ON THE TRANSFER FUNCTION MODELING OF FLEXIBLE STRUCTURES WITH DISTRIBUTED DAMPING

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Abstract

The authors have suggested the use of distributed passive damping via constrained viscoelastic layer surface treatments, as a means of augmenting active controllers for flexible structures. This enhances system stability and provides justification for the use of low order dynamic models and controllers. In previous work a colocated transfer function based upon experimental measurements was employed in root locus analysis of a system with passive damping. Here we present a method of deriving colocated and non-colocated transfer functions for flexible systems with distributed damping. The (sandwich) flexible structure is modeled as though it was constructed of an equivalent homogeneous material exhibiting linear viscoelastic behavior. Transfer functions are obtained through solution of a modified form of the Bernoulli-Euler equation which is derived using the standard constitutive relationship between stress and strain for viscoelastic materials in place of Hook's law. General observations are made with regard to open loop pole and zero locations for colocated and non-colocated transfer functions. The effect of joint damping is also considered.

Submitted on May 15, 1986 for presentation at the 1986 ASME Winter Annual Meeting.

Introduction

The authors have suggested augmenting active controllers for flexible structures by adding passive mechanical damping to flexible members. In previous work [1,2] it was shown that this approach improves robustness and eases the burden of active control. The analysis performed in [1] was based upon a dynamic model established through experimental identification and considered the case of colocated control only. In this paper we discuss the analytical development of transfer function models for damped systems exhibiting linear viscoelastic behavior.

System Configuration

The system under consideration, illustrated in Figure 1, is a single link single axis arm which rotates in the horizontal plane about a pinned end upon application of a control torque $\tau(t)$. The flexible member is a long slender beam of length L with uniformly distributed damping. The beam is fixed to a pinned hub with inertia J and payload of mass m is mounted on the opposite end. Damping of the pinned joint is represented by a rotary viscous dashpot with damping coefficient b. This configuration is viewed as being representative of lightweight, large payload capacity manipulator arms. Many of the concepts involved however, should easily extend to the larger class of systems comprised of structures with flexible members in general.

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Figure 1. Arm Configuration and Notation (top view).

Distributed Damping Treatment

There are several methods of adding passive damping to flexible structures. The authors have employed constrained viscoelastic layer damping treatments in experiments with a system such as the one illustrated in Figure 1. This type of treatment is particularly well suited to beam-like structures with open surface area available on which to apply the treatment, a common characteristic of efficient structures. This method involves sandwiching a thin film of viscoelastic material between the flexible member's surface and an elastic constraining layer. This concept is illustrated in Figure 2. Materials with high tensile stiffness provide the most effective constraining layers. When elastic deflection occurs, shear induced plastic deformation is imposed in the viscoelastic layer, which in turn produces energy dissipation thereby providing the desired mechanical damping effect. The damping provided by a given constrained layer treatment can be optimized [3] with respect to some frequency of interest (e.g. in the vicinity of the lowest frequency uncontrolled modes) by cutting the constraining layer into sections of "optimal length". This concept has been employed by the authors [4] with great success.

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Figure 2. Sectioned constrained viscoelastic layer damping treatment

The now standard method of characterizing systems with viscoelastic damping treatment involves the use of the concept of a complex elastic modulus [5], whereby both the energy storage and dissipation properties of a material are represented by a complex group of the form $E_\omega(1+j\eta_\omega)$ where E_ω is designated the storage modulus, η_ω is the loss factor and both are dependent of frequency ω. There are a number of drawbacks with using this approach in connection with the development of dynamic models for controller design. Most importantly, the strict permissibility of the complex modulus approach relies upon the assumption of simple harmonic motion. Hence the complex modulus should not be used in connection with problems of a transient nature. The authors support the use of an alternative approached based upon the constitutive differential equation describing the relationship between stress and strain in a viscoelastic material.

Differential Equation

Here we seek a differential equation of motion representing transverse displacement of a flexible beam-like structure incorporating sectioned constrained viscoelastic layer damping. The equations derived are valid for small transverse deflections $u(x,t)$ and small deviations of the hub angle $\theta(t)$ from an operating point. The combined beam and damping treatment system is modeled as a beam composed of an equivalent homogeneous material exhibiting linear viscoelastic behavior. The assumption that the beam's viscoelastic properties are constant throughout its length L will normally be justifiable when the constraining layer is sectioned into segments that are much shorter than the beam length however, these properties may vary substantially in x when the constraining layer is continuous.

