = \log \frac{1}{B} \int_0^D \left[ \left( \frac{1}{1-D_{11}} \right)^2 + 2 \left( \frac{1}{1-\eta D_{11}} \right)^2 \right]^{-\lambda/2} dD_{11} - k \log \sigma \quad (133)
\]

Hence, having selected \( \lambda, B \) is constrained by this equation. Such a constraint was not, however, imposed on \( B \) in the first year of this work (although the values used are relatively close to the constrained values) since all experiments were to be conducted at the same isochronous stress level. By imposing the constraint, rupture would probably not be predicted until approximately 1100 hours which agrees more readily with the experiment.

In summary, history GT-4 has emphasized the refinements necessary in the creep strain rate equation, the order of the anisotropic damage representation, and determination of \( B \) and \( \lambda \) values.

One additional result of the experimental program regards the analysis of the two uniaxial specimens mentioned earlier. It is quite interesting to note
that the angle $\theta$ determined for wedge cracking in the uniaxial case is only $-8.3^\circ$, indicating that that maximum principal damage value is more nearly collinear with the loading axis than for the proportionally loaded biaxial Specimen GT-1. Since only two uniaxial specimens were sectioned and examined, it is likely that the angle of $8.3^\circ$ is not of high confidence level. Accuracy of the angle is more highly dependent on sample size than is accuracy of $\eta$ since orientation is particularly important for calculation of the shear component of damage.

Discussion of Practical Implementation of Simple Second Order Tensor Approach

In this section, matrix forms of the damage rate equations will be written for those more familiar with matrix structural analysis. In the full nine dimensional damage space, we may write equation (106) as

\[
\begin{bmatrix}
D_{11} & D_{12} & D_{13} \\
D_{12} & D_{22} & D_{23} \\
D_{13} & D_{23} & D_{33}
\end{bmatrix} = B[\sigma^*(\sigma)]^k \psi^{1/2} \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

\[
\sum_{k=1}^{3} (1 - \eta) \begin{bmatrix}
[\nu_1^{(k)} \nu_2^{(k)}]
[\nu_1^{(k)} \nu_3^{(k)}]
[\nu_2^{(k)} \nu_3^{(k)}]
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \sigma_{12} \sigma_{13} \\
\sigma_{22} \sigma_{23} \\
\sigma_{33}
\end{bmatrix} \times
\begin{array}{c}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{array}
\]
\[
\begin{bmatrix}

\nu_1^{(k)} \\
\nu_2^{(k)} \\
\nu_3^{(k)}
\end{bmatrix}
\]

(134)

where \(\nu_i^{(k)}\) are the \(i\)th components of the unit vector in the direction of the \(k\)th principal stress (\(k\)th eigenvector), \(\sigma\) is the engineering stress, and

\[\Psi = (\phi : \phi) = \text{tr}[[\phi]^T[\phi]] \quad (135)\]

where

\[
[\phi] = \begin{bmatrix}
(1-D_{11}) & -D_{12} & -D_{13} \\
-D_{12} & (1-D_{22}) & -D_{23} \\
-D_{13} & -D_{23} & (1-D_{33})
\end{bmatrix}
\]

(136)

In equation (134), \(\langle F\rangle = F\) if \(F \geq 0\); \(\langle F\rangle = 0\) if \(F < 0\).

The simple deviatoric form of the creep strain rate may be written

\[
\begin{bmatrix}
\dot{\epsilon}_{11} & \dot{\epsilon}_{12} & \dot{\epsilon}_{13} \\
\dot{\epsilon}_{22} & \dot{\epsilon}_{23} & \dot{\epsilon}_{33}
\end{bmatrix}
= \frac{3}{2} \left(\frac{\alpha}{A}\right) \left[\frac{1 + (c(D_{11} + D_{22} + D_{33}))}{\bar{\sigma}}\right] \frac{1}{n} \left[1 + (c(D_{11} + D_{22} + D_{33}))\right]^{n-1}
\]

(137)
Determination of Constants $B$, $k$, $\lambda$, $\eta$, $A$, $n$, $c$, $m$:

Constants $A$ and $n$ may be determined from uniaxial tests at multiple levels since

$$\epsilon^c = \left( \frac{\sigma_{11}}{A} \right)^n \quad (138)$$

in uniaxial secondary creep.

Exponent $k$ may be identified as the magnitude of the slope of a log $t_R$ versus log $\sigma_{11}$ plot at the desired temperature obtained from uniaxial creep tests.

Isotropic damage weighting factor $\eta$ may be determined from a uniaxial creep test by

$$\eta = \frac{D_{22}}{D_{11}} \quad (139)$$

where $D_{11}$ is the longitudinal damage and $D_{22}$ is the transverse damage. These damage components can be determined from micrographs as

$$D_{11} = \frac{1}{C_T} \sum_{k=1}^{N} n_1(k) n_1(k) \Delta S_g(k) \quad (140)$$
\[ D_{22} = \frac{1}{L_T} \sum_{k=1}^{N} n_1(k)n_2(k)\Delta S_g(k) \]

where \( n_1(k) \) and \( n_2(k) \) are the components of the unit vector normal to the \( k \)th damaged grain boundary segment in the specimen longitudinal and transverse directions, respectively; \( L_T \) is the total grain boundary length and \( \Delta S_g(k) \) is the length of the \( k \)th grain boundary micro-discontinuity (e.g. wedge crack, cavity, coalesced cavities).

