Project No. E-25-664 (R-6036-0A0)

Project Director: G. L. Main

Sponsor: AFOSR/Bolling AFB, D.C.

Type Agreement: Grant: 85-0375

Award Period: From 09/01/85 To 08/31/88
   (Performance) 11/30/86
   (Reports)

Sponsor Amount:
   Estimated: $  
   Funded: $ 97,568

Cost Sharing Amount: $ N/A

Title: True Asymptotic Plasma - Sheath Matching With an Asymptotically Correct Collisional Pre-Sheath

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Defense Priority Rating: N/A

Military Security Classification: N/A
(or) Company/Industrial Proprietary: N/A

Restrictions

See Attached N/A

Supplemental Information Sheet for Additional Requirements.

Travel: Foreign travel must have prior approval — Contact OCA in each case. Domestic travel requires sponsor approval where total will exceed greater of $500 or 125% of approved proposal budget category.

Equipment: Title vests with Georgia Tech. for proposed equipment.

Comments:

Copies To: SPONSOR'S I. D. NO. 02.104.001.85.013

Project Director
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Reports Coordinator (OCA)
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Project File
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NOTICE OF PROJECT CLOSEOUT

Date 7/18/89

Project No. E-25-664
Center No. R6036-0A0

Project Director G. L. Main
School/Lab ME

Sponsor AIR FORCE

Contract/Grant No. AFOSR-85-0375
GTRC XX GIT

Prime Contract No. N/A

Title True Asymptotic Plasma - Sheath Matching with Asymptotically Correct Collisional

Effective Completion Date 10/31/88 (Performance) 1/31/89 (Reports)

Closeout Actions Required:

- None
- Final Invoice or Copy of Last Invoice
- Final Report of Inventions and/or Subcontracts
- Government Property Inventory & Related Certificate
- Classified Material Certificate
- Release and Assignment
- Other

Includes Subproject No(s). 

project Under Main Project No.

Continues Project No. Continued by Project No.

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Other
True Asymptotic Plasma - Sheath Matching with an
Asymptotically Correct Collisional Presheath

by

Geoffrey L. Main

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Atlanta, Georgia 303032

Submitted to

Air Force Office of Scientific Research
Bolling Air Force Base
Washington, D.C., 20332-6448

AFOSR technical Officer:
Dr. Robert Vondra

March, 1986
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INTRODUCTION

Current analytical models of Thermionic Convertors break the gap into two regions: the neutral plasma and the collisionless sheaths. Such models are reasonably accurate for the standard ignited mode but not necessary accurate or even approximately true under the following conditions, all of which are of interest for advanced thermionic conversion:

1. Ion emitting or reflecting emitter and collector surfaces,
2. Non-ignited pulsed modes, and
3. Knudsen mode convertors in which the gap size is on the order of an ion mean-free-path or less.

The key to analytical work for these conditions is a tractable theory for a collisional non-Maxwellian plasma in the thermionic convertor. Under the present grant substantial theoretical progress has been made toward this goal as covered in the attached paper, “Asymptotically Collisional Presheaths.”

SUMMARY OF PROGRESS AND RESEARCH FORECAST

To this point under the grant, two thermionic convertor codes have been converted from PL1 into FORTRAN so that they may run on the Georgia Tech Cyber 990 and IBM PC-ATs. The FORTRAN versions are currently being tested and will shortly be modified to include collisional presheath models resulting from the above mentioned work.

A working collisional presheath theory has been developed is prepared for insertion into the sheath boundary conditions for the ignited mode thermionic convertor. It is expected that both thermionic convertor codes will be working and tested in FORTRAN by May, 1986 ( one is isothermal and the other is a full time dependent implicit code ). During summer 1986 the collisional presheath effects on the ignited mode thermionic convertor will be explored.

The collisional presheath theory has turned out to be unexpectedly general in nature, therefore we propose to broaden the research effort to develop a code which will handle the second and third areas mentioned in the introduction. Discussions regarding collaboration
on these areas are ongoing with Rasor Associates, of Sunnyvale, California.

PERSONNEL

In addition to the Principal Investigator, Dr Geoffrey L. Main, the following personnel are employed: Gregory L. Ridderbusch, graduate student, David Hamm, graduate student, Leo Taske, undergraduate, Warren Coleman, undergraduate.

Two more Georgia Tech undergraduate students who are in the top 10% of their class plan to join the thermionics research effort as of fall 1986 as graduate students.

FACILITIES

Under this grant two IBM PC/AT computer systems and a Talaris 800 laser printer have been acquired. The PC/ATs are used for FORTRAN programming and for document and graphics production. This report was produced using one of the PC/ATs and the laser printer.

FORTRAN code which cannot be efficiently run on the PC/ATs is uploaded from the PC/ATs to the campus CYBER 855 via a direct connection on the Georgia Tech Network.

PAPERS AND PUBLICATIONS

Under the grant period, two conference papers have been presented or are to be presented: “Asymptotically Correct Collisional Presheaths,” by G L. Main at the Gaseous Electronics Conference, in Monterey, California during October 1985 and “Pulsed Mode Thermionic Convertor Theory Incorporating a Collisional Presheath and a Collisionless Sheath” by G. L. Main and G. L. Ridderbusch at the IEEE conference on plasma science in Saskatoon, Saskatchewan, Canada during June 1986.
Three journal papers relating to this grant have also been submitted: 1) "Asymptotically Correct Collisional Presheaths" by G. L. Main has been submitted to the Physics of Fluids and results directly from the research conducted under the grant, 2) "A General Solution Condition for Collisionless Sheaths" by G. L. Main and S. H. Lam has been submitted to the Journal of Plasma Physics, and 3) "Effects of Emitter Sheath Ion Reflection and Trapped Ions on Thermionic Convertor Performance Using an Isothermal Electron Model" by G. L. Main submitted to the Journal of Applied Physics. Copies of these three papers are attached.
RESEARCH PROGRESS AND FORECAST REPORT
AFOSR GRANT 85-0375

True Asymptotic Plasma - Sheath Matching with an
Asymptotically Correct Collisional Presheath

by

Geoffrey L. Main

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Georgia Institute of Technology
Atlanta, Georgia 30332

Submitted to

Air Force Office of Scientific Research
Bolling Air Force Base
Washington, D.C., 20332-6448

AFOSR technical Officer:
Dr. Julian M. Tishkoff

March, 1987
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INTRODUCTION

Current analytical models of Thermionic Convertors break the gap into two regions: the neutral plasma and the collisionless sheaths. Such models are reasonably accurate for the standard ignited mode but not necessary accurate or even approximately true under the following conditions, all of which are of interest for advanced thermionic conversion:

1. Ion emitting or reflecting emitter and collector surfaces,
2. Non-ignited pulsed modes, and
3. Knudsen mode convertors in which the gap size is on the order of an ion mean-free-path or less.

Thus the present work on collisional presheaths is carried out. In addition there are other important areas of application of the present work including plasma etching of semiconductors, plasma diodes for ion beam production for inertial confinement fusion, plasma-surface interactions in rail guns and plasma-surface interactions in general.

SUMMARY OF PROGRESS AND RESEARCH FORECAST

During this report period, Aug. 31, 1986 until the present, a time dependent thermionic convertor code incorporating the the asymptotic presheath work (see the annual report of Aug. 31, 1986) has been developed and is currently be tested.

Also substantial additional presheath theoretical work has been done with Fokker-Planck collision terms to account for diffusion of ions in velocity space. This makes the collisional presheath theory much more broadly applicable. The theorertical work is attached as an appendix.
PERSONNEL

In addition to the Principal Investigator, Dr. Geoffrey L. Main, the following personnel are working or have worked on the grant research during the grant period: Gregory L. Ridderbusch, graduate student, David Hamm, graduate student, Roger Bouwmans, graduate student, Jeffrey P. Dansereau, graduate student, Warren Coleman, undergraduate.

FACILITIES

Under this report period one additional IBM PC/AT computer system has been acquired. The PC/ATs have proved invaluable and are used for FORTRAN programming and for document and graphics production. This report was produced using one of the PC/ATs and the laser printer.

FORTRAN code which cannot be efficiently run on the PC/ATs is uploaded from the PC/ATs to the campus CYBER 855 via a direct connection on the Georgia Tech Network.

PAPERS AND PUBLICATIONS


In addition, a third journal paper has been submitted (to IEEE trans. on Plasma Science) entitled “A Three Scale Uniform Asymptotic Approximation to a Collisional Presheath and a Collisionless Double Sheath with Ion Reflection,” by G. L. Main and G. L. Ridderbusch.
ASYMPTOTICALLY CORRECT COLLISIONAL PRESHEATHS

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Abstract: Few exact solutions for collisional presheaths exist because of the difficulty of simultaneously satisfying both the collisional Boltzmann equation and the Poisson equation. The exact solutions that do exist are for very specialized collision terms such as constant cross section charge exchange with cold neutrals. The present paper presents an asymptotic method which is applicable to a variety of collision terms and is applied in particular to constant collision frequency charge exchange with non-cold neutrals. Constant collision frequency and constant cross section collision with cold neutrals results are also presented. The first order terms for the presheath potential rise and ion distribution functions are calculated and it is shown that second and higher order terms can be calculated using a multi-exponential expansion for presheath potential rise. The first order cold neutral constant cross section results correspond well to the exact solution. The calculated presheath potential rises are of the order expected from the Bohm criterion, and in some of the specialized cold neutral cases, exactly $\frac{4T_e}{3}$. The presheath potential rise is reduced by a neutral plasma potential gradient which accelerates ions toward the presheath. In all
cases the collisional presheath is asymptotically matched to both the neutral plasma and the collisionless sheath.

**PACS: 52.40.Hf, 52.75.Fk**
I. INTRODUCTION

The majority of plasma-surface interaction work matches a neutral plasma to a collisionless sheath without detailed consideration of a collisional presheath. However, the collisional presheath structure is of great interest. Sheath theory beginning with Bohm\textsuperscript{1} tends to assume that the plasma ion distribution is cold so that a minimum presheath potential rise may be calculated which makes the collisionless sheath self consistent. Harrison and Thompson\textsuperscript{2} generalize the Bohm criterion to non-cold ion distributions, however the result is sensitive to the density of the low energy tail of the ion distribution which in turn is strongly affected by the collisional presheath. And, a second difficulty in the absence of a collisional presheath is that the collisionless sheath and the surface beyond it may return no ions or a non-thermal distribution of ions which the collisional presheath must match to the neutral plasma region.

Some exact solutions exist for presheaths; notably Ecker and Kanne\textsuperscript{3} and Riemann\textsuperscript{4} who derive exact solutions for collision terms based on charge exchange with cold neutrals and Emmert et al\textsuperscript{5} who derive an exact collisionless solution in which there is an ionization source. In the present paper an asymptotically correct collisional presheath theory is developed which can be applied to a less restrictive range of collision terms. Potential in the presheath is expanded as a multi-exponential series and the distribution functions are expanded in terms of presheath potential rise. First order approximations are calculated for both constant collision frequency and constant cross section charge exchange collisions.
II. FIRST ORDER ASYMPTOTIC POTENTIAL FORMULATION

In this section it is assumed that the potential in the collisional presheath is of the form

\[ U = U_0 + \Delta U = \alpha x + e^{\beta x} \]  

(1)

where \( U_0 = \alpha x \) is the assumed linear potential in the neutral plasma and \( \Delta U = e^{\beta x} \) is the additional potential rise in the collisional presheath as shown in fig.1. In this paper the convention used is that \( U = q\phi \) where \( q \) is the electron charge and \( \phi \) is potential in electron volts so that \( U \) has units of energy. In addition, potential is defined in the reverse of the usual sign convention so that increasing potential repels electrons. With these conventions, the Boltzmann equation can be written as

\[ \frac{dU}{dx} \{ v \frac{\partial f}{\partial U} \pm \frac{1}{m} \frac{\partial f}{\partial v} \} \equiv \{ \frac{\partial f}{\partial t} \}. \]  

(2)

In equation (2) and those following, the \( \pm \) denotes the sign of the charged species in question; the upper sign referring to positively charged ions and the lower sign referring to electrons. The Boltzmann equation is expressed in terms of \( \Delta U \) which will be the expansion variable in the presheath:

\[ v\beta \Delta U \frac{\partial f}{\partial \Delta U} \pm \frac{1}{m} (\beta \Delta U + \alpha) \frac{\partial f}{\partial v} = \{ \frac{\partial f}{\partial t} \}. \]  

(3)

The distribution function is then expanded as

\[ f = f_0(v) + \Delta U f_1(v) + \Delta U^2 f_2(v) + ... \]  

(4)
so that the derivatives are

$$\frac{\partial f}{\partial \Delta U} = f_1(v) + 2\Delta U f_2(v) + 3\Delta U^2 f_3(v) + ...$$  \hspace{1cm} (5)

and

$$\frac{\partial f}{\partial v} = \frac{\partial f_0}{\partial v}(v) + \Delta U \frac{\partial f_1}{\partial v}(v) + \Delta U^2 \frac{\partial f_2}{\partial v}(v) + ...$$  \hspace{1cm} (6)

Substitution of (5) and (6) into the Boltzmann equation (3) yields the terms:

$$1: \quad \pm \frac{\alpha}{m} \frac{\partial f_0}{\partial v}(v) = \left[ \left\{ \frac{\partial f}{\partial t} \right\} \right]_1$$

$$\Delta U: \quad n_0 f_1(v) \pm \beta \frac{\partial f_0}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_1}{\partial v}(v) = \left[ \left\{ \frac{\partial f}{\partial t} \right\} \right]_{\Delta U}$$

$$\Delta U^2: \quad 2n_0 f_2(v) \pm \beta \frac{\partial f_1}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_2}{\partial v}(v) = \left[ \left\{ \frac{\partial f}{\partial t} \right\} \right]_{\Delta U^2}$$

$$\vdots$$

$$\Delta U^n: \quad n_0 f_n(v) \pm \beta \frac{\partial f_{n-1}}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_n}{\partial v}(v) = \left[ \left\{ \frac{\partial f}{\partial t} \right\} \right]_{\Delta U^n}.$$  \hspace{1cm} (7d)

The quantity \(\beta\), representing the presheath potential rise, is determined from the Poisson equation,

$$\frac{d^2U}{dx^2} = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_i(v, \Delta U) \, dv - \int_{-\infty}^{\infty} f_e(v, \Delta U) \, dv \right]$$  \hspace{1cm} (8)

where \(q\) is the electron charge. It is assumed that the ions are singly ionized for simplicity.

The Poisson equation (8) is expanded as

$$\beta^2 \Delta U = 4\pi q^2 \left[ \left( \int_{-\infty}^{\infty} f_{i1}(v) \, dv - \int_{-\infty}^{\infty} f_{e1}(v) \, dv \right) \Delta U + \left( \int_{-\infty}^{\infty} f_{i2}(v) \, dv - \int_{-\infty}^{\infty} f_{e2}(v) \, dv \right) \Delta U^2 + ... \right]$$  \hspace{1cm} (9)

where charge neutrality at \(\Delta U = 0\) has eliminated the terms containing \(f_{i0}\) and \(f_{e0}\):

$$n_0 = \int_{-\infty}^{\infty} f_{i0}(v) \, dv = \int_{-\infty}^{\infty} f_{e0}(v) \, dv.$$
The quantity \( n_0 \) is the neutral plasma density of the asymptotic presheath, not of the neutral plasma.

### III. First Order Solution with a Constant Collision Frequency Charge Exchange Collision Term

The constant collision frequency charge exchange collision term is modeled as

\[
\left\{ \frac{\partial f_i}{\partial t} \right\}_c = \frac{1}{\tau_{nn}} \left\{ f_n(v) \int_{-\infty}^{\infty} f_i(u) \, du - f_i(v) \int_{-\infty}^{\infty} f_n(u) \, du \right\}
\]  

(10)

where \( f_n(v) \) is the neutral distribution and \( \tau \) is the collision time. Previous work has assumed cold neutrals and results in an integral equation which is solvable only for constant collision cross section\(^4\).

#### A. Zero Plasma Potential Gradient (\( \alpha = 0 \))

In this case equations 7a,b,c-n become

\[
1: \quad 0 = \frac{1}{\tau_{nn}} \left[ f_n(v) \int_{-\infty}^{\infty} f_i0(u) \, du - f_i0(v) \int_{-\infty}^{\infty} f_n(u) \, du \right]
\]

\[\Delta U : \quad \nu \beta f_{i1}(v) + \frac{\beta}{m} \frac{\partial f_{i0}(v)}{\partial v} = \frac{1}{\tau_{nn}} \left[ f_n(v) \int_{-\infty}^{\infty} f_{i1}(u) \, du - f_{i1}(v) \int_{-\infty}^{\infty} f_n(u) \, du \right]\]

\[\vdots\]

\[\Delta U^n : \quad n \nu \beta f_{in}(v) + \frac{\beta}{m} \frac{\partial f_{i(n-1)}(v)}{\partial v} = \frac{1}{\tau_{nn}} \left[ f_n(v) \int_{-\infty}^{\infty} f_{in}(u) \, du - f_{in}(v) \int_{-\infty}^{\infty} f_n(u) \, du \right].\]

Under the assumption that the neutral distribution is Maxwellian \( f_n(v) = n_n \sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{m v^2}{2kT}\right) \).
the solution to (11) is

\[ f_{00} = C f_n(v) \]
\[ f_{11}(v) = \frac{1}{kT} f_{00}(v) \]
\[ \vdots \]
\[ f_{0n}(v) = \frac{1}{nkT} f_{1(n-1)}(v). \]

Thus

\[ f_i(v, \Delta U) = C e^{\left( \frac{2i}{a} \right)} f_n(v) \]  \hspace{1cm} (13)

which is the expected result. In this case the mean ion velocity is zero throughout the collisional presheath since charge exchange collisions conserve ions and the mean ion velocity in the neutral plasma is zero. Thus, if \( \alpha = 0 \), constant collision frequency charge exchange collisions do not shift the ion distribution upward in velocity. This presheath can be matched to a collisionless sheath only if the collisionless sheath returns all of the ions entering it from the collisional presheath.

With electron density assumed to follow

\[ n_e(\Delta U) = n_0 e^{\left( \frac{-qU}{kT_e} \right)} \]

the Poisson equation (9) yields to the first order

\[ \beta^2 = 4\pi q^2 n_0 \left( \frac{1}{kT} + \frac{1}{kT_e} \right) \]

which is the length scale of the Debye length. Thus for \( \alpha = 0 \) the collisional presheath is not distinct from the collisionless sheath since there is no separate collisional presheath length scale.
B. Non-Zero Plasma Potential Gradient \((\alpha \neq 0)\)

Under this condition there is a net flux of ions from the plasma into the sheath which allows the construction of a collisional presheath that accelerates the ions and depopulates the ion distribution of returning ions. Thus the collisional presheath may be correctly matched to collisionless sheath which returns no ions. In this case (7a) and (7b) can be written as

\[
\frac{\alpha}{m} \frac{\partial f_0}{\partial v} (v) = \frac{1}{\tau n_n} [f_n(v)n_0 - n_n f_0(v)]
\]

\[(14a)\]

\[
v_0 \beta f_1(v) + \frac{\beta}{m} \frac{\partial f_0}{\partial v} + \frac{\alpha}{m} \frac{\partial f_1}{\partial v} (v) = \frac{1}{\tau n_n} [f_n(v)n_1 - n_n f_1(v)].
\]

\[(14b)\]

The solution to these are

\[
f_0(v) = n_0 e^{-\frac{m}{\alpha \tau} \sqrt{\frac{m}{2 \pi k T}} \int_{-\infty}^{v} e^{-\frac{m u^2}{2 k T} + \frac{m}{\alpha \tau} u} \, du}
\]

\[(15)\]

and

\[
f_1(v) = \exp \left( -\frac{\beta m v^2}{2\alpha} - \frac{m v}{\alpha \tau} \right) \left[ \int_{0}^{v} \exp \left( \frac{\beta m u^2}{2\alpha} + \frac{m u}{\alpha \tau} \right) \left( \frac{n_1 m}{\alpha \tau} \sqrt{\frac{m}{2 \pi k T}} e^{-\frac{m u^2}{2 k T} - \frac{m}{\alpha \tau} u} - \frac{\beta}{\alpha} \frac{\partial f_0}{\partial v} (u) \right) \, du + C \right]
\]

\[(16)\]

where

\[
n_0 = \int_{-\infty}^{\infty} f_0(v) \, dv
\]

\[(17)\]

and

\[
n_1 = \int_{-\infty}^{\infty} f_1(v) \, dv.
\]

\[(18)\]

The constant of integration in (15) has been set so that \(f_0\) goes to zero at \(-\infty\); \(f_0\) goes to zero at \(\infty\) regardless of the constant of integration. Equation (17) is immediately satisfied.
by (15). The constant of integration $C$ in (16) must be set so that (18), which represents self-consistency, is satisfied. It can be seen from (16) that $f_1$ goes to zero at $-\infty$ and $\infty$ regardless of the constant $C$. From (18), then

$$C = \frac{n_1 \left(1 - \int_{-\infty}^{\infty} \exp \left(-\frac{\beta m u^2}{2a} - \frac{m u}{a r} \right) f_0 \exp \left(\frac{\beta m u^2}{2a} + \frac{m u}{a r} \right) \sqrt{\frac{m}{2\pi k T}} \exp \left(-\frac{m u^2}{2k T} \right) \, du \, dv \right)}{\sqrt{\frac{2\pi m}{m\beta} e^{\frac{m}{2\beta r^2}}} \left( \int_{-\infty}^{\infty} \exp \left(-\frac{\beta m u^2}{2a} - \frac{m u}{a r} \right) f_0 \frac{\beta}{a} \exp \left(\frac{\beta m u^2}{2a} + \frac{m u}{a r} \right) \frac{\partial f_0}{\partial u} (u) \, du \, dv \right)}.$$  

The exponential sheath rise, $\beta_1$, is determined from the Poisson equation under the simplifying assumption that

$$n_e = \int_{-\infty}^{\infty} f_e (v) \, dv = n_0 \exp \left(-\frac{\Delta U}{kT_e} \right).$$  

One might expect that the approximation should be $n_e = \exp(-U/kT_e)$, however this cannot be true in the asymptotic presheath because $n_e$ must approach $n_0$ as $U$ approaches negative infinity. With (20) the Poisson equation (9) to the first order becomes

$$\beta^2 = 4\pi q^2 n_1 \left( n_1 + \frac{n_0}{kT_e} \right).$$  

Since the ion density is only calculated to the first order, the same will be done for the electron density in (20).

To obtain a particular solution it is assumed here that the collisionless sheath to which the collisional presheath is joined at $\Delta U = \Delta U^*$ returns no ions. In particular,

$$\int_{-\infty}^{0} f_0 (v, \Delta U^*) \, dv = 0$$  

or

$$\int_{-\infty}^{0} f_0 (v) \, dv + \Delta U^* \int_{-\infty}^{0} f_1 (v) \, dv = 0.$$  

(23)
and
\[ \int_{-\infty}^{0} v f(v, \Delta U^*) \, dv = 0 \] (24)
or
\[ \int_{-\infty}^{0} v f_0(v) \, dv + \Delta U^* \int_{-\infty}^{0} v f_1(v) \, dv = 0. \] (25)

Because the approximation is only first order, it is not possible to impose the condition that \( f(v) \) is uniformly zero for returning ions. Equations (23) and (25) represent zero returning ion density and zero returning ion flux. When higher order terms are included, the conditions of zero returning ion momentum flux, zero returning ion energy flux and so forth can be applied in succession. Equations (21), (23) and (25) are solved for \( n_1, \beta \) and \( \Delta U^* \) with all other quantities assumed constant. Equation (21) immediately satisfies the Bohm criterion at \( \Delta U = \Delta U^* \) for the first order approximation

\[ n_1 + \frac{n_0}{kT_e} > 0. \] (26)

The Poisson equation (21) can be written as

\[ \beta^2 \lambda_D^2 = 1 + kT_e \frac{n_1}{n_0} \] (27)

where

\[ \lambda_D = \sqrt{\frac{kT_e}{4\pi q^2 n_0}} \] (28)
is the Debye length. It is expected that the length scale of the presheath should be of the order \( \beta = 1/\lambda_i \) where \( \lambda_i \) is the ion mean free path. In the circumstance that the Debye length is small compared to the ion mean free path, the product \( \beta^2 \lambda_D^2 \) is small and

\[ n_1 = -\frac{n_0}{kT_e} \] (29)
The neutral plasma region is matched to the collisional presheath also at $\Delta U = \Delta U^*$ as shown in figure 1 to produce a three scale uniform asymptotic solution. In particular, assuming constant collision frequencies, the momentum equations become

$$\left[ kT_i - \frac{m_i \Gamma_i^2}{n^2} \right] \frac{dn}{dx} = n \frac{dU}{dx} - \frac{m_i \Gamma_i}{\tau}$$ (30)

and

$$\left[ kT_e - \frac{m_e \Gamma_e^2}{n^2} \right] \frac{dn}{dx} = -n \frac{dU}{dx} - \frac{m_e \Gamma_e}{\tau_e}$$ (31)

where

$$n = n_0 \left( 1 - \frac{\Delta U^*}{kT_e} \right)$$ (32)

$$\frac{dn}{dx} = -n_0 \beta \frac{\Delta U^*}{kT_e}$$ (33)

and

$$\frac{dU}{dx} = \alpha + \beta \Delta U^*.$$ (34)

The quantity $n$ is the plasma density at the matching point $\Delta U^*$ and $\Gamma_i$ and $\Gamma_e$ are respectively the ion and electron net fluxes.

Nondimensionalization results in:

$$A = \frac{\alpha \tau}{m} \sqrt{\frac{m}{2kT}}$$ (35)

$$B = \frac{\beta kT}{\alpha}$$ (36)

$$R_e = \frac{T_e}{T}$$ (37)

$$\omega = \sqrt{\frac{m}{2kT}} v_i$$ (38)

where $A$ and $R_e$ are the parameters and $B$ is a function of $A$ and $R_e$. The quantity $A$ represents the nondimensional asymptotic presheath potential gradient, $B$ represents the
nondimensional exponential presheath rise, $R_e$ is the electron to neutral temperature ratio, and $\omega$ is the nondimensional velocity. The distribution functions can then be written as

$$F_0(\omega, A) = \frac{f_0(v)}{n_0 \sqrt{\frac{m}{2kT}}} = \frac{\exp \left( - \frac{w}{\sqrt{\pi} A} \right)}{\sqrt{\pi} A} \int_{-\infty}^{\omega} \exp \left( - \xi^2 + \frac{\xi}{A} \right) d\xi$$  \hspace{1cm} (39)

and

$$F_1(\omega, A, B) = \frac{f_1(v)}{n_0 \sqrt{\frac{m}{2kT}}} = \frac{\exp \left( - B \omega^2 - \frac{\omega}{A} \right)}{\sqrt{\pi}} \left[ \int_{-\infty}^{\omega} \exp \left( B \xi^2 + \xi \frac{1}{A} \exp \left( - \xi^2 \right) - \frac{1}{A} \exp \left( - \xi^2 \right) \right) d\xi + C \right]$$  \hspace{1cm} (40)

where

$$C = \frac{\left( 1 - \frac{1}{\sqrt{\pi} A} \int_{-\infty}^{\infty} \exp \left( - B \omega^2 - \frac{\omega}{A} \right) \int_{0}^{\omega} \exp \left( B \xi^2 + \xi \frac{1}{A} \exp \left( - \xi^2 \right) - \frac{1}{A} \exp \left( - \xi^2 \right) \right) d\xi \, d\omega \right)}{\sqrt{\frac{1}{B} \exp \left( \frac{1}{4B \xi^2} \right)}}$$

Thus (23) and (25) become

$$\int_{-\infty}^{0} F_0(\omega, A) \, d\omega + \frac{\Delta U^*}{kT_e} \int_{-\infty}^{0} F_1(\omega, A, B) \, d\omega = 0 \hspace{1cm} (42)$$

and

$$\int_{-\infty}^{0} \omega F_0(\omega, A) \, d\omega + \frac{\Delta U^*}{kT_e} \int_{-\infty}^{0} \omega F_1(\omega, A, B) \, d\omega = 0 \hspace{1cm} (43)$$

Figure 2 presents the presheath potential rise $\Delta U^*$ and the nondimensional exponential rise $B$ as function of the nondimensional asymptotic presheath potential gradient $A$ for a range of electron to neutral temperature ratios $R_e$. As would be intuitively expected, the presheath potential rise decreases with increasing $A$. Figure 3 presents the ion distribution functions at the neutral plasma - collisional presheath interface, $F_0(\omega)$, the first order...
correction to the distribution function, \( F_1(\omega) \), and the resulting distribution function at the collisional presheath - collisionless sheath interface, \( F_0(\omega) + \Delta U^* F_1(\omega) \). Although the resulting distribution is not uniformly zero for \( \omega < 0 \), its net returning density and flux are zero by (42) and (43). It is expected that higher order corrections to the distribution function and potential with the corresponding application of higher order moment conditions of zero returning momentum, energy and so forth will converge the returning distribution function toward a uniform zero.

In the limit of cold neutrals, the constant collision frequency charge exchange solution is considerably simplified. Equations (14a) and (14b) become

\[
\frac{\alpha \partial f_0}{m \partial v}(v) = \frac{1}{\tau n_n} \left[ n_n \delta_n(v)n_0 - n_n f_0(v) \right] \tag{44}
\]

and

\[
v \beta f_1(v) + \frac{\beta \partial f_0}{m \partial v} + \frac{\alpha \partial f_1}{m \partial v}(v) = \frac{1}{\tau n_n} \left[ n_n \delta_n(v)n_1 - n_n f_1(v) \right]. \tag{45}
\]

The solution to (44) and (45) are

\[
f_0(v) = \begin{cases} 
    n_0 \left( \frac{m}{\alpha r} \right) \exp \left( - \frac{mv}{\alpha r} \right), & v > 0 \\
    0, & v < 0
\end{cases} \tag{46}
\]

and

\[
f_1(v) = \begin{cases} 
    \exp \left( - \frac{\beta mv^2}{2\alpha} - \frac{mv}{\alpha r} \right) \left[ C^+ + \frac{\beta n_0}{\alpha} \left( \frac{m}{\alpha r} \right)^2 \int_0^v \exp \left( \frac{\beta mu^2}{2\alpha} \right) du \right], & v > 0 \\
    \exp \left( - \frac{\beta mv^2}{2\alpha} - \frac{mv}{\alpha r} \right) \left[ C^- \right], & v < 0
\end{cases} \tag{47}
\]

such that

\[
C^+ - C^- = \frac{m}{\alpha r} \left( n_1 - \frac{\beta}{\alpha} n_0 \right). \tag{48}
\]

Equation 46 immediately satisfies \( n_0 = \int_{-\infty}^{\infty} f_0(v) \, dv \). No returning ions implies that

\[
C^- = 0 \tag{49}
\]
and

\[ G^+ = \frac{m}{\alpha \tau} \left( n_1 - \frac{\beta}{\alpha} n_0 \right) \]  

(50)
since \( f_0 \) on \( v < 0 \) is already zero. The final condition is then that \( n_1 = \int_0^\infty f_1(v) \, dv \) or

\[ n_1 = \int_0^\infty \exp \left( -\frac{\beta \mu v^2}{2\alpha} - \frac{mv}{\alpha \tau} \right) \left[ \frac{n_1 m}{\alpha \tau} - \frac{\beta n_0 m}{\alpha \tau} + \frac{\beta n_0}{\alpha} \left( \frac{m}{\alpha \tau} \right)^2 \int_0^v \exp \left( \frac{\beta \mu u^2}{2\alpha} \right) \, du \right] \, dv. \]  

(51)
The application of \( n_1 = -\frac{n_0}{kT_e} \) yields

\[ \frac{\beta kT_e}{\alpha} = \frac{1 - \int_0^\infty \exp \left( -\frac{\beta \sigma_t^2 \xi^2}{2m} - \xi \right) \, d\xi}{\int_0^\infty \exp \left( -\frac{\beta \sigma_t^2 \xi^2}{2m} - \xi \right) \left( 1 - \int_0^\xi \exp \left( \frac{\beta \sigma_t^2 \eta^2}{2m} \right) \, d\eta \right) \, d\xi}. \]  

(52)
In this case, \( \Delta U^* \) is defined by

\[ f_0(0^+) + \Delta U^* f_1(0^+) = 0 \]  

(53)
which yields

\[ \Delta U^* \frac{kT_e}{kT_e} = \frac{1}{\frac{\beta kT_e}{\alpha} + 1} \]  

(54)
as expected. In the limit of \( \frac{\beta \sigma_t^2}{2m} \to 0 \) we have

\[ \frac{\beta kT_e}{\alpha} = 1 \]  

(55)
and

\[ \Delta U^* = \frac{kT_e}{2} \]  

(56)
which corresponds to the Bohm criterion. Figure 4 presents the variation of \( B = \frac{\beta kT_e}{\alpha} \) with \( \frac{\beta \sigma_t^2}{2m} \) for the cold neutral case. A particular \( \beta \) for the parameters can be conveniently found by drawing a line from the origin with slope \( \frac{2m kT_e}{\sigma_t^2 \alpha} \) so that the intersection is the solution. Figure 5 presents an example cold neutral ion distribution. Examination of the
ion distribution function at \( v = 0 \) shows that the slope is discontinuous. This is because the neutral source is delta function at \( v = 0 \) and the charge exchange rate varies inversely with \( v \). Consequently, this presheath cannot be matched to a collisionless sheath at \( \Delta U = \Delta U^* \). In fact there can be no collisionless sheath because of the strength of the charge exchange term at \( v = 0 \). The complete structure of this sheath-presheath must be found by calculating the higher order terms of the collisional presheath.

IV. FIRST ORDER SOLUTION WITH A QUASI-CONSTANT CROSS SECTION COLLISION TERM

First order asymptotic solutions can also be developed for a quasi-constant cross section collision term

\[
\left\{ \frac{\partial f}{\partial t} \right\}_c = \sigma \left[ \int_{-\infty}^{\infty} f_n(v)f(u) |v-u| du - \int_{-\infty}^{\infty} f(v)f_n(u) |v-u| du \right].
\] (57)

This collision term is not really constant cross section because it is a one dimensional representation which does not take into account average velocities in the other two dimensions. However, this collision term corresponds to that commonly called constant cross section. The application of this term leads to a set of integro-differential equations which can be at least approximately solved, and in the cold neutral case it leads to readily soluble first order differential equations. The cold neutral case presented here corresponds to that which can be solved exactly (Riemann\(^4\)). Unfortunately, though, the exact solution method is
not extensible to non-cold neutrals. The cold neutral collision term is

$$\left\{ \frac{\partial f}{\partial t} \right\}_c = \sigma_{n_n} \delta(v) \int_{-\infty}^{\infty} f(u) |u| du - \sigma f(v)n_n |v|,$$  \hspace{1cm} (58)

and the zero order Boltzmann equation term (7a) becomes

$$\frac{\alpha}{m} \frac{\partial f_0}{\partial v}(v) = \sigma_{n_n} \delta(v) \int_{-\infty}^{\infty} f_0(u) |u| du - \sigma_{n_n} |v| f_0(v),$$  \hspace{1cm} (59)

for which the solution is

$$f_0(v) = \begin{cases} \frac{n_0 \sqrt{2}}{\pi} \sqrt{\frac{\sigma m n_n}{\alpha}} \exp\left(-\frac{\sigma m n_n v^2}{2\alpha}\right), & v > 0 \\ 0, & v < 0 \end{cases}.$$  \hspace{1cm} (60)

The first order Boltzmann term is

$$v \beta f_1(v) + \frac{\beta}{m} \frac{\partial f_0}{\partial v}(v) + \frac{\alpha}{m} \frac{\partial f_1}{\partial v}(v) = \sigma_{n_n} \delta(v) \int_{-\infty}^{\infty} f_1(u) |u| du - \sigma_{n_n} |v| f_1(v),$$  \hspace{1cm} (61)

for which the solution is

$$f_1(v) = \begin{cases} \exp\left(-\frac{1}{2} \frac{\beta m}{\alpha} \frac{\sigma m n_n}{\alpha} v^2\right) \left[ \frac{n_0}{} \sqrt{2} \left(\sqrt{\frac{\sigma m n_n}{\alpha}}\right)^3 \left(\exp\left(\frac{\beta m v^2}{\alpha}\right) - 1\right) + C^+ \right], & v > 0 \\ \exp\left(-\frac{1}{2} \frac{\beta m}{\alpha} \frac{\sigma m n_n}{\alpha} v^2\right) \left[ C^- \right], & v < 0 \end{cases}.$$  \hspace{1cm} (62)

The jump condition at \( v = 0 \) must be satisfied in (61),

$$C^+ - C^- = \frac{\sigma m n_n}{\alpha} \int_{-\infty}^{\infty} f_1(u) |u| du - \frac{\beta}{\alpha} n_0 \sqrt{2} \left(\sqrt{\frac{\sigma m n_n}{\alpha}}\right).$$  \hspace{1cm} (63)

No returning ions, \( C^- = 0 \) and the application of (63) to (62) yields

$$C^+ = -n_0 \frac{\beta}{\alpha} \sqrt{\frac{2}{\pi}} \sqrt{\frac{\sigma m n_n}{\alpha}}.$$  \hspace{1cm} (64)

The collisional presheath - collisionless sheath boundary \( \Delta U^* \) is again

$$0 = f(0^+) = f_0(0^+) + \Delta U^* f_1(0^+)$$  \hspace{1cm} (65).
which yields

$$\frac{\Delta U^*}{kT_e} = \frac{\alpha}{\beta kT_e}. \quad (66)$$

Equation (62) is integrated to

$$n_1 = \int_{-\infty}^{\infty} f_1(v) dv = \frac{n_0 n_s \sigma}{\alpha} \left[ 1 - \sqrt{1 + \frac{\beta}{\sigma n_n}} \right] \quad (67)$$

and applied to the Poisson equation (8) to produce

$$\left( \frac{\beta}{\sigma n_n} \right)^2 = \left( \frac{4\pi n_0^2}{kT_e} \right)^2 \left( \frac{1}{\sigma n_n} \right)^2 \left[ \frac{n_s \sigma kT_e}{\alpha} \left( 1 - \sqrt{1 + \frac{\beta}{\sigma n_n}} \right) + 1 \right]. \quad (68)$$

Under the assumption that the Debye length is short compared to the ion mean free path,

$$\left( \frac{4\pi n_0^2}{kT_e} \right)^2 \left( \frac{1}{\sigma n_n} \right)^2 \gg 1,$$

equation (68) results in

$$\frac{\beta}{\sigma n_n} = \frac{\alpha}{n_s \sigma kT_e} \left( 2 + \frac{\alpha}{n_s kT_e} \right) \quad (69)$$

and

$$\frac{\Delta U^*}{kT_e} = \frac{1}{2 + \frac{\alpha}{n_s \sigma kT_e}}. \quad (70)$$

The Bohm criterion is satisfied at $\Delta U^*$ to the first order by virtue of (68). And interestingly the presheath potential rise for $\alpha = 0$ is exactly that required by the cold ion Bohm criterion.

Figure 6 presents an example cold neutral ion distribution.

V. CONCLUSIONS

It has been shown that approximate collisional presheath solutions can be obtained for a variety of collision terms. In particular the constant collision frequency case has been
solved approximately whereas previous attempts at exact solutions have found this case intractable. In addition it has been shown that higher order corrections can be made a regular and tractable fashion. Also the return of ions from the collisionless sheath can be treated.

ACKNOWLEDGMENTS

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APPENDIX
MULTI-EXPONENTIAL FORMULATION

In the previous sections we have calculated only the first order terms in the ion distribution and presheath potential rise. Also we have implicitly made the same first order approximation for electrons,

\[ n_e = n_0 \left(1 - \frac{\Delta U}{kT_e}\right). \]  \hspace{1cm} (A1)

A complete multiexponential expansion can also be constructed that correctly calculates the second and higher order terms. Potential in the presheath is

\[ U = U_0 + \Delta U + a_2 \Delta U^2 + a_3 \Delta U^3 + ... \]  \hspace{1cm} (A2)

where \( U_0 = \alpha x \) and \( \Delta U = \exp(\beta x) \). Thus

\[ \frac{dU}{dx} = \alpha + \beta \Delta U + 2\beta a_2 \Delta U^2 + 3\beta a_3 \Delta U^3 + ... \]  \hspace{1cm} (A3)

and

\[ \frac{d(\Delta U)}{dU} = \frac{\beta \Delta U}{\alpha + \beta \Delta U + 2\beta a_2 \Delta U^2 + 3\beta a_3 \Delta U^3 + ...} \]  \hspace{1cm} (A4)

which transforms the Boltzmann equation

\[ \frac{dU}{dx} \left[ v \frac{\partial f}{\partial \Delta U} (v) \frac{\partial \Delta U}{\partial U} \pm \frac{1}{m} \frac{\partial f}{\partial v} (v) \right] = \left\{ \frac{\partial f}{\partial t} \right\}_e \]  \hspace{1cm} (A5)

into

\[ v \beta \Delta U \frac{\partial f}{\partial \Delta U} (v) \pm \frac{1}{m} \left( \alpha + \beta \Delta U + 2\beta a_2 \Delta U^2 + ... \right) \frac{\partial f}{\partial v} (v) = \left\{ \frac{\partial f}{\partial t} \right\}_e \]  \hspace{1cm} (A6)
or

$$1: \pm \frac{\alpha}{m} \frac{\partial f_0}{\partial t}(v) = \left[ \left\{ \frac{\partial f}{\partial t} \right\}_c \right]_1 \tag{A7a}$$

$$\Delta U: \quad u \beta f_1(v) \pm \frac{\beta}{m} \frac{\partial f_0}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_1}{\partial v}(v) = \left[ \left\{ \frac{\partial f}{\partial t} \right\}_c \right]_1 \Delta U \tag{A7b}$$

$$\Delta U^2: \quad 2u \beta f_2(v) \pm 2z \frac{\partial f_0}{m} \frac{\partial f_0}{\partial v}(v) \pm \frac{\beta}{2} \frac{\partial f_1}{\partial v}(v) \pm \frac{\alpha}{2} \frac{\partial f_2}{\partial v}(v) = \left[ \left\{ \frac{\partial f}{\partial t} \right\}_c \right]_1 \Delta U^2 \tag{A7c}$$

$$\Delta U^n: \quad n u \beta f_n(v) \pm \frac{\beta}{m} \frac{\partial f_0}{\partial v}(v) \pm \frac{(n-1) \beta}{m} \frac{\partial f_1}{\partial v}(v) \pm \ldots \pm \frac{\beta}{m} \frac{\partial f_{n-1}}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_n}{\partial v}(v)$$

$$= \left[ \left\{ \frac{\partial f}{\partial t} \right\}_c \right]_1 \Delta U^n \tag{A7d}$$

The Poisson equation (8) becomes

$$\beta^2 \Delta U + (2\beta)^2 a_2 \Delta U^2 + (3\beta)^2 a_3 \Delta U^3 + \ldots$$

$$= 4\pi q^2 \left[ \Delta U \left( \int_{-\infty}^{\infty} f_{t1}(v) \, dv - \int_{-\infty}^{\infty} f_{t2}(v) \, dv \right) \right.$$

$$+ \Delta U^2 \left( \int_{-\infty}^{\infty} f_{t2}(v) \, dv - \int_{-\infty}^{\infty} f_{t2}(v) \, dv \right)$$

$$+ \ldots \right]. \tag{A8}$$
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EFFECTS OF EMITTER SHEATH ION REFLECTION
AND TRAPPED IONS ON THERMIONIC CONVERTOR
PERFORMANCE USING AN ISOTHERMAL ELECTRON MODEL

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Abstract: This paper couples exact collisionless sheath calculations to an isothermal electron model of a thermionic convertor. The emitter sheath structure takes into account reflected ions, trapped ions and surface emission ions. It is shown that the lessening the net loss of ions at the emitter in the ignited mode by these phenomena degrades performance. In addition it is shown that when the emitter returns too many of the ions, the arc is extinguished because there is insufficient resistive heating to maintain the necessary plasma electron temperature for ionization. These results suggest that the ignited mode cannot be improved much. However, non-ignited modes in which the electron temperature remains low, such as the pulsed mode do not suffer from this adverse behavior.
1. INTRODUCTION

Emitter sheath phenomena are important in thermionic energy convertors because the emitter sheath forms the emitter boundary condition for the plasma in the gap by controlling both the ion loss rate and the loss rate of hot (3000 K) plasma electrons to the emitter. This paper examines two expected emitter sheath phenomena and their effects on convertor performance: reflection of ions coming from the plasma by a double emitter sheath, and ions trapped in the double emitter sheath. The authors have previously suggested that ion reflection might improve thermionic energy convertor performance and subsequently shown that ion reflection at the emitter is likely to degrade the performance in the ignited mode and, in addition, that trapped ions in a double emitter sheath are also likely to degrade performance in the ignited mode. Lundgren has also shown this with simplified ion and electron dynamics. In the present paper the effects of emitter ion reflection and ion trapping in the ignited mode are calculated using exact electron and ion dynamics in the collisionless (except for ion trapping) sheaths. The electrons entering the sheaths from the plasma are assumed to have a Maxwellian distribution, but no assumptions are made about the returning electrons and the electron density in the sheath is calculated exactly. The ions entering the sheaths from the plasma are not assumed cold, but are given the correct ion temperature and shifted in velocity according to a generalization of the Bohm criterion.

Both ion reflection and trapped ions in the emitter sheath reduce the normalized (by plasma density) net ion loss rate to the emitter. Also, both of these phenomena raise the normalized plasma density adjacent to the emitter. The higher plasma density at the emitter causes a greater increase in the loss of hot plasma electron energy to the emitter than the corresponding decrease in the loss of ionization energy (carried by the ions) to the emitter. Therefore these emitter sheath phenomena increase arc-drop. Within the limitations of the present isothermal thermionic convertor formulation, all three of these phenomena (which become significant at low currents) steepen the current-voltage characteristic. At low current densities, the present theory shows that the collector sheath height decreases, resulting in a larger electron diffusion velocity than can be justified for the continuum model used in the plasma region. The result of lower performance at lower
current is in agreement with experimental studies. At some current density which depends strongly on the emitter sheath conditions, the ignited mode is no longer self-sustaining and the arc is extinguished.

Figure 1 is a schematic diagram of the cesium diode convertor. The emitter is heated externally to temperature \( T_E \) which is typically 1750 K and the collector is cooled to temperature \( T_C \) which is typically 750 K. The gap space, \( d \), or convertor length, which is typically 0.25 mm, separates the emitter from the collector. The cesium reservoir, which is sometimes imbedded in the collector, is kept at temperature, \( T_R \), to maintain the desired cesium pressure (typically 1 torr) in the gap. The electrical load is connected across the emitter and collector to produce power.

2. THE ISOTHERMAL ELECTRON FORMULATION

In this section the isothermal thermionic convertor formulation is developed. The formulation is similar to that of Lam\(^7\) but is generalized to eliminate the assumption of high sheaths which has previously been used to simplify the electron dynamics. Since both low emitter and low collector sheath heights are encountered as a consequence of ion reflection and trapped ions, the assumption of Boltzmann plasma electron distributions at the plasma-sheath interface must be abandoned. At both the emitter and collector the low sheaths return few plasma electrons, leaving the distributions largely one sided. Furthermore, at the emitter sheath emitted electrons must be taken into account. Thus the ratio of electrons moving toward the sheath to the total density of electrons at the sheath edge is not 1/2, as in the Boltzmann assumption.

In fig. 2 we define the potentials in the convertor. All of the potentials are nondimensionalized by emitter temperature as follows:

\[
x = \frac{q\phi}{kT_E}
\]

(1)

where

\( x = \) nondimensional potential,

\( \phi = \) potential,
\( q = \) electron charge,
\( k = \) Boltzmann constant,
\( T_E = \) emitter temperature.

We also use the following terminology for various potentials in the convertor:

\( \Phi_E = \) emitter work function,
\( \Delta \chi = \) back sheath height,
\( \Delta \chi_s = \) reflective potential,
\( \chi_E = \) emitter sheath height,
\( \Delta \chi_p = \) plasma potential drop,
\( V_d = \) arc drop,
\( \chi_C = \) collector sheath height,
\( \Phi_C = \) collector work function,
\( V_{out} = \) convertor output voltage.

Inspection of fig. 2 yields immediately the following relations

\[
V_d = V_{out} - (\Phi_E - \Phi_C) - \Delta \chi, \quad (2)
\]

\[
V_d = (\chi_C - \chi_E) - \Delta \chi_p. \quad (3)
\]

The Richardson current density of electrons from the emitter is

\[
J_R = 120 T_E^2 (k^2) \exp(-\Phi_E). \quad (4)
\]

The emitted current density which crosses the emitter sheath potential peak into the convertor plasma region is

\[
J_E = \begin{cases} 
J_R \exp(-\Delta \chi), & \Delta \chi > 0, \\
J_R, & \Delta \chi \leq 0.
\end{cases} \quad (5)
\]

We also define the net current density through the convertor, \( J \), and the normalized current density,

\[
j = \frac{J}{J_E} \quad (6)
\]
Electron temperature is nondimensionalized as

$$\tau = \frac{T_e}{T_E}$$  \hspace{1cm} (7)

where $T_e$ is the plasma electron temperature which, in this section, is constant by the isothermal assumption. Finally we have the thermal speeds,

$$a_e = \sqrt{\frac{8kT_e}{\pi m}}$$  \hspace{1cm} (8)

$$a_E = \sqrt{\frac{8kT_E}{\pi m}}$$  \hspace{1cm} (9)

The isothermal formulation is developed from here in the same way as the general formulation except that we take full advantage of the isothermal assumption by looking only at the global conservation equations instead of the local ones used in the general formulation. We then assume that the transport properties, collision frequencies, and the ionization source coefficient are constant across the convertor because of the isothermal assumption. Also we find only the steady state solution. We carry out this development by deriving the global conservation equations for the isothermal case (current, momentum, and electron energy) and then reducing these to a set of three simultaneous equations in the variables $\tau$, $x_E$, and $x_C$. In some cases the actual calculations are carried out using different variables when $x_E$ or $x_C$ are small or zero. In the case, for instance, of a single ion repelling emitter sheath we use $j$ because $x_E$ is zero. These equations are nonlinear and solved numerically using a positive definite Newton's method.

First we consider conservation of current. The collector is assumed to emit nothing, therefore at the collector plasma-sheath interface we have

$$J = \frac{a_e a_1 n(1)}{2} e^{-x_E}$$  \hspace{1cm} (10)

where $a_1$ is the fraction of the total plasma density at the collector sheath which is moving toward the collector and $n(1)$ is the total plasma density at the plasma-collector sheath interface. Because we continue to assume that the plasma electron distribution there is Maxwellian, we can write $a_1$ as

$$a_1 = \frac{1}{1 + \frac{2}{\sqrt{\pi}} \int_0^{x_E} e^{-u^2} du}$$  \hspace{1cm} (11)

which takes into account the plasma electrons reflected by the collector sheath. We still assume that the plasma electron distribution coming into the collector sheath is Maxwellian
and that it does not have any velocity shift because the sheath is expected to be electron repelling. In the limit of a high collector sheath, $a_1 = 1/2$ and we have a fully Boltzmann distribution of electrons at the collector sheath edge. The situation at the emitter is more complex because the emitted electrons must be taken into account. We have the back scattered current density, $J_{BS}$, which is the plasma electron current density moving into the emitter,

$$J_{BS} = \frac{n(0)a_0}{2}\exp(-\frac{x_E}{r})$$

(12)

where $n(0)$ is the total plasma density at the emitter sheath-plasma interface and $a_0$ is the fraction of total plasma density at the interface moving toward the emitter.

Continuity of electron current demands

$$J_E = J_{BS} + J$$

(13)

which can be written as

$$J_E = J \left(1 + \frac{n(0)a_0}{n(1)a_1}\exp\left(\frac{X_C - X_E}{r}\right)\right).$$

(14)

This can be rewritten using eqs.3 and 6 as

$$j = \frac{1}{1 + \frac{n(0)a_0}{n(1)a_1}\exp\left(\frac{Y_a + \Delta X_a}{r}\right)}.$$

(15)

The quantity $a_0$ can be written as

$$a_0 = \sqrt{\frac{\pi}{2}}Q \left(\frac{1}{j} - 1\right)\exp\left(\frac{x_E}{r}\right)$$

(16)

where

$$Q = \frac{\langle u_e \rangle_0}{\sqrt{kT_e/m}}$$

is the electron Mach number at the emitter. This is just an application of eq.13.

Electron energy conservation is developed by considering energy exchange with the emitter and collector and energy lost to ionization. Power carried into the plasma by emitted electrons is

$$P_E = J_E (2 + \Phi_E + \Delta \chi) \frac{kT_e}{q}.$$

(17)

Power returned to the emitter is

$$P_{BS} = (J_E - J)(2r + \Phi_E + \Delta \chi) \frac{kT_e}{q}.$$

(18)
Power flowing into the collector is

\[ P_C = J(2r + \Phi_E + V_d) \frac{kT_E}{q} \]  \hspace{1cm} (19)

Ionization power loss is

\[ P_{ion} = J_{ion}V_f \frac{kT_E}{q} \]  \hspace{1cm} (20)

where \( J_{ion} \) is the total ion current into both the emitter and collector, and \( V_f \) is the first ionization energy. Conservation of electron energy is

\[ P_E = P_{BS} + P_C + P_{ion} \]  \hspace{1cm} (21)

which can be reduced to

\[ r = 1 - \frac{1}{2} j V_d - \frac{1}{2} j_i V_f \]  \hspace{1cm} (22)

where \( j_i = J_{ion}/J_E \). In the ignited mode \( r \) is generally about 2 (\( T_E = 1500 \) K and \( T_e = 3000 \) K), consequently the arc-drop, \( V_d \), is negative. In other words the high plasma electron temperature is generated by resistance heating.

Finally, we consider electron and ion momentum. From electron momentum conservation, we find the potential drop in the plasma region. By adding the electron and ion momentum equations as in the general case, we find our diffusion equation and boundary conditions to which the sheaths contribute flux terms. When we introduce the ionization source term into this, we have the complete formulation. Electron momentum conservation is

\[ 0 = -\frac{dp_e}{dz} - qn \frac{d\psi}{dz} - \frac{a_{mn}n_{ue}}{\lambda_e}, \]  \hspace{1cm} (23)

where \( \lambda_e \) is electron mean free path. Using \( p_e = n kT_e \) and \( J = qn u_e \), we can rearrange eq.23 into

\[ J = -\frac{q\lambda_e}{ma_e}(kT_e \frac{dn}{dz} + nq \frac{d\psi}{dz}). \]  \hspace{1cm} (24)

This can be further reduced by dividing by \( J_E \) and using \( \xi = z/d \) where \( d \) is the convertor gap thickness:

\[ j = -\frac{\pi \lambda_e}{4d \sqrt{n}} \left( \frac{dn}{d\xi} + n \frac{dX}{d\xi} \right). \]  \hspace{1cm} (25)

Integration of this equation from the emitter sheath interface to the collector sheath interface yields

\[ \Delta x_p = \tau \ln \left( \frac{n(1)}{n(0)} \right) + jR \]  \hspace{1cm} (26)
where

\[ R = \frac{4}{\pi} \frac{d}{\lambda_e \sqrt{\tau}} \int_0^1 \frac{n_E}{n(\xi)} d\xi. \]  \hspace{1cm} (27)

The quantity \( R \) is the normalized plasma resistance.

The ion and electron momentum equations can be written

\[ kT_e \frac{d\rho}{dx} = -q_n \frac{d\phi}{dx} - \frac{mnu_e a_e}{\lambda_e} \frac{d}{dx} \]
\[ kT_i \frac{d\rho}{dx} = q_n \frac{d\phi}{dx} - \frac{Mnu_i a_i}{\lambda_i} \frac{d}{dx} \]  \hspace{1cm} (28)

where \( \lambda_i \) is ion mean free path and \( a_i \) is ion thermal speed,

\[ a_i = \sqrt{\frac{8\pi T_E}{\pi M}} \]

Addition of eqs.28 yields

\[ (kT_e + kT_E) \frac{d\rho}{dx} = -\left( \frac{a_e m}{\lambda_e} u_e + \frac{a_i M}{\lambda_i} u_i \right) n, \]  \hspace{1cm} (29)

which is ambipolar diffusion. Equation 29 is differentiated to become

\[ (kT_e + kT_E) \frac{d^2\rho}{dx^2} + \frac{a_e m}{\lambda_e} \frac{d}{dx} (nu_e) + \frac{a_i M}{\lambda_i} \frac{d}{dx} (nu_i) = 0. \]  \hspace{1cm} (30)

We assume recombination is negligible and the ionization source term is

\[ \frac{d}{dx} (u_e n) = \frac{d}{dx} (u_i n) = S n. \]  \hspace{1cm} (31)

Using eqs.31 in eq.30 yields

\[ \frac{d^2\rho}{d\xi^2} + \left[ \left( \frac{a_e m}{\lambda_e} + \frac{a_i M}{\lambda_i} \right) \frac{S d}{dx} \right] n = 0. \]  \hspace{1cm} (32)

Equation 29 taken at the boundaries of the plasma (at the emitter and collector sheath interfaces) forms the plasma boundary conditions

\[ \frac{d\rho}{d\xi} \left|_{\xi=0} \right. = \beta_0 n_0, \quad \frac{d\rho}{d\xi} \left|_{\xi=1} \right. = \beta_1 n_1 \]  \hspace{1cm} (33)

where

\[ \beta_0 = \frac{-d}{kT_e + kT_E} \left( \frac{a_e m}{\lambda_e} u_{e0} + \frac{a_i M}{\lambda_i} u_{i0} \right), \]
\[ \beta_1 = \frac{d}{kT_e + kT_E} \left( \frac{a_e m}{\lambda_e} u_{e1} + \frac{a_i M}{\lambda_i} u_{i1} \right). \]  \hspace{1cm} (34)

Equation 32 is written as

\[ \frac{d^2\rho}{d\xi^2} + A^2(\tau) n = 0 \]  \hspace{1cm} (35)
where

\[ A^2(\tau) = d^2S\left(\frac{a_n m}{\lambda_e} + \frac{a_i M}{\lambda_i}\right) \]  

(36)

where \( A(\tau) \) is the ionization coefficient and is found from consideration of ionization kinetics. Its solution for \( n \) is

\[ n(\xi) = B \sin(A\xi + C) \]  

(37)

where \( B \) and \( C \) are constants of integration and \( A = A(\tau) \). The quantities \( \beta_0 \) and \( \beta_1 \), which are the boundary conditions for eq. 37, can be written as functions of \( \tau, x_E, x_C \) and \( \Delta x_s \),

\[ \beta_0 = \beta_0(\tau, x_E, x_C, \Delta x_s), \]
\[ \beta_1 = \beta_1(\tau, x_E, x_C, \Delta x_s). \]

(38)

When there is no reflection, \( \beta_0 \) and \( \beta_1 \) are both large, i.e.,

\[ \beta_0 = O\left(\frac{d}{\lambda_i}\right), \beta_1 = O\left(\frac{d}{\lambda_i}\right). \]

Significant reflection on the emitter side reduces \( \beta_0 \) and it may indeed attain negative values for sufficiently strong reflection.

The density equation (eq. 35) with the boundary conditions \( \beta_0 \) and \( \beta_1 \) is a linear eigenvalue problem; its solution yields \( A \) and \( C \) as functions of \( \beta_0 \) and \( \beta_1 \). The calculated results are shown in fig. 3. Since \( A(\tau) \) is function of \( \tau \) from the ionization kinetics, the value of \( \tau \) is thus determined by a function of \( \beta_0 \) and \( \beta_1 \). The plasma resistance, \( R \), also can be expressed in terms of functions of \( \beta_0 \) and \( \beta_1 \) through \( A \) and \( C \) using eq. 27:

\[ R = \frac{4}{\sqrt{2\pi}} \frac{d}{\lambda_e} \frac{\sin C}{A} \ln\left(\frac{\tan\left(\frac{A+C}{2}\right)}{\tan\left(\frac{C}{2}\right)}\right). \]

(39)

The sheath results which provide \( j, Q, \beta_0 \) and \( \beta_1 \), complete the isothermal formulation. The results are summarized below. The quantities \( \beta_0, \beta_1, Q \) and \( j \) are found from the sheath calculations as functions of \( \tau, x_E, x_C \), and \( \Delta x_s \), i.e.,

\[ \beta_0 = \beta_0(\tau, x_E, x_C, \Delta x_s), \]
\[ \beta_1 = \beta_1(\tau, x_E, x_C, \Delta x_s), \]
\[ Q = Q(\tau, x_E, x_C, \Delta x_s), \]
\[ j = j(\tau, x_E, x_C, \Delta x_s). \]

From the eigenvalue problem for the plasma density we then find

\[ A(\tau) = A(\beta_0, \beta_1). \]

(40)
From the continuity equation for current we find

$$\chi_C - \chi_E = r \ln\left(\frac{\sin(A + C)}{\sin(C)}\right) + r \ln\left(\frac{\alpha_1}{\alpha_0}\right) + r \ln\left(\frac{1}{j} - 1\right).$$ (41)

And from the electron momentum equation we find

$$\chi_C - \chi_E = r \ln\left(\frac{\sin(A + C)}{\sin(C)}\right) + jR + \frac{2(r - 1)}{j} - \frac{j_i V_{ji}}{j}.$$ (42)

These three previous equations determine $\chi_E$, $\chi_C$ and $r$ when $\Delta x$, is given. This set of equations is valid for all $\Delta x$. Even in the case of $\Delta x, \leq 0$ when there is no reflection, the calculations differ from previous isothermal calculations because the Boltzmann assumption on the electrons is not used as indicated by the presence of $\alpha_0$ and $\alpha_1$.

3. CALCULATED RESULTS FOR ION REFLECTION AND TRAPPED IONS

In this section we develop isothermal solutions for the thermionic convertor with the emitter sheath phenomena of ion reflection, trapped ions and surface emission ions included. Emitter sheath effects on thermionic convertor performance can be divided into two categories: 1) changes in net ion flux rate into the sheath which affect plasma density directly, and 2) changes in sheath potential distribution which affect the exchange of "hot" plasma electrons for "cold" emitter ions directly. A decreased influx of ions into the sheath, which occurs for all three emitter sheath phenomena, increases the plasma density at the neutral plasma-emitter sheath interface. Theoretical intuition suggests that an increased plasma density at the emitter would benefit performance by reducing resistance through the plasma and therefore reducing arc-drop. However, this is not the case. While the plasma density at the emitter increases slightly, plasma density at the collector decreases. Consequently total resistance increases.

All three of these phenomena increase in significance as net current density through the convertor is reduced. Each of the these reduces the net ion loss rate to the emitter and consequently increases arc-drop (therefore degrading performance at low current densities). This increase in arc-drop is in agreement with the same tendency in the experimental results. However, the experimental results also show a plateau (of low arc-drop) at low current density. This plateau occurs at a current density corresponding
to significant surface ion emission and is therefore thought to occur as surface emission replaces volume ionization as the dominant source of plasma ions. Unfortunately, the theoretical calculations cannot be carried into this region because the collisionless collector sheath matching (to the neutral plasma) fails.

To provide a realistic framework for presenting the results, we consider the convertor conditions shown as case 1 in table 1. Case 2 is shown because it has the largest surface emission of any typical thermionic convertor operating condition (because the work function is high and the temperature is also high). Instead of presenting case 2 separately, we demonstrate the effects of surface emission in case 1 by increasing the surface emission by a factor of 100 thereby bringing it up to the level in case 2. The net current density at which surface emission becomes significant can be estimated by multiplying \( J_{es+} \) by the square root of the ion to electron mass ratio (approximately 500). In case 1 this means that surface emission becomes significant at \( J = 0.01 \text{amps/cm}^2 \) while in case 2 significant surface emission begins at \( J = 1.0 \text{amps/cm}^2 \).

4. EFFECTS OF ION REFLECTION

In this section we discuss the isothermal results for case 1 with ion reflection, but without trapped ions and with the small amount of surface emission ions of case 1. Figure 4 is the C-V diagram for this case.

The dotted line extending upward from point A is the single electron repelling emitter sheath solution. However, we have not taken recombination or the Schottky effect into account in this isothermal formulation which are expected to become important at current densities near \( J_R \). The interest of this paper begins at point A, where the single sheath doubles over. Between points A and B, where the back sheath height, \( \Delta X \), is less than the sheath height, \( X_E \), the emitter sheath is non-reflecting. In this region the sheath heights, \( X_E \) and \( X_C \), remain constant while the plasma density is proportional to net current, \( J \) (the normalized plasma density \( n_e/J \) is constant). Only the back sheath height, \( \Delta X \), changes and the C-V curve in this region is Boltzmann (the arc-drop is constant). Beginning at point B and continuing to point C, the double emitter sheath reflects plasma ions because the back sheath is larger than the front sheath, in other words the reflective potential, \( \Delta X_s = \Delta X - X_E \),
is positive. The result is that net ion loss rate, \( \bar{u} \), decreases and that arc-drop increases. The dotted curve BD is the same double sheath except that it assumes no ions are reflected; therefore \( \bar{u} \) is constant and arc-drop is constant. The two curves BC and BD are almost indistinguishable because the increase in arc-drop is small until the net current density is extremely small. The reason for this is that the shift speed is approximately \( u_s = 2 \) and therefore a large increase in reflective potential is required to change \( \bar{u} \) significantly (the half reflection point is \( \Delta x_s = 4.0 \) or approximately \( J = J_R \exp(-4) = 0.4 \text{amp/cm}^2 \)).

The curve EF is the single electron repelling emitter sheath case. It is the limiting case for large amounts of trapped ions in which the double sheath peak has been completely suppressed by the trapped ions. For this case the emitter sheath solution gives \( u_s = 0 \). This curve is not topologically connected to the curve ABC; it will be shown in section 5 that trapped ions move ABC toward the single ion repelling sheath case. The curve is much steeper (a faster increase in arc-drop) in this case because \( u_s = 0 \) (the half point in ion reflection is approximately \( J = 8 \text{amp/cm}^2 \)). Curve EG is the single ion repelling case assuming no reflection and is therefore a Boltzmann line with constant arc-drop.

At points F and C the solutions fail at the collector. The explanation for this failure is best given by examining figs. 5, 6, 7 and 8.

Figure 5 is the normalized plasma density through the convertor gap. The highest curve with no reflection, \( \Delta x_s = 0 \), has the largest plasma density at the collector but the lowest plasma density at the emitter. Ion reflection, which decreases the ion loss rate to the emitter, raises the plasma density at the emitter but lowers the plasma density at the collector. The lower plasma density at the collector forces a smaller collector sheath height to pass the net current density. This can be seen from eq.10. Figure 6 is the potential through the convertor under the same reflection conditions as in fig. 5. In fig. 6 the first two spaces on the left make up the double emitter sheath, and the last space on the right is the collector sheath. The region between the two sheaths is the neutral plasma region. In the no reflection case, it can be seen that the potential has a pronounced well in the middle. This is the result of the large plasma density in the middle. As reflection increases, this well disappears on the collector side of the plasma because resistive drop there (due to low plasma density) increases to the degree that it is greater than the ambipolar rise (due to decreasing density toward the collector). Simultaneously with plasma potential gradient at the collector becoming negative, the collector sheath goes
toward zero height. Figure 7 shows the critical collector sheath quantities as the collector sheath failure occurs. Collector sheath height, $x_C$, goes toward zero, the shift speed, $u_{ec}$, goes toward negative infinity, and the ion loss rate to the collector, $T_{ic}$, is driven to zero.

Figure 8 shows the changes in the emitter sheath height, ion shift speed and ion loss rate. When the collector sheath failure occurs, the ion loss rate to the collector is zero ($u_e = 0$) and the corresponding plasma ion distribution at the collector is bunched at zero velocity ($u_{sc} = -\infty$). While the mathematics hold self-consistently until $u_e = 0$, the physics is clearly poor at this point because $u_e = 0$ demands that the plasma ions at the collector have zero energy (zero temperature and zero mean velocity). An estimate of when the physics becomes poor is $u_e = 0$. At this point the net ion loss rate is close to the thermal speed.

A second physical difficulty that occurs with collector sheath failure is that the electron Mach number there, $Q_e$, (from eq. 10) becomes

$$Q_e = \sqrt{\frac{2}{\pi}}$$

because the collector sheath height approaches zero (actually about .001). In the present continuum formulation of the plasma region, it was assumed in eq. 13 that $Q_e$ is small so that the electron momentum term, $u_e du_e/dz$ can be neglected.

One could take the solution below the collector sheath failure point if $u_e$ could attain negative values or if $Q_e$ could attain values larger than $\sqrt{2/\pi}$. There is no physical basis for assuming that $u_e$ can become negative since the collector emits nothing. However, there is a physical basis for allowing $Q_e$ to be larger than $\sqrt{2/\pi}$, (an electron distribution shift) as can be seen in fig. 6: the potential drop nearing the collector becomes progressively more electron accelerating as the collector sheath fails and therefore the electron distribution should be shifted as the ion distribution is in an electron repelling sheath. However, this would clearly invalidate the assumption that the electron momentum term is negligible. Therefore the momentum term must be added to explore further in this direction and this has not been done because of the resulting complexity in the equations.

Comparison of fig. 7 to fig. 8 at the collector sheath failure point ($\Delta x_c = 2.5, u_e = 0$) shows that the ion loss rate to the emitter is positive. At this point the plasma is still ignited and generating ions as can be seen from figs. 9 and 10. The ionization coefficient, A, has dropped by 50%, but the plasma electron temperature has dropped by only 5%. Finally, we note in fig. 11 that the normalized plasma resistance, $R$, has risen by almost 100%. This is responsible for the increase in arc-drop and the decrease in performance. Plasma
resistance increases in response to reflection because the loss of plasma electron energy
to the emitter is more important than the loss of ionization energy to the emitter. Ion
reflection at the emitter increases the normalized plasma density there, and consequently
increases the normalized loss of plasma electron energy there. The basis of this can be
seen from conservation of electron energy (eq. 22),
\[ \tau = 1 - \frac{1}{2} jV_d - \frac{1}{2} \frac{j_i V_{fi}}{j_i V_{di}}. \] (43)
The ion energy loss term is generally small compared to the electron energy loss term:
\[ \frac{\frac{1}{2} j_i V_{fi}}{\frac{1}{2} j_i V_{di}} = \frac{J_i V_{fi}}{j_i V_{di}} \approx O(0.02). \] (44)
Therefore, we take the electron energy equation as:
\[ \tau = 1 - \frac{1}{2} jV_d. \] (45)
Since \( \tau \) is nearly constant (because of the ionization kinetics), the product \( jV_d \) is nearly
constant. Ion reflection decreases \( j \) (because the normalized plasma density increases) and
therefore increases arc-drop, \( V_d \) (makes \( V_d \) a more negative number).

If the equations are reformulated in such a way as to be valid past the collector sheath
failure point, then we can expect to eventually see a decrease in arc-drop and a low current
plateau as the electron temperature approaches 1 (the ignited plasma is extinguished and
the ionization source is surface emission). This can be seen from eq. 43. However, as
we see, the collector failure occurs before \( \tau \) has dropped more than 5%. Consequently we
do not see any plateau or decrease in arc-drop as net current density is decreased in the
present calculations.