Linear Viscoelasticity. The relationship between stress σ and strain ε for a purely elastic linear material is given by Hook's law:

$$\sigma = E\varepsilon \qquad (1)$$

where the constant of proportionality E is Young's modulus of elasticity. This relationship is employed in the derivation of equations of motion (e.g. Bernoulli-Euler Equations) for systems constructed of linear elastic materials. In considering the general class of materials comprised of those exhibiting linear viscoelastic behavior the relationship between stress and strain is represented by a linear partial differential equation of arbitrary order:

$$\sigma + a_1\frac{\partial\sigma}{\partial t} + a_2\frac{\partial^2\sigma}{\partial t^2} + a_3\frac{\partial^3\sigma}{\partial t^3} + \ldots + a_n\frac{\partial^n\sigma}{\partial t^n} \tag{2}$$

$$= E\left[\varepsilon + b_1\frac{\partial\sigma}{\partial t} + b_2\frac{\partial^2\sigma}{\partial t^2} + b_3\frac{\partial^3\sigma}{\partial t^3} + \ldots + b_m\frac{\partial^m\sigma}{\partial t^m}\right]$$

where t is time and a_i and b_i are constants.

<u>Derivation of Modified Bernoulli-Euler Equation</u>. A derivation following a familiar development of the Bernoulli-Euler equation [6], employing a force and moment balance on a differential element of a beam constructed of material obeying equation 2 (Figure 3) produces a modified form of the equation that accommodates beams constructed of a homogeneous, linear viscoelastic material. As is customary for "simple beam theory" we employ the assumptions that the rotation of a differential element of the beam is insignificant compared to the transverse translation and the shear deformation is small in relation to the bending deformation.

The forces and moments acting on a beam element of length dx are depicted in Figure 3. Here V designates shear force, M designates moment and f(x,t) is a distributed loading. From elementary mechanics an expression for the normal strain in an element of an initially straight beam under pure bending is given by:

$$\varepsilon(x,y,t) = y\frac{\partial^2 u(x,t)}{\partial x^2} \tag{3}$$

for small deflections, where u(x,t) is transverse deflection and y is distance from the beam neutral axis.

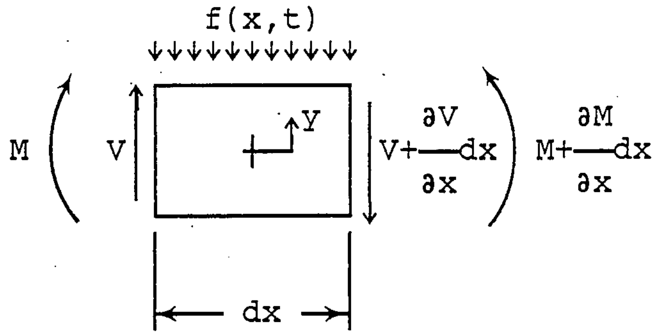

Figure 3. Forces and moments on differential beam element.

Equilibrium requires that the internal moment due to σ must equal the external moment M.

$$M = \int_A y\sigma(x,y,t)dA \tag{4}$$

Combining equations 2, 3 and 4 and employing the definition of area moment of inertia I yields:

$$\sum_{i=0}^{n} a_i \frac{\partial^i M}{\partial t^i} = EI \sum_{j=0}^{m} b_j \frac{\partial^{2+j} u(x,t)}{\partial x^2 \partial t^j} \tag{5}$$

where $a_0=b_0=1$. Summing the forces acting on the beam element and the moments about it's centroid, eliminating terms of second or higher order and neglecting inertia torque due to rotation of the element yields the following expressions:

$$\frac{\partial V}{\partial x} + \rho A \frac{\partial^2 u(x,t)}{\partial t^2} = f(x,t) \quad \text{and} \quad \frac{\partial M}{\partial x} = V \tag{6}$$