Constants \( B \) and \( \varepsilon \) are properly determined from interrupted uniaxial tests in addition to ruptured specimens at the same stress level. Consider a test interrupted at a known \( (t/t_R) \) value; \( \varepsilon \) may be found by

\[ \left( \frac{t}{t_R} \right) = \int_{0}^{D_t} \frac{\left[ \left( \frac{1}{1-D_{11}} \right)^2 + 2 \left( \frac{1}{1-\eta D_{11}} \right)^2 \right]^{-\lambda/2}}{D_{11}} dD_{11} \]

where \( D_c \) is the measured longitudinal damage at rupture, \( D_t \) is the measured longitudinal damage at time \( t \) of interruption. All quantities in equation (142) are known except for \( \lambda \). Then, having solved for \( \lambda \), \( B \) can be determined, i.e.

\[ B = \frac{1}{t_R(\sigma_{11})^k} \int_{0}^{D_c} \frac{\left[ \left( \frac{1}{1-D_{11}} \right)^2 + 2 \left( \frac{1}{1-\eta D_{11}} \right)^2 \right]^{-\lambda/2}}{D_{11}} dD_{11} \]

Constants \( c \) and \( m \) can then be adjusted to fit the secondary-tertiary uniaxial response by simultaneous integration of the damage rate equation and creep strain rate equation.
It should be noted that the damage at rupture, $D_c$, is in general a function of stress level as expressed in equation (109). A reasonable choice for the rupture criterion is

$$\max_{\text{all } n} \left[ (n_i D_i n_j)^r (n_k \sigma_k n_k) \right] = \text{constant}$$

(144)

which can be written for the uniaxial case as

$$\sigma_{11} D_c^r = \text{constant}$$

(145)

where $r$ is a constant determined by measuring axial damage in uniaxial tests at multiple stress levels.

Comparison With Linear Time Fraction Rule:

The linear time fraction rule may be stated as

$$\sum_{j=1}^{M} \left( \frac{t_j}{t_{R_j}} \right) = 1$$

(146)

Clearly, this rule implies that damage is isotropic. Rotation of the principal stresses, for example, at a constant isochronous stress results in a predicted rupture time from the linear rule equal to the uniaxial rupture time. The continuum damage rule will produce the same result only if $\eta = 1$. However, for $\eta < 1$ the predicted rupture time increases nonlinearly in the continuum damage approach during rotation of principal stresses.

Two features of the continuum damage model play a key role in relating damage fraction to time fraction for multiple stress level tests (variable loading histories). These features are the rupture criterion and exponent $z$. 

-66-
Assuming damage is isotropic (i.e. $\eta = 1$), the damage rate equation may be integrated for uniaxial loading to give uniaxial damage as a function of time fraction:

$$D_{11} = 1 - \left[1 - \left(1 - D_c \right)^{(1+\xi)} \left(\frac{t}{t_R}\right) \right]^{\frac{1}{1+\xi}}$$  \hspace{1cm} (147)

It is very important to note that if damage is assumed constant at rupture, i.e. $D_c = \text{constant}$, and if $\xi = \text{constant}$ then $D_{11}$ is a unique function of time fraction independent of stress level. The linear time fraction rule may then be stated in terms of the damage parameter as

$$\sum_{j=1}^{M} \frac{\left\{1 - \left(1 - D_j \right)^{(1+\xi)}\right\}}{\left\{1 - \left(1 - D_c \right)^{(1+\xi)}\right\}} = 1$$ \hspace{1cm} (148)

where $D_j$ is the value of $D_{11}$ at time $t_j$. No stress level sequence effects are described by this approach. Note, though, that the damage ratio $(D_j/D_c)$ is not summed to one.

Stress level sequence effects can, in general, be very important in creep. It is well-documented [34] that the grain boundary damage accumulated at low stress levels can be significantly larger than at higher stress levels. Hence, low-high sequences can be more damaging than high-low sequences at a given temperature [34]. In contrast to the current work, Chaboche introduced sequence effects by making exponent $\xi$ [14] a function of applied stress while maintaining the rupture criterion $D_c = 1$. However, the stress level dependence may be more rationally introduced through a rupture criterion such as equation (144) which recognizes the contribution of applied stress to the rupture process. In fact, if an inverse relationship exists between applied
stress and damage, as in equation (144), then the ratio of damage at any selected time fraction of a lower stress level test to that of a higher stress level test will be greater than unity; this result, of course, is equivalent to that of the $\epsilon = \epsilon (\sigma)$ approach taken by Chaboche. This result is easily obtained from equation (147). In this case, however, coefficient B must be made a function of applied stress to ensure that a Monkman-Grant (or other applicable) relationship is maintained between applied stress and rupture time.

The importance of the rupture criterion in a mechanistically accurate formulation of creep damage is now established. It should be noted that all aforementioned statements regarding sequence effects apply as well to the anisotropic damage case. For clarity of presentation, the isotropic damage case is discussed since closed form solutions exists for the integrals.

It is interesting to note that in the current formulation with $\epsilon = \text{constant}$, materials which obey a linear time fraction criterion are defined by $D_C = \text{constant at rupture}$; i.e. a stress-level independent rupture criterion. Other rupture criterion lead to nonlinear time fraction rules. It is reasonable to expect, if the damage parameter is based on physical damage, that proper correlation of the growth of this damage and the rupture event will result in a mechanistically based damage accumulation rule, whether expressed in terms of the damage parameter or time fraction.

**Suggested Further Developments**

The results of this study have suggested several important areas which require further attention. These areas are briefly listed as follows:

(A) Interrupted tests need to be run to determine damage growth exponent $\epsilon$ accurately.
(B) Damage must be measured for both uniaxial and biaxial tests at different effective stress levels and stress states to investigate appropriate stress-level and damage-dependent rupture criteria.

(C) The measured directional distribution of damage should be examined for biaxial proportional and nonproportional tests to determine if second order tensor representation is accurate. More general anisotropic (higher even rank tensor) damage framework should be developed for materials which exhibit a high degree of damage anisotropy (i.e. low $\eta$). Current framework may be sufficient for medium to high degrees of isotropy of damage (i.e. $\sim 0.4 \leq \eta \leq 1$). Also, coupling of creep and plastic deformation may lead to necessity of introducing the effects of plastic deformation during a load reversal or rotation on subsequent creep, particularly since the damage rate after such a reversal depends on matrix power law creep.