5. EFFECTS OF TRAPPED IONS

Figure 12 shows the effect of trapped ions on the C-V characteristics. In this section
the trapped ion distribution is assumed to have the temperature of plasma ion distribution,
and that 100% trapped ions \( f_{tr} = 1.0 \) completes the plasma ion thermal distribution.

Curve AHIJ is the C-V characteristic for \( f_{tr} = 0.10 \). At point A there cannot be any
trapped ions since the back sheath height, \( \Delta x \), is zero. Therefore the trapped C-V merges
into the non-trapped curve there. The actual amount of trapped ions on the $f_{tr} = 0.10$ curve increases from zero at point A to the full 10% of a thermal distribution at point H where the back sheath height, $\Delta X$, is equal to the sheath height, $X_E$. The shift speed increases on AH from 1.95 to 3.00. This corresponds to what is seen in fig. 3.4.8 where $\Delta X < X_E$.

The rise in shift speed has been limited to 3.00. This limit is placed on the shift speed because a sheath with height of about 1.0 should not have a pre-sheath region capable of shifting the entire distribution so far. In fact limiting the shift speed is equivalent to increasing the cut-off speed for the ion distribution function.

The arc-drop decreases as result of the increase in $u_*$ and the consequent increase in the net ion loss rate to the emitter. A "hump" can be seen on AH where the shift speed hits 3.00. The arc-drop is lowest on this "hump" because the shift speed is at its maximum of 3.00. Between points H and I the back sheath height remains equal to the sheath height, $\Delta X - X_E = X_* = 0$. On this segment, $u_*$ decreases to 1.25, therefore increasing arc-drop.

From point I to point J, the shift speed remains constant at 1.25 and the ion loss rate decreases because of reflection. The other trapped cases, $f_{tr} = 0.2, 0.3$ and 0.4 have not been connected because they hit the 3.00 maximum shift speed much sooner than in the $f_{tr} = 0.1$ case.

Point J is the collector sheath failure point. Each of the $f_{tr} = 0.2, 0.3$ and 0.4 curves begins at $\Delta X_* = 0$ and ends at the collector sheath failure point. It should be noted that each of the trapped ion curves fails at a higher current than the last because the shift speed is lower.

6. EFFECTS OF EMITTER SURFACE EMISSION

Figure 12 shows the effect of surface emission on the $f_{tr} = 0.10$ curve; surface emission is added by multiplying the actual small amount of surface emission in case 1 by a factor of 100. This brings the surface emission up to the level in case 2, making it significant at $J = 1.0$ amp/cm$^2$. It can be seen that surface emission increases arc-drop; it does so in exactly the same way as reflection or trapped ions do - it decreases the net loss rate of ions to the emitter.
7. COMPARISON WITH EXPERIMENTAL RESULTS AND CONCLUSIONS

Figure 13 superimposes the isothermal results of fig. 12 on the experimental results for a cesium reservoir temperature of 551 K which produces a 1 torr neutral cesium pressure. The experimental results are from\(^8\). The point of this comparison is that the steepness of the C-V characteristic in the experimental convertors can be explained by a decreasing ion loss rate to the emitter. We have shown that all three of the expected emitter sheath phenomena decrease the ion loss rate to the emitter. We cannot calculate the amount of trapped ions in a collisionless sheath without knowledge of the collisional processes. However, the experimental C-V suggests that if the amount of trapped ions \((f_t)\) increases from 0% at \(J = 14\) amps/cm\(^2\) (the double sheath formation point) to 10% at \(J = 2\) amps/cm\(^2\) then the steepness could result from trapped ions reducing the ion loss rate to the emitter. Since these percentages are based on a thermal distribution of ions, they seem physically reasonable. Unfortunately, the collector sheath failure prevents us from going to the point in the calculations where \(r\) drops enough to make surface emission the source of ions.

The experimental curve is nearly a constant .05 volts below the isothermal result \((f_t = 0.10)\) except at high current densities and at the "hump". Comparison of the curves at high current density is not valid since neither the Schottky effect nor recombination have been included. The Schottky effect is important above 12 amps/cm\(^2\) in this case because the emitter sheath is single electron repelling (to the plasma) and therefore puts a strong electric field against the emitter with the appropriate sign. Recombination is also potentially important because the plasma density scales with current density and at high current densities the plasma density in the middle of the convertor approaches the Saha density. The .05 volt difference may or may not be explained by a discrepancy in the assumed collector work function. At 800 K the collector emits essentially nothing and therefore any change in the collector work function directly affects output voltage. If the collector work function were in fact 1.65 volts instead of 1.60 volts then the isothermal result would lie nearly on top of the experimental result. We have not adjusted the assumed collector work function so as to illustrate the importance of it and therefore the importance of the surface physics of the adsorbed cesium layer. The "hump" should not be taken as an expected experimental result since it results from the interaction of the trapped ions.
with the plasma-emitter sheath interface. Instead it should be taken as a second reason (in addition to the cut-off of the ion distribution) for further study of the matching region between the collisionless sheath and the neutral plasma.

8. ACKNOWLEDGMENTS

This work was supported by the Air Force Office of Scientific Research.
CASE 1  

\[ T_E = 1500 \text{ K} \]  \[ T_E = 1750 \text{ K} \]  
\[ T_C = 750 \text{ K} \]  \[ T_C = 750 \text{ K} \]  
\[ p_{ce} = 1 \text{ torr} \]  \[ p_{ce} = 1 \text{ torr} \]  
\[ d = 10 \text{ mil} \]  \[ d = 10 \text{ mil} \]  
\[ \phi_E = 2.12 \text{ eV} \]  \[ \phi_E = 2.67 \text{ eV} \]  
\[ \phi_C = 1.60 \text{ eV} \]  \[ \phi_C = 1.73 \text{ eV} \]  
\[ J_R = 20 \text{ amp/cm}^2 \]  \[ J_R = 7.57 \text{ amp/cm}^2 \]  
\[ J_{cs+} = 1.80 \times 10^{-5} \text{ amp/cm}^2 \]  \[ J_{cs+} = 2.10 \times 10^{-3} \text{ amp/cm}^2 \]  

CASE 2  

\[ T_E = 1750 \text{ K} \]  
\[ T_C = 750 \text{ K} \]  
\[ p_{ce} = 1 \text{ torr} \]  
\[ d = 10 \text{ mil} \]  
\[ \phi_E = 2.67 \text{ eV} \]  
\[ \phi_C = 1.73 \text{ eV} \]  
\[ J_R = 7.57 \text{ amp/cm}^2 \]  
\[ J_{cs+} = 2.10 \times 10^{-3} \text{ amp/cm}^2 \]  

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\( n \left( \frac{kT_e}{m} \right) \)

\( x/d \)

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\( \Delta x_s = 2.5 \)
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A GENERAL SOLUTION CONDITION FOR COLLISIONLESS SHEATHS

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Abstract

A general solution condition for collisionless sheaths is developed. Previous work has assumed that the Bohm criterion or the generalized Bohm criterion assures a self-consistent sheath solution. This paper shows that for non-monotonic collisionless sheath structures, such as double sheaths containing trapped ions, the generalized Bohm criterion is a necessary but not a sufficient condition. The general solution condition developed is always sufficient and the generalized Bohm criterion is shown to be special case of it. The general solution condition is applied to a double emitter sheath containing trapped ions. First, it is shown that the low energy part of the plasma ion distribution coming into the sheath cannot be neglected as claimed in some analyses, because the shift in mean ion velocity through the presheath (generalized Bohm speed) depends strongly on low energy ions. Second, it is shown that the presence of trapped ions moves the point of critical self-consistency away from the collisionless sheath-neutral plasma asymptotic match and into the collisionless sheath. Consequently, both the sheath structure and the generalized Bohm speed depend on the amount of trapped ions. Thus collisional effects may dominate the structure of a presumably collisionless sheath through the trapping mechanism and the collisional pre-sheath which determines the low energy ion component entering the collisionless sheath.

1. Introduction

In this paper the collisionless sheath formulation for arbitrary potential structures is developed and a general solution condition is found. Originally, the collisionless sheath formulation was developed by Langmuir [1]. Bohm [2] showed that a minimum ion speed (of plasma ions coming into the sheath) was necessary to construct a self-consistent collisionless sheath. However, Bohm's analysis assumes, among other things, that the ion distribution is monoenergetic (cold). Harrison and Thompson [3] removed the assumption of cold ions and derive a generalized Bohm criterion. However,
Harrison and Thompson's analysis assumes that the critical self-consistency point (which necessitates the generalized Bohm criterion) occurs at the plasma-sheath interface where the collisionless sheath is asymptotically matched to the continuum quasi-neutral plasma. Also Harrison and Thompson circumvent difficulties with low velocity ions by assuming that \( <v^2> = <v^{-2}>^{-1} \) where the brackets indicate averages over the ion distribution function and \( v \) is velocity. In this paper, we show that the point of critical self-consistency does not necessarily occur at the plasma-sheath interface, and that the resulting general solution condition is substantially affected by the location of the critical point. In the case of trapped ions, the critical point may occur in the middle of sheath (well away from the neutral plasma in terms of potential rise). Under this condition the asymptotic match then exists in the middle of the collisionless sheath, such that the sheath length would be infinite unless the general solution condition is oversatisfied.

Using the generalized solution condition, this paper treats as an example the double (nonmonotonic) sheath structure that is expected to be important at an emitting electrode in such applications as thermionic energy convertors. This paper does not consider such complexities as magnetic field or a curved surface. We assume instead that the magnetic pressure is low and that the radius of curvature of the plate is large compared to both the mean-free-paths and the Debye length. Consideration of magnetic effects can be found in Chodura [4] and consideration of curvature effects can be found in many references under the guise of probe theory including Lam [5]. Also this paper does not consider the pre-sheath structure which must exist between the Maxwellian plasma and the collisionless sheath. Instead the effect of different ion distributions (various cut-off velocities) coming from the pre-sheath is shown. It is apparent from the results in this paper that the pre-
sheath region is important to the double sheath structure; however little work
exists as yet on the pre-sheath. See, for instance, references [6,7,8,9].

The simplest demonstration of the Bohm criterion can be constructed as
follows. Since the sheath region is assumed to be collisionless, we can write
the Poisson equation as

$$\frac{d^2\phi}{dx^2} = 4\pi q(n_i(\phi) - n_e(\phi)) \tag{1}$$

where $\phi$ is potential, $x$ is distance into the sheath, $q$ is electron
charge, $n_i(\phi)$ is ion density, and $n_e(\phi)$ is electron density. In this
equation and all that follow in this paper, the sense of electric potential is
the reverse of the usual; increasing potential repels electrons. Equation 1
can be non-dimensionalized as

$$\frac{d^2\chi}{d\zeta^2} = F(\chi) \tag{2}$$

where $\chi = q\phi / kT_R$, $\zeta = x / \lambda_D$, $\lambda_D = \sqrt{\frac{kT_R}{4\pi q^2 n_0}}$

and where $F(\chi) = \frac{n_i(\chi) - n_e(\chi)}{n_0}$.

$T_R$ is a reference temperature usually taken to be the heavy particle
temperature and $n_0$ is a reference plasma density usually taken as the plasma
density at the neutral plasma-sheath interface. $\lambda_D$ is the Debye length.

At the neutral plasma-sheath interface we define $\chi = 0$, and we have charge
neutrality there; consequently $F(0) = 0$. Also, we have $\frac{d\chi}{d\zeta} \bigg|_{\chi=0} = 0$ because the
neutral plasma cannot support a strong electric field. The ion and electron
density parts of $F(\chi)$ can be modeled in this simple case as
\[ F_i(x) = \frac{u_{\text{mono}}}{\sqrt{u_{\text{mono}}^2 + x}} \quad \text{and} \quad F_e(x) = e^{-x/\tau} \]  

The ion distribution is assumed to be cold and \( u_{\text{mono}} = \sqrt{\frac{2kT_e}{M}} \)

where \( u_{\text{mono}} \) is the ion distribution speed, \( k \) is Boltzmann's constant and \( M \) is ion mass. The electron distribution is assumed to be Maxwellian where \( \tau = T_e / T_R \) and \( T_e \) is electron temperature. Under these assumptions, the Bohm criterion requires

\[ \frac{\partial F_i(x)}{\partial x} - \frac{\partial F_e(x)}{\partial x} \bigg|_{x=0} \geq 0 \]  

since otherwise no self-consistent solution to eq. 2 exists at \( x = 0 \).

Equation 4 applied to eq. 3 yields

\[ u_{\text{mono}} \geq \sqrt{\frac{kT_e}{M}} \]  

which is the standard Bohm result.

This simple model is not sufficient for all applications because it assumes the following:

1) that the sheath height, \( X_E \), is large so that the Boltzmann electron distribution approximation is valid,
2) that the ion distribution is cold, and
3) that there are no other species involved besides the electrons and ions from the plasma.

In such applications as a thermionic energy convertor (TEC), accurate collisionless sheath results are desired, yet all three of these assumptions
are violated. In addition, an emitter sheath (particularly in a TEC) may be
doubled over and therefore also contain trapped ions. Exact collisionless
sheath calculations carried out under these conditions require that the simple
Bohm criterion (eq. 4) be generalized. A warm ion distribution requires the
generalization of Harrison and Thompson [3] in which the Bohm criterion
becomes
\[
\frac{\partial F_i(x)}{\partial x} \bigg|_{x=0} = \int f_i(v) \frac{1}{2v^2} dv \leq \frac{M}{2kT_e}
\]
where the variable of integration, v, is velocity in the x direction. It can
be seen immediately, however, that the integral in eq. 6 does not converge
unless conditions are imposed on \( f_i(v) \). Namely, the ion velocity
distribution, \( f_i(v) \), must go to zero sufficiently fast near \( v=0 \) (be cut
off). In general one would expect the ion distribution entering the
collisionless sheath from the collisional presheath to be cut off because the
collisional processes in the presheath will not resupply sufficient low
velocity ions to replace those accelerated toward the sheath [6,7]. We do not
deal with two or three dimensional effects which may effect the cutoff [10].
Since the form of the ion distribution entering the sheath is not known, we
have assumed for the purposes of this paper that it is a shifted Maxwellian
with the neutral temperature and cutoff below a prescribed velocity. If the
integral in eq. 6 is undefined, then the expansion of \( F(x) \) as a Taylor
series has failed because \( \frac{\partial F_i}{\partial x} \bigg|_{x=0} \) does not exist.

In the Appendix the complete asymptotic expansion of \( F(x) \) for
small \( x \) is presented, which is
\[
F_i(x) = \int f_i(v)dv - x^{1/2}f(0) + O(x\ln x) \quad .
\]
The $\chi^{1/2}$ term does not appear in the electron distribution expansion,

$$F_e(\chi) = \int f_e(v) \, dv + O(\chi \ln \chi)$$

(8)

(The terms represented by $O(\chi \ln \chi)$ can be found in Appendix A and are omitted here for convenience.) We have, then

$$F(\chi) = -\chi^{1/2} \, f(0) + O(\chi \ln \chi).$$

(9)

Therefore $f(0)$ must equal zero since otherwise $F(\chi) < 0$ for some $\chi > 0$ and we could not then construct a self-consistent solution. Thus even when the full asymptotic expansion is carried out, the ion distribution must be cut off off. (It will be shown later in fig. 5 that the dependence of the Bohm criterion on the cut-off is weak.)

While the Bohm criterion, or the generalized version is a necessary condition for a self-consistent sheath structure, it is not always sufficient. In particular, trapped ions in a double sheath move the point of critical self-consistency away from the neutral plasma-sheath into the collisionless sheath itself. For this reason, we must develop a general solution condition, of which the generalized Bohm criterion is a special case.

2. The General Solution Condition

Before proceeding to discuss the possible sheath configurations and their specific equations, we develop the general solution condition which applies to all collisionless sheaths to preclude consideration of non-self-consistent solutions. Asymptotically matching the collisionless sheath to the quasi-
neutral plasma always requires some condition on the ion distribution function coming into the sheath from the plasma. In the past this had been the Bohm criterion which assumed cold plasma ions and required the monoenergetic plasma ion distribution to be shifted up in velocity to

\[ U_{\text{mono}} = \sqrt{\frac{kT_e}{M}}. \]

In this section it will be shown that a general condition for solving the collisionless sheath with a neutral plasma-sheath interface exists and that the Bohm criterion and other local (at the plasma-sheath interface) matching conditions are special cases of the general condition. We show that local matching is a necessary but not sufficient condition on the sheath solution; that a point of critical self-consistency exists which is not necessarily at the neutral plasma-sheath interface but may appear anywhere in the sheath. In the absence of trapped ions or surface emission ions, as is the case with most past calculations, local matching (in which the point of critical self-consistency is at neutral plasma-sheath interface) proves to be also sufficient. Further, we show, in the case of no trapped ions or surface emission ions, that for finite temperature the ion distribution must have no zero velocity ions when it enters the collisionless sheath.

A self-consistent sheath solution is found by integrating the non-dimensional Poisson equation from the last section,

\[ \frac{d^2 x}{d\xi^2} = F(x). \]  \hspace{1cm} (10)

The specific forms of \( F(x) \) are developed in the next section but we need not know them until we wish to evaluate specific cases. By convention \( x = 0 \) at
the plasma-sheath interface and increasing $\chi$ repels electrons. To construct a non-trivial solution we integrate eq. 10 as follows:

$$\frac{dx}{d\zeta} \frac{d^2x}{d\zeta^2} = F(\chi) \frac{dx}{d\zeta}, \quad (11)$$

$$\frac{1}{2} \left( \frac{dx}{d\zeta} \right)^2 \bigg|_{\zeta_1}^{\zeta_2} = \int_{\zeta_1}^{\zeta_2} F(\chi) \frac{dx}{d\zeta} d\zeta, \quad (12)$$

$$\frac{1}{2} \left( \frac{dx}{d\zeta} \right)^2 \bigg|_{x_1}^{x_2} = \int_{x_1}^{x_2} F(\chi) d\chi. \quad (13)$$

Transforming eq. 12 into eq. 13 implicitly assumes $\chi(\zeta)$ is monotonic on the domain $(\zeta_1, \zeta_2)$. Since at the plasma-sheath interface, we assume charge neutrality and zero electric field, we have $F(0) = 0$ and $dx/d\zeta = 0$. Therefore we can write,

$$\frac{1}{2} \left( \frac{dx}{d\zeta} \right)^2 = \int_{0}^{x} F(\chi) d\chi, \quad (14)$$

and,

$$\sqrt{2 \int_{0}^{x} F(\chi) d\chi} = \pm d\zeta. \quad (15)$$

Equation 15 is the method for constructing non-trivial solutions. From eq. 15 and our previous observation about converting eq. 12 into eq. 13 we can see that the inequality,

$$\int_{0}^{x} F(\chi) d\chi > 0, \quad 0 < \chi < \chi_m \quad (16)$$
must hold where $x_m$ is the first maximum or minimum $x$ reaches in the sheath since otherwise $\chi_0$ is not monotonic. If we attempt to construct a solution which does not meet this condition, then eq. 15 causes $\chi_0$ to double back before reaching its full sheath height. The point where eq. 16 meets the equality for $0 < x < x_m$ is called point of critical self consistency.

From the general solution condition (eq. 16), we can derive necessary local (at the plasma-sheath interface) matching conditions, of which the generalized Bohm criterion is a special case. We can expand $F(x)$ around $x = 0$ as

$$F(x) = \sum_{n=1}^{\infty} a_n x^{n/2} - \sum_{n=2}^{\infty} b_n x^{n/2} \ln x$$

$$F(x) = \sum_{n=1}^{\infty} a_n x^{n/2} - \sum_{n=2}^{\infty} b_n x^{n/2} \ln x$$

since $F(x)$ may be represented asymptotically in this form for cases we intend to develop. Then eq. 17 may be integrated,

$$\int_{0}^{\infty} F(x) dx = \sum_{n=1}^{\infty} \frac{a_n x^{n/2+1}}{n + 1}$$

$$\int_{0}^{\infty} F(x) dx = \sum_{n=1}^{\infty} \frac{a_n x^{n/2+1}}{n + 1}$$

$$- \sum_{n=2}^{\infty} \frac{b_n x^{n/2+1} (\ln x^{n/2+1} - 1)}{(n/2+1)^2}$$

$$\int_{0}^{\infty} F(x) dx = \sum_{n=1}^{\infty} \frac{a_n x^{n/2+1}}{n + 1}$$

$$- \sum_{n=2}^{\infty} \frac{b_n x^{n/2+1} (\ln x^{n/2+1} - 1)}{(n/2+1)^2}$$

The asymptotic order of the terms in this equation is $a_1, b_2, a_2, b_3, a_3$ .... To insure that at least local matching is satisfied, the first non-zero
term must be positive. We call this the generalized local matching. The usual Bohm criterion assumes \( F(x) \) is expandable as,

\[
F(x) = 0 + \frac{\partial F}{\partial x} \bigg|_{x=0} x + \frac{\partial^2 F}{\partial x^2} \bigg|_{x=0} \frac{x^2}{2!} + \ldots
\]  

(19)

in which case the criterion is

\[
\frac{a_2}{a_x} = \frac{\partial F}{\partial x} \bigg|_{x=0} > 0.
\]  

(20)

When only cold ions are considered, the Bohm criterion (eq. 20) is sufficient, but when finite temperature ions are considered, we must apply the generalized local matching. If trapped or surface emission ions are present, we find that local matching is not sufficient and that the full solution condition must be applied.

3. The Double Emitter Sheath with Trapped Ions

The double emitter sheath is of particular interest to the TEC and other devices in which thermionic emission is substantial. With sufficient thermionic emission, the emitter sheath will double over as shown in fig. 2 such that it repels electrons from both the emitter and the plasma. In addition the double sheath may trap ions in its "well" and it may reflect ions coming from the plasma if the reflective potential \( \Delta x_s \), is greater than zero.

In order to analyze the double emitter sheath, we need models for the distribution functions of ions and electrons coming into the sheath, and we also need a model for the trapped ion distribution in the double sheath. Figures 3 and 4 show the distribution functions and definitions.

As shown in fig. 3 the emitted electrons are assumed to leave the plate
with a Maxwellian distribution at $T_E$, the plate temperature. The electrons coming into the sheath from the plasma are assumed to have a Maxwellian distribution at $T_e$, the plasma electron temperature. The ion distribution coming from the plasma is assumed to be a shifted Maxwellian distribution where $U_s$ is the shift velocity and $U_{cut}$ is the cut-off velocity. It is also assumed to have plate temperature, $T_E$. It should be noted here that the shift speed, $U_s$, becomes the Bohm speed in the limit of zero ion temperature. The trapped ion distribution (at $x = x_E$) is assumed to be Maxwellian with temperature $T_E$, and cut off at $-\sqrt{x_E}$ and $\sqrt{x_E}$, as shown in Fig. 4. Under these assumptions, the density functions ($F_i(x)$ and $F_e(x)$) for ions and electrons on the plasma side of the sheath can be written as

$$F_i(x) = \frac{n_o}{\sqrt{\pi}} \int_{-x_E}^{x_E} \frac{e^{-u^2}}{u} \frac{udu}{\sqrt{u^2 + x_E}} + \frac{\sqrt{\Delta x-x_E}}{\sqrt{\pi}} \int_{-x_E}^{x_E} \frac{e^{-u^2}}{u} \frac{udu}{\sqrt{u^2 + x_E}}$$

$$+ \frac{2f_{tr}^e}{\sqrt{\pi}} \int_{-\sqrt{x_E}}^{\sqrt{x_E}} \frac{e^{-u^2}}{u} \frac{udu}{\sqrt{u^2 + x_E}}$$

and

$$F_e(x) = n_o \frac{2}{\sqrt{\pi}} \int_{-x_E}^{x_E} \frac{e^{-u^2}}{u} \frac{udu}{\sqrt{u^2 - x_E}} + \frac{2}{\sqrt{\pi}} \int_{-x_E}^{x_E} \frac{e^{-u^2}}{u} \frac{udu}{\sqrt{u^2 - x_E}}$$

$$+ n_e \frac{2}{\sqrt{\pi}} \int_{0}^{2 \sqrt{x_E}} \frac{e^{-u^2}}{u} \frac{udu}{\sqrt{u^2 + (x_E-x)}}$$

In eqs. 21 and 22 the density functions $F_i(x)$ and $F_e(x)$ are normalized by $n_o$ (the plasma density at the neutral plasma-sheath interface) and therefore the
densities $f_{\text{plasma}}$, $f_{\text{tr}}$, $n_0$ and $n_E$ are likewise normalized. In eqs. 21 and 22 potential is non-dimensionalized by

$$\chi = \frac{q\phi}{kT_E}.$$ 

In the ion density expression (eq. 21) the second integral is the reflected ion term which is a mirror image of the lower velocity part of the first term. The quantity $u$ is the non-dimensional ion velocity at the sheath edge ($\chi = 0$). The third integral in eq. 21 is the trapped ion distribution where $u$ is the non-dimensional ion velocity where $u$ is the non-dimensional ion velocity at the sheath potential peak ($\chi = \chi_E$). Also the non-dimensionalization of $u$ is $u = \sqrt{M/2kT_E} U$ where $U$ is velocity. We have assumed $T_i = T_E$. In the electron density expression (eq. 22) the first integral is the incoming plasma electron term, the second integral is the returning plasma electron term and the third integral is the emitted electron term. The non-dimensional velocities, $u$, are $u = \sqrt{m/2kT_e} U$ in the first and second integrals of eq. 22 and $u = \sqrt{m/2kT_E} U$ in the third integral of eq. 22. In the first and second integrals, $u$ is at the sheath edge ($\chi = 0$) and in the third integral, $u$ is at the sheath peak. The quantity $\tau = T_e/T_E$ is the electron temperature ratio.

The quantities $f_{\text{plasma}}$, $f_{\text{tr}}$, $n_E$ and $n_0$ are densities, all of which are normalized by $n_0$, the plasma density at the plasma-sheath interface. The quantity $f_{\text{plasma}}$ represents the density of ions coming into the sheath from the plasma, $f_{\text{tr}}$ represents the density of ions trapped in the double sheath, $n_E$ represents the density of electrons coming from the emitter that cross the sheath potential peak, and $n_0^-$ represents the density of electrons coming from the plasma into the sheath. The quantities $u_{\text{cut}}$ and $u_s$ are the normalized
cut-off speed and shift speed:

\[ u_{\text{cut}} = \frac{U_{\text{cut}}}{\sqrt{\frac{2kT_E}{M}}} \quad \text{and} \quad u_s = \frac{U_s}{\sqrt{\frac{2kT_E}{M}}} \]

The solution to this sheath is found by treating the following quantities as parameters:

\[ f_{\text{tr}}, u_{\text{cut}}, x_E, \Delta x_s, \tau \]

and the remaining quantities as variables which will be found:

\[ f_{\text{plasma}}, n_E, n_0, u_s \]

The four equations which produce the solution are:

\[ F_i(0) = 1 \quad , \quad (23) \]

\[ F_e(0) = 1 \quad , \quad (24) \]

\[ x_E \]

\[ \int_0^{x_E} (F_i(x) - F_e(x)) dx = 0 \quad , \quad (25) \]

\[ \min \left[ \int_0^{x_E} (F_i(x) - F_e(x)) dx; 0 < x < x_E \right] = 0 \quad . \quad (26) \]

Equations 23 and 24 account for charge neutrality at the plasma-sheath interface and that the plasma density is \( n_0 \) there. Equation 25 requires that the sheath have zero electric field at the potential peak, \( x_E \) (because the
potential gradient at the peak must be zero). Equation 26 is the general solution condition, which insure that the solution is self-consistent. It replaces the Bohm criterion and in fact contains the Bohm criterion as a special case as shown in section two.

As noted, the cut-off velocity, $u_{\text{cut}}$, is a parameter in the collisionless sheath formulation, and is in fact determined by collisional processes in a pre-sheath region that must exist between the collisionless sheath and the neutral plasma. Previous collisionless sheath formulations have gotten around this difficulty by using the monoenergetic ion assumption illustrated in Section 1. Figure 5 shows the effect of $u_{\text{cut}}$ on $u_s$ under typical double sheath conditions.

For comparison, the monoenergetic Bohm speed is shown as a dotted line which obviously does not depend on $u_{\text{cut}}$. In this paper we do not calculate exactly what the $u_{\text{cut}}$ should be, but instead note that the effect of $u_{\text{cut}}$ on the shift speed, $u_s$, is relatively weak, and in subsequent calculations set value for $u_{\text{cut}}$ at

$$u_{\text{cut}} = (.1) u_{\text{mono}}\ ,$$

(27)

on the grounds that any physically reasonable value of the cut-off should be on the order of 10% of the monoenergetic Bohm speed. Support for this can be found in Emmert [6], and Riemann [7]. In many applications, using this approximation would produce a perfectly acceptable result. However it should be noted that a good approximate value for $u_{\text{cut}}$ derived from the transition region would produce a very accurate results in $u_s$ since the dependence of $u_s$ on $u_{\text{cut}}$ is essentially logarithmic. It should be noted from fig. 5 that $u_s$ is
approximately twice the monoenergetic Bohm speed under the assumption of eq.
27. In order for the shift speed, \( u_s \), to equal the monoenergetic Bohm speed,
\( u_{\text{mono}} \), the cut-off velocity, \( u_{\text{cut}} \), would have to be set at \((.8)u_{\text{mono}}\). This
seems physically unrealistic, and therefore we expect that a non-monoenergetic
ion distribution should produce a shift speed, \( u_s \), significantly higher than
\( u_{\text{mono}} \).

The introduction of trapped ions into the double sheath moves the point
of critical self-consistency away from the neutral plasma-sheath interface
into the sheath as shown in figs. 6 and 7.

Trapped ions have a substantial effect on the sheath solution. Figure 6
shows the general solution condition for the double sheath with \( \chi_E = 2.0 \), \( \tau = 2.0 \)
and for \( f_{\text{tr}} = 0.0 \) and \( f_{\text{tr}} = 0.1 \). In the no trapped ion case (\( f_{\text{tr}} = 0.0 \)) we
have local matching (in which the point of critical self-consistency occurs at
the neutral plasma-sheath interface, \( \chi = 0 \)). As trapped ions are added, the
point of critical self-consistency moves into the sheath, and at \( f_{\text{tr}} = 0.1 \) has
moved to the middle of the sheath (\( \chi = 0.6 \)). Figure 7 shows the sheath
structure (\( \chi \) versus \( \zeta = x/\lambda_0 \) where \( x \) is distance into the sheath). The two
cases shown in fig. 7 correspond to those in fig. 6. The point of critical
self-consistency on \( f_{\text{tr}} = 0.1 \) in fig. 7 should actually be an asymptotic match
of the upper part of sheath to the lower part. However, we have made the
slope finite for display purposes. This asymptotic "flat spot"
at \( \chi \approx 0.6 \) should not be troublesome because it can be removed by raising \( u_s \)
above the minimum required by the general solution condition. The existence
of such an asymptotic region in the middle of collisionless sheath is
counterintuitive, but not completely unexpected since it was previously only
by assumption that critical self-consistency occurred at the plasma-sheath
interface. Under ordinary circumstances, i.e., a single sheath which has no
trapped ions, the matching occurs at the plasma-sheath interface because the ion density as a function of $x$ ($F_i(x)$) is well behaved. This is obviously not the case with trapped ions present in a double sheath. The asymptotic "flat spot" raises an interesting question about collisionless sheath width. If $u_s$ is not much greater than the minimum required for a match, then the collisionless sheath would be collisional because of its length. Therefore if the sheath is to remain collisionless, $u_s$ must be large enough to make the sheath short compared to the mean-free-paths. Figure 8 shows the trapped ion effect on shift speed $u_s$ for various sheath heights $x_E$. It is significant that $u_s$ drops rapidly with increasing amounts of trapped ions. The curves terminate where the self-consistent sheath solution is no longer possible because the amount of trapped ions added prevents charge neutrality at the plasma-sheath interface. It should be noted that $u_s$ is ion distribution shift speed, not the mean speed $\overline{u}$ of the ions into the sheath. Therefore when $u_s$ is negative, the ion distribution is concentrated at low velocities and $\overline{u}$ is low but not negative.

4. Conclusions

A general solution for collisionless sheaths is important in any case where the sheath structure is nonmonotonic, or for some other reason one suspects that the point of critical self-consistency does not occur at the plasma sheath interface. As demonstrated with the example of a double emitter sheath, the generalized Bohm criterion is not sufficient to ensure a self-consistent sheath structure because the presence of trapped ions moves the point of critical self-consistency away from the plasma-sheath interface and into the sheath itself. The strong effect of trapped ions and ion distribution cut-off velocity on shift speed $u_s$ shows that, even in a
collisionless sheath, the collisional effects cannot be ignored. In an application such as the TEC, we need to determine the collisional mechanisms for both ion trapping, and the pre-sheath region since otherwise the ion distribution shift speed can only be estimated within a factor of two.

In addition the appearance of the asymptotic flat spot in the middle of the collisionless sheath implies that the sheath width is not unambiguously determined by collisionless processes. Therefore it seems unavoidable that collisional effects are important in presumably collisionless sheaths. In particular, an understanding of how the collisional presheath modifies the incoming plasma ion distribution seems necessary and also an understanding of the trapping mechanism in the case of the double emitter sheath seems necessary. The Bohm criterion or generalizations of it produce, in effect, estimates of potential rise through the collisional presheath based on collisionless self-consistency, and cannot be expected to yield accurate results for complex sheath structures. While many applications require only estimates of sheath potential rise and consequent ion acceleration through the sheath, such applications as TECs require accurate knowledge of sheath structure.

5. Acknowledgements

This work was supported by AFOSR Contract 83-0048.
REFERENCES


APPENDIX: ASYMPTOTIC EXPANSIONS

The asymptotic expansions of

\[ F_i(x) = \int_0^\infty f_i(u) \frac{u \, du}{\sqrt{u^2 + x}} \quad (1) \]

and

\[ F_e(x) = \int_0^\infty f_e(u) \frac{u \, du}{\sqrt{u^2 - x}} \quad (2) \]

are

\[ F_i(x) = \int_0^\infty f_i(u) \frac{u \, du}{\sqrt{u^2 + x}} = \]

\[ + x \left[ \frac{f^{(1)}(0)}{4} - \frac{1}{2} \int_0^\infty f^{(3)}(u) u \ln(2u) \, du \right] + o(x^{3/2}) \quad (3) \]
and

\[ F_e(x) = \int_{\sqrt{x}}^{\infty} f_e(u) \frac{udu}{\sqrt{u^2 - x}} = \]

\[ \int_{0}^{\infty} f_e(u) du + x \ln x \left[ - \frac{f'(0)}{4} \right] \]

\[ + x \left[ - \frac{f''(0)}{4} + \frac{1}{2} \int_{0}^{\infty} f^{(3)}(u) u \ln(2u) du \right] \quad (4) \]

\[ + O(x^2). \]

These asymptotic expansions are developed by splitting the integrals into two parts and expanding \( f_e(u) \) or \( f_i(u) \) into a Taylor series on the lower part and expanding

\[ \frac{u}{\sqrt{u^2 - x}} \quad \text{or} \quad \frac{u}{\sqrt{u^2 + x}} \]

on the upper part. The two pieces can be recombined analytically to produce eq. 3 and 4. Details of analysis are not presented here, however the results are valid if \( f_i(u) \) or \( f_e(u) \) can be expanded as a Taylor series on some interval \([0, k]\) where \( k > 0 \). This condition is generally easy to satisfy, and if necessary, the analysis can be extended for non-Taylor expandable \( f_i(u) \) and \( f_e(u) \) if some appropriate convergent series can be found, on \([0, k]\).
If the functions \( f_i(u) \) and \( f_e(u) \) are cut-off (\( f_i(u) \) or \( f_e(u) = 0 \) for \( 0 < u < u_{\text{cut}} \)) then eqs. 3 and 4 reduce to

\[
\int_{u_{\text{cut}}}^{\infty} f_i(u) \frac{u du}{\sqrt{u^2 + x}} = 0 (x^2) 
\]

and

\[
\int_{u_{\text{cut}}}^{\infty} f_i(u) du - x \left[ \int_{u_{\text{cut}}}^{\infty} f_i(u) \frac{1}{u^2} du \right] + 0 (x^2) 
\]

These are, of course, also the results attained by differentiation of the well behaved integrands.
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Figure 4. Ion Velocity Distribution at the Potential Peak ($x = x_E$).
Figure 5. Shift Speed as a Function of Cutoff Speed for $f_{tr} = 0.0$. 

- $\Delta x_s = 0.0$
- $\tau = 2.0$
- $x_e = 1.0$

Bohm Speed
Figure 6. Solution Condition with Trapped Ions
Figure 7. Sheath Structure with Trapped Ions
Figure 8. Shift Speed with Trapped Ions
RESEARCH PROGRESS AND FORECAST REPORT
AFOSR GRANT 85-0375

True Asymptotic Plasma - Sheath Matching with an
Asymptotically Correct Collisional Presheath

by

Geoffrey L. Main

School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

Submitted to

Air Force Office of Scientific Research
Bolling Air Force Base
Washington, D.C., 20332-6448

AFOSR technical Officer:
Dr. Julian M. Tishkoff

March, 1987
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INTRODUCTION

Current analytical models of Thermionic Convertors break the gap into two regions: the neutral plasma and the collisionless sheaths. Such models are reasonably accurate for the standard ignited mode but not necessary accurate or even approximately true under the following conditions, all of which are of interest for advanced thermionic conversion:

1. Ion emitting or reflecting emitter and collector surfaces,
2. Non-ignited pulsed modes, and
3. Knudsen mode convertors in which the gap size is on the order of an ion mean-free-path or less.

Thus the present work on collisional presheaths is carried out. In addition there are other important areas of application of the present work including plasma etching of semiconductors, plasma diodes for ion beam production for inertial confinement fusion, plasma-surface interactions in rail guns and plasma-surface interactions in general.

SUMMARY OF PROGRESS AND RESEARCH FORECAST

During this report period, Aug. 31, 1986 until the present, a time dependent thermionic convertor code incorporating the the asymptotic presheath work (see the annual report of Aug. 31, 1986) has been developed and is currently be tested.

Also substantial additional presheath theoretical work has been done with Fokker-Planck collision terms to account for diffusion of ions in velocity space. This makes the collisional presheath theory much more broadly applicable. The theoretical work is attached as an appendix.
PERSONNEL

In addition to the Principal Investigator, Dr Geoffrey L. Main, the following personnel are working or have worked on the grant research during the grant period: Gregory L. Ridderbusch, graduate student, David Hamm, graduate student, Roger Bouwmans, graduate student, Jeffrey P. Dansereau, graduate student, Warren Coleman, undergraduate.

FACILITIES

Under this report period one additional IBM PC/AT computer system has been acquired. The PC/ATs have proved invaluable and are used for FORTRAN programming and for document and graphics production. This report was produced using one of the PC/ATs and the laser printer.

FORTRAN code which cannot be efficiently run on the PC/ATs is uploaded from the PC/ATs to the campus CYBER 855 via a direct connection on the Georgia Tech Network.

PAPERS AND PUBLICATIONS

Since the preceding annual report (Aug. 31, 1986) two submitted papers have been accepted for publication: 1) "Asymptotically Correct Collisional Presheaths" by G. L. Main to appear in Physics of Fluids (expected in the June 1987 volume), and 2) "Effects of Emitter Sheath Ion Reflection and Trapped Ions on Thermionic Convertor Performance Using an Isothermal Electron Model" by G. L. Main and S. H. Lam to appear in IEEE trans. on Plasma Science (expected in the June 1987 volume).

In addition, a third journal paper has been submitted (to IEEE trans. on Plasma Science) entitled "A Three Scale Uniform Asymptotic Approximation to a Collisional Presheath and a Collisionless Double Sheath with Ion Reflection," by G. L. Main and G. L. Ridderbusch.
Current analytical models of Thermionic Convertors break the gap into two regions: the neutral plasma and the collisionless sheaths. Such models are reasonably accurate for the standard ignited mode but not necessarily accurate or even approximately true under the following conditions, all of which are of interest for advanced thermionic conversion:

1. Ion emitting or reflecting emitter and collector surfaces,
2. Non-ignited pulsed modes, and
3. Knudsen mode convertors in which the gap size is on the order of an ion mean-free-path or less.

The key to analytical work for these conditions is a tractable theory for a collisional non-Maxwellian plasma in the thermionic convertor. Under the present grant substantial theoretical progress has been made toward this goal as covered in the attached paper, "Asymptotically Collisional Presheaths."
INSTRUCTIONS FOR PREPARATION OF REPORT DOCUMENTATION PAGE

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If the report is in a foreign language and the title is given in both English and a foreign language, list the foreign language title first, followed by the English title enclosed in parentheses. If part of the text is in English, list the English title first followed by the foreign language title enclosed in parentheses. If the title is given in more than one foreign language, use a title that reflects the language of the text. If both the text and titles are in a foreign language, the title should be translated, if possible, unless the title is also the name of a foreign periodical. Transliterations of often used foreign alphabets (see Appendix A of MIL-STD-847B) are available from DTIC if possible, this set of terms should be selected so that the terms individually and as a group will remain UNCLASSIFIED without losing meaning. However, priority must be given to specifying proper subject terminology rather than making the set of terms appear "UNCLASSIFIED." Each term on classified reports must be immediately followed by its security classification, enclosed in parentheses.

For reference on standard terminology the "DTIC Retrieval and Indexing Terminology" DRIT-1979, AD-4068 500, and the DoD "Thesaurus of Engineering and Scientific Terms (TEST) 1968, AD-672 000, may be useful.

Block 12. Personal Author(s): Give the complete name(s) of the author(s) in this order: last name, first name, and middle name. In addition, list the affiliation of the authors if it differs from that of the performing organization.

List all authors. If the document is a compilation of papers, it may be more useful to list the authors with the titles of their papers as a contents note in the abstract in Block 19. If appropriate, the names of editors and compilers may be entered in this block.

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ANNUAL RESEARCH REPORT
AFOSR GRANT 85-0375

True Asymptotic Plasma - Sheath Matching with an
Asymptotically Correct Collisional Presheath

Principal Investigator: Geoffrey L. Main

School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, Georgia 303032

Submitted to

Air Force Office of Scientific Research
Bolling Air Force Base
Washington, D.C., 20332-6448

AFOSR technical Officer:
Dr. Robert Vondra

September, 1986
INTRODUCTION

Current analytical models of Thermionic Convertors break the gap into two regions: the neutral plasma and the collisionless sheaths. Such models are reasonably accurate for the standard ignited mode but not necessarily accurate or even approximately true under the following conditions, all of which are of interest for advanced thermionic conversion:

1. Ion emitting or reflecting emitter and collector surfaces,
2. Non-ignited pulsed modes, and
3. Knudsen mode convertors in which the gap size is on the order of an ion mean-free-path or less.

The key to analytical work for these conditions is a tractable theory for a collisional non-Maxwellian plasma in the thermionic convertor. Under the present grant substantial theoretical progress has been made toward this goal as covered in the attached paper, “Asymptotically Collisional Presheaths.”

SUMMARY OF PROGRESS

To this point under the grant, two thermionic convertor codes have been converted from PL1 into FORTRAN so that they may run on the Georgia Tech Cyber 990 and IBM PC-ATs.

The collisional presheath theory has been inserted into the sheath boundary conditions for the ignited mode thermionic convertor and the results are now being examined.

The collisional presheath theory has turned out to be unexpectedly general in nature, therefore we propose to broaden the research effort to develop a code which will handle the second and third areas mentioned in the introduction.
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PAPERS, PUBLICATIONS AND RESEARCH RESULTS p. 4
PERSONNEL

In addition to the Principal Investigator, Dr Geoffrey L. Main, the following personnel are employed:

Gregory L. Ridderbusch, graduate student,
David Hamm, graduate student,
Roger Bouwmanns, graduate student,
Jeffrey Dansereau, graduate student,
Warren Coleman, undergraduate.

FACILITIES

Under this grant two IBM PC/AT computer systems and a Talaris 800 laser printer have been acquired. The PC/ATs are used for FORTRAN programming and for document and graphics production. This report was produced using one of the PC/ATs and the laser printer.

FORTRAN code which cannot be efficiently run on the PC/ATs is uploaded from the PC/ATs to the campus CYBER 855 via a direct connection on the Georgia Tech Network.

PAPERS, PUBLICATIONS AND RESEARCH RESULTS

Under the grant period, two conference papers have been presented or are to be presented: “Asymptotically Correct Collisional Presheaths,” by G L. Main at the Gaseous Electronics Conference, in Monterey, California during October 1985 and “Pulsed Mode Thermionic Convertor Theory Incorporating a Collisional Presheath and a Collisionless Sheath” by G. L. Main and G. L. Ridderbusch at the IEEE conference on plasma science in Saskatoon, Saskatchewan, Canada during June 1986.

Three journal papers relating to this grant are under review: 1) “Asymptotically Cor-
rect Collisional Presheaths" by G. L. Main has been submitted to the Physics of Fluids and results directly from the research conducted under the grant, 2) "A General Solution Condition for Collisionless Sheaths" by G. L. Main and S. H. Lam has been submitted to the Journal of Plasma Physics, and 3) "Effects of Emitter Sheath Ion Reflection and Trapped Ions on Thermionic Convertor Performance Using an Isothermal Electron Model" by G. L. Main submitted to the IEEE Transactions on Plasma Science. These papers follow.

Furthermore, two more papers are nearing completion. The first, entitled "A Uniform Three Scale Asymptotic Solution for a Collisional Presheath and Collisionless Sheath with Ion Reflection," by G. L. Main and G. L. Ridderbusch is an extension of the first of the first paper that allows comprehensive and accurate treatment of most emitter-plasma structures. We believe it will be interest to far more plasma-electrode interactions than just the TEC. The second, as yet to be given a precise title, covers the results of our complete TEC theory (including the presheaths) and is authored by G. L. Main, G. L. Ridderbusch and D. Hamm.

Finally, two invited talks are to be given by G. L. Main on the TEC and its plasma physics at Cornell University in Ithaca, N.Y. on Oct. 1, 1986 and at the University of Central Florida in Orlando on Oct. 16, 1986.
True Asymptotic Plasma-Sheath Matching with an Asymptotically Correct Collisional Presheath

Geoffrey L. Main

Annual Research Report

FROM 9/1/86 to 8/31/87

87 September 30

5

Thermionic Energy Convertors, Plasmas, Sheaths, Presheaths

This report covers work done on plasma sheaths and presheaths as applicable to Thermionic Energy Convertors (TECs) during the period 1 September 1986 to 1 August 1987. A code modelling the TEC has been completed with the sheath and presheath work incorporated. Also additional work on presheaths with Ikker-Planck Collision terms has been done.
Dear Dr. Birkan:

Enclosed is the annual report for AFOSR grant 85-0375. I have included two rather lengthy appendices that cover this year’s work in detail and summarized the work and purposes in the body of the report.

Although the core of the research is on plasma sheaths and presheaths (plasma-electrode interactions), the main application is to Thermionic Energy Conversion.

I and other people concerned with Thermionic Energy Conversion as applicable to space power (for electric propulsion and other uses requiring large amounts of power in space) would like to meet with you to discuss basic research in the area. I will call you to try to arrange a meeting.

Sincerely,

Geoffrey L. Main
Assistant Professor
True Asymptotic Plasma - Sheath Matching with an Asymptotically Correct Collisional Presheath

Principal Investigator: Geoffrey L. Main

School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, Georgia 303032

Submitted to

Air Force Office of Scientific Research
Bolling Air Force Base
Washington, D.C., 20332-6448

AFOSR technical Officer:
Dr. Mitat Birkan

September, 1987
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INTRODUCTION

The work under this grant on plasma presheaths, which form a transition region between the collisionless electrode sheaths and the plasma, is directed toward the problems of the Thermionic Energy Convertor (TEC). Figure 1 shows a schematic of a TEC in a reactor core for space power applications and the basic physics. Cesium is put the gap between the emitter and collector for two purposes: first, to ionize and neutralize the space charge so that a useful electron current density can pass (10 - 100 amps/square cm), and second to reduce the electrode work functions by adsorption of cesium. Of the plasma physics of the the cesium filled gap of the TEC, the plasma-electrode interactions are the most significant part because these regions form boundary conditions which control the plasma density and temperatures of the entire gap. Thus the research under this grant has been directed toward the study of collisional presheaths which form the layer adjacent to an electrode on the order of one ion mean free path thick. However, the research pursued under this grant is not limited in applicability to TECs but is of interest to plasma-surface interactions in general. Other applications include electric propulsion where electrode erosion is a problem and not fully understood and more generally any plasma-surface interaction.

SUMMARY OF PROGRESS

During the past year, two lines of progress have been made: First, the work of the previous year on asymptotically correct collisional presheaths (which resulted in the paper of the same name, Phys. Fluids, June 1987) has been incorporated into a full time dependent code for the plasma dynamics of the TEC called PREDCOR. And, second, an approximate time dependent theory for presheath with Coulomb collisions (a Fokker-Planck collision term) has been developed.

The Time Dependent TEC Code. There are two reasons that the time dependent be-
havior of a TEC is of particular interest: first, the efficiency may be improved by periodically raising the plasma density with a voltage pulse applied to the TEC so that the plasma density need not be maintained by a high plasma electron temperature. It turns out that energy cost of creating the ions by a pulse of voltage is minimal, but creating them by maintaining a high plasma electron temperature (3000K) is expensive because the plasma electrons are cooled by the emitter electrons. Second, if the TEC can be pulsed in such a way as to reset the voltage over each cycle,

$$\int V \, dt = 0 \quad \text{over each cycle,}$$  \hspace{1cm} (1)

then the TEC can be inductively coupled to put out a much higher voltage. The raw TEC output voltage of about .7V with a current density of about 10 amps/square cm results in most current designs calling for the Thermionic Fuel Elements to be put in series flashlight style to increase the voltage. However, this arrangement requires the sheath insulators shown in figure 1 to separate to up 50 Volts from the liquid metal coolant along with other difficult electrical insulating problems in the reactor core. The inductive coupling concept eliminates all of these difficult insulating problems in the core. Rasor[1] shows the tremendous advantages of inductive coupling. Figure 2 shows the advantages of pulsing and the advantages of inductive coupling.

So far, we have studied the pulsing TEC without completely quenching the plasma and breaking it down again during each cycle. The results are that efficiency does not seem to increase and that we cannot reset the voltage during each cycle (the details of these results are contained in appendix A). These are not surprising results and lead us to conclude that for either efficiency improvement or inductive coupling that the plasma must be completely quenched during each cycle or that its conductivity must be vastly reduced (below what naturally occurs in the non-quenched oscillation). We are not yet able to model the complete quenching and breakdown of the plasma.

The quenching of the plasma is an inherently slow process because the ions must diffuse out to the electrodes. This has led to a new idea which we are studying in the final year of this grant: the addition of an electronegative gas to the cesium plasma so that the conductivity could be greatly reduced without diffusing out the cesium ions. The sheath and presheath theory for this is being developed.
The Fokker-Planck Collision Term Presheath Work. This work has been pursued to account for ion diffusion in velocity space (in the collisional presheath) which results from the preponderance of small angle collisions between charged particles. To our great surprise, we have tentatively reached the conclusion that the collisional presheath is necessarily a large angle collision phenomena. This means that even if the bulk of the plasma is dominated by small angle collisions as in a fully ionized plasma, that the collisional presheath transition to the collisionless sheath and wall is controlled by the minority of large angle collisions. We continue to study this result as it can have a very significant impact on the presheath structure. The details of this work are contained in appendix B, which is the thesis of Jeffrey P. Dansereau completed under this grant.

REFERENCES


PAPERS AND PUBLICATIONS

1. The paper entitled “Asymptotically Correct Collisional Presheaths,” by G L. Main, Phys. Fluids, June 1987, has been produced under this grant.

2. “A General Solution Condition for Collisionless Sheaths” by G. L. Main and S. H. Lam has been accepted by the Journal of Plasma Physics, and is to appear.


Schematic of Thermionic Convertors In Core

Thermionic Convertor Physics

Figure 1
Thermionic cells
without inductive coupling

\[ V \approx 50 \text{v} \]

50v insulation needed

Reactor core

Thermionic reactor with inductive coupling (from Rasor [1])

This condition is difficult to achieve:
low current density and large negative voltage

\[ \int V \, dt = 0 \]

Figure 2
APPENDIX A

INTRODUCTION

The time dependent behavior of a Thermionic Energy Converter (TEC) as part of an electrical circuit is analyzed using a computer code that models the neutral plasma region and the electrode boundaries. It has been shown [1] that the TEC will naturally oscillate with constant current when the loss of ions to the collector is large. When the TEC is operating with constant current, large amounts of current are being forced through the TEC to raise the plasma density through each cycle. This occurs when the plasma density is low and therefore resistance is high. This dissipates electrical energy since the electrical energy used to raise the density of the plasma is much larger than the energy required to raise the plasma density by ionization. It is anticipated that the performance of a TEC can be improved by connecting it to a pulsing circuit such that the current is not high during the raising of the plasma density.

The condition of constant current that is examined in [1] essentially means that the TEC is connected in series to an infinitely large inductor. Therefore, a circuit was examined in which a TEC is connected in series to an inductor and a resistor (see figure 1). The amount of inductance was varied to examine the changes in operating characteristics and performance of the TEC.

A proposed circuit for the TEC that combines pulsing with inductive coupling to the benefit of efficiency and power conditioning is given in [2]. The possibilities of this circuit were examined using the desired current profile as input to the computer code.

The code used to analyze the performance of the TEC (PREDCOR) uses fluid mechanics type equations to model the neutral plasma region and a fast numerical model of a three asymptotic analysis of the double emitter collisionless sheath-collisional presheath-neutral plasma which produces the ion distribution function in the plasma-electrode region. The result is a full time dependent model of the TEC plasma. The code is written in FORTRAN 77 and was run on an IBM PC-AT. A copy of the code is provided in Appendix A.

PREDCOR requires certain input conditions to model the behavior of the plasma in the TEC. The conditions for this work are:

- Emitter Temperature: 1700 K
- Collector Temperature: 900 K
- Emitter Work Function: 2.54 eV
- Collector Work Function: 1.63 eV
- Inter-Electrode Space: 0.254 mm
- Cesium Pressure: 2 torr
INDUCTOR CIRCUIT ANALYSIS

The circuit used in this analysis is shown in figure 1. It consists of a TEC connected in series to an inductor and a resistor. This circuit was chosen to examine the behavior of a TEC with varying amounts of inductance and comparing to the case of infinite inductance (constant current). This circuit allows the current to vary as the voltage varies. A resistance of 0.2 ohms was used for the analysis shown. This value was chosen to match the average output voltage under the condition of constant current. The value L shown in the plots is a normalized inductance used to simplify reporting of the data. The actual value of inductance in henrys is L multiplied by 0.016. The value 0.016 occurs from the electron characteristic time in the TEC which is 0.016 micro-seconds. The time shown is a nondimensional time with the electron characteristic time used to nondimensionalize the time.

The circuit was solved numerically using a central difference technique to give fourth order accuracy. The circuit was incorporated into PREDCOR as an external subroutine. PREDCOR gave the subroutine information at discrete time steps and the subroutine solved for the current in the circuit. It was necessary to greatly modify PREDCOR to allow the current to vary. It was necessary to have an initial current to solve the problem. In all the data an initial current of 4.0 amps per square centimeter was used. Solutions were obtained for values of L of 100, 500, 1000, 10000, 100000, and 1000000.

Figure 2 shows the variation of average output power of the TEC with varying inductance. It shows that as the inductance is increased the power output of the converter increases and as expected when L becomes larger and larger it approaches the performance of the TEC with constant current (infinite inductance). Figure 3 shows the voltage profiles of the elements in the circuits for L=1000 and figure 4 shows them for L=1000000. Examination of figures 3 and 4 along with figures 12 and 21, which show the current profiles for the two cases, indicates the reason that the power output increases as the inductance increases. The smaller inductance allows the current to change more rapidly than the high inductance which keeps the current essentially constant. Studying figure 3 with figure 12 shows that with the smaller inductance the current varies with the voltage such that when the voltage is low the current is high and when the voltage is high the current is low. This characteristic reduces the output power from the case where the current is constant.

Figures 5 thru 24 show the operating characteristics of the TEC for values of L of 100, 500, 1000, 10000, 100000, 1000000 and for the case of constant current. As expected, when the inductance gets larger the operating characteristics of the TEC approach those of the case of constant current. At smaller inductances the oscillations in voltage, current, and power are larger than they are at larger inductances. This is expected since the smaller inductances allow the characteristics to change more rapidly than they will at large inductances. An interesting result is that there appears to be a critical value of the inductance in
order for the TEC to be stable. This is expected since the operation of the TEC indicates a natural inductance in the converter. Figures 5, 6, and 7 show that operation of the converter is unstable (growing oscillations) at an L of 100 (1.6 henrys). Figures 8, 9, and 10 show the results for a L of 500 (8 henrys). This appears to be right on the edge of the stability limit as the oscillations begin to grow slightly toward the end of the run. Therefore, the critical inductance for the conditions run is slightly greater than 8 henrys. At an inductance lower than this the TEC will become unstable and will not operate. This critical value of inductance occurs because if the inductance is below a critical value, the changes that are allowed in the characteristics of the circuit are too rapid for the reaction time of the plasma in the TEC. The densities and temperatures in the plasma cannot change fast enough to keep up with what the circuit is attempting to allow. This shows that inductance must be part of any circuit used in operation with a TEC.
PULSING CIRCUIT ANALYSIS

An inductive output coupling circuit is proposed in [2]. In this analysis it was attempted to analyze the operation of this type of circuit using PREDCOR. To accomplish this, the desired current waveform was used as input to the program and the other operating characteristics determined from this. The desired current and voltage profiles are shown in figure 25.

Figures 26 thru 31 show attempts to achieve these profiles using PREDCOR and to compare the power output of these cases to that of constant current. Figure 25 shows that the desired waveforms are to have high current with a low voltage and a low current with a large negative voltage. In running the tests, instantaneous changes in the current are not possible, therefore the waveforms used in the analysis are not square waves but have a finite slope between maximums and minimums. It was not numerically possible to have the current as low as is desired in the proposed circuit therefore the current had to be held at a higher value for its minimums.

Figures 26, 27, 28, and 29 show four different cases that were run. These show that even though it is possible to force the voltage negative it was not possible to achieve the large negative voltages desired. This indicates that it may be difficult to shut down the TEC by using a quenching pulse [2].

The results obtained show that with the input current waveforms used the average power output is slightly less than it is with constant current (see figures 30 and 31).
CONCLUSIONS

The results presented here do not improve the power output of the TEC. However, they give indications of what is necessary to improve the performance of a TEC and show some interesting characteristics. One needs output characteristics such that the current is high when the voltage is high and the current low when the voltage is low to improve performance. The circuit with an inductor and a resistor in series with a TEC shows that the best performance is achieved with a very large inductance (constant current). The data also indicates that there is a critical value of inductance below which operation of a TEC is unstable. The results of this analysis indicate that a different circuit, possibly one with inductance, must be used to improve the performance of a TEC.

The second part of the analysis, though far from being conclusive, shows that it may be difficult to achieve the shutdown (no current flowing) of a TEC by using a quenching voltage pulse. The operating characteristics that such operation would provide are desirable to improve the performance of a TEC. A method to achieve these operating characteristics could be to mix an electronegative gas such as sulfur hexafluoride with the cesium vapor in the converter. The electronegative gas would absorb the free electrons in the plasma at low electron temperatures, effectively shutting the converter down. At higher temperatures the electrons would have enough energy to break free of the electronegative gas and the converter would be operating. The use of an electronegative gas in a TEC to improve the performance of the TEC is the subject of further study.
REFERENCES


FIGURE 1

INDUCTIVE CIRCUIT
PREDCOR RESULTS

POWER OUTPUT VS. INDUCTANCE

\[ \text{Log}(L) \]

\[ P_{\text{avg}} \text{ (W/sq cm)} \]

\[ 2.48 \quad 2.5 \quad 2.52 \quad 2.54 \quad 2.56 \quad 2.58 \quad 2.6 \quad 2.62 \quad 2.64 \quad 2.66 \quad 2.68 \quad 2.7 \quad 2.72 \quad 2.74 \]

\[ 2 \quad 4 \quad 6 \quad 8 \]

\text{FIGURE 2}
PREDCOR RESULTS
VOLTAGES IN CIRCUIT: L = 1000

Figure 3
PREDCOR RESULTS

VOLTAGES IN CIRCUIT: L = 1000000

Figure 4-
PREDCOR RESULTS

R = 0.2 : L = 100.0

Figure 5
PREDCOR RESULTS

R = 0.2 : L = 100.0

CURRENT (amps/sq cm)

TIME (Thousands)

FIGURE 6
PREDCOR RESULTS

$R = 0.2 : L = 500$

Figure 8
PREDCOR RESULTS

$R = 0.2 : L = 500$

CURRENT (amps/sq cm)

0 2 4
(Thousands)

TIME

FIGURE 9
PREDCOR RESULTS

$R = 0.2 : L = 500$

Figure 10
PREDCOR RESULTS

\[ R = 0.2 : L = 1000 \]

**Figure 11**
PREDCOR RESULTS

$R = 0.2 : L = 1000$

CURRENT (amps/sq cm)

(Thousands) TIME

FIGURE 12
PREDCOR RESULTS

$R = 0.2 : L = 1000$

Figure 13
PREDCOR RESULTS

R = 0.2 : L = 10000

VOLTAGE

(Thousands)

TIME

FIGURE 14
PREDCOR RESULTS

$R = 0.2 : L = 10000$

![Graph of CURRENT vs TIME]

**Figure 15**
PREDCOR RESULTS

R = 0.2 : L = 10000

Figure 16
PREDCOR RESULTS

$R = 0.2 : L = 100000$

**Figure 17**

Graph showing voltage over time with a logarithmic scale for voltage.
PREDCOR RESULTS

\( R = 0.2 : L = 100000 \)

**Figure 18**
PREDCOR RESULTS

$R = 0.2 : L = 100000$

Figure 19
PREDCOR RESULTS

$R = 0.2 : L = 1000000$

**Figure 20**
PREDCOR RESULTS

R = 0.2 : L = 1000000

CURRENT (amps/sq cm)

TIME (Thousands)

Figure 21
PREDCOR RESULTS

$R = 0.2 : L = 1000000$

Figure 22
PREDCOR RESULTS
CONSTANT CURRENT 4.0

TIME

(Figures 23)
PREDCOR RESULTS
CONSTANT CURRENT 4.0

Figure 29
Figura 25

Desired Voltage and Current Waveforms in Pulsing-Inductively Coupled Circuit
PREDCOR RESULTS: VOLTAGE AND CURRENT

731A: Pout = 1.76

Figure 26
PREDCOR RESULTS: VOLTAGE AND CURRENT

$84C: P_{out} = 2.53$

Figure 27
PREDCOR RESULTS : VOLTAGE AND CURRENT

87A : Pout = 2.53

Figure 28
PREDCOR RESULTS: VOLTAGE AND CURRENT

85A: Pout = 2.90

Figure 29
PREDCOR RESULTS

CONSTANT CURRENT 4.125

Figure 30
PREDCOR RESULTS

CONSTANT CURRENT 4.5 : Pout = 3.02

Figure 3
APPENDIX B

AN ANALYTIC - NUMERICAL SCHEME FOR A COLLISIONAL
FOKKER - PLANCK TIME DEPENDENT SHEATH - PRESHEATH
STRUCTURE

A THESIS
Presented to
The Faculty of the Division of Graduate Studies
By
Jeffrey Paul Dansereau

In Partial Fulfillment
of the Requirements for the Degree
Master of Science in Mechanical Engineering

Georgia Institute of Technology
September, 1987
AN ANALYTIC - NUMERICAL SCHEME FOR A COLLISIONAL
FOKKER - PLANCK TIME DEPENDENT SHEATH - PRESHEATH
STRUCTURE

Approved:

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Date approved by chairman ______
ACKNOWLEDGMENTS

I would like to thank Geoffrey L. Main for all of his help and encouragement during the development of this project. This work was supported by the Air Force Office of Scientific Research grant 85-0375.
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NOMENCLATURE

$a$ - Inverse square of the particle thermal velocity
$a_1, a_2, \ldots$ - Coefficients of the potential structure
$a, b$ - Integral limits
$A$ - First derived Fokker Planck collision function
$B$ - Second derived Fokker Planck collision function
$d_i(r)$ - An $m \times 1$ matrix
$D(r)$ - An $m \times 1$ solution matrix
$e$ - Electron charge
$E$ - Electric field strength
$f$ - Particle distribution function in velocity space
$F$ - Nondimensional particle distribution function in velocity space
$g$ - Collision function
$h$ - Grid spacing width
$k$ - Boltzmann's constant
$L$ - System dimension
$m$ - Number of points in velocity space
$m$ - Particle mass
$M$ - Reduced mass
$n$ - Plasma density
$n_R$ - Reference density
$q$ - Elementary charge
$R$ - Radial velocity component
$t$ - Time
$T_i$ - Ion temperature
$T_e$ - Electron temperature
$T(r)$ - An $m \times m$ matrix
$U$ - Potential
$v$ - Velocity vector
$v_i(r)$ - An $m \times 1$ matrix
$V(r)$ - An $m \times 1$ solution matrix
$x$ - Position vector
$x$ - Axial position component
$z$ - Axial velocity component
$Z$ - Ionization level
$Z$ - Nondimensional axial velocity component
$\alpha$ - Potential gradient across the plasma
$\beta$ - Exponential coefficient of the presheath rise
$\Gamma$ - Fokker - Planck coefficient
$\Delta U$ - Potential expansion parameter
$\varepsilon_0$ - Permittivity of free space
$\xi, \eta, \xi$ - Variables of integration
$\theta$ - Velocity rotation angle in cylindrical coordinates
$\lambda_D$ - Debye length
$\lambda_i$ - Mean free path
$\Lambda$ - Coulomb logarithm
$\tau$ - Nondimensional time
$\Phi$ - Electric potential
$\varphi$ - Nondimensional coefficients of the potential structure

Subscripts:

$c$ - Collision term
$e$ - Electron
$i$ - Ion
$n$ - Order of expansion

Superscripts:

$i$ - Particle species with which collisions occur
$*$ - Nondimensional
SUMMARY

The Maxwellian sources and charge exchange terms used to model particle interactions in current presheath models do not represent the Coulomb collisions taking place in fully ionized plasmas. These models approximate the collisional effects in the presheaths of partially ionized plasmas but are used to implicitly extrapolate the interesting case of fully ionized plasmas. The present study uses a Fokker-Planck collision term which models the limit of the small angle Coulomb collisions that occur in fully ionized plasmas. Normally these small angle collisions dominate the particle interactions of fully ionized plasmas. The Boltzmann equation coupled with the Fokker-Planck term, and the Poisson equation have been expanded using an exponential asymptotic technique. These equations have been solved numerically to determine the time dependent evolution of the presheath. The results presented show the presheath potential structure and particle distribution in velocity space. The model produces a self-consistent and accurate potential structure. The particle velocity distribution in the presheath has the correct acceleration of ions toward the wall but because the Fokker-Planck collision term only models the limit of small angle collisions it is unable to clear the particle distribution of returning ions. The collisional processes become dominated by the effects of the large angle collisions as the Debye sheath edge is approached. This study has found that a presheath model which describes the Coulomb collisions occurring in a fully ionized plasma must account for both the small angle and the large angle particle collisions to explain the clearing out of returning ions that must exist for the transition to an absorbing wall.
CHAPTER I

INTRODUCTION

The interaction of man and plasma, in some form, exists at almost all levels of society. A plasma is an ionized gas that has a collective behavior in an electromagnetic field. Plasmas exist in everyday devices like fluorescent lights, neon signs, and electric arc welders. An understanding of the basic behavior and interaction of plasmas is essential to the advancement of all current plasma applications and to the discovery of new applications. This thesis involves the study of how a plasma interacts with the walls and surfaces with which it comes in contact.

Why is it important to understand plasma-wall interactions? Two basic reasons answer this question. First, a plasma has a strong effect on any surfaces it comes in contact with. The high temperature plasma can erode or destroy any surface quickly pitting and changing a wall which may need to maintain a particular profile or surface condition. Secondly, the wall affects the characteristics of the plasma. A surface can have a profound effect on the plasma depending on the amount and rate at which it can absorb energy. Examples of situations in which plasma-wall interactions are of importance include:

- Diverter plates in magnetic confinement fusion reactors.
- The rails in a plasma rail gun.
- Any body (like the space shuttle) upon reentry to the atmosphere.
- Plasma switches.
- Plasma etching.
Almost every other use or occurrence of plasmas.

This study is primarily applicable to fully ionized plasmas. The hot temperatures necessary to produce fully ionized plasmas occur only in situations like on the surfaces of diverter plates in Tokamak fusion reactors.

The development of a mathematical model to represent the plasma-wall interaction region, or sheath, and a numerical solution to this model is the focus of this thesis. An understanding of the interaction between the plasma and the wall is achieved with a time dependent solution to the sheath region. If the potential structure and the particle velocity distributions are known for every location in the sheath then the energy going into the surface can be determined. In this way the results of this study can be used as a boundary condition for problems involving plasma characteristics and for problems involving the surface physics of plasma devices.
CHAPTER II

BACKGROUND

A plasma will naturally maintain itself in a neutral and field free state. Application of forces and processes that try to alter the equilibrium are resisted by the plasma. A surface within a plasma that is not at the same potential as the plasma will be shielded from the remainder of the plasma by a sheath. The outer edge of this sheath is nearly at the plasma potential. Bohm\cite{1} first came up with a criterion to determine the extent of the sheath. Bohm modeled the sheath region as completely collisionless. He also considered that the transition region from this collisionless sheath to the plasma was too small to be important.

More recent work has been done to describe this transition, or presheath, region. Self\cite{2} has an exact solution to the sheath equation and has shown that the collisionless sheath makes a transition directly to the neutral plasma in the limit as $\frac{\lambda_D}{L} \to 0$, where $\lambda_D$ is the Debye length and $L$ is the plasma dimension. Emmert et al.\cite{3} has determined a presheath structure based on the assumption of a Maxwellian source of ions to model the particle collisions. The solution to this model shows that the transition point from the sheath to the presheath has a finite electric field strength. Bissell and Johnson\cite{4} have performed a similar solution using a Maxwellian source of ions. In contrast to Emmert et al., Bissell and Johnson have found that the electric field strength becomes infinite at the sheath edge. This solution agrees with the fluid and cold ion models. In a recent paper Bissell\cite{5} shows that Emmert obtained a finite electric field strength because the Maxwellian source term used produced no ions at the point of zero velocity. Bissell and Johnson used a more realistic
Maxwellian source that produced ions at the zero point in velocity space for their solution.

Another approach to the problem involves the use of a charge exchange term to model the particle collisions. Riemann\textsuperscript{[6]} has produced results using this technique. In a recent paper by Main\textsuperscript{[7]} a charge exchange model is used to obtain a solution to the presheath potential structure and particle distribution. This model involves an asymptotic approximation of the plasma equations. The Boltzmann and Poisson equations are asymptotically expanded and then solved analytically when combined with a charge exchange model of the particle collisions.