Here ρ is the material density, A designates the beam cross sectional area and I is the area moment of inertia corresponding to the cross section. Combining equations 5 and 6 and then Laplace transforming the result yields the following Laplace domain version of the Bernoulli-Euler equation, which is modified to accommodate viscoelastic material behavior:

$$E^*(s)I\frac{\partial^4 U(x,s)}{\partial x^4} + \rho A s^2 U(x,s) = F(x,s) \qquad (7)$$

where

$$E^*(s) = E\left[\frac{1+b_1 s+b_2 s^2+ \ldots +b_m s^m}{1+a_1 s+a_2 s^2+ \ldots +a_n s^n}\right] \qquad (8)$$

is a transfer function describing the viscoelastic character of the material. We shall designate $E^*(s)$ as the elastic modulus operator because it replaces E in the equation of motion and in the boundary conditions. Here s is the Laplace operator and U(x,s) and F(x,s) are the Laplace transformed transverse displacement and distributed load variables { u(x,t) and f(x,t) } respectively.

Comparison of Simple Viscoelastic Models. To illustrate the effect of increasing viscoelastic model order we compare solutions of equation 7 using three different simple viscoelastic models. The boundary conditions considered are for a simply supported beam driven by a torque applied at one end. Bode amplitude plots for the (colocated) transfer function between input torque and angular velocity of the driven end provide a basis for comparison. It is instructive to visualize the constitutive relationship (2) as having been generated by an equivalent system of linear springs and dashpots arranged so as to duplicate the behavior of the material under consideration. The following simple examples demonstrate this concept.

1.) Hook's Law Model

The behavior of materials exhibiting Hookean behavior (1) may be visualized as equivalent to a spring of stiffness E with applied force σ and displacement ε.

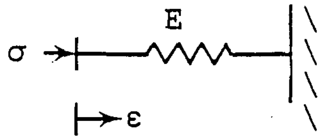

Figure 4 Network equivalent to linear elasticity

In practice the Hook's law model is often used to find natural frequencies of a structure then the eigenvalues are modified slightly, to reflect the inherent damping observed in real structures due to the internal characteristics of the material and various external effects such as air drag.

The customary representation of this "structural damping" provides damping ratio which is equal for each of the oscillatory natural frequencies. The Bode amplitude plot of Figure 5 represents a simply supported beam using the Hook's law material model with a constant damping ratio of 0.01 added. The parameters are chosen so that the first vibration frequency occurs at π^2 hertz.

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Figure 5 Hook's law model with structural damping added.

2.) Kelvin-Voight Model

The Kelvin-Voight model of viscoelastic behavior represents the stress as a weighted sum of strain and strain rate:

$$\sigma = E\left[\varepsilon + b\frac{\partial \varepsilon}{\partial t} \right] \qquad (9)$$

The equivalent network representation of equation 9 consists of a spring of stiffness E in parallel with a dashpot with damping coefficient bE.

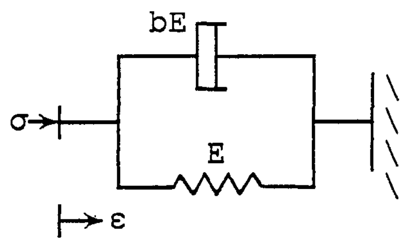

Figure 6 Equivalent network for Kelvin-Voight model

The parameters used to generate Figure 7 were selected so that the frequency and damping of the first vibration mode match those of Case #1. This model predicts that the damping ratios increase strongly with frequency. This result is not consistent with the observed behavior of physical systems. It is of interest to note that the complex modulus may be viewed as a frequency domain version of the Kelvin-Voight model with parameters that vary with frequency so as to match experimental results.

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Figure 7 Kelvin-Voight model.

3.) A case that is of interest in the present work is the "three parameter model" which is an idealized model[1] for constrained layer damping treatments. The equivalent network consists of two springs and one dashpot in the following arrangement:

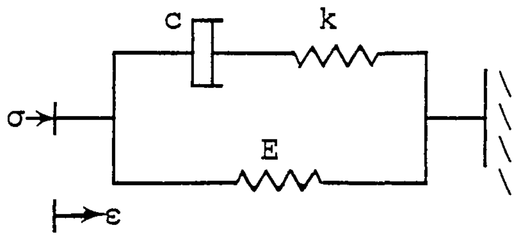

Figure 8. A three parameter model

Casting the constitutive relationship between stress and strain in the form of equation 2 we arrive at the following form:

$$\sigma + a\frac{\partial\sigma}{\partial t} = E\left[\ \varepsilon + b\frac{\partial\varepsilon}{\partial t}\ \right] \tag{10}$$

where a=c/k and b=(1+k/E)c/k.