(D) A stress-level-dependent creep damage rate equation based on void growth and coalescence should recognize the relative roles of grain boundary diffusion and power law creep, including regimes of coupling between these two mechanisms. This coupling could be expressed through the appropriate modified form of the current damage state in the isothermal damage rate equation, e.g.

$$\dot{D} = \left[ B_1 (\sigma^*)^k \{ (\dot{\varepsilon} : \dot{\varepsilon})^{k/2} - (\dot{\varepsilon} : \dot{\varepsilon})^{-1/2} \} + B_2 \{ (D_{kk})^{-1/2}/\ln(1/D_{kk}) \} \sigma_1 \right] \eta + (1 - \eta) \sum_{j=1}^{3} \nu_j \otimes \nu \nu(j) M(j) \right \} (149)$$

where $B_1$ and $B_2$ are functions of effective stress. Of course, other micro-mechanical formulations can be appropriate, depending on
whether diffusion is constrained or unconstrained, grain boundary or matrix, etc. The problem is exacerbated by the fact that actual components may be loaded at low stress levels conducive to diffusion dominated damage growth, while experiments are conducted at higher stress levels, due to time constraints, where power law void growth occurs.

It is also necessary to acknowledge the presence of unstable microstructures when such exist. Precipitated carbides on grain boundaries can serve as void initiation sites; a high area fraction of carbides may actually retard void growth. Such microstructural "aging" effects may require the addition of a scalar state variable and associated evolution equation representative of area fraction of grain boundary void initiation sites. Obviously, this state variable would couple with the damage rate (void growth) equation. This approach would seemingly be necessary only for unstable microstructures.

(E) Solution of general coupled thermo-viscoplastic problems admitting damage requires a proper constitutive equation for growth of all internal variables including damage. Proper growth equations should satisfy the Clausius-Duhem inequality with the generalized thermodynamic forces related to the rate of conjugate internal variables through a viscous/damage potential function. This function can be constructed in an inverse manner, proceeding from the phenomenological growth laws to the potential function, invoking normality as a heuristic postulate. The anisotropy of damage rate would therefore be embedded in the damage potential.
Conclusions

The major achievement of this study has been the physical linkage of the continuum creep damage approach to grain boundary damage. It is the first study known to the author which compares predicted tensorial damage with measured values. The following is a list of key results of this investigation.

1. A generalization of isochronous stress and continuum damage concepts has been made to include multiaxial nonproportional loading.

2. Material tests necessary to determine isothermal model constants and parameters have been identified.

3. Good agreement has been obtained for both rupture time and physical damage between predicted and measured results on two proportionally loaded biaxial specimens.

4. The second rank tensor-based definition of damage is an approximation of the physical damage distribution when the principal axes of stress rotate. Further work must address the accuracy of this approximation.

5. Wedge crack damage in the proportionally loaded type 304 stainless steel specimens of this study was comprised of nearly equal contribution of isotropic and anisotropic components, with an orientation of the principal values of damage coincident with those of the stress tensor. Investigation of cavitation damage was inconclusive due to the great extent of cavitation and grain boundary sliding at rupture. It is apparent that a definition of damage based on area fraction of coalesced voids or wedge cracks results in a value of damage much less than unity at rupture (~0.2), in agreement with prior micro-mechanical studies. Hence, the rupture criterion is most aptly expressed in a "psuedo-fracture" manner as a function of both accumulated grain boundary damage and stress level.
6. Further required refinements include use of a unified creep-plasticity theory to properly model creep-plasticity interaction, use of an isochronous overstress when power law creep drives void growth, and more precise experimental determination of $B$ and $\ell$.

Implementation of these refinements should substantially improve the capability to model cumulation of damage under nonproportional loading. Future work will address this implementation.
TABLE 1
Predicted and Measured Quantities for Specimens GT-1, 2, 3, 4

<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>D_{11}</td>
<td>0.63</td>
<td>0.31</td>
<td>0.63</td>
<td>0.24</td>
<td>0.67</td>
<td>0.15</td>
<td>0.57</td>
<td>0.20</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>D_{12}</td>
<td>0.071</td>
<td>0.037</td>
<td>-0.071</td>
<td>-0.016</td>
<td>-0.048</td>
<td>0.0014</td>
<td>0.082</td>
<td>0.0035</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D_{22}</td>
<td>0.42</td>
<td>0.24</td>
<td>0.42</td>
<td>0.15</td>
<td>0.44</td>
<td>0.096</td>
<td>0.40</td>
<td>0.13</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D_{1}</td>
<td>0.65</td>
<td>0.326</td>
<td>0.65</td>
<td>0.24</td>
<td>0.68</td>
<td>0.15</td>
<td>0.60</td>
<td>0.20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D_{2}</td>
<td>0.40</td>
<td>0.224</td>
<td>0.40</td>
<td>0.15</td>
<td>0.43</td>
<td>0.096</td>
<td>0.37</td>
<td>0.13</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\eta = D_{2}/D_{1})</td>
<td>0.61</td>
<td>0.69</td>
<td>0.61</td>
<td>0.62</td>
<td>0.63</td>
<td>0.64</td>
<td>0.61</td>
<td>0.65</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\theta^{(+ccw)})</td>
<td>17.0°</td>
<td>23.0°</td>
<td>-17.0°</td>
<td>-9.9°</td>
<td>-11.5°</td>
<td>1.5°</td>
<td>22.0°</td>
<td>2.9°</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(e_{11c})</td>
<td>0.18</td>
<td>0.155</td>
<td>0.18</td>
<td>0.146</td>
<td>0.19</td>
<td>0.119</td>
<td>0.24</td>
<td>0.181</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(e_{12c})</td>
<td>0.091</td>
<td>0.097</td>
<td>-0.091</td>
<td>-0.080</td>
<td>-0.036</td>
<td>-0.126</td>
<td>0.175</td>
<td>-0.009</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>t_R (hrs)</td>
<td>1020</td>
<td>892</td>
<td>1020</td>
<td>1173</td>
<td>1030</td>
<td>1398</td>
<td>440</td>
<td>1230</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Definitions:

\(D_1, D_2\) = maximum and minimum in-plane principal damage values, respectively.