All of these sheath and presheath solutions have modeled particle collisions by large instantaneous changes in particle velocity. These models do not represent the Coulomb collisions occurring in the presheath of a fully ionized plasma.

The current study extends the asymptotic solution presented by Main\textsuperscript{[7]} to include a Fokker - Planck collision term instead of the charge exchange term. Unlike the previous collision terms used, the Fokker - Planck term describes the Coulomb collisions that exist within a fully ionized plasma. The addition of the Fokker - Planck term necessitates the use of numerical techniques, rather than analytical techniques, to obtain a solution. In using the Fokker - Planck term the collision processes are being modeled directly. The model developed obtains the time dependent evolution of the presheath for a fully ionized plasma.
CHAPTER III

MODEL FORMULATION

3.1 Concepts

In order to have a complete understanding of the problem at hand certain concepts need to be presented which will help in understanding the overall structure of the model.

1) Debye Length ($\lambda_D$) - The shielding distance beyond which the particle charge effect is weak. This is the natural charge separation distance. Negatively charged particles become surrounded by positively charged particles and vice versa, thus, balancing the overall charge at any point (see figure 3.1). There is a point beyond which a particle is not effected by the specific charge but responds to the influence of the entire plasma. The thermal effects in the plasma become dominant over the electric field strength.

2) Mean Free Path ($\lambda_i$) - The average distance a particle travels before its trajectory has been altered by ninety degrees. The mean free path is a function of the density of the plasma. The denser the plasma the shorter the mean free path. For the plasma under consideration in this study $\lambda_i >> \lambda_D$.

3) The coordinate system used to describe the plasma - The coordinate system used in the model is known phase space. In this system any point is described using three position coordinates and three velocity coordinates. Any orthogonal coordinate system, cartesian, cylindrical, spherical, can be used to describe both the position and the velocity components.
4) Distributions and Distribution Functions - Plasmas are studied in a collective sense. The motion of the entire plasma and not individual particles is described by the model. Therefore, the velocity of the plasma at any given location must be described by a distribution. The distribution function describes the overall particle velocity distribution.

5) Potential - In a plasma the wall potential is greater than the neutral plasma potential. The lighter, thus, faster electrons are absorbed by the wall faster than the heavier and slower ions. A net positive charge exists near the wall, increasing the potential (see figure 3.2). The potential at the physical interface between the wall and the plasma is dependent on the rate at which ions are absorbed by the surface. In this study \( U = -e\Phi \) where \( e \) is the electron charge and \( \Phi \) is electric potential in electron volts so that \( U \) has units of energy. The addition of the negative sign defines potential in the reverse of the usual sign convention so that increasing potential repels electrons.

6) Collision Possibilities using the Fokker-Planck Collision term - To describe the overall structure of the sheath the various collision possibilities must be included in a comprehensive model. The Fokker-Planck term describes the four major collision possibilities.

1) Ion - Ion
2) Ion - Electron
3) Electron - Ion
4) Electron - Electron

The collision model does not take into account three body collisions. Three body collisions are very rare, as such, the model is not hampered by the lack of
terms to describe these collisions.

3.2 Wall Region Model

The model of the Plasma - Wall region can be broken into three areas.

1) Neutral Plasma Region \( O(L) \) - The neutral plasma region represents the majority of the system and can be considered to have a physical width that is on the order of the overall dimension of the system, \( L \). This region is considered to be fully collisional. The velocity distribution is near Maxwellian and as such can be modeled by fluid type equations (see figure 3.3).

2) Debye Sheath Region \( O(\lambda_D) \) - This region is a very thin area directly adjacent to the wall. Its width is considered to be on the order of a Debye length and since \( \lambda_i \gg \lambda_D \) no collisions are expected in this region. This collisionless sheath was first modeled by Langmuir\(^8\) and Bohm\(^1\) and is considered very well known and understood.

3) Collisional Presheath Region \( O(\lambda_i) \) - This is a transition region between the collisional neutral plasma and the collisionless Debye sheath region. It is considered to have a physical width on the order of a mean free path. Therefore, collisions are expected but at the same time the region cannot be considered fully collisional.

The potential must transition from a lower level in the neutral plasma to a higher level at the wall. The goal of this study has been to obtain a time dependent model of the evolution of the presheath region which asymptotically approaches the known potential in both the neutral plasma and in the Debye sheath region.

In order to show the validity of the three region model an example of Debye sheath width in relation to the overall wall region is appropriate. For this example
average hydrogen fusion plasma characteristics have been assumed:

\[ T_i = 10^5 K \]
\[ n_0 = 10^{20} m^{-3} \]

If the Debye length within this plasma is calculated an order of magnitude value for the Debye sheath width is determined. An appropriate equation for the Debye length in meters is:\[^9\]

\[ \lambda_D = 69 \left( \frac{T_i}{n_0} \right)^{\frac{1}{2}} \quad (3.1) \]

From this equation:

\[ \lambda_D = 2.18 \times 10^{-6} m \]

The overall sheath width is on the order of a mean free path. An appropriate equation for the mean free path in meters is:\[^10\]

\[ \lambda_i = 1.2 \times 10^{-4} \frac{1}{Z^2} \left( \frac{T_i}{\epsilon} \right)^2 \left( \frac{n_i}{10^{20}} \right)^{-1} \quad (3.2) \]

For a singly ionized plasma \( Z = 1 \). Using this equation and the above example plasma characteristics the mean free path can be calculated.

\[ \lambda_i = 1.14 \times 10^{-2} m \]

This is an order of magnitude estimate value for the width of the entire sheath region. Since the Debye sheath width is on the order of a Debye length it can be seen that the collisionless sheath is very thin in comparison with the entire wall region.

In order to obtain an idea of the importance of the electric field in the wall region an order of magnitude analysis is useful. The magnitude of the electric field is proportional to the thermal energy per length scale.

\[ E \sim \frac{kT_i}{x} \quad (3.3) \]
In the sheath region the length scale is the Debye length.

\[ E \sim \frac{kT_i}{\lambda_D} \] (3.4)

Therefore, in this region the electric field is very significant since \( \lambda_D \) is very small. The collisional effects are small in comparison, and can be neglected.

In the presheath region the length scale is the mean free path.

\[ E \sim \frac{kT_i}{\lambda_i} \] (3.5)

Therefore, the electric field strength is on the order of the collisional effects making both important factors within this region.

In the neutral plasma region the length scale is the overall system dimension.

\[ E \sim \frac{kT_i}{L} \] (3.6)

Therefore, the electric field is very weak and can be neglected in comparison with the collisional effects.

### 3.3 Presheath Model

#### 3.3.1 Equations Describing the Collective Behavior of a Plasma

The primary equation used to describe the behavior of a plasma is the Boltzmann equation. The Boltzmann equation represents the collective motion of many charged particles moving in an electromagnetic field\[11\].

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{1}{m} \mathbf{E} \cdot \nabla \mathbf{v} = \left\{ \frac{\partial f}{\partial t} \right\}_c
\] (3.7)

Where the + sign is for ion particles and the - sign is for electrons. In the Boltzmann equation \( f \) is the particle distribution function, and is defined such that \( n = \int_{-\infty}^{\infty} f \, dv \).
The quantity \( t \) is time, \( v \) is a velocity vector, \( x \) is a position vector, \( m \) is the particle mass, and \( U \) is the potential. The term on the right hand side represents particle collisions and can take on various forms depending on the model being used.

Another important equation in describing the behavior of a plasma is the Poisson equation. The Poisson equation is the elementary definition of potential as the collective effect of charged particles on a point\(^7\).

\[
\frac{d^2U}{dx^2} = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_i(v, \Delta U) \, dv - \int_{-\infty}^{\infty} f_e(v, \Delta U) \, dv \right] \tag{3.8}
\]

Where \( q \) is the elementary charge, the subscript \( i \) refers to ions and the subscript \( e \) refers to electrons. The term in brackets is the ion - electron density difference. The potential is the driving force in the Boltzmann equation. The Poisson equation relates the potential to the particle distribution.

To complete the set of equations necessary for a full description of the presheath a collision term must be chosen to model the particle interactions. This study uses the Fokker - Planck collision term to model the particle collisions. The Fokker - Planck term represents the right hand side of the Boltzmann equation\(^11\).

\[
\left\{ \frac{\partial f}{\partial t} \right\}_e = \Gamma \left[ - \frac{\partial}{\partial v_i} \left( m \frac{\partial}{\partial v_i} \nabla^2 g \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_k} \left( f \frac{\partial^2 g}{\partial v_i \partial v_k} \right) \right] \tag{3.9}
\]

Where,

\[
\Gamma = \frac{\alpha^2 q^2 \ln \Lambda}{4\pi \varepsilon_0 m^2} \tag{3.10}
\]

\[
g(v) = \int f(v') \mid v - v' \mid dv' \tag{3.11}
\]

\[
M = \frac{mm'}{m + m'} \tag{3.12}
\]

Where \( M \) is called the reduced mass. No superscript refers to the particle species undergoing the collisions and the superscript \( i \) refers to the particle species with which the collision occurs.
The Fokker–Planck equation describes the Coulomb collision between two charged particles. Certain restrictions and assumptions are made when using the Fokker–Planck collision term. First, it best describes fully ionized plasmas. The collision term models charged particle interaction and is most accurate for plasmas with few neutrals. This situation occurs only on the hottest of plasma surfaces like the divertor plates in Tokamak fusion reactors.

The second restriction involves the type of collisions that the Fokker–Planck term models. The overwhelming majority of particle collisions lead to only small deflections in the particle trajectories. The Fokker–Planck term describes the limit of these small angle deflections. Finally, the model does not take into account three body, and higher order, collisions.

3.3.2 Solution Conditions

The three equations presented in the previous section in conjunction with the asymptotic forms of potential and velocity distribution provide the necessary information to determine the presheath structure if two additional conditions are met.

First, if the equations are written in cylindrical coordinates the particle velocity distribution is axially symmetric. There is no theta, \( \theta \), dependence of the velocity distribution. Cylindrical coordinates are used for both the velocity and the position. The \( z \) direction is perpendicular to the wall (see Figure 3.4) with the positive direction being defined into the wall. The coordinates \( R, \theta, z \) have been used in velocity space for convenience.

The second condition for a solution to these equations involves an assumption of the particle velocity distribution parallel to the surface. For this model the radial velocity distribution has been assumed to take the form of a Maxwellian distribution. In addition, the temperature in the radial direction has been assumed
to be uniform and constant at all 'x' locations. Thus, the radial velocity distribution is constant for any position. Figure 3.5 is a schematic of these conditions. Note that at any 'x' location and rotation angle, \( \theta \), the radial velocity distribution is constant and follows a Maxwellian distribution. This represents the conditions of uniform temperature and axial symmetry, throughout the wall region. Figure 3.5 also shows a representation of the point of no returning ions. This is the point where the presheath transitions to the collisionless sheath.

The conditions of uniform temperature and radial Maxwellian distribution although good approximations are not exact models of the real situation.

The overall problem reduces to one dimension, the \( z \) direction, with the above conditions. The \( \theta \) dependence having been removed by the axial symmetry and the radial dependence having been removed by the Maxwellian assumption. This one dimensional problem can be solved by straightforward numerical techniques.

### 3.3.3 Expansion of the Boltzmann Equation

The presheath model involves the expansion of the potential and velocity into asymptotic approximations. The potential is assumed to follow an exponential asymptotic form.

\[
U = U_0 + a_1 \Delta U + a_2 \Delta U^2 + \cdots
\]  

(3.13) 

where

\[
U_0 = \alpha x \quad \text{and} \quad \Delta U = e^{\delta x}
\]  

(3.14) 

\( a_1, a_2 \ldots \) are parameters which describe the potential structure. Alpha, \( \alpha \), is non-zero for a non-zero potential gradient in the neutral plasma. \( \Delta U \) is called the potential expansion parameter.

The particle distribution in velocity space is a function of potential and can be
similarly expanded.

\[ f(v, \Delta U) = f_0(v) + \Delta U f_1(v) + \Delta U^2 f_2(v) + \cdots \]  

The Boltzmann equation can be written as

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{1}{m} \frac{\partial}{\partial v} \left( \frac{\partial f}{\partial x} \right) = \left\{ \frac{\partial f}{\partial t} \right\}_c \]  

where \( \pm \) sign is respectively positive for ion particles and negative for electrons. The potential \( U \) has units of energy and is defined as shown in figure 3.3 such that \( U = -e\Phi \) where \( \Phi \) is electric potential. The Boltzmann equation can be expanded using equations 3.13 and 3.14. In addition since the solution is one dimensional in velocity space the velocity derivatives reflect only the \( 'z' \) direction. The following expansions are used.

\[ \frac{\partial f}{\partial t} = \frac{\partial f_0}{\partial t} + \Delta U \frac{\partial f_1}{\partial t} + \Delta U^2 \frac{\partial f_2}{\partial t} + \cdots \]  

\[ \frac{\partial f}{\partial (\Delta U)} = f_1 + 2\Delta U f_2 + 3\Delta U^2 f_3 + \cdots \]  

\[ \frac{\partial (\Delta U)}{\partial x} = \beta \Delta U \]  

\[ \frac{\partial U}{\partial x} = \alpha + \beta a_1 \Delta U + 2\beta a_2 \Delta U^2 + 3\beta a_3 \Delta U^3 + \cdots \]  

\[ \frac{\partial f}{\partial x} = \frac{\partial f_0}{\partial x} + \Delta U \frac{\partial f_1}{\partial x} + \Delta U^2 \frac{\partial f_2}{\partial x} + \cdots \]  

Using these expansions the Boltzmann equation can be broken down by order in \( \Delta U \) assuming the collision term can be likewise broken down:

\[ 1: \quad \frac{\partial f_0}{\partial t} \pm \frac{\alpha}{m} \frac{\partial f_0}{\partial x} = \left\{ \frac{\partial f}{\partial t} \right\}_c \]  

\[ \Delta U: \quad \frac{\partial f_1}{\partial t} + \beta z f_1 \pm \frac{\alpha}{m} \frac{\partial f_1}{\partial x} \pm \beta a_1 \frac{\partial f_0}{\partial x} = \left\{ \frac{\partial f}{\partial t} \right\}_c \]  

\[ \Delta U^2: \quad \frac{\partial f_2}{\partial t} + 2\beta z f_2 \pm \frac{\alpha}{m} \frac{\partial f_2}{\partial x} \pm \beta a_1 \frac{\partial f_1}{\partial x} \pm \frac{2\beta a_2}{m} \frac{\partial f_0}{\partial x} = \left\{ \frac{\partial f}{\partial t} \right\}_c \]  

\[ \vdots \]

\[ \Delta U^n: \quad \frac{\partial f_n}{\partial t} + n\beta z f_n \pm \frac{\alpha}{m} \frac{\partial f_n}{\partial x} \pm \beta a_1 \frac{\partial f_{n-1}}{\partial x} \pm \frac{2\beta a_2}{m} \frac{\partial f_{n-2}}{\partial x} \pm \cdots \pm \frac{(n-1)\beta a_{n-1}}{m} \frac{\partial f_1}{\partial x} \pm n\beta a_n \frac{\partial f_0}{\partial x} = \left\{ \frac{\partial f}{\partial t} \right\}_c \]  

\[ \Delta U^n \]
The above exponential expansion is the only known way to break apart the Boltzmann equation. To complete the expansion the Poisson and Fokker-Planck terms must be similarly broken down.

At this point it is appropriate to nondimensionalize the expansion of the Boltzmann equation. The following nondimensional quantities have been used.

\[ F_i = \frac{f_i}{n_R a^{3/2}} \quad \text{for} \quad 0 \leq i \leq n \]  \hspace{1cm} (3.23a)
\[ \tau = \frac{n_R a}{a^{1/2}} t \]  \hspace{1cm} (3.23b)
\[ Z = \sqrt{a'} z \]  \hspace{1cm} (3.23c)
\[ \varphi_{-1} = \frac{\alpha a}{mn_R} \]  \hspace{1cm} (3.23d)
\[ \varphi_0 = \frac{\beta}{n_R} \]  \hspace{1cm} (3.23e)
\[ \varphi_i = \frac{\beta a a}{mn_R} \Gamma \quad \text{for} \quad 1 \leq i \leq n \]  \hspace{1cm} (3.23f)

Where \( n_R \) is a reference density, \( a \) and \( a' \) are the inverse square of the thermal velocities of the colliding particles, \( a = \frac{m}{2kT} \) and \( a' = \frac{m'}{2k'T'} \). The quantities \( \varphi_n \) represent the potential structure of the presheath. \( F_i \) is nondimensionalized such that \( 1 = \int_{-\infty}^{\infty} F dZ \). Using these nondimensionalizations, the expansion of the Boltzmann equations becomes

\[ \frac{\partial F_0}{\partial \tau} + \varphi_{-1} \frac{\partial F_0}{\partial Z} = \left[ \left\{ \frac{\partial F}{\partial \tau} \right\}_c \right]_1 \]  \hspace{1cm} (3.24a)
\[ \Delta U : \frac{\partial F_1}{\partial \tau} + \varphi_0 Z F_1 \pm \varphi_1 \frac{\partial F_1}{\partial Z} = \left[ \left\{ \frac{\partial F}{\partial \tau} \right\}_c \right]_{\Delta U} \]  \hspace{1cm} (3.24b)
\[ \Delta U^2 : \frac{\partial F_2}{\partial \tau} + 2\varphi_0 Z F_2 \pm \varphi_1 \frac{\partial F_1}{\partial Z} \pm \varphi_1 \frac{\partial F_1}{\partial Z} \pm 2\varphi_2 \frac{\partial F_0}{\partial Z} = \left[ \left\{ \frac{\partial F}{\partial \tau} \right\}_c \right]_{\Delta U^2} \]  \hspace{1cm} (3.24c)
\[ \vdots \]
\[ \Delta U^n : \frac{\partial F_n}{\partial \tau} + n\varphi_0 Z F_n \pm \varphi_{-1} \frac{\partial F_n}{\partial Z} \pm \varphi_1 \frac{\partial F_{n-1}}{\partial Z} \pm 2\varphi_2 \frac{\partial F_{n-2}}{\partial Z} \pm \cdots \pm (n-1)\varphi_{n-2} \frac{\partial F_2}{\partial Z} \pm n\varphi_0 \frac{\partial F_0}{\partial Z} = \left[ \left\{ \frac{\partial F}{\partial \tau} \right\}_c \right]_{\Delta U^n} \]  \hspace{1cm} (3.24d)
3.3.4 Expansion of the Poisson Equation

The Poisson equation,

\[
\frac{d^2 U}{dz^2} = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_i(v, \Delta U) \, dv - \int_{-\infty}^{\infty} f_e(v, \Delta U) \, dv \right]
\]

is broken down using the same technique as the Boltzmann equation. Since:

\[
\frac{\partial^2 U}{\partial z^2} = \beta^2 a_1 \Delta U + 4\beta^2 a_2 \Delta U^2 + 9\beta^2 a_3 \Delta U^3 + \cdots
\]

\[
\int_{-\infty}^{\infty} f(v, \Delta U) \, dv = \int_{-\infty}^{\infty} f_0(v) \, dv + \int_{-\infty}^{\infty} f_1(v) \, dv + \Delta U \int_{-\infty}^{\infty} f_2(v) \, dv + \cdots
\]

Using these expansions the Poisson equation can be broken down by order in \( \Delta U \).

\[
1: \quad 0 = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_0(v) \, dv - \int_{-\infty}^{\infty} f_{e0}(v) \, dv \right]
\]

\[
\Delta U: \quad \beta^2 a_1 = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_1(v) \, dv - \int_{-\infty}^{\infty} f_{e1}(v) \, dv \right]
\]

\[
\Delta U^2: \quad 4\beta^2 a_2 = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_2(v) \, dv - \int_{-\infty}^{\infty} f_{e2}(v) \, dv \right]
\]

\[
\Delta U^n: \quad n^2 \beta^2 a_n = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_{in}(v) \, dv - \int_{-\infty}^{\infty} f_{ein}(v) \, dv \right]
\]

The assumption of Boltzmann electrons is made to enable the numerical calculations to proceed with time steps on the order of an ion characteristic time. In the asymptotic presheath the Boltzmann electron assumption becomes

\[
n_e = n_0 e^{-[a_1 \Delta U + a_2 \Delta U^2 + \cdots + a_n \Delta U^n]} e^{\frac{1}{2e}}
\]

where \( n_e \) is the electron density, \( T_e \) is the electron temperature, and \( n_0 \) is the electron density in the asymptotic presheath at \( \Delta U = 0 \). Expansion of (3.28) in terms of \( \Delta U \) yields

\[
n_e = \left[ n_0 + \Delta U \left( -\frac{a_1}{kT_e} n_0 \right) + \Delta U^2 \left( -\frac{a_2}{kT_e} + \frac{1}{2} \frac{a_1^2}{(kT_e)^2} \right) n_0 \right] + \Delta U^3 \left( -\frac{a_3}{kT_e} + \frac{a_1 a_2}{(kT_e)^2} - \frac{1}{6} \frac{a_1^3}{(kT_e)^3} \right) n_0 + \cdots
\]
With the assumption that $\lambda_D \ll \lambda_i$, the Poisson equation reduces to equating electron and ion densities in order of $\Delta U$. The Poisson and Boltzmann electron equations are nondimensionalized in the same manner as the Boltzmann equation. The nondimensionalized Poisson equation (3.27a-d) is combined with the Boltzmann electron equation (3.28) to become

$$\int_{-\infty}^{\infty} F_0 dZ = \eta e^{\left[ \phi_1 + \ldots + \phi_n \right]} \frac{1}{\phi_0} \left( \frac{2T_i}{T_e} \right)$$

(3.30a)

$$\int_{-\infty}^{\infty} F_1 dZ = \eta e^{\left[ \phi_1 + \ldots + \phi_n \right]} \frac{1}{\phi_0} \left[ - \frac{\phi_1}{\phi_0} \left( \frac{2T_i}{T_e} \right) \right]$$

(3.30b)

$$\int_{-\infty}^{\infty} F_2 dZ = \eta e^{\left[ \phi_1 + \ldots + \phi_n \right]} \frac{1}{\phi_0} \left[ - \frac{\phi_2}{\phi_0} \left( \frac{2T_i}{T_e} \right) + \frac{1}{2} \left( \frac{\phi_1}{\phi_0} \right)^2 \left( \frac{2T_i}{T_e} \right)^2 \right]$$

(3.30c)

$$\int_{-\infty}^{\infty} F_3 dZ = \eta e^{\left[ \phi_1 + \ldots + \phi_n \right]} \frac{1}{\phi_0} \left[ - \frac{\phi_3}{\phi_0} \left( \frac{2T_i}{T_e} \right) + \frac{\phi_1 \phi_2}{\phi_0 \phi_0} \left( \frac{2T_i}{T_e} \right)^2 - \frac{1}{6} \left( \frac{\phi_1}{\phi_0} \right)^3 \left( \frac{2T_i}{T_e} \right)^3 \right]$$

(3.30d)

where $\eta = \frac{n_i}{n_e}$ and may be specified as a function of time. This equation can be used to solve for the potential structure at each time step. $T_i$ is the ion temperature and $T_e$ is the electron temperature.

### 3.3.5 Expansion of the Fokker - Planck Term

The Fokker - Planck term must also be expanded in order of $\Delta U$ but first must be put into cylindrical coordinates. In addition, the assumption of axial symmetry must be accounted for in the term. This can be accomplished by expanding each term in the general Fokker - Planck term.

$$\left\{ \frac{\partial f}{\partial t} \right\}_e = \Gamma \left[ - \frac{\partial}{\partial v_i} \left( f \frac{m}{2M} \frac{\partial}{\partial v_i} \nabla^2 g \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_k} \left( f \frac{\partial^2 g}{\partial v_i \partial v_k} \right) \right]$$

(3.31)

The first term can be rewritten:

$$- \frac{m}{2M} \left( \nabla \cdot f \nabla \left( \nabla^2 g \right) \right)$$

(3.31)

$$\nabla^2 g = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial g}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 g}{\partial \theta^2} + \frac{\partial^2 g}{\partial z^2}$$

(3.32)
Axial symmetry eliminates all $\theta$ dependence, eliminating the middle term. In addition, with the assumption of a Maxwellian radial velocity distribution the problem has been reduced to one dimension. Thus,

$$\nabla(\nabla^2 g) = \frac{\partial^3 g}{\partial z^3} \quad (3.33)$$

and the first term reduces to:

$$-\frac{\partial}{\partial z} \left( f \frac{m}{2M} \frac{\partial^3 g}{\partial z^3} \right) \quad (3.34)$$

The second term in the Fokker-Planck equation,

$$\frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_k} \left( f \frac{\partial^2 g}{\partial v_i \partial v_k} \right),$$

can be reduced directly to the one dimensional case.

$$\frac{1}{2} \frac{\partial^2}{\partial z^2} \left( f \frac{\partial^2 g}{\partial z^2} \right) \quad (3.35)$$

Therefore, the Fokker-Planck term in one dimensional cylindrical coordinates is:

$$\frac{1}{\Gamma} \left\{ \frac{\partial f}{\partial t} \right\}_c = \frac{1}{2} \frac{\partial^2}{\partial z^2} \left( f \frac{\partial^2 g}{\partial z^2} \right) - \frac{\partial}{\partial z} \left( f \frac{m}{2M} \frac{\partial^3 g}{\partial z^3} \right) \quad (3.36)$$

In order to get a complete collision model the function $g$ must also be converted to the appropriate coordinate system.

$$g(v) = \int_{-\infty}^{\infty} f(v') \mid v - v' \mid dv'$$

This definition can be reduced for the one dimensional case.

$$g(z) = \int_{-\infty}^{\infty} f(\eta) \mid z - \eta \mid d\eta \quad (3.37)$$

or, written another way

$$g(z) = \int_{-\infty}^{\infty} f(z + \xi) \mid \xi \mid d\xi \quad (3.38)$$
With this definition the Fokker-Planck term becomes:

$$\frac{1}{\Gamma} \left\{ \frac{\partial f}{\partial t} \right\}_c = \frac{1}{2} \frac{\partial^2}{\partial z^2} \left( f \int_{-\infty}^{\infty} \frac{\partial^2}{\partial z^2} \left( f(z + \xi) \right) | \xi | d\xi \right) - \frac{\partial}{\partial z} \left( f \frac{m}{2M} \int_{-\infty}^{\infty} \frac{\partial^3}{\partial z^3} \left( f(z + \xi) \right) | \xi | d\xi \right)$$ (3.39)

For brevity let \( \frac{\partial}{\partial z} \) denote a derivative with respect to \( z \). Using this notation the Fokker-Planck term becomes:

$$\frac{1}{\Gamma} \left\{ \frac{\partial f}{\partial t} \right\}_c = \frac{1}{2} \frac{\partial^2}{\partial z^2} \left( f \int_{-\infty}^{\infty} f''(z + \xi) | \xi | d\xi \right) - \frac{\partial}{\partial z} \left( f \frac{m}{2M} \int_{-\infty}^{\infty} f'''(z + \xi) | \xi | d\xi \right)$$ (3.40)

The Fokker-Planck term is nondimensionalized using the same variables as the Boltzmann equation.

$$\left\{ \frac{\partial F}{\partial \tau} \right\}_c = \frac{\partial^2}{\partial Z^2} \left( F \frac{1}{2 \sigma} \int_{-\infty}^{\infty} F''(Z + \xi) | \xi | d\xi \right) + \frac{\partial}{\partial Z} \left( F \frac{m}{2M \sigma} \int_{-\infty}^{\infty} F'''(Z + \xi) | \xi | d\xi \right)$$ (3.41)

Let:

$$A(F) = \frac{1}{2 \sigma} \int_{-\infty}^{\infty} F''(Z + \xi) | \xi | d\xi$$ (3.42)

$$B(F) = \frac{m}{2M \sigma} \int_{-\infty}^{\infty} F'''(Z + \xi) | \xi | d\xi$$ (3.43)

The Fokker-Planck term can be written in a compact form using these defined functions.

$$\left\{ \frac{\partial F}{\partial \tau} \right\}_c = \frac{\partial^2}{\partial Z^2} (FA(F)) + \frac{\partial}{\partial Z} (FB(F))$$ (3.44)

The Fokker-Planck term can be expanded by order in \( \Delta U \) using the same technique as the Boltzmann equation.

$$\left\{ \frac{\partial F}{\partial \tau} \right\}_c = \left[ \frac{\partial^2}{\partial Z^2} (F_0A(F_0)) + \frac{\partial}{\partial Z} (F_0B(F_0)) \right] + \Delta U \left[ \frac{\partial^2}{\partial Z^2} (F_1A(F_0)) + \frac{\partial}{\partial Z} (F_1B(F_0)) + \frac{\partial^2}{\partial Z^2} (F_0A(F_1)) + \frac{\partial}{\partial Z} (F_0B(F_1)) \right] + \cdots$$

$$+ \Delta U^n \left[ \sum_{m=0}^{n} \left( \frac{\partial^2}{\partial Z^2} (F_{n-m}A(F_m)) + \frac{\partial}{\partial Z} (F_{n-m}B(F_m)) \right) \right]$$ (3.45)
This expansion can be combined directly with the expansion of the Boltzmann equation.

### 3.3.6 Solution Approach

To obtain a time dependent solution to the Boltzmann equation with the Fokker-Planck collision term a numerical technique is necessary. The solution presented has been limited to ion-ion collisions because these collisions represent the majority of the collisional energy and momentum transfer within the presheath. The positive signs in the Boltzmann equation must be applied for ion-ion collisions. The ratio of particle mass to reduced mass, \( \frac{m_i}{M} \), must be equal to two for like particles. The ratio of the inverse squares of the particle thermal velocities, \( \frac{1}{v_i^2} \), must be one for like particle collisions. The nondimensional Boltzmann equation reduces to the following form.

\[
\frac{\partial F_i}{\partial \tau} - \frac{\partial^2}{\partial Z^2} \left( F_i A(F_0) \right) - \frac{\partial}{\partial Z} \left( F_i B(F_0) \right) + \varphi_0 i Z F_i + \varphi_{-1} \frac{\partial F_i}{\partial Z} = - \sum_{m=1}^{i} \varphi_m m \frac{\partial F_{i-m}}{\partial Z} + \sum_{m=1}^{i} \left( \frac{\partial^2}{\partial Z^2} \left( F_{i-m} A(F_m) \right) + \frac{\partial}{\partial Z} \left( F_{i-m} B(F_m) \right) \right)
\]

where the summations are taken to be zero if \( i = 0 \). The functions \( A(F_m) \) and \( B(F_m) \) can be written:

\[
A(F) = \frac{1}{2} \int_{-\infty}^{\infty} F''(Z + \phi) \phi | d\phi \tag{3.47}
\]

\[
B(F) = - \int_{-\infty}^{\infty} F''(Z + \phi) \phi | d\phi \tag{3.48}
\]

The 'r' equations in the expansion are solved to obtain the time dependent particle velocity distribution. The Poisson equation is employed at each time step to obtain the potential structure. The ratio of the higher order equations with respect to the first order equation eliminates the \( \eta e^{[\phi_1 + \cdots + \phi_n]} \left( \frac{1}{\eta} \right) \) term from the Poisson equation. Using this technique each successive component of the potential...
structure can be determined from the previous components.

\[
\frac{\int_{-\infty}^{\infty} F_1 \, dZ}{\int_{-\infty}^{\infty} F_0 \, dZ} = - \frac{\varphi_1}{\varphi_0} \left( \frac{2T_i}{T_e} \right) \quad (3.49a)
\]

\[
\frac{\int_{-\infty}^{\infty} F_2 \, dZ}{\int_{-\infty}^{\infty} F_0 \, dZ} = - \frac{\varphi_2}{\varphi_0} \left( \frac{2T_i}{T_e} \right) + \frac{1}{2} \left( \frac{\varphi_1}{\varphi_0} \right)^2 \left( \frac{2T_i}{T_e} \right)^2 \quad (3.49b)
\]

\[
\frac{\int_{-\infty}^{\infty} F_3 \, dZ}{\int_{-\infty}^{\infty} F_0 \, dZ} = - \frac{\varphi_3}{\varphi_0} \left( \frac{2T_i}{T_e} \right) + \frac{\varphi_1 \varphi_2}{\varphi_0 \varphi_0} \left( \frac{2T_i}{T_e} \right)^2 - \frac{1}{6} \left( \frac{\varphi_1}{\varphi_0} \right)^3 \left( \frac{2T_i}{T_e} \right)^3 \quad (3.49c)
\]

Using these equations the particle velocity distribution and the potential structure of the presheath are determined as a function of time.

Of interest in this study is the point at which there are no returning ions. This is the presheath - sheath interface. This occurs when the net ion flux away from the wall is zero. To calculate this point in the presheath it is necessary to obtain a value for the potential expansion parameter \( \Delta U \) such that when the overall particle distribution is reconstructed from the various terms in the expansion no returning ions are present. Thus, at the critical point of no returning ions the model determines the total particle velocity distribution, the potential structure, and a value for the potential expansion parameter. From the potential structure and the potential expansion parameter the presheath height at the point of no returning ions can be determined (see figure 3.5).
CHAPTER IV

NUMERICAL TECHNIQUE

4.1 Problem Approach

The solution of the Boltzmann equation, as written in equation 3.46, coupled with the Poisson equation (3.49) is the goal of the numerical procedure.

The general approach is to solve the Boltzmann equation for the particle velocity distribution using a partially implicit, partially explicit scheme. Each step in time the Boltzmann equation is solved using some results from the previous time step. In equation 3.46 the left hand side is solved implicitly while the right hand side is solved explicitly.

\[
\frac{\partial F_i}{\partial t} - \frac{\partial^2}{\partial Z^2} \left( F_i A(F_0) \right) - \frac{\partial}{\partial Z} \left( F_i B(F_0) \right) + \varphi_0 i Z F_i + \varphi^{-1} \frac{\partial F_i}{\partial Z} \\
= - \sum_{m=1}^{i} \varphi_m m \frac{\partial F_{i-m}}{\partial Z} + \sum_{m=1}^{i} \left( \frac{\partial^2}{\partial Z^2} \left( F_{i-m} A(F_m) \right) + \frac{\partial}{\partial Z} \left( F_{i-m} B(F_m) \right) \right)
\]

(3.46)

The left hand side of this equation can be put in a matrix form.

\[
\begin{bmatrix}
T(t)
\end{bmatrix}
\begin{bmatrix}
F_i(t + \Delta t)
\end{bmatrix}
\]

(4.1)

In this form the matrix \(T(t)\) is an \(m \times m\) matrix created from the left hand side of the Boltzmann equation. The quantity \(m\) is the number of divisions in the velocity space \(Z\) chosen for the numerical scheme. The matrix \(T(t)\) is computed from \(A(F_0(t))\) and \(B(F_0(t))\). The values of these derived functions are taken from the solution to the particle velocity distribution at the previous time step, \(t\). Numerical derivatives are used to represent the partial derivatives in the equation. This procedure produces a diagonal matrix where all elements except those on an odd number of centered
diagonals are zero. The number of diagonals reflects the order of accuracy in the solution. Diagonal matrices of this form are easily and quickly inverted. The $F_i(r + \Delta r)$ matrix is an $m \times 1$ matrix of unknowns that represents the particle velocity distribution at the current time step. 'i' equations of this form can be written corresponding to the number of terms in the expansion.

The right hand side of the Boltzmann equation can also be put in a matrix form.

$$
\phi_1 \begin{bmatrix} v_1(r) \\ \vdots \\ v_n(r) \end{bmatrix} + \phi_{n+1} \begin{bmatrix} \phi_1(r) \\ \vdots \\ \phi_n(r) \end{bmatrix} + \begin{bmatrix} d_1(r) \end{bmatrix}
$$

(4.2)

The scalar $\phi_n$ values are unknowns and represent the nondimensional coefficients in the asymptotic potential structure of the presheath.

The $v_1(r)$ matrices are $m \times 1$ matrices which are comprised of the partial derivatives of the velocity distribution at the previous time step, $r$. They represent the first summation on the right hand side of the Boltzmann equation.

$$
\sum_{m=1}^{i} \phi_m \left( -m \frac{\partial F_{i-m}}{\partial Z} \right)
$$

The $d_i$ matrix is an $m \times 1$ matrix comprised of the second summation on the right hand side of the Boltzmann equation.

$$
\sum_{m=1}^{i} \left( \frac{\partial^2}{\partial Z^2} \left( F_{i-m} A(F_m) \right) + \frac{\partial}{\partial Z} \left( F_{i-m} B(F_m) \right) \right)
$$

All values of the distribution and the functions $A(F_m)$ and $B(F_m)$ are taken at the previous time step, $r$.

Putting together equations 4.1 and 4.2 a matrix form of the Boltzmann equation is created that can be solved for the particle velocity distribution.

$$
\begin{bmatrix} T(r) \\ F_i(r + \Delta r) \end{bmatrix} = \phi_1 \begin{bmatrix} v_1(r) \\ \vdots \\ v_n(r) \end{bmatrix} + \phi_{n+1} \begin{bmatrix} \phi_1(r) \\ \vdots \\ \phi_n(r) \end{bmatrix} + \begin{bmatrix} d_1(r) \end{bmatrix}
$$

(4.3)
'i' equations of this form can be written corresponding to the number of terms in the asymptotic expansion being used.

These equations are quickly inverted to obtain the particle velocity distribution at the current time step.

\[
F_i(r + \Delta r) = \varphi_1 V_i^1(r) + \cdots + \varphi_n V_i^n(r) + D_i(r) \quad (4.4)
\]

Where the \( V_n \) and \( D \) matrices represent the \( m \times 1 \) solution matrix to the inversion of the \( T(r) \) matrix with the corresponding \( v_n \) or \( d \) matrix.

The particle velocity distribution is obtained from this equation using the \( \varphi_n \) values from the previous time step.

Equation 3.49 is employed to obtain the \( \varphi_n \) values at the current time step.

\[
\begin{align*}
\frac{\int_{-\infty}^{\infty} F_1 \, dZ}{\int_{-\infty}^{\infty} F_0 \, dZ} &= -\varphi_1 \left( \frac{2T_1}{T_e} \right) \\
\frac{\int_{-\infty}^{\infty} F_2 \, dZ}{\int_{-\infty}^{\infty} F_0 \, dZ} &= -\varphi_2 \left( \frac{2T_1}{T_e} \right) + \frac{1}{2} \left( \frac{\varphi_1}{\varphi_0} \right)^2 \left( \frac{2T_1}{T_e} \right)^2 \\
\frac{\int_{-\infty}^{\infty} F_3 \, dZ}{\int_{-\infty}^{\infty} F_0 \, dZ} &= -\varphi_3 \left( \frac{2T_1}{T_e} \right) + \frac{\varphi_1 \varphi_2}{\varphi_0 \varphi_0} \left( \frac{2T_1}{T_e} \right)^2 - \frac{1}{6} \left( \frac{\varphi_1}{\varphi_0} \right)^3 \left( \frac{2T_1}{T_e} \right)^3
\end{align*}
\]

(3.49a) (3.49b) (3.49c)

The new distributions are integrated numerically and the \( \varphi_1 \) through \( \varphi_n \) scalars are determined consecutively. The value of \( \varphi_0 \) is input and is a nondimensional representation of the coefficient \( \beta \) in the potential expansion parameter, \( \Delta U \).

For each time step, the overall particle velocity distribution can be determined at any location from the original expansion once it has been nondimensionalized.

\[
F(Z, \Delta U^*) = F_0(Z) + \Delta U^* F_1(Z) + \Delta U^{*2} F_2(Z) + \cdots \quad (4.5)
\]

since,

\[
\Delta U^* = e^{\varphi_0 z^*} \quad (4.6)
\]
Where the * quantities are nondimensional. \( x \) has been nondimensionalized with respect to an ion mean free path.

\[
x^* = \frac{x}{n_R \Gamma}
\]

The potential structure of the presheath is determined from the nondimensional form of the original expansion of potential.

\[
U^* = U^*_0 + \varphi_1 \Delta U^* + \varphi_2 \Delta U^{*2} + \cdots
\]  \hspace{1cm} (4.7)

Using this procedure the time dependent evolution of the presheath is obtained.

The point of no returning ions occurs where the integral of the left half plane of the total particle velocity distribution is zero.

\[
0 = \int_{-\infty}^{0} F(Z)\,dZ
\]  \hspace{1cm} (4.8)

This equation can be rewritten using the expansion of the particle distribution.

\[
0 = \int_{-\infty}^{0} F_0(Z)\,dZ + \Delta U^* \int_{-\infty}^{0} F_1(Z)\,dZ + \Delta U^{*2} \int_{-\infty}^{0} F_2(Z)\,dZ + \cdots
\]  \hspace{1cm} (4.9)

Equation 4.9 can be solved for \( \Delta U^* \) Since the particle distributions are now known as a function of time and velocity. The entire solution at the point of no returning ions is known with this last piece of information.

4.2 Numerical Integration, Differentiation, and Matrix Inversion

In order to obtain a solution to the potential structure and particle distribution in the presheath it is necessary to develop the applicable mathematical tools. The primary techniques needed are integration, differentiation, and matrix inversion.
4.2.1 Numerical Integration

Throughout the solution integration is computed using a Simpson’s $\frac{1}{3}$ rule technique\cite{13}.

$$\int_{a}^{b} f(x)dx = \frac{h}{3} \left( f_1 + 4f_2 + 2f_3 + 4f_4 + 2f_5 + \cdots + 2f_{n-1} + 4f_n + f_{n+1} \right) \quad (4.10)$$

Where $h$ is the spacing between the points and $f_1$ through $f_{n+1}$ represent the function values at each point. This procedure has a global error of $O(h^4)$. If the step size is chosen appropriately this procedure is very accurate.

4.2.2 Numerical Differentiation

The technique for determining numerical differentiation is a second order accurate scheme. This reduces the number of computations while maintaining high accuracy. Second order accurate numerical differentiation requires that only three points be known. Thus, the 'T(τ)' matrix contains only three diagonals. If third order accuracy was used the 'T(τ)' matrix would require five diagonals to represent the five points needed for the differentiation. In addition, to maintain uniformity a central difference technique is desirable on as many points as possible. The greater the number of points needed for each derivative the more points that require forward or backward difference techniques (rather than the central difference technique).

Below is a list of the techniques used to obtain derivatives\cite{12}.

Central Difference

$$\frac{\partial F}{\partial x} = \frac{F(x + 1) - F(x - 1)}{2h} \quad (4.11a)$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{F(x + 1) - 2F(x) + F(x - 1)}{h^2} \quad (4.11b)$$

$$\frac{\partial^3 F}{\partial x^3} = \frac{F(x + 2) - 2F(x + 1) + 2F(x - 1) - F(x - 2)}{2h^3} \quad (4.11c)$$
Forward Difference

\[
\frac{\partial F}{\partial x} = \frac{-F(x+2) + 4F(x+1) - 3F(x)}{2h} \quad (4.12a)
\]

\[
\frac{\partial^2 F}{\partial x^2} = \frac{F(x+2) - 2F(x+1) + F(x)}{h^2} \quad (4.12b)
\]

\[
\frac{\partial^3 F}{\partial x^3} = \frac{F(x+3) - 3F(x+2) + 3F(x+1) - F(x)}{h^3} \quad (4.12c)
\]

Backward Difference

\[
\frac{\partial F}{\partial x} = \frac{3F(x) - 4F(x-1) + 3F(x-2)}{2h} \quad (4.13a)
\]

\[
\frac{\partial^2 F}{\partial x^2} = \frac{F(x) - 2F(x-1) + F(x-2)}{h^2} \quad (4.13b)
\]

\[
\frac{\partial^3 F}{\partial x^3} = \frac{F(x) - 3F(x-1) + 3F(x-2) - F(x-3)}{h^3} \quad (4.13c)
\]

Where \( h \) is the grid spacing. The derivatives are being taken about point \( x \).

It is worth noting that the third derivative equations require up to five points. There is no second order accurate numerical third derivative representation. These equations are third order accurate. This does not affect the \( T(r) \) matrix in that it contains no third derivatives. The solution procedure requires third derivatives only in the determination of the function \( B(F_i(r)) \).

These equations are used throughout the solution for derivatives with respect to velocity, \( Z \), and time, \( r \).

4.2.3 Matrix Inversion

In order to obtain a solution a procedure for inverting a diagonal matrix is necessary. The procedure used will invert any centered diagonal matrix. For the second order accurate case the matrix in question is tridiagonal. The procedure uses Gaussian elimination on all terms below the center diagonal and then through back substitution determines the solution vector. This technique can quickly invert a \( 200 \times 200 \) tridiagonal matrix.
4.3 Obtaining a Solution

As in any numerical model certain restraints and conditions must be met to obtain an accurate solution. This model requires some form of input distribution function and in order to obtain higher order terms must also have a perturbation applied to the potential structure. In addition, certain numerical techniques have been used to remove instabilities in the model.

4.3.1 Initial Distribution

To model the presheath region an initial particle velocity distribution that conforms to a Maxwellian profile has been used. This profile represents the distribution that naturally occurs in the neutral plasma region. The idea is that the time dependent evolution of the distribution will change from a Maxwellian at time zero to a shifted new form as the presheath is entered. The Maxwellian profile is initially given to the zero order term having set the initial conditions of all higher order terms to zero.

If the potential structure of the presheath is not perturbed in some manner then the model represents the neutral plasma region and the particle velocity distribution remains Maxwellian (as it should). If, however, a small perturbation in the potential structure is added (ie. a nonzero $a_1, a_2 \cdots$) then the model readjusts to describe the presheath region. In this manner the model is used to give the time dependent evolution of the presheath.

4.3.2 Instability Damping

By the nature of the implicit - explicit technique being employed certain numerical problems are expected to appear. This model is no exception. Two techniques have been used to remove these instabilities.
The most important thing to do to avoid numerical problems in a scheme of this nature is to ensure that as much as possible of the solution is computed implicitly. In addition, once some new data has been calculated it should be applied to any new calculations immediately.

In this model each term in the expansion of the particle distribution function uses the new data already determined in calculating all of the lower order terms. Once $F_0$ is determined that information is used in calculating $F_1$. This idea is repeated for the higher order terms.

A second method applied to the model to eliminate oscillatory instabilities that start on a very small scale and grow is the application of a very weak averaging scheme to the particle distribution functions. Each point in the distribution is weakly averaged with the points on either side.

$$F(Z) = \frac{0.025F(Z + 1) + F(Z) + 0.025F(Z - 1)}{1.05} \tag{4.14}$$

This technique, although necessary, has the negative effect of falsely increasing the energy in the system by spreading the distribution slightly (see figure 5.1). The change is very small and can be considered insignificant with respect to the overall solution.

### 4.4 Program Structure

The entire program has been written in FORTRAN and can be run on either an IBM PC AT or on the CYBER mainframe. The code has been written in a segmented manner that easily allows one section to be altered without having to alter other sections. The overall structure of the program consists of three initialization programs, three input data files, the main program, and three output data files. The main program contains a driver and seventeen subroutines. Several of the sub-
routines perform operations that are used throughout the main code. Figure 4.1 is a diagram of the structure of the program. The flexibility of the code is derived from the generalized subroutine structure and the ability to enter a variety of input variables. The driver keeps track of time and maintains the overall operating structure of the solution while the subroutines perform the necessary manipulations. Below is a list of the function of each program, data file and subroutine.

AVE - Subroutine to smooth distributions by averaging.

CONSERV - Subroutine to determine conservation of energy, momentum, and particles.

CONSOUT - Conservation output data file.

CRF - Particle distribution initialization program.

CRPHI - Potential structure initialization program.

DENSITY - Subroutine to solve for a new presheath structure.

FDATA - Initial particle distribution data file.

FD1 - Subroutine to find first derivatives.

FD2 - Subroutine to find second derivatives.

FD3 - Subroutine to find third derivatives.

FINDA - Subroutine to determine 'A' function.

FINDB - Subroutine to determine 'B' function.

FPINIT - Primary initialization program.

FPOUT - Output particle distribution data file.

FPSHETH - Main program driver.

GETAB - Subroutine to make A and B function vectors.

INITDAT - Initialization data file.

MAKED - Subroutine to make d matrix.
MAKEF - Subroutine to read initial particle distribution.

MAKEPHI - Subroutine to read initial potential structure.

MAKET - Subroutine to make T matrix.

MAKEV - Subroutine to make \( v \) matrix.

MODIAG - Subroutine to invert diagonal matrices.

PHIDAT - Initial potential structure data file.

PHIDOUT - Output potential structure data file.

SIMPS - Subroutine to perform Simpson's rule integration.

TOT - Subroutine to obtain total distribution at point of no returning ions.
CHAPTER V

RESULTS

The ratio of ion temperature to electron temperature, \( \frac{T_i}{T_e} \), has been set to one half throughout these results. There is little effect on the particle distribution or potential structure if the temperature ratio is changed to other values. The electron temperature is expected to be higher than the ion temperature in the presheath since electrons absorb energy from electric and magnetic fields faster than ions and other large particles.

Through repeated test runs of the model it was found that fifty-one points in velocity space were enough to provide high accuracy and produce good results. The range of points in velocity space has been truncated to \( \pm 5 \) nondimensional units. The results show that at \( \pm 5 \) the distribution is near zero, substantiating the truncation.

A time step of 0.2 nondimensional times was found to keep the solution accurate. Three nondimensional units in time were sufficient to produce stable results.

It was found that the magnitude of the higher order terms in the particle velocity distribution drop off very rapidly. Thus, the higher order terms have very little impact on the shape of the potential or of the particle distribution.

To understand the effects of a quiescent plasma interacting with a surface the potential gradient in the neutral plasma has been set to zero. To accomplish this the \( \alpha \) term in the expansion of potential has been set to zero.

\[
U = U_0 + a_1 \Delta U + a_2 \Delta U^2 + \ldots
\]

(3.13)
where

\[ U_0 = \alpha x \quad \text{and} \quad \Delta U = e^{\beta z} \quad (3.14) \]

A stable solution exists only for a specific critical value of the exponential coefficient, \( \beta \), which represents the scale of the presheath. The quantity \( \beta \) is nondimensionalized as \( \varphi_0 = \frac{\beta}{n_R \Gamma} \), where \( n_R \Gamma \) is an ion mean free path. It is expected, as shown in section 3.2, that the critical value should be on the order of a mean free path. It was found that \( \frac{\beta}{n_R \Gamma} = 0.4 \) produces the most nearly stable results. The distributions become unstable for values greater or less than 0.4. The small remaining instability at \( \frac{\beta}{n_R \Gamma} = 0.4 \) can be attributed to the inexact nature of the numerical solution.

The results presented here are first order and produce a complete picture of the structure of the presheath because the higher order terms collective contribution is more than an order of magnitude smaller. Figures 5.1 and 5.2 are plots of the zeroth and first order expansions of the particle distribution in velocity space. The zero order term remains Maxwellian because the potential gradient in the neutral plasma is zero. The first order term of the distribution obtains a profile that has roughly the shape (but not magnitude) of the negative first derivative of the zeroth order solution. The potential expansion parameter at the point of no returning ions is determined for each time step. Using the particle distribution functions and the known potential expansion parameter together produce the overall particle velocity distribution at the point of no returning ions, the presheath-sheath interface. Figure 5.3 shows this distribution.

The positive shift in the total distribution is as expected for the presheath. The ions are being pulled into the wall. The particle distribution for velocities away from the wall is zero for the case of no returning ions. The point of no returning ions exists where the particle distribution for velocities away from the wall integrates
to zero. Figure 5.3 shows that the ion distribution becomes negative for velocities away from the wall. A negative particle distribution cannot exist physically. The addition of higher order terms does not correct the problem because the expansion drops off so quickly that any higher order terms have no impact on the shape of the distribution. The problem is fundamental to the type of collision term being applied in the model. The Fokker - Planck term only models the limit of small angle collisions. However, large angle collisions become important in the presheath.

The first term of the nondimensionalized potential structure, $\varphi_1$, has been initially perturbed to $1.0 \times 10^{-4}$ to obtain the results presented in figures 5.1, 5.2 and 5.3. Perturbing the potential structure provides the model with the nonequilibrium condition necessary to initiate the time dependent development of the presheath. The strength of the initial perturbation is not significant to obtaining an accurate particle distribution and potential structure of the presheath. Figures 5.4, 5.5, and 5.6 are the result of an initial perturbation of $1.0 \times 10^{-3}$ and figures 5.7, 5.8 and 5.9 are the result of an initial perturbation of $1.0 \times 10^{-6}$. Comparing these results show that the magnitude of the initial perturbation only affects the scale of the first order term and has no effect on the overall particle distribution in velocity space.

Figure 5.10 is a plot of the position of the point of no returning ions, the presheath - sheath interface, as a function of time for the three solutions. Since no source of ions exists in the model the relative position of the plasma with respect to the surface changes as a function of time. The wall is moving into the plasma, or the plasma is moving into the wall, at the rate at which the wall is absorbing ions. The three solutions have different magnitudes but follow the same profile. The strength of the perturbation controls the relative position of the zero point.

Figure 5.11 is a plot of the potential structure of the presheath obtained from
the three solutions as a function of position. The data for the potential structure has been taken from the solution at a nondimensional time of two. Note that the affect of the different initial perturbation values is to cause a shift in the relative position of the potential but has no effect on the shape of the potential or on the strength of the potential at the point of no returning ions. Changing the perturbation strength alters the location of the zero point but not its shape. The stronger the perturbation the further the zero point is moved from the surface. The horizontal line in the plot depicts the presheath height at the point of no returning ions. The vertical lines show the position of the point of no returning ions.

A time dependent plot of the presheath height at the point of no returning ions is presented in figure 5.12. This plot shows that the time evolution of the sheath height approaches smoothly to a nearly constant value of 0.16. All three solutions fall on the same curve. This shows that the strength of the perturbation does not affect the results obtained.
CHAPTER VI

CONCLUSIONS

The solution obtained is an accurate representation of the time dependent development of the Fokker - Planck presheath. The model produces a precise potential structure, however, the distribution of returning ions breaks down in the presheath. An oscillation develops in the negative tail of the distribution, as seen in figures 5.3, 5.6, and 5.9. This oscillation cannot be removed by including additional terms to the expansion. In addition, the sheath height of 0.16 determined at the point of no returning ions is roughly an order of magnitude smaller than expected. Both of these conditions lead to the conclusion that the Fokker - Planck collision term does not represent the type of collisions that remove the returning ions in the presheath. This breakdown is due to the failure of the Fokker - Planck collision term to model the large angle collisions that take place within the presheath. The Fokker - Planck term is effective at modeling the collisions present in the center of the plasma but breaks down in the presheath. The primary mechanism behind clearing out the returning ions from within the presheath is not particle diffusion as represented by small angle deflections but rather the large velocity changes caused by large angle collisions. Since the Fokker - Planck term models particle collisions that represent the limit of small angle collisions it is inadequate at describing the mechanisms controlling the ion velocity distribution moving away from the wall. The solutions obtained using a Maxwellian distribution by Bissell and Johnson[4] and Emmert et al.[5] and those obtained using a charge exchange collision model by Riemann[6] and Main[7] effectively include the large angle collisions since they model the collisions
by instantaneous changes in particle velocity and position. These collision models do not represent the Coulomb collisions taking place in a fully ionized plasma. They do not represent the collision processes but only approximate the collisional effects.

This Fokker - Planck presheath model produces a self-consistent and precise potential structure. The particle velocity distribution in the presheath has the correct acceleration of ions toward the wall but because the Fokker - Planck collision term only models the limit of small angle collisions it is unable to clear the particle distribution of returning ions. The effect of not modeling the large angle collisions is that the particle distribution for returning ions is accurate only in the initial section of the presheath where the collisional processes are dominated by particle diffusion. The collisional processes become dominated by the effects of the large angle collisions as the interface between the presheath and the Debye sheath is approached. Only by including a collision term which accounts for these large angle collisions can a presheath model produce a particle velocity distribution that accurately models the condition of no returning ions. This study has found that a presheath model which describes the Coulomb collisions occurring in a fully ionized plasma must account for both the small angle collisions and the large angle collisions.
REFERENCES


FIGURES

Figure 3.1 Debye Shielding

Figure 3.2 Wall Potential
Figure 3.3 Wall Region Model
Figure 3.4 Coordinate System in Velocity Space
Figure 3.5 Radial, Symmetric, and No Returning Ions Conditions
Figure 4.1 Program Diagram
Figure 5.1 Zero Order Particle Velocity Distribution, $\varphi_1 = 1.0 \times 10^{-4}$
Figure 5.2 First Order Particle Velocity Distribution, $\varphi_1 = 1.0 \times 10^{-4}$
Figure 5.3 Total Particle Velocity Distribution, $\varphi_1 = 1.0 \times 10^{-4}$
Figure 5.4 Zero Order Particle Velocity Distribution, $\phi_1 = 1.0 \times 10^{-3}$
Figure 5.5 First Order Particle Velocity Distribution, $\varphi_1 = 1.0 \times 10^{-3}$
Figure 5.6 Total Particle Velocity Distribution, $\varphi_1 = 1.0 \times 10^{-3}$
Figure 5.7 Zero Order Particle Velocity Distribution, $\varphi_i = 1.0 \times 10^{-5}$
Figure 5.8 First Order Particle Velocity Distribution, $\phi_1 = 1.0 \times 10^{-5}$
Figure 5.9 Total Particle Velocity Distribution, \( \phi_1 = 1.0 \times 10^{-5} \)
Figure 5.10 Time Dependent Position of the Potential
Figure 5.11 Potential Structure of the Presheath
Figure 5.12 Time Dependent Structure of the Sheath Height
APPENDIX: PROGRAM LISTING

* THIS PROGRAM WAS WRITTEN BY JEFFREY P. DANSEREAU *
* THIS VERSION WAS LAST UPDATED ON 7/17/87 *

C
C THIS PROGRAM IS A TIME DEPENDENT MODEL OF THE SHEATH -
C PRESHEATH OF A PLASMA. IT USES A FOKKER-PLANCK COLLISION
C TERM WITH ONLY COULOMB COLLISIONS. BELOW IS A LIST OF
C THE VARIABLES AND THEIR MEANING
C
C T - DIAGONAL REPRESENTATION OF T MATRIX. 1ST IS DIAGONALS.
C 2ND IS M VEL POS.
C V - 3-D MATRIX OF V COMPONENTS. 1ST POS. IS F'S, 2ND IS
C N PHI POS., 3RD IS M VEL POS.
C VV - 3-D MATRIX OF V VALUES AFTER INVERSION WITH T MATRIX.
C POS. ARE SAME AS V MATRIX WITH ADDITIONAL ROW FOR BC'S.
C D - 2-D MATRIX OF D VALUES. 1ST POS. IS F'S, 2ND IS M VEL POS.
C DD - 2-D MATRIX OF D VALUES AFTER INVERSION WITH T MATRIX
C PHI - VECTOR OF PHI VALUES
C F - 2-D MATRIX OF DENSITY FUNCTIONS, 1ST POS. IS THE n F'S
C THAT ARE BEING USED. (n=N-1).  F(1,X) = F0 ECT... THE
C 2ND POS IS THE M VEL POS.
C TSTEP - VALUE OF DELTA T AS TIME IS STEPPED THROUGH
C VSTEP - VALUE OF DELTA V AS VEL. SPACE IS STEPPED THROUGH
C RTIME - CURRENT VALUE OF NON-DIM. TIME
C ETA - CURRENT VEL.
C NETA - ETA PARAMETER IN POISSON EQN
C TOTE - RATIO OF T TO Te(T OVER Te)
C SM - SMALL M(m) IN F-P EQN
C BM - BIG M(M) IN F-P EQN
C AA - a IN F-P EQN
C AP - a' IN F-P EQN
C M - NUMBER OF DIV. IN VEL. SPACE
C N - NUMBER OF PHI VALUES
C ND - NUMBER OF DIAGONALS IN T MATRIX
C TIME - INTEGER VALUE IN TIME LOOP
C TEND - RTIME TO FINISH SIMULATION
C L - INTEGER TIME VALUE TO END SIMULATION
C FLAG1 - FLAG TO PRINT OR NOT PRINT MATRICES (99 TO PRINT)
C FLAG2 - FLAG TO PRINT OR NOT PRINT INTERMEDIATE MATRICES
C (99 TO PRINT)
C TRBLE - FLAG TO PRINT TROUBLE STATEMENT IF MATRIX IS NOT
C INVERTABLE
C B,SOLN - INTERMEDIATE VALUES OF VARIOUS FUNCTIONS
C X,Y,Z - INTEGER COUNTERS
C
C THIS PROGRAM STEPS THROUGH TIME SOLVING THE BOLTZMANN EQN FOR
C VALUES OF THE EXPANDED DENSITY FUNCTION
C
REAL T(6,8,202),V(6,6,202),B(202),SOLN(202),TEND
REAL VV(6,6,202),D(6,202),DD(6,202),PHI(6),L
REAL F(6,202),TSTEP,RTIME,SM,BM,AA,AP,VSTEP,ETA,NETA,TOTE
REAL FTOTAL(202),SO,DU,DUPHI
INTEGER X,Y,Z,M,N,ND,TIME,TRBLE,FLAG1,FLAG2,FLAG3,P,PSTEP,SKIP
C
READ IN PARAMETERS AND PRINT THEM
C
OPEN(UNIT=2,FILE='INITDAT.DAT',STATUS='OLD')
READ (2,705) M,NETA,TOTE,AA,AP,SM,FLAG3,SKIP,SO
READ (2,707) BM,VSTEP,N,ND,TEND, TSTEP,FLAG 1,FLAG2
CLOSE(UNIT=2)
OPEN(UNIT=1,FILE='PHI.OUT',STATUS='UNKNOWN')
OPEN(UNIT=3,FILE='FPSHETH.OUT',STATUS='UNKNOWN')
WRITE(3,717) '*****************************************************************************

717 FORMAT(A,A)
WRITE(3,711) 'FOKKER - PLANCK SIMULATION OUTPUT'
WRITE(3,717) '*****************************************************************************

710 FORMAT(A,A)
WRITE(3,715) 'NUMBER OF VEL. STEPS - M',M
WRITE(3,715) 'NUMBER OF PHI VALUES - N',N
WRITE(3,715) 'NUMBER OF DIAGONALS IN T MATRIX - ND', ND
WRITE(3,720) 'VALUE OF ETA IN POISSON EQN - NETA', NETA
WRITE(3,720) 'VALUE OF T OVER TE - TOTE', TOTE
WRITE(3,720) 'SIZE OF EACH VEL. STEP - VSTEP', VSTEP
WRITE(3,720) 'SIZE OF EACH TIME STEP - TSTEP', TSTEP
WRITE(3,720) 'VALUE OF ENDING TIME FOR SIMULATION', TEND
WRITE(3,720) 'VALUE OF A PARAMETER IN F-P EQN - AA', AA
WRITE(3,720) 'VALUE OF A PRIME PARAMETER IN F-P EQN - AP', AP
WRITE(3,720) 'VALUE OF SMALL M IN F-P EQN - SM', SM
WRITE(3,720) 'VALUE OF BIG M IN F-P EQN - BM', BM
WRITE(3,716) 'FLAG TO PRINT PRIMARY MATRICES(99 TO PRINT)', + ' - FLAG1', FLAG1
WRITE(3,716) 'FLAG MAKE FDATA AND PHIDAT NEW FINAL VALUES', + ' (99 = YES) - FLAG3', FLAG3
WRITE(3,715) 'TIME SKIP FOR PRINT OF F FILE - SKIP', SKIP

716  FORMAT(5X,A,A,2X,I3)
WRITE(3,710)    '
710  FORMAT(15X,A,//)
711  FORMAT(8X,A,//)
715  FORMAT(5X,A,2X,I3)
720  FORMAT(5X,A,2X,F9.5)
C
C MAKE NON TIME DEPENDENT QUANTITIES AND INITIAL APPROXIMATIONS
C TO F'S, PHI'S, AN THE BC
C
DU=0.0
DUPHI1=0.0
CALL MAKEF(M,F)
CALL MAKEPHI(N,RTIME,PHI)
CALL TOT(F,FTOTAL,PHI,N,M,VSTEP,DU,DUPHI1)
TT=0.0
CALL CONSERV(F,VSTEP,TT,M)
C
C ****************************
C
C START MAIN TIME LOOP
C
C ****************************
C
C COMPUTE END LOOP TIME L
C
L=TEND/TSTEP
PSTEP=0
C
DO 40 TIME=1,INT(L)+1
C PRINT RESULTS OF CURRENT TIME STEP
C
PSTEP=PSTEP+1
IF (TIME.LE.3) GOTO 1000
IF (PSTEP.GE.SKIP) THEN
  PSTEP=0
  GOTO 1000
ENDIF
GOTO 1001
1000  WRITE(3,350) 'TIME = ',RTIME,'TSTEP = ',TIME-1
      DO 210 Z=1,N
          WRITE(3,250) 'PHI(',Z-2,') = ',PHI(Z)
      210 CONTINUE
      WRITE(3,351) 'DU = ',DU,'DU*PHI(1) = ',DUPHI1
      WRITE(3,711) 
      ETA=-5.0
      DO 220 X=1,M
          WRITE(3,300) X,ETA,F(1,X),F(2,X),F(3,X),FTOTAL(X)
          ETA=ETA+VSTEP
      220 CONTINUE
1001  WRITE(3,710) 
      WRITE(1,813) TIME,RTIME,PHI(1),PHI(2),PHI(3),DU,DUPHI1
813 FORMAT(I3,1X,F7.4,1X,E13.6,1X,E13.6,1X,E13.6,1X,E13.6,
      1X,E13.6)
      IF (TIME.GT.INT(L)) GOTO 40
      WRITE(*,946) 'CURRENTLY IN TIME STEP','TIME','OF',INT(L)
946 FORMAT(5X,A,I4,2X,A,I4)
C
C MAKE TIME DEPENDENT QUANTITIES T MATRIX, V MATRIX, D MATRIX
C
DO 445 JJ=1,2
CALL MAKET(ND,M,F,VSTEP,SM,BM,AA,AP,TSTEP,SO,PHI,N,T)
CALL MAKEV(N,M,F,VSTEP,V)
CALL MAKED(M,F,VSTEP,TSTEP,SM,BM,AA,AP,N,PHI,D)

C
C INVERT T MATRIX WITH V VECTORS, MAKING VV MATRIX
C
DO 50 X=1,N-1
   DO 60 Y=3,N
      DO 70 Z=1,M
         B(Z)=V(X,Y,Z)
      70 CONTINUE
   CALL MODIAG(M,ND,T,B,SOLN,TRBLE,X)
   IF (TRBLE.EQ.999) THEN
      WRITE(3,*) 'MATRIX HAS NO SOLUTION'
      TRBLE=0
   ENDIF
   DO 80 Z=1,M
      VV(X,Y,Z)=SOLN(Z)
   80 CONTINUE
  60 CONTINUE
50 CONTINUE

C
C INVERT T MATRIX WITH D VECTORS, MAKING DD MATRIX
C
DO 90 X=1,N-1
   DO 100 Y=1,M
      B(Y)=D(X,Y)
   100 CONTINUE
   CALL MODIAG(M,ND,T,B,SOLN,TRBLE,X)
   IF (TRBLE.EQ.999) THEN
      WRITE(3,*) 'MATRIX HAS NO SOLUTION'
      TRBLE=0
   ENDIF
   DO 110 Y=1,M
      DD(X,Y)=SOLN(Y)
   110 CONTINUE
90 CONTINUE

C
C GET NEW F VALUES FROM NEW PHI VALUES
C
DO 160 X=1,N-1
   IF ((JJ.EQ.1).AND.(X.EQ.2)) GOTO 160
IF ((JJ.EQ.2).AND.(X.EQ.1)) GOTO 160
DO 165 Z=1,M
    F(X,Z)=0.0
165 CONTINUE
DO 170 Y=3,N
    DO 180 Z=1,M
        F(X,Z)=F(X,Z)+PHI(Y)*VV(X,Y,Z)
180 CONTINUE
170 CONTINUE
DO 190 Z=1,M
    F(X,Z)=F(X,Z)+DD(X,Z)
190 CONTINUE
160 CONTINUE
CALL AVE(F,M,N,JJ)
C
445 CONTINUE
C
C GET NEW PHI VALUES
C
CALL DENSITY(N,M,PHI,F,TOTE,VSTEP)
C
MAKE TOTAL DENSITY
C
CALL TOT(F,FTOTAL,PHI,N,M,VSTEP,DU,DUPHI1)
C
TT=RTIME+TSTEP
CALL CONSERV(F,VSTEP,TT,M)
C
INCREASE REAL TIME TO NEXT POSITION
C
RTIME=RTIME+TSTEP
C
CONTINUE TIME LOOP
C
40 CONTINUE
C *****************************************************
C
END TIME LOOP
C
C *****************************************************
C FORMAT STATEMENTS FOR PRINTS
C
250 FORMAT(5X,A,I4,A,1X,E13.6)
275 FORMAT(2X,A,2X,A,9X,A,12X,A,12X,A,12X,A,10X,A)
300 FORMAT(3,1X,F5.2,1X,E13.6,1X,E13.6,1X,E13.6,1X,E13.6,1X,E12.5)
350 FORMAT(5X,A,14X,A,10X,A)
351 FORMAT(5X,A,E13.6,5X,A,E13.6)
CLOSE(UNIT=3)
IF (FLAG3.EQ.99) THEN
   OPEN(UNIT=4,FILE='FDATA.DAT',STATUS='UNKNOWN')
   DO 265 X=1,M
      WRITE(4,266) F(1,X),F(2,X),F(3,X),F(4,X)
   266 FORMAT(E13.6,1X,E13.6,1X,E13.6,1X,E13.6)
   265 CONTINUE
   CLOSE(UNIT=1)
   OPEN(UNIT=8,FILE='PHIDAT.DAT',STATUS='UNKNOWN')
   DO 267 X=1,N
      WRITE(8,268) RTIME
      WRITE(8,268) PHI(X)
   268 FORMAT(E13.6)
   267 CONTINUE
ENDIF
STOP
END

REAL F(6,202),ETA,VSTEP,NETA,TOTE,B,PI,C
INTEGER X,M,Z
OPEN(UNIT=3,FILE='FDATA.DAT',STATUS='UNKNOWN')
ETA=-5.0
PRINT*, 'INPUT M,VSTEP'
READ(*,*) M,VSTEP
N=M-(M-1)/2
DO 10 X=1,N
   F(1,X)=EXP(-(ETA**2))
   F(2,X)=0.0
   F(3,X)=0.0
   F(4,X)=0.0
   ETA=ETA+VSTEP
10 CONTINUE
   Z=1
   DO 30 X=M,N+1,-1
      DO 40 Y=1,4
         F(1,X)=F(1,Z)
      40       CONTINUE
   Z=Z+1
30 CONTINUE
   DO 20 X=1,M
      WRITE(3,100) F(1,X),F(2,X),F(3,X),F(4,X)
   100   FORMAT(E13.6,1X,E13.6,1X,E13.6,1X,E13.6)
20 CONTINUE
STOP
END