\(\theta\) = angle, measured positive counter-clockwise at the specimen outer surface, from the longitudinal axis to the direction of \(D_1\).
<table>
<thead>
<tr>
<th>Standard Deviation</th>
<th>Specimen Identification</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>GT-2</td>
</tr>
<tr>
<td>SD(D₁)</td>
<td>0.17</td>
</tr>
<tr>
<td>SD(D₂)</td>
<td>0.12</td>
</tr>
<tr>
<td>SD(θ)</td>
<td>44.1°</td>
</tr>
<tr>
<td># Micrographs</td>
<td>42</td>
</tr>
</tbody>
</table>

Note: Number of grains per micrograph = 30-60.
REFERENCES


Fig. 1  Comparison of minimum creep rates at 593°C for several types of uniaxial tests.
Fig. 2  Constant load uniaxial creep curve at 593°C and 23.0 ksi.
Fig. 3 Flowchart of computer program used to iteratively fit the uniaxial creep curve.
damage(49) = .9850499

c change ?

\[ b = 1.539 \times 10^{-17} \]
\[ l = 4.8 \]
\[ ita = 0.61 \]
\[ c = 1.35 \]
\[ m = 2.25 \]
\[ .440855 \]

Fig. 4 Overlay of experimental creep curve and output of model.
Fig. 5 Typical micrographs for determination of wedge crack (top) and cavitation (bottom) damage tensors.
Fig. 6 Specimen transverse cross-section, illustrating radial damage gradient.
Fig. 7 Flowchart of computer program used to integrate coupled deformation-damage equations for the biaxial histories of this study.
Note: All stresses in ksi

Fig. 8 Axial-torsional nominal stress loading histories for (a) Specimen GT-1, (b) Specimen GT-2, (c) Specimen GT-3, and (d) Specimen GT-4. Axial and shear stress magnitudes for specimens GT-1, 2, 3 are 22.73 ksi and 7.58 ksi, respectively, which results in an isochronous stress of 25.54 ksi.
Fig. 9  Evolution of predicted damage and creep strain components versus $t/t_R$ for Specimen GT-1.
Fig. 10 Evolution of predicted damage and creep strain components versus $t/t_R$ for Specimen GT-2.
Fig. 11 Experimental curves for axial displacement (top) and relative angular rotation (bottom) versus time for the gage length of Specimen GT-1.
Fig. 12 Experimental curves for axial displacement (top) and relative angular rotation (bottom) versus time for the gage length of Specimen GT-2.
Fig. 13 Axial-torsional nominal stress loading history of Specimen GT-3.
Fig. 14 Evolution of predicted damage and creep strain components versus \( t/t_R \) for Specimen GT-3.
Fig. 15 Experimental curves for axial displacement (top) and relative angular rotation (bottom) versus time for the gage length of Specimen GT-3.
Fig. 16 Axial-torsional nominal stress loading history of Specimen GT-4.
APPENDIX

LISTING OF COMPUTER PROGRAMS

I. Wedge crack damage tensor analysis program.

II. Cavitation damage tensor analysis program.

III. Coupled deformation-damage integration program.
this program computes the 2nd order wedge crack damage tensor.

input file is generated by program wedge.for.

PRINT "input name of wedge crack data file:" : INPUT A$

***** definitions *****

gblen = total grain boundary length
bl = segment boundary length
bn1 = x dir. normal vector comp. to segment boundary
bn2 = y dir. normal vector comp. to segment boundary

OPEN A$ FOR INPUT AS #1

GBLEN=0 : ' initialize grain boundary length = 0

INPUT #1,BL

PRINT BL

IF BL=0 THEN 72

GBLEN=GBLEN+BL

INPUT #1,BN1,BN2

PRINT BN1,BN2

TNORM=SQRT(BN1^2+BN2^2) ; BN1=BN1/TNORM ; BN2=BN2/TNORM

GOTO 50

PRINT "grain boundary length =" ; GBLEN

PRINT "press any key to continue:" : INPUT H$

INPUT #1,BL : PRINT BL

D11=0! : D22=0! : D12=0!

' input wedge crack information
' and compute wedge crack damage tensor.

wlen = wedge crack length
wn1= x dir. normal vector comp. to wedge crack
wn2= y dir. normal vector comp. to wedge crack
dw11 = damage component 11
dw22 = damage component 22
dw12 = damage component 12

INPUT #1,WLEN

PRINT WLEN

TNORM=SQRT(WN1^2+WN2^2) ; WN1=WN1/TNORM ; WN2=WN2/TNORM

T11=WN1^2 : T22=WN2^2 : T12=WN1*WN2

T11=T11*WLEN : T22=T22*WLEN : T12=T12*WLEN

D11=D11+T11 : D22=D22+T22 : D12=D12+T12

DW11=(1!/GBLEN)*D11 : DW22=(1!/GBLEN)*D22 : DW12=(1!/GBLEN)*D12

PRINT DW11,DW22,DW12

IF WLEN <> -9999 THEN 90

D1=(DW11+DW22)/2!+SQR((DW11-DW22)^2/4+DW12^2/4)

D3=(DW11+DW22)/2!-SQR((DW11-DW22)^2/4+DW12^2/4)

PRINT "d1=","d3=" ; D3

END

CLOSE #1
this program computes the 2nd order cavitation damage tensor.
input file is generated by program cavity.for.
PRINT "input name of cavitation data file:": INPUT A$

***** definitions *****
gblen = total grain boundary length
bl = cavitated segment boundary length
bn1 = x dir. normal vector comp. to segment boundary
bn2 = y dir. normal vector comp. to segment boundary
fcav = fraction of segment boundary cavitated.
nseg = total number of cavitated segments

GBLEN=0
OPEN A$ FOR INPUT AS #1
D11=0 : D12=0 : D22=0!
INPUT #1,NSEG
FOR I=1 TO NSEG
  INPUT #1,BL
  GBLEN=GBLEN+BL
  TNORM=SQR(BN1^2+BN2^2) : BN1=BN1/TNORM : BN2=BN2/TNORM
  INPUT #1,FCAV
  D11=D11+FCAV*BL*BN1^2 : D22=D22+FCAV*BL*BN2^2 : D12=D12+FCAV*BN1*BN2
  DC11=(1/GBLEN)*D11 : DC22=(1/GBLEN)*D22 : DC12=(1/GBLEN)*D12
PRINT DC11,DC22,DC12
  dcll = damage component 11
dc22 = damage component 22
dc12 = damage component 12
NEXT I
END
PROGRAM START (INPUT, OUTPUT, TAPE5=INPUT, TAPE6=OUTPUT)
REAL KAPPA, LAMBDA, M, NU, INEDAM
COMMON /HOKY1/ ALPHA, BETA, B, KAPPA, LAMBDA, ETA, SIGMA(3,3)
COMMON /HOKY2/ DAMAGE(3,3), DAMRATE(3,3), NU(3,3,3), SSTAR
COMMON /HOKY3/ INEDAM(3,3), FI(3,3)
COMMON /HOKY4/ SEI(3), SP(3), SIGBAR, SIFRIM(3,3)
COMMON /HOKY5/ EIVEC(3,3), TEN(3,3), VEC(3), M(3)
COMMON /HOKY6/ ERROR, MITER
COMMON /HOKY7/ STRAIN(3,3), STRATE(3,3), DKK
COMMON /HOKY8/ ACONST, CTIMES, POWERM, POWERN
DATA ALPHA, BETA/1.0859, 0.2893/
DATA B, KAPPA, LAMBDA/1.44021E-17, 8.5551, 4.8/
DATA ACONST, CTIMES, POWERM, POWERN/47.66, 1.49, 2.9, 12.5/
SIGMA(1,1)=0.0
SIGMA(1,2)=19.45
SIGMA(1,3)=0.0
SIGMA(2,1)=19.45
SIGMA(2,2)=0.0
SIGMA(2,3)=0.0
SIGMA(3,1)=0.0
SIGMA(3,2)=0.0
SIGMA(3,3)=0.0
SIGM=(SIGMA(1,1)+SIGMA(2,2)+SIGMA(3,3))/3.
DO 243 IU=1,3
DO 243 IL=1,3
SIFRIM(IU,IL)=SIGMA(IU,IL)-SIGM
IF(IU.NE.IL) SIFRIM(IU,IL)=SIGMA(IU,IL)
243 CONTINUE
ETA=.65
INTEVAL=49
ERROR=1.E-5
MITER=20
DELTIME=10
C , H E A D I N G - I N P U T - D A T A
WRITE(6,30)
30 FORMAT(10X,"SIGMA")
DO 50 I=1,3
50 WRITE(6,100) (SIGMA(I,J),J=1,3)
100 FORMAT(3(2X,E15.7,5X))
WRITE(6,110) ALPHA, BETA
WRITE(6,120) B, KAPPA, LAMBDA
120 FORMAT(5X,E15.7,5X,F14.7,5X,F14.7)
WRITE(6,205) ETA
205 FORMAT(2X," ETA = ",F5.2)
C , I N I T I A L I Z E
666 DO 10 I=1,3
10 DO 10 J=1,3
10 STRAIN(I,J)=0.0
C , C
CALL STPNU
C
I=1
TIME=(I-1)*DELTIME
467 WRITE(6,200) TIME
200 FORMAT(5X,"TIME = ",F10.5)
DO 40 LLLL=1,3
40 WRITE(6,210) (DAMAGE(LLLL,JJ),JJ=1,3)
DO 42 IN=1,3
    WRITE(6,210) (STRAIN(IN,IM),IM=1,3)
210    FORMAT(3X,E15.7)
IF(MOD(I-1,INTEVAL).EQ.0) GO TO 700
750    DKK=DAMAGE(1,1)+DAMAGE(2,2)+DAMAGE(3,3)
    CALL RUNGE(1,DELTIME)
    CALL BIJAIJ(EIVEC,DAMAGE,VALUE)
    CALL RUNGE2(1,DELTIME,1)
    IF(VALUE.GE.1.) GO TO 999
    I=I+1
    TIME=(I-1)*DELTIME
    GO TO 467
700    SIGMA(1,2)=SIGMA(1,2)
    SIGMA(2,1)=SIGMA(2,1)
    SIGM=(SIGMA(1,1)+SIGMA(2,2)+SIGMA(3,3))/3.
    DO 710 IU=1,3
    DO 710 IL=1,3
        SIFRIM(IU,IL)=SIGMA(IU,IL)-SIGM
        IF(IU.NE.IL) SIFRIM(IU,IL)=SIGMA(IU,IL)
710    CONTINUE
    CALL STPNU
    GO TO 750
999    CONTINUE
333    CONTINUE
STOP
END
C SUBROUTINE RUNGE-KUTTA
SUBROUTINE RUNGE(IN,H)
REAL KAPPA,LAMBDA,NU,INEDAM,M
DIMENSION ERRY(3,3)
DIMENSION A(4),B(4),C(4),D(4),CDX(4)
COMMON /HOKY1/ALPHA,BETA,BI,KAPPA,LAMBDA,ETA,SIGMA(3,3)
COMMON /HOKY2/Y(3,3),YPRIME(3,3),NU(3,3,3),SSTAR
COMMON /HOKY3/INEDAM(3,3),FI(3,3)
COMMON /HOKY4/SEI(3),SP(3),SIGBAR,SIFRIM(3,3)
COMMON /HOKY5/EIVEC(3,3),TEN(3,3),VEC(3),M(3)
COMMON /HOKY6/ERROR,MITER
COMMON /HOKY7/STRAIN(3,3),STRATE(3,3),DKK
COMMON /HOKY8/ACONST,CTIMES,POWERM,POWERN
IF(IN.NE.1) GO TO 2100
IN=0
CDX(1)=0.0
A(1)=0.5
B(1)=2.0
C(1)=1.5
D(1)=0.5
CDX(2)=0.5
A(2)=1.-SQRT(0.5)
B(2)=1.0
C(2)=3*(1.-SQRT(0.5))
D(2)=1.-SQRT(0.5)
CDX(3)=0.0
A(3)=1.+SQRT(0.5)
B(3)=1.0
C(3)=3*(1.+SQRT(0.5))
D(3)=1.+SQRT(0.5)
CDX(4)=0.5
A(4)=1./6
B(4)=2.0
C(4)=0.5
D(4)=0.5
  DO 2000 I=1,3
  DO 2000 II=1,3
2000 CONTINUE
2100 CONTINUE
DO 2300 I=1,4
CALL DDOT2
  DO 2200 J=1,3
  DO 2200 JJ=1,3
  ERRY(J,JJ)=0.0
2200 CONTINUE
2300 CONTINUE
RETURN
END
C
SUBROUTINE RUNGE-2
SUBROUTINE RUNGE2(IN,H,IJ)
REAL KAPPA,LAMBDA,NU,INEDAM,M
DIMENSION ERRY(3,3)
DIMENSION A(4),B(4),C(4),D(4),CDX(4)
COMMON /HOKY1/ALPHA,BETA,BI,KAPPA,LAMBDA,ETA,SIGMA(3,3)
COMMON /HOKY2/DAMAGE(3,3),DAMRATE(3,3),NU(3,3,3),SSTAR
COMMON /HOKY3/INEDAM(3,3),FI(3,3)
COMMON /HOKY4/SEI(3),SP(3),SIGBAR,SIFRIM(3,3)
COMMON /HOKY5/EIVEC(3,3),TEN(3,3),VEC(3),H(3)
COMMON /HOKY6/ERROR,MITER
COMMON /HOKY7/Y(3,3),YPRIME(3,3),DKK
COMMON /HOKY8/ACONST,CTIMES,POWERM,POWERN
CDX(1)=0.0
  A(1)=0.5
  B(1)=2.0
  C(1)=1.5
  D(1)=0.5
CDX(2)=0.5
  A(2)=1.0-SQRT(0.5)
  B(2)=1.0
  C(2)=3*(1.-SQRT(0.5))
  D(2)=1.0-SQRT(0.5)
CDX(3)=0.0
  A(3)=1.+SQRT(0.5)
  B(3)=1.0
  C(3)=3*(1.+SQRT(0.5))
  D(3)=1.+SQRT(0.5)
CDX(4)=0.5
  A(4)=1./6
  B(4)=2.0
  C(4)=0.5
  D(4)=0.5
  DO 2000 I=1,3
  DO 2000 II=1,3
  ERRY(I,II)=0.0
2000 CONTINUE
2100 CONTINUE
DO 2300 I=1,4
CALL DDOT2(IJ)
  DO 2200 J=1,3
  DO 2200 JJ=1,3
  ERRY(J,JJ)=0.0
2200 CONTINUE
2300 CONTINUE
RETURN
END
SUBROUTINE STRAIN-RATE
SUBROUTINE EDOT2(IJ)
REAL KAPPA,LAMBDA,M,NU,INEDAM
COMMON /HOKY1/ALPHA,BETA,B,KAPPA,LAMBDA,ETA,SIGMA(3,3)
COMMON /HOKY2/DAMAGE(3,3),DAMRATE(3,3),NU(3,3,3),SSTAR
COMMON /HOKY3/INEDAM(3,3),FI(3,3)
COMMON /HOKY4/SEI(3),SP(3),SIGBAR,SIFRIM(3,3)
COMMON /HOKY5/EIVEC(3,3),TEN(3,3),VEC(3),M(3)
COMMON /HOKY6/ERROR,MITER
COMMON /HOKY7/STRAIN(3,3),STRATE(3,3),DKK
COMMON /HOKY8/ACONST,CTIMES,POWERM,POWERN
DUM1=(SIGBAR/ACONST)**POWERN
DUM2=(1.+(CTIMES*DKK)**POWERM)
DO 10 I=1,3
DO 10 J=1,3
STRATE(I,J)=1.5*DUM1*DUM2*(SIFRIM(I,J)/SIGBAR)
10 CONTINUE
RETURN
END