REAL PHI(6),RTIME
INTEGER N,X
N=5
RTIME=0.0
WRITE(*,*) 'INPUT PHI(-1)'
READ(*,200) PHI(1)
WRITE(*,*) 'INPUT PHI(0)'
READ(*,200) PHI(2)
200 FORMAT(F10.5)
WRITE(*,*) 'INPUT PHI(1)'
READ(*,200) PHI(3)
DO 10 X=4,N
   PHI(X)=0.0
10 CONTINUE
OPEN(UNIT=9,FILE='PHIDAT.DAT',STATUS='UNKNOWN')
WRITE(9,120) RTIME
DO 20 X=1,N
   WRITE(9,120) PHI(X)
20 CONTINUE
RETURN
END
SUBROUTINE CONSERV(F,VSTEP,T,M)
REAL F(6,202),F0(202),VSTEP,ZF(202),Z2F(202),ENER,MOM,DEN,T,ETA
INTEGER X,M
ETA=-5.0
DO 10 X=1,M
   F0(X)=F(1,X)
   ZF(X)=F(1,X)*ETA
   Z2F(X)=ZF(X)*ETA
   ETA=ETA+VSTEP
10 CONTINUE
CALL SIMPS(F0,M,VSTEP,DEN)
CALL SIMPS(ZF,M,VSTEP,MOM)
CALL SIMPS(Z2F,M,VSTEP,ENER)
OPEN(UNIT=9,FILE='CONSERV.OUT',STATUS='UNKNOWN')
WRITE(9,100) T,DEN,MOM,ENER
100 FORMAT(2X,F8.5,1X,E13.6,1X,E13.6,1X,E13.6)
RETURN
END

SUBROUTINE DENSITY(N,M,PHI,F,TOTE,VSTEP)
REAL F(6,202),PHI(6),TOTE,VSTEP,G(202),R(4),N0(6)
INTEGER M,N,X,Y
DO 10 X=1,N-1
   DO 20 Y=1,M
      G(Y)=F(X,Y)
20 CONTINUE
   CALL SIMPS(G,M,VSTEP,N0(X))
10 CONTINUE
R(1)=N0(2)/N0(1)
R(2)=N0(3)/N0(1)
R(3)=N0(4)/N0(1)
PHI(3)=PHI(2)*R(1)/(2.0*TOTE)
PHI(4)=PHI(2)*(((PHI(3)/PHI(2))**2)*((2.0*TOTE)**2)/2.0)-R(2))
   + /(2.0*TOTE)
PHI(5)=PHI(2)*(((2.0*TOTE)**2)*PHI(3)*PHI(4)/(PHI(2)**2))
   + -(((2.0*TOTE)**3)*((PHI(3)/PHI(2))**3)/6.0)-R(3))/(2.0*TOTE)
RETURN
END
SUBROUTINE MAKEF(M,F)
REAL F(6,202)
INTEGER X,M
OPEN(UNIT=8,FILE='FDATA.DAT',STATUS='UNKNOWN')
DO 10 X=1,M
   READ(8,100) F(1,X),F(2,X),F(3,X),F(4,X)
100 FORMAT(E13.6,1X,E13.6,1X,E13.6,1X,E13.6)
10 CONTINUE
RETURN
END

SUBROUTINE MAKEPHI(N,RTIME,PHI)
REAL PHI(6),RTIME
INTEGER N,X
OPEN(UNIT=9,FILE='PHIDAT.DAT',STATUS='UNKNOWN')
READ(9,120) RTIME
DO 20 X=1,N
   READ(9,120) PHI(X)
120 FORMAT(E13.6)
20 CONTINUE
RETURN
END

SUBROUTINE MAKEV(N,M,F,VSTEP,V)
REAL F(6,202),V(6,6,202),VSTEP,FD1(6),Z,ETA
INTEGER N,M,X,Y,R,S
ETA=-5.0
DO 10 X=1,M
   CALL FD1(F,N,M,X,VSTEP,FD1)
   IF ((X.EQ.1).OR.(X.EQ.M)) THEN
      DO 15 Y=1,N-1
         DO 17 R=3,N
            V(Y,R,X)=0.0
50 CONTINUE
55 CONTINUE
50 CONTINUE
55 CONTINUE
   ELSE
      DO 20 Y=1,N-1
         CALL FD1(F,N,M,X,VSTEP,FD1)
         CALL FD1(V(6,Y,X),VSTEP,FD1)
      END
Z = 1.0
S = 1
DO 30 R = 3, N
   IF (S.GT.(Y-1)) THEN
      V(Y, R, X) = 0.0
   ELSE
      V(Y, R, X) = -Z*FD1(Y - R + 2)
   ENDIF
   S = S + 1
   Z = Z + 1.0
30    CONTINUE
20    CONTINUE
ENDIF
ETA = ETA + VSTEP
10    CONTINUE
RETURN
END

SUBROUTINE MAkED(M, F, VSTEP, TSTEP, SM, BM, AA, AP, N, PHI, D)
REAL VSTEP, TSTEP, SM, BM, AA, AP, F(6,202), D(6,202), A(6,202), PHI(6)
REAL FD1(6), FD2(6), AH(202), BH(202), B(6,202)
REAL ETA, GD1, GD2
INTEGER N, M, X, Y, Z
DO 10 X = 1, N - 1
   CALL GETAB(F, VSTEP, M, SM, BM, AA, AP, AH, BH, X)
   DO 15 Y = 1, M
      A(X, Y) = AH(Y)
      B(X, Y) = BH(Y)
15    CONTINUE
10    CONTINUE
DO 17 Y = 1, N - 1
   D(Y, M) = 0.0
   D(Y, 1) = 0.0
17    CONTINUE
ETA = -5.0 + VSTEP
DO 20 X = 2, M - 1
   DO 30 Y = 1, N - 1
      D(Y, X) = 0.0
30    CONTINUE
   DO 40 Z = 1, Y
40    CONTINUE
IF (Z.LT.Y-1) THEN
    GD2=(A(Z+1,X+1)*F(Y-Z,X+1)-2.0*
        A(Z+1,X)*F(Y-Z,X)+A(Z+1,X-1)*F(Y-Z,X-1))/(VSTEP**2)
    GD1=(B(Z+1,X+1)*F(Y-Z,X+1)-B(Z+1
        ,X-1)*F(Y-Z,X-1))/(2.0*VSTEP)
    D(Y,X)=D(Y,X)+GD2+GD1
ENDIF

CONTINUE
D(Y,X)=D(Y,X)+F(Y,X)/TSTEP
CONTINUE
ETA=ETA+VSTEP
CONTINUE

RETURN
END

SUBROUTINE FD1(F,N,M,X,VSTEP,FD1)
REAL F(6,202),FD1(6),VSTEP
INTEGER Y,X,N,M
DO 10 Y=1,N-1
    IF (X.LT.2) THEN
        FD1(Y)=(-F(Y,X+2)+4.0*F(Y,X+1)-3.0*F(Y,X))/(2.0*VSTEP)
    ELSE IF (X.GE.M-1) THEN
        FD1(Y)=(3.0*F(Y,X)-4.0*F(Y,X-1)+F(Y,X-2))/(2.0*VSTEP)
    ELSE
        FD1(Y)=(F(Y,X+1)-F(Y,X-1))/(2.0*VSTEP)
    ENDIF
10 CONTINUE
RETURN
END

SUBROUTINE MAKET(ND,M,F,VSTEP,SM,BM,AA,AP,TSTEP,SO,PHI,N,T)
REAL VSTEP,TSTEP,SM,BM,AA,AP,F(6,202),T(6,8,202),A(202)
REAL ETA,B(202),PHI(6),P
INTEGER ND,M,X,Y,N,Z
CALL GETAB(F,VSTEP,M,SM,BM,AA,AP,A,B,1)
P=0.0
DO 5 Z=1,N-1
ETA=-5.0
DO 10 X=1,M
  IF ((X.EQ.M).OR.(X.EQ.1)) THEN
    T(Z,2,X)=0.0
    T(Z,3,X)=1.0
    T(Z,4,X)=0.0
  ELSE
    T(Z,2,X)=((-A(X-1)/VSTEP)+B(X-1)/2.0)/VSTEP
    T(Z,2,X)=T(Z,2,X)-PHI(1)/(2.0*VSTEP)
    T(Z,3,X)=1.0/TSTEP+(2.0*A(X)/(VSTEP**2))
    T(Z,3,X)=T(Z,3,X)+PHI(2)*P*ETA
    T(Z,4,X)=((-A(X+1)/VSTEP-B(X+1)/2.0)/VSTEP)
    T(Z,4,X)=T(Z,4,X)+PHI(1)/(2.0*VSTEP)
  ENDIF
  ETA=ETA+VSTEP
10 CONTINUE
P=P+1.0
5 CONTINUE
RETURN
END

SUBROUTINE GETAB(F,VSTEP,M,SM,BM,AA,AP,A,B,Y)
  REAL A(202),B(202),AA,AP,F(6,202),VSTEP,Z,SM,BM,E(202)
  REAL P(202),H(202),G(202),K(202),DB,DA,S1,S2,S3,S4,S5
  INTEGER M,X,Y,N
  Z=-5.0
  DO 10 X=1,M
    CALL FINDA(F,Z,VSTEP,AA,AP,M,A(X),Y)
    CALL FINDB(F,Z,VSTEP,AA,AP,SM,BM,M,B(X),Y)
    Z=Z+VSTEP
 10 CONTINUE
  ETA=-5.0
  DO 20 X=1,M
    H(X)=B(X)*F(Y,X)
    G(X)=A(X)*F(Y,X)
    K(X)=ETA*B(X)*F(Y,X)
    P(X)=ETA*F(Y,X)
    E(X)=F(Y,X)
    ETA=ETA+VSTEP
 20 CONTINUE
20 CONTINUE
CALL SIMPS(H,M,VSTEP,S1)
CALL SIMPS(G,M,VSTEP,S2)
CALL SIMPS(K,M,VSTEP,S3)
CALL SIMPS(P,M,VSTEP,S4)
CALL SIMPS(E,M,VSTEP,S5)
IF (S5.EQ.0.0) RETURN
DB=-S1/S5
DA=-(S2-S3-DB*S4)/S5
DO 30 X=1,M
   B(X)=B(X)+DB
   A(X)=A(X)+DA
30 CONTINUE
RETURN
END

SUBROUTINE FINDA(F,Z,VSTEP,AA,AP,M,A,Y)
REAL F(6,202),Z,VSTEP,AA,AP,A,ETA,SOLN,H(202),FD2
INTEGER X,M,Y
ETA=-5.0
DO 10 X=1,M
   CALL FD2(F,M,X,VSTEP,FD2,Y)
   H(X)=FD2*ABS(ETA-Z)
   ETA=ETA+VSTEP
10 CONTINUE
CALL SIMPS(H,M,VSTEP,SOLN)
A=(SOLN*AA)/(2.0*AP)
RETURN
END

SUBROUTINE FINDB(F,Z,VSTEP,AA,AP,SM,BM,B,Y)
REAL F(6,202),Z,VSTEP,AA,AP,B,ETA,G2,SOLN
REAL J(202),FD3,SM,BM
INTEGER X,M,Y
ETA=-5.0
DO 10 X=1,M
   CALL FD3(F,M,X,VSTEP,FD3,Y)
10 CONTINUE
CALL SIMPS(H,M,VSTEP,SOLN)
A=(SOLN*AA)/(2.0*AP)
RETURN
END
J(X) = FD3*ABS(ETA-Z)  
ETA = ETA + VSTEP  

10 CONTINUE  
CALL SIMPS(J,M,VSTEP,SOLN)  
B = -(AA/AP) * (SM/(2.0*BM)) * SOLN  
RETURN  
END  

SUBROUTINE SIMPS(F,N,H,RESULT)  
REAL F(202),H,RESULT  
INTEGER N,NPANEL,NHALF,NBEGIN,NEND  
NPANEL = N - 1  
NHALF = NPANEL / 2  
NBEGIN = 1  
RESULT = 0.0  
IF ((NPANEL - 2*NHALF) .NE. 0) THEN  
   RESULT = 3.0*H/8.0*(F(1) + 3.0*F(2) + 3.0*F(3) + F(4))  
   NBEGIN = 4  
   IF (N.EQ.4) RETURN  
ENDIF  
RESULT = RESULT + H/3.0*(F(NBEGIN) + 4.0*F(NBEGIN + 1) + F(N))  
NBEGIN = NBEGIN + 2  
IF (NBEGIN .EQ. 4) RETURN  
NEND = N - 2  
DO 10 I = NBEGIN, NEND, 2  
   RESULT = RESULT + H/3.0*(2.0*F(I) + 4.0*F(I+1))  
10 CONTINUE  
RETURN  
END  

SUBROUTINE FD2(F,M,X,VSTEP,FD2,Y)  
REAL F(6,202),FD2,VSTEP  
INTEGER X,M,Y  
IF (X.LE.1) THEN  
   FD2 = (F(Y,X+2) - 2.0*F(Y,X+1) + F(Y,X)) / (VSTEP**2)  
ELSE IF (X.GE.M) THEN  
   FD2 = (F(Y,X) - 2.0*F(Y,X-1) + F(Y,X-2)) / (VSTEP**2)  
ELSE  
   IF (X.EQ.1) THEN  
      FD2 = (F(Y,X) - 2.0*F(Y,X-1) + F(Y,X-2)) / (VSTEP**2)  
   ELSE IF (X.LE.M-1) THEN  
      FD2 = (F(Y,X-1) - 2.0*F(Y,X-2) + F(Y,X-3)) / (VSTEP**2)  
   ENDIF  
ENDIF
ELSE
FD2=(F(Y,X+1)-2.0*F(Y,X)+F(Y,X-1))/(VSTEP**2)
ENDIF
RETURN
END

SUBROUTINE FD3(F,M,X,VSTEP,FD3,Y)
REAL F(6,202),FD3,VSTEP
INTEGER X,M,Y
IF (X.LE.2) THEN
FD3=(F(Y,X+3)-3.0*F(Y,X+2)+3.0*F(Y,X+1)-F(Y,X))/(VSTEP**3)
ELSE IF (X.GE.M-1) THEN
FD3=(F(Y,X)-3.0*F(Y,X-1)+3.0*F(Y,X-2)-F(Y,X-3))/(VSTEP**3)
ELSE
FD3=(F(Y,X+2)-2.0*F(Y,X+1)+2.0*F(Y,X-1)-F(Y,X-2)) /
+(2.0*(VSTEP**3))
ENDIF
RETURN
END

SUBROUTINE MODIAG(M,D,MATRIX,C,SOLN,TRBLE,N)
REAL A(8,202),B,SOLN(202),MATRIX(6,8,202),C(202)
INTEGER M,D,W,X,Y,Z,TRBLE,N,I,MN1
TRBLE=0
DO 10 X=1,D+2
DO 20 Y=1,M
IF (X.EQ.D+2) THEN
A(X,Y)=C(Y)
ELSE
A(X,Y)=MATRIX(N,X,Y)
ENDIF
20 CONTINUE
10 CONTINUE
DO 30 I=2,M
A(2,I)=A(2,I)/A(3,I-1)
A(3,I)=A(3,I)-A(2,I)*A(4,I-1)
A(5,I)=A(5,I)-A(2,I)*A(5,I-1)
30 CONTINUE
30 CONTINUE
DO 80 X=1,M
  IF (A(3,X).EQ.0) THEN
    PRINT*, 'MATRIX HAS NO SOLUTION'
    TRBLE=999
    GOTO 999
  ENDIF
80 CONTINUE
MN1=M-1
A(5,M)=A(5,M)/A(3,M)
DO 40 I=MN1,1,-1
  A(5,I)=(A(5,I)-A(4,I)*A(5,I+1))/A(3,I)
40 CONTINUE
DO 50 X=1,M
  SOLN(X)=A(5,X)
50 CONTINUE
999 RETURN
END

SUBROUTINE AVE(F,M,N,JJ)
REAL F(6,202)
INTEGER X,N,M,Z,JJ
DO 443 Z=1,N-1
  IF (JJ.EQ.1).AND.(Z.EQ.2)) GOTO 443
  IF ((JJ.EQ.2).AND.(Z.EQ.1)) GOTO 443
  DO 444 X=2,M-1
    F(Z,X)=(.025*F(Z,X-1)+F(Z,X)+.025*F(Z,X+1))/1.05
  444 CONTINUE
443 CONTINUE
RETURN
END

SUBROUTINE TOT(F,FTOTAL,PHI,N,M,VSTEP,DU,DUPHI1)
REAL F(6,202),FTOTAL(202),PHI(6),VSTEP,G(202),N0(6),DU
REAL X1,X2,TOL,F1,F2,XERR
INTEGER M,N,X,Y,NLIM
L=(M+1)/2
DO 10 X=1,N-1
   DO 20 Y=1,L
      G(Y)=F(X,Y)
   CONTINUE
   CALL SIMPS(G,L,VSTEP,NO(X))
10 CONTINUE
X1=0.0
X2=1.0E6
TOL=.0001
NLIM=50
30 CALL FCN(NO,X1,N,F1)
CALL FCN(NO,X2,N,F2)
IF (F1*F2.GT.0.0) THEN
   X2=X2*10.0
   IF (X2.GT.1.0E15) THEN
      WRITE(*,*) 'NO SIGN CHANGE UP TO 1E15'
      RETURN
   ENDIF
   GOTO 30
ENDIF
DO 40 J=1,NLIM
   DU=(X1+X2)/2.0
   CALL FCN(NO,DU,N,FR)
   XERR=ABS(X1-X2)/2.0
   IF (XERR.LE.TOL) GOTO 1000
   IF (ABS(FR).LE.TOL) GOTO 1000
   IF (FR*X1.GT.0.0) THEN
      X1=DU
      F1=FR
   ELSE
      X2=DU
      F2=FR
   ENDIF
40 CONTINUE
WRITE(*,*) 'NLIM EXCEEDED'
RETURN
1000 DO 50 X=1,M
   FTOTAL(X)=0.0
   DO 60 Y=1,N-1
      FTOTAL(X)=FTOTAL(X)+DU**(Y-1)*F(Y,X)
   60 CONTINUE
ASYMPTOTICALLY CORRECT COLLISIONAL PRESHEATHS

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Abstract: Few exact solutions for collisional presheaths exist because of the difficulty of simultaneously satisfying both the collisional Boltzmann equation and the Poisson equation. The exact solutions that do exist are for very specialized collision terms such as constant cross section charge exchange with cold neutrals. The present paper presents an asymptotic method which is applicable to a variety of collision terms and is applied in particular to constant collision frequency charge exchange with non-cold neutrals. Constant collision frequency and constant cross section collision with cold neutrals results are also presented. The first order terms for the presheath potential rise and ion distribution functions are calculated and it is shown that second and higher order terms can be calculated using a multi-exponential expansion for presheath potential rise. The first order cold neutral constant cross section results correspond well to the exact solution. The calculated presheath potential rises are of the order expected from the Bohm criterion, and in some of the specialized cold neutral cases, exactly $\frac{kT_e}{2}$. The presheath potential rise is reduced by a neutral plasma potential gradient which accelerates ions toward the presheath. In all cases the collisional presheath is asymptotically matched to both the neutral plasma and the collisionless sheath.

PACS: 52.40.Hf, 52.75.Fk
1. INTRODUCTION

The majority of plasma-surface interaction work matches a neutral plasma to a collisionless sheath without detailed consideration of a collisional presheath. In many applications, especially those such as thermionic energy convertors that require exact knowledge of the presheath-sheath structure, the collisional presheath structure is of great interest. Sheath theory beginning with Bohm[1] tends to assume that the plasma ion distribution is cold so that a minimum presheath potential rise may be calculated which makes the collisionless sheath self consistent. Harrison and Thompson[2] generalize the Bohm criterion to non-cold ion distributions, however the result is sensitive to the density of the low energy tail of the ion distribution which in turn is strongly affected by the collisional presheath. A second difficulty in the absence of a collisional presheath is that the collisionless sheath and the surface beyond it may return no ions or a non-thermal distribution of ions which the collisional presheath must match to the neutral plasma region.

Some exact solutions exist for presheaths; notably Ecker and Kanne[3] and Riemann[4] who derive exact solutions for collision terms based on charge exchange with cold neutrals and Emmert et al[5] who derive an exact collisionless solution in which there is an ionization source. In the present paper an asymptotically correct collisional presheath theory is developed which can be applied to a less restrictive range of collision terms. Potential in the presheath is expanded as a multi-exponential series and the distribution functions are expanded in terms of presheath potential rise. First order approximations are calculated for both constant collision frequency and constant cross section charge exchange collisions.

2. FIRST ORDER ASYMPTOTIC POTENTIAL FORMULATION

In this section it is assumed that the potential in the collisional presheath is of the form

\[ U = U_0 + \Delta U = \alpha x + e^{\beta x} \]  

(1)

where \( U_0 = \alpha x \) is the assumed linear potential in the neutral plasma and \( \Delta U = e^{\beta x} \) is the additional potential rise in the collisional presheath as shown in fig.1. In this paper the
convention used is that \( U = q \phi \) where \( q \) is the electron charge and \( \phi \) is potential in electron volts so that \( U \) has units of energy. In addition, potential is defined in the reverse of the usual sign convention so that increasing potential repels electrons. With these conventions, the Boltzmann equation can be written as

\[
\frac{dU}{dx} \{ v \frac{\partial f}{\partial U} \pm \frac{1}{m} \frac{\partial f}{\partial v} \} = \{ \frac{\partial f}{\partial t} \} e. \tag{2}
\]

In equation 2 and those following, the \( \pm \) denotes the sign of the charged species in question; the upper sign referring to positively charged ions and the lower sign referring to electrons. It is assumed that the distribution function in the neutral plasma \( (p_x < 0) \) is independent of potential, therefore the Boltzmann equation is expressed in terms of \( \Delta U \) which will be the expansion variable in the presheath:

\[
v \beta \Delta U \frac{\partial f}{\partial \Delta U} \pm \frac{1}{m} (\beta \Delta U + \alpha) \frac{\partial f}{\partial v} = \{ \frac{\partial f}{\partial t} \} e. \tag{3}
\]

The distribution function is then expanded as

\[
f = f_0(v) + \Delta U f_1(v) + \Delta U^2 f_2(v) + \ldots \tag{4}
\]

so that the derivatives are

\[
\frac{\partial f}{\partial \Delta U} = f_1(v) + 2 \Delta U f_2(v) + 3 \Delta U^2 f_3(v) + \ldots \tag{5}
\]

and

\[
\frac{\partial f}{\partial v} = \frac{\partial f_0}{\partial v}(v) + \Delta U \frac{\partial f_1}{\partial v}(v) + \Delta U^2 \frac{\partial f_2}{\partial v}(v) + \ldots \tag{6}
\]

Substitution of (4) and (5) into the Boltzmann equation (3) yields the terms:

\[
1: \quad \pm \frac{\alpha}{m} \frac{\partial f_0}{\partial v}(v) = \{ \{ \frac{\partial f}{\partial t} \} e \}_1. \tag{7a}
\]

\[
\Delta U: \quad v \beta f_1(v) \pm \frac{\beta}{m} \frac{\partial f_0}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_1}{\partial v}(v) = \{ \{ \frac{\partial f}{\partial t} \} e \}_\Delta U. \tag{7b}
\]

\[
\Delta U^2: \quad 2v \beta f_2(v) \pm \frac{\beta}{m} \frac{\partial f_1}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_2}{\partial v}(v) = \{ \{ \frac{\partial f}{\partial t} \} e \}_\Delta U^2. \tag{7c}
\]

\[
\vdots
\]

\[
\Delta U^n: \quad n v \beta f_n(v) \pm \frac{\beta}{m} \frac{\partial f_{n-1}}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_n}{\partial v}(v) = \{ \{ \frac{\partial f}{\partial t} \} e \}_\Delta U^n. \tag{7n}
\]

The quantity \( \beta \), representing the presheath potential rise, is determined from the Poisson equation,

\[
\frac{d^2U}{dx^2} = 4 \pi q^2 \left[ \int_{-\infty}^{\infty} f_i(v, \Delta U) \, dv - \int_{-\infty}^{\infty} f_e(v, \Delta U) \, dv \right] \tag{8}
\]
where \( q \) is the electron charge. It is assumed that the ions are singly ionized for simplicity. The Poisson equation (8) is expanded as

\[
\beta^2 \Delta U = 4\pi q^2 \left[ \left( \int_{-\infty}^{\infty} f_{i1}(v) \, dv - \int_{-\infty}^{\infty} f_{i0}(v) \, dv \right) \Delta U + \left( \int_{-\infty}^{\infty} f_{e2}(v) \, dv - \int_{-\infty}^{\infty} f_{e0}(v) \, dv \right) \Delta U^2 + \ldots \right] \quad (9)
\]

where charge neutrality at \( \Delta U = 0 \) has eliminated the terms containing \( f_{i0} \) and \( f_{e0} \):

\[
n_0 = \int_{-\infty}^{\infty} f_{i0}(v) \, dv = \int_{-\infty}^{\infty} f_{e0}(v) \, dv.
\]

The quantity \( n_0 \) is the neutral plasma density.

### 3. First Order Solution with a Constant Collision Frequency Charge Exchange Collision Term

The constant collision frequency charge exchange collision term is modeled as

\[
\left\{ \frac{\partial f_i}{\partial t} \right\}_c = \frac{1}{\tau_n} \left\{ f_n(v) \int_{-\infty}^{\infty} f_{i}(u) \, du - f_i(v) \int_{-\infty}^{\infty} f_n(u) \, du \right\}
\]

(10)

where \( f_n(v) \) is the neutral distribution and \( \tau \) is the collision time. Previous work has assumed cold neutrals and results in an integral equation which is solvable only for constant collision cross section[4].

#### A. Zero Plasma Potential Gradient (\( \alpha = 0 \))

In this case equations 7a,b,c-n become

\[
1: \quad 0 = \frac{1}{\tau_n} \left[ f_n(v) \int_{-\infty}^{\infty} f_{i0}(u) \, du - f_{i0}(v) \int_{-\infty}^{\infty} f_n(u) \, du \right]
\]

\[
\Delta U : \quad v\beta f_{i1}(v) + \frac{\beta}{m} \frac{\partial f_{i0}}{\partial v}(v) = \frac{1}{\tau_n} \left[ f_n(v) \int_{-\infty}^{\infty} f_{i1}(u) \, du - f_{i1}(v) \int_{-\infty}^{\infty} f_n(u) \, du \right]
\]

\[
\vdots
\]

\[
\Delta U^n : \quad n v\beta f_{in}(v) + \frac{\beta}{m} \frac{\partial f_{i(n-1)}}{\partial v}(v) = \frac{1}{\tau_n} \left[ f_n(v) \int_{-\infty}^{\infty} f_{in}(u) \, du - f_{in}(v) \int_{-\infty}^{\infty} f_n(u) \, du \right].
\]

(11)
Under the assumption that the neutral distribution is Maxwellian \( f_n(v) = n_n \sqrt{\frac{m}{2\pi kT}} \exp \left( -\frac{mv^2}{2kT} \right) \), the solution to (11) is

\[
\begin{align*}
  f_{i0} &= C f_n(v) \\
  f_{i1}(v) &= \frac{1}{kT} f_{i0}(v) \\
  &\vdots \\
  f_{in}(v) &= \frac{1}{nkT} f_{i(n-1)}(v).
\end{align*}
\]

(12)

Thus

\[
 f_i(v, \Delta U) = C e^{(\frac{\Phi U}{kT})} f_n(v)
\]

(13)

which is the expected result. In this case the mean ion velocity is zero throughout the collisional presheath since charge exchange collisions conserve ions and the mean ion velocity in the neutral plasma is zero. Thus, if \( \alpha = 0 \), constant collision frequency charge exchange collisions do not shift the ion distribution upward in velocity. This presheath can be matched to a collisionless sheath only if the collisionless sheath returns all of the ions entering it from the collisional presheath.

With electron density assumed to follow

\[
 n_e(\Delta U) = e^{(\frac{\Delta \Phi}{kT_e})}
\]

the Poisson equation (9) yields

\[
 \rho^2 = 4\pi q^2 n_0 \left( \frac{1}{kT} + \frac{1}{kT_e} \right)
\]

which is the length scale of the Debye length. Thus for \( \alpha = 0 \) the collisional presheath is not distinct from the collisionless sheath since there is no separate collisional presheath length scale.

**B. Non-Zero Plasma Potential Gradient (\( \alpha \neq 0 \))**

Under this condition there is a flux of ions from the plasma into the sheath which allows the construction of a collisional presheath that accelerates the ions and depopulates the ion distribution of returning ions. Thus the collisional presheath may be correctly
matched to collisionless sheath which returns no ions. In this case (7a) and (7b) can be written as

\[
\frac{\alpha}{m} \frac{\partial f_0}{\partial v}(v) = \frac{1}{\tau_n} \left[ f_n(v) n_0 - n_n f_0(v) \right] \quad (14a)
\]

\[
v \beta f_1(v) + \frac{\beta}{m} \frac{\partial f_0}{\partial v} + \frac{\alpha}{m} \frac{\partial f_1}{\partial v}(v) = \frac{1}{\tau_n} \left[ f_n(v) n_1 - n_n f_1(v) \right]. \quad (14b)
\]

The solution to these are

\[
f_0(v) = n_0 e^{-\frac{m \beta}{2 \alpha}} \sqrt{\frac{m}{2 \pi kT}} \int_{-\infty}^{\infty} e^{-\frac{m \beta^2}{8 \alpha^2} + \frac{m \beta}{2 \alpha} u} \, du \quad (15)
\]

and

\[
f_1(v) = \exp \left( -\frac{\beta m v^2}{2 \alpha} - \frac{m v}{\alpha} \right) \left[ \int_0^v \exp \left( \frac{\beta m^2}{2 \alpha} + \frac{m \alpha}{2 \alpha^2} \right) \left( \frac{n_1 m}{n_0} \sqrt{\frac{m}{2 \pi kT}} e^{-\frac{m \beta^2}{8 \alpha^2} - \frac{m \beta}{2 \alpha} u} \right) du + C \right] \quad (16)
\]

where

\[
n_0 = \int_{-\infty}^{\infty} f_0(v) \, dv \quad (17)
\]

and

\[
n_1 = \int_{-\infty}^{\infty} f_1(v) \, dv. \quad (18)
\]

The constant of integration in (15) has been set so that \( f_0 \) goes to zero at \(-\infty\); \( f_0 \) goes to zero at \( \infty \) regardless of the constant of integration. Equation (17) is immediately satisfied by (15). The constant of integration \( C \) in (16) must be set so that (18), which represents self-consistency, is satisfied. It can be seen from (16) that \( f_1 \) goes to zero at \(-\infty\) and \( \infty \) regardless of the constant \( C \). From (18), then

\[
C = \frac{n_1 \left( 1 - \int_{-\infty}^{\infty} \exp \left( -\frac{\beta m^2}{2 \alpha} - \frac{m v}{\alpha} \right) \int_0^v \exp \left( \frac{\beta m^2}{2 \alpha} + \frac{m \alpha}{2 \alpha^2} \right) \frac{m}{n_0} \sqrt{\frac{m}{2 \pi kT}} \exp \left( -\frac{m \beta^2}{8 \alpha^2} \right) du \, dv \right)}{\sqrt{\frac{2 \pi \alpha}{m \beta} e^{2 \alpha \beta^2}}} + \frac{\int_{-\infty}^{\infty} \exp \left( -\frac{\beta m^2}{2 \alpha} - \frac{m v}{\alpha} \right) \int_0^v \frac{\beta}{\alpha} \exp \left( \frac{\beta m^2}{2 \alpha} + \frac{m \alpha}{2 \alpha^2} \right) f_0(v) \, du \, dv}{\sqrt{\frac{2 \pi \alpha}{m \beta} e^{2 \alpha \beta^2}}} \quad (19)
\]

The exponential sheath rise, \( \beta \), is determined from the Poisson equation under the simplifying assumption that

\[
n_\phi = \int_{-\infty}^{\infty} f_\phi(v) \, dv = n_0 \exp \left( -\frac{\Delta U}{k T_\phi} \right). \quad (20)
\]

With (20) the Poisson equation (9) to the first order becomes

\[
\beta^2 = 4 \pi q^2 \left( n_1 + \frac{n_0}{k T_\phi} \right). \quad (21)
\]
Since the ion density is only calculated to the first order, the same will be done for the electron density in (20).

To obtain a particular solution it is assumed here that the collisionless sheath to which the collisional presheath is joined at \( \Delta U = \Delta U^* \) returns no ions. In particular,

\[
\int_{-\infty}^{0} f(v, \Delta U^*) \, dv = 0 \tag{22}
\]

or

\[
\int_{-\infty}^{0} f_0(v) \, dv + \Delta U^* \int_{-\infty}^{0} f_1(v) \, dv = 0. \tag{23}
\]

and

\[
\int_{-\infty}^{0} v f(v, \Delta U^*) \, dv = 0 \tag{24}
\]

or

\[
\int_{-\infty}^{0} v f_0(v) \, dv + \Delta U^* \int_{-\infty}^{0} v f_1(v) \, dv = 0. \tag{25}
\]

Because the approximation is only first order, it is not possible to impose the condition that \( f(v) \) is uniformly zero for returning ions. Equations (23) and (25) represent zero returning ion density and zero returning ion flux. When higher order terms are included, the conditions of zero returning ion momentum flux, zero returning ion energy flux and so forth can be applied in succession. Solving (21), (23) and (25) for \( n_1, \beta \) and \( \Delta U^* \) closes the equations. Equation (20) is also the Bohm criterion at \( \Delta U = \Delta U^* \) for the first order approximation

\[
n_1 + \frac{n_0}{kT_e} > 0. \tag{26}
\]

The Poisson equation (21) can be written as

\[
\beta^2 \lambda_D^2 = 1 + kT_e \frac{n_1}{n_0} \tag{27}
\]

where

\[
\lambda_D = \sqrt{\frac{kT_e}{4\pi q^2 n_0}} \tag{28}
\]

is the Debye length. It is expected that the length scale of the presheath should be of the order \( \beta = 1/\lambda_i \) where \( \lambda_i \) is the ion mean free path. In the circumstance that the Debye length is small compared to the ion mean free path, the product \( \beta^2 \lambda_D^2 \) is small and

\[
n_1 = -\frac{n_0}{kT_e}. \tag{29}
\]
Nondimensionalization results in:

$$A = \frac{\pi r}{m} \sqrt{\frac{m}{2kT}}$$  \hspace{1cm} (30)$$

$$B = \frac{\beta kT}{\alpha}$$  \hspace{1cm} (31)$$

$$R_e = \frac{T_e}{T}$$  \hspace{1cm} (32)$$

$$\omega = \sqrt{\frac{m}{2kT} v},$$  \hspace{1cm} (33)$$

where $A$ and $R_e$ are the parameters and $B$ is a function of $A$ and $R_e$. The distribution functions can then be written as

$$F_0(\omega, A) = \frac{f_0(v)}{n_0 \sqrt{\frac{\pi m}{2kT}}} = \frac{\exp\left(-\frac{\omega}{\sqrt{A}}\right)}{\sqrt{\pi A}} \int_{-\infty}^{\omega} \exp\left(-\xi^2 + \frac{\xi}{A}\right) d\xi$$  \hspace{1cm} (34)$$

and

$$F_1(\omega, A, B) = \frac{f_1(v)}{n_0 \sqrt{\frac{\pi m}{2kT}}}$$

$$= \frac{\exp\left(-\frac{B\omega^2 - \omega}{\sqrt{A}}\right)}{\sqrt{\pi}} \left[ \int_0^\omega \exp\left(B\xi^2 + \frac{\xi}{A}\right) \left( -\frac{1}{A} \exp\left(-\xi^2\right) - R_e B \left(\frac{1}{A} \exp\left(-\xi^2\right) - \frac{1}{A^2} \exp\left(-\frac{\xi}{A}\right) \int_{-\infty}^\xi \exp\left(-\eta^2 + \frac{\eta}{A}\right) d\eta\right) \right] d\xi + C \right]$$  \hspace{1cm} (35)$$

where

$$C = -\frac{\left(1 - \frac{\frac{1}{\sqrt{\pi A}} \int_{-\infty}^{\infty} \exp\left(-B\omega^2 - \frac{\omega}{\sqrt{A}}\right) \int_0^\omega \exp\left(B\xi^2 + \frac{\xi}{A} - \xi^2\right) d\xi d\omega\right)}{\frac{1}{B} \exp\left(\frac{1}{4B^2a^2}\right)}$$

$$+ \frac{R_e B \int_{-\infty}^{\infty} \exp\left(-B\omega^2 - \frac{\omega}{\sqrt{A}}\right) \int_0^\omega \exp\left(B\xi^2 + \frac{\xi}{A}\right) \left(\frac{1}{A} \exp\left(-\xi^2\right) - \frac{1}{A^2} \exp\left(-\frac{\xi}{A}\right) \int_{-\infty}^\xi \exp\left(-\eta^2 + \frac{\eta}{A}\right) d\eta\right) d\xi d\omega}{\sqrt{\frac{1}{B} \exp\left(\frac{1}{4B^2a^2}\right)}}.$$  \hspace{1cm} (36)$$

Thus (23) and (25) become

$$\int_{-\infty}^{0} F_0(\omega, A) d\omega + \frac{\Delta U^*}{kT_e} \int_{-\infty}^{0} F_1(\omega, A, B) d\omega = 0$$  \hspace{1cm} (37)$$

and

$$\int_{-\infty}^{0} \omega F_0(\omega, A) d\omega + \frac{\Delta U^*}{kT_e} \int_{-\infty}^{0} \omega F_1(\omega, A, B) d\omega = 0.$$  \hspace{1cm} (38)$$

Figure 2 presents the presheath potential rise $\Delta U^* / kT_e$ and the nondimensional exponential rise $B$ as function of the nondimensional neutral plasma gradient $A$ for a range of electron to neutral temperature ratios $R_e$. As would be intuitively expected, the presheath potential rise decreases with increasing $A$. Figure 3 presents the ion distribution functions.
at the neutral plasma - collisional presheath interface, $F_0(\omega)$, the first order correction to the distribution function, $F_1(\omega)$, and the resulting distribution function at the collisional presheath - collisionless sheath interface, $F_0(\omega) + \Delta U^* F_1(\omega)$. Although the resulting distribution is not uniformly zero for $\omega < 0$, its net returning density and flux are zero by (37) and (38). It is expected that higher order corrections to the distribution function and potential with the corresponding application of higher order moment conditions of zero returning momentum, energy and so forth will converge the returning distribution function toward a uniform zero.

In the limit of cold neutrals, the constant collision frequency charge exchange solution is considerably simplified. Equations (14a) and (14b) become

$$\frac{\alpha \partial f_0}{m \partial v}(v) = \frac{1}{r_n} \left[ n_n \delta_n(v)n_0 - n_n f_0(v) \right]$$

and

$$v\beta f_1(v) + \frac{\beta \partial f_0}{m \partial v} + \frac{\alpha \partial f_1}{m \partial v}(v) = \frac{1}{r_n} \left[ n_n \delta_n(v)n_1 - n_n f_1(v) \right].$$

The solution to (39) and (40) are

$$f_0(v) = \begin{cases} 
  n_0 \left( \frac{m}{\alpha \tau} \right) \exp \left( -\frac{mv}{\alpha \tau} \right), & v > 0 \\
  0, & v < 0 
\end{cases}$$

and

$$f_1(v) = \begin{cases} 
  \exp \left( -\frac{\beta mv^2}{2\alpha} - \frac{mv}{\alpha \tau} \right) \left[ C^+ + \frac{\beta}{\alpha} n_0 \left( \frac{m}{\alpha \tau} \right)^2 \int_0^v \exp \left( \frac{\beta mu^2}{2\alpha} \right) du \right], & v > 0 \\
  \exp \left( -\frac{\beta mv^2}{2\alpha} - \frac{mv}{\alpha \tau} \right) \left[ C^- \right], & v < 0 
\end{cases}$$

such that

$$C^+ - C^- = \frac{m}{\alpha \tau} \left( n_1 - \frac{\beta}{\alpha} n_0 \right).$$

Equation 41 immediately satisfies $n_0 = \int_{-\infty}^{\infty} f_0(v) dv$. No returning ions implies that

$$C^- = 0$$

and

$$C^+ = \frac{m}{\alpha \tau} \left( n_1 - \frac{\beta}{\alpha} n_0 \right)$$

since $f_0$ on $v < 0$ is already zero. The final condition is then that $n_1 = \int_{-\infty}^{\infty} f_1(v) dv$ or

$$n_1 = \int_{-\infty}^{\infty} \exp \left( -\frac{\beta mv^2}{2\alpha} - \frac{mv}{\alpha \tau} \right) \left[ n_1 \frac{m}{\alpha \tau} - \frac{\beta n_0}{\alpha \tau} + \frac{\beta n_0}{\alpha \tau} \left( \frac{m}{\alpha \tau} \right)^2 \int_{-\infty}^{v} \exp \left( \frac{\beta mu^2}{2\alpha} \right) du \right] dv.$$
The application of \( n_1 = -\frac{\partial}{\partial t} \) yields

\[
\frac{\beta kT_e}{\alpha} = \frac{1 - \int_0^\infty \exp \left( -\frac{\beta a r^2}{2m} \xi^2 - \xi \right) d\xi}{\int_0^\infty \exp \left( -\frac{\beta a r^2}{2m} \xi^2 - \xi \right) \left( 1 - \int_0^\xi \exp \left( \frac{\beta a r^2}{2m} \eta^2 \right) d\eta \right) d\xi}.
\]

(47)

In this case, \( \Delta U^* \) is defined by

\[
f_0(0^+) + \Delta U^* f_1(0^+) = 0
\]

(48)

which yields

\[
\frac{\Delta U^*}{kT_e} = \frac{1}{\frac{\beta kT_e}{\alpha} + 1}
\]

(49)

as expected. In the limit of \( \frac{\beta a r^2}{2m} \to 0 \) we have

\[
\frac{\beta kT_e}{\alpha} = 1
\]

(50)

and

\[
\Delta U^* = \frac{kT_e}{2}
\]

(51)

which corresponds to the Bohm criterion. Figure 4 presents the variation of \( B = \frac{\beta kT_e}{\alpha} \) with \( \frac{\beta a r^2}{2m} \) for the cold neutral case. A particular \( \beta \) for the parameters can be conveniently found by drawing a line from the origin with slope \( \frac{2m kT_e}{\beta a r^2} \) so that the intersection is the solution. Figure 5 presents an example cold neutral ion distribution.

4. FIRST ORDER SOLUTION WITH A QUASI-CONSTANT CROSS SECTION COLLISION TERM

First order asymptotic solutions can also be developed for a quasi-constant cross section collision term

\[
\left\{ \frac{\partial f}{\partial t} \right\}_e = \sigma \left[ \int_{-\infty}^\infty f_n(v)f(u)|v-u|du - \int_{-\infty}^\infty f(v)f_n(u)|v-u|du \right].
\]

(52)

This collision term is not really constant cross section because it is a one dimensional representation which does not take into account average velocities in the other two dimensions. However, this collision term corresponds to that commonly called constant cross section. The application of this term leads to a set of integro-differential equations which can be at
least approximately solved, and in the cold neutral case it leads to readily soluble first order differential equations. The cold neutral case presented here corresponds to that which can be solved exactly (Riemann [4]). Unfortunately, though, the exact solution method is not extensible to non-cold neutrals. The cold neutral collision term is

$$\left\{ \frac{\partial f}{\partial t} \right\}_c = \sigma n_n \delta(v) \int_{-\infty}^{\infty} f(u) |u| du - \sigma f(v)n_n |v|,$$  \hspace{1cm} (53)

and the zero order Boltzmann equation term (7a) becomes

$$\frac{\alpha}{m} \frac{\partial f_0}{\partial v}(v) = \sigma n_n \delta(v) \int_{-\infty}^{\infty} f_0(u) |u| du - \sigma n_n |v| f_0(v),$$  \hspace{1cm} (54)

for which the solution is

$$f_0(v) = \begin{cases} \frac{n_0}{\sqrt{\pi}} \frac{\sigma m n_n}{\alpha} \exp \left( - \frac{\sigma m n_n}{2\alpha} v^2 \right), & v > 0 \\ 0, & v < 0 \end{cases}.$$  \hspace{1cm} (55)

The first order Boltzmann term is

$$v \beta f_1(v) + \frac{\beta}{m} \frac{\partial f_0}{\partial v}(v) + \frac{\alpha}{m} \frac{\partial f_1}{\partial v}(v) = \sigma n_n \delta(v) \int_{-\infty}^{\infty} f_1(u) |u| du - \sigma n_n |v| f_1(v),$$  \hspace{1cm} (56)

for which the solution is

$$f_1(v) = \begin{cases} \exp \left( - \frac{1}{2} \left( \frac{\beta m}{\alpha} + \frac{\sigma m n_n}{\alpha} \right) v^2 \right) \left[ \frac{n_0}{\sqrt{\pi}} \left( \frac{1}{\sqrt{\frac{\sigma m n_n}{\alpha}}} \right)^3 \left( \exp \left( \frac{\beta m v^2}{\alpha} \right) - 1 \right) + C^+ \right], & v > 0 \\ \exp \left( - \frac{1}{2} \left( \frac{\beta m}{\alpha} - \frac{\sigma m n_n}{\alpha} \right) v^2 \right) \left[ C^- \right], & v < 0 \end{cases}.$$  \hspace{1cm} (57)

The jump condition at \( v = 0 \) must be satisfied in (56),

$$C^+ - C^- = \frac{\sigma m n_n}{\alpha} \int_{-\infty}^{\infty} f_1(u) |u| du - \frac{\beta}{\alpha} n_0 \frac{\sqrt{2}}{\pi} \frac{\sqrt{\sigma m n_n}}{\alpha}.$$  \hspace{1cm} (58)

No returning ions, \( C^- = 0 \) and the application of (58) to (57) yields

$$C^+ = -n_0 \frac{\beta}{\alpha} \frac{\sqrt{2}}{\pi} \frac{\sqrt{\sigma m n_n}}{\alpha}.$$  \hspace{1cm} (59)

The collisional presheath - collisionless sheath boundary \( \Delta U^* \) is again

$$0 = f(0^+) = f_0(0^+) + \Delta U^* f_1(0^+)$$  \hspace{1cm} (60)

which yields

$$\frac{\Delta U^*}{kT_e} = \frac{\alpha}{\beta k T_e}.$$  \hspace{1cm} (61)
Equation (57) is integrated to

\[ n_1 = \int_{-\infty}^{\infty} f_1(u) \, dv = \frac{n_0 n_e \sigma}{\alpha} \left[ 1 - \sqrt{1 + \frac{\beta}{\sigma_n}} \right] \]  

(62)

and applied to the Poisson equation (8) to produce

\[ \left( \frac{\beta}{\sigma_n} \right)^2 = \left( \frac{4 \pi q^2 n_0}{k T_e} \right)^2 \left( \frac{1}{\sigma n} \right)^2 \left[ \frac{n_e \sigma k T_e}{\alpha} \left( 1 - \sqrt{1 + \frac{\beta}{\sigma n}} \right) + 1 \right]. \]  

(63)

Under the assumption that the Debye length is short compared to the ion mean free path,

\[ \frac{\beta}{\sigma_n} \ll 1, \]  

equation (63) results in

\[ \frac{\beta}{\sigma_n} = \frac{\alpha}{n_e \sigma k T_e} \left( 1 + \frac{\alpha}{n_e \sigma k T_e} \right), \]  

(64)

and

\[ \frac{\Delta U^*}{k T_e} = \frac{1}{2 + \frac{\alpha}{n_e \sigma k T_e}}. \]  

(65)

The Bohm criterion is satisfied at \( \Delta U^* \) to the first order by virtue of (63). And interestingly the presheath potential rise for \( \alpha = 0 \) is exactly that required by the cold ion Bohm criterion. Figure 6 presents an example cold neutral ion distribution.

5. MULTI-EXPONENTIAL FORMULATION

In the previous sections we have calculated only the first order terms in the ion distribution and presheath potential rise. Also we have implicitly made the same first order approximation for electrons,

\[ n_e = n_0 \left( 1 - \frac{\Delta U}{k T_e} \right). \]

(66)

A complete multiexponential expansion can also be constructed that correctly calculates the second and higher order terms. Potential in the presheath is

\[ U = U_0 + \Delta U + a_2 \Delta U^2 + a_3 \Delta U^3 + ... \]

(67)

where \( U_0 = \alpha x \) and \( \Delta U = \exp(\beta x) \). Thus

\[ \frac{dU}{dx} = \alpha + \beta \Delta U + 2 \beta a_2 \Delta U^2 + 3 \beta a_3 \Delta U^3 + ... \]

(68)
and
\[
\frac{d(\Delta U)}{dU} = \frac{\beta \Delta U}{\alpha + \beta \Delta U + 2\beta a_2 \Delta U^2 + 3\beta a_3 \Delta U^3 + ...}
\]
which transforms the Boltzmann equation
\[
\frac{dU}{d\tau} \left[ v \frac{\partial f}{\partial \Delta U}(v) \frac{\partial \Delta U}{\partial U} \pm \frac{1}{m} \frac{\partial f}{\partial v}(v) \right] = \left\{ \frac{\partial f}{\partial t} \right\}_c
\]
into
\[
v\beta \Delta U \frac{\partial f}{\partial \Delta U}(v) \pm \frac{1}{m} \left( \alpha + \beta \Delta U + 2\beta a_2 \Delta U^2 + ... \right) \frac{\partial f}{\partial v}(v) = \left\{ \frac{\partial f}{\partial t} \right\}_c
\]
or
\[
1:\quad \pm \frac{\alpha}{m} \frac{\partial f_0}{\partial v}(v) = \left[ \left\{ \frac{\partial f}{\partial t} \right\}_c \right]_1
\]
\[
\Delta U:\quad v \beta f_1(v) \pm \frac{\beta}{m} \frac{\partial f_0}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_1}{\partial v}(v) = \left[ \left\{ \frac{\partial f}{\partial t} \right\}_c \right]_{\Delta U}
\]
\[
\Delta U^2:\quad 2v \beta f_2(v) \pm \frac{2\beta a_2}{m} \frac{\partial f_0}{\partial v}(v) \pm \frac{\beta}{m} \frac{\partial f_1}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_2}{\partial v}(v) = \left[ \left\{ \frac{\partial f}{\partial t} \right\}_c \right]_{\Delta U^2}
\]
\[
\Delta U^n:\quad n v \beta f_n(v) \pm \frac{n \beta a_n}{m} \frac{\partial f_0}{\partial v}(v) \pm \frac{(n-1) \beta a_{n-1}}{m} \frac{\partial f_1}{\partial v}(v) \pm ... \pm \frac{\beta}{m} \frac{\partial f_{n-1}}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_n}{\partial v}(v)
\]
\[
= \left[ \left\{ \frac{\partial f}{\partial t} \right\}_c \right]_{\Delta U^n}.
\]

The Poisson equation (8) becomes
\[
\beta^2 \Delta U + (2\beta)^2 a_2 \Delta U^2 + (3\beta)^2 a_3 \Delta U^3 + ... = 4\pi q^2 \left[ \Delta U \left( \int_{-\infty}^{\infty} f_{i1}(v) \, dv - \int_{-\infty}^{\infty} f_{e1}(v) \, dv \right) 
+ \Delta U^2 \left( \int_{-\infty}^{\infty} f_{i2}(v) \, dv - \int_{-\infty}^{\infty} f_{e2}(v) \, dv \right)
+ ... \right].
\]

6. CONCLUSIONS

It has been shown that approximate collisional presheath solutions can be obtained for a variety of collision terms. In particular the constant collision frequency case has been solved approximately whereas previous attempts at exact solutions have found this case intractable. In addition it has been shown that higher order corrections can be made a
regular and tractable fashion. Also the return of ions from the collisionless sheath can be treated.

ACKNOWLEDGMENTS

This work was supported by the Air Force Office of Scientific Research grant 85-0375. Thanks are also due to Gregory Ridderbusch who produced the numerical results.

REFERENCES


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6. Constant Cross Section Ion Distribution with Cold Neutrals.
Figure 1. Asymptotically Correct Potential in the Collisional Presheath.
Figure 1. Asymptotically Correct Potential in the Collisional Presheath.

\[ U = U_0 + \Delta U \]

\[ \Delta U^* \]

neutral plasma — collisional presheath — collisionless sheath
Figure 2. Constant Collision Frequency Presheath Rise
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Abstract: This paper couples exact collisionless sheath calculations to an isothermal electron model of a thermionic convertor. The emitter sheath structure takes into account reflected ions, trapped ions and surface emission ions. It is shown that the lessening the net loss of ions at the emitter in the ignited mode by these phenomena degrades performance. In addition it is shown that when the emitter returns to many of the ions, the arc is extinguished because there is insufficient resistive heating to maintain the necessary plasma electron temperature for ionization. These results suggest that the ignited mode cannot be improved much. However, non-ignited modes in which the electron temperature remains low, such as the pulsed mode do not suffer from this adverse behavior.
1. INTRODUCTION

Emitter sheath phenomena are important in thermionic energy converters because the emitter sheath forms the emitter boundary condition for the plasma in the gap by controlling both the ion loss rate and the loss rate of hot (3000 K) plasma electrons to the emitter. This paper examines two expected emitter sheath phenomena and their effects on converter performance: reflection of ions coming from the plasma by a double emitter sheath, and ions trapped in the double emitter sheath. The authors have previously suggested that ion reflection might improve thermionic energy converter performance [1] and and subsequently shown that ion reflection at the emitter is likely to degrade the performance in the ignited mode and, in addition, that trapped ions in a double emitter sheath are also likely to degrade performance in the ignited mode [2]. Lundgren [3,4] has also shown this with simplified ion and electron dynamics. In the present paper the effects of emitter ion reflection and ion trapping in the ignited mode are calculated using exact electron and ion dynamics in the collisionless (except for ion trapping) sheaths. The electrons entering the sheaths from the plasma are assumed to have a Maxwellian distribution, but no assumptions are made about the returning electrons and the electron density in the sheath is calculated exactly. The ions entering the sheaths from the plasma are not assumed cold, but are given the correct ion temperature and shifted in velocity according to a generalization of the Bohm criterion [5,6].

Both ion reflection and trapped ions in the emitter sheath reduce the normalized (by plasma density) net ion loss rate to the emitter. Also, both of these phenomena raise the normalized plasma density adjacent to the emitter. The higher plasma density at the emitter causes a greater increase in the loss of hot plasma electron energy to the emitter than the corresponding decrease in the loss of ionization energy (carried by the ions) to the emitter. Therefore these emitter sheath phenomena increase arc-drop. Within the limitations of the present isothermal thermionic converter formulation, all three of these phenomena (which become significant at low currents) steepen the current-voltage characteristic. At low current densities, the present theory shows that the collector sheath height decreases, resulting in a larger electron diffusion velocity than can be justified for the continuum model used in the plasma region. The result of lower performance at lower
current is in agreement with experimental studies. At some current density which depends strongly on the emitter sheath conditions, the ignited mode is no longer self-sustaining and the arc is extinguished.

Figure 1 is a schematic diagram of the cesium diode convertor. The emitter is heated externally to temperature $T_E$ which is typically 1750 K and the collector is cooled to temperature $T_C$ which is typically 750 K. The gap space, $d$, or convertor length, which is typically .25 mm, separates the emitter from the collector. The cesium reservoir, which is sometimes imbedded in the collector, is kept at temperature, $T_R$, to maintain the desired cesium pressure (typically 1 torr) in the gap. The electrical load is connected across the emitter and collector to produce power.

2. THE ISOTHERMAL ELECTRON FORMULATION

In this section the isothermal thermionic convertor formulation is developed. The formulation follows closely the notation of Lam[7] but is generalized to eliminate the assumption of high sheaths which has previously been used to simplify the electron dynamics. Since both low emitter and low collector sheath heights are encountered as a consequence of ion reflection and trapped ions, the assumption of Boltzmann plasma electron distributions at the plasma-sheath interface must be abandoned. At both the emitter and collector the low sheaths return few plasma electrons, leaving the distributions largely one sided. Furthermore, at the emitter sheath emitted electrons must be taken into account. Thus the ratio of electrons moving toward the sheath to the total density of electrons at the sheath edge is not $1/2$, as in the Boltzmann assumption.

In fig. 2 we define the potentials in the convertor. All of the potentials are nondimensionalized by emitter temperature as follows:

$$\chi = \frac{q\phi}{kT_E}$$

(1)

where

$\chi = $ nondimensional potential,

$\phi = $ potential,
\[ q = \text{electron charge}, \]
\[ k = \text{Boltzmann's constant}, \]
\[ T_E = \text{emitter temperature}. \]

We also use the following terminology for various potentials in the convertor:

\[ \Phi_E = \text{emitter work function}, \]
\[ \Delta \chi = \text{back sheath height}, \]
\[ \Delta \chi_s = \text{reflective potential}, \]
\[ \chi_E = \text{emitter sheath height}, \]
\[ \Delta \chi_p = \text{plasma potential drop}, \]
\[ V_d = \text{arc-drop}, \]
\[ \chi_C = \text{collector sheath height}, \]
\[ \Phi_C = \text{collector work function}, \]
\[ V_{out} = \text{convertor output voltage}. \]

Inspection of fig. 2 yields immediately the following relations

\[ V_d = V_{out} - (\Phi_E - \Phi_C) - \Delta \chi, \quad (2) \]
\[ V_d = (\chi_C - \chi_E) - \Delta \chi_p. \quad (3) \]

The Richardson current density of electrons from the emitter is

\[ J_R \left( \frac{\text{amps}}{\text{cm}^2} \right) = 120 T_E^2 K^2 \exp(-\Phi_E). \quad (4) \]

The emitted current density which crosses the emitter sheath potential peak into the convertor plasma region is

\[ J_E = J_R \exp(-\Delta \chi), \quad \Delta \chi > 0, \]
\[ J_E = J_R, \quad \Delta \chi \leq 0. \quad (5) \]

We also define the net current density through the convertor, \( J \), and the normalized current density,

\[ j = \frac{J}{J_E} \quad (6) \]
Electron temperature is nondimensionalized as

$$\tau = \frac{T_e}{T_E}$$  \hspace{1cm} (7)

where \(T_e\) is the plasma electron temperature which, in this section, is constant by the isothermal assumption. Finally we have the thermal speeds,

$$a_e = \sqrt{\frac{8kT_e}{\pi m}}$$  \hspace{1cm} (8)

$$a_E = \sqrt{\frac{8kT_E}{\pi m}}$$  \hspace{1cm} (9)

The isothermal formulation is developed from here in the same way as the general formulation except that we take full advantage of the isothermal assumption by looking only at the global conservation equations instead of the local ones used in the general formulation. We then assume that the transport properties, collision frequencies, and the ionization source coefficient are constant across the convertor because of the isothermal assumption. Also we find only the steady state solution. We carry out this development by deriving the global conservation equations for the isothermal case (current, momentum, and electron energy) and then reducing these to a set of three simultaneous equations in the variables \(\tau, \chi_E,\) and \(\chi_C\). * These equations are nonlinear and solved numerically using a positive definite Newton’s method.

First we consider conservation of current. The collector is assumed to emit nothing, therefore at the collector plasma-sheath interface we have

$$J = \frac{a_e a_1 n(1)}{2} e^{-\frac{\chi_E}{2}}$$  \hspace{1cm} (10)

where \(a_1\) is the fraction of the total plasma density at the collector sheath which is moving toward the collector and \(n(1)\) is the total plasma density at the plasma-collector sheath interface. Because we continue to assume that the plasma electron distribution there is Maxwellian, we can write \(a_1\) as

$$a_1 = \frac{1}{1 + \frac{2}{\sqrt{\pi}} \int_0^{\chi_E} e^{-u^2} du}$$  \hspace{1cm} (11)

* In some cases the actual calculations are carried out using different variables when \(\chi_E\) or \(\chi_C\) are small or zero. In the case, for instance, of a single ion repelling emitter sheath we use \(j\) because \(\chi_E\) is zero.
which takes into account the plasma electrons reflected by the collector sheath. We still assume that the plasma electron distribution coming into the collector sheath is Maxwellian and that it does not have any velocity shift because the sheath is expected to be electron repelling. In the limit of a high collector sheath, $\alpha_1 = 1/2$ and we have a fully Boltzmann distribution of electrons at the collector sheath edge. The situation at the emitter is more complex because the emitted electrons must be taken into account. We have the back scattered current density, $J_{BS}$, which is the plasma electron current density moving into the emitter,

$$J_{BS} = \frac{n(0)\alpha_0}{2} \exp\left(-\frac{X_E}{\tau}\right)$$  \hspace{1cm} (12)$$

where $n(0)$ is the total plasma density at the emitter sheath-plasma interface and $\alpha_0$ is the fraction of total plasma density at the interface moving toward the emitter.

Continuity of electron current demands

$$J_E = J_{BS} + J$$  \hspace{1cm} (13)$$

which can be written as

$$J_E = J(1 + \frac{n(0)\alpha_0}{n(1)\alpha_1} \exp\left(\frac{X_N^b - X_E}{\tau}\right))$$  \hspace{1cm} (14)$$

This can be rewritten using eqs.3 and 6 as

$$j = \frac{1}{1 + \frac{n(0)\alpha_0}{n(1)\alpha_1} \exp\left(\frac{X_N^b + \Delta X^b}{\tau}\right)}$$  \hspace{1cm} (15)$$

The quantity $\alpha_0$ can be written as

$$\alpha_0 = \sqrt{\frac{\pi}{2}} Q \left(\frac{1}{J} - 1\right) \exp\left(\frac{X_E}{\tau}\right)$$  \hspace{1cm} (16)$$

where

$$Q = \frac{(u_e)_0}{\sqrt{kT_e/m}}$$

is the electron Mach number at the emitter. This is just an application of eq.13.

Electron energy conservation is developed by considering energy exchange with the emitter and collector and energy lost to ionization. Power carried into the plasma by emitted electrons is

$$P_E = J_E(2 + \Phi_E + \Delta \chi) \frac{kT_E}{q}.$$  \hspace{1cm} (17)$$

Power returned to the emitter is

$$P_{BS} = (J_E - J)(2r + \Phi_E + \Delta \chi) \frac{kT_E}{q}.$$  \hspace{1cm} (18)$$
Power flowing into the collector is

\[ P_C = J(2\tau + \Phi_E + V_d)\frac{kT_E}{q}. \]  \hspace{1cm} (19)

Ionization power loss is

\[ P_{\text{ion}} = J_{\text{ion}}V_{fi}\frac{kT_E}{q}, \]  \hspace{1cm} (20)

where \( J_{\text{ion}} \) is the total ion current into both the emitter and collector, and \( V_{fi} \) is the first ionization energy. Conservation of electron energy is

\[ P_E = P_{BS} + P_C + P_{\text{ion}} \]  \hspace{1cm} (21)

which can be reduced to

\[ \tau = 1 - \frac{1}{2}jV_d - \frac{1}{2}j_iV_{fi} \]  \hspace{1cm} (22)

where \( j_i = J_{\text{ion}}/J_E \). In the ignited mode \( \tau \) is generally about 2 (\( T_E = 1500 \) K and \( T_e = 3000 \) K), consequently the arc-drop, \( V_d \), is negative. In other words the high plasma electron temperature is generated by resistance heating.

Finally, we consider electron and ion momentum. From electron momentum conservation, we find the potential drop in the plasma region. By adding the electron and ion momentum equations as in the general case, we find our diffusion equation and boundary conditions to which the sheaths contribute flux terms. When we introduce the ionization source term into this, we have the complete formulation. Electron momentum conservation is

\[ 0 = -\frac{dp_e}{dx} - qn\frac{d\psi}{dx} - \frac{a_eqnu_e}{\lambda_e}, \]  \hspace{1cm} (23)

where \( \lambda_e \) is electron mean free path. Using \( p_e = nkT_e \) and \( J = qnu_e \), we can rearrange eq.23 into

\[ J = -\frac{q\lambda_e}{ma_e}(kT_e\frac{dn}{dx} + nq\frac{d\psi}{dx}). \]  \hspace{1cm} (24)

This can be further reduced by dividing by \( J_E \) and using \( \xi = x/d \) where \( d \) is the convertor gap thickness:

\[ j = -\frac{\pi \lambda_e}{4d}\frac{1}{\sqrt{n_E}}(r\frac{dn}{d\xi} + n\frac{d\chi}{d\xi}). \]  \hspace{1cm} (25)

Integration of this equation from the emitter sheath interface to the collector sheath interface yields

\[ \Delta \chi_P = \tau \ln\left(\frac{n(1)}{n(0)}\right) + jR \]  \hspace{1cm} (26)
where
\[ R = \frac{4}{\pi} \frac{d}{\lambda_e} \sqrt{\int_0^1 \frac{n_e}{n(\xi)} d\xi}. \] (27)

The quantity \( R \) is the normalized plasma resistance.

The ion and electron momentum equations can be written
\[
kT_e \frac{dn}{dx} = -q_n \frac{d\phi}{dx} - \frac{m_n u_e a_i}{\lambda_e} \quad \text{(28)}
\]
\[
kT_i \frac{dn}{dx} = q_n \frac{d\phi}{dx} - \frac{M n u_i a_i}{\lambda_i} \quad \text{(28)}
\]
where \( \lambda_i \) is ion mean free path and \( a_i \) is ion thermal speed,
\[ a_i = \sqrt{\frac{8\pi T_e}{\pi M}}. \]

Addition of eqs.28 yields
\[
(kT_e + kT_E) \frac{dn}{dx} = -\left( \frac{a_m}{\lambda_e} u_e + \frac{a_i M}{\lambda_i} u_i \right) n, \]
which is ambipolar diffusion. Equation 29 is differentiated to become
\[
(kT_e + kT_E) \frac{d^2 n}{dx^2} + \frac{a_m}{\lambda_e} \frac{d}{dx} (n u_e) + \frac{a_i M}{\lambda_i} \frac{d}{dx} (n u_i) = 0. \] (30)

We assume recombination is negligible and the ionization source term is
\[
\frac{d}{dx} (u_e n) = \frac{d}{dx} (u_i n) = S n. \] (31)

Using eqs.31 in eq.30 yields
\[
\frac{d^2 n}{dx^2} + \left[ \left( \frac{a_m}{\lambda_e} + \frac{a_i M}{\lambda_i} \right) S d^2 \right] n = 0. \] (32)

Equation 29 taken at the boundaries of the plasma (at the emitter and collector sheath interfaces) forms the plasma boundary conditions
\[
\left. \frac{dn}{d\xi} \right|_{0} = \beta_0 n_0, \quad \left. \frac{dn}{d\xi} \right|_{1} = \beta_1 n_1 \] (33)

where
\[
\beta_0 = -\frac{-d}{kT_e + kT_E} \left( \frac{a_m}{\lambda_e} u_e + \frac{a_i M}{\lambda_i} u_i \right), \quad \beta_1 = \frac{d}{kT_e + kT_E} \left( \frac{a_m}{\lambda_e} u_e + \frac{a_i M}{\lambda_i} u_i \right). \] (34)

Equation 32 is written as
\[
\frac{d^2 n}{d\xi^2} + A^2(r) n = 0 \] (35)
where
\[ A^2(r) = d^2 S \left( \frac{\alpha_m}{\lambda_e} + \frac{\alpha_i M}{\lambda_i} \right) \]  
(30)
where \( A(r) \) is the ionization coefficient and is found from consideration of ionization kinetics. Its solution for \( n \) is
\[ n(\xi) = B \sin(A\xi + C) \]  
(37)
where \( B \) and \( C \) are constants of integration and \( A = A(r) \). The quantities \( \beta_0 \) and \( \beta_1 \), which are the boundary conditions for eq.37, can be written as functions of \( \tau, x_E, x_C \) and \( \Delta x_s \),
\[ \beta_0 = \beta_0(\tau, x_E, x_C, \Delta x_s), \]
\[ \beta_1 = \beta_1(\tau, x_E, x_C, \Delta x_s). \]
(38)
When there is no reflection, \( \beta_0 \) and \( \beta_1 \) are both large, i.e.,
\[ \beta_0 = O\left(\frac{d}{\lambda_e}\right), \beta_1 = O\left(\frac{d}{\lambda_i}\right). \]

Significant reflection on the emitter side reduces \( \beta_0 \) and it may indeed attain negative values for sufficiently strong reflection.

The density equation (eq. 35) with the boundary conditions \( \beta_0 \) and \( \beta_1 \) is a linear eigenvalue problem; its solution yields \( A \) and \( C \) as functions of \( \beta_0 \) and \( \beta_1 \). The calculated results are shown in fig. 3. Since \( A(r) \) is function of \( r \) from the ionization kinetics, the value of \( r \) is thus determined by a function of \( \beta_0 \) and \( \beta_1 \). The plasma resistance, \( R \), also can be expressed in terms of functions of \( \beta_0 \) and \( \beta_1 \) through \( A \) and \( C \) using eq.27:
\[ R = \frac{4}{\sqrt{2\pi}} \frac{d}{\lambda_e} \frac{\sin C}{A} \ln \left( \frac{\tan \left( \frac{A+C}{2} \right)}{\tan \left( \frac{C}{2} \right)} \right). \]  
(39)

The sheath results which provide \( j, Q, \beta_0 \) and \( \beta_1 \), complete the isothermal formulation. The results are summarized below. The quantities \( \beta_0, \beta_1, Q \) and \( j \) are found from the sheath calculations as functions of \( \tau, x_E, x_C, \) and \( \Delta x_s \), i.e.,
\[ \beta_0 = \beta_0(\tau, x_E, x_C, \Delta x_s), \]
\[ \beta_1 = \beta_1(\tau, x_E, x_C, \Delta x_s), \]
\[ Q = Q(\tau, x_E, x_C, \Delta x_s), \]
\[ j = j(\tau, x_E, x_C, \Delta x_s). \]

From the eigenvalue problem for the plasma density we then find
\[ A(r) = A(\beta_0, \beta_1). \]  
(40)
From the continuity equation for current we find

$$\chi_C - \chi_E = r \ln \left( \frac{\sin(A + C)}{\sin(C)} \right) + r \ln \frac{\alpha_1}{\alpha_0} + r \ln \left( \frac{1}{j} - 1 \right).$$

(41)

And from the electron momentum equation we find

$$\chi_C - \chi_E = r \ln \left( \frac{\sin(A + C)}{\sin(C)} \right) + jR + 2 \frac{(r - 1)}{j} - \frac{\dot{V}_f}{j}.$$  

(42)

These three previous equations determine $\chi_E$, $\chi_C$ and $r$ when $\Delta \chi$ is given. This set of equations is valid for all $\Delta \chi$. Even in the case of $\Delta \chi \leq 0$ when there is no reflection, the calculations differ from previous isothermal calculations because the Boltzmann assumption on the electrons is not used as indicated by the presence of $\alpha_0$ and $\alpha_1$.