SUBROUTINE FIFI(FI:FI L/2)
SUBROUTINE FIFI(A,B,C)
DIMENSION A(3,3)

VARIABLES
A INVERSE DAMAGE TENSOR (3,3)
B SCALAR VALUE
C MATERIAL CONSTANT

BT=0.
DO 10 I=1,3
DO 10 J=1,3
BT=BT+A(I,J)*A(I,J)
10 B=BT**(C/2)
RETURN
END

SUBROUTINE SUBSTITUTION
SUBROUTINE SUBN(A,B,N)
DIMENSION A(3,3),B(3,3)

VARIABLES
A ORIGINAL MATRIX (N,N)
B DUPLICATING MATRIX (N,N)
N DIMENSION OF MATRIX

DO 10 I=1,N
DO 10 J=1,N
B(I,J)=A(I,J)
10 RETURN
END

SUBROUTINE MULTIPLICATION
SUBROUTINE MPLY(A,B,C,N)
DIMENSION A(3,3),B(3,3),C(3,3)

VARIABLES
A PREMULTIPLYING MATRIX (N,N)
B POSTMULTIPLYING MATRIX (N,N)
C A X B (N,N)
N DIMENSION

DO 10 I=1,N
DO 10 J=1,N
C(I,J)=0.1
10 RETURN
END
DO 10 K=1,N
C(I,J)=C(I,J)+A(I,K)*B(K,J)
RETURN
END

SUBROUTINE IDENTITY - B
SUBROUTINE INEGAB(B,IMB)
REAL IMB(3,3)
DIMENSION B(3,3)

C
VARIABLES
B DAMAGE TENSOR (3,3)
IMB I - B (3,3)

DO 10 I=1,3
DO 10 J=1,3
IF(J.EQ.I) THEN
IMB(I,J)=1.-B(I,J)
ELSE
IMB(I,J)=-B(I,J)
ENDIF
10 CONTINUE
RETURN
END

SUBROUTINE INVERSE
SUBROUTINE INVS(H,HINVS,N)
DIMENSION H(3,3),HINVS(3,3),A(3,3),B(3,3)

C
VARIABLES
H ORIGINAL MATRIX (N,N)
HINVS INVERSED MATRIX (N,N)
N DIMENSION

REQUIRED SUBROUTINES
1) SUBN
2) MPLY

C
CHECK IDENTITY MATRIX
IF(N.EQ.3) GO TO 300
IF(H(1,1).NE.H(2,2)) GO TO 450
IF(H(1,2).NE.H(2,1)) GO TO 450
IF(H(1,2).NE.0.0) GO TO 450
DO 210 I=1,N
DO 210 J=1,N
210 HINVS(I,J)=H(I,J)/(H(1,1)*H(2,2))
GO TO 999
300 IF(H(1,1).NE.H(2,2)) GO TO 450
IF(H(1,1).NE.H(3,3)) GO TO 450
IF(H(1,2).NE.H(1,3)) GO TO 450
IF(H(2,1).NE.H(2,3)) GO TO 450
IF(H(3,1).NE.H(3,2)) GO TO 450
IF(H(1,2).NE.H(2,1)) GO TO 450
IF(H(1,2).NE.H(3,1)) GO TO 450
IF(H(1,2).NE.0.0) GO TO 450
DO 310 I=1,N
DO 310 J=1,N
310 HINVS(I,J)=H(I,J)/(H(1,1)*H(2,2) * H(3,3))
GO TO 999
450 CALL SUBN(H,A,N)
NM1=N-1
DO 10 I=1,NM1
SUM=0.
DO 11 K=1,N
SUM=SUM+A(K,K)
11 SUM=SUM/I
DO 12 J=1,N

\begin{verbatim}
A(J,J)=A(J,J)-SUM
IF(I.EQ.NM1) CALL SUBN(A,HINVS,N)
CALL MPLY(H,A,B,N)
DO 13 I=1,N
DO 13 J=1,N
13 HINVS(I,J)=HINVS(I,J)/A(1,1)
RETURN
999 CONTINUE
RETURN
END
C SUBROUTINE EIGEN-VALUE
SUBROUTINE EIGEN(H,EIGENS,ERROR,MITER,N)
DIMENSION B(4),C(5),DBDA(4),H(3,3),EIGENS(3)
DATA B(1),DBDA(1)/1.0,0.0/
C VARIABLES
C H ORIGINAL MATRIX (N,N)
C EIGENS EIGENVALUES OF ORIGINAL MATRIX (N)
C ERROR MAXIMUM ERROR-RANGE OF EIGENVALUES
C MITER MAXIMUM ITERATION COUNTS
C N DIMENSION
C REQUIRED SUBROUTINES
C 1) COEFF
DUM01=H(1,1)*(H(2,3)**2 - H(2,2) * H(3,3))
DUM02=H(1,2)*(H(1,2)*H(3,3) - H(1,3)*H(2,3))
DUM03=H(1,3)*(H(1,3)*H(2,2) - H(1,2)*H(2,3))
DUM0 = DUM01 + DUM02 + DUM03
DUM11=H(1,1)*H(2,2) + H(2,2)*H(3,3) + H(3,3) * H(1,1)
DUM12=H(1,2)**2 + H(2,3)**2 + H(3,1)**2
DUM1 = DUM11 - DUM12
DUM2 = -(H(1,1) + H(2,2) + H(3,3))
C CONDITIONS
IF(ABS(DUM0) .LT.1.E-40) THEN
  IF(ABS(DUM1).LT.1.E-40) THEN
    EIGENS(1)=0.0
    EIGENS(2)=0.0
    EIGENS(3)= - DUM2
  ELSE
    EIGENS(1)=0.0
    EIGENS(2)=(-DUM2-SQRT(DUM2**2-4.*DUM1))/2.
    EIGENS(3)=( - DUM2+SQRT(DUM2**2-4.*DUM1))/2.
  ENDIF
ELSE
  C(1)=0.
  C(2) =1.
  C(3)=C(3)/C(2)
  C(4)=C(4)/C(2)
  C(5)=C(5)/C(2)
  I=N+1
ENDIF ELSE
  C(1)=0.
  C(2)=1.
  C(3)=C(3)/C(2)
  C(4)=C(4)/C(2)
  C(5)=C(5)/C(2)
  I=N+1
ENDIF ELSE
  C(1)=0.
  C(2)=1.
  C(3)=C(3)/C(2)
  C(4)=C(4)/C(2)
  C(5)=C(5)/C(2)
  I=N+1
ENDIF
A=0.
IM1=I-1

DO 40 LI=1,MITER
  DO 41 J=2,I
    B(J)=C(J+1)-A*B(J-1)
  41 DBDA(J)=-A*DBDA(J-1)-B(J-1)
    DA=-B(1)/DBDA(1)
    A=A+DA
    IF(ABS(DA)-ERROR) 11,11,40
  40 CONTINUE

DO 42 J1=1,IM1
\end{verbatim}
C(J1+1) = B(J1)
EIGENS(I-1) = -1./A
IF(I.EQ.2) GO TO 12
I = I - 1
GO TO 10

CONTINUE
ENDIF
RETURN
END

SUBROUTINE SORTING
SUBROUTINE SORT(A,B,N)
DIMENSION A(3,3), B(3,3)

C
VARIABLES
A ORIGINAL SERIES OF VALUES (N)
B ASCENDINGLY SORTED SERIES OF VALUES (N)

DO 10 I = 1, N - 1
DO 10 J = I + 1, N
IF(A(I).LE.A(J)) THEN
BIG = A(J)
SMALL = A(I)
A(I) = BIG
A(J) = SMALL
ELSE
ENDIF
CONTINUE
10

DO 20 K = 1, N
B(K) = A(K)
CONTINUE
RETURN
END

SUBROUTINE EIGEN-VECTOR
SUBROUTINE EIVECTR(A,B,EI,N)
DIMENSION A(3,3), B(3,3), EI(3), C(3,3), X(3)

C
VARIABLES
A ORIGINAL MATRIX (N,N)
B EIGENVECTORS N X (N)
EI EIGENVALUES (N)
N DIMENSION

REQUIRED SUBROUTINES
1) HOMO

DO 20 K = 1, N
DO 10 I = 1, N
DO 10 J = 1, N
IF(J.EQ.I) THEN
C(I,J) = A(I,J) - EI(K)
ELSE
C(I,J) = A(I,J)
ENDIF
10 CONTINUE

IF(C(2,2).EQ.0.0) GO TO 55
IF((C(1,3).EQ.0.0).AND.(C(2,3).EQ.0.0)) THEN
X(1) = 1.
X(2) = -C(1,1)/C(1,2)
ELSE
CALL HOMO(C,X,N)
ENDIF
GO TO 77
55 CALL HOMO(C,X,N)
77 XX = 0.