3. CALCULATED RESULTS FOR ION REFLECTION AND TRAPPED IONS

In this section we develop isothermal solutions for the thermionic convertor with the emitter sheath phenomena of ion reflection, trapped ions and surface emission ions included. Emitter sheath effects on thermionic convertor performance can be divided into two categories: 1) changes in net ion flux rate into the sheath which affect plasma density directly, and 2) changes in sheath potential distribution which affect the exchange of “hot” plasma electrons for “cold” emitter ions directly. A decreased influx of ions into the sheath, which occurs for all three emitter sheath phenomena, increases the plasma density at the neutral plasma- emitter sheath interface. Theoretical intuition suggests that an increased plasma density at the emitter would benefit performance by reducing resistance through the plasma and therefore reducing arc-drop. However, this is not the case. While the plasma density at the emitter increases slightly, plasma density at the collector decreases. Consequently total resistance increases.

All three of these phenomena increase in significance as net current density through the convertor is reduced. Each of the these reduces the net ion loss rate to the emitter and consequently increases arc-drop (therefore degrading performance at low current densities). This increase in arc-drop is in agreement with the same tendency in the experimental results. However, the experimental results also show a plateau (of low arc-drop) at low current density. This plateau occurs at a current density corresponding
to significant surface ion emission and is therefore thought to occur as surface emission replaces volume ionization as the dominant source of plasma ions. Unfortunately, the theoretical calculations cannot be carried into this region because the collisionless collector sheath matching (to the neutral plasma) fails.

To provide a realistic framework for presenting the results, we consider the convertor conditions shown as case 1 in table 1.

<table>
<thead>
<tr>
<th>CASE 1</th>
<th>CASE 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_E = 1500$ K</td>
<td>$T_E = 1750$ K</td>
</tr>
<tr>
<td>$T_C = 750$ K</td>
<td>$T_C = 750$ K</td>
</tr>
<tr>
<td>$p_{cs} = 1$ torr</td>
<td>$p_{cs} = 1$ torr</td>
</tr>
<tr>
<td>$d = 10$ mil</td>
<td>$d = 10$ mil</td>
</tr>
<tr>
<td>$\phi_E = 2.12$ eV</td>
<td>$\phi_E = 2.67$ eV</td>
</tr>
<tr>
<td>$\phi_C = 1.60$ eV</td>
<td>$\phi_C = 1.73$ eV</td>
</tr>
<tr>
<td>$J_R = 20$ amp/cm$^2$</td>
<td>$J_R = 7.57$ amp/cm$^2$</td>
</tr>
<tr>
<td>$J_{e^+} = 1.80 \times 10^{-5}$ amp/cm$^2$</td>
<td>$J_{e^+} = 2.10 \times 10^{-3}$ amp/cm$^2$</td>
</tr>
</tbody>
</table>

Table 1. Isothermal Solution Conditions

Case 2 is shown because it has the largest surface emission of any typical thermionic convertor operating condition (because the work function is high and the temperature is also high). Instead of presenting case 2 separately, we demonstrate the effects of surface emission in case 1 by increasing the surface emission by a factor of 100 thereby bringing it up to the level in case 2. The net current density at which surface emission becomes significant can be estimated by multiplying $J_{e^+}$ by the square root of the ion to electron mass ratio (approximately 500). In case 1 this means that surface emission becomes significant at $J = 0.01$ amp/cm$^2$ while in case 2 significant surface emission begins at $J = 1.0$ amp/cm$^2$. 
Effects of Ion Reflection

In this section we develop the isothermal results for case 1 with ion reflection, but without trapped ions and with the small amount of surface emission ions of case 1. Figure 4 is the C-V diagram for this case.

The dotted line extending upward from point A is the single electron repelling emitter sheath solution. However, we have not taken recombination into account in this isothermal formulation which are expected to become important at current densities near $J_R$. Therefore the curve above point A should be treated as qualitative and not quantitative.

The interest of this paper begins at point A, where the single sheath doubles over. Between points A and B, where the back sheath height, $\Delta X$, is less than the sheath height, $\chi E$, the emitter sheath is non-reflecting. In this region the sheath heights, $\chi E$ and $\chi C$, remain constant while the plasma density is proportional to net current, $J$ (the normalized plasma density $n_c/J$ is constant). Only the back sheath height, $\Delta X$, changes and the C-V curve in this region is Boltzmann (the arc-drop is constant). Beginning at point B and continuing to point C, the double emitter sheath reflects plasma ions because the back sheath is larger than the front sheath, in other words the reflective potential, $\Delta X = \Delta X - \chi E$, is positive. The result is that net ion loss rate, $\bar{u}$, decreases and that arc-drop increases. The dotted curve BD is the same double sheath except that it assumes no ions are reflected; therefore $\bar{u}$ is constant and arc-drop is constant. The two curves BC and BD are almost indistinguishable because the increase in arc-drop is small until the net current density is extremely small. The reason for this is that the shift speed is approximately $u_s = 2$ and therefore a large increase in reflective potential is required to change $\bar{u}$ significantly (the half reflection point is $\Delta X_s = 4.0$ or approximately $J = J_R \exp(-4) = 0.4 \text{amp/cm}^2$).

The curve EF is the single electron repelling emitter sheath case. It is the limiting case for large amounts of trapped ions in which the double sheath peak has been completely suppressed by the trapped ions. For this case the emitter sheath solution gives $u_s = 0$ [6]. This curve is not topologically connected to the curve ABC; it will be shown in subsection 2 that trapped ions move ABC toward the single ion repelling sheath case. The curve is much steeper (a faster increase in arc-drop) in this case because $u_s = 0$ (the half point in ion reflection is approximately $J = 8 \text{amp/cm}^2$). Curve EG is the single ion repelling case
assuming no reflection and is therefore a Boltzmann line with constant arc-drop.

At points F and C the solutions fail at the collector. The explanation for this failure is best given by examining figs. 5, 6, 7 and 8. Figure 5 is the normalized plasma density through the convertor gap.

Figure 5. Normalized Plasma Density with Reflection

The highest curve with no reflection, \( \Delta x_s = 0 \), has the largest plasma density at the collector but the lowest plasma density at the emitter. Ion reflection, which decreases the ion loss rate to the emitter, raises the plasma density at the emitter but lowers the plasma density at the collector. The lower plasma density at the collector forces a smaller collector sheath height to pass the net current density. This can be seen from eq.10. Figure 6 is the potential through the convertor under the same reflection conditions as in fig. 5. In fig. 6 the first two spaces on the left make up the double emitter sheath, and the last space on the right is the collector sheath. The region between the two sheaths is the neutral plasma region. In the no reflection case, it can be seen that the potential has a pronounced well in the middle. This is the result of the large plasma density in the middle. As reflection increases, this well disappears on the collector side of the plasma because resistive drop
there (due to low plasma density) increases to the degree that it is greater than the ambipolar rise (due to decreasing density toward the collector). Simultaneously with plasma potential gradient at the collector becoming negative, the collector sheath goes toward zero height. Figure 7 shows the critical collector sheath quantities as the collector sheath failure occurs. Collector sheath height, $X_C$ goes toward zero, the shift speed, $u_{sc}$, goes toward negative infinity, and the ion loss rate to the collector, $u_C$, is driven to zero. Figure 8 shows the changes in the emitter sheath height, ion shift speed and ion loss rate. When the collector sheath failure occurs, the ion loss rate to the collector is zero ($u_c = 0$) and the corresponding plasma ion distribution at the collector is bunched at zero velocity ($u_c = -\infty$). While the mathematics hold self-consistently until $u_c = 0$, the physics is clearly poor at this point because $u_c = 0$ demands that the plasma ions at the collector have zero energy (zero temperature and zero mean velocity). An estimate of when the physics becomes poor is $u_c = 0$. At this point the net ion loss rate is close to the thermal speed. A second physical difficulty that occurs with collector sheath failure is that the electron Mach number there, $Q_e$ (from eq. 10) becomes

$$Q_e = \frac{\sqrt{2}}{\pi}$$

because the collector sheath height approaches zero (actually about .001). In the present continuum formulation of the plasma region, it was assumed in eq. 13 that $Q_e$ is small so that the electron momentum term, $u_e du_e/dx$ can be neglected.

One could take the solution below the collector sheath failure point if $u_c$ could attain negative values or if $Q_e$ could attain values larger than $\sqrt{2/\pi}$. There is no physical basis for assuming that $u_c$ can become negative since the collector emits nothing. However, there is a physical basis for allowing $Q_e$ to be larger than $\sqrt{2/\pi}$, (an electron distribution shift) as can be seen in fig. 6: the potential drop nearing the collector becomes progressively more electron accelerating as the collector sheath fails and therefore the electron distribution should be shifted as the ion distribution is in an electron repelling sheath. However, this would clearly invalidate the assumption that the electron momentum term is negligible. Therefore the momentum term must be added to explore further in this direction and this has not been done because of the resulting complexity in the equations.

Comparison of fig. 7 to fig. 8 at the collector sheath failure point ($\Delta X_s = 2.5, u_c = 0$) shows that the ion loss rate to the emitter is positive. At this point the plasma is still ignited and generating ions as can be seen from figs. 9 and 10. The ionization coefficient, $A$, has
dropped by 50%, but the plasma electron temperature has dropped by only 5%. Finally, we note in fig. 11 that the normalized plasma resistance, R, has risen by almost 100%. This is responsible for the increase in arc-drop and the decrease in performance. Plasma resistance increases in response to reflection because the loss of plasma electron energy to the emitter is more important than the loss of ionization energy to the emitter. Ion reflection at the emitter increases the normalized plasma density there, and consequently increases the normalized loss of plasma electron energy there. The basis of this can be seen from conservation of electron energy (eq. 22),

$$\tau = 1 - \frac{1}{2} j V_d - \frac{1}{2} J V_{fi}. \quad (43)$$

The ion energy loss term is generally small compared to the electron energy loss term:

$$\frac{\frac{1}{2} J V_{fi}}{\frac{1}{2} j V_d} = \frac{J V_{fi}}{J V_d} \approx O(0.02). \quad (44)$$

Therefore, we take the the electron energy equation as:

$$\tau = 1 - \frac{1}{2} j V_d. \quad (45)$$

Since $\tau$ is nearly constant (because of the ionization kinetics), the product $j V_d$ is nearly constant. Ion reflection decreases $j$ (because the normalized plasma density increases) and therefore increases arc-drop, $V_d$ (makes $V_d$ a more negative number).

If the equations are reformulated in such a way as to be valid past the collector sheath failure point, then we can expect to eventually see a decrease in arc-drop and a low current plateau as the electron temperature approaches 1 (the ignited plasma is extinguished and the ionization source is surface emission). This can be seen from eq. 43. However, as we see, the collector failure occurs before $\tau$ has dropped more than 5%. Consequently we do not see any plateau or decrease in arc-drop as net current density is decreased in the present calculations.

4. EFFECTS OF TRAPPED IONS

Figure 12 shows the effect of trapped ions on the C-V characteristics. In this section the trapped ion distribution is assumed to have the temperature of plasma ion distribution, and that 100% trapped ions ($f_{tr} = 1.0$) completes the plasma ion thermal distribution.
Curve AHIJ is the C-V characteristic for $f_{tr} = 0.10$. At point A there cannot be any trapped ions since the back sheath height, $\Delta x$, is zero. Therefore the trapped C-V merges into the non-trapped curve there. The actual amount of trapped ions on the $f_{tr} = 0.10$ curve increases from zero at point A to the full 10% of a thermal distribution at point H where the back sheath height, $\Delta x$, is equal to the sheath height, $x_E$. The shift speed increases on AH from 1.95 to 3.00. This corresponds to what is seen in fig. 3.4.8 where $\Delta x < x_E$.

The rise in shift speed has been limited to 3.00. This limit is placed on the shift speed because a sheath with height of about 1.0 should not have a pre-sheath region capable of shifting the entire distribution so far. In fact limiting the shift speed is equivalent to increasing the cut-off speed for the ion distribution function.

The arc-drop decreases as result of the increase in $u_*$ and the consequent increase in the net ion loss rate to the emitter. A “hump” can be seen on AH where the shift speed hits 3.00. The arc-drop is lowest on this “hump” because the shift speed is at its maximum of 3.00. Between points H and I the back sheath height remains equal to the sheath height, $\Delta x - x_E = x_s = 0$. On this segment, $u_*$ decreases to 1.25, therefore increasing arc-drop.

From point I to point J, the shift speed remains constant at 1.25 and the ion loss rate decreases because of reflection. The other trapped cases, $f_{tr} = 0.2, 0.3, 0.4$ have not been connected because they hit the 3.00 maximum shift speed much sooner than in the $f_{tr} = 0.1$ case.

Point J is the collector sheath failure point. Each of the $f_{tr} = 0.2, 0.3, 0.4$ curves begins at $\Delta x_s = 0$ and ends at the collector sheath failure point. It should be noted that each of the trapped ion curves fails at a higher current than the last because the shift speed is lower.

5. EFFECTS OF EMITTER SURFACE EMISSION

In this section the effect of surface emission is discussed. Figure 13 adds the effect of surface emission to fig. 12. On the $f_{tr} = 0.10$ curve, surface emission is added by multiplying the actual small amount of surface emission in case 1 by a factor of 100. This brings the surface emission up to the level in case 2, making it significant at $J = 1.0$ amp/cm². It can
be seen that surface emission increases arc-drop; it does so in exactly the same way as reflection or trapped ions do - it decreases the net loss rate of ions to the emitter. Also the collector sheath failure occurs at point K in exactly the same way as for reflection.

6. COMPARISON WITH EXPERIMENTAL RESULTS AND CONCLUSIONS

Figure 14 superimposes the isothermal results of fig. 13 on the experimental results for a cesium reservoir temperature of 551 K which produces a 1 torr neutral cesium pressure. The experimental results are from [8]. The point of this comparison is that the steepness of the C-V characteristic in the experimental convertors can be explained by a decreasing ion loss rate to the emitter. We have shown that all three of the expected emitter sheath phenomena decrease the ion loss rate to the emitter. We cannot calculate the amount of trapped ions in a collisionless sheath without knowledge of the collisional processes. However, the experimental C-V suggests that if the amount of trapped ions \( f_{tr} \) increases from 0% at \( J = 14 \text{ amps/cm}^2 \) (the double sheath formation point) to 10% at \( J = 2 \text{ amps/cm}^2 \) then the steepness could result from trapped ions reducing the ion loss rate to the emitter. Since these percentages are based on a thermal distribution of ions, they seem physically reasonable. Unfortunately, the collector sheath failure prevents us from going to the point in the calculations where \( r \) drops enough to make surface emission the source of ions.

The experimental curve is nearly a constant .05 volts below the isothermal result \( f_{tr} = 0.10 \) except at high current densities and at the "hump". Comparison of the curves at high current density is not valid since neither the Schottky effect nor recombination have been included. The Schottky effect is important above 12 \text{ amps/cm}^2 in this case because the emitter sheath is single electron repelling (to the plasma) and therefore puts a strong electric field against the emitter with the appropriate sign. Recombination is also potentially important because the plasma density scales with current density and at high current densities the plasma density in the middle of the convertor approaches the Saha density. The .05 volt difference may or may not be explained by a discrepancy in the assumed collector work function. At 800 K the collector emits essentially nothing and therefore any change in the collector work function directly affects output voltage. If the collector work function were in fact 1.65 volts instead of 1.60 volts then the isothermal
result would lie nearly on top of the experimental result. We have not adjusted the assumed collector work function so as to illustrate the importance of it and therefore the importance of the surface physics of the adsorbed cesium layer. The “hump” should not be taken as an expected experimental result since it results from the interaction of the trapped ions with the plasma-emitter sheath interface. Instead it should be taken as a second reason (in addition to the cut-off of the ion distribution) for further study of the matching region between the collisionless sheath and the neutral plasma.

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Emitter $T_E = 1750^\circ K$

Collector $T_C = 750^\circ K$

CESIUM RESERVOIR $T_R$
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Figure 14. Isothermal vs. Experimental C-V Diagrams
A general solution condition for collisionless sheaths is developed. Previous work has assumed that the Bohm criterion or the generalized Bohm criterion assures a self-consistent sheath solution. This paper shows that for non-monotonic collisionless sheath structures, such as double sheaths containing trapped ions, the generalized Bohm criterion is a necessary but not a sufficient condition. The general solution condition developed is always sufficient and the generalized Bohm criterion is shown to be special case of it. The general solution condition is applied to a double emitter sheath containing trapped ions. First, it is shown that the low energy part of the plasma ion distribution coming into the sheath cannot be neglected as claimed in some analyses, because the shift in mean ion velocity through the presheath (generalized Bohm speed) depends strongly on low energy ions. Second, it is shown that the presence of trapped ions moves the point of critical self-consistency away from the collisionless sheath-neutral plasma asymptotic match and into the collisionless sheath. Consequently, both the sheath structure and the generalized Bohm speed depend on the amount of trapped ions. Thus collisional effects may dominate the structure of a presumably collisionless sheath through the trapping mechanism and the collisional pre-sheath which determines the low energy ion component entering the collisionless sheath.

1. Introduction

In this paper the collisionless sheath formulation for arbitrary potential structures is developed and a general solution condition is found. Originally, the collisionless sheath formulation was developed by Langmuir [1]. Bohm [2] showed that a minimum ion speed (of plasma ions coming into the sheath) was necessary to construct a self-consistent collisionless sheath. However, Bohm's analysis assumes, among other things, that the ion distribution is monoenergetic (cold). Harrison and Thompson [3] removed the assumption of cold ions and derive a generalized Bohm criterion. However,
Harrison and Thompson's analysis assumes that the critical self-consistency point (which necessitates the generalized Bohm criterion) occurs at the plasma-sheath interface where the collisionless sheath is asymptotically matched to the continuum quasi-neutral plasma. Also Harrison and Thompson circumvent difficulties with low velocity ions by assuming that $\langle v^2 \rangle = \langle v^{-2} \rangle ^{-1}$ where the brackets indicate averages over the ion distribution function and $v$ is velocity. In this paper, we show that the point of critical self-consistency does not necessarily occur at the plasma-sheath interface, and that the resulting general solution condition is substantially affected by the location of the critical point. In the case of trapped ions, the critical point may occur in the middle of sheath (well away from the neutral plasma in terms of potential rise). Under this condition the asymptotic match then exists in the middle of the collisionless sheath, such that the sheath length would be infinite unless the general solution condition is oversatisfied.

Using the generalized solution condition, this paper treats as an example the double (nonmonotonic) sheath structure that is expected to be important at an emitting electrode in such applications as thermionic energy convertors. This paper does not consider such complexities as magnetic field or a curved surface. We assume instead that the magnetic pressure is low and that the radius of curvature of the plate is large compared to both the mean-free-paths and the Debye length. Consideration of magnetic effects can be found in Chodura [4] and consideration of curvature effects can be found in many references under the guise of probe theory including Lam [5]. Also this paper does not consider the pre-sheath structure which must exist between the Maxwellian plasma and the collisionless sheath. Instead the effect of different ion distributions (various cut-off velocities) coming from the pre-sheath is shown. It is apparent from the results in this paper that the pre-
sheath region is important to the double sheath structure; however little work exists as yet on the pre-sheath. See, for instance, references [6,7,8,9].

The simplest demonstration of the Bohm criterion can be constructed as follows. Since the sheath region is assumed to be collisionless, we can write the Poisson equation as

\[
\frac{d^2 \phi}{dx^2} = 4\pi q(n_i(\phi) - n_e(\phi)) 
\]

where \( \phi \) is potential, \( x \) is distance into the sheath, \( q \) is electron charge, \( n_i(\phi) \) is ion density, and \( n_e(\phi) \) is electron density. In this equation and all that follow in this paper, the sense of electric potential is the reverse of the usual; increasing potential repels electrons. Equation 1 can be non-dimensionalized as

\[
\frac{d^2 \chi}{dc^2} = F(\chi) 
\]

where \( \chi = q\phi/kT_R \), \( c = x/\lambda_D \), \( \lambda_D = \sqrt{\frac{kT_R}{4\pi q^2 n_o}} \)

and where \( F(\chi) = \frac{n_i(\chi) - n_e(\chi)}{n_o} \).

\( T_R \) is a reference temperature usually taken to be the heavy particle temperature and \( n_o \) is a reference plasma density usually taken as the plasma density at the neutral plasma-sheath interface. \( \lambda_D \) is the Debye length.

At the neutral plasma-sheath interface we define \( \chi = 0 \), and we have charge neutrality there; consequently \( F(0) = 0 \). Also, we have \( \frac{dx}{dc} \bigg|_{\chi=0} = 0 \) because the neutral plasma cannot support a strong electric field. The ion and electron density parts of \( F(\chi) \) can be modeled in this simple case as
\[ F_{i}(x) = \frac{u_{\text{mono}}}{\sqrt{u_{\text{mono}}^{2} + x}} \quad \text{and} \quad F_{e}(x) = e^{-x/\tau} \quad (3) \]

The ion distribution is assumed to be cold and \( u_{\text{mono}} = \sqrt{\frac{2kT_{e}}{M}} \)

where \( u_{\text{mono}} \) is the ion distribution speed, \( k \) is Boltzmann's constant and \( M \) is ion mass. The electron distribution is assumed to be Maxwellian where \( \tau = T_{e}/T_{R} \) and \( T_{e} \) is electron temperature. Under these assumptions, the Bohm criterion requires

\[ \frac{\partial F_{i}(x)}{\partial x} - \frac{\partial F_{e}(x)}{\partial x} \bigg|_{x=0} \geq 0 \quad (4) \]

since otherwise no self-consistent solution to eq. 2 exists at \( x = 0 \).

Equation 4 applied to eq. 3 yields

\[ u_{\text{mono}} \geq \sqrt{\frac{kT_{e}}{M}} \quad (5) \]

which is the standard Bohm result.

This simple model is not sufficient for all applications because it assumes the following:

1) that the sheath height, \( X_{E} \), is large so that the Boltzmann electron distribution approximation is valid,

2) that the ion distribution is cold, and

3) that there are no other species involved besides the electrons and ions from the plasma.

In such applications as a thermionic energy convertor (TEC), accurate collisionless sheath results are desired, yet all three of these assumptions
are violated. In addition, an emitter sheath (particularly in a TEC) may be doubled over and therefore also contain trapped ions. Exact collisionless sheath calculations carried out under these conditions require that the simple Bohm criterion (eq. 4) be generalized. A warm ion distribution requires the generalization of Harrison and Thompson [3] in which the Bohm criterion becomes

\[
\frac{\partial F_i(x)}{\partial x} \bigg|_{x=0} = \int f_i(v) \frac{1}{2v^2} \, dv \leq \frac{M}{2kT_e} \quad (6)
\]

where the variable of integration, \(v\), is velocity in the \(x\) direction. It can be seen immediately, however, that the integral in eq. 6 does not converge unless conditions are imposed on \(f_i(v)\). Namely, the ion velocity distribution, \(f_i(v)\), must go to zero sufficiently fast near \(v=0\) (be cut off). In general one would expect the ion distribution entering the collisionless sheath from the collisional presheath to be cut off because the collisional processes in the presheath will not resupply sufficient low velocity ions to replace those accelerated toward the sheath [6,7]. We do not deal with two or three dimensional effects which may effect the cutoff [10]. Since the form of the ion distribution entering the sheath is not known, we have assumed for the purposes of this paper that it is a shifted Maxwellian with the neutral temperature and cutoff below a prescribed velocity. If the integral in eq. 6 is undefined, then the expansion of \(F(x)\) as a Taylor series has failed because \(\frac{\partial F_i}{\partial x} \bigg|_{x=0}\) does not exist.

In the Appendix the complete asymptotic expansion of \(F(x)\) for small \(x\) is presented, which is

\[
F_i(x) = \int f_i(v) \, dv - x^{1/2} f(0) + O(x \ln x) \quad (7)
\]
The $x^{1/2}$ term does not appear in the electron distribution expansion,

$$F_e(x) = \int_0^\infty f_e(v) \, dv + O(x \ln x) \quad (8)$$

(The terms represented by $O(x \ln x)$ can be found in Appendix A and are omitted here for convenience.) We have, then

$$F(x) = -x^{1/2} f(0) + O(x \ln x) \quad (9)$$

Therefore $f(0)$ must equal zero since otherwise $F(x) < 0$ for some $x > 0$ and we could not then construct a self-consistent solution. Thus even when the full asymptotic expansion is carried out, the ion distribution must be cut off. (It will be shown later in fig. 5 that the dependence of the Bohm criterion on the cut-off is weak.)

While the Bohm criterion, or the generalized version is a necessary condition for a self-consistent sheath structure, it is not always sufficient. In particular, trapped ions in a double sheath move the point of critical self-consistency away from the neutral plasma-sheath into the collisionless sheath itself. For this reason, we must develop a general solution condition, of which the generalized Bohm criterion is a special case.

2. The General Solution Condition

Before proceeding to discuss the possible sheath configurations and their specific equations, we develop the general solution condition which applies to all collisionless sheaths to preclude consideration of non-self-consistent solutions. Asymptotically matching the collisionless sheath to the quasi-
neutral plasma always requires some condition on the ion distribution function coming into the sheath from the plasma. In the past this had been the Bohm criterion which assumed cold plasma ions and required the monoenergetic plasma ion distribution to be shifted up in velocity to

\[ U_{\text{mono}} = \sqrt{\frac{kT_e}{M}}. \]

In this section it will be shown that a general condition for solving the collisionless sheath with a neutral plasma-sheath interface exists and that the Bohm criterion and other local (at the plasma-sheath interface) matching conditions are special cases of the general condition. We show that local matching is a necessary but not sufficient condition on the sheath solution; that a point of critical self-consistency exists which is not necessarily at the neutral plasma-sheath interface but may appear anywhere in the sheath. In the absence of trapped ions or surface emission ions, as is the case with most past calculations, local matching (in which the point of critical self-consistency is at neutral plasma-sheath interface) proves to be also sufficient. Further, we show, in the case of no trapped ions or surface emission ions, that for finite temperature the ion distribution must have no zero velocity ions when it enters the collisionless sheath.

A self-consistent sheath solution is found by integrating the non-dimensional Poisson equation from the last section,

\[ \frac{d^2x}{d\zeta^2} = F(x). \]  \hspace{1cm} (10)

The specific forms of \(F(x)\) are developed in the next section but we need not know them until we wish to evaluate specific cases. By convention \(x = 0\) at
the plasma-sheath interface and increasing \( x \) repels electrons. To construct a non-trivial solution we integrate eq. 10 as follows:

\[
\frac{d^2x}{d\zeta^2} = F(x) \frac{dx}{d\zeta}, \tag{11}
\]

\[
\frac{1}{2} (\frac{dx}{d\zeta})^2 \left|^{\zeta_2}_{\zeta_1} \right. = \int_{\zeta_1}^{\zeta_2} F(x) \frac{dx}{d\zeta} d\zeta, \tag{12}
\]

\[
\frac{1}{2} (\frac{dx}{d\zeta})^2 \left|^{\xi_2}_{\xi_1} \right. = \int_{\xi_1}^{\xi_2} F(x) dx. \tag{13}
\]

Transforming eq. 12 into eq. 13 implicitly assumes \( \chi(\zeta) \) is monotonic on the domain \((\zeta_1, \zeta_2)\). Since at the plasma-sheath interface, we assume charge neutrality and zero electric field, we have \( F(0) = 0 \) and \( \frac{dx}{d\zeta} = 0 \). Therefore we can write,

\[
\frac{1}{2} (\frac{dx}{d\zeta})^2 = \int_{0}^{x} F(x) dx \tag{14}
\]

and,

\[
\frac{dx}{\sqrt{2 \int_{0}^{x} F(x) dx}} = \pm d\zeta. \tag{15}
\]

Equation 15 is the method for constructing non-trivial solutions. From eq. 15 and our previous observation about converting eq. 12 into eq. 13 we can see that the inequality,

\[
\int_{0}^{x} F(x) dx > 0, 0 < x < x_m \tag{16}
\]
must hold where $x_m$ is the first maximum or minimum $x$ reaches in the sheath since otherwise $x(\zeta)$ is not monotonic. If we attempt to construct a solution which does not meet this condition, then eq. 15 causes $x(\zeta)$ to double back before reaching its full sheath height. The point where eq. 16 meets the equality for $0 < x < x_m$ is called point of critical self consistency.

From the general solution condition (eq. 16), we can derive necessary local (at the plasma-sheath interface) matching conditions, of which the generalized Bohm criterion is a special case. We can expand $F(x)$ around $x = 0$ as

$$F(x) = \sum_{n=1}^{\infty} a_n x^{n/2} - \sum_{n=2}^{\infty} b_n x^{n/2+1}$$

(17)

since $F(x)$ may be represented asymptotically in this form for cases we intend to develop. Then eq. 17 may be integrated,

$$\int_0^x F(x)dx = \sum_{n=1}^{\infty} \frac{a_n}{n/2 + 1} x^{n/2+1}$$

(18)

$$- \sum_{n=2}^{\infty} \frac{b_n}{(n/2+1)^2} x^{n/2+1} (\ln(x^{n/2+1}) - 1).$$

The asymptotic order of the terms in this equation is $a_1, b_2, a_2, b_3, a_3 \ldots$. To insure that at least local matching is satisfied, the first non-zero
term must be positive. We call this the generalized local matching. The usual Bohm criterion assumes $F(x)$ is expandable as,

$$F(x) = 0 + \frac{\partial F}{\partial x} \bigg|_{x=0} x + \frac{\partial^2 F}{\partial x^2} \bigg|_{x=0} \frac{x^2}{2!} + \ldots$$  

(19)

in which case the criterion is

$$a_2 = \frac{\partial F}{\partial x} \bigg|_{x=0} > 0 \ .$$  

(20)

When only cold ions are considered, the Bohm criterion (eq. 20) is sufficient, but when finite temperature ions are considered, we must apply the generalized local matching. If trapped or surface emission ions are present, we find that local matching is not sufficient and that the full solution condition must be applied.

3. **The Double Emitter Sheath with Trapped Ions**

The double emitter sheath is of particular interest to the TEC and other devices in which thermionic emission is substantial. With sufficient thermionic emission, the emitter sheath will double over as shown in fig. 2 such that it repels electrons from both the emitter and the plasma. In addition the double sheath may trap ions in its "well" and it may reflect ions coming from the plasma if the reflective potential $\Delta x_s$, is greater than zero.

In order to analyze the double emitter sheath, we need models for the distribution functions of ions and electrons coming into the sheath, and we also need a model for the trapped ion distribution in the double sheath. Figures 3 and 4 show the distribution functions and definitions.

As shown in fig. 3 the emitted electrons are assumed to leave the plate
with a Maxwellian distribution at $T_E$, the plate temperature. The electrons coming into the sheath from the plasma are assumed to have a Maxwellian distribution at $T_e$, the plasma electron temperature. The ion distribution coming from the plasma is assumed to be a shifted Maxwellian distribution where $U_s$ is the shift velocity and $U_{\text{cut}}$ is the cut-off velocity. It is also assumed to have plate temperature, $T_E$. It should be noted here that the shift speed, $U_s$, becomes the Bohm speed in the limit of zero ion temperature. The trapped ion distribution (at $x=x_E$) is assumed to be Maxwellian with temperature $T_E$, and cut off at $-x_E$ and $x_E$, as shown in fig. 4. Under these assumptions, the density functions ($F_i(x)$ and $F_e(x)$) for ions and electrons in the sheath can be written as

$$F_i(x) = \frac{f_{\text{plasma}}}{\sqrt{\pi}} \left\{ \int_{-\infty}^{x} e^{-\frac{(u-u_s)^2}{u_{\text{cut}}}} \frac{udu}{\sqrt{u^2+x^2}} + \int_{-\infty}^{\Delta x} e^{-\frac{(u-u_s)^2}{u_{\text{cut}}}} \frac{udu}{\sqrt{u^2+x^2}} \right\}$$

$$+ \frac{2f_{\text{tre}}}{\sqrt{\pi}} \left\{ \int_{-\infty}^{x_E} e^{-\frac{u^2}{u_{\text{cut}}}} \frac{udu}{\sqrt{u^2+x^2}} \right\}$$

(21)

and

$$F_e(x) = n_o \left\{ \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{u_{\text{cut}}}} \frac{udu}{\sqrt{u^2-x^2/\tau}} + \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{u_{\text{cut}}}} \frac{udu}{\sqrt{u^2-x^2/\tau}} \right\}$$

$$+ n_E \left\{ \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\frac{u^2}{u_{\text{cut}}}} \frac{udu}{\sqrt{u^2+(x_E-x)^2}} \right\}$$

(22)

In eqs. 21 and 22 the density functions $F_i(x)$ and $F_e(x)$ are normalized by $n_o$ (the plasma density at the neutral plasma-sheath interface) and therefore the
densities \( f_{\text{plasma}} \), \( f_{tr} \), \( n_o^- \) and \( n_E \) are likewise normalized. In eqs. 21 and 22 potential is non-dimensionalized by

\[
\chi = \frac{q\phi}{kT_E}.
\]

In the ion density expression (eq. 21) the second integral is the reflected ion term which is a mirror image of the lower velocity part of the first term. The quantity \( u \) is the non-dimensional ion velocity at the sheath edge \( (\chi = 0) \). The third integral in eq. 21 is the trapped ion distribution where \( u \) is the non-dimensional ion velocity \( u \) is the non-dimensional ion velocity at the sheath potential peak \( (\chi = \chi_E) \). Also the non-dimensionalization of \( u \) is \( u = \frac{M}{2kT_E} U \) where \( U \) is velocity. We have assumed \( T_i = T_E \). In the electron density expression (eq. 22) the first integral is the incoming plasma electron term, the second integral is the returning plasma electron term and the third integral is the emitted electron term. The non-dimensional velocities, \( u \), are \( u = \frac{m}{2kT_e} U \) in the first and second integrals of eq. 22 and \( u = \frac{m}{2kT_E} U \) in the third integral of eq. 22. In the first and second integrals, \( u \) is at the sheath edge \( (\chi = 0) \) and in the third integral, \( u \) is at the sheath peak. The quantity \( \tau = T_e/T_E \) is the electron temperature ratio.

The quantities \( f_{\text{plasma}} \), \( f_{tr} \), \( n_E \) and \( n_o^- \) are densities, all of which are normalized by \( n_o \), the plasma density at the plasma-sheath interface. The quantity \( f_{\text{plasma}} \) represents the density of ions coming into the sheath from the plasma, \( f_{tr} \) represents the density of ions trapped in the double sheath, \( n_E \) represents the density of electrons coming from the emitter that cross the sheath potential peak, and \( n_o^- \) represents the density of electrons coming from the plasma into the sheath. The quantities \( u_{\text{cut}} \) and \( u_s \) are the normalized
cut-off speed and shift speed:

\[ u_{\text{cut}} = \sqrt{\frac{U_{\text{cut}}}{2kT_E}}, \quad u_s = \sqrt{\frac{U_s}{2kT_E}}. \]

The solution to this sheath is found by treating the following quantities as parameters:

\[ f_{tr}, u_{\text{cut}}, x_E, \Delta x_s, \tau, \]

and the remaining quantities as variables which will be found:

\[ f_{\text{plasma}}, n_E, n_0, u_s. \]

The four equations which produce the solution are:

\[ F_i(0) = 1, \quad (23) \]

\[ F_e(0) = 1, \quad (24) \]

\[ x_E \]
\[ \int_0^{x_E} (F_i(x) - F_e(x))dx = 0, \quad (25) \]

\[ \min \left[ \int_0^{x_E} (F_i(x) - F_e(x))dx ; 0 < x < x_E \right] = 0. \quad (26) \]

Equations 23 and 24 account for charge neutrality at the plasma-sheath interface and that the plasma density is \( n_0 \) there. Equation 25 requires that the sheath have zero electric field at the potential peak, \( x_E \) (because the
potential gradient at the peak must be zero). Equation 26 is the general solution condition, which insures that the solution is self-consistent. It replaces the Bohm criterion and in fact contains the Bohm criterion as a special case as shown in section two.

As noted, the cut-off velocity, $u_{\text{cut}}$, is a parameter in the collisionless sheath formulation, and is in fact determined by collisional processes in a pre-sheath region that must exist between the collisionless sheath and the neutral plasma. Previous collisionless sheath formulations have gotten around this difficulty by using the monoenergetic ion assumption illustrated in Section 1. Figure 5 shows the effect of $u_{\text{cut}}$ on $u_s$ under typical double sheath conditions.

For comparison, the monoenergetic Bohm speed is shown as a dotted line which obviously does not depend on $u_{\text{cut}}$. In this paper we do not calculate exactly what the $u_{\text{cut}}$ should be, but instead note that the effect of $u_{\text{cut}}$ on the shift speed, $u_s$, is relatively weak, and in subsequent calculations set value for $u_{\text{cut}}$ at

$$u_{\text{cut}} = (10) u_{\text{mono}},$$

(27)

on the grounds that any physically reasonable value of the cut-off should be on the order of 10% of the monoenergetic Bohm speed. Support for this can be found in Emmert [6], and Riemann [7]. In many applications, using this approximation would produce a perfectly acceptable result. However it should be noted that a good approximate value for $u_{\text{cut}}$ derived from the transition region would produce a very accurate results in $u_s$ since the dependence of $u_s$ on $u_{\text{cut}}$ is essentially logarithmic. It should be noted from fig. 5 that $u_s$ is
approximately twice the monoenergetic Bohm speed under the assumption of eq. 27. In order for the shift speed, $u_s$, to equal the monoenergetic Bohm speed, $u_{\text{mono}}$, the cut-off velocity, $u_{\text{cut}}$, would have to be set at $(0.8)u_{\text{mono}}$. This seems physically unrealistic, and therefore we expect that a non-monoenergetic ion distribution should produce a shift speed, $u_s$, significantly higher than $u_{\text{mono}}$.

The introduction of trapped ions into the double sheath moves the point of critical self-consistency away from the neutral plasma-sheath interface into the sheath as shown in figs. 6 and 7.

Trapped ions have a substantial effect on the sheath solution. Figure 6 shows the general solution condition for the double sheath with $\chi_E=2.0, \tau=2.0$ and for $f_{\text{tr}} = 0.0$ and $f_{\text{tr}} = 0.1$. In the no trapped ion case ($f_{\text{tr}} = 0.0$) we have local matching (in which the point of critical self-consistency occurs at the neutral plasma-sheath interface, $\chi = 0$). As trapped ions are added, the point of critical self-consistency moves into the sheath, and at $f_{\text{tr}} = 0.1$ has moved to the middle of the sheath ($\chi = 0.5$). Figure 7 shows the sheath structure ($\chi$ versus $\zeta = x/\lambda_D$ where $x$ is distance into the sheath). The two cases shown in fig. 7 correspond to those in fig. 6. The point of critical self-consistency on $f_{\text{tr}} = 0.1$ in fig. 7 should actually be an asymptotic match of the upper part of sheath to the lower part. However, we have made the slope finite for display purposes. This asymptotic "flat spot" at $\chi \approx 0.6$ should not be troublesome because it can be removed by raising $u_s$ above the minimum required by the general solution condition. The existence of such an asymptotic region in the middle of collisionless sheath is counterintuitive, but not completely unexpected since it was previously only by assumption that critical self-consistency occurred at the plasma-sheath interface. Under ordinary circumstances, i.e., a single sheath which has no
trapped ions, the matching occurs at the plasma-sheath interface because the ion density as a function of $x$ ($F_i(x)$) is well behaved. This is obviously not the case with trapped ions present in a double sheath. The asymptotic "flat spot" raises an interesting question about collisionless sheath width. If $u_s$ is not much greater than the minimum required for a match, then the collisionless sheath would be collisional because of its length. Therefore if the sheath is to remain collisionless, $u_s$ must be large enough to make the sheath short compared to the mean-free-paths. Figure 8 shows the trapped ion effect on shift speed $u_s$ for various sheath heights $x_E$. It is significant that $u_s$ drops rapidly with increasing amounts of trapped ions. The curves terminate where the self-consistent sheath solution is no longer possible because the amount of trapped ions added prevents charge neutrality at the plasma-sheath interface. It should be noted that $u_s$ is ion distribution shift speed, not the mean speed $\bar{u}$ of the ions into the sheath. Therefore when $u_s$ is negative, the ion distribution is concentrated at low velocities and $\bar{u}$ is low but not negative.

4. Conclusions

A general solution for collisionless sheaths is important in any case where the sheath structure is nonmonotonic, or for some other reason one suspects that the point of critical self-consistency does not occur at the plasma sheath interface. As demonstrated with the example of a double emitter sheath, the generalized Bohm criterion is not sufficient to ensure a self-consistent sheath structure because the presence of trapped ions moves the point of critical self-consistency away from the plasma-sheath interface and into the sheath itself. The strong effect of trapped ions and ion distribution cut-off velocity on shift speed $u_s$ shows that, even in a
collisionless sheath, the collisional effects cannot be ignored. In an application such as the TEC, we need to determine the collisional mechanisms for both ion trapping, and the pre-sheath region since otherwise the ion distribution shift speed can only be estimated within a factor of two.

In addition the appearance of the asymptotic flat spot in the middle of the collisionless sheath implies that the sheath width is not unambiguously determined by collisionless processes. Therefore it seems unavoidable that collisional effects are important in presumably collisionless sheaths. In particular, an understanding of how the collisional presheath modifies the incoming plasma ion distribution seems necessary and also an understanding of the trapping mechanism in the case of the double emitter sheath seems necessary. The Bohm criterion or generalizations of it produce, in effect, estimates of potential rise through the collisional presheath based on collisionless self-consistency, and cannot be expected to yield accurate results for complex sheath structures. While many applications require only estimates of sheath potential rise and consequent ion acceleration through the sheath, such applications as TECs require accurate knowledge of sheath structure.

5. Acknowledgements

This work was supported by AFOSR Contract 83-0048.
REFERENCES

APPENDIX: ASYMPTOTIC EXPANSIONS

The asymptotic expansions of

\[
F_1(x) = \int_0^\infty f_1(u) \frac{udu}{\sqrt{u^2 + x}} \tag{1}
\]

and

\[
F_e(x) = \int \frac{f_e(u) \frac{udu}{\sqrt{u^2 - x}}}{\sqrt{x}} \tag{2}
\]

are

\[
F_1(x) = \lim_{n \to \infty} \int_0^\infty f_1(u) \frac{udu}{\sqrt{u^2 + x}} = 
\]

\[
\int_0^\infty f_1(u) \, du - x^{1/2} \left[ f_1(0) \right] + x \ln(x) \left[ \frac{f(1)(0)}{4} \right] 
\]

\[
+ x \left[ \frac{f(1)(0)}{4} - \frac{1}{2} \int f(3)(u) \ln(2u) \, du \right] 
\]

\[
+ O(x^{3/2}) \quad (3)
\]
\[ F_e(x) = \int_{\sqrt{x}}^\infty \frac{f_e(u)}{\sqrt{u^2 - x}} \, du = \int_0^\infty f_e(u) \, du + x \ln x \left[ - \frac{f'(0)}{4} \right] + x \left[ - \frac{f'(0)}{4} + \frac{1}{2} \int_0^\infty f''(u) \, u \ln(2u) \, du \right] + O(x^2). \]

These asymptotic expansions are developed by splitting the integrals into two parts and expanding \( f_e(u) \) or \( f_i(u) \) into a Taylor series on the lower part and expanding \( \frac{u}{\sqrt{u^2 - x}} \) or \( \frac{u}{\sqrt{u^2 + x}} \) on the upper part. The two pieces can be recombined analytically to produce eq. 3 and 4. Details of analysis are not presented here, however the results are valid if \( f_i(u) \) or \( f_e(u) \) can be expanded as a Taylor series on some interval \([0, k]\) where \( k > 0 \). This condition is generally easy to satisfy, and if necessary, the analysis can be extended for non-Taylor expandable \( f_i(u) \) and \( f_e(u) \) if some appropriate convergent series can be found, on \([0, k]\).
If the functions \( f_i(u) \) and \( f_e(u) \) are cut-off \( (f_i(u) \text{ or } f_e(u) = 0 \text{ for } 0 < u < u_{\text{cut}}) \) then eqs. 3 and 4 reduce to

\[
\int_{u_{\text{cut}}}^{\infty} f_i(u) \frac{udu}{\sqrt{u^2 + x}} =
\]

\[
\int_{u_{\text{cut}}}^{\infty} f_i(u) du - x [\int_{u_{\text{cut}}}^{\infty} f_i(u) \frac{1}{2u^2} du] + O(\chi^2)
\] (5)

and

\[
\int_{u_{\text{cut}}}^{\infty} f_e(u) \frac{udu}{\sqrt{u^2 - x}} = \int_{u_{\text{cut}}}^{\infty} f_e(u) du + x [\int_{u_{\text{cut}}}^{\infty} f(u) \frac{1}{2u^2} du]
\] (6)

\[
+ O(\chi^2).
\]

These are, of course, also the results attained by differentiation of the well behaved integrands.
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$\Delta \chi_s = 0.0$

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AN ASYMPTOTIC APPROXIMATION FOR A TIME DEPENDENT
COLLISIONAL PRESHEATH WITH A FOKKER-PLANCK
COLLISION TERM

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I. INTRODUCTION

The Boltzmann and Poisson equations for a Fokker-Planck Collisional Presheath are broken down by assuming a multi-exponential expansion for potential. With the assumption that the ion and electron distributions perpendicular to the surface are Maxwellian with uniform temperatures, the Boltzmann equation becomes a set of integro-differential equations in the velocity space that are fairly benign (they can be advanced through time like a parabolic PDE). A particular partially implicit numerical scheme allows the boundary conditions (such as no returning ions and specified $n$ and $\frac{1}{n} \frac{dn}{dx}$ at the plasma interface) to be imposed as linear constraints on the presheath potential structure at each time step. With the additional assumption of Boltzmann electrons, the scheme can be advanced through time with steps on the order of an ion characteristic time requiring on the order
of \((mn)^2\) operations at each time step where \(m\) is the number of divisions in velocity space (typically 20) and \(n\) is the number of terms in the expansion of potential (typically 5).

II. ASYMPTOTIC FORMULATION WITH FOKKER-PLANCK COLLISION TERMS

The potential is assumed to be of the form

\[ U = U_0 + a_1 \Delta U + a_2 \Delta U^2 + \cdots \]  \(1\)

where

\[ U_0 = \alpha x \quad \text{and} \quad \Delta U = e^{\beta x}, \]  \(2\)

and the distribution functions are expanded as

\[ f(v, \Delta U) = f_0(v) + \Delta U f_1(v) + \Delta U^2 f_2(v) + \cdots. \]  \(3\)

The Boltzmann equation can be written as

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial (\Delta U)} \frac{\partial (\Delta U)}{\partial x} \pm \frac{1}{m} \frac{\partial U}{\partial x} \frac{\partial f}{\partial v} = \left\{ \frac{\partial f}{\partial t} \right\},
\]  \(4\)

where \(\pm\) sign is respectively upper for ion particles and lower for electrons. The potential \(U\) has units of energy and is defined as shown in fig.1 such that \(U = -e\phi\) where \(\phi\) is electric potential. Since

\[ \frac{\partial (\Delta U)}{\partial x} = \beta \Delta U, \]  \(5\)

\[ \frac{\partial U}{\partial x} = \alpha + \beta a_1 \Delta U + 2\beta a_2 \Delta U^2 + 3\beta a_3 \Delta U^3 + \cdots, \]  \(6\)
$$\frac{\partial f}{\partial v} = \frac{\partial f_0}{\partial v} + \Delta U \frac{\partial f_1}{\partial v} + \Delta U^2 \frac{\partial f_2}{\partial v} + \cdots, \quad (7)$$

$$\frac{\partial f}{\partial (\Delta U)} = f_1 + 2\Delta U f_2 + 3\Delta U^2 f_3 + \cdots, \quad (8)$$

The Boltzmann equation can be broken down by order in $\Delta U$ assuming the collision term can be likewise broken down:

\[ 1: \quad \frac{\partial f_0}{\partial t} \pm \frac{\alpha}{m} \frac{\partial f_0}{\partial v} = \left[ \{ \frac{\partial f}{\partial t} \} \right]_c \quad (9a) \]

\[ \Delta U: \quad \frac{\partial f_1}{\partial t} + \beta v f_1 \pm \frac{\alpha}{m} \frac{\partial f_1}{\partial v} \pm \frac{\beta a_1}{m} \frac{\partial f_0}{\partial v} = \left[ \{ \frac{\partial f}{\partial t} \} \right]_{\Delta U} \quad (9b) \]

\[ \Delta U^2: \quad \frac{\partial f_2}{\partial t} + 2\beta v f_2 \pm \frac{2\beta a_1}{m} \frac{\partial f_1}{\partial v} \pm \frac{\beta a_2}{m} \frac{\partial f_0}{\partial v} = \left[ \{ \frac{\partial f}{\partial t} \} \right]_{\Delta U^2} \quad (9c) \]

\[ \vdots \]

\[ \Delta U^n: \quad \frac{\partial f_n}{\partial t} + n\beta v f_n \pm \frac{\alpha}{m} \frac{\partial f_n}{\partial v} \pm \frac{\beta a_1}{m} \frac{\partial f_{n-1}}{\partial v} \pm \frac{2\beta a_2}{m} \frac{\partial f_{n-2}}{\partial v} \pm \cdots \pm \frac{(n-1)\beta a_{n-1}}{m} \frac{\partial f_1}{\partial v} \pm \frac{n\beta a_n}{m} \frac{\partial f_0}{\partial v} = \left[ \{ \frac{\partial f}{\partial t} \} \right]_{\Delta U^n}. \quad (9d) \]

The Poisson equation

\[ \frac{d^2 U}{d x^2} = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_1(v, \Delta U) dv - \int_{-\infty}^{\infty} f_{e1}(v, \Delta U) dv \right] \quad (10) \]

can similarly be broken down as

\[ 1: \quad 0 = \int_{-\infty}^{\infty} f_{01}(v) dv - \int_{-\infty}^{\infty} f_{e01}(v) dv \quad (11a) \]

\[ \Delta U: \quad \beta^2 a_1 = \int_{-\infty}^{\infty} f_{11}(v) dv - \int_{-\infty}^{\infty} f_{e11}(v) dv \quad (11b) \]

\[ \Delta U^2: \quad \beta^2 a_2 = \int_{-\infty}^{\infty} f_{12}(v) dv - \int_{-\infty}^{\infty} f_{e12}(v) dv \quad (11c) \]

\[ \vdots \]

\[ \Delta U^n: \quad \beta^2 a_n = \int_{-\infty}^{\infty} f_{1n}(v) dv - \int_{-\infty}^{\infty} f_{en}(v) dv. \quad (11d) \]
The Fokker-Planck collision term for Coulomb collisions

\[
\left\{ \frac{\partial f}{\partial t} \right\}_c = \Gamma \left[ \frac{-\partial}{\partial v_i} \left( f \frac{m}{2M} \frac{\partial}{\partial v_i} \nabla^2 g \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_k} \left( f \frac{\partial^2 g}{\partial v_i \partial v_k} \right) \right]
\]

where

\[
\Gamma = \frac{q^2 q^2 \ln \Lambda}{4 \pi \varepsilon_0 m^2},
\]

\[
g(v) = \int f(v') \vert v - v' \vert \, dv',
\]

\[
M = \frac{m m'}{m + m'}
\]

and where no superscript refers to the particle species undergoing the collisions and the superscript \( i \) refers to the particle species with which the collision occurs, can be written as

\[
\left\{ \frac{\partial f}{\partial t} \right\}_c = \Gamma \left[ \frac{-m}{2M} \nabla \cdot (f \nabla (\nabla^2 g)) + \nabla f \cdot \nabla (\nabla^2 g) + \frac{1}{2} f \nabla^2 \nabla^2 g + \frac{1}{2} \frac{\partial^2 f}{\partial v_i \partial v_k} \cdot \frac{\partial^2 g}{\partial v_i \partial v_k} \right].
\]

In polar coordinates with axial symmetry (no \( \theta \) dependence):

\[
\frac{1}{2} \frac{\partial^2 f}{\partial v_i \partial v_k} \cdot \frac{\partial^2 g}{\partial v_i \partial v_k} = \frac{1}{2} \frac{\partial^2 f}{\partial v^2} \frac{\partial^2 g}{\partial v^2} + \frac{1}{2} \frac{\partial^2 f}{\partial z^2} \frac{\partial^2 g}{\partial z^2} + \frac{1}{2} \frac{\partial^2 f}{\partial R \partial z} \frac{\partial^2 g}{\partial R \partial z} + \frac{1}{2} \frac{\partial^2 f}{\partial R^2} \frac{\partial^2 g}{\partial R^2}
\]

\[
\nabla^2 g = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial g}{\partial R} \right) + \frac{\partial^2 g}{\partial z^2},
\]

\[
\nabla \nabla^2 g = \varepsilon_R \left( \frac{\partial}{\partial R} \left( \frac{\partial}{\partial R} \left( \frac{\partial g}{\partial R} \right) \right) + \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \left( \frac{\partial g}{\partial z} \right) \right) + \partial^2 g \right)
\]

and

\[
\nabla^2 \nabla^2 g = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial}{\partial R} \left( \frac{\partial g}{\partial R} \right) \right) + \frac{\partial}{\partial R} \left( \frac{\partial}{\partial R} \left( \frac{\partial^2 g}{\partial R^2} \right) \right) + \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \left( \frac{\partial^2 g}{\partial z^2} \right) \right) + \frac{\partial^4 g}{\partial z^4}.
\]

For brevity the polar coordinates symbols \( R, \theta \) and \( z \) refer to velocity.

Under the assumption that the radial distributions are Maxwellian with constant temperatures,

\[
f(R, z) = f_s(z) e^{-\alpha R^2}
\]
where \( a \) is based on the temperature of the species undergoing the collisions and when \( a' \) replaces \( a \), \( a' \) is based on the temperature of the species with which the collisions occur.

The function \( g \) can then be written as

\[
g(R, z) = \int_{-\infty}^{\infty} f_z(z + \xi) \int_0^{2\pi} \int_0^\infty e^{-a'(R^2+r^2+2Rr \cos(\varphi-\theta))} \sqrt{r^2 + \xi^2} r dr d\varphi d\xi. \tag{22}
\]

In this case and in what follows, distribution functions integrated to form \( g \) are of the the species. The derivatives of \( g(R, z) \) at \( R = 0 \) are then

\[
\frac{\partial g^n(0, z)}{\partial R^n} = \int_{-\infty}^{\infty} f_z(z + \xi) \frac{\partial^n G}{\partial R^n} \bigg|_{R=0} d\xi \tag{23}
\]

where

\[
G(R, \xi) = \int_0^{2\pi} \int_0^\infty e^{-a'(R^2+r^2+2Rr \cos(\varphi-\theta))} \sqrt{r^2 + \xi^2} r dr d\varphi. \tag{24}
\]

Then

\[
G(0, \xi) = \int_0^\infty 2\pi e^{-a' r^2} \sqrt{r^2 + \xi^2} r dr, \tag{25}
\]

\[
\frac{\partial G}{\partial R} \bigg|_{R=0} = 0, \tag{26}
\]

\[
\frac{\partial^2 G}{\partial R^2} \bigg|_{R=0} = \int_0^\infty 4\pi e^{-a' r^2} (a'^2 r^4 - a' r^2) \sqrt{r^2 + \xi^2} r dr, \tag{27}
\]

\[
\frac{\partial^3 G}{\partial R^3} \bigg|_{R=0} = 0, \tag{28}
\]

\[
\frac{\partial^4 G}{\partial R^4} \bigg|_{R=0} = \int_0^\infty e^{-a' r^2} (12\pi a'^4 r^4 - 48\pi a'^2 r^2 + 24\pi a'^2) \sqrt{r^2 + \xi^2} r dr. \tag{29}
\]

Therefore \( g \) can be expressed as

\[
g(R) = g(0) + \frac{R^2}{2!} \frac{\partial^2 g(0)}{\partial R^2} + \frac{R^4}{4!} \frac{\partial^4 g(0)}{\partial R^4} + O(R^6) \tag{30}
\]

so at \( R = 0 \),

\[
\frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial g}{\partial R} \right) = 2 \frac{\partial^2 g}{\partial R^2} + \frac{2R^2}{3} \frac{\partial^4 g}{\partial R^4} + O(R^4), \tag{31}
\]
\[
\frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial g}{\partial R} \right) \right) = \frac{4}{3} R \frac{\partial^4 g}{\partial R^4} + O(R^2), \tag{32}
\]

\[
\frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial g}{\partial R} \right) \right) \right) = \frac{8}{3} \frac{\partial^4 g}{\partial R^4} + O(R). \tag{33}
\]

Also then at \( R = 0, \)

\[
\nabla^2 \nabla^2 g = \frac{8}{3} \frac{\partial^4 g}{\partial R^4} + 4 \frac{\partial^4 g}{\partial R^2 \partial z^2} + \frac{\partial^4 g}{\partial z^4}, \tag{34}
\]

\[
\nabla \nabla^2 g = \varepsilon_R(0) + \varepsilon_z \left( 2 \frac{\partial^3 g}{\partial R^2 \partial z} + \frac{\partial^3 g}{\partial z^3} \right). \tag{35}
\]

Since \( f(R, z) = f_z(z) e^{-a'R^2} \) and since \( \frac{\partial^2 g}{\partial R^2} = \frac{1}{R} \frac{\partial g}{\partial R} \) in the limit \( R \to 0, \)

\[
\frac{1}{2} \frac{\partial^2 f}{\partial \nu_i \partial \nu_k} \frac{\partial^2 g}{\partial \nu_i \partial \nu_k} = \frac{1}{2} \frac{\partial^2 f}{\partial z^2} \frac{\partial^2 g}{\partial z^2} - \sqrt{a} (\sqrt{a} + \sqrt{a'}) f \frac{\partial^2 g}{\partial R^2}. \tag{36}
\]

Therefore the collision term at \( R = 0 \) can be written as

\[
\frac{1}{\Gamma} \left[ \frac{\partial f}{\partial t} \right]_c = \frac{\partial^2 f}{\partial z^2} \left[ \frac{1}{2} \frac{\partial^2 g}{\partial z^2} \right] \\
+ \frac{\partial f}{\partial z} \left[ \left( 2 \frac{\partial^3 g}{\partial R^2 \partial z} + \frac{\partial^3 g}{\partial z^3} \right) \left( 1 - \frac{m}{2M} \right) \right] \\
+ f \left[ \frac{1}{2} \left( 8 \frac{\partial^4 g}{\partial R^4} + 2 \frac{\partial^4 g}{\partial R^2 \partial z^2} + \frac{\partial^4 g}{\partial z^4} \right) \left( 1 - \frac{m}{2M} \right) - \sqrt{a} (\sqrt{a} + \sqrt{a'}) \frac{\partial^2 g}{\partial R^2} \right]. \tag{37}
\]

Invoking the nondimensionalization \( \mathcal{R} = \sqrt{a} R \) and \( Z = \sqrt{a'} z \) and defining the functions

\[
\mathcal{G}_0(\eta) = \int_0^\infty e^{-u^4} 2\pi \sqrt{u^2 + \eta^2} \, u \, du \tag{38}
\]

\[
\mathcal{G}_2(\eta) = \int_0^\infty e^{-u^4} 4\pi (u^2 - 1) \sqrt{u^2 + \eta^2} \, u \, du \tag{39}
\]

\[
\mathcal{G}_4(\eta) = \int_0^\infty e^{-u^4} (2\pi u^4 - 48\pi u^2 + 24\pi) \sqrt{u^2 + \eta^2} \, u \, du \tag{40}
\]
results in

\[
\frac{1}{\Gamma} \left\{ \frac{\partial f}{\partial t} \right\}_e = \frac{\partial^2 f}{\partial Z^2} \left[ \frac{1}{2} \int_{-\infty}^{\infty} f''_{\pm}(Z + \eta) g_0(\eta) d\eta \left( \frac{a}{a'} \right) \right] \\
+ \frac{\partial f}{\partial Z} \left[ \left( 2 \int_{-\infty}^{\infty} f'_{\pm}(Z + \eta) g_2(\eta) d\eta + \int_{-\infty}^{\infty} f'''_{\pm}(Z + \eta) g_0(\eta) d\eta \right) \left( 1 - \frac{m}{2M} \right) \sqrt{\frac{a}{a'}} \right] \\
+ f \left[ \frac{1}{2} \left( \int_{-\infty}^{\infty} f_{\pm}(Z + \eta) g_4(\eta) d\eta + 2 \int_{-\infty}^{\infty} f'''_{\pm}(Z + \eta) g_2(\eta) d\eta + \int_{-\infty}^{\infty} f''''_{\pm}(Z + \eta) g_0(\eta) d\eta \right) \left( 1 - \frac{m}{2M} \right) \\
- \left( \frac{a}{a'} + \sqrt{\frac{a}{a'}} \right) \int_{-\infty}^{\infty} f_{\pm}(Z + \eta) g_2(\eta) d\eta \right].
\] (41)

From (41) it can be seen that the collision term can be expressed as a bilinear operator

\[
\frac{1}{\Gamma} \left\{ \frac{\partial f}{\partial t} \right\}_e = \frac{\partial^2 f}{\partial Z^2} A(f) + \frac{\partial f}{\partial Z} B(f) + f C(f).
\] (42)

It is also quasi-linear in the sense that \( A(f), B(f) \) and \( C(f) \) average out any local disturbances in \( f \). Inserting the expansion (3) for \( f \) into (43) yields:

\[
\frac{1}{\Gamma} \left\{ \frac{\partial f}{\partial t} \right\}_e = \left[ \frac{\partial^2 f_0}{\partial Z^2} A(f_0) + \frac{\partial f_0}{\partial Z} B(f_0) + f_0 C(f_0) \right] \\
+ \Delta U \left[ \frac{\partial^2 f_1}{\partial Z^2} A(f_0) + \frac{\partial f_1}{\partial Z} B(f_0) + f_1 C(f_0) + \frac{\partial^2 f_0}{\partial Z^2} A(f_1) + \frac{\partial f_0}{\partial Z} B(f_1) + f_0 C(f_1) \right] \\
+ \cdots \\
+ \Delta U^n \left[ \sum_{m=0}^{n} \left( \frac{\partial^2 f_{n-m}}{\partial Z^2} A(f_m) + \frac{\partial f_{n-m}}{\partial Z} B(f_m) + f_{n-m} C(f_m) \right) \right].
\] (43)

III. THE TIME DEPENDENT SHEATH-PRESHEATH STRUCTURE FOR ION-ION FOKKER-PLANCK COLLISIONS AND BOLTZMANN ELECTRONS

WITH NO RETURNING IONS

The Boltzmann electron assumption is made to enable the numerical calculations to proceed with time steps on the order of an ion characteristic time. In the asymptotic
In the asymptotic presheath the Boltzmann electron assumption becomes

\[ n_e = n_0 e^{-\left[a_1 \Delta U + a_2 \Delta U^2 + \cdots + a_n \Delta U^n\right] \frac{1}{kT_e}} \]  

(44)

where \( n_e \) is the electron density, \( T_e \) is the electron temperature, and \( n_0 \) is the electron density in the asymptotic presheath at \( \Delta U = 0 \). Expansion of (44) in terms of \( \Delta U \) yields

\[ n_e = \left[n_0 + \Delta U \left(- \frac{a_1}{kT_e} n_0\right) + \Delta U^2 \left(- \frac{a_2}{kT_e} + \frac{1}{2} \left( \frac{a_2}{kT_e} \right)^2\right) n_0\right] + \Delta U^3 \left(- \frac{a_3}{kT_e} + \frac{a_1 a_2}{(kT_e)^2} - \frac{1}{6} \left( \frac{a_3}{kT_e}\right)^3\right) n_0 + \cdots. \]

(45)

The assumption is made that \( \lambda_D \ll \lambda_i \) where \( \lambda_D \) is the Debye length and \( \lambda_i \) is an ion mean free path so that the Poisson equation (11a-d) reduces to equating electron and ion densities in order of \( \Delta U \). The Boltzmann equations (9a-d) are nondimensionalized in the following way:

\[ F_i = \frac{f_i}{n_R a^{3/2}} \quad \text{for} \quad 0 \leq i \leq n, \]

(46)

\[ \tau = \frac{n_R \Gamma}{a^{1/2}}, \]

(47)

\[ \varphi_{-1} = \frac{n a}{mn_R \Gamma}, \]

(48)

\[ \varphi_0 = \frac{\beta}{n_R \Gamma}, \]

(49)

\[ \varphi_i = \frac{\beta a a_{i-1}}{mn_R \Gamma} \quad \text{for} \quad 1 \leq i \leq n. \]

(50)

The quantities \( \varphi_{-1} \) through \( \varphi_n \) represent the potential structure of the presheath. Using these nondimensionalizations, the Boltzmann equations become

\[ \frac{\partial F_i}{\partial \tau} - \frac{\partial^2 F_i}{\partial Z^2} A(F_0) - \frac{\partial F_i}{\partial Z} B(F_0) - F_i C(F_0) \]

\[ = - \varphi_{0i} Z F_i - \varphi_{-1} \frac{\partial F_i}{\partial Z} - \sum_{m=1}^i \varphi_{m} \frac{\partial F_{i-m}}{\partial Z} \]

\[ + \sum_{m=1}^i \left( \frac{\partial^2 F_{i-m}}{\partial Z^2} A(F_m) + \frac{\partial F_{i-m}}{\partial Z} B(F_m) + F_{i-m} C(F_m) \right) \]

(51)
where the summations are taken to be zero if $i = 0$. The Poisson equation (11a-d) and (46), nondimensionalized, become

\begin{align}
\int_{-\infty}^{\infty} F_0 dZ &= \eta \left[ \varphi_1 + \cdots + \varphi_n \right] \frac{1}{\varphi_0} \left( \frac{2T}{T_e} \right) \\
\int_{-\infty}^{\infty} F_1 dZ &= \eta \left[ \varphi_1 + \cdots + \varphi_n \right] \frac{1}{\varphi_0} \left( \frac{2T}{T_e} \right) + \frac{1}{2} \left( \frac{\varphi_1}{\varphi_0} \right)^2 \left( \frac{2T}{T_e} \right)^2 \\
\int_{-\infty}^{\infty} F_2 dZ &= \eta \left[ \varphi_1 + \cdots + \varphi_n \right] \frac{1}{\varphi_0} \left( \frac{2T}{T_e} \right) + \frac{\varphi_1 \varphi_2}{\varphi_0 \varphi_0} \left( \frac{2T}{T_e} \right)^2 - \frac{1}{6} \left( \frac{\varphi_1}{\varphi_0} \right)^3 \left( \frac{2T}{T_e} \right)^3 \\
\int_{-\infty}^{\infty} F_3 dZ &= \eta \left[ \varphi_1 + \cdots + \varphi_n \right] \frac{1}{\varphi_0} \left( \frac{2T}{T_e} \right) + \frac{\varphi_1 \varphi_2}{\varphi_0 \varphi_0} \left( \frac{2T}{T_e} \right)^2 - \frac{1}{6} \left( \frac{\varphi_1}{\varphi_0} \right)^3 \left( \frac{2T}{T_e} \right)^3
\end{align}

where $\eta = \frac{n}{n_0}$ and may be specified as a function of time.

The equations (51) for the distribution functions are advanced through time by implicit-explicit scheme that allows the boundary conditions to be applied as linear constraints that determine $\varphi_{-1}$ through $\varphi_n$ at each time step. In matrix form, (51) are

\begin{equation}
\begin{bmatrix}
T(r) & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\begin{bmatrix}
F_i(r + \Delta r) \\
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix}
= \varphi_{-1} \begin{bmatrix} v_{i-1}(r) \end{bmatrix} + \varphi_0 \begin{bmatrix} v_0^0(r) \end{bmatrix} + \cdots + \varphi_n \begin{bmatrix} v_n^0(r) \end{bmatrix} + \begin{bmatrix} d_i(r) \end{bmatrix}
\end{equation}

for $0 \leq i \leq n$ where $T(r)$ is an $m \times m$ matrix and the others are $m \times 1$ matrices. The quantity $m$ is the number of divisions in the velocity space $Z$ chosen for the numerical scheme. The matrix $T(r)$ is computed from $A(F_0(r))$, $B(F_0(r))$ and $C(F_0(r))$ and has zero elements except for an odd number of diagonals that reflect the order of accuracy (normally 5 diagonals are used to achieve forth order accuracy) and the bottom row which is used to invoke the Poisson equations (52a-d). The sparse nature of $T(r)$ enables quick inversion (of order $m$ operations) of (53) into

\begin{equation}
\begin{bmatrix}
F_i(r + \Delta r) \\
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix}
= \varphi_{-1} \begin{bmatrix} V_{i-1}(r) \end{bmatrix} + \varphi_0 \begin{bmatrix} V_0^0(r) \end{bmatrix} + \cdots + \varphi_n \begin{bmatrix} V_n^0(r) \end{bmatrix} + \begin{bmatrix} D_i(r) \end{bmatrix}
\end{equation}

where $\varphi_{-1}$ through $\varphi_n$ are still undermined scalars.
The boundary conditions imposed are that 1) there are no ions returning from the surface and 2) \( \frac{1}{n} \frac{dn}{dz} \) is imposed as a function of time. The no returning ions condition cannot be imposed exactly because the distribution function is approximated by the \( n + 1 \) parts. Instead the conditions

\[
\int_{-\infty}^{0} Z^i (F_0 + F_1 + \cdots + F_n) dZ \quad \text{for} \quad 0 \leq i \leq n
\]

are applied to represent zero returning density, zero returning flux, zero returning momentum and so forth up to \( n + 1 \) conditions. The boundary condition on \( \frac{1}{n} \frac{dn}{dz} \) at \( z = 0 \) becomes (from (5), (6) and (45))

\[
\xi = \frac{-kT \rho a}{mnR^{n+1} d \rho} \frac{1}{n} \frac{dn}{dz} = \varphi_1 + 2\varphi_2 + \cdots + n\varphi_n
\]

where \( \xi \) is imposed. The boundary conditions (55) and (56) are written in matrix form as

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots \\
0 & BC & \ddots & \ddots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
\xi \\
F_0 \\
\vdots \\
F_n
\end{bmatrix} =
\begin{bmatrix}
\xi \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

where \( BC \) is an \((n + 1) \times (n + 1)\) matrix of \( m \)-vector elements computed from (55). Also the elements \( F_0 \) through \( F_n \) are \( m \)-vectors. The remainder of the elements are scalars. The \( m \)-vectors are multiplied as inner products to form scalars. The distributions \( F_i \) at the advanced time step (54) and \( \xi \) can be represented as

\[
\begin{bmatrix}
\xi \\
F_0 \\
\vdots \\
F_n
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & \cdots & n \\
V_0^{-1} & V_0 & V_0^1 & \cdots & V_0^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
V_n^{-1} & V_n & V_n^1 & \cdots & V_n^n
\end{bmatrix} \begin{bmatrix}
\varphi^{-1} \\
\varphi_0 \\
\vdots \\
\varphi_n
\end{bmatrix} + \begin{bmatrix}
0 \\
D_0 \\
\vdots \\
D_n
\end{bmatrix}
\]

where again \( F_0 \) through \( F_n \) and the \( V \)'s are \( m \)-vectors and the other elements are scalars.

Equations (57) and (58) are combined to form
\[
\begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ldots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & V_n \\
\end{bmatrix}
\begin{bmatrix}
V_0^{-1} & V_0^0 & \ldots & V_0^n \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
V_n^{-1} & V_n^0 & \ldots & V_n^n \\
\end{bmatrix}
\begin{bmatrix}
\phi_{-1} \\
\vdots \\
\phi_0 \\
\vdots \\
\phi_n \\
\end{bmatrix}
+ 
\begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ldots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & D_n \\
\end{bmatrix}
\begin{bmatrix}
\zeta \\
\vdots \\
D_0 \\
\vdots \\
0 \\
\end{bmatrix}
\]  

which is inverted to find \( \phi_{-1} \) through \( \phi_n \) and complete the advance to the next time step.

ACKNOWLEDGMENTS

This work was supported by the Air Force Office of Scientific Research grant 85-0375.

REFERENCES


True Asymptotic Plasma - Sheath Matching with an Asymptotically Correct Collisional Presheath

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Submitted to

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June, 1989

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This report covers work done on plasma sheaths and presheaths as applicable to Thermionic Energy Convertors (TECs) during the period 1 September 1985 to 31 October 1988. A code modelling the TEC has been completed with the plasma sheath and presheath work incorporated. Also additional work on presheaths using Fokker-Planck Collision terms has been done.
True Asymptotic Plasma - Sheath Matching with an Asymptotically Correct Collisional Presheath

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INTRODUCTION

The work under this grant on plasma presheaths, which form a transition region between the collisionless electrode sheaths and the plasma, is directed toward the problems of the Thermionic Energy Convertor (TEC). Figure 1 shows a schematic of a TEC in a reactor core for space power applications and the basic physics. Cesium is put the gap between the emitter and collector for two purposes: first, to ionize and neutralize the space charge so that a useful electron current density can pass (10 - 100 amps/square cm), and second to reduce the electrode work functions by adsorption of cesium. Of the plasma physics of the cesium filled gap of the TEC, the plasma-electrode interactions are the most significant part because these regions form boundary conditions which control the plasma density and temperatures of the entire gap. Thus the research under this grant has been directed toward the study of collisional presheaths which form the layer adjacent to an electrode on the order of one ion mean free path thick. However, the research pursued under this grant is not limited in applicability to TECs but is of interest to plasma-surface interactions in general. Other applications include electric propulsion where electrode erosion is a problem and not fully understood and more generally any plasma-surface interaction.

This report includes the asymptotic presheath theory developed, and is preceded by the basic theory of the Thermionic Energy Convertor (TEC) and is followed by the application of the theory to a time dependent model of the TEC in the program called TEC. As shown in the TEC results, the agreement with experiment is good except in the low current regime of the TEC where an unexplained disagreement remains. This is still a puzzle.
BASIC TEC THEORY

The basic theory of the TEC is set forth in the following paper published under this grant.
Effects of Emitter Sheath Ion Reflection and Trapped Ions on Thermionic Converter Performance Using an Isothermal Electron Model

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Effects of Emitter Sheath Ion Reflection and Trapped Ions on Thermionic Converter Performance Using an Isothermal Electron Model

GEOFFREY L. MAIN and S. H. LAM

Abstract—This paper couples exact collisionless sheath calculations to an isothermal electron model of a thermionic converter. The emitter sheath structure takes into account reflected ions, trapped ions, and surface emission ions. It is shown that lessening the net loss of ions at the emitter in the ignited mode by these phenomena degrades performance. In addition, it is shown that when the emitter returns too many of the ions, the arc is extinguished because there is insufficient resistive heating to maintain the necessary plasma electron temperature for ionization. These results suggest that the ignited mode cannot be improved much. However, nonignited modes in which the electron temperature remains low, such as the pulsed mode, do not suffer from this adverse behavior.

I. INTRODUCTION

EMITTER sheath phenomena are important in thermionic energy converters because the emitter sheath forms the emitter boundary condition for the plasma in the gap by controlling both the ion loss rate and the loss rate of hot (3000 K) plasma electrons to the emitter. This paper examines two expected emitter sheath phenomena and their effects on converter performance: reflection of ions coming from the plasma by a double emitter sheath, and ions trapped in the double emitter sheath. The authors have previously suggested that ion reflection might improve thermionic energy converter performance [1] and have subsequently shown that ion reflection at the emitter is likely to degrade the performance in the ignited mode and, in addition, that trapped ions in a double emitter sheath are also likely to degrade performance in the ignited mode [2]. Lundgren [3], [4] has also shown this with simplified ion and electron dynamics. In the present paper the effects of emitter ion reflection and ion trapping in the ignited mode are calculated using exact electron and ion dynamics in the collisionless (except for ion trapping) sheaths. The electrons entering the sheaths from the plasma are assumed to have a Maxwellian distribution, but no assumptions are made about the returning electrons, and the electron density in the sheath is calculated exactly. The ions entering the sheaths from the plasma are not assumed cold, but are given the correct ion temperature and shifted in velocity according to a generalization of the Bohm criterion [5], [6].

Both ion reflection and trapped ions in the emitter sheath reduce the normalized (by plasma density) net ion loss rate to the emitter. Also, both of these phenomena raise the normalized plasma density adjacent to the emitter. The higher plasma density at the emitter causes a greater increase in the loss of hot plasma electron energy to the emitter than the corresponding decrease in the loss of ionization energy (carried by the ions) to the emitter. Therefore, these emitter sheath phenomena increase arc-drop. Within the limitations of the present isothermal thermionic converter formulation, all three of these phenomena (which become significant at low currents) steepen the current-voltage characteristic. At low current densities, the present theory shows that the collector sheath height decreases, resulting in a larger electron diffusion velocity than can be justified for the continuum model used in the plasma region. The result of lower performance at lower current is in agreement with experimental studies. At some current density which depends strongly on the emitter sheath conditions, the ignited mode is no longer self-sustaining and the arc is extinguished.

Fig. 1 is a schematic diagram of the cesium diode converter. The emitter is heated externally to temperature $T_e$ which is typically 1750 K or higher, and the collector is cooled to temperature $T_c$ which is typically 900–1100 K. The gap space $d$, or converter length, which is typically 0.25 mm, separates the emitter from the collector. The cesium reservoir, which is sometimes imbedded in the collector, is kept at temperature $T_k$ to maintain the desired cesium pressure (typically 1 to 2 torr) in the gap. The electrical load is connected across the emitter and collector to produce power.

II. THE ISOTHERMAL ELECTRON FORMULATION

In this section the isothermal thermionic converter formulation is developed. The formulation is similar to that of Lam [7] but is generalized to eliminate the assumption of high sheaths which has previously been used to simplify the electron dynamics. Since both low-emitter and low-collector sheath heights are encountered as a consequence of ion reflection and trapped ions, the assumption
of Boltzmann plasma electron distributions at the plasma-sheath interface must be abandoned. At both the emitter and collector, the low sheaths return few plasma electrons, leaving the distributions largely one sided. Furthermore, at the emitter sheath emitted electrons must be taken into account. Thus the ratio of electrons moving toward the sheath to the total density of electrons at the sheath edge is not \( \frac{1}{2} \), as in the Boltzmann assumption.

In Fig. 2 we define the potentials in the converter. All of the potentials are nondimensionalized by emitter temperature as follows:

\[
\chi = \frac{\phi}{kT_E}
\]

where

- \( \chi \) nondimensional potential,
- \( \phi \) potential,
- \( q \) electron charge,
- \( k \) Boltzmann constant,
- \( T_E \) emitter temperature.

We also use the following terminology for various potentials in the converter:

- \( \Phi_E \) emitter work function,
- \( \Delta \) back sheath height,
- \( \Delta \chi \) reflective potential,
- \( \Delta \chi_E \) emitter sheath height,
- \( \Delta \chi_P \) plasma potential drop,
- \( V_d \) arc-drop,
- \( \chi_c \) collector sheath height,
- \( \Phi_c \) collector work function, and
- \( V_{out} \) converter output voltage.

Inspection of Fig. 2 immediately yields the following relations:

\[
V_d = V_{out} - (\Phi_E - \Phi_c) - \Delta \chi
\]

\[
V_d = (\chi_c - \chi_E) - \Delta \chi_P.
\]

The Richardson current density of electrons from the emitter is

\[
J_R \left( \frac{A}{cm^2} \right) = 120 \frac{T_E^2}{K^2} (K^2) \exp(-\Phi_E).
\]

The emitted current density which crosses the emitter sheath potential peak into the converter plasma region is

\[
J_E = J_R \exp(-\Delta \chi), \quad \Delta \chi > 0
\]

\[
J_E = J_R, \quad \Delta \chi \leq 0.
\]

We also define the net current density through the converter \( J \) and the normalized current density \( j = \frac{J}{J_E} \).

We have assumed for convenience that the ion contribution to net current is negligible because the cesium-ion-to-electron-mass ratio is enormous. Ions will typically contribute no more than 1 percent of the net current. Electron temperature is nondimensionalized as

\[
\tau = \frac{T_e}{T_E}
\]

where \( T_e \) is the plasma electron temperature which, in this section, is constant by the isothermal assumption. Finally, we have the thermal speeds:

\[
a_e = \sqrt{8kT_e/\pi m}
\]

\[
a_E = \sqrt{8kT_E/\pi m}.
\]

The isothermal formulation is developed from here in the same way as the general formulation except that we take full advantage of the isothermal assumption by looking only at the global conservation equations instead of the local ones used in the general formulation. We then assume that the transport properties, collision frequencies, and the ionization source coefficient are constant across the converter because of the isothermal assumption. Also, we find only the steady-state solution. We carry out this development by deriving the global conservation equations for the isothermal case (current, momentum, and electron energy) and then reducing these to a set of three simultaneous equations in the variables \( \tau, \chi_E, \) and \( \chi_c \). In some cases the actual calculations are carried out
using different variables when \( \chi_E \) or \( \chi_C \) are small or zero. In the case, for instance, of a single ion-repelling emitter sheath we use \( j \) because \( \chi_E \) is zero. These equations are nonlinear and solved numerically using a positive definite Newton’s method.

First, we consider conservation of current. The collector is assumed to emit nothing; therefore, at the plasma-collector sheath interface we have

\[
J = \frac{\alpha_1 n(1)}{2} e^{-\chi_C/\tau} \tag{10}
\]

where \( \alpha_1 \) is the fraction of the total plasma density at the collector sheath which is moving toward the collector and \( n(1) \) is the total plasma density at the plasma-collector sheath interface. Because we continue to assume that the part of the plasma electron distribution coming into the collector sheath is Maxwellian, we can write \( \alpha_1 \) as

\[
\alpha_1 = \frac{1}{1 + \frac{2}{\sqrt{\pi}} \int_0^{\chi_C/\tau} e^{-u^2} du} \tag{11}
\]

which takes into account the plasma electrons reflected by the collector sheath. We still assume that the plasma electron distribution coming into the collector sheath is Maxwellian and that it does not have any velocity shift because the sheath is expected to be electron repelling. In the limit of a high collector sheath, \( \alpha_1 = 1/2 \) and we have a fully Boltzmann distribution of electrons at the collector sheath edge. The situation at the emitter is more complex because the emitted electrons must be taken into account. We have the backscattered current density \( J_{BS} \) which is the plasma electron current density moving into the emitter:

\[
J_{BS} = \frac{n(0) a_e a_0}{2} \exp \left( -\frac{\chi_E}{\tau} \right) \tag{12}
\]

where \( n(0) \) is the total plasma density at the emitter sheath-plasma interface and \( a_0 \) is the fraction of total plasma density at the interface moving toward the emitter.

Continuity of electron current demands

\[
J_r = J_e \tag{13}
\]

which can be written as

\[
J_e = J \left( 1 + \frac{n(0) a_0}{n(1) \alpha_1} \exp \left( \frac{\chi_C - \chi_E}{\tau} \right) \right) \tag{14}
\]

This can be rewritten using (3) and (6) as

\[
\alpha_0 = \sqrt{\frac{\pi}{2}} Q \left( \frac{1}{j} - 1 \right) \exp \left( \frac{\chi_E}{\tau} \right) \tag{16}
\]

where

\[
Q = \frac{(u_e)_{0}}{\sqrt{kT_e / m}} \tag{23}
\]

is the electron Mach number at the emitter. This is just an application of (13).

Electron energy conservation is developed by considering energy exchange with the emitter and collector and energy lost to ionization. Power carried into the plasma by emitted electrons is

\[
P_E = J_e \left( 2 + \Phi_e + \Delta \chi \right) \frac{kT_e}{q} \tag{17}
\]

Power returned to the emitter is

\[
P_{BS} = (J_e - J)(2\tau + \Phi_e + \Delta \chi) \frac{kT_e}{q} \tag{18}
\]

Power flowing into the collector is

\[
P_C = J(2\tau + \Phi_e + V_d + \Delta \chi) \frac{kT_e}{q} \tag{19}
\]

Ionization power loss is

\[
P_{ion} = J_{ion} V_i \frac{kT_e}{q} \tag{20}
\]

where \( J_{ion} \) is the total ion current into both the emitter and collector, and \( V_i \) is the first ionization energy. Conservation of electron energy is

\[
P_e = P_E + P_C + P_{ion} \tag{21}
\]

which can be reduced to

\[
\tau = 1 - \frac{1}{2} j V_d - \frac{1}{2} j V_i \tag{22}
\]

where \( j = J_{ion} / J_e \). In the ignited mode \( \tau \) is generally about 2 (\( T_e = 1750 \text{ K} \) and \( T_C = 3000 \text{ K} \); consequently the arc-drop \( V_d \) is negative. In other words, the high plasma electron temperature is generated by resistance heating.

Finally, we consider electron and ion momentum. From electron momentum conservation, we find the potential drop in the plasma region. By adding the electron and ion momentum equations as in the general case, we find our diffusion equation and boundary conditions to which the sheaths contribute flux terms. When we introduce the ionization source term into this, we have the complete formulation. Electron momentum conservation is

\[
0 = -\frac{dp_e}{dx} - qn \frac{dv}{dx} - \frac{a_e m u_e}{\lambda_e} \tag{23}
\]

where \( \lambda_e \) is electron mean-free path. Using \( p_e = n kT_e \) and \( J = q n u_e \), we can rearrange (23) into

\[
J = -\frac{q \lambda_e}{m u_e} \left( kT_e \frac{dn}{dx} + q n \frac{dv}{dx} \right) \tag{24}
\]
This can be further reduced by dividing by $J_F$ and using $\xi = x/d$ where $d$ is the converter gap thickness:

$$j = -\frac{2}{4d}\frac{\lambda}{\sqrt{\tau n_k}} \left( \frac{d}{d\xi} \left( \tau \frac{dn}{d\xi} + n \frac{dn}{d\xi} \right) \right).$$

(25)

Integration of this equation from the emitter sheath interface to the collector sheath interface yields

$$\Delta \chi_p = \tau \ln \left( \frac{n(1)}{n(0)} \right) + jR$$

(26)

where

$$R = \frac{4}{\lambda} \sqrt{\tau} \int_0^1 \frac{n_k}{n(\xi)} d\xi.$$  

(27)

The quantity $R$ is the normalized plasma resistance.

The ion and electron momentum equations can be written

$$kT_e \frac{dn}{dx} = -q \frac{d\varphi}{dx} - \frac{mmu_e}{\lambda_e}$$

(28a)

$$kT_i \frac{dn}{dx} = qn \frac{d\varphi}{dx} - \frac{mmu_i}{\lambda_i}$$

(28b)

where $\lambda$ is ion mean-free path and $a$, is ion thermal speed:

$$a_i = \sqrt{\frac{8kT_i}{\pi M}}.$$  

Addition of (28a) and (28b) yields

$$(kT_e + kT_i) \frac{dn}{dx} = -(m \lambda_e a_e + m \lambda_i a_i) n$$

(29)

which is ambipolar diffusion. Equation (29) is differentiated to become

$$(kT_e + kT_i) \frac{d^2n}{dx^2} + \frac{m \lambda_e a_e}{\lambda_e} \frac{d}{dx} (n x) + \frac{m \lambda_i a_i}{\lambda_i} \frac{d}{dx} (n x) = 0.$$  

(30)

We assume recombination is negligible and the ionization source term is

$$\frac{d}{dx} (u_n) = \frac{d}{dx} (n u_n) = S n.$$  

(31)

Using (31) in (30) yields

$$\frac{d^2n}{d\xi^2} + \left( \frac{m \lambda_e a_e}{\lambda_e} + \frac{m \lambda_i a_i}{\lambda_i} \right) S d \left| n \right| = 0.$$  

(32)

Equation (29) taken at the boundaries of the plasma (at the emitter and collector sheath interfaces) forms the plasma boundary conditions

$$\left( \frac{dn}{d\xi} \right)_0 = \beta_0 n_0$$

$$\left( \frac{dn}{d\xi} \right)_1 = \beta_1 n_1$$

(33)

where

$$\beta_0 = -\frac{d}{kT_e + kT_i} \left( \frac{a_m}{\lambda_e} u_{e0} + \frac{a_M}{\lambda_i} u_{i0} \right)$$

$$\beta_1 = \frac{d}{kT_e + kT_i} \left( \frac{a_m}{\lambda_e} u_{e1} + \frac{a_M}{\lambda_i} u_{i1} \right).$$

(34)

Equation (32) is written as

$$\frac{d^2n}{d\xi^2} + A^2(\eta n) = 0$$

(35)

where

$$A(\eta) = d^2S \left( \frac{a_m}{\lambda_e} + \frac{a_M}{\lambda_i} \right)$$

(36)

where $A(\eta)$ is the ionization coefficient and is found from consideration of ionization kinetics of the cesium according to Lawless [8]. Its solution for $n$ is

$$n(\xi) = B \sin (A \xi + B)$$

(37)

where $B$ and $C$ are constants of integration and $A = A(\eta)$. The quantities $\beta_0$ and $\beta_1$, which are the boundary conditions for (37), can be written as functions of $\tau$, $\chi_F$, $\chi_c$, and $\Delta \chi_s$:

$$\beta_0 = \beta_0(\tau, \chi_F, \chi_c, \Delta \chi_s)$$

$$\beta_1 = \beta_1(\tau, \chi_F, \chi_c, \Delta \chi_s).$$

(38)

When there is no reflection, $\beta_0$ and $\beta_1$ are both large, i.e.:

$$\beta_0 = O\left( \frac{d}{n} \right), \quad \beta_1 = O\left( \frac{d}{n} \right).$$

Significant reflection on the emitter side reduces $\beta_0$ and it may indeed attain negative values for sufficiently strong reflection.

The density equation (35) with the boundary conditions $\beta_0$ and $\beta_1$ is a linear eigenvalue problem; its solution yields $A$ and $C$ as functions of $\beta_0$ and $\beta_1$. The calculated results are shown in Fig. 3. Since $A(\eta)$ is a function of $\eta$ from the ionization kinetics, the value of $\tau$ is thus determined by a function of $\beta_0$ and $\beta_1$. The plasma resistance $R$ also can be expressed in terms of functions of $\beta_0$ and $\beta_1$ through $A$ and $C$ using (27):

$$R = \frac{4}{\lambda} \sqrt{\tau} \frac{d}{\lambda} \left( \frac{\tan (\frac{A + C}{2})}{\tan (\frac{C}{2})} \right).$$

(39)

The sheath results which provide $J$, $Q$, $\beta_0$, and $\beta_1$, complete the isothermal formulation. The sheath theory used is an exact solution to the Poisson equation and collisionless Boltzmann equation for warm plasma ion distribution with a 10 percent of Bohm speed cutoff velocity to approximate the effect of the collisional presheath [6]. The results are summarized below. The quantities $\beta_0$, $\beta_1$, $Q$,
and \( j \) are found from the sheath calculations as functions of \( \tau, x_F, x_C, \) and \( \Delta x_i \), i.e.:

\[
\begin{align*}
\beta_0 &= \beta_0(\tau, x_F, x_C, \Delta x_i) \\
\beta_1 &= \beta_1(\tau, x_F, x_C, \Delta x_i) \\
Q &= Q(\tau, x_F, x_C, \Delta x_i) \\
j &= j(\tau, x_F, x_C, \Delta x_i).
\end{align*}
\]

From the eigenvalue problem for the plasma density we then find

\[
A(\tau) = A(\beta_0, \beta_1). \tag{40}
\]

From the continuity equation for current we find

\[
\begin{align*}
x_C - x_F &= \tau \ln \left( \frac{\sin(A + C)}{\sin(C)} \right) \\
&\quad + \tau \ln \frac{\alpha_1}{\alpha_0} + \tau \ln \left( \frac{1}{j} - 1 \right) \tag{41}
\end{align*}
\]

and from the electron momentum equation we find

\[
\begin{align*}
x_C - x_C &= \tau \ln \left( \frac{\sin(A + C)}{\sin(C)} \right) \\
&\quad + jR + \frac{2(\tau - 1)}{j} - jV_\beta. \tag{42}
\end{align*}
\]

These three previous equations determine \( x_F, x_C, \) and \( \tau \) when \( \Delta x_i \) is given. This set of equations is valid for all \( \Delta x_i \). Even in the case of \( \Delta x_i \leq 0 \) when there is no reflection, the calculations differ from previous isothermal calculations because the Boltzmann assumption on the electrons is not used as indicated by the presence of \( \alpha_0 \) and \( \alpha_1 \).

### III. Calculated Results for Ion Reflection and Trapped Ions

In this section we develop isothermal solutions for the thermionic converter with the emitter sheath phenomena of ion reflection, trapped ions, and surface emission ions included. Emitter sheath effects on thermionic converter performance can be divided into two categories: 1) changes in net ion flux rate into the sheath which affect plasma density directly; and 2) changes in sheath potential distribution which affect the exchange of "hot" plasma electrons for "cold" emitter ions directly. A decreased influx of ions into the sheath, which occurs for all three emitter sheath phenomena, increases the plasma density at the neutral plasma emitter sheath interface. Theoretical intuition suggests that an increased plasma density at the emitter would benefit performance by reducing resistance through the plasma and therefore reducing arc-drop. However, this is not the case. While the plasma density at the emitter increases slightly, plasma density at the collector decreases. Consequently, total resistance increases.

All three of these phenomena increase in significance as the net current density through the converter is reduced. Each of these reduces the net ion loss rate to the emitter and consequently increases arc-drop (therefore, degrading performance at low current densities). This increase in arc-drop is in agreement with the same tendency in the experimental results. However, the experimental results also show a plateau (of low arc-drop) at low current density. This plateau occurs at a current density corresponding to significant surface ion emission and is therefore thought to occur as surface emission replaces volume ionization as the dominant source of plasma ions. Unfortunately, the theoretical calculations cannot be carried into this region because the collisionless collector sheath matching (to the neutral plasma) fails.

To provide a realistic framework for presenting the results, we consider the converter conditions shown as case 1 in Table I. Case 2 is shown because it has the largest surface emission of any typical thermionic converter operating condition (because the work function is high and the temperature is also high). Instead of presenting case 2 separately, we demonstrate the effects of surface emis-
son in case 1 by increasing the surface emission by a factor of 100 thereby bringing it up to the level in case 2. The net current density at which surface emission becomes significant can be estimated by multiplying \( J_{e_1} \) by the square root of the ion to electron mass ratio (approximately 500). In case 1, this means that surface emission becomes significant at \( J = 0.01 \text{ A/cm}^2 \) while in case 2 significant surface emission begins at \( J = 1.0 \text{ A/cm}^2 \).

IV. EFFECTS OF ION REFLECTION

In this section we discuss the isothermal results for case 1 with ion reflection, but without trapped ions and with the small amount of surface emission ions of case 1. Fig. 4 is the \( CV \) diagram for this case.

The dotted line extending upward from point A is the single electron-repelling emitter sheath solution. However, we have not taken recombination or the Schottky effect into account in this isothermal formulation which are expected to become important at current densities near \( J_k \). The interest of this paper begins at point A, where the single sheath doubles over. Between points A and B, where the back sheath height \( \Delta X_e \) is less than the sheath height \( \Delta X_e \), the emitter sheath is nonreflecting. In this region the sheath heights \( \Delta X_e \) and \( \Delta X_c \) remain constant while the plasma density is proportional to net current \( J \) (the normalized plasma density \( n_c J \) is constant). Only the back sheath height \( \Delta X_e \) changes and the \( CV \) curve in this region is Boltzmann (the arc-drop is constant). Beginning at point B and continuing to point C, the double emitter sheath reflects plasma ions because the back sheath is larger than the front sheath; in other words, the reflective potential \( \Delta X_e = \Delta X_c - \Delta X_e \) is positive. The result is that net ion loss rate into the sheath \( u \) decreases and that arc-drop increases. The quantity \( u \) is defined as the mean ion velocity into the sheath normalized by the Bohm speed, \( \sqrt{kT_e/M} \). The dotted curve \( BD \) is the same double sheath except that it assumes no ions are reflected; therefore, \( u \) is constant and arc-drop is constant. The two curves \( BC \) and \( BD \) are almost indistinguishable because the increase in arc-drop is small until the net current density is extremely small. The reason for this is that the shift speed is approximately \( u_s = 2 \), and, therefore, a large increase in reflective potential is required to change \( u \) significantly (the half-reflection point is \( \Delta X_e = 4.0 \) or approximately \( J = J_k \exp (-4) = 0.4 \text{ A/cm}^2 \)). The shift speed \( u_s \) is defined as the velocity at the peak of the incoming ion distribution again normalized by the Bohm speed.

The curve \( EF \) is the single electron-repelling emitter sheath case. It is the limiting case for large amounts of trapped ions in which the double sheath peak has been completely suppressed by the trapped ions. For this case, the emitter sheath solutions gives \( u_s = 0 \). This curve is not topologically connected to the curve \( ABC \); it will be shown in Section V that trapped ions move \( ABC \) toward the single ion-repelling sheath case. The curve is much steeper (a faster increase in arc-drop) in this case because \( u_s = 0 \) (the half-point in ion reflection is approximately \( J = 8 \text{ A/cm}^2 \)). Curve \( EG \) is the single ion-repelling case assuming no reflection and is therefore a Boltzmann line with constant arc-drop.

At points \( F \) and \( C \) the solutions fail at the collector. The explanation for this failure is best given by examining Figs. 5-8.

Fig. 5 is the normalized plasma density through the converter gap. The highest curve with no reflection \( \Delta X_e = 0 \) has the largest plasma density at the collector but the lowest plasma density at the emitter. Ion reflection, which decreases the ion loss rate to the emitter, raises the plasma density at the emitter but lowers the plasma density at the collector. The lower plasma density at the collector forces a smaller collector sheath height to pass the net current density. This can be seen from (10). Fig. 6 is the potential through the converter under the same reflection conditions as in Fig. 5. In Fig. 6 the first two spaces on the left make up the double emitter sheath, and the last space on the
right is the collector sheath. The region between the two sheaths is the neutral plasma region. In the no-reflection case, it can be seen that the potential has a pronounced well in the middle. This is the result of the large plasma density in the middle. As reflection increases, this well disappears on the collector side of the plasma because resistive drop there (due to low plasma density) increases to the degree that it is greater than the ambipolar rise (due to decreasing density toward the collector). Simultaneously with plasma potential gradient at the collector becoming negative, the collector sheath goes toward zero height. Fig. 7 shows the critical collector sheath quantities as the collector sheath failure occurs. Collector sheath height $x_c$ goes toward zero, the shift speed $u_{sc}$ goes toward negative infinity, and the ion loss rate to the collector $\bar{u}_c$ is driven to zero. The two preceding quantities $\bar{u}_c$ and $u_{sc}$ are defined at the collector sheath as $\bar{u}$ and $u$, were at the emitter sheath. Fig. 8 shows the changes in the emitter sheath height, ion shift speed and ion loss rate. When the collector sheath failure occurs, the ion loss rate to the collector is zero ($\bar{u}_c = 0$) and the corresponding plasma ion distribution at the collector is bunched at zero velocity ($u_{sc} = -\infty$). While the mathematics hold self-consistantly until $\bar{u}_c = 0$, the physics is clearly poor at this point because $\bar{u}_c = 0$ demands that the plasma ions at the collector have zero energy (zero temperature and zero mean velocity). An estimate of when the physics becomes poor is $u_5$ at this point the net ion loss rate is close to the thermal speed. A second physical difficulty that occurs with collector sheath failure is that the electron Mach number there $Q_e$ (from (10)) becomes

$$Q_e = \frac{2}{\sqrt{\pi}}$$

because the collector sheath height approaches zero (actually about 0.001). In the present continuum formulation of the plasma region, it was assumed in (13) that $Q_e$ is small so that the electron momentum term $\bar{u}_c du_{sc}/dx$ can be neglected.

One could take the solution below the collector sheath failure point if $\bar{u}_c$ could attain negative values or if $Q_e$ could attain values larger than $\sqrt{2}/\pi$. There is no physical basis for assuming that $\bar{u}_c$ can become negative since the collector emits nothing. However, there is a physical basis for allowing $Q_e$ to be larger than $\sqrt{2}/\pi$ (an electron distribution shift) as can be seen in Fig. 6: the potential drop nearing the collector becomes progressively more electron accelerating as the collector sheath fails, and, therefore, the electron distribution should be shifted as the ion distribution is in an electron-repelling sheath. However, this would clearly invalidate the assumption that the electron momentum term is negligible. Therefore, the momentum term must be added to explore further in this direction and this has not been done because of the resulting complexity in the equations.
Comparison of Fig. 7 to Fig. 8 at the collector sheath failure point ($\Delta x_s = 2.5, u_e = 0$) shows that the ion loss rate to the emitter is positive. At this point the plasma is still ignited and generating ions as can be seen from Figs. 9 and 10. The ionization coefficient $A$ has dropped by 50 percent, but the plasma electron temperature has dropped by only 5 percent. Finally, we note in Fig. 11 that the normalized plasma resistance $R$ has risen by almost 100 percent. This is responsible for the increase in arc-drop and the decrease in performance. Plasma resistance increases in response to reflection because the loss of plasma electron energy to the emitter is more important than the loss of ionization energy to the emitter. Ion reflection at the emitter increases the normalized plasma density there, and consequently increases the normalized loss of plasma electron energy there. The basis of this can be seen from conservation of electron energy (22):

$$\tau = 1 - \frac{1}{2} jV_d - \frac{1}{2} jV_h.$$  (43)

The ion energy loss term is generally small compared to the electron energy loss term:

$$\frac{\frac{1}{2} jV_h}{\frac{1}{2} jV_d} = \frac{J_h V_h}{J_d V_d} = O(0.02).$$  (44)

Therefore, we take the electron energy equation as

$$\tau = I - \frac{1}{2} jV_d.$$  (45)

Since $\tau$ is nearly constant (because of the ionization kinetics), the product $jV_d$ is nearly constant. Ion reflection decreases $j$ (because the normalized plasma density increases) and therefore increases arc-drop $V_d$ (makes $V_d$ a more negative number).

If the equations are reformulated in such a way as to be valid past the collector sheath failure point, then we can eventually expect to see a decrease in arc-drop and a low-current plateau as the electron temperature approaches 1 (the ignited plasma is extinguished and the ionization source is surface emission). This can be seen from (43). However, as we see, the collector failure occurs before $\tau$ has dropped more than 5 percent. Consequently, we do not see any plateau or decrease in arc-drop as net current density is decreased in the present calculations.

V. EFFECTS OF TRAPPED IONS

Fig. 12 shows the effect of trapped ions on the $CV$ characteristics. In this section the trapped ion distribution is assumed to have the temperature of plasma ion distribution, and 100-percent trapped ions ($f_{tr} = 1.0$) is defined to complete the ion distribution at the double emitter sheath peak such that one has a Maxwellian distribution there. Based on physical reasoning about the trapping mechanism, one expects on the order of 10. Also, some trapping calculations have been done for approximate sheath formulations [9], [10] which support this.

Curve $AHIJ$ is the $CV$ characteristic for $f_{tr} = 0.10$. At point $A$ there cannot be any trapped ions since the back sheath height $\Delta x$ is zero. Therefore, the trapped $CV$ merges into the nontrapped curve there. The actual amount of trapped ions on the $f_{tr} = 0.10$ curve increases from zero at point $A$ to the full 10 percent of a thermal distribution at point $H$ where the back sheath height $\Delta x$ is equal to the sheath height $x_E$. The shift speed increases on $AH$ from 1.95 to 3.00. This corresponds to what is seen in Fig. 12 where $\Delta x < x_E$.  

Fig. 9. Ionization coefficient $A$ and $C$.

Fig. 10. Plasma electron temperature.

Fig. 11. Normalized plasma resistance.
The rise in shift speed has been limited to 3.00. This limit is placed on the shift speed because a sheath with height of about 1.0 should not have a presheath region capable of shifting the entire distribution so far. In fact, limiting the shift speed is equivalent to increasing the cut-off speed for the ion distribution function.

The arc-drop decreases as a result of the increase in $u_s$ and the consequent increase in the net ion loss rate to the emitter. A "hump" can be seen on $AH$ where the shift speed hits 3.00. The arc-drop is lowest on this "hump" because the shift speed is at its maximum of 3.00. Between points $H$ and $I$ the back sheath height remains equal to the sheath height, $\Delta x = x_e = x_i = 0$. On this segment, $u_i$ decreases to 1.25, therefore increasing arc-drop.

From point $I$ to point $J$, the shift speed remains constant at 1.25 and the ion loss rate decreases because of reflection. The other trapped cases $f_r = 0.2, 0.3,$ and $0.4$ have not been connected because they hit the 3.00 maximum shift speed much sooner than in the $f_r = 0.1$ case.

Point $J$ is the collector sheath failure point. Each of the $f_r = 0.2, 0.3,$ and $0.4$ curves begins at $\Delta x_i = 0$ and ends at the collector sheath failure point. It should be noted that each of the trapped ion curves fails at a higher current than the last because the shift speed is lower.

VI. EFFECTS OF EMITTER SURFACE EMISSION

Fig. 12 shows the effect of surface emission on the $f_r = 0.10$ curve: surface emission is added by multiplying the actual small amount of surface emission in case 1 by a factor of 100. This brings the surface emission up to the level in case 2, making it significant at $J = 1.0 \text{ A/cm}^2$. It can be seen that surface emission increases arc-drop; it does so in exactly the same way as reflection or trapped ions do—it decreases the net loss rate of ions to the emitter.

VII. COMPARISON WITH EXPERIMENTAL RESULTS AND CONCLUSIONS

Fig. 13 superimposes the isothermal results of Fig. 12 on the experimental results for a cesium reservoir temperature of 551 K which produces a 1-torr neutral cesium pressure. The experimental results are from [11]. The point of this comparison is that the steepness of the $CV$ characteristic in the experimental converters can be explained by a decreasing ion loss rate to the emitter. We have shown that all three of the expected emitter sheath phenomena decrease the ion loss rate to the emitter. We cannot calculate the amount of trapped ions in a collisionless sheath without knowledge of the collisional processes. However, the experimental $CV$ suggests that if the amount of trapped ions ($f_r$) increases from 0 percent at $J = 14 \text{ A/cm}^2$ (the double sheath formation point) to 10 percent at $J = 2 \text{ A/cm}^2$, then the steepness could result from trapped ions reducing the ion loss rate to the emitter. Since these percentages are based on a thermal distribution of ions, they seem physically reasonable. Unfortunately, the collector sheath failure prevents us from going to the point in the calculations where $r$ drops enough to make surface emission the source of ions.

The experimental curve is nearly a constant 0.05 V below the isothermal result ($f_r = 0.10$) except at high current densities and at the "hump." Comparison of the curves at high current density is not valid since neither the Schottky effect nor recombination has been included. The Schottky effect is important above 12 A/cm$^2$ in this case because the emitter sheath is single electron repelling (to the plasma) and therefore puts a strong electric field against the emitter with the appropriate sign. Recombination is also potentially important because the plasma density scales with current density, and at high current densities the plasma density in the middle of the converter approaches the Saha density. The 0.05-V difference may or may not be explained by a discrepancy in the assumed collector work function. At 750 K the collector emits essentially nothing and therefore any change in the collector work function directly affects output voltage. If the collector work function were in fact 1.65 instead of 1.60 V, then the isothermal result would lie nearly on top of the
experimental result. We have not adjusted the assumed collector work function so as to illustrate the importance of it and therefore the importance of the surface physics of the adsorbed cesium layer. The "hump" should not be taken as an expected experimental result since it results from the interaction of the trapped ions with the plasma-emitter sheath interface. Instead it should be taken as a second reason (in addition to the cutoff of the ion distribution) for further study of the matching region between the collisionless sheath and the neutral plasma.

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ASYMPTOTIC SHEATH THEORY

The Asymptotic Sheath Theory developed under this grant is set forth in the following paper.
Asymptotically correct collisional presheaths

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pp. 1800-1809
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(Received 16 April 1986; accepted 19 February 1987)

Few exact solutions for collisional presheaths exist because of the difficulty of simultaneously satisfying both the collisional Boltzmann equation and the Poisson equation. The exact solutions that do exist are for very specialized collision terms such as constant cross-section charge exchange with cold neutrals. The present paper presents an asymptotic method which is applicable to a variety of collision terms and is applied in particular to constant collision frequency charge exchange with noncold neutrals. Constant collision frequency and constant cross-section collision with cold neutral results are also presented. The first-order terms for the presheath potential rise and ion distribution functions are calculated and it is shown that second- and higher-order terms can be calculated using a multiexponential expansion for presheath potential rise. The first-order cold neutral constant cross-section results correspond well to the exact solution. The calculated presheath potential rises are of the order expected from the Bohm criterion, and in some of the specialized cold neutral cases, exactly $kT_e/2$. The presheath potential rise is reduced by a neutral plasma potential gradient which accelerates ions toward the presheath. In all cases the collisional presheath is asymptotically matched to both the neutral plasma and the collisionless sheath.

I. INTRODUCTION

The majority of plasma–surface interaction work matches a neutral plasma to a collisionless sheath without detailed consideration of a collisional presheath. However, the collisional presheath structure is of great interest. Sheath theory, beginning with Bohm, tends to assume that the plasma ion distribution is cold so that a minimum presheath potential rise may be calculated, which makes the collisionless sheath self-consistent. Harrison and Thompson generalize the Bohm criterion to noncold ion distributions; however, the result is sensitive to the density of the low energy tail of the ion distribution, which in turn is strongly affected by the collisional presheath. And, a second difficulty in the absence of a collisional presheath is that the collisionless sheath and the surface beyond it may return no ions or a nonthermal distribution of ions which the collisional presheath must match to the neutral plasma region.

Some exact solutions exist for presheaths; notable is the work of Ecker and Kanne and Rieman, who derive exact solutions for collision terms based on charge exchange with cold neutrals and Emmert et al., who derive an exact collisionless solution in which there is an ionization source. In the present paper an asymptotically correct collisional presheath theory is developed which can be applied to a less restrictive range of collision terms. Potential in the collisional presheath is expanded as a multiexponential series and the distribution functions are expanded in terms of presheath potential rise. First-order approximations are calculated for both constant collision frequency and constant cross-section charge exchange collisions.

II. FIRST-ORDER ASYMPTOTIC POTENTIAL FORMULATION

In this section it is assumed that the potential in the collisional presheath is of the form

$$U = U_0 + \Delta U = ax + e^{\Delta x},$$

where $U_0 = ax$ is the assumed linear potential in the neutral plasma and $\Delta U = e^{\Delta x}$ is the additional potential rise in the collisional presheath, as shown in Fig. 1. In this paper the convention used is that $U = q\phi$, where $q$ is the electron charge and $\phi$ is potential in electron volts so that $U$ has units of energy. In addition, potential is defined in the reverse of the usual sign convention so that increasing potential repels electrons. With these conventions, the Boltzmann equation can be written as

$$\frac{dU}{dx} \left( v \frac{df}{dU} \pm \frac{1}{m} \frac{df}{dv} \right) = \left( \frac{df}{dt} \right).$$

In Eq. (2) and those following, the $\pm$ denotes the sign of the charged species in question; the upper sign refers to positively charged ions and the lower sign to electrons. The Boltzmann equation is expressed in terms of $\Delta U$, which will be the expansion variable in the presheath:

$$v\beta U \frac{df}{d\Delta U} \pm \frac{1}{m} (\beta\Delta U + \alpha) \frac{df}{dv} = \left( \frac{df}{dt} \right).$$

The distribution function is then expanded as

$$f = f_0(v) + \Delta U f_1(v) + \Delta^2 f_2(v) + \cdots,$$

so that the derivatives are

$$\frac{df}{d\Delta U} = f_1(v) + 2\Delta U f_2(v) + 3\Delta U^2 f_3(v) + \cdots,$$

and

$$\frac{df}{dv} = \frac{df_0}{dv}(v) + \Delta U \frac{df_1}{dv}(v) + \Delta U^2 \frac{df_2}{dv}(v) + \cdots.$$
The quantity $\beta$, representing the presheath potential rise, is determined from the Poisson equation
\[ \frac{d^2U}{dx^2} = 4\pi q^2 \left( \int_{-\infty}^{\infty} f_i(v,AU)dv - \int_{-\infty}^{\infty} f_0(v,AU)dv \right), \]
where $q$ is the electron charge. It is assumed that the ions are singly ionized for simplicity. The Poisson equation (8) is expanded as
\[
\beta^2 AU = 4\pi q^2 \left[ \left( \int_{-\infty}^{\infty} f_0(v)dv - \int_{-\infty}^{\infty} f_0(v)dv \right) \Delta U + \left( \int_{-\infty}^{\infty} f_0(v)dv - \int_{-\infty}^{\infty} f_0(v)dv \right) \Delta U^2 + \cdots \right],
\]
where charge neutrality at $AU = 0$ has eliminated the terms containing $f_0$ and $f_0$. The quantity $n_0$ is the neutral plasma density of the asymptotic presheath, not of the neutral plasma.

**III. FIRST-ORDER SOLUTION WITH A CONSTANT COLLISION FREQUENCY CHARGE EXCHANGE COLLISION TERM**

The constant collision frequency charge exchange collision term is modeled as
\[
\frac{df_{\alpha}}{dt} = \frac{1}{\tau n_0} \left( f_{\alpha}(v) \int_{-\infty}^{\infty} f_0(u)du - f_0(v) \int_{-\infty}^{\infty} f_{\alpha}(u)du \right),
\]
where $f_0(v)$ is the neutral distribution and $\tau$ is the collision time. Previous work has assumed cold neutrals and results in an integral equation which is solvable only for constant collision cross section.

**A. Zero plasma potential gradient ($\alpha = 0$)**

In this case Eqs. (7) become
\[
\begin{align*}
1: & \quad \pm \frac{\alpha}{m} \frac{\partial f_0}{\partial v}(v) = \left[ \frac{\partial f_{\alpha}}{\partial t} \right]_{\Delta U}, \\
\Delta U: & \quad \mp \beta \frac{\partial f_{\alpha}}{m} \frac{\partial f_0}{\partial v}(v) = \left[ \frac{\partial f_{\alpha}}{\partial t} \right]_{\Delta U}, \\
\Delta U^2: & \quad 2\beta \frac{\partial f_{\alpha}}{m} \frac{\partial f_0}{\partial v}(v) = \left[ \frac{\partial f_{\alpha}}{\partial t} \right]_{\Delta U}, \\
\Delta U^n: & \quad \beta \frac{\partial f_{\alpha}}{m} \frac{\partial f_0}{\partial v}(v) = \left[ \frac{\partial f_{\alpha}}{\partial t} \right]_{\Delta U^2}.
\end{align*}
\]

Under the assumption that the neutral distribution is Maxwellian $f_0(v) = \sqrt{n_0/2\pi kT} \exp(-mv^2/2kT)$, the solution to (11) is
\[
\begin{align*}
f_0 &= C f_{\alpha}(v), \\
f_{\alpha}(v) &= (1/kT)f_{\alpha}(v), \\
\vdots \\
f_n(v) &= (1/nkT)f_{\alpha(n-1)}(v).
\end{align*}
\]
Thus
\[
f_{\alpha}(v,\Delta U) = Ce^{(\Delta U/kT)}f_{\alpha}(v),
\]
which is the expected result. In this case the mean ion velocity is zero throughout the collisional presheath since charge exchange collisions conserve ions and the mean ion velocity in the neutral plasma is zero. Thus, if $\alpha = 0$, constant collision frequency charge exchange collisions do not shift the ion distribution upward in velocity. This presheath can be matched to a collisionless sheath only if the collisionless sheath returns all the ions entering it from the collisional presheath.
The Poisson equation (9) yields, to first order,

$$\beta^2 = 4\pi e^2 n_0 (1/kT_e + 1/kT_i),$$

which is the length scale of the Debye length. Thus for \( \alpha = 0 \) the collisional presheath is not distinct from the collisionless sheath since there is no separate collisional presheath length scale.

### B. Nonzero plasma potential gradient (\( \alpha \neq 0 \))

Under this condition there is a net flux of ions from the plasma into the sheath, which allows the construction of a collisional presheath that accelerates the ions and depopulates the ion distribution of returning ions. Thus the collisional presheath may be correctly matched to the collisionless sheath which returns no ions. In this case (7a) and (7b) can be written as

$$\frac{\alpha}{m} \frac{\partial f_0}{\partial \nu} \left( \nu \right) = \frac{1}{\tau_n} \left[ f_0 \left( \nu \right) n_0 - n \nu f_0 \left( \nu \right) \right],$$

and

$$\alpha \theta f_i \left( \nu \right) + \frac{\beta}{m} \frac{\partial f_0}{\partial \nu} + \frac{\alpha}{m} \frac{\partial f_i}{\partial \nu} \left( \nu \right) = \frac{1}{\tau_n} \left[ f_0 \left( \nu \right) n_1 - n \nu f_i \left( \nu \right) \right].$$

The solution to Eqs. (14) are

$$f_0 \left( \nu \right) = n_0 \exp \left( \frac{-\mu \nu}{\alpha \tau} \right) \exp \left( \frac{m}{\alpha \nu kT} \right) \int_{-\infty}^{\infty} \exp \left( \frac{-\mu^2}{2kT} \right) du$$

and

$$f_i \left( \nu \right) = \exp \left( \frac{-\beta \mu^2}{\alpha \tau} \right) \left[ \int_{-\infty}^{\infty} \exp \left( \frac{\beta \mu^2}{2 \alpha \tau} + \frac{\mu \nu}{\alpha \tau} \right) \left( \frac{m}{\alpha \nu kT} \right) \exp \left( \frac{\mu^2}{2kT} \right) \right] du + C,$$

where

$$n_0 = \int_{-\infty}^{\infty} f_0 \left( \nu \right) d\nu$$

and

$$n_1 = \int_{-\infty}^{\infty} f_i \left( \nu \right) d\nu.$$

The constant of integration in (15) has been set so that \( f_0 \) goes to zero at \(-\infty\); \( f_0 \) goes to zero at \( \infty \) regardless of the constant of integration. Equation (17) is immediately satisfied by (15). The constant of integration \( C \) in (16) must be set so that (18), which represents self-consistency, is satisfied. It can be seen from (16) that \( f_i \) goes to zero at \(-\infty \) and \( \infty \) regardless of the constant \( C \). From (18), then

$$C = n_1 \left[ 1 - \int_{-\infty}^{\infty} \exp \left( \frac{-\beta \mu^2}{\alpha \tau} \right) \left( \frac{m}{\alpha \nu kT} \right) \exp \left( \frac{\mu^2}{2kT} \right) du \right]$$

$$\times \left[ \frac{2\alpha}{m} \exp \left( \frac{m}{2 \alpha \nu kT} \right) \right]^{-1} + \left[ \int_{-\infty}^{\infty} \exp \left( \frac{-\beta \mu^2}{\alpha \tau} \right) \left( \frac{m}{\alpha \nu kT} \right) \exp \left( \frac{\mu^2}{2kT} \right) du \right]$$

where

$$\beta^2 = 4\pi e^2 \left( n_1 + n_0 / kT_e \right).$$

The exponential sheath rise \( \beta \) is determined from the Poisson equation under the simplifying assumption that

$$n_1 = \int_{-\infty}^{\infty} f_i \left( \nu \right) d\nu = n_0 \exp \left( \frac{\Delta U}{kT_e} \right).$$

One might expect that the approximation should be \( n_1 = n_0 \exp \left( -U / kT_e \right) \); however, this cannot be true in the asymptotic presheath because \( n_1 \) must approach \( n_0 \) as \( U \) approaches negative infinity. With (20) the Poisson equation (9) to first order becomes

$$\beta^2 = 4\pi e^2 \left( n_1 + n_0 / kT_e \right).$$

Since the ion density is only calculated to first order, the same will be done for the electron density in (20).

To obtain a particular solution it is assumed here that the collisionless sheath to which the collisional presheath is joined at \( \Delta U = \Delta U^* \) returns no ions. In particular,

$$\int_{-\infty}^{0} f(v, \Delta U^*) d\nu = 0.$$
or
\[ \int_{-\infty}^{0} f_0(v)dv + \Delta U^* \int_{-\infty}^{0} f_1(v)dv = 0 \]  
(23)
and
\[ \int_{-\infty}^{0} v f(v,\Delta U^*)dv = 0, \]  
(24)

or
\[ \int_{-\infty}^{0} v f_0(v)dv + \Delta U^* \int_{-\infty}^{0} v f_1(v)dv = 0. \]  
(25)

Because the approximation is only first order, it is not possible to impose the condition that \( f(v) \) is uniformly zero for returning ions. Equations (23) and (25) represent zero returning ion density and zero returning ion flux. When higher-order terms are included, the conditions of zero returning ion momentum flux, zero returning ion energy flux, etc., can be applied in succession. Equations (21), (23), and (25) are solved for \( n_1, \beta, \) and \( \Delta U^* \), with all other quantities assumed constant. Equation (21) immediately satisfies the Bohm criterion at \( \Delta U = \Delta U^* \) for the first-order approximation
\[ n_1 + n_0 / k T_e > 0. \]  
(26)
The Poisson equation (21) can be written as
\[ \beta^2 \lambda_D^2 = 1 + k T_e (n_1 / n_0), \]  
(27)
where
\[ \lambda_D = \sqrt{k T_e / 4 \pi q^2 n_0} \]  
(28)
is the Debye length. It is expected that the length scale of the presheath should be of the order \( \beta = 1 / \lambda_i \), where \( \lambda_i \) is the ion mean free path. In the circumstance that the Debye length is small compared to the ion mean free path, the product \( \beta^2 \lambda_D^2 \) is small and
\[ n_1 = - n_0 / k T_e. \]  
(29)
The neutral plasma region is matched to the collisional presheath also at \( \Delta U = \Delta U^* \), as shown in Fig. 1, to produce a three-scale uniform asymptotic solution. In particular, assuming constant collision frequencies, the momentum equations become
\[ \left( k T_i - \frac{m, \Gamma_i}{n^2} \right) \frac{dn}{dx} = n \frac{dU}{dx} - \frac{m, \Gamma_i}{\tau} \]  
(30)
and
\[ \left( k T_e - \frac{m, \Gamma_e}{n^2} \right) \frac{dn}{dx} = - n \frac{dU}{dx} - \frac{m, \Gamma_e}{\tau_e}, \]  
(31)
where

FIG. 1. Asymptotically correct potential in the collisional presheath.
The quantity $n$ is the plasma density at the matching point $\Delta U^*/kT_e$, and $\Gamma_i$ and $\Gamma_e$ are, respectively, the ion and electron net fluxes. Nondimensionalization results in

$$A = (\alpha \tau/m) \sqrt{m/2kT}, \quad B = \beta kT/\alpha,$$

$$R_s = T_e/T,$$

$$\omega = \sqrt{m/2kT} v,$$

where $A$ and $R_s$ are the parameters and $B$ is a function of $A$ and $R_s$. The quantity $A$ represents the nondimensional asymptotic presheath potential gradient, $B$ represents the nondimensional exponential presheath rise, $R_s$ is the electron to neutral temperature ratio, and $\omega$ is the nondimensional velocity. The distribution functions can then be written as

$$F_0(\omega, A) = \frac{f_0(v)}{n_0 \sqrt{m/2kT}} \int_0^\infty \exp\left(-\frac{\xi^2}{A}\right) d\xi$$

and

$$F_1(\omega, A, B) = \frac{f_1(v)}{(n_0 kT_e) \sqrt{m/2kT}} \int_0^\infty \exp\left(-\frac{\omega - \frac{\xi}{A}}{A}\right) \left[\int_0^\infty \exp\left(B \xi^2 + \frac{\xi}{A}\right) d\xi\right] \left[\int_0^\infty \exp\left(-\frac{\xi^2}{A}\right) d\xi\right] d\xi + C,$$

where

$$C = \left[1 - \frac{1}{\sqrt{\pi A}} \int_0^\infty \exp\left(-B \omega - \frac{\omega}{A}\right) \int_0^\infty \exp\left(B \xi^2 + \frac{\xi}{A} - \frac{\xi^2}{A}\right) d\xi d\omega \right] \left[\frac{1}{B} \exp\left(\frac{1}{4BA^2}\right)\right]^{-1}$$

$$+ \frac{R_s B}{\sqrt{\pi A}} \int_0^\infty \exp\left(-B \omega - \frac{\omega}{A}\right) \int_0^\infty \exp\left(B \xi^2 + \frac{\xi}{A}\right) d\xi d\omega \left[\int_0^\infty \exp\left(\frac{1}{4BA^2}\right)\right]^{-1} \times \left[\frac{1}{A} \int_0^\infty \exp\left(-\frac{\xi^2}{A}\right) d\xi \int_0^\infty \exp\left(-\eta^2 + \frac{\eta}{A}\right) d\eta \right] d\xi d\omega \left[\frac{1}{B} \exp\left(\frac{1}{4BA^2}\right)\right]^{-1}.$$

Thus (23) and (25) become

$$\int_{-\infty}^0 F_0(\omega, A) d\omega + \frac{\Delta U^*}{kT_e} \int_{-\infty}^0 F_1(\omega, A, B) d\omega = 0,$$

and

$$\int_{-\infty}^0 \omega F_0(\omega, A) d\omega + \frac{\Delta U^*}{kT_e} \int_{-\infty}^0 \omega F_1(\omega, A, B) d\omega = 0.$$

Figure 2 presents the presheath potential rise $\Delta U^*/kT_e$ and the nondimensional exponential rise $B$ as a function of the nondimensional asymptotic presheath potential gradient $A$ for a range of electron to neutral temperature ratios $R_s$. As would be intuitively expected, the presheath potential rise decreases with increasing $A$. Figure 3 presents the ion distribution functions at the neutral plasma-collisional presheath interface $F_0(\omega)$, the first-order correction to the distribution function $F_1(\omega)$, and the resulting distribution function at the collisional presheath-collisionless sheath interface $F_0(\omega) + \Delta U^* F_1(\omega)$. Although the resulting distribution is not uniformly zero for $\omega < 0$, its net returning density and flux are zero by (42) and (43). It is expected that higher-order corrections to the distribution function and potential with the corresponding application of higher-order moment conditions of zero returning momentum, energy, etc., will converge the returning distribution function toward a uniform zero.

In the limit of cold neutrals, the constant collision frequency charge exchange solution is considerably simplified. Equations (14a) and (14b) become
The solutions to (44) and (45) are

\[ f_0(v) = \begin{cases} \frac{n_0(m/\alpha \tau)\exp(-mv/\alpha \tau)}{\tau n_n}, & v > 0, \\ 0, & v < 0, \end{cases} \] (46)

and

\[ f_1(v) = \begin{cases} \exp\left(\frac{-\beta \nu^2}{2\alpha^2} - \frac{mv}{\alpha \tau}\right)\left[C + \frac{\beta}{\alpha} \frac{n_0(m/\alpha \tau)}{\tau n_n} \int_0^K \exp\left(\frac{\beta \nu^2}{2\alpha}\right)du\right], & v > 0, \\ \exp\left(\frac{-\beta \nu^2}{2\alpha^2} - \frac{mv}{\alpha \tau}\right)(C^-), & v < 0, \end{cases} \] (47)
such that
\[ C^+ - C^- = (m/\alpha \tau) [n_i - (\beta/\alpha) n_0]. \] (48)

Equation (46) immediately satisfies \( n_0 = \int_{-\infty}^{\infty} f_0(v) dv \). No returning ions implies that
\[ C^- = 0 \] (49)
and
\[ C^+ = (m/\alpha \tau) [n_i - (\beta/\alpha) n_0] \] (50)
since \( f_0 \) on \( v < 0 \) is already zero. The final condition is then that \( n_1 = \int_{-\infty}^{\infty} f_1(v) dv \), or

\[ n_1 = \int_{-\infty}^{\infty} \exp\left( -\frac{\beta m v^2}{2\alpha} \right) \left[ n_i \frac{m}{\alpha \tau} - \frac{\beta n_0}{\alpha} \frac{m}{\alpha \tau} + \frac{\beta n_0}{\alpha} \left( \frac{m}{\alpha \tau} \right)^2 \right] \left( \frac{m}{2\alpha} \right) \exp\left( \frac{\beta m v}{2\alpha} \right) dv. \] (51)

The application of \( n_i = \frac{-1}{\nu/kT} \) yields
\[ \frac{\beta k T_e}{\alpha} = \left[ 1 - \int_{-\infty}^{\infty} \exp\left( -\frac{\beta a \tau^2 \xi^2}{2m} - \xi \right) d\xi \right] \left[ \int_{-\infty}^{\infty} \exp\left( -\frac{\beta a \tau^2 \xi^2}{2m} - \xi \right) \left[ 1 - \int_{0}^{t} \exp\left( \frac{\beta a \tau^2 \eta^2}{2m} \right) d\eta \right] d\xi \right]^{-1}. \] (52)

In this case, \( \Delta U^* \) is defined by
\[ f_0(0^-) + \Delta U^* f_1(0^+) = 0, \] (53)
which yields
\[ \Delta U^*/kT_e = 1/(\beta k T_e/\alpha + 1), \] (54)
as expected. In the limit of \( \beta a \tau^2/2m \to 0 \) we have
\[ \beta k T_e/\alpha = 1 \] (55)
and
\[ \Delta U^* = kT_e/2, \] (56)
which corresponds to the Bohm criterion. Figure 4 presents the variation of \( B = \beta k T_e/\alpha \), with \( \beta a \tau^2/2m \) for the cold neutral case. A particular \( \beta \) for the parameters can be conveniently found by drawing a line from the origin, with slope \( 2m k T_e/\tau^2 \alpha^2 \), so that the intersection is the solution. Figure 5 presents an example cold neutral ion distribution. Examination of the ion distribution function at \( v = 0 \) shows that the slope is discontinuous. This is because the neutral source is a delta function at \( v = 0 \). It appears that the Bohm criterion cannot be satisfied at \( \Delta U^* \) because the integral \( \int_{-\infty}^{\infty} \left[ f(v) / v^2 \right] dv \) is singular; however, the use of this integral in the Bohm criterion assumes that the ions accelerated are not replaced. In this case the ions accelerated from \( v = 0 \) are replaced by ions from the cold neutral distribution which, of course, is a delta function at \( v = 0 \).

IV. FIRST-ORDER SOLUTION WITH A QUASICONSTANT CROSS-SECTION COLLISION TERM

First-order asymptotic solutions can also be developed for a quasicontant cross-section collision term
\[ \langle \frac{\partial f}{\partial t} \rangle_{\epsilon} = \sigma \left( \int_{-\infty}^{\infty} f_n(v) f(u) |v - u| du - \int_{-\infty}^{\infty} f(u) f_n(u) |v - u| du \right). \] (57)
This collision term is not really constant cross section because it is a one-dimensional representation which does not take into account average velocities in the other two dimensions. However, this collision term corresponds to that commonly called constant cross section. The application of this term leads to a set of integro-differential equations which can be at least approximately solved, and in the cold neutral case it leads to readily soluble first-order differential equations. The cold neutral case presented here corresponds to that which can be solved exactly (Riemann*). Unfortunately, though, the exact solution method is not extensible to noncold neutrals. The cold neutral collision term is

\[ \frac{df}{dt}(v) = \sigma n \delta(v) \int_{-\infty}^{\infty} f(u) |u| du - \sigma f n |v| \]

and the zero-order Boltzmann equation term (7a) becomes

\[ \frac{a}{m} \frac{df}{dv}(v) = \sigma n \delta(v) \int_{-\infty}^{\infty} f_0(u) |u| du - \sigma n |v| f_0(v), \]

for which the solution is

\[ f_0(v) = \begin{cases} n_0 \sqrt{\frac{2}{\pi}} \sqrt{\frac{\sigma mn}{a}} \exp\left(-\frac{\sigma mn}{2a} v^2\right), & v > 0, \\ 0, & v < 0. \end{cases} \]

The first-order Boltzmann term is

\[ u \beta f_1(v) + \frac{\beta}{m} \frac{df_1}{dv}(v) + \frac{a}{m} \frac{df_1}{dv}(v) = \sigma n \delta(v) \int_{-\infty}^{\infty} f_1(u) |u| du - \sigma n |v| f_1(v), \]

for which the solution is

\[ f_1(v) = \begin{cases} \exp\left[-\frac{1}{2} (\beta m a + \sigma mn a)^2 \right] \left[n_0 \sqrt{\frac{2}{\pi}} \left(\frac{\sigma mn}{a}\right)^{3/2} \exp\left(-\frac{\beta m v^2}{2a}\right) - 1 \right] + C^+, & v > 0, \\ \exp\left[-\frac{1}{2} (\beta m a - \sigma mn a)^2 \right] (C^-), & v < 0. \end{cases} \]

The jump condition at \( v = 0 \) must be satisfied in (61):

\[ C^+ - C^- = \frac{\delta mn n}{a} \int_{-\infty}^{\infty} f_1(u) |u| du - \frac{\beta}{a} n_0 \sqrt{\frac{2}{\pi}} \sqrt{\frac{\sigma mn}{a}}. \]

No returning ions, \( C^- = 0 \), and the application of (63) to (62) yields

\[ C^+ = -n_0 (\beta/a) \sqrt{2/\pi} \sqrt{\sigma mn a}. \]

The collisional presheath-collisionless sheath boundary \( \Delta U^* \) is again
FIG. 6. Constant cross section ion distribution with cold neutrals.

\[ 0 = f(0^+) = f_0(0^+) + \Delta U* f_1(0^+) \]

which yields

\[ \Delta U*/kT_e = \alpha/\beta kT_e. \]  \hfill (65)

Equation (62) is integrated to

\[ n_1 = \int_{-\infty}^{\infty} f_1(v)dv = \frac{n_0 n_e \sigma}{\alpha} \left( 1 - \sqrt{1 + \frac{\beta}{\sigma n_e}} \right) \]  \hfill (66)

and applied to the Poisson equation (8) to produce

\[ \left( \frac{\beta}{\sigma n_e} \right)^2 = \left( \frac{4\pi q^2 n_0}{kT_e} \right)^2 \left( \frac{1}{\sigma n_e} \right)^2 \left[ \frac{n_e \sigma kT_e}{\alpha} \left( 1 - \sqrt{1 + \frac{\beta}{\sigma n_e}} \right) + 1 \right]. \]  \hfill (67)

Under the assumption that the Debye length is short compared to the ion mean free path,

\[ (4\pi q^2 n_0/kT_e)^2 (1/\sigma n_e)^2 \gg 1. \]

Eq. (68) results in

\[ \beta/\sigma n_e = \alpha/n_e \sigma kT_e \left( 2 + \alpha/n_e \sigma kT_e \right) \]  \hfill (68)

and

\[ \Delta U*/kT_e = 1/(2 + \alpha/n_e \sigma kT_e). \]  \hfill (69)

The Bohm criterion is satisfied at \( \Delta U^* \) to the first order by virtue of (68). And interestingly, the presheath potential rise for \( \alpha = 0 \) is exactly that required by the cold ion Bohm criterion. Figure 6 presents the results for cold neutrals with \( \alpha/n_e \sigma kT_e = 1 \). From the ion distribution at \( \Delta U^* \), the mean ion velocity into the sheath can be determined to be \( \bar{v} = 1.06 \sqrt{kT_e/m_i} \), while the exact solution of Riemann gives \( \bar{v} = 1.27 \sqrt{kT_e/m_i} \); thus the first-order asymptotic result appears close.

V. CONCLUSIONS

It has been shown that approximate collisional presheath solutions can be obtained for a variety of collision terms. In particular the constant collision frequency case has been solved approximately, whereas previous attempts at exact solutions have found this case intractable. In addition, it has been shown that higher-order corrections can be made a regular and tractable fashion. Also the return of ions from the collisionless sheath can be treated.

ACKNOWLEDGMENTS

Thanks are due to Gregory Ridderbusch who produced the numerical results. This work was supported by the Air Force Office of Scientific Research Grant No. 85-0375.

APPENDIX: MULTIEXPONENTIAL FORMULATION

In the previous sections we have calculated only the first-order terms in the ion distribution and presheath potential rise. Also, we have implicitly made the same first-order approximation for electrons:
\[ n_e = n_0 (1 - \Delta U / kT_e). \]  \hspace{1cm} (A1)

A complete multiexponential expansion can also be constructed that correctly calculates the second- and higher-order terms. Potential in the presheath is

\[ U = U_0 + \Delta U + a_2 \Delta U^2 + a_3 \Delta U^3 + \cdots, \]  \hspace{1cm} (A2)

where \( U_0 = ax \) and \( \Delta U = \exp(\beta x) \). Thus

\[ \frac{dU}{dx} = \alpha + \beta \Delta U + 2\beta a_2 \Delta U^2 + 3\beta a_3 \Delta U^3 + \cdots \]  \hspace{1cm} (A3)

and

\[ \frac{d(\Delta U)}{dU} = \frac{\beta a_2 \Delta U}{\alpha + \beta \Delta U + 2\beta a_2 \Delta U^2 + 3\beta a_3 \Delta U^3 + \cdots}, \]  \hspace{1cm} (A4)

which transforms the Boltzmann equation

\[ \frac{dU}{dx} \left( v \frac{\partial f}{\partial \Delta U} \frac{\partial \Delta U}{\partial U} \pm \frac{1}{m} \frac{\partial f}{\partial v} (v) \right) = \left( \frac{\partial f}{\partial t} \right) \epsilon, \]  \hspace{1cm} (A5)

into

\[ u \beta \Delta U \frac{\partial f}{\partial \Delta U} (v) \pm \frac{1}{m} (\alpha + \beta \Delta U + 2\beta a_2 \Delta U^2 + \cdots) \frac{\partial f}{\partial v} (v) = \left( \frac{\partial f}{\partial t} \right) \epsilon, \]  \hspace{1cm} (A6)

or

\[ \text{I: } \pm \frac{\alpha}{m} \frac{\partial f}{\partial v} (v) = \left[ \left( \frac{\partial f}{\partial t} \right) \epsilon \right], \]  \hspace{1cm} (A7a)

\[ \Delta U: u \beta f_1 (v) \pm \frac{\beta}{m} \frac{\partial f}{\partial v} (v) \pm \frac{\alpha}{m} \frac{\partial f_1}{\partial v} (v) = \left[ \left( \frac{\partial f}{\partial t} \right) \epsilon \right] \Delta U, \]  \hspace{1cm} (A7b)

\[ \Delta U^2: 2u \beta f_2 (v) \pm \frac{2\beta a_2}{m} \frac{\partial f}{\partial v} (v) \pm \frac{\beta}{m} \frac{\partial f_1}{\partial v} (v) \pm \frac{\alpha}{m} \frac{\partial f_2}{\partial v} (v) = \left[ \left( \frac{\partial f}{\partial t} \right) \epsilon \right] \Delta U^2, \]  \hspace{1cm} (A7c)

\[ \vdots \]

\[ \Delta U^n: n u \beta f_n (v) \pm \frac{n \beta a_n}{m} \frac{\partial f}{\partial v} (v) \pm \frac{(n - 1) \beta a_{n-1}}{m} \frac{\partial f_{n-1}}{\partial v} (v) \pm \cdots \pm \frac{\beta}{m} \frac{\partial f_2}{\partial v} (v) \pm \frac{\alpha}{m} \frac{\partial f_n}{\partial v} (v) = \left[ \left( \frac{\partial f}{\partial t} \right) \epsilon \right] \Delta U^n. \]  \hspace{1cm} (A7d)

The Poisson equation (8) becomes

\[ \beta \Delta U + (2\beta)^2 a_2 \Delta U^2 + (3\beta)^2 a_3 \Delta U^3 + \cdots \]

\[ = 4\pi q^2 \left[ \Delta U \left( \int_{-\infty}^{\infty} f_n (v) dv - \int_{-\infty}^{\infty} f_{n+1} (v) dv \right) \right. \]

\[ + \Delta U^2 \left( \int_{-\infty}^{\infty} f_2 (v) dv - \int_{-\infty}^{\infty} f_3 (v) dv \right) + \cdots. \]  \hspace{1cm} (A8)

\[ ^1 \text{D. Bohm, in Characteristics of Electrical Discharges in Magnetic Fields, edited by A. Guthrie and R. Wakering (McGraw-Hill, New York, 1949), p. 77.} \]


\[ ^4 \text{K. U. Riemann, Phys. Fluids 24, 2163 (1981).} \]

THE TEC PROGRAM RESULTS

The TEC program results shown here incorporate the asymptotic presheath work and give good agreement except at low current density. This disagreement is still not understood.
Comparison of TEC Program Results to Experimental Data


\[ T_E = 1700 \text{ K} \]
\[ T_C = 773 \text{ K} \]
\[ T_R = 567 \text{ K} \]
\[ \Phi_E = 2.642 \text{ eV} \]
\[ \Phi_C = 1.630 \text{ eV} \]
\[ d = 10 \text{ mils} \]

Ideal Diode Characteristic

Experimental Curve

Boltzman Line

TEC program

Calculated Curve

remaining disagreement

Output Voltage (volts)
TEC INITIAL DATA SUMMARY

PHYSICAL OPERATING CONDITIONS-----

EMITTER TEMPERATURE (TE) = 1700.0 KELVIN
COLLECTOR TEMPERATURE (TC) = 773.0 KELVIN
EMITTER WORK FUNCTION (EWF) = 2.642 EV
COLLECTOR WORK FUNCTION (CWF) = 1.630 EV
CONVERTOR PRESSURE (PN) = 1.541 TORR
GAP THICKNESS (D) = 0.254 MM
OPERATING CURRENT (J) = 2.000 AMPS/CM^2

TEC FUNCTION SETTINGS-----

DIAGNOSTIC LEVEL (CHKDOT) = 1
RESTART SEQUENCE (OFILE) = 0
POINT DENSITY (N) = 11

PHYSICAL PARAMETERS EVALUATED-----

RICHARDSON CURRENT (JRIC) = 0.51E+01 AMPS/CM^2
REFERENCE DENSITY (NR) = 0.10E+15 1/CM^3
CHARACTERISTIC TIME (TCHAR) = 0.0208 SECS*E-06
NONDIM CURRENT (I) = 0.0196
NONDIM EMISSION (ENR) = 0.008(NRIC/NR)
KNUDSEN NUMBER (KN) = 0.0791
SQRT(MASS RATIO) (SMR) = 0.0020
MEAN FREE PATH RATIO (LAMDAR) = 0.3344

Voltage

T in Characteristic Time 10usec

0.00 100.00 200.00 300.00 400.00 500.00
TEC INITIAL DATA SUMMARY

PHYSICAL OPERATING CONDITIONS------

EMITTER TEMPERATURE (TE) = 1700.0 KELVIN
COLLECTOR TEMPERATURE (TC) = 773.0 KELVIN
EMITTER WORK FUNCTION (EWF) = 2.642 EV
COLLECTOR WORK FUNCTION (CWF) = 1.630 EV
CONVERTOR PRESSURE (PN) = 1.541 TORR
GAP THICKNESS (D) = 0.254 MM
OPERATING CURRENT (J) = 2.000 AMPS/CM^2

TEC FUNCTION SETTINGS------

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REFERENCE DENSITY (NR) = 0.10E+15 1/CM^3
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NODIM CURRENT (I) = 0.0196
NODIM EMISSION (ENR) = 0.008 (NRIC/NR)
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MEAN FREE PATH RATIO (LAMDAR) = 0.3344

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PAPERS AND PUBLICATIONS

1. The paper entitled "Asymptotically Correct Collisional Presheaths," by G L. Main, Phys. Fluids, June 1987, has been produced under this grant.