DO 15 JJ = 1, N
XX=XX+X(JJ)*X(JJ)
XX=SQRT(XX)
DO 30 KK=1,N
B(K,KK)=X(KK)/XX
30 CONTINUE
RETURN
END

SUBROUTINE HOMOGENEOUS
SUBROUTINE HOMO(A,X,N)
DIMENSION A(3,3),B(3,3),BINVS(3,3),X(3),Y(3)
C
VARIABLES
C
A COEFFICIENT MATRIX (N,N)
C
X SOLUTION VECTOR (N)
C
N DIMENSION
C
REQUIRED SUBROUTINES
C
1) INVS
C
2) MPLY
C
3) SUBN
C
X(N)=1.
NM1=N-1
DO 40 I=1,NM1
Y(I)=-A(I,N)
DO 40 J=1,NM1
B(I,J)=A(I,J)
40 IF(NM1.EQ.1) BINVS(1,1)=1./B(1,1)
IF(NM1.EQ.1) GO TO 10
CALL INVS(B,BINVS,NM1)
10 DO 41 I=1,NM1
X(I)=0.
DO 41 J=1,NM1
41 X(I)=X(I)+BINVS(I,J)*Y(J)
RETURN
END

SUBROUTINE OUT-DOT(VECTOR)
SUBROUTINE OUTVEC(A,B)
DIMENSION A(3),B(3,3)
C
VARIABLES
C
A VECTOR TO MAKE TENSOR (3)
C
B TENSOR (3,3)
DO 10 1=1,3
DO 10 J=1,3
10 B(I,J)=A(I)*A(J)
RETURN
END

SUBROUTINE STRESS - STAR
SUBROUTINE STPNU
REAL INEDAM,NU,K,KAPPA,LAMBDA
COMMON /HOKY1/ALPHA,BETA,B,KAPPA,LAMBDA,ETA,SIGMA(3,3)
COMMON /HOKY2/DAMAGE(3,3),DAMRATE(3,3),NU(3,3,3),SSTAR
COMMON /HOKY3/INEDAM(3,3),FI(3,3)
COMMON /HOKY4/SEI(3,3),SP(3),SIGBAR,SIFRIM(3,3)
COMMON /HOKY5/EIVEC(3,3),TEN(3,3),VEC(3),M(3)
COMMON /HOKY6/ERROR,MITER
COMMON /HOKY7/STRAIN(3,3),STRATE(3,3),DKK
COMMON /HOKY8/ACONST,CTIMES,POWERM,POWERN
CALL EIGEN(SIGMA,SEI,ERROR,MITER,3)
CALL SORT(SEI,SP,3)
J1=SIGMA(1,1)+SIGMA(2,2)+SIGMA(3,3)
S1=SP(1)-J1/3.
T1=(SP(1)-SP(2))**2
T2 = (SP(2) - SP(3)) ** 2
T3 = (SP(3) - SP(1)) ** 2
SIGBAR = SQRT((T1 + T2 + T3) / 2.)
SS = SQRT(SP(1)**2 + SP(2)**2 + SP(3)**2)
DUM1 = ((2./3.) * (SIGBAR / S1)) ** ALPHA
DUM2 = BETA * (J1 / SS - 1.)
SSTAR = 1.5 * S1 * DUM1 * EXP(DUM2)

C
C
C !!! FIND NU TENSOR !!!
CALL EIVECTR(SIGMA, EIVEC, SP, 3)
DO 10 I = 1, 3
    DO 20 J = 1, 3
        20 VEC(J) = EIVEC(I, J)
    CALL OUTVEC(VEC, TEN)
    DO 10 K = 1, 3
        DO 10 L = 1, 3
            10 NU(I, K, L) = TEN(K, L)
    RETURN
END

SUBROUTINE DAMAGE_RATE(1)
REAL LAMBDA, NU, KAPPA, INEDAM, M
COMMON /HOKY1/ ALPHA, BETA, B, KAPPA, LAMBDA, ETA, SIGMA(3, 3)
COMMON /HOKY2/ DAMAGE(3, 3), DAMRATE(3, 3), NU(3, 3, 3), SSTAR
COMMON /HOKY3/ INEDAM(3, 3), FI(3, 3)
COMMON /HOKY4/ SEI(3), SP(3), SIGBAR, SIFRIM(3, 3)
COMMON /HOKY5/ EIVEC(3, 3), TEN(3, 3), VEC(3), M(3)
COMMON /HOKY6/ ERROR, MITER
COMMON /HOKY7/ STRAIN(3, 3), STRATE(3, 3), DKK
COMMON /HOKY8/ ACONST, CTIMES, POWERM, POWERN
CALL INEGAB(DAMAGE, INEDAM)
CALL INVS(INEDAM, FI, 3)
CALL FIFI(FI, FISQ, LAMBDA)

DUM1 = B * ((SSTAR)**KAPPA) * FISQ
CALL HEAVY(SIGMA, SP(1), M, EIVEC)
DO 10 I = 1, 3
    DO 10 J = 1, 3
        SUM2 = 0.0
        DO 5 K = 1, 3
            5 SUM2 = SUM2 + NU(K, I, J) * M(K)
        IF (J .EQ. 1) THEN
            DAMRATE(I, J) = DUM1 * (ETA + (1. - ETA) * SUM2)
        ELSE
            DAMRATE(I, J) = DUM1 * (1. - ETA) * SUM2
        ENDIF
    10 CONTINUE
RETURN
END

SUBROUTINE BIJAIAJ(A, B, P)
DIMENSION A(3, 3), B(3, 3)
P = 0.0
DO 10 J = 1, 3
    DO 10 K = 1, 3
        10 P = P + B(J, K) * A(1, J) * A(1, K)
RETURN
END

SUBROUTINE HEAVYSIDE - FUNCTION
SUBROUTINE HEAVY(SIGMA, STRESS, M, EIVEC)
REAL M
DIMENSION SIGMA(3,3),EIVEC(3,3),M(3)
DO 10 I=1,3
   M(I)=0.0
   DO 5 J=1,3
      DO 5 K=1,3
         5 M(I)=M(I)+EIVEC(I,J)*SIGMA(J,K)*EIVEC(I,K)
         M(I)=M(I)/STRESS
      IF(M(I).GE.0.) THEN
         M(I)=M(I)
      ELSE
         M(I)=0.
      ENDIF
   CONTINUE
10 RETURN
END