3. "A Uniform Three Scale Asymptotic Solution for a Collisional Presheath and Collisionless Sheath with Ion Reflection," by G. L. Main and G. L. Ridderbusch is to appear in IEEE Trans. on Plasma Science, and has been produced under this grant.
APPENDIX A - Fokker Planck Collisions Presheath

This theory has been developed under this grant and is found to be applicable to fully ionized plasmas but was not incorporated into the Thermionic Convertor work due to its computational complexity.
AN ANALYTIC - NUMERICAL SCHEME FOR A COLLISIONAL
FOKKER - PLANCK TIME DEPENDENT SHEATH - PRESHEATH
STRUCTURE

A THESIS
Presented to
The Faculty of the Division of Graduate Studies
By
Jeffrey Paul Dansereau

In Partial Fulfillment
of the Requirements for the Degree
Master of Science in Mechanical Engineering

Georgia Institute of Technology
September, 1987
AN ANALYTIC - NUMERICAL SCHEME FOR A COLLISIONAL 
FOKKER - PLANCK TIME DEPENDENT SHEATH - PRESHEATH 
STRUCTURE 

Approved: 

______________________________
Geoffrey L. Main, Chairman 

______________________________
J. Narl Davidson 

______________________________
Thomas L. Eddy 

Date approved by chairman ______
ACKNOWLEDGMENTS

I would like to thank Geoffrey L. Main for all of his help and encouragement during the development of this project. This work was supported by the Air Force Office of Scientific Research grant 85-0375.
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NOMENCLATURE

$a$ - Inverse square of the particle thermal velocity
$a_1, a_2, \cdots$ - Coefficients of the potential structure
$a, b$ - Integral limits
$A$ - First derived Fokker Planck collision function
$B$ - Second derived Fokker Planck collision function
$d_i(r)$ - An $m \times 1$ matrix
$D(r)$ - An $m \times 1$ solution matrix
$e$ - Electron charge
$E$ - Electric field strength
$f$ - Particle distribution function in velocity space
$F$ - Nondimensional particle distribution function in velocity space
$g$ - Collision function
$h$ - Grid spacing width
$k$ - Boltzmann's constant
$L$ - System dimension
$m$ - Number of points in velocity space
$m$ - Particle mass
$M$ - Reduced mass
$n$ - Plasma density
$n_R$ - Reference density
$q$ - Elementary charge
$R$ - Radial velocity component
$t$ - Time
$T_i$ - Ion temperature
$T_e$ - Electron temperature
$T(r)$ - An $m \times m$ matrix
$U$ - Potential
$v$ - Velocity vector
$v_i(r)$ - An $m \times 1$ matrix
$V(r)$ - An $m \times 1$ solution matrix
$z$ - Position vector
$z$ - Axial position component
$z$ - Axial velocity component
$Z$ - Ionization level
$Z$ - Nondimensional axial velocity component
$\alpha$ - Potential gradient across the plasma
$\beta$ - Exponential coefficient of the presheath rise
$\Gamma$ - Fokker-Planck coefficient
$\Delta U$ - Potential expansion parameter
$\varepsilon_0$ - Permittivity of free space
$\zeta, \eta, \xi$ - Variables of integration
$\theta$ - Velocity rotation angle in cylindrical coordinates
$\lambda_D$ - Debye length
$\lambda_i$ - Mean free path
$\Lambda$ - Coulomb logarithm
$\tau$ - Nondimensional time
$\Phi$ - Electric potential
$\varphi$ - Nondimensional coefficients of the potential structure

Subscripts:
$e$ - Collision term
$e$ - Electron
$i$ - Ion
$n$ - Order of expansion

Superscripts:
$\text{i}$ - Particle species with which collisions occur
$\ast$ - Nondimensional
SUMMARY

The Maxwellian sources and charge exchange terms used to model particle interactions in current presheath models do not represent the Coulomb collisions taking place in fully ionized plasmas. These models approximate the collisional effects in the presheaths of partially ionized plasmas but are used to implicitly extrapolate the interesting case of fully ionized plasmas. The present study uses a Fokker-Planck collision term which models the limit of the small angle Coulomb collisions that occur in fully ionized plasmas. Normally these small angle collisions dominate the particle interactions of fully ionized plasmas. The Boltzmann equation coupled with the Fokker-Planck term, and the Poisson equation have been expanded using an exponential asymptotic technique. These equations have been solved numerically to determine the time dependent evolution of the presheath. The results presented show the presheath potential structure and particle distribution in velocity space. The model produces a self-consistent and accurate potential structure. The particle velocity distribution in the presheath has the correct acceleration of ions toward the wall but because the Fokker-Planck collision term only models the limit of small angle collisions it is unable to clear the particle distribution of returning ions. The collisional processes become dominated by the effects of the large angle collisions as the Debye sheath edge is approached. This study has found that a presheath model which describes the Coulomb collisions occurring in a fully ionized plasma must account for both the small angle and the large angle particle collisions to explain the clearing out of returning ions that must exist for the transition to an absorbing wall.
CHAPTER I

INTRODUCTION

The interaction of man and plasma, in some form, exists at almost all levels of society. A plasma is an ionized gas that has a collective behavior in an electromagnetic field. Plasmas exist in everyday devices like fluorescent lights, neon signs, and electric arc welders. An understanding of the basic behavior and interaction of plasmas is essential to the advancement of all current plasma applications and to the discovery of new applications. This thesis involves the study of how a plasma interacts with the walls and surfaces with which it comes in contact.

Why is it important to understand plasma-wall interactions? Two basic reasons answer this question. First, a plasma has a strong effect on any surfaces it comes in contact with. The high temperature plasma can erode or destroy any surface quickly pitting and changing a wall which may need to maintain a particular profile or surface condition. Secondly, the wall affects the characteristics of the plasma. A surface can have a profound effect on the plasma depending on the amount and rate at which it can absorb energy. Examples of situations in which plasma-wall interactions are of importance include:

- Diverter plates in magnetic confinement fusion reactors.
- The rails in a plasma rail gun.
- Any body (like the space shuttle) upon reentry to the atmosphere.
- Plasma switches.
- Plasma etching.
Almost every other use or occurrence of plasmas.

This study is primarily applicable to fully ionized plasmas. The hot temperatures necessary to produce fully ionized plasmas occur only in situations like on the surfaces of diverter plates in Tokamak fusion reactors.

The development of a mathematical model to represent the plasma-wall interaction region, or sheath, and a numerical solution to this model is the focus of this thesis. An understanding of the interaction between the plasma and the wall is achieved with a time dependent solution to the sheath region. If the potential structure and the particle velocity distributions are known for every location in the sheath then the energy going into the surface can be determined. In this way the results of this study can be used as a boundary condition for problems involving plasma characteristics and for problems involving the surface physics of plasma devices.
CHAPTER II

BACKGROUND

A plasma will naturally maintain itself in a neutral and field free state. Application of forces and processes that try to alter the equilibrium are resisted by the plasma. A surface within a plasma that is not at the same potential as the plasma will be shielded from the remainder of the plasma by a sheath. The outer edge of this sheath is nearly at the plasma potential. Bohm\textsuperscript{[1]} first came up with a criterion to determine the extent of the sheath. Bohm modeled the sheath region as completely collisionless. He also considered that the transition region from this collisionless sheath to the plasma was too small to be important.

More recent work has been done to describe this transition, or presheath, region. Self\textsuperscript{[2]} has an exact solution to the sheath equation and has shown that the collisionless sheath makes a transition directly to the neutral plasma in the limit as $\lambda_D \rightarrow 0$, where $\lambda_D$ is the Debye length and $L$ is the plasma dimension. Emmert et al.\textsuperscript{[3]} has determined a presheath structure based on the assumption of a Maxwellian source of ions to model the particle collisions. The solution to this model shows that the transition point from the sheath to the presheath has a finite electric field strength. Bissell and Johnson\textsuperscript{[4]} have performed a similar solution using a Maxwellian source of ions. In contrast to Emmert et al., Bissell and Johnson have found that the electric field strength becomes infinite at the sheath edge. This solution agrees with the fluid and cold ion models. In a recent paper Bissell\textsuperscript{[5]} shows that Emmert obtained a finite electric field strength because the Maxwellian source term used produced no ions at the point of zero velocity. Bissell and Johnson used a more realistic
Maxwellian source that produced ions at the zero point in velocity space for their solution.

Another approach to the problem involves the use of a charge exchange term to model the particle collisions. Riemann[6] has produced results using this technique. In a recent paper by Main[7] a charge exchange model is used to obtain a solution to the presheath potential structure and particle distribution. This model involves an asymptotic approximation of the plasma equations. The Boltzmann and Poisson equations are asymptotically expanded and then solved analytically when combined with a charge exchange model of the particle collisions.

All of these sheath and presheath solutions have modeled particle collisions by large instantaneous changes in particle velocity. These models do not represent the Coulomb collisions occurring in the presheath of a fully ionized plasma.

The current study extends the asymptotic solution presented by Main[7] to include a Fokker - Planck collision term instead of the charge exchange term. Unlike the previous collision terms used, the Fokker - Planck term describes the Coulomb collisions that exist within a fully ionized plasma. The addition of the Fokker - Planck term necessitates the use of numerical techniques, rather than analytical techniques, to obtain a solution. In using the Fokker - Planck term the collision processes are being modeled directly. The model developed obtains the time dependent evolution of the presheath for a fully ionized plasma.
CHAPTER III

MODEL FORMULATION

3.1 Concepts

In order to have a complete understanding of the problem at hand certain concepts need to be presented which will help in understanding the overall structure of the model.

1) Debye Length ($\lambda_D$) - The shielding distance beyond which the particle charge effect is weak. This is the natural charge separation distance. Negatively charged particles become surrounded by positively charged particles and vice versa, thus, balancing the overall charge at any point (see figure 3.1). There is a point beyond which a particle is not effected by the specific charge but responds to the influence of the entire plasma. The thermal effects in the plasma become dominant over the electric field strength.

2) Mean Free Path ($\lambda_t$) - The average distance a particle travels before its trajectory has been altered by ninety degrees. The mean free path is a function of the density of the plasma. The denser the plasma the shorter the mean free path. For the plasma under consideration in this study $\lambda_t >> \lambda_D$.

3) The coordinate system used to describe the plasma - The coordinate system used in the model is known phase space. In this system any point is described using three position coordinates and three velocity coordinates. Any orthogonal coordinate system, cartesian, cylindrical, spherical, can be used to describe both the position and the velocity components.
4) Distributions and Distribution Functions - Plasmas are studied in a collective sense. The motion of the entire plasma and not individual particles is described by the model. Therefore, the velocity of the plasma at any given location must be described by a distribution. The distribution function describes the overall particle velocity distribution.

5) Potential - In a plasma the wall potential is greater than the neutral plasma potential. The lighter, thus, faster electrons are absorbed by the wall faster than the heavier and slower ions. A net positive charge exists near the wall, increasing the potential (see figure 3.2). The potential at the physical interface between the wall and the plasma is dependent on the rate at which ions are absorbed by the surface. In this study \( U = -e\phi \) where \( e \) is the electron charge and \( \phi \) is electric potential in electron volts so that \( U \) has units of energy. The addition of the negative sign defines potential in the reverse of the usual sign convention so that increasing potential repels electrons.

6) Collision Possibilities using the Fokker-Planck Collision term - To describe the overall structure of the sheath the various collision possibilities must be included in a comprehensive model. The Fokker-Planck term describes the four major collision possibilities.

1) Ion - Ion

2) Ion - Electron

3) Electron - Ion

4) Electron - Electron

The collision model does not take into account three body collisions. Three body collisions are very rare, as such, the model is not hampered by the lack of
terms to describe these collisions.

3.2 Wall Region Model

The model of the Plasma - Wall region can be broken into three areas.

1) Neutral Plasma Region \( O(L) \) - The neutral plasma region represents the majority of the system and can be considered to have a physical width that is on the order of the overall dimension of the system, \( L \). This region is considered to be fully collisional. The velocity distribution is near Maxwellian and as such can be modeled by fluid type equations (see figure 3.3).

2) Debye Sheath Region \( O(\lambda_D) \) - This region is a very thin area directly adjacent to the wall. Its width is considered to be on the order of a Debye length and since \( \lambda_i \gg \lambda_D \) no collisions are expected in this region. This collisionless sheath was first modeled by Langmuir\[8\] and Bohm\[1\] and is considered very well known and understood.

3) Collisional Presheath Region \( O(\lambda_i) \) - This is a transition region between the collisional neutral plasma and the collisionless Debye sheath region. It is considered to have a physical width on the order of a mean free path. Therefore, collisions are expected but at the same time the region cannot be considered fully collisional.

The potential must transition from a lower level in the neutral plasma to a higher level at the wall. The goal of this study has been to obtain a time dependent model of the evolution of the presheath region which asymptotically approaches the known potential in both the neutral plasma and in the Debye sheath region.

In order to show the validity of the three region model an example of Debye sheath width in relation to the overall wall region is appropriate. For this example
average hydrogen fusion plasma characteristics have been assumed:

\[ T_i = 10^5 K \]

\[ n_0 = 10^{20} m^{-3} \]

If the Debye length within this plasma is calculated an order of magnitude value for the Debye sheath width is determined. An appropriate equation for the Debye length in meters is:\[^9\]

\[ \lambda_D = 69 \left( \frac{T_i}{n_0} \right)^{\frac{1}{3}} \]  \hspace{1cm} (3.1)

From this equation:

\[ \lambda_D = 2.18 \times 10^{-6} m \]

The overall sheath width is on the order of a mean free path. An appropriate equation for the mean free path in meters is:\[^10\]

\[ \lambda_i = 1.2 \times 10^{-4} \frac{1}{Z^4} \left( \frac{T_i}{\varepsilon} \right)^2 \left( \frac{n_i}{10^{20}} \right)^{-1} \]  \hspace{1cm} (3.2)

For a singly ionized plasma \( Z = 1 \). Using this equation and the above example plasma characteristics the mean free path can be calculated.

\[ \lambda_i = 1.14 \times 10^{-2} m \]

This is an order of magnitude estimate value for the width of the entire sheath region. Since the Debye sheath width is on the order of a Debye length it can be seen that the collisionless sheath is very thin in comparison with the entire wall region.

In order to obtain an idea of the importance of the electric field in the wall region an order of magnitude analysis is useful. The magnitude of the electric field is proportional to the thermal energy per length scale.

\[ E \sim \frac{kT_i}{z} \]  \hspace{1cm} (3.3)
In the sheath region the length scale is the Debye length.

\[ E \sim \frac{kT_i}{\lambda_D} \]  

(3.4)

Therefore, in this region the electric field is very significant since \( \lambda_D \) is very small. The collisional effects are small in comparison, and can be neglected.

In the presheath region the length scale is the mean free path.

\[ E \sim \frac{kT_i}{\lambda_i} \]  

(3.5)

Therefore, the electric field strength is on the order of the collisional effects making both important factors within this region.

In the neutral plasma region the length scale is the overall system dimension.

\[ E \sim \frac{kT_i}{L} \]  

(3.6)

Therefore, the electric field is very weak and can be neglected in comparison with the collisional effects.

### 3.3 Presheath Model

#### 3.3.1 Equations Describing the Collective Behavior of a Plasma

The primary equation used to describe the behavior of a plasma is the Boltzmann equation. The Boltzmann equation represents the collective motion of many charged particles moving in an electromagnetic field\[11\].

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{1}{m} \frac{\partial U}{\partial x} \frac{\partial f}{\partial v} = \left\{ \frac{\partial f}{\partial t} \right\}_c
\]  

(3.7)

Where the + sign is for ion particles and the – sign is for electrons. In the Boltzmann equation \( f \) is the particle distribution function, and is defined such that \( n = \int_{-\infty}^{\infty} f dv \).
The quantity $t$ is time, $v$ is a velocity vector, $x$ is a position vector, $m$ is the particle mass, and $U$ is the potential. The term on the right hand side represents particle collisions and can take on various forms depending on the model being used.

Another important equation in describing the behavior of a plasma is the Poisson equation. The Poisson equation is the elementary definition of potential as the collective effect of charged particles on a point:

$$\frac{\partial^2 U}{\partial x^2} = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_i(v, \Delta U) \, dv - \int_{-\infty}^{\infty} f_e(v, \Delta U) \, dv \right]$$

(3.8)

Where $q$ is the elementary charge, the subscript $i$ refers to ions and the subscript $e$ refers to electrons. The term in brackets is the ion - electron density difference. The potential is the driving force in the Boltzmann equation. The Poisson equation relates the potential to the particle distribution.

To complete the set of equations necessary for a full description of the presheath a collision term must be chosen to model the particle interactions. This study uses the Fokker - Planck collision term to model the particle collisions. The Fokker - Planck term represents the right hand side of the Boltzmann equation:

$$\left\{ \frac{\partial f}{\partial t} \right\}_e = \Gamma \left[ -\frac{\partial}{\partial v_i} \left( f \frac{m}{2M} \frac{\partial}{\partial v_i} \nabla^2 g \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_k} \left( f \frac{\partial^2 g}{\partial v_i \partial v_k} \right) \right]$$

(3.9)

Where,

$$\Gamma = \frac{g^2 q^2 \ln \Lambda}{4\pi^2 m^2}$$

(3.10)

$$g(v) = \int f(v') \left| v - v' \right| \, dv'$$

(3.11)

$$\mathcal{M} = \frac{mm'}{m + m'}$$

(3.12)

Where $\mathcal{M}$ is called the reduced mass. No superscript refers to the particle species undergoing the collisions and the superscript $i$ refers to the particle species with which the collision occurs.
The Fokker - Planck equation describes the Coulomb collision between two charged particles. Certain restrictions and assumptions are made when using the Fokker - Planck collision term. First, it best describes fully ionized plasmas. The collision term models charged particle interaction and is most accurate for plasmas with few neutrals. This situation occurs only on the hottest of plasma surfaces like the diverter plates in Tokamak fusion reactors.

The second restriction involves the type of collisions that the Fokker - Planck term models. The overwhelming majority of particle collisions lead to only small deflections in the particle trajectories. The Fokker - Planck term describes the limit of these small angle deflections. Finally, the model does not take into account three body, and higher order, collisions.

3.3.2 Solution Conditions

The three equations presented in the previous section in conjunction with the asymptotic forms of potential and velocity distribution provide the necessary information to determine the presheath structure if two additional conditions are met.

First, if the equations are written in cylindrical coordinates the particle velocity distribution is axially symmetric. There is no theta, $\theta$, dependence of the velocity distribution. Cylindrical coordinates are used for both the velocity and the position. The 'z' direction is perpendicular to the wall (see Figure 3.4) with the positive direction being defined into the wall. The coordinates $R, \theta, z$ have been used in velocity space for convenience.

The second condition for a solution to these equations involves an assumption of the particle velocity distribution parallel to the surface. For this model the radial velocity distribution has been assumed to take the form of a Maxwellian distribution. In addition, the temperature in the radial direction has been assumed
to be uniform and constant at all \( z \) locations. Thus, the radial velocity distribution is constant for any position. Figure 3.5 is a schematic of these conditions. Note that at any \( z \) location and rotation angle, \( \theta \), the radial velocity distribution is constant and follows a Maxwellian distribution. This represents the conditions of uniform temperature and axial symmetry, throughout the wall region. Figure 3.5 also shows a representation of the point of no returning ions. This is the point where the presheath transitions to the collisionless sheath.

The conditions of uniform temperature and radial Maxwellian distribution although good approximations are not exact models of the real situation.

The overall problem reduces to one dimension, the \( z \) direction, with the above conditions. The \( \theta \) dependence having been removed by the axial symmetry and the radial dependence having been removed by the Maxwellian assumption. This one dimensional problem can be solved by straight forward numerical techniques.

### 3.3.3 Expansion of the Boltzmann Equation

The presheath model involves the expansion of the potential and velocity into asymptotic approximations. The potential is assumed to follow an exponential asymptotic form.

\[
U = U_0 + a_1 \Delta U + a_2 \Delta U^2 + \cdots
\]  

\[ (3.13) \]

where

\[
U_0 = \alpha z \quad \text{and} \quad \Delta U = e^{\beta z}
\]  

\[ (3.14) \]

\( a_1, a_2 \ldots \) are parameters which describe the potential structure. Alpha, \( \alpha \), is non-zero for a non-zero potential gradient in the neutral plasma. \( \Delta U \) is called the potential expansion parameter.

The particle distribution in velocity space is a function of potential and can be
similarly expanded.

\[ f(v, \Delta U) = f_0(v) + \Delta U f_1(v) + \Delta U^2 f_2(v) + \cdots \]  

(3.15)

The Boltzmann equation can be written as

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial (\Delta U)} + \frac{1}{m} \frac{\partial U}{\partial \nu} \frac{\partial f}{\partial \nu} = \{ \frac{\partial f}{\partial t} \}_e
\]

(3.16)

where ± sign is respectively positive for ion particles and negative for electrons.

The potential \( U \) has units of energy and is defined as shown in figure 3.3 such that \( U = -e\Phi \) where \( \Phi \) is electric potential. The Boltzmann equation can be expanded using equations 3.13 and 3.14. In addition since the solution is one dimensional in velocity space the velocity derivatives reflect only the 'z' direction. The following expansions are used.

\[
\frac{\partial f}{\partial t} = \frac{\partial f_0}{\partial t} + \Delta U \frac{\partial f_1}{\partial t} + \Delta U^2 \frac{\partial f_2}{\partial t} + \cdots
\]

(3.17)

\[
\frac{\partial f}{\partial (\Delta U)} = f_1 + 2\Delta U f_2 + 3\Delta U^2 f_3 + \cdots
\]

(3.18)

\[
\frac{\partial (\Delta U)}{\partial z} = \beta \Delta U
\]

(3.19)

\[
\frac{\partial U}{\partial z} = \alpha + \beta a_1 \Delta U + 2\beta a_2 \Delta U^2 + 3\beta a_3 \Delta U^3 + \cdots
\]

(3.20)

\[
\frac{\partial f}{\partial z} = \frac{\partial f_0}{\partial z} + \Delta U \frac{\partial f_1}{\partial z} + \Delta U^2 \frac{\partial f_2}{\partial z} + \cdots
\]

(3.21)

Using these expansions the Boltzmann equation can be broken down by order in \( \Delta U \) assuming the collision term can be likewise broken down:

\[
1: \quad \frac{\partial f_0}{\partial t} \pm \frac{a}{m} \frac{\partial f_0}{\partial z} = \left[ \{ \frac{\partial f}{\partial t} \}_e \right]_1
\]

(3.22a)

\[
\Delta U : \quad \frac{\partial f_1}{\partial t} + \beta z f_1 \pm \frac{a}{m} \frac{\partial f_1}{\partial z} \pm \frac{\beta a_1}{m} \frac{\partial f_0}{\partial z} = \left[ \{ \frac{\partial f}{\partial t} \}_e \right]_{\Delta U}
\]

(3.22b)

\[
\Delta U^2 : \quad \frac{\partial f_2}{\partial t} + 2\beta z f_2 \pm \frac{a}{m} \frac{\partial f_2}{\partial z} \pm \frac{\beta a_1}{m} \frac{\partial f_1}{\partial z} \pm \frac{2\beta a_2}{m} \frac{\partial f_0}{\partial z} = \left[ \{ \frac{\partial f}{\partial t} \}_e \right]_{\Delta U^2}
\]

(3.22c)

\[
\Delta U^n : \quad \frac{\partial f_n}{\partial t} + n \beta z f_n \pm \frac{a}{m} \frac{\partial f_n}{\partial z} \pm \frac{\beta a_1}{m} \frac{\partial f_{n-1}}{\partial z} \pm \frac{2\beta a_2}{m} \frac{\partial f_{n-2}}{\partial z} \pm \cdots \pm \frac{(n-1)\beta a_{n-1}}{m} \frac{\partial f_1}{\partial z} \pm \frac{n \beta a_n}{m} \frac{\partial f_0}{\partial z} = \left[ \{ \frac{\partial f}{\partial t} \}_e \right]_{\Delta U^n}
\]

(3.22d)
The above exponential expansion is the only known way to break apart the Boltzmann equation. To complete the expansion the Poisson and Fokker-Planck terms must be similarly broken down.

At this point it is appropriate to nondimensionalize the expansion of the Boltzmann equation. The following nondimensional quantities have been used.

\[ F_i = \frac{f_i}{n_R a^{3/2}} \text{ for } 0 \leq i \leq n \]  
\[ r = \frac{n_R \Gamma}{a^{1/2} t} \]  
\[ Z = \sqrt{\alpha} z \]  
\[ \varphi_{-1} = \frac{aa}{mn_R \Gamma} \]  
\[ \varphi_0 = \frac{\beta}{n_R \Gamma} \]  
\[ \varphi_i = \frac{\beta a a}{mn_R \Gamma} \text{ for } 1 \leq i \leq n \]

Where \( n_R \) is a reference density, \( a \) and \( a' \) are the inverse square of the thermal velocities of the colliding particles, \( a = \frac{m}{2kT} \) and \( a' = \frac{m'}{2kT'} \). The quantities \( \varphi_{-1} \) through \( \varphi_n \) represent the potential structure of the presheath. \( F_i \) is nondimensionalized such that \( 1 = \int_{-\infty}^{\infty} F dZ \). Using these nondimensionalizations, the expansion of the Boltzmann equations becomes

\[ 1: \frac{\partial F_0}{\partial r} + \varphi_{-1} \frac{\partial F_0}{\partial Z} = \left( \frac{\partial F}{\partial r} \right)_e \]  
\[ \Delta U: \frac{\partial F_1}{\partial r} + \varphi_0 Z F_1 + \varphi_{-1} \frac{\partial F_1}{\partial Z} = \left( \frac{\partial F}{\partial r} \right)_e \Delta U \]  
\[ \Delta U^2: \frac{\partial F_2}{\partial r} + 2\varphi_0 Z F_2 + \varphi_{-1} \frac{\partial F_2}{\partial Z} = \left( \frac{\partial F}{\partial r} \right)_e \Delta U^2 \]  
\[ \vdots \]
\[ \Delta U^n: \frac{\partial F_n}{\partial r} + n \varphi_0 Z F_n + \varphi_{-1} \frac{\partial F_n}{\partial Z} = \left( \frac{\partial F}{\partial r} \right)_e \Delta U^n \]  
\[ \cdots + (n-1)\varphi_{n-1} \frac{\partial F_{n-1}}{\partial Z} + n \varphi_n \frac{\partial F_0}{\partial Z} = \left( \frac{\partial F}{\partial r} \right)_e \Delta U^n \]
3.3.4 Expansion of the Poisson Equation

The Poisson equation,

$$\frac{d^2 U}{dx^2} = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_i(u, \Delta U) \, dv - \int_{-\infty}^{\infty} f_e(u, \Delta U) \, dv \right] \tag{3.8}$$

is broken down using the same technique as the Boltzmann equation. Since:

$$\frac{\partial^2 U}{\partial x^2} = \beta^2 a_1 \Delta U + 4\beta^2 a_2 \Delta U^2 + 9\beta^2 a_3 \Delta U^3 + \cdots \tag{3.25}$$

$$\int_{-\infty}^{\infty} f(u, \Delta U) \, dv = \int_{-\infty}^{\infty} f_0(u) \, dv + \Delta U \int_{-\infty}^{\infty} f_1(u) \, dv + \Delta U^2 \int_{-\infty}^{\infty} f_2(u) \, dv + \cdots \tag{3.26}$$

Using these expansions the Poisson equation can be broken down by order in $\Delta U$.

1: \quad 0 = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_{i0} \, dv - \int_{-\infty}^{\infty} f_{e0}(u) \, dv \right] \tag{3.27a}

$\Delta U$: \quad \beta^2 a_1 = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_{i1}(v) \, dv - \int_{-\infty}^{\infty} f_{e1}(v) \, dv \right] \tag{3.27b}

$\Delta U^2$: \quad 4\beta^2 a_2 = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_{i2}(v) \, dv - \int_{-\infty}^{\infty} f_{e2}(v) \, dv \right] \tag{3.27c}

\vdots

$\Delta U^n$: \quad n^2 \beta^2 a_n = 4\pi q^2 \left[ \int_{-\infty}^{\infty} f_{in}(v) \, dv - \int_{-\infty}^{\infty} f_{en}(v) \, dv \right] \tag{3.27d}

The assumption of Boltzmann electrons is made to enable the numerical calculations to proceed with time steps on the order of an ion characteristic time. In the asymptotic presheath the Boltzmann electron assumption becomes

$$n_e = n_0 e^{-(a_1 \Delta U + a_2 \Delta U^2 + \cdots + a_n \Delta U^n) kT_e} \tag{3.28}$$

where $n_e$ is the electron density, $T_e$ is the electron temperature, and $n_0$ is the electron density in the asymptotic presheath at $\Delta U = 0$. Expansion of (3.28) in terms of $\Delta U$ yields

$$n_e = [n_0] + \Delta U \left[ - \frac{a_1}{kT_e} n_0 \right] + \Delta U^2 \left[ \left( - \frac{a_2}{kT_e} + \frac{1}{2} \frac{a_2^2}{(kT_e)^2} \right) n_0 \right]$$

$$+ \Delta U^3 \left[ \left( - \frac{a_3}{kT_e} + \frac{a_1 a_2}{(kT_e)^2} - \frac{1}{6} \frac{a_2^3}{(kT_e)^3} \right) n_0 \right] + \cdots \tag{3.29}$$
With the assumption that $\lambda_D < \lambda_i$ the Poisson equation reduces to equating electron and ion densities in order of $\Delta U$. The Poisson and Boltzmann electron equations are nondimensionalized in the same manner as the Boltzmann equation. The nondimensionalized Poisson equation (3.27a-d) is combined with the Boltzmann electron equation (3.28) to become

$$
\int_{-\infty}^{\infty} F_0 \, dZ = \eta e^{[\nu_1 + \cdots + \nu_n]} \frac{d}{dx} \left( \frac{\nu_n}{2} \right)
$$

$$
\int_{-\infty}^{\infty} F_1 \, dZ = \eta e^{[\nu_1 + \cdots + \nu_n]} \frac{d}{dx} \left( \frac{\nu_n}{2} \right) \left[ - \frac{\nu_1}{\nu_0} \left( \frac{2T_i}{T_e} \right) \right]
$$

$$
\int_{-\infty}^{\infty} F_2 \, dZ = \eta e^{[\nu_1 + \cdots + \nu_n]} \frac{d}{dx} \left( \frac{\nu_n}{2} \right) \left[ - \frac{\nu_2}{\nu_0} \left( \frac{2T_i}{T_e} \right) + \frac{1}{2} \left( \frac{\nu_1}{\nu_0} \right)^2 \left( \frac{2T_i}{T_e} \right) \right]
$$

$$
\int_{-\infty}^{\infty} F_3 \, dZ = \eta e^{[\nu_1 + \cdots + \nu_n]} \frac{d}{dx} \left( \frac{\nu_n}{2} \right) \left[ - \frac{\nu_3}{\nu_0} \left( \frac{2T_i}{T_e} \right) + \frac{\nu_1 \nu_2}{\nu_0 \nu_0} \left( \frac{2T_i}{T_e} \right)^2 - \frac{1}{6} \left( \frac{\nu_1}{\nu_0} \right)^3 \left( \frac{2T_i}{T_e} \right) \right]
$$

where $\eta = \frac{a}{a_0}$ and may be specified as a function of time. This equation can be used to solve for the potential structure at each time step. $T_i$ is the ion temperature and $T_e$ is the electron temperature.

### 3.3.5 Expansion of the Fokker – Planck Term

The Fokker – Planck term must also be expanded in order of $\Delta U$ but first must be put into cylindrical coordinates. In addition, the assumption of axial symmetry must be accounted for in the term. This can be accomplished by expanding each term in the general Fokker – Planck term.

$$
\left\{ \frac{\partial f}{\partial t} \right\} = \Gamma \left[ - \frac{\partial}{\partial w_i} (f \frac{m}{2M} \frac{\partial}{\partial w_i} \nabla^2 g) + \frac{1}{2} \frac{\partial^2}{\partial w_i \partial w_k} (f \frac{\partial^2 g}{\partial w_i \partial w_k}) \right] \quad (3.9)
$$

The first term can be rewritten:

$$
- \frac{m}{2M} (\nabla \cdot f \nabla (\nabla^2 g))
$$

$$
\nabla^2 g = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial g}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 g}{\partial \theta^2} + \frac{\partial^2 g}{\partial z^2}
$$

$$
\partial R \mathcal{L}
$$
Axial symmetry eliminates all \( \theta \) dependence, eliminating the middle term. In addition, with the assumption of a Maxwellian radial velocity distribution the problem has been reduced to one dimension. Thus,

\[
\nabla (\nabla^2 g) = \frac{\partial^3 g}{\partial z^3}
\]

(3.33)

and the first term reduces to:

\[
-\frac{\partial}{\partial z} \left( f \frac{m}{2M} \frac{\partial^3 g}{\partial z^3} \right)
\]

(3.34)

The second term in the Fokker–Planck equation,

\[
\frac{1}{2} \frac{\partial^2}{\partial z^2} \left( f \frac{\partial^2 g}{\partial u_i \partial u_k} \right)
\]

can be reduced directly to the one dimensional case.

\[
\frac{1}{2} \frac{\partial^2}{\partial z^2} \left( f \frac{\partial^2 g}{\partial z^2} \right)
\]

(3.35)

Therefore, the Fokker–Planck term in one dimensional cylindrical coordinates is:

\[
\frac{1}{\Gamma} \left\{ \frac{\partial f}{\partial t} \right\}_z = \frac{1}{2} \frac{\partial^2}{\partial z^2} \left( f \frac{\partial^2 g}{\partial z^2} \right) - \frac{\partial}{\partial z} \left( f \frac{m}{2M} \frac{\partial^3 g}{\partial z^3} \right)
\]

(3.36)

In order to get a complete collision model the function \( g \) must also be converted to the appropriate coordinate system.

\[
g(v) = \int_{-\infty}^{\infty} f(v') |v - v'| dv'
\]

This definition can be reduced for the one dimensional case.

\[
g(z) = \int_{-\infty}^{\infty} f(\eta) |z - \eta| d\eta
\]

(3.37)

or, written another way

\[
g(z) = \int_{-\infty}^{\infty} f(z + \xi) |\xi| d\xi
\]

(3.38)
With this definition the Fokker-Planck term becomes:

$$\frac{1}{\Gamma} \left\{ \frac{\partial f}{\partial t} \right\}_c = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( f \int_{-\infty}^{\infty} \frac{\partial^2}{\partial z^2} (f(z+\xi)) \mid \xi \mid d\xi \right) - \frac{\partial}{\partial z} \left( f \frac{m}{2M} \int_{-\infty}^{\infty} \frac{\partial^3}{\partial z^3} (f(z+\xi)) \mid \xi \mid d\xi \right) \tag{3.39}$$

For brevity let \( \dot{f} \) denote a derivative with respect to \( z \). Using this notation the Fokker-Planck term becomes:

$$\frac{1}{\Gamma} \left\{ \frac{\partial f}{\partial t} \right\}_c = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( f \int_{-\infty}^{\infty} f''(z+\xi) \mid \xi \mid d\xi \right) - \frac{\partial}{\partial z} \left( f \frac{m}{2M} \int_{-\infty}^{\infty} f'''(z+\xi) \mid \xi \mid d\xi \right) \tag{3.40}$$

The Fokker-Planck term is nondimensionalized using the same variables as the Boltzmann equation.

$$\left\{ \frac{\partial F}{\partial t} \right\}_c = \frac{\partial^2}{\partial z^2} \left( F \frac{1}{a'} \int_{-\infty}^{\infty} F''(Z+\xi) \mid \xi \mid d\xi \right) + \frac{\partial}{\partial z} \left( F \frac{m}{2M} a' \int_{-\infty}^{\infty} F'''(Z+\xi) \mid \xi \mid d\xi \right) \tag{3.41}$$

Let:

$$A(F) = \frac{1}{a'} \int_{-\infty}^{\infty} F''(Z+\xi) \mid \xi \mid d\xi \tag{3.42}$$

$$B(F) = \frac{m}{2M} a' \int_{-\infty}^{\infty} F'''(Z+\xi) \mid \xi \mid d\xi \tag{3.43}$$

The Fokker-Planck term can be written in a compact form using these defined functions.

$$\left\{ \frac{\partial F}{\partial t} \right\}_c = \frac{\partial^2}{\partial z^2} (FA(F)) + \frac{\partial}{\partial z} (FB(F)) \tag{3.44}$$

The Fokker-Planck term can be expanded by order in \( \Delta U \) using the same technique as the Boltzmann equation.

$$\left\{ \frac{\partial F}{\partial t} \right\}_c = \left[ \frac{\partial^2}{\partial z^2} (F_0A(F_0)) + \frac{\partial}{\partial z} (F_0B(F_0)) \right]$$

$$+ \Delta U \left[ \frac{\partial^2}{\partial z^2} (F_1A(F_0)) + \frac{\partial}{\partial z} (F_1B(F_0)) + \frac{\partial^2}{\partial z^2} (F_0A(F_1)) + \frac{\partial}{\partial z} (F_0B(F_1)) \right]$$

$$+ \cdots$$

$$+ \Delta U^n \left[ \sum_{m=0}^{n} \left( \frac{\partial^2}{\partial z^2} (F_{n-m}A(F_m)) + \frac{\partial}{\partial z} (F_{n-m}B(F_m)) \right) \right] \tag{3.45}$$
This expansion can be combined directly with the expansion of the Boltzmann equation.

3.3.6 Solution Approach

To obtain a time dependent solution to the Boltzmann equation with the Fokker-Planck collision term a numerical technique is necessary. The solution presented has been limited to ion-ion collisions because these collisions represent the majority of the collisional energy and momentum transfer within the presheath. The positive signs in the Boltzmann equation must be applied for ion-ion collisions. The ratio of particle mass to reduced mass, $\frac{m}{\mu}$, must be equal to two for like particles. The ratio of the inverse squares of the particle thermal velocities, $\frac{1}{v_i^2}$, must be one for like particle collisions. The nondimensional Boltzmann equation reduces to the following form.

$$\frac{\partial F_i}{\partial t} - \frac{\partial^2}{\partial Z^2} \left[ F_i A(F_0) \right] - \frac{\partial}{\partial Z} \left[ F_i B(F_0) \right] + \varphi_0 i Z F_i + \varphi_i \frac{\partial F_i}{\partial Z}$$

$$= - \sum_{m=1}^{i} \varphi_{m-m} \frac{\partial F_{i-m}}{\partial Z} + \sum_{m=1}^{i} \left( \frac{\partial^2}{\partial Z^2} \left[ F_{i-m} A(F_{m}) \right] + \frac{\partial}{\partial Z} \left[ F_{i-m} B(F_{m}) \right] \right)$$

where the summations are taken to be zero if $i = 0$. The functions $A(F_m)$ and $B(F_m)$ can be written:

$$A(F) = \frac{1}{2} \int_{-\infty}^{\infty} F''(Z + \xi) \mid \xi \mid d\xi$$

$$B(F) = - \int_{-\infty}^{\infty} F''(Z + \xi) \mid \xi \mid d\xi$$

The 'i' equations in the expansion are solved to obtain the time dependent particle velocity distribution. The Poisson equation is employed at each time step to obtain the potential structure. The ratio of the higher order equations with respect to the first order equation eliminates the $\eta^{\{\nu_1 + \cdots + \nu_m\}} \frac{1}{\mu_0} \left( \frac{T_e}{T_i} \right)$ term from the Poisson equation. Using this technique each successive component of the potential
structure can be determined from the previous components.

\[
\frac{\int_{-\infty}^{\infty} F_1 \, dZ}{\int_{-\infty}^{\infty} F_0 \, dZ} = -\frac{\varphi_1}{\varphi_0} \left(\frac{2T_i}{T_e}\right) \tag{3.49a}
\]

\[
\frac{\int_{-\infty}^{\infty} F_2 \, dZ}{\int_{-\infty}^{\infty} F_0 \, dZ} = -\frac{\varphi_0}{\varphi_0} \left(\frac{2T_i}{T_e}\right) + \frac{1}{2} \left(\frac{\varphi_1}{\varphi_0}\right)^2 \left(\frac{2T_i}{T_e}\right)^2 \tag{3.49b}
\]

\[
\frac{\int_{-\infty}^{\infty} F_3 \, dZ}{\int_{-\infty}^{\infty} F_0 \, dZ} = -\frac{\varphi_0}{\varphi_0} \left(\frac{2T_i}{T_e}\right) + \frac{\varphi_1}{\varphi_0} \frac{\varphi_2}{\varphi_0} \left(\frac{2T_i}{T_e}\right)^2 \frac{1}{6} \left(\frac{\varphi_1}{\varphi_0}\right)^3 \left(\frac{2T_i}{T_e}\right)^3 \tag{3.49c}
\]

Using these equations the particle velocity distribution and the potential structure of the presheath are determined as a function of time.

Of interest in this study is the point at which there are no returning ions. This is the presheath - sheath interface. This occurs when the net ion flux away from the wall is zero. To calculate this point in the presheath it is necessary to obtain a value for the potential expansion parameter \(\Delta U\) such that when the overall particle distribution is reconstructed from the various terms in the expansion no returning ions are present. Thus, at the critical point of no returning ions the model determines the total particle velocity distribution, the potential structure, and a value for the potential expansion parameter. From the potential structure and the potential expansion parameter the presheath height at the point of no returning ions can be determined (see figure 3.5).
CHAPTER IV

NUMERICAL TECHNIQUE

4.1 Problem Approach

The solution of the Boltzmann equation, as written in equation 3.46, coupled with the Poisson equation (3.49) is the goal of the numerical procedure.

The general approach is to solve the Boltzmann equation for the particle velocity distribution using a partially implicit, partially explicit scheme. Each step in time the Boltzmann equation is solved using some results from the previous time step. In equation 3.46 the left hand side is solved implicitly while the right hand side is solved explicitly.

\[
\frac{\partial F_i}{\partial t} - \frac{\partial^2}{\partial z^2} \left( F_i A(F_0) \right) - \frac{\partial}{\partial z} \left( F_i B(F_0) \right) + \varphi_0 z F_i + \varphi^{-1} \frac{\partial F_i}{\partial z} = \sum_{m=1}^{i} \varphi_m \frac{\partial F_{i-m}}{\partial z} + \sum_{m=1}^{i} \left( \frac{\partial^2}{\partial z^2} \left( F_{i-m} A(F_m) \right) + \frac{\partial}{\partial z} \left( F_{i-m} B(F_m) \right) \right) \quad (3.46)
\]

The left hand side of this equation can be put in a matrix form.

\[
\begin{bmatrix}
T(\tau)
\end{bmatrix}
\begin{bmatrix}
F_i(\tau + \Delta\tau)
\end{bmatrix}
\]

(4.1)

In this form the matrix \( T(\tau) \) is an \( m \times m \) matrix created from the left hand side of the Boltzmann equation. The quantity \( m \) is the number of divisions in the velocity space \( Z \) chosen for the numerical scheme. The matrix \( T(\tau) \) is computed from \( A(F_0(\tau)) \) and \( B(F_0(\tau)) \). The values of these derived functions are taken from the solution to the particle velocity distribution at the previous time step, \( \tau \). Numerical derivatives are used to represent the partial derivatives in the equation. This procedure produces a diagonal matrix where all elements except those on an odd number of centered
diagonals are zero. The number of diagonals reflects the order of accuracy in the solution. Diagonal matrices of this form are easily and quickly inverted. The \( F_i(r + \Delta r) \) matrix is an \( m \times 1 \) matrix of unknowns that represents the particle velocity distribution at the current time step. 'i' equations of this form can be written corresponding to the number of terms in the expansion.

The right hand side of the Boltzmann equation can also be put in a matrix form.

\[
\varphi_1 \begin{bmatrix} v_1^i(r) \end{bmatrix} + \cdots + \varphi_n \begin{bmatrix} v_n^i(r) \end{bmatrix} + \begin{bmatrix} d_i(r) \end{bmatrix} \tag{4.2}
\]

The scalar \( \varphi_n \) values are unknowns and represent the nondimensional coefficients in the asymptotic potential structure of the presheath.

The \( v_i^o(r) \) matrices are \( m \times 1 \) matrices which are comprised of the partial derivatives of the velocity distribution at the previous time step, \( r \). They represent the first summation on the right hand side of the Boltzmann equation.

\[
\sum_{m=1}^{i} \varphi_m \left( -m \frac{\partial F_{i-m}}{\partial z} \right)
\]

The \( d_i \) matrix is an \( m \times 1 \) matrix comprised of the second summation on the right hand side of the Boltzmann equation.

\[
\sum_{m=1}^{i} \left( \frac{\partial^2}{\partial z^2} \left( F_i - m A(F_m) \right) + \frac{\partial}{\partial z} \left( F_{i-m} B(F_m) \right) \right)
\]

All values of the distribution and the functions \( A(F_m) \) and \( B(F_m) \) are taken at the previous time step, \( r \).

Putting together equations 4.1 and 4.2 a matrix form of the Boltzmann equation is created that can be solved for the particle velocity distribution.

\[
\begin{bmatrix} T(r) \\ F_i(r + \Delta r) \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \cdots \\ \varphi_n \end{bmatrix} \begin{bmatrix} v_1^i(r) \\ \cdots \\ v_n^i(r) \end{bmatrix} + \begin{bmatrix} d_i(r) \end{bmatrix} \tag{4.3}
\]
't' equations of this form can be written corresponding to the number of terms in
the asymptotic expansion being used.

These equations are quickly inverted to obtain the particle velocity distribution
at the current time step.

\[
\begin{bmatrix}
F_i(r + \Delta r)
\end{bmatrix} = \varphi_1 \begin{bmatrix} V_i^1(r) \end{bmatrix} + \cdots + \varphi_n \begin{bmatrix} V_i^n(r) \end{bmatrix} + \begin{bmatrix} D_i(r) \end{bmatrix}
\] (4.4)

Where the \( V_n \) and \( D \) matrices represent the \( m \times 1 \) solution matrix to the inversion
of the \( T(r) \) matrix with the corresponding \( v_n \) or \( d \) matrix.

The particle velocity distribution is obtained from this equation using the \( \varphi_n \)
values from the previous time step.

Equation 3.49 is employed to obtain the \( \varphi_n \) values at the current time step.

\[
\begin{align*}
\int_{-\infty}^{\infty} F_1 dZ &= -\frac{\varphi_1}{\varphi_0} \left( \frac{2T_i}{T_e} \right) \\
\int_{-\infty}^{\infty} F_2 dZ &= -\frac{\varphi_2}{\varphi_0} \left( \frac{2T_i}{T_e} \right) + \frac{1}{2} \left( \frac{\varphi_1}{\varphi_0} \right)^2 \left( \frac{2T_i}{T_e} \right)^2 \\
\int_{-\infty}^{\infty} F_3 dZ &= -\frac{\varphi_3}{\varphi_0} \left( \frac{2T_i}{T_e} \right) + \frac{\varphi_1 \varphi_2}{\varphi_0 \varphi_0} \left( \frac{2T_i}{T_e} \right)^2 - \frac{1}{6} \left( \frac{\varphi_1}{\varphi_0} \right)^3 \left( \frac{2T_i}{T_e} \right)^3
\end{align*}
\] (3.49a)(3.49b)(3.49c)

The new distributions are integrated numerically and the \( \varphi_1 \) through \( \varphi_n \) scalars
are determined consecutively. The value of \( \varphi_0 \) is input and is a nondimensional
representation of the coefficient \( \beta \) in the potential expansion parameter, \( \Delta U \).

For each time step, the overall particle velocity distribution can be determined
at any location from the original expansion once it has been nondimensionalized.

\[
F(Z, \Delta U^*) = F_0(Z) + \Delta U^* F_1(Z) + \Delta U^{*2} F_2(Z) + \cdots
\] (4.5)

since,

\[
\Delta U^* = e^{\varphi_0 \Delta z^*}
\] (4.6)
Where the • quantities are nondimensional. \( x \) has been nondimensionalized with respect to an ion mean free path.

\[
x^* = \frac{x}{n_R \Gamma}
\]

The potential structure of the presheath is determined from the nondimensional form of the original expansion of potential.

\[
U^* = U_0^* + \varphi_1 \Delta U^* + \varphi_2 \Delta U^{*2} + \cdots
\]  
(4.7)

Using this procedure the time dependent evolution of the presheath is obtained.

The point of no returning ions occurs where the integral of the left half plane of the total particle velocity distribution is zero.

\[
0 = \int_{-\infty}^{0} F(Z) dZ
\]  
(4.8)

This equation can be rewritten using the expansion of the particle distribution.

\[
0 = \int_{-\infty}^{0} F_0(Z) dZ + \Delta U^* \int_{-\infty}^{0} F_1(Z) dZ + \Delta U^{*2} \int_{-\infty}^{0} F_2(Z) dZ + \cdots
\]  
(4.9)

Equation 4.9 can be solved for \( \Delta U^* \) Since the particle distributions are now known as a function of time and velocity. The entire solution at the point of no returning ions is known with this last piece of information.

4.2 Numerical Integration, Differentiation, and Matrix Inversion

In order to obtain a solution to the potential structure and particle distribution in the presheath it is necessary to develop the applicable mathematical tools. The primary techniques needed are integration, differentiation, and matrix inversion.
4.2.1 Numerical Integration

Throughout the solution integration is computed using a Simpson's \( \frac{1}{3} \) rule technique\[12\].

\[
\int_a^b f(x) \, dx = \frac{h}{3} \left( f_1 + 4f_2 + 2f_3 + 4f_4 + 2f_5 + \cdots + 2f_{n-1} + 4f_n + f_{n+1} \right) \tag{4.10}
\]

Where \( h \) is the spacing between the points and \( f_1 \) through \( f_{n+1} \) represent the function values at each point. This procedure has a global error of \( O(h^4) \). If the step size is chosen appropriately this procedure is very accurate.

4.2.2 Numerical Differentiation

The technique for determining numerical differentiation is a second order accurate scheme. This reduces the number of computations while maintaining high accuracy. Second order accurate numerical differentiation requires that only three points be known. Thus, the \( 'T(r)' \) matrix contains only three diagonals. If third order accuracy was used the \( 'T(r)' \) matrix would require five diagonals to represent the five points needed for the differentiation. In addition, to maintain uniformity a central difference technique is desirable on as many points as possible. The greater the number of points needed for each derivative the more points that require forward or backward difference techniques ( rather than the central difference technique).

Below is a list of the techniques used to obtain derivatives\[12\].

Central Difference

\[
\frac{\partial F}{\partial x} = \frac{F(x + 1) - F(x - 1)}{2h} \quad \tag{4.11a}
\]

\[
\frac{\partial^2 F}{\partial x^2} = \frac{F(x + 1) - 2F(x) + F(x - 1)}{h^2} \quad \tag{4.11b}
\]

\[
\frac{\partial^3 F}{\partial x^3} = \frac{F(x + 2) - 2F(x + 1) + 2F(x - 1) - F(x - 2)}{2h^3} \quad \tag{4.11c}
\]
Forward Difference

\[
\frac{\partial F}{\partial x} = \frac{-F(z + 2) + 4F(z + 1) - 3F(z)}{2h} \tag{4.12a}
\]

\[
\frac{\partial^2 F}{\partial x^2} = \frac{F(z + 2) - 2F(z + 1) + F(z)}{h^2} \tag{4.12b}
\]

\[
\frac{\partial^3 F}{\partial x^3} = \frac{F(z + 3) - 3F(z + 2) + 3F(z + 1) - F(z)}{h^3} \tag{4.12c}
\]

Backward Difference

\[
\frac{\partial F}{\partial x} = \frac{3F(z) - 4F(z - 1) + 3F(z - 2)}{2h} \tag{4.13a}
\]

\[
\frac{\partial^2 F}{\partial x^2} = \frac{F(z) - 2F(z - 1) + F(z - 2)}{h^2} \tag{4.13b}
\]

\[
\frac{\partial^3 F}{\partial x^3} = \frac{F(z) - 3F(z - 1) + 3F(z - 2) - F(z - 3)}{h^3} \tag{4.13c}
\]

Where \( h \) is the grid spacing. The derivatives are being taken about point \( z \).

It is worth noting that the third derivative equations require up to five points. There is no second order accurate numerical third derivative representation. These equations are third order accurate. This does not affect the 'T(r)' matrix in that it contains no third derivatives. The solution procedure requires third derivatives only in the determination of the function \( B(F_i(r)) \).

These equations are used throughout the solution for derivatives with respect to velocity, \( Z \), and time, \( r \).

4.2.3 Matrix Inversion

In order to obtain a solution a procedure for inverting a diagonal matrix is necessary. The procedure used will invert any centered diagonal matrix. For the second order accurate case the matrix in question is tridiagonal. The procedure uses Guassian elimination on all terms below the center diagonal and then through back substitution determines the solution vector. This technique can quickly invert a \( 200 \times 200 \) tridiagonal matrix.
4.3 Obtaining a Solution

As in any numerical model certain restraints and conditions must be met to obtain an accurate solution. This model requires some form of input distribution function and in order to obtain higher order terms must also have a perturbation applied to the potential structure. In addition, certain numerical techniques have been used to remove instabilities in the model.

4.3.1 Initial Distribution

To model the presheath region an initial particle velocity distribution that conforms to a Maxwellian profile has been used. This profile represents the distribution that naturally occurs in the neutral plasma region. The idea is that the time dependent evolution of the distribution will change from a Maxwellian at time zero to a shifted new form as the presheath is entered. The Maxwellian profile is initially given to the zero order term having set the initial conditions of all higher order terms to zero.

If the potential structure of the presheath is not perturbed in some manner then the model represents the neutral plasma region and the particle velocity distribution remains Maxwellian (as it should). If, however, a small perturbation in the potential structure is added (i.e. a nonzero \( a_1, a_2 \cdots \)) then the model readjusts to describe the presheath region. In this manner the model is used to give the time dependent evolution of the presheath.

4.3.2 Instability Damping

By the nature of the implicit - explicit technique being employed certain numerical problems are expected to appear. This model is no exception. Two techniques have been used to remove these instabilities.
The most important thing to do to avoid numerical problems in a scheme of this nature is to ensure that as much as possible of the solution is computed implicitly. In addition, once some new data has been calculated it should be applied to any new calculations immediately.

In this model each term in the expansion of the particle distribution function uses the new data already determined in calculating all of the lower order terms. Once \( F_0 \) is determined that information is used in calculating \( F_1 \). This idea is repeated for the higher order terms.

A second method applied to the model to eliminate oscillatory instabilities that start on a very small scale and grow is the application of a very weak averaging scheme to the particle distribution functions. Each point in the distribution is weakly averaged with the points on either side.

\[
F(Z) = \frac{0.025F(Z + 1) + F(Z) + 0.025F(Z - 1)}{1.05}
\]  

(4.14)

This technique, although necessary, has the negative effect of falsely increasing the energy in the system by spreading the distribution slightly (see figure 5.1). The change is very small and can be considered insignificant with respect to the overall solution.

4.4 Program Structure

The entire program has been written in FORTRAN and can be run on either an IBM PC AT or on the CYBER mainframe. The code has been written in a segmented manner that easily allows one section to be altered without having to alter other sections. The overall structure of the program consists of three initialization programs, three input data files, the main program, and three output data files. The main program contains a driver and seventeen subroutines. Several of the sub-
routines perform operations that are used throughout the main code. Figure 4.1 is a diagram of the structure of the program. The flexibility of the code is derived from the generalized subroutine structure and the ability to enter a variety of input variables. The driver keeps track of time and maintains the overall operating structure of the solution while the subroutines perform the necessary manipulations. Below is a list of the function of each program, data file and subroutine.

AVE - Subroutine to smooth distributions by averaging.
CONSERV - Subroutine to determine conservation of energy, momentum, and particles.
CONSOUT - Conservation output data file.
CRF - Particle distribution initialization program.
CRPHI - Potential structure initialization program.
DENSITY - Subroutine to solve for a new presheath structure.
FDATA - Initial particle distribution data file.
FD1 - Subroutine to find first derivatives.
FD2 - Subroutine to find second derivatives.
FD3 - Subroutine to find third derivatives.
FINDA - Subroutine to determine 'A' function.
FINDB - Subroutine to determine 'B' function.
FPINIT - Primary initialization program.
FPOUT - Output particle distribution data file.
FPSHETH - Main program driver.
GETAB - Subroutine to make A and B function vectors.
INITDAT - Initialization data file.
MAKED - Subroutine to make d matrix.
MAKEF - Subroutine to read initial particle distribution.
MAKEPHI - Subroutine to read initial potential structure.
MAKET - Subroutine to make $T$ matrix.
MAKEV - Subroutine to make $v$ matrix.
MODIAG - Subroutine to invert diagonal matrices.
PHIDAT - Initial potential structure data file.
PHIDOUT - Output potential structure data file.
SIMPS - Subroutine to perform Simpson's rule integration.
TOT - Subroutine to obtain total distribution at point of no returning ions.
CHAPTER V

RESULTS

The ratio of ion temperature to electron temperature, $\frac{T_i}{T_e}$, has been set to one half throughout these results. There is little effect on the particle distribution or potential structure if the temperature ratio is changed to other values. The electron temperature is expected to be higher than the ion temperature in the presheath since electrons absorb energy from electric and magnetic fields faster than ions and other large particles.

Through repeated test runs of the model it was found that fifty-one points in velocity space were enough to provide high accuracy and produce good results. The range of points in velocity space has been truncated to $\pm 5$ nondimensional units. The results show that at $\pm 5$ the distribution is near zero, substantiating the truncation.

A time step of 0.2 nondimensional times was found to keep the solution accurate. Three nondimensional units in time were sufficient to produce stable results.

It was found that the magnitude of the higher order terms in the particle velocity distribution drop off very rapidly. Thus, the higher order terms have very little impact on the shape of the potential or of the particle distribution.

To understand the effects of a quiescent plasma interacting with a surface the potential gradient in the neutral plasma has been set to zero. To accomplish this the $a$ term in the expansion of potential has been set to zero.

$$U = U_0 + a_1 \Delta U + a_2 \Delta U^2 + \cdots$$  \hspace{1cm} (3.13)
where

\[ U_0 = \alpha x \quad \text{and} \quad \Delta U = e^{bs} \quad (3.14) \]

A stable solution exists only for a specific critical value of the exponential coefficient, \( b \), which represents the scale of the presheath. The quantity \( b \) is nondimensionalized as \( \varphi_0 = \frac{e}{nR} \), where \( nR \) is an ion mean free path. It is expected, as shown in section 3.2, that the critical value should be on the order of a mean free path. It was found that \( \frac{e}{nR} = 0.4 \) produces the most nearly stable results. The distributions become unstable for values greater or less than 0.4. The small remaining instability at \( \frac{e}{nR} = 0.4 \) can be attributed to the inexact nature of the numerical solution.

The results presented here are first order and produce a complete picture of the structure of the presheath because the higher order terms collective contribution is more than an order of magnitude smaller. Figures 5.1 and 5.2 are plots of the zeroth and first order expansions of the particle distribution in velocity space. The zero order term remains Maxwellian because the potential gradient in the neutral plasma is zero. The first order term of the distribution obtains a profile that has roughly the shape (but not magnitude) of the negative first derivative of the zeroth order solution. The potential expansion parameter at the point of no returning ions is determined for each time step. Using the particle distribution functions and the known potential expansion parameter together produce the overall particle velocity distribution at the point of no returning ions, the presheath - sheath interface. Figure 5.3 shows this distribution.

The positive shift in the total distribution is as expected for the presheath. The ions are being pulled into the wall. The particle distribution for velocities away from the wall is zero for the case of no returning ions. The point of no returning ions exists where the particle distribution for velocities away from the wall integrates
to zero. Figure 5.3 shows that the ion distribution becomes negative for velocities away from the wall. A negative particle distribution cannot exist physically. The addition of higher order terms does not correct the problem because the expansion drops off so quickly that any higher order terms have no impact on the shape of the distribution. The problem is fundamental to the type of collision term being applied in the model. The Fokker-Planck term only models the limit of small angle collisions. However, large angle collisions become important in the presheath.

The first term of the nondimensionalized potential structure, $\varphi_1$, has been initially perturbed to $1.0 \times 10^{-4}$ to obtain the results presented in figures 5.1, 5.2 and 5.3. Perturbing the potential structure provides the model with the nonequilibrium condition necessary to initiate the time dependent development of the presheath. The strength of the initial perturbation is not significant to obtaining an accurate particle distribution and potential structure of the presheath. Figures 5.4, 5.5, and 5.6 are the result of an initial perturbation of $1.0 \times 10^{-3}$ and figures 5.7, 5.8 and 5.9 are the result of an initial perturbation of $1.0 \times 10^{-5}$. Comparing these results show that the magnitude of the initial perturbation only affects the scale of the first order term and has no effect on the overall particle distribution in velocity space.

Figure 5.10 is a plot of the position of the point of no returning ions, the presheath-sheath interface, as a function of time for the three solutions. Since no source of ions exists in the model the relative position of the plasma with respect to the surface changes as a function of time. The wall is moving into the plasma, or the plasma is moving into the wall, at the rate at which the wall is absorbing ions. The three solutions have different magnitudes but follow the same profile. The strength of the perturbation controls the relative position of the zero point.

Figure 5.11 is a plot of the potential structure of the presheath obtained from
the three solutions as a function of position. The data for the potential structure has been taken from the solution at a nondimensional time of two. Note that the affect of the different initial perturbation values is to cause a shift in the relative position of the potential but has no effect on the shape of the potential or on the strength of the potential at the point of no returning ions. Changing the perturbation strength alters the location of the zero point but not its shape. The stronger the perturbation the further the zero point is moved from the surface. The horizontal line in the plot depicts the presheath height at the point of no returning ions. The vertical lines show the position of the point of no returning ions.

A time dependent plot of the presheath height at the point of no returning ions is presented in figure 5.12. This plot shows that the time evolution of the sheath height approaches smoothly to a nearly constant value of 0.16. All three solutions fall on the same curve. This shows that the strength of the perturbation does not affect the results obtained.
CHAPTER VI

CONCLUSIONS

The solution obtained is an accurate representation of the time dependent development of the Fokker-Planck presheath. The model produces a precise potential structure, however, the distribution of returning ions breaks down in the presheath. An oscillation develops in the negative tail of the distribution, as seen in figures 5.3, 5.6, and 5.9. This oscillation cannot be removed by including additional terms to the expansion. In addition, the sheath height of 0.16 determined at the point of no returning ions is roughly an order of magnitude smaller than expected. Both of these conditions lead to the conclusion that the Fokker-Planck collision term does not represent the type of collisions that remove the returning ions in the presheath. This breakdown is due to the failure of the Fokker-Planck collision term to model the large angle collisions that take place within the presheath. The Fokker-Planck term is effective at modeling the collisions present in the center of the plasma but breaks down in the presheath. The primary mechanism behind clearing out the returning ions from within the presheath is not particle diffusion as represented by small angle deflections but rather the large velocity changes caused by large angle collisions. Since the Fokker-Planck term models particle collisions that represent the limit of small angle collisions it is inadequate at describing the mechanisms controlling the ion velocity distribution moving away from the wall. The solutions obtained using a Maxwellian distribution by Bissell and Johnson\[4\] and Emmert et al.[5] and those obtained using a charge exchange collision model by Riemann\[6\] and Main\[7\] effectively include the large angle collisions since they model the collisions
by instantaneous changes in particle velocity and position. These collision models
do not represent the Coulomb collisions taking place in a fully ionized plasma. They
do not represent the collision processes but only approximate the collisional effects.

This Fokker - Planck presheath model produces a self-consistent and precise
potential structure. The particle velocity distribution in the presheath has the
correct acceleration of ions toward the wall but because the Fokker - Planck collision
term only models the limit of small angle collisions it is unable to clear the particle
distribution of returning ions. The effect of not modeling the large angle collisions
is that the particle distribution for returning ions is accurate only in the initial
section of the presheath where the collisional processes are dominated by particle
diffusion. The collisional processes become dominated by the effects of the large
angle collisions as the interface between the presheath and the Debye sheath is
approached. Only by including a collision term which accounts for these large
angle collisions can a presheath model produce a particle velocity distribution that
accurately models the condition of no returning ions. This study has found that a
presheath model which describes the Coulomb collisions occurring in a fully ionized
plasma must account for both the small angle collisions and the large angle collisions.
REFERENCES


FIGURES

Figure 3.1 Debye Shielding

Figure 3.2 Wall Potential
Figure 3.3 Wall Region Model
Figure 3.4 Coordinate System in Velocity Space
Maxwellian Radial Velocity Distribution

$T = \text{Constant}$

Neutral Plasma Distribution
Near Presheath Distribution
Distribution at the Point of no returning ions

Figure 3.5 Radial, Symmetric, and No Returning Ions Conditions
Figure 4.1 Program Diagram
Nondimensional Axial Velocity (Z)

Figure 5.1 Zero Order Particle Velocity Distribution. $\phi_r = 1.0 \times 10^{-4}$
Figure 5.2 First Order Particle Velocity Distribution, \( \phi_1 = 1.0 \times 10^{-4} \)
Figure 5.3 Total Particle Velocity Distribution, $\varphi_1 = 1.0 \times 10^{-4}$
Figure 5.4 Zero Order Particle Velocity Distribution, $\varphi_1 = 1.0 \times 10^{-3}$
Figure 5.5 First Order Particle Velocity Distribution, $\phi_1 = 1.0 \times 10^{-3}$
Figure 5.6 Total Particle Velocity Distribution, $\varphi_1 = 1.0 \times 10^{-3}$
Figure 5.7 Zero Order Particle Velocity Distribution, \( \varphi_1 = 1.0 \times 10^{-5} \)
Figure 5.8 First Order Particle Velocity Distribution, $\varphi_1 = 1.0 \times 10^{-6}$
Figure 5.9 Total Particle Velocity Distribution, $\varphi_1 = 1.0 \times 10^{-5}$
Figure 5.10 Time Dependent Position of the Potential

Initial Perturbation

- \( \psi = 1.0 \times 10^{-3} \)
- \( \psi = 1.0 \times 10^{-4} \)
- \( \psi = 1.0 \times 10^{-5} \)
Sheath Height At the Point of No Returning Ions

Figure 5.11 Potential Structure of the Presheath
Figure 5.12 Time Dependent Structure of the Sheath Height
APPENDIX: PROGRAM LISTING

* THIS PROGRAM WAS WRITTEN BY JEFFREY P. DANSEREAU

* THIS VERSION WAS LAST UPDATED ON 7/17/87

T - DIAGONAL REPRESENTATION OF T MATRIX. 1ST IS DIAGONALS.
2ND IS M VEL POS.

V - 3-D MATRIX OF V COMPONENTS. 1ST POS. IS F'S, 2ND IS
N PHI POS., 3RD IS M VEL POS.

VV - 3-D MATRIX OF V VALUES AFTER INVERSION WITH T MATRIX.
POS. ARE SAME AS V MATRIX WITH ADDITIONAL ROW FOR BC'S.

D - 2-D MATRIX OF D VALUES. 1ST POS. IS F'S, 2ND IS M VEL POS.

DD - 2-D MATRIX OF D VALUES AFTER INVERSION WITH T MATRIX

PHI - VECTOR OF PHI VALUES

F - 2-D MATRIX OF DENSITY FUNCTIONS, 1ST POS. IS THE n F'S
THAT ARE BEING USED. (n=N-1). F(1,X)= F0 ECT... THE
2ND POS IS THE M VEL POS.

TSTEP - VALUE OF DELTA T AS TIME IS STEPPED THROUGH

VSTEP - VALUE OF DELTA V AS VEL. SPACE IS STEPPED THROUGH

RTIME - CURRENT VALUE OF NON-DIM. TIME

ETA - CURRENT VEL.

NETA - ETA PARAMETER IN POISSON EQN

TOTE - RATIO OF T TO Te(T OVER Te)

SM - SMALL M(m) IN F-P EQN

BM - BIG M(M) IN F-P EQN

AA - a IN F-P EQN

AP - a' IN F-P EQN

M - NUMBER OF DIV. IN VEL. SPACE

N - NUMBER OF PHI VALUES

ND - NUMBER OF DIAGONALS IN T MATRIX
C TIME - INTEGER VALUE IN TIME LOOP
C TEND - RTIME TO FINISH SIMULATION
C L - INTEGER TIME VALUE TO END SIMULATION
C FLAG1 - FLAG TO PRINT OR NOT PRINT MATRICES(99 TO PRINT)
C FLAG2 - FLAG TO PRINT OR NOT PRINT INTERMEDIATE MATRICES
C (99 TO PRINT)
C TRBLE - FLAG TO PRINT TROUBLE STATEMENT IF MATRIX IS NOT
C INVERTABLE
C B,SOLN - INTERMEDIATE VALUES OF VARIOUS FUNCTIONS
C X,Y,Z - INTEGER COUNTERS
C
C THIS PROGRAM STEPS THROUGH TIME SOLVING THE BOLTZMANN EQN FOR
C VALUES OF THE EXPANDED DENSITY FUNCTION
C
REAL T(6,8,202),V(6,6,202),B(202),SOLN(202),TEND
REAL VV(6,6,202),D(6,202),DD(6,202),PHI(6),L
REAL F(6,202),TSTEP,RTIME,SM,BM,AA,AP,VSTEP,ETA,NETA,TOTE
REAL FTOTAL(202),SO,DU,DUPH1
INTEGER X,Y,Z,M,N,ND,TIME,TRBLE,FLAG1,FLAG2,FLAG3,P,PSTEP,SKIP
C
C READ IN PARAMETERS AND PRINT THEM
C
OPEN(UNIT=2,FILE='INITDAT.DAT',STATUS='OLD')
READ(2,705) M,NETA,TOTE,AA,AP,SM,FLAG3,SKIP,S0
READ(2,707) BM,VSTEP,N,ND,TEND,TSTEP,FLAG1,FLAG2
CLOSE(UNIT=2)
705 FORMAT(I4,1X,F6.4,1X,F6.4,1X,F6.4,1X,F6.4,1X,F6.4,1X,F6.4,1X,F6.4,1X,F13,1X,
       /)
707 FORMAT(I4,1X,F6.4,1X,F6.4,1X,F6.4,1X,F6.4,1X,F6.4,1X,F6.4,1X,F6.4,1X,F6.4,1X,F13,1X)
OPEN(UNIT=1,FILE='PHI.OUT',STATUS='UNKNOWN')
OPEN(UNIT=3,FILE='FPSHETH.OUT',STATUS='UNKNOWN')
WRITE(3,717)'******************************************************************************
+******
717 FORMAT(A,A)
WRITE(3,711)'FOKKER - PLANCK SIMULATION OUTPUT'
WRITE(3,715)'******************************************************************************
+******
WRITE(3,710)'INPUT PARAMETERS'
WRITE(3,715)'NUMBER OF VEL. STEPS - M',M
WRITE(3,715)'NUMBER OF PHI VALUES - N',N
WRITE(3,715) 'NUMBER OF DIAGNALS IN T MATRIX - ND',ND
WRITE(3,720) 'VALUE OF ETA IN POISSON EQN - NETA',NETA
WRITE(3,720) 'VALUE OF T OVER TE - TOTE',TOTE
WRITE(3,720) 'SIZE OF EACH VEL. STEP - VSTEP',VSTEP
WRITE(3,720) 'SIZE OF EACH TIME STEP - TSTEP',TSTEP
WRITE(3,720) 'VALUE OF ENDING TIME FOR SIMULATION',TEND
WRITE(3,720) 'VALUE OF A PARAMETER IN F-P EQN - AA',AA
WRITE(3,720) 'VALUE OF A PRIME PARAMETER IN F-P EQN - AP',AP
WRITE(3,720) 'VALUE OF SMALL M IN F-P EQN - SM',SM
WRITE(3,720) 'VALUE OF BIG M IN F-P EQN - BM',BM
WRITE(3,716) 'FLAG TO PRINT PRIMARY MATRICES(99 TO PRINT)',
     + ' - FLAGI',FLAGI
WRITE(3,716) 'FLAG MAKE FDATA AND PHIDAT NEW FINAL VALUES',
     + ' (99 = YES) - FLAG3',FLAG3
WRITE(3,715) 'TIME SKIP FOR PRINT OF F FILE - SKIP',SKIP
716 FORMAT(5X,A,A,2X,I3)
WRITE(3,710) 
710 FORMAT(15X,A,//)
711 FORMAT(8X,A,/) 
715 FORMAT(5X,A,2X,I3)
720 FORMAT(5X,A,2X,F9.5)
C
C MAKE NON TIME DEPENDENT QUANTITIES AND INITIAL APPROXIMATIONS
C TO F'S, PHI'S, AN THE BC
C
DU=0.0
DUPHI1=0.0
CALL MAKEF(M,F)
CALL MAKEPHI(N,RTIME,PHI)
CALL TOT(F,FTOTAL,PHI,N,M,VSTEP,DU,DUPHI1)
TT=0.0
CALL CONSERV(F,VSTEP,TT,M)
C
C ******************************************************************************
C
C START MAIN TIME LOOP
C
C ******************************************************************************
C

C COMPUTE END LOOP TIME L

L=TEND/TSTEP
PSTEP=0

DO 40 TIME=1,INT(L)+1

C PRINT RESULTS OF CURRENT TIME STEP

PSTEP=PSTEP+1
IF (TIME.LE.3) GOTO 1000
IF (PSTEP.GE.SKIP) THEN
   PSTEP=0
   GOTO 1000
ENDIF
GOTO 1001

1000 WRITE(3,350) 'TIME = ',RTIME,'TSTEP = ',TIME-1
   DO 210 Z=1,N
      WRITE(3,250) 'PHI(',Z-2,') = ',PHI(Z)
   210 CONTINUE
   WRITE(3,351) 'DU = ',DU,'DU*PHI(1) = ',DUPHI1
   WRITE(3,711) 
   WRITE(3,275) 'M','ETA','F0','F1','F2','F3','FTOTAL'
   ETA=-5.0
   DO 220 X=1,M
      WRITE(3,300) X,ETA,F(1,X),F(2,X),F(3,X),FTOTAL(X)
      ETA=ETA+VSTEP
   220 CONTINUE

1001 WRITE(3,710) 

IF (TIME.GT.INT(L)) GOTO 40
WRITE(*,946) 'CURRENTLY IN TIME STEP',TIME,'OF',INT(L)

946 FORMAT(5X,A,I4,2X,A,I4)

C MAKE TIME DEPENDENT QUANTITIES T MATRIX, V MATRIX, D MATRIX

DO 445 JJ=1,2
   CALL MAKET(ND,M,F,VSTEP,SM,BM,AA,AP,TSTEP,SO,PHI,N,T)
   CALL MAKEV(N,M,F,VSTEP,V)
CALL Maked(M,F,Vstep, Tstep, SM, BM, AA, AP, N, PHI, D)

C

INVERT T MATRIX WITH V VECTORS, MAKING VV MATRIX

C

DO 50 X=1,N-1
   DO 60 Y=3,N
      DO 70 Z=1,M
         B(Z)=V(X,Y,Z)
      70 CONTINUE
   60 CONTINUE
50 CONTINUE

C

INVERT T MATRIX WITH D VECTORS, MAKING DD MATRIX

C

DO 90 X=1,N-1
   DO 100 Y=1,M
      B(Y)=D(X,Y)
100 CONTINUE

C

GET NEW F VALUES FROM NEW PHI VALUES

C

DO 160 X=1,N-1
   IF ((JJ.EQ.1).AND.(X.EQ.2)) GOTO 160
IF (JJ.EQ.2).AND.(X.EQ.1)) GOTO 160
DO 165 Z=1,M
   F(X,Z)=0.0
165 CONTINUE
DO 170 Y=3,N
   DO 180 Z=1,M
      F(X,Z)=F(X,Z)+PHI(Y)*VV(X,Y,Z)
180 CONTINUE
170 CONTINUE
DO 190 Z=1,M
   F(X,Z)=F(X,Z)+DD(X,Z)
190 CONTINUE
160 CONTINUE
CALL AVE(F,M,N,JJ)
C
445 CONTINUE
C
C GET NEW PHI VALUES
C
CALL DENSITY(N,M,PHI,F,TOTE,VSTEP)
C
C MAKE TOTAL DENSITY
C
CALL TOT(F,FTOTAL,PHI,N,M,VSTEP,DU,DUPHI1)
C
TT=RTIME+TSTEP
CALL CONSERV(F,VSTEP,TT,M)
C
C INCREASE REAL TIME TO NEXT POSITION
C
RTIME=RTIME+TSTEP
C
C CONTINUE TIME LOOP
C
40 CONTINUE
C
********************************************************************************
C
C END TIME LOOP
C
C
********************************************************************************
FORMAT STATEMENTS FOR PRINTS

250 FORMAT(5X,A,I4,A,1X,E13.6)
275 FORMAT(2X,A,2X,A,9X,A,12X,A,12X,A,12X,A,10X,A)
300 FORMAT(3I1X,F5.2,1X,E13.6,1X,E13.6,1X,E13.6,1X,E13.6,1X,E13.6,1X,E12.5)
350 FORMAT(5X,A,F7.4,10X,A,15X,)
351 FORMAT(5X,A,E13.6,5X,A,E13.6)
CLOSE(UNIT=3)
IF (FLAG3.EQ.99) THEN
  OPEN(UNIT=4,FILE='FDATA.DAT',STATUS='UNKNOWN')
  DO 265 X=1,M
    WRITE(4,266) F(1,X),F(2,X),F(3,X),F(4,X)
    266 FORMAT(E13.6,1X,E13.6,1X,E13.6,1X,E13.6)
  CONTINUE
  CLOSE(UNIT=1)
  OPEN(UNIT=8,FILE='PHIDAT.DAT',STATUS='UNKNOWN')
  DO 267 X=1,N
    WRITE(8,268) RTIME
    WRITE(8,268) PHI(X)
    268 FORMAT(E13.6)
  CONTINUE
ENDIF
STOP
END

REAL F(6,202),ETA,VSTEP,NETA,TOTE,B,PI,C
INTEGER X,M,Z
OPEN(UNIT=3,FILE='FDATA.DAT',STATUS='UNKNOWN')
ETA=-5.0
PRINT *, 'INPUT M,VSTEP'
READ(*,*) M,VSTEP
N=M-(M-1)/2
DO 10 X=1,N
  F(1,X)=EXP(-ETA**2))
  F(2,X)=0.0
  F(3,X)=0.0
  F(4,X)=0.0
  ETA=ETA+VSTEP
REAL PHI(6), RTIME
INTEGER N,X
N=5
RTIME=0.0
WRITE(*,*) 'INPUT PHI(-1)'
READ(*,200) PHI(1)
WRITE(*,*) 'INPUT PHI(0)'
READ(*,200) PHI(2)
200 FORMAT(F10.5)
WRITE(*,*) 'INPUT PHI(1)'
READ(*,200) PHI(3)
DO 10 X=4,N
   PHI(X)=0.0
10 CONTINUE
OPEN(UNIT=9,FILE='PHIDAT.DAT',STATUS='UNKNOWN')
WRITE(9,120) RTIME
DO 20 X=1,N
   WRITE(9,120) PHI(X)
20 FORMAT(E13.6)
20 CONTINUE
RETURN
END
SUBROUTINE CONSERV(F, VSTEP, T, M)
REAL F(6,202), F0(202), VSTEP, ZF(202), Z2F(202), ENER, MOM, DEN, T, ETA
INTEGER X, M
ETA = -5.0
DO 10 X = 1, M
    F0(X) = F(1, X)
    ZF(X) = F(1, X) * ETA
    Z2F(X) = ZF(X) * ETA
    ETA = ETA + VSTEP
10 CONTINUE
CALL SIMPS(F0, M, VSTEP, DEN)
CALL SIMPS(ZF, M, VSTEP, MOM)
CALL SIMPS(Z2F, M, VSTEP, ENER)
OPEN(UNIT=9, FILE='CONSERV.OUT', STATUS='UNKNOWN')
WRITE(9, 100) T, DEN, MOM, ENER
100 FORMAT(2X, F8.5, 1X, E13.6, 1X, E13.6, 1X, E13.6)
RETURN
END

SUBROUTINE DENSITY(N, M, PHI, F, TOTE, VSTEP)
REAL F(6,202), PHI(6), TOTE, VSTEP, G(202), R(4), NO(6)
INTEGER M, N, X, Y
DO 10 X = 1, N-1
    DO 20 Y = 1, M
        G(Y) = F(X, Y)
    20 CONTINUE
    CALL SIMPS(G, M, VSTEP, NO(X))
10 CONTINUE
R(1) = NO(2) / NO(1)
R(2) = NO(3) / NO(1)
R(3) = NO(4) / NO(1)
PHI(3) = PHI(2) * R(1) / (2.0 * TOTE)
PHI(4) = PHI(2) * (((PHI(3) / PHI(2))**2)*((2.0*TOTE)**2)/2.0 - R(2)) +
        PHI(5) = PHI(2) * (((2.0*TOTE)**2)*PHI(3)*PHI(4)/(PHI(2)**2)) +
        PHI(3) = PHI(2)**3 * ((PHI(3)/PHI(2))**3)/6.0 - R(3))/(2.0*TOTE)
RETURN
END
SUBROUTINE MAKEF(M,F)
REAL F(6,202)
INTEGER X,M
OPEN(UNIT=8,FILE='FDATA.DAT',STATUS='UNKNOWN')
DO 10 X=1,M
   READ(8,100) F(1,X),F(2,X),F(3,X),F(4,X)
100   FORMAT(E13.6,1X,E13.6,1X,E13.6,1X,E13.6)
10 CONTINUE
RETURN
END

SUBROUTINE MAKEPHI(N,RTIME,PHI)
REAL PHI(6),RTIME
INTEGER N,X
OPEN(UNIT=9,FILE='PHIDAT.DAT',STATUS='UNKNOWN')
READ(9,120) RTIME
DO 20 X=1,N
   READ(9,120) PHI(X)
120   FORMAT(E13.6)
20 CONTINUE
RETURN
END

SUBROUTINE MAKEV(N,M,F,VSTEP,V)
REAL F(6,202),V(6,6,202),VSTEP,FD1(6),Z,ETA
INTEGER N,M,X,Y,R,S
ETA=-5.0
DO 10 X=1,M
   CALL FD1(F,N,M,X,VSTEP,FD1)
   IF ((X.EQ.1).OR.(X.EQ.M)) THEN
      DO 15 Y=1,N-1
         DO 17 R=3,N
            V(Y,R,X)=0.0
17      CONTINUE
15   CONTINUE
   ELSE
      DO 20 Y=1,N-1
108
Z=1.0
S=1
DO 30 R=3,N
   IF (S.GT.(Y-1)) THEN
      V(Y,R,X)=0.0
   ELSE
      V(Y,R,X)=-Z*FD1(Y-R+2)
   ENDIF
   S=S+1
   Z=Z+1.0
30    CONTINUE
20    CONTINUE
ENDIF
ETA=ETA+VSTEP
10 CONTINUE
RETURN
END

SUBROUTINE MAKED(M,F,VSTEP,TSTEP,SM,BM,AA,AP,N,PHI,D)
REAL VSTEP,TSTEP,SM,BM,AA,AP,F(6,202),D(8,202),A(6,202), PHI(6)
REAL FD1(6),FD2(6),AH(202),BH(202),B(6,202)
REAL ETA,GD1,GD2
INTEGER N,M,X,Y,Z
DO 10 X=1,N-1
   CALL GETAB(F,VSTEP,M,SM,BM,AA,AP,AH,BH,X)
   DO 15 Y=1,M
      A(X,Y)=AH(Y)
      B(X,Y)=BH(Y)
15    CONTINUE
10 CONTINUE
DO 17 Y=1,N-1
   D(Y,M)=0.0
   D(Y,1)=0.0
17 CONTINUE
ETA=-5.0+VSTEP
DO 20 X=2,M-1
   DO 30 Y=1,N-1
      D(Y,X)=0.0
   DO 40 Z=1,Y

109
IF (Z.LE.Y-1) THEN
  GD2=(A(Z+1,X+1)*F(Y-Z,X+1)-2.0*
    A(Z+1,X)*F(Y-Z,X)+A(Z+1,X-1)*F(Y-Z,X-1))/(VSTEP**2)
  GD1=(B(Z+1,X+1)*F(Y-Z,X+1)-B(Z+1
    ,X-1)*F(Y-Z,X-1))/(2.0*VSTEP)
  D(Y,X)=D(Y,X)+GD2+GD1
ENDIF
40  CONTINUE
  D(Y,X)=D(Y,X)+F(Y,X)/TSTEP
30  CONTINUE
  ETA=ETA+VSTEP
20  CONTINUE
RETURN
END

SUBROUTINE FD1(F,N,M,X,VSTEP,FD1)
REAL F(6,202),FD1(6),VSTEP
INTEGER Y,X,N,M
DO 10 Y=1,N-1
  IF (X.LE.2) THEN
    FD1(Y)=(-F(Y,X+2)+4.0*F(Y,X+1)-3.0*F(Y,X))/(2.0*VSTEP)
  ELSE IF (X.GE.M-1) THEN
    FD1(Y)=(3.0*F(Y,X)-4.0*F(Y,X-1)+F(Y,X-2))/(2.0*VSTEP)
  ELSE
    FD1(Y)=(F(Y,X+1)-F(Y,X-1))/(2.0*VSTEP)
  ENDIF
10  CONTINUE
RETURN
END

SUBROUTINE MAKET(ND,M,F,VSTEP,SM,BM,AA,AP,TSTEP,SO,PHI,N,T)
REAL VSTEP,TSTEP,SM,BM,AA,AP,F(6,202),T(6,8,202),A(202)
REAL ETA,B(202),PHI(6),P
INTEGER ND,M,X,Y,N,Z
CALL GETAB(F,VSTEP,M,SM,BM,AA,AP,A,B,1)
P=0.0
DO 5 Z=1,N-1

ETA=-5.0
DO 10 X=1,M
   IF ((X.EQ.M).OR.(X.EQ.1)) THEN
      T(Z,2,X)=0.0
      T(Z,3,X)=1.0
      T(Z,4,X)=0.0
   ELSE
      T(Z,2,X)=((-A(X-1)/VSTEP)+B(X-1)/2.0)/VSTEP
      T(Z,2,X)=T(Z,2,X)-PHI(1)/(2.0*VSTEP)
      T(Z,3,X)=1.0/TSTEP+(2.0*A(X)/(VSTEP**2))
      T(Z,3,X)=T(Z,3,X)+PHI(2)*P*ETA
      T(Z,4,X)=((-A(X+1)/VSTEP-B(X+1)/2.0)/VSTEP)
      T(Z,4,X)=T(Z,4,X)+PHI(1)/(2.0*VSTEP)
   ENDIF
   ETA=ETA+VSTEP
10 CONTINUE
P=P+1.0
5 CONTINUE
RETURN
END

SUBROUTINE GETAB(F,VSTEP,M,SM,BM,AA,AP,A,B,Y)
REAL A(202),B(202),AA,AP,F(6,202),VSTEP,Z,SM,BM,E(202)
REAL P(202),H(202),G(202),K(202),DB,DA,S1,S2,S3,S4,S5
INTEGER M,X,Y,N
Z=-5.0
DO 10 X=1,M
   CALL FINDA(F,Z,VSTEP,AA,AP,M,A(X),Y)
   CALL FINDB(F,Z,VSTEP,AA,AP,SM,BM,M,B(X),Y)
   Z=Z+VSTEP
10 CONTINUE
ETA=-5.0
DO 20 X=1,M
   H(X)=B(X)*F(Y,X)
   G(X)=A(X)*F(Y,X)
   K(X)=ETA*B(X)*F(Y,X)
   P(X)=ETA*F(Y,X)
   E(X)=F(Y,X)
   ETA=ETA+VSTEP
20 CONTINUE
CONTINUE
CALL SIMPS(H,M,VSTEP,S1)
CALL SIMPS(G,M,VSTEP,S2)
CALL SIMPS(K,M,VSTEP,S3)
CALL SIMPS(P,M,VSTEP,S4)
CALL SIMPS(E,M,VSTEP,S5)
IF (S5.EQ.0.0) RETURN
DB=-S1/S5
DA=-(S2-S3-DB*S4)/S5
DO 30 X=1,M
   B(X)=B(X)+DB
   A(X)=A(X)+DA
30 CONTINUE
RETURN
END

SUBROUTINE FINDA(F,Z,VSTEP,AA,AP,M,A,Y)
REAL F(6,202),Z,VSTEP,AA,AP,A,ETA,SOLN,H(202),FD2
INTEGER X,M,Y
ETA=-5.0
DO 10 X=1,M
   CALL FD2(F,M,X,VSTEP,FD2,Y)
   H(X)=FD2''ABS(ETA-Z)
   ETA=ETA+VSTEP
10 CONTINUE
CALL SIMPS(H,M,VSTEP,SOLN)
A=(SOLN*AA)/(2.0*AP)
RETURN
END

SUBROUTINE FINDB(F,Z,VSTEP,AA,AP,SM,BM,M,B,Y)
REAL F(6,202),Z,VSTEP,AA,AP,B,ETA,G2,SOLN
REAL J(202),FD3,SM,BM
INTEGER X,M,Y
ETA=-5.0
DO 10 X=1,M
   CALL FD3(F,M,X,VSTEP,FD3,Y)
J(X) = FD3 * ABS(ETA - Z)
ETA = ETA + VSTEP

10 CONTINUE
CALL SIMPS(J, M, VSTEP, SOLN)
B = -(AA/AP) * (SM/(2.0*BM)) * SOLN
RETURN
END

SUBROUTINE SIMPS(F, N, H, RESULT)
REAL F(202), H, RESULT
INTEGER N, NPANEL, NHALF, NBEGIN, NEND
NPANEL = N - 1
NHALF = NPANEL / 2
NBEGIN = 1
RESULT = 0.0
IF (NPANEL - 2*NHALF) .NE. 0 THEN
   RESULT = 3.0 * H / 8.0 * (F(1) + 3.0 * F(2) + 3.0 * F(3) + F(4))
   NBEGIN = 4
   IF (N .EQ. 4) RETURN
ENDIF
RESULT = RESULT + H / 3.0 * (F(NBEGIN) + 4.0 * F(NBEGIN + 1) + F(N))
NBEGIN = NBEGIN + 2
IF (NBEGIN .EQ. 4) RETURN
NEND = N - 2
DO 10 I = NBEGIN, NEND, 2
   RESULT = RESULT + H / 3.0 * (2.0 * F(I) + 4.0 * F(I + 1))
10 CONTINUE
RETURN
END

SUBROUTINE FD2(F, M, X, VSTEP, FD2, Y)
REAL F(6, 202), FD2, VSTEP
INTEGER X, M, Y
IF (X .LE. 1) THEN
   FD2 = (F(Y, X + 2) - 2.0 * F(Y, X + 1) + F(Y, X)) / (VSTEP ** 2)
ELSE IF (X .GE. M) THEN
   FD2 = (F(Y, X) - 2.0 * F(Y, X - 1) + F(Y, X - 2)) / (VSTEP ** 2)
ELSE
ELSE
    FD2=(F(Y,X+1)-2.0*F(Y,X)+F(Y,X-1))/(VSTEP**2)
ENDIF
RETURN
END

SUBROUTINE FD3(F,M,X,VSTEP,FD3,Y)
REAL F(6,202),FD3,VSTEP
INTEGER X,M,Y
IF (X.LE.2) THEN
    FD3=(F(Y,X+3)-3.0*F(Y,X+2)+3.0*F(Y,X+1)-F(Y,X))/(VSTEP**3)
ELSE IF (X.GE.M-1) THEN
    FD3=(F(Y,X)-3.0*F(Y,X-1)+3.0*F(Y,X-2)-F(Y,X-3))/(VSTEP**3)
ELSE
    FD3=(F(Y,X+2)-2.0*F(Y,X+1)+2.0*F(Y,X-1)-F(Y,X-2))/
    + (2.0*(VSTEP**3))
ENDIF
RETURN
END

SUBROUTINE MODIAG(M,D,MATRIX,C,SOLN,TRBLE,N)
REAL A(8,202),B,SOLN(202),MATRTX(6,8,202),C(202)
INTEGER M,D,W,X,Y,Z,TRBLE,N,I,MN1
TRBLE=0
DO 10 X=1,D+2
    DO 20 Y=1,M
        IF (X.EQ.D+2) THEN
            A(X,Y)=C(Y)
        ELSE
            A(X,Y)=MATRIX(N,X,Y)
        ENDIF
    20 CONTINUE
10 CONTINUE
DO 30 I=2,M
    A(2,I)=A(2,I)/A(3,I-1)
    A(3,I)=A(3,I)-A(2,I)*A(4,I-1)
    A(5,I)=A(5,I)-A(2,I)*A(5,I-1)
30 CONTINUE
30 CONTINUE
DO 80 X=1,M
  IF (A(3,X).EQ.0) THEN
    PRINT*,'MATRIX HAS NO SOLUTION'
    TRBLE=999
    GOTO 999
  ENDIF
80 CONTINUE
MN1=M-1
A(5,M)=A(5,M)/A(3,M)
DO 40 I=MN1,1,-1
  A(5,I)=(A(5,I)-A(4,I)*A(5,I+1))/A(3,I)
40 CONTINUE
DO 50 X=1,M
  SOLN(X)=A(5,X)
50 CONTINUE
999 RETURN
END

SUBROUTINE AVE(F,M,N,JJ)
REAL F(6,202)
INTEGER X,N,M,Z,JJ
DO 443 Z=1,N-1
  IF (.NOT.(Z.EQ.1).AND.(Z.EQ.2)) GOTO 443
  IF (.NOT.(Z.EQ.1).AND.(Z.EQ.2)) GOTO 443
  DO 444 X=2,M-1
    F(Z,X)=(.025*F(Z,X-1)+F(Z,X)+.025*F(Z,X+1))/1.05
444 CONTINUE
443 CONTINUE
RETURN
END

SUBROUTINE TOT(F,FTOTAL,PHI,N,M,VSTEP,DU,DUPHI1)
REAL F(6,202),FTOTAL(202),PHI(6),VSTEP,G(202),NO(6),DU
REAL X1,X2,TOL,F1,F2,XERR
INTEGER M,N,X,Y,NLIM
L=(M+1)/2
DO 10 X=1,N-1
   DO 20 Y=1,L
      G(Y)=F(X,Y)
   20 CONTINUE
   CALL SIMPS(G,L,VSTEP,NO(X))
10 CONTINUE

X1=0.0
X2=1.0E6
TOL=.0001
NLIM=50
30 CALL FCN(NO,X1,N,F1)
CALL FCN(NO,X2,N,F2)
IF (F1*F2.GT.0.0) THEN
   X2=X2*10.0
   IF (X2.GT.1.0E15) THEN
      WRITE(*,*),'NO SIGN CHANGE UP TO 1E15'
      RETURN
   ENDIF
   GOTO 30
ENDIF
GOTO 30
ENDIF
DO 40 J=1,NLIM
   DU=(X1+X2)/2.0
   CALL FCN(NO,DU,N,FR)
   XERR=ABS(X1-X2)/2.0
   IF (XERR.LE.TOL) GOTO 1000
   IF (ABS(FR).LE.TOL) GOTO 1000
   IF (FR*X1.GT.0.0) THEN
      X1=DU
      F1=FR
   ELSE
      X2=DU
      F2=FR
   ENDIF
40 CONTINUE
WRITE(*,*),'NLIM EXCEEDED'
RETURN
1000 DO 50 X=1,M
      FTOTAL(X)=0.0
      DO 60 Y=1,N-1
         FTOTAL(X)=FTOTAL(X)+DU**(Y-1)*F(Y,X)
         60 CONTINUE
   50 CONTINUE
APPENDIX B - The TEC program
C

C******************************************************************
C
C TEC DATA INITIALIZATION ACCESS CODE
C WRITTEN BY GREGORY L RIDDERBUSCH
C AUGUST 1986
C******************************************************************

REAL ARRAY(9)
INTEGER IPARAM(5)
WRITE(*,103)
OPEN(UNIT=2,FILE='PINDATA.DAT',STATUS='OLD')
READ(2,101) (ARRAY(I),I=1,9)
READ(2,102) (IPARAM(I),I=1,5)
10 WRITE(*,104) (ARRAY(I),I=1,7)
WRITE(*,105)
READ(*,*) IVALUE
IF (IVALUE .EQ. 0) THEN
   GOTO 20
ELSE
   WRITE(*,106)
   READ(*,*) VALUE
   ARRAY(IVALUE)=VALUE
   GOTO 10
ENDIF
20 WRITE(*,107) (IPARAM(I),I=1,5)
WRITE(*,110) ARRAY(8),ARRAY(9)
WRITE(*,105)
READ(*,*) 'VALUE
IF (IVALUE .EQ. 0) THEN
   GOTO 30
ELSE
   WRITE(*,108)
   IF (IVALUE .EQ. 6 .OR. IVALUE .EQ. 7) THEN
      READ(*,*) VALUE
      ARRAY(IVALUE+2)=VALUE
      GOTO 20
   ELSE
      READ(*,*) INEW
      IPARAM(IVALUE)=INEW
      GOTO 20
   ENDIF
ENDIF
30 REWIND(2)
WRITE(2,101) (ARRAY(I),I=1,9)
WRITE(2,102) (IPARAM(I),I=1,5)
CLOSE(2)
WRITE(*,109)
STOP

C

101 FORMAT(F8.1/F8.1/F6.3/F6.3/F6.3/F6.3/F7.2/F5.2/F6.1)
102 FORMAT(I1/I1/I3/I3/I3)
103 FORMAT(1X,'******************************************************************'/
   &' TEC OPERATING CONDITIONS'/
   &1X,'******************************************************************')
104 FORMAT(/4X,'CURRENT CONVERTOR OPERATING SETTINGs:'//
   &7X,'1. EMITTER TEMPERATURE: ',F8.1,' KELVIN'/,
   &7X,'2. COLLECTOR TEMPERATURE: ',F8.1,' KELVIN'/,
   &7X,'3. EMITTER WORK FUNCTION: ',F6.2,' EV'/,
   &7X,'4. COLLECTOR WORK FUNCTION: ',F6.2,' EV'/,
   &7X,'5. CONVERTOR PRESSURE: ',F6.2,' TORR'/,
   &7X,'6. GAP THICKNESS: ',F6.2,' MM'/)
OPERATING CURRENT: \,F7.2, AMPS/CM^2\)

ENTER ID NUMBER OF VALUE TO BE CHANGED, 0=None: \)

ENTER NEW OPERATING SETTING: \)

CURRENT TEC FUNCTION SETTINGS:

1. DIAGNOSTIC LEVEL: \,I3, (0—RESTRICTED OUTPUT)/
   (1—FULL OUTPUT)/
   (2—ENABLE SHEATH DIAGNOSTICS)/
   (3—ENABLE DOT DIAGNOSTICS)/
2. RESTART SEQUENCE: \,I3, (0—DEFAULT STARTUP VALUES)/
   (1—RESTART WITH PREVIOUS VALUES)/
3. STEPS BETWEEN PRINTS: \,I3/
4. POINT DENSITY: \,I3, (11,21,31,...151)/
5. LOTUS SKIP FACTOR: \,I3, (1..99)'

ENTER NEW FUNCTIONAL SETTING: \)

6. TIME STEP: \,F5.2, (0.1,0.2,0.3,0.4,0.5)/
7. END TIME: \,F6.1, (1.0,2.0,...,10.0)'

END
**DOS FILE PRED1.FOR**

**PROGRAM TEC**

```fortran
REAL CNE, ENE, ECHI, CCHI, EALPHA, CALPHA, LAMNEB, LAMTAU
REAL DTP, T2, AN, AT, CN, CT, BN, BT, RE, KN, TCHAR, PN, DT
REAL SMR, LAMDAR, NR, TE, TC, ENR, I, ARECN, EGND, ELOSSB, NNR
REAL TAU(0:150), NEB(0:150), DELTAT, SN, ST, PI, CA, CSAHA, DZ
REAL DTUNDZ, MUI(0:150), RMUR, TAUN(0:150), EMISS, TIME2, LCCHI
REAL NDOT1(0:150), TDOT1(0:150), NNB(0:150), TIME1, JNET, CV(0:150)
REAL FYEN, IVD, IKN, EGRADA, PHIB, EWF, CWF, D, ESOURCE(0:150)
INTEGER N, IDEN, EFIX, CFIX, CHKDOT, ICOUNT, NSTEPS, C, PC, LS, LC, EO, EC

C

COMMON /PRED/ CA, CSAHA, DT, DTUNDZ, DZ, EGND, ELOSSB, ENR, I, IDEN, KN
COMMON /PRED/ LAMDAR, LAMNEB, LAMTAU, MUI, N, NNB, NNR, NR, PI, RE, RMUR
COMMON /PRED/ TAUN, EFIX, CFIX, CHKDOT, FYEN, ENE, ECHI, CCHI, CNE
COMMON /PRED/ EALPHA, CALPHA, CV, ESOURCE, LCCHI

C data nsteps /1/

OPEN (UNIT=2, FILE='PINDATA.DAT', STATUS='OLD')
OPEN (UNIT=4, FILE='EXOUT.DAT', STATUS='UNKNOWN')
OPEN (UNIT=7, FILE='LOTUS1.DAT', STATUS='UNKNOWN')
OPEN (UNIT=8, FILE='PREDRES.DAT', STATUS='OLD')
OPEN (UNIT=9, FILE='PRNTOUT.DAT', STATUS='UNKNOWN')

C...SET ABBREVIATED PRNTDAT SAMPLING POINTS WITH PC
C...SET LOTUS SKIP COUNTER WITH LC

CALL INITIAL(TE, TC, EWF, CWF, PN, NSTEPS, DTP, T2, AN, AT,
& CN, CT, BN, BT, TCHAR, SMR, ARECN, DELTAT, SN, ST, TAU, NEB, LS, pc)
c=pc
LC=LS
EO=100*LS
EC=EO

C IF (CHKDOT .EQ. 3)
& OPEN (UNIT=9, FILE='D:PDOTDIAG.DAT', STATUS='UNKNOWN')
IF (CHKDOT .GT. 1)
& OPEN (UNIT=11, FILE='D:PSTHDIAG.DAT', STATUS='UNKNOWN')

C-------------------------------------------
EGRADA=0.0
ICALCS=INT(T2/DTP+0.001)
WRITE(*,76)
DO 30 ICOUNT=0,ICALCS
WRITE(*,77) (icount),100*(icount)/ICALCS
TIME1=(icount)*DTP
TIME2=(icount+1)*DTP
CALL PREDCOR(TIME1,TIME2,TAU,NEB,NSTEPS,TE,TC,TDOT1,NDOT1,
& PN,ARECN,TCHAR,AN,AT,BN,BT,CN,CT,IVD,SMR,EGRADA,PHIB,JNET)
VOUT=EWF + (PHIB + IVD/I + LCCHI)*TE/11600 - CWF
IF (C .EQ. PC) THEN
WRITE (9,67) TIME1,VOUT
IF (CHKDOT .EQ. 0) THEN
WRITE (9,70)
ELSE
EMISS=ENR/NEB(I1)
WRITE (9,72) ENE, ECHI, EALPHA, CNE, LCCHI+CCHI, CALPHA, PHIB,
```

120
IVD/I,EMISS
WRITE(9,73) (I1,NDOT1(I1),NEB(I1),
TDOT1(I1),TAU(I1),I1=0,N+1)
ENDIF
IF (EFIX .EQ. 1) WRITE(9,74)
IF (CFIX .EQ. 1) WRITE(9,75)
C=0
ENDIF
C=C+1
IF (LC .EQ. LS) THEN
WRITE(7,78)TIME1,VOUT,ENE,JNET,ECHI,CCHI,EALPHA,CALPHA,
PHIB,IVD/I,EMISS,NEB(1),NEB(11),TAU(1),TAU(11)
LC=0
ENDIF
LC=LC+1
IF (EC .EQ. EO) THEN
WRITE(4,*) TIME1
WRITE(4,71) (I1,NEB(I1),TAU(I1),I1=1,N)
EC=0
ENDIF
EC=EC+1
30 CONTINUE
C********OUTPUT STARTUP VALUES TO PREDRES.DAT
WRITE(8,69) ENE,CNE,ECHI,CCHI,EALPHA,CALPHA,N
WRITE(8,68) (NEB(I1),TAU(I1),I1=0,N+1)
REWIND(8)
CLOSE(8)
STOP
C
67 FORMAT(/,**RESULTS AT TIME = ',F8.2,4X,**OPERATING VOLTAGE=',F6.3)
68 FORMAT(2F8.3)
69 FORMAT(F9.4/F9.4/F8.3/F8.3/F9.4/F9.4/I3)
70 FORMAT(3X,'#',NEB(#),TAU(#)'/
&3X,----------------------')
71 FORMAT(3X,I3,4X,F6.3,7X,F5.2)
72 FORMAT(3X,'ENE =',F8.3,4X,'ECHI=',F8.3,4X,'EALPHA=',F8.3/
&3X,'CCHI=',F8.3,4X,'CALPHA=',F8.3/
&3X,'PHIB=',F8.3,4X,'VD=',F8.3,4X,'EMISS=',F8.3/
&3X,'#',NDOT(#),NEB(#),TDOT(#),TAU(#)'/
&3X,'----------------------')
73 FORMAT(3X,I3,4X,F7.4,4X,F6.3,7X,F7.4,4X,F5.2)
74 FORMAT('***AT LEAST ONE UNPHYSICAL EMITTER BC WAS INVOKED.')
75 FORMAT('***AT LEAST ONE UNPHYSICAL COLLECTOR BC WAS INVOKED.')
76 FORMAT('/1X,** TEC CALCULATIONS BEGIN...(WAIT)...')
77 FORMAT(4X,'ITERATION #',I3,' **COMPLETION--',I3,' %')
78 FORMAT(16E13.6)
END
C*******************************************************************************
SUBROUTINE PREDCOR(T1,T2,TAU,NEB,NSTEPS,TE,TC,TDOT1,NDOT1,
+ PN,ARECN,TCHAR,AN,AT,BN,BT,CN,CT,IVD,SMR,EGRADA,PHIB,JNET)
C*******************************************************************************
REAL T1,T2,DT,AN,AT,BN,BT,CN,CT,IVD,LAMTAU,LAMNEB,CNE
REAL TAU (0:150),NEB (0:150),TDOT1 (0:150),NDOT1 (0:150),ENE
REAL ECHI,CCHI,EALPHA,CALPHA,MUI (0:150),I,DZ,TCHAR,LCCHI
REAL TE,TC,DTAUNDZ,PN,ENR,NR,EGNDB,ELOSSB,RE,SMR,LMARD
REAL RMUR,KN,NNR,ARECN,PI,CA,CSAHA,NNB (0:150),TAUN (0:150)
REAL MSOURCE (0:150),ESOURCE (0:150),CV (0:150),MUEA (0:150)
REAL NDOT2 (0:150),TDOT2 (0:150),TTILDA (0:150),NTILDA (0:150)
REAL J,IVD,AVD,FYEN,EGRADA,PHIB,JNET
INTEGER N, CFIX, EFIX, IDEN, CHKDOT, NSTEPS

COMMON /PRED/ CA, CSAHA, DT, DTAUNDZ, DZ, EGNDDB, ELOSSB, ENR, I, IDEN, KN
COMMON /PRED/ LAMDAR, LAMNEB, LAMTAU, MUI, N, NNB, NNR, NR, PI, RE, RMUR
COMMON /PRED/ TAUN, EFIX, CFIX, CHKDOT, FYEN, ENE, ECHI, CCHI, CNE
COMMON /PRED/ EALPHA, CALPHA, CV, ESOURCE, LCCHI

C

C*****HANDLE EXCEPTIONAL CONDITIONS
DZ=1.0/(N-1)
DT=(T2-T1)/NSTEPS
CFIX=0
EFIX=0

C*****SET NEUTRAL TEMPERATURE AND DENSITY
IF (TE.EQ.TC)THEN
  DO 10 I1=0,N+1
  TAUN(I1)=1.0
10 CONTINUE
ELSE
  DO 20 I1=0,N+1
  TAUN(I1)=1.0+(TC/TE-1.0)*(I1-1.0)/(N-1)
20 CONTINUE
ENDIF

NNR=965.5E16*PN/TE
DO 30 I1=0,N+1
  NNB(I1)=1.0/TAUN(I1)
30 CONTINUE

DTAUNDZ=TAUN(N)-TAUN(1)

C*****SET TRANSPORT PARAMETERS
RMUR=LAMDAR*SMR
DO 40 I1=0,N+1
  MUI(I1)=SQRT(TAUN(I1))
40 CONTINUE

C*****SET IONIZATION AND SAHA PARAMETERS
CA=0.41283*ARECN*TCHAR*(NR/1.0E14)**2*(TE/1500)**(-4.5)
CSAHA=LOG((1.4027E20*NNR/NR/NR)*(TE/1500)**1.5)

C-----PREDICTOR STEP
IF (CHKDOT.EQ.3) WRITE(9,81)
CALL DOT(NDOT1,TDOT1,NEB,TAU,EGRADA,PHIB,JNET)
DO 45 I1=0,N+1
  NTILDA(I1)=NEB(I1)+AN*DT*NDOT1(I1)
  TTILDA(I1)=TAU(I1)+AT*DT*TDOT1(I1)
45 CONTINUE

C-----CORRECTOR STEP
IF (AN.EQ.0.0.AND.AT.EQ.0.0) THEN
  DO 50 I1=0,N+1
    NDOT2(I1)=0.0
    TDOT2(I1)=0.0
 50 CONTINUE
  GOTO 55
ELSE
  IF (CHKDOT.EQ.3) WRITE(9,82)
  CALL DOT(NDOT2,TDOT2,NTILDA,TTILDA,EGRADA,PHIB,JNET)
ENDIF

DO 60 I1=0,N+1
  NEB(I1)=NEB(I1)+DT*(BN*NDOT1(I1)+CN*NDOT2(I1))
  TAU(I1)=TAU(I1)+DT*(BT*TDOT1(I1)+CT*TDOT2(I1))
60 CONTINUE

70 CONTINUE
C**** UPDATE TIME DERIV.S, IMAGE POINTS, AND FIND PLASMA POWER GAIN

CV(0)=0.0
CV(N+1)=0.0
ESOURCE(0)=0.0
ESOURCE(N+1)=0.0
IF (CHKDOT.EQ.3) WRITE(9,83)
CALL DOT(NDOT1,TDOT1,NEB,TAU,EGRADA,PHIB,JNET)
IIVD=0.0
DO 75 I1=2, (N-1)
   IIVD=IIVD+(ESOURCE(I1)-CV(I1)*TDOT1(I1))
75 CONTINUE
AVD=0.5*(ESOURCE(1)-CV(1)*TDOT1(1))+0.5*(ESOURCE(N)-CV(N)
   +*TDOT1(N))
IVD=(AVD+IIVD)*DZ+2.0*I*(TAU(1)-TAU(N))-(NEB(1)*ENE/KN)
   +*(TAU(1)-1)
RETURN

C
81 FORMAT(///,'CALL DOT FOR FIRST TIME.',///)
82 FORMAT(///,'CALL DOT FOR SECOND TIME.',///)
83 FORMAT(///,'CALL DOT FOR LAST TIME.',///)
END
SUBROUTINE DOT(NEBDOT, TAU, EGRADA, PHIB, JNET)

REAL A, ALPHA(0:150), BETA(0:150), CA, CALPHA, CHI, CDETA, CNE, SIGMA
REAL CTA, CTZ, CV(0:150), CONVECT, D21B, D32B, DT, DZ, DGDU, DELU
REAL DETAP, DTAUNDZ, DZ, EGRADA, PHIB, BIGU, JRC, CGRADA

COMMON /PRED/ CA, CSAHA, DT, DTAUNDZ, DZ, EGRADA, ELOSSB, ENR, I, IDEN, KN
COMMON /PRED/ LAMDAR, LAMNEB, LAMTAU, MUI, N, NNB, NNR, NR, PI, RE, RMUR
COMMON /PRED/ TAUN, EFIX, CFIX, CHKDOT, FYEN, ENE, ECHI, CCHI, CNE
COMMON /PRED/ EALPHA, CALPHA, CV, ESOURCE, LCCHI

F(1) = 5.74E-3
F(2) = 1.40E-3
F(3) = 2.3
F(4) = 0.2
F(5) = 2.70E-2
F(6) = 5.74E-3
F(7) = 4.24E-2
F(8) = 2.82
F(9) = 0.0
F(10) = 11.607
F(11) = 0.0
F(12) = 27.04

C****SET THERMAL & ELECTRICAL CONDUCTIVITIES AT 0+ (E) & 1- (C)
IF(TAU(1).LT.0.1) THEN
   TAU(1) = 0.1
   EFIX = 1
ENDIF
IF(TAU(N).LT.0.1) THEN
   TAU(N) = 0.1
   CFIX = 1
ENDIF
IF(RE.EQ.0.5) THEN
   DO 10 I1 = 0, N+1
      MUEA(I1) = TAUN(I1)
   10 CONTINUE
ENDIF
IF(RE.EQ.0.0) THEN
   DO 20 I1 = 0, N+1
      MUEA(I1) = TAUN(I1)/SQRT(TAU(I1))
   20 CONTINUE
ENDIF
IF(RE.EQ.-0.5) THEN
   DO 30 I1 = 0, N+1
      MUEA(I1) = TAUN(I1)/TAU(I1)
   30 CONTINUE
ENDIF
30 CONTINUE
ELSE
   DO 40 I1=0,N+1
   MUEA(I1)=TAUN(I1)*(TAU(I1)**(RE-0.5))
40 CONTINUE
ENDIF
DO 50 I1=0,N+1
   K(I1)=((RE+2.0)/FYEN)*MUEA(I1)*NEB(I1)*TAU(I1)
   PC(I1)=NEB(I1)*(TAU(I1)+TAU(I1))
50 CONTINUE
DETA=ALOG(K(2)/K(1))*DZ/(K(2)-K(1))
DETAP=ALOG(K(2)/K(1))*DZ/(K(2)-K(1))
C****Determine emitter sheath*****
JNET=(I*KN*1.595769)/(SQRT(TAU(1)))*NEB(1)
CALL SHEATH(JNET,ENR/NEB(1),TAU(1),ECHI,PHIB,EALPHA,ENE)
C-----FIND EMITTER (0+) DERIVATIVES FROM B.C.
   IF(ECHI.LE.1E-5.OR.ECHI.GE.20) EFIX=1
   ETETA=(TAU(1)-1)*ENE*NEB(1)/KN-I*(ECHI-TAU(1)/2)
   ETZ=ETETA/K(1)
   EPCZ=(SQRT(PI/8/EALPHA)/LAMDAR/KN)*NEB(1)/MUI(1)-I/MUEA(1)
   ENZ=(EPCZ-NEB(1)*(ETZ+DTAUNDZ))/(TAU(1)+TAU(1))
C****Solve collector sheath
CFLAG=0
CNE=0.0
U=1.0
GU=G(U,NEB(N),TAU(N),I,KN)
DELU=0.05
DO 80 I1=1,50
   DGDU=(G(U+DELU,NEB(N),TAU(N),I,KN)-GU)/DELU
   DELTAU=-GU/DGDU
   U=U+DELTAU
   GU=G(U,NEB(N),TAU(N),I,KN)
   IF (ABS(GU).LE.0.001) THEN
      CCHI=U*TAU(N)
      GOTO 85
   ENDIF
8 1) CONTINUE
CCHI = 0.0
WRITE(*,205)
IF (CHKDOT .GT. 1) WRITE(11,205)
C****Determine derivatives at collector (1-) FROM B.C.
85 IF (CHKDOT .GT. 1)
   &WRITE(11,206) EGRADA,ALPHA2,TAU(1),ENR/NEB(1),
   &JNET,ECHI,PHIB,ENE,EALPHA,CCHI,CALPHA,IFLAG,RCODE
   DPH = -CCHI/TAU(N)
   IF(DPH.LT.0.0000001) DPH = 0.0000001
   CALPHA=(1.0/TAU(N))*(3.14159265/2)*((1.0 + ERF(SQRT(DPH)))
   + - (1.0/2.0) *(1.0 + ERF(SQRT(4.0*DPH))) )**2 / ( EXP(-DPH)
   + - (1.0/4.0)*EXP(-4.0*DPH) )**2
   LCCHI = 0.0
   IF(CCHI.LT.0.0) THEN
      LCCHI = CCHI
   CCHI = 0.0
ENDIF
CTETA=-I*(CCHI-2.0)
CTZ=CTETA/K(N)
CDETA=ALOG(K(N)/K(N-1))*DZ/(K(N)-K(N-1))
NBCOA=TAU(1)+TAUN(1)
NBCUB=SQRT(PI/EALPHA/8.0)/LAMDAR/KN/MUI(1)-ENE*NEB(1)/K(1)*
   (TAU(1)-1.0)-DTAUNDZ
NBC0C=I*NEB(1)/K(1)* (ECHI-TAU(1)/2.0)-I/MUEA(1)
NBC1A=TAU(N)+TAU(N)
NBC1B=SQR(PI/CALPHA/8.0)/LAMDAR/KN/MUI(N)-CNE*NEB(N)/K(N)*
+ (TAU(N)-1.0)*DTAUNDZ
NBC1C=-I*NEB(N)/K(N)* (CCHI-TAU(N)/2.0)-I/MUEA(N)
TBC0A=1.0
TBC0B=ENE*NEB(1)/KN+I/2.0
TBC0C=-ENE*NEB(1)/KN-I*ECHI
TBC1A=-1.0
TBC1B=CNE*NEB(N)/KN-I/2.0
TBC1C=-CNE*NEB(N)/KN+I*CCHI
NEBA(0)=(NBC0A*LAMNEB)/(2.0*DZ)
NEBB(0)=-LAMNEB*NBC0B
NEBC(0)=-NBC0A*LAMNEB/(2.0*DZ)
NEBV(0)=NBC0B*(1.0-LAMNEB)*NEB(1)+NBC0C-(1.0-LAMNEB)*
+ NBCOA*(NEB(2)-NEB(0))/(2.0*DZ)
TAUA(0)=TBCOA*LAMTAU/(2.0*DETAP)
TAUB(0)=-LAMTAU*TBC0B
TAUC(0)=TBCOA*LAMTAU/(2.0*DETAP)
TAUV(0)=TBC0B*(1.0-LAMTAU)*TAU(1)+TBC0C-(1.0-LAMTAU)*
+ TBCOA*(TAU(2)-TAU(0))/(2.0*DETAP)
NEBA(N+1)=(NBC1A*LAMNEB)/(2.0*DZ)
NEBB(N+1)=-LAMNEB*NBC1B
NEBC(N+1)=-NBC1A*LAMNEB/(2.0*DZ)
NEBV(N+1)=NBC1B*(1.0-LAMNEB)*NEB(N)+NBC1C-(1.0-LAMNEB)*
+ NBC1A*(NEB(N+1)-NEB(N-1))/(2.0*DZ)
TAUA(N+1)=TBC1A*LAMTAU/(2.0*CDETA)
TAUB(N+1)=-LAMTAU*TBC1B
TAUC(N+1)=-TBC1A*LAMTAU/(2.0*CDETA)
TAUV(N+1)=TBC1B*(1.0-LAMTAU)*TAU(N)+TBC1C-(1.0-LAMTAU)*
+ TBC1A*(TAU(N+1)-TAU(N-1))/(2.0*CDETA)

C*****INITIALIZE GAMMAP & QKP FOR LOOP
MURS=MUI(2)/MUEA(2)+MUI(1)/MUEA(1)*(1-2*DZ*
+ (0.5-RE)*ETZ/TAU(1)-0.5*DTAUNDZ/TAUN(1))
GAMMAP=0.5*(((MUI(1)+MUI(2))*(PC(1)-PC(0))
+ ))/DZ+I*MURS)
QKP=(TAU(1)-TAU(0))/DETA
MUIS=MUI(1)+MUI(2)

DO 100 J=1,N
C-----UPDATE FOR NEW J
GAMMAP=GAMMAP
QKM=QKP
DETA=DETA
MUISOLD=MUIS
MURSOLD=MURS
IF (J.NE.N) THEN
DETA=ALOG(K(J+1)/K(J))*DZ/(K(J+1)-K(J))
MUIS=MUI(J)+MUI(J+1)
MURS=MUI(J)/MUEA(J)+MUI(J+1)/MUEA(J+1)
ELSE
MURS=MURS+2*DZ*(MUI(N)/MUEA(N))**((0.5-RE)*CTZ/TAU(N)
+ **(0.5*DZ*CTZ/TAU(N))
ENDIF
C-----FIND AMBIPOLAR FLUX AT J+1/2
GAMMAP=0.5*((MUIS*(PC(J+1)-PC(J)))/DZ+I*MURS)
C-----FIND MASS SOURCE AT J
A=CA/TAU(J)**4.5
NES2=NNB(J)*TAU(J)**1.5*EXP(CSAHA-EGNDB/TAU(J))
D21B=F(7)**(1+1.5*EXP(CSAHA-EGNDB/TAU(J))

124
D32B = F(2) * EXP(F(3) / TAU(J))
IB = A * NES2 * (1 + F(1) / NEB(J)) / (1 + D21B * (1 + D32B / NEB(J)) / NEB(J))
P0 = 1 + (F(4) / NEB(J)) * (1 + F(5) / NEB(J)) / (1 + F(6) / NEB(J))
NUE = NEB(J) * IB / NES2
MSOURCE(J) = NEB(J) * IB * (1 - P0 * NUE)
  IF (IDEN.EQ.1) MSOURCE(J) = NEB(J) * A * NES2
NEBDOT(J) = RMUR * (GAMMA - GAMMA) / DZ + MSOURCE(J)
NA(J) = RMUR * MUIS * (TAU(J+1) + TAU(J+1)) / 2.0 / DZ**2
NB(J) = RMUR * MUISOLD * (TAU(J) + TAU(J)) / 2.0 / DZ**2
NC(J) = RMUR * MUISOLD * (TAU(J-1) + TAU(J-1)) / 2.0 / DZ**2
ND(J) = I* (MURS - MURSOLD) * RMUR / DZ / 2.0
  + NEB(J) * IB * (1.0 + SQRT(P0 / NES2)) * NEB(J)
C
+ NS(J) = IB * (1.0 - P0 * NUE)
NS(J) = -NEB(J) * IB * (1.0 + SQRT(P0 / NES2)) * NEB(J) * SQRT(P0 / NES2)
  IF (IDEN.EQ.1) NS(J) = A * NES2
NEBA(J) = -DT * NA(J) * LAMNEB
NEBB(J) = 1.0 + DT * NB(J) * LAMNEB - DT * NS(J) * LAMNEB
NEBC(J) = -DT * NC(J) * LAMNEB
NEBV(J) = NEB(J) + DT * NA(J) * (1.0 - LAMNEB) * NEB(J+1) - DT * NB(J) * (1.0 - LAMNEB) * NEB(J-1) + DT * ND(J) + DT * NS(J) * (1.0 - LAMNEB) * NEB(J)
KDE = K(J) * (DETA + DETAP) / 2
QKP = (TAU(J+1) - TAU(J)) / DETAP
CONVECT = (1.5) * I * (DETA * QKP + DETAP * QKM) / (2 * KDE)
SIGMA = NEB(J) * MUEA(J)
POHMIC = I * (I / SIGMA + TAU(J) * (NEB(J+1) - NEB(J-1))
  / (2 * DZ * NEB(J)))
PB = (F(9) * NNR / NR) * EXP(-F(10) / TAU(J))
CV(J) = 1.5 * NEB(J) + NNB(J) * (F(10) * PB + F(12) * PB / NUE)
  / (TAU(J) * TAU(J))
ESOURCE(J) = -ELOSSB * MSOURCE(J)
  - NNB(J) * PB * (2 * NUE * NEBDOT(J) / NEB(J))
TAUDOT(J) = ((QKP - QKM) / KDE + CONVECT + POHMIC + ESOURCE(J))
  + (CV(J)
TA(J) = 1.0 / (DETA + DETAP * KDE * CV(J))
TB(J) = (1.0 / DETAP + 1.0 / DETA) / KDE / CV(J)
TC(J) = 1.0 / DETA / KDE / CV(J)
TS(J) = (CONVECT + POHMIC + ESOURCE(J)) / CV(J)
TAU(J) = -DT * LAMTAU * TAU(J)
TAUB(J) = 1.0 + DT * LAMTAU * TB(J)
TAUC(J) = -DT * LAMTAU * TC(J)
TAUV(J) = TAU(J) + DT * (1.0 - LAMTAU) * TAU(J) * TAU(J+1)
  + DT * (1.0 - LAMTAU) * TB(J) * TAU(J)
  + DT * (1.0 - LAMTAU) * TC(J) * TAU(J-1)
  + TS(J) * DT
IF (CHKDOT.EQ.3) WRITE(9,201) J, NEB(J), J, TAU(J),
  J, MSOURCE(J), J, PB, PBP, A, D21B, D32B, P0, IB,
  + NUE, NES2, QKP, GAMMAP, DETAP, MURS
CONTINUE
NEBB(N+1) = NEBA(N+1)
TAUC(0) = TAUC(0) - TAUC(1) * TAUA(0) / TAUA(1)
TAUB(0) = TAUB(0) - TAUB(1) * TAUA(0) / TAUA(1)
TAUV(0) = TAUV(0) - TAUV(1) * TAUA(0) / TAUA(1)
TAUA(0) = TAUB(0)
TAUB(0) = TAUC(0)
TAUA(N+1) = TAUA(N+1) - TAUA(N) * TAUC(N+1) / TAUC(N)
TAUB(N+1) = TAUB(N+1) - TAUB(N) * TAUC(N+1) / TAUC(N)
TAUV(N+1) = TAUV(N+1) - TAUV(N) * TAUC(N+1) / TAUC(N)
TAUC(N+1) = TAUB(N+1)
TAUB(N+1) = TAUA(N+1)

ALPHA(0) = -NEBA(0) / NEBB(0)
DO 110 I1 = 1, N
   ALPHA(I1) = -NEBA(I1) / (NEBC(I1) * ALPHA(I1-1) + NEBB(I1))
110 CONTINUE

ALPHA(N+1) = 0.0
BETA(0) = NEBV(0) / NEBB(0)
DO 120 I1 = 1, N+1
   BETA(I1) = (NEBV(I1) - NEBC(I1) * BETA(I1-1)) / (NEBC(I1) * ALPHA(I1-1) + NEBB(I1))
120 CONTINUE

NEBU(N+1) = BETA(N+1)
DO 130 I1 = N, 0, -1
   NEBU(I1) = ALPHA(I1) * NEBU(I1+1) + BETA(I1)
130 CONTINUE

ALPHA(0) = -TAUA(0) / TAUB(0)
DO 140 I1 = 1, N
   ALPHA(I1) = -TAUA(I1) / (TAUC(I1) * ALPHA(I1-1) + TAUB(I1))
140 CONTINUE

ALPHA(N+1) = 0.0
BETA(0) = TAUV(0) / TAUB(0)
DO 150 I1 = 1, N+1
   BETA(I1) = (TAUV(I1) - TAUC(I1) * BETA(I1-1)) / (TAUC(I1) * ALPHA(I1-1) + TAUB(I1))
150 CONTINUE

TAUU(N+1) = BETA(N+1)
DO 160 I1 = N, 0, -1
   TAUU(I1) = ALPHA(I1) * TAUU(I1+1) + BETA(I1)
160 CONTINUE

DO 170 I1 = 0, N+1
   TAUUDOT(I1) = (TAUU(I1) - TAU(I1)) / DT
   NEBDOT(I1) = (NEBU(I1) - NEB(I1)) / DT
170 CONTINUE

IF (CHKDOT.EQ.3) THEN
   WRITE(9,202) (I1, NEBDOT(I1), I1, TAUUDOT(I1), I1=0, N+1)
   WRITE(9,203) (I1, ALPHA(I1), I1, BETA(I1), I1=0, N+1)
   WRITE(9,204) (I1, NEBU(I1), I1, TAUU(I1), I1=0, N+1)
ENDIF
ENDIF
RETURN

C
201 FORMAT ('NEB(', I2, ',') = ', F8.3, ', TAU(', I2, ',') = ', F8.3,
          & MSOURCE(', I2, ') = ', F8.3/, ' J = ', I2, ', ' PB = ', F8.3,
          & P0 = ', F8.3, ', IB = ', F8.3/, ' NUE = ', F8.3, ', NES2 = ', F8.3/
          & KQP = ', F8.3, ', GAMMAP = ', F8.3/, ' DETAP = ', F8.3, ', MURS = ', F8.3)
202 FORMAT ('NEBDOT(', I2, ',') = ', F8.3, ', TAUUDOT(', I2, ',') = ', F8.3)
203 FORMAT ('ALPHA(', I2, ',') = ', F8.3, ', BETA(', I2, ',') = ', F8.3)
204 FORMAT ('NEBU(', I2, ',') = ', F8.3, ', TAUU(', I2, ',') = ', F8.3)
205 FORMAT ('//2X, 'COLLECTOR SHEATH FAILED TO CONVERGE', //)
206 FORMAT ('//1X, 'EGRADA=', F7.3, 3X, 'ALPHA2=', F7.3/)
&1X,'RLE=' ,F5.2,3X,'EMISS=' ,F9.1,3X,'JNET=' ,F7.4/
&1X,'ECHI=' ,F7.4,3X,'PHIB=' ,F7.3,1X,'ENE=' ,F10.3,3X,
&'EALPHA=' ,F7.4/1X,'CCHIP=' ,F7.3,3X,'CALPHA=' ,F7.4/
&'IFLAG=' ,I1,3X,'RCODE=' ,A1)

END

FUNCTION G (UX, NEB, TAU, I, KN)

REAL UX, NEB, TAU, I, KN, SQX, GX
DOUBLE PRECISION ERF

IF (UX .LE. 0.0) THEN
  SQX = 0.0
ELSE
  SQX = SQRT(UX)
ENDIF

G = UX * LOG(1.0 + ERF(SQX)) - LOG(NEB * SQRT(TAU) / I / KN / 2.0)

RETURN
END

FUNCTION ERF(X)

DOUBLE PRECISION X1, SUM, ERF, ERFX, P(3,6), Q(3,6), PI, TSUM1, TSUM2
INTEGER IPOWER(3)
DATA IPOWER/2,1,-2/
PI = 3.1415926535898
TSUM1 = 0.0
TSUM2 = 0.0
X1 = X
IF (X1 .LT. 0.0) X1 = -X1
IF (X1 .GT. 5.93) THEN
  ERFX = 1.0
  GOTO 1240
END IF
IF (X1 .EQ. 0.0) THEN
  ERF = 0.0
  GOTO 1245
ELSE
  IFLAG = 1
END IF
IF (X1 .LE. 4.0 .AND. X1 .GE. 0.47) IFLAG = 2
IF (X1 .GT. 4.0) IFLAG = 3
DO 1205 J = 1, 6
  TSUM1 = TSUM1 + P(IFLAG,J) * (X1 ** (IPOWER(IFLAG)*(J-1)))
  TSUM2 = TSUM2 + Q(IFLAG,J) * (X1 ** (IPOWER(IFLAG)*(J-1)))
1205 CONTINUE
SUM = TSUM1 / TSUM2
GOTO (1210, 1220, 1230), IFLAG
1210 ERFX = X1 * SUM
  GOTO 1240
1220 ERFX = 1.0 - DEXP(-X1 * X1) * SUM
  GOTO 1240
1230 ERFX = 1.0 - DEXP(-X1 * X1) / X1 * (1.0 / DSQRT(PI) + (1.0 / (X1 * X1) * SUM))
1240 ERF = ERFX
IF (X .LT. 0.0) ERF = -ERFX
1245 RETURN

C*** P(IFLAG,J) Q(IFLAG,J) ***
DATA P(1,1)/3.20937758913847D+03/, P(1,2)/3.774852376853D+02/
DATA P(1,3)/1.1386415415105D+02/, P(1,4)/3.16112374387057/
DATA P(1,5)/1.85777706184603D-01/, P(1,6)/0.0/
DATA Q(1,1)/2.84423683343917D+03/, Q(1,2)/1.2826165077372D+03/
DATA Q(1,3)/2.44024637934444D+02/, Q(1,4)/2.36012909523441D+01/
DATA Q(1,5)/1.0/, Q(1,6)/0.0/, P(2,1)/2.289892851659D+01/
DATA P(2,2)/2.6094746956075D+01/, P(2,3)/1.457189596926D+01/
DATA P(2,4)/4.2677201070898/, P(2,5)/5.6437160686381D-01/
DATA P(2,6)/-6.0858151959688D-06/, Q(2,1)/2.289895749891D+01/
DATA Q(2,2)/5.1933570687552D+01/, Q(2,3)/5.0273202863803D+01/
DATA Q(2,4)/2.6288795758761D+01/, Q(2,5)/7.568482293618/
DATA Q(2,6)/1.0/, P(3,1)/-6.5874916152983D-04/
DATA P(3,2)/-1.60837851487423D-02/, P(3,3)/-1.2578172611123D-01/
DATA P(3,4)/-3.60344899949804D-01/, P(3,5)/-3.05326634961232D-01/
DATA P(3,6)/-1.63153871373021D-02/, Q(3,1)/2.3352049762687D-03/
DATA Q(3,2)/6.05183413124413D-02/, Q(3,3)/5.27905102951428D-01/
DATA Q(3,4)/1.87295284992346/, Q(3,5)/2.56852019228982/, Q(3,6)/1/
END
**SUBROUTINE INITIAL(TE, TC, EWF, CWF, PN, NSTEPS, DTP, T2, AN, AT, & CN, CT, BN, BT, TCHAR, SMR, ARECN, DELTAT, SN, ST, TAU, NEB, LS, pc)**

**REAL CA, CSAHA, CNE, ENE, ECHI, CCHI, EALPHA, CALPHA, LAMNEB, LAMTAU**
**REAL DT, DTAUNDZ, DTP, T2, AN, AT, CN, CT, BN, BT, RE, KN, TCHAR, PN**
**REAL SMR, LAMDAE, LAMNEB, EWF, CWF, J, RMUR, LCCHI**
**REAL EALPHA, CALPHA, CV, ARECN, DELTAT, SN, ST, TAU, NEB, LS, pc**

**COMMON /PRED/ CA, CSAHA, DT, DTAUNDZ, DTP, T2, AN, AT, CN, CT, BN, BT, RE, KN, TCHAR, PN**
**COMMON /PRED/ LAMDAE, LAMNEB, EWF, CWF, J, RMUR, LCCHI**
**COMMON /PRED/ EALPHA, CALPHA, CV, ARECN, DELTAT, SN, ST, TAU, NEB, LS, pc**

**WRITE(*,150)**
**C*****READ FILE INDATA.DAT**
**READ(2,151) TE, TC, EWF, CWF, PN, D, J, DTP, T2, CHKDOT, OFILE, pc, N, LS**
**REWIND(2)**
**CLOSE(2)**
**C*****READ FILE PRECOR.DAT**
**CALL DATAINT**
**C*****SET NUMERICAL PARAMETERS, RECOMPILATION REQUIRED TO CHANGE**
**AN=0.5**
**AT=0.5**
**CN=0.5**
**CT=0.5**
**BN=1.0-CN**
**BT=1.0-CN**
**PI=3.1415926**
**IDEN=0**
**RE=0.0**
**FYEN=1.0**
**NR=1.0E14**
**DELTAT=DTP/NSTEPS**
**LAMNEB=1.0**
**LAMTAU=1.0**

**C*****SET TRANSPORT PROPERTIES**
**LAMDAE=1.0/32.3/PN**
**LAMDAI=1.0/96.6/PN**
**LAMDAR=LAMDAE/LAMDAI**
**KN=LAMDAE/D**
**TCHAR=D/(KN*3.75*(TE**.5))**
**SMR=1.0/492.2**

**C*****EXECUTE OFILE SELECTION SETTING**
**IF (OFILE .EQ. 1) THEN**
**READ(8,152) ENE, CNE, ECHI, CCHI, EALPHA, CALPHA, N**
**READ(8,154) (NEB(I1), TAU(I1), I1=0,N+1)**
**REWIND(8)**
**ELSE**
**ENE=0.8**
**CNE=0.8**
**ECHI=3.0**
**CCHI=3.0**
**EALPHA=0.5**
CALPHA=0.5
DO 10 I1=0,N+1
NEB(I1)=I1
NEBCAL(I1)=(NEB(I1)-1)/(N-1)
NEB(I1)=4.0*(KN+NEBCAL(I1)*(1-NEBCAL(I1)))
TAU(I1)=2700/TE
10 CONTINUE
ENDIF
C*****MISCELANEOUS DEFAULTS
ARECN=0.31
EGNDB=3.896/8.609E-05/TE
ELOSSB=EGNDB
SN=DELTAT*(N-1)**2*3*LAMDA*SMR*(BN+CN)
ST=DELTAT*(N-1)**2*0.667*(RE+2)**2*(RE+0.5)*(CT+CT)/FYEN
C*****NODIMENSIONALIZE CURRENT AND CALCULATE RICHARDSON EMISSION
VALUE=EXP(-11600.0*EWF/TE)
ENR=(7.676E+14*(TE)**1.5*VALUE)/NR
I=J / (KN*NR*(3.1265322E-13)*SQRT(TE))
JRIC=120.0*TE*TE*(EXP(-11600.0*EWF/TE))
C*****OUTPUT INITIALIZATION DATA TO PRNTOUT.DAT
WRITE(9,155) TE,TC,EWF,CWF,PN,D,J,CHKDOT,OFILE,N,
& JRIC,SN,TCHAR,I,ENR,KN,SMR,LAMDAR,
& NSTEPS,T2,DELTAT,DTP,LS
IF (CHKDOT.GT.1) THEN
WRITE(9,160) ECHI,ENE,EALPHA,CCHI,ARECN,EGNDB,IDEN,FYEN,RE
& BN,BT,CN,CT,SN,ST,LAMNEB,LAMTAU,ELOSSB,
& ARECN,EGNDB,RE
WRITE(9,159) (I1,NEB(I1),I1,TAU(I1),I1=1,N)
ENDIF
WRITE (9, 161)
C*****CHECK FOR UNREAL CURRENT CONDITION J/JR > 1.0
IF (2*KN*I/ENR .GE. 1.0) THEN
WRITE (*, 153)
STOP
ENDIF
RETURN
C
C--------------------------------------------------------------
150 FORMAT (1X,'******************************'/
& 1X,'***** TEC START *****'/
& 1X,'****** TEC INITIAL DATA SUMMARY'/'
& 1X,'PHYSICAL OPERATING CONDITIONS----'/
& 1X,'EMITTER TEMPERATURE (TE)='F8.1,' KELVIN'/
& 1X,'COLLECTOR TEMPERATURE (TC)='F8.1,' KELVIN'/
& 1X,'EMITTER WORK FUNCTION (EWF)='F6.3,' EV'/
& 1X,'COLLECTOR WORK FUNCTION (CWF)='F6.3,' EV'/
& 1X,'CONVERTOR PRESSURE (PN)='F6.3,' TORR'/
& 1X,'OPERATING CURRENT (J)='F7.3,' AMPS/CM^2'/
& 1X,'TEC FUNCTION SETTINGS----'/
& 1X,'DIAGNOSTIC LEVEL (CHKDOT)= ',I1/
&1X,' RESTART SEQUENCE  (OFILE)= ',I1/
&1X,' POINT DENSITY     (N)=',I3//
&1X,' PHYSICAL PARAMETERS EVALUATED-----'/
&1X,' RICHARDSON CURRENT  (JRIC)=',E9.2,' AMPS/CM^2'/
&1X,' REFERENCE DENSITY    (NR)=',E9.2,' 1/CM^3'/
&1X,' CHARACTERISTIC TIME  (TCHAR)= ',F7.4,' SECS*E-06'/
&1X,' NONDIM CURRENT     (I)= ',F7.4/
&1X,' NONDIM EMISSION     (ENR)= ',F8.3,' (NRIC/NR)'/
&1X,' KNUDSEN NUMBER     (KN)= ',F7.4/
&1X,' SQRT(MASS RATIO)    (SMR)= ',F7.4/
&1X,' MEAN FREE PATH RATIO (LAMDAR)= ',F7.4/
&1X,' TIME SETTINGS-----'/4X,'NSTEPS=',I3/
&4X,' LSF=',I3)
159 FORMAT(4X,'NEB(',I2,')=',F8.3,' TAU(',I2,')=',F8.3)
160 FORMAT(/1X,'ADVANCED DIAGNOSTIC OUTPUT-----'/
&4X,'ECHI=',F5.1,' ENE=',F7.4,' EALPHA=',F7.4/
&4X,'CCHI=',F5.1,' CNE=',F7.4,' CALPHA=',F7.4/
&4X,'AN=',F7.4,' AT=',F7.4/4X,'BN=',F7.4,' BT=',F7.4/
&4X,'CN=',F7.4,' CT=',F7.4/4X,'SN=',F7.4,' ST=',F7.4/
&4X,'LAMNEB=',F5.2/4X,'LAMTAU=',F5.2/4X,'ELOSSB=',F6.3/
&4X,'ARECN=',F6.3/4X,'EGNDB=',F6.3/4X,'IDEN=',I1/
&4X,'FYEY=',F5.2/4X,'RE=',F5.2/
&1X,'STARTUP DENSITY AND TEMPERATURE RATIOS-----')
161 FORMAT(/1X,'----------------------------------------'
&'---------------------')
END