The three parameter (Figure 8) model's coefficients were selected so that the frequency and damping of the first vibration mode match those of the previous cases and so the damping ratio of the third mode was 0.06 (comparable to our experimental results with passive damping for a particular system [1]). In this case the damping ratio reaches a maximum at frequencies in the vicinity of 100 Hz. This behavior typifies systems treated with the section length optimized, constrained layer treatment.

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Figure 9. Three parameter "ideal" constrained layer system.

The models presented are intended only as examples of the modulus operator method. Clearly by considering a large number of springs and dashpots a system equivalent to equation 2 can be created. In practice, one would chose

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1/ This model can be derived by assuming that 1.) the constraining layer and beam are purely elastic 2.) the damping layer is purely viscous and 3.) the constraining layer section length is allowed to approach zero while the damping layer thickness is varied in the proper proportion so as to maintain the optimal damping condition as per [3]. The model is discussed in [9].

the parameters to fit experimental results, setting the model order as appropriate to provide the desired fit.

Transfer Functions

In this section transfer functions are developed to express relationships between the control input and the various outputs of interest. The procedure used follows Schmitz [7] and Breakwell [8].

In the interest of designing and evaluating a controller for the system of Figure 1 the following outputs may be of interest:

$\theta(t)$ = angular displacement at x=0
$u(L,t)$ = transverse beam tip deflection
$\sigma(0,t)$ = beam surface strain at x=0
$\sigma(x_i,t)$ = beam surface strain at $x=x_i$

We define the variable T(s) as the Laplace transform of the input torque $\tau(t)$ and let F(x,s)=0. The boundary conditions for equation 7 associated with the system of Figure 1 are analogous to those for an equivalent purely elastic system modified by replacing E with $E^*(s)$. The complete system may be expressed in the following compact form:

$$U_{xxxx}(x,s) + \frac{s^2 \rho A}{E^*(s)I} U(x,s) = 0 \tag{11}$$

subject to the boundary conditions:

$$U(0,s) = 0 \tag{12}$$

$$U_{xx}(0,s) - \frac{bs+Js^2}{E^*(s)I} U_x(o,s) = \frac{-T(s)}{E^*(s)I} \tag{13}$$

$$U_{xx}(L,s) = 0 \tag{14}$$

$$U_{xxx}(L,s) - \frac{ms^2}{E^*(s)I} U(L,s) = 0 \tag{15}$$

The subscript x's denote derivatives with respect to x. In order to avoid excessive notation we introduce:

$$\beta^4 = \frac{-s^2 \rho A}{E^*(s)I} \tag{16}$$

The s^2 in boundary conditions 13 and 15 is conveniently eliminated by substituting β as follows.

$$\frac{ms^2}{E^*(s)I} = \frac{-\beta^4 m}{\rho A} = -\underline{m}\beta^4 \qquad (17)$$

$$\frac{Js^2}{E^*(s)I} = \frac{-J\beta^4}{\rho A} = -\underline{J}\beta^4 \qquad (18)$$

The elimination of s in the equations is desirable in the interest of book keeping and to simplify the problem of root finding. Note that the s associated with joint damping is not easily eliminated. A method that we call "boundary condition feedback" will be introduced in the next section to handle this term. The general solution to equation 7 is:

$$U(x,s) = A_1 \sin\beta x + A_2 \sinh\beta x + A_3 \cos\beta x + A_4 \cosh\beta x \qquad (19)$$

where the A_i's are generally functions of s. The application of boundary condition 12 yields $A_4 = -A_3$. The remaining coefficients are best handled using Cramer's rule. Upon applying 13 through 15 and simplifying the resulting matrix form is:

$$\begin{bmatrix} S_\beta & -Sh_\beta & C_\beta + Ch_\beta \\ -C_\beta + \beta\underline{m}S_\beta & Ch_\beta + \beta\underline{m}Sh_\beta & S_\beta - Sh_\beta + \beta\underline{m}(C_\beta - Ch_\beta) \\ \underline{J}\beta^3 - \dfrac{bs}{E^*(s)I\beta} & \underline{J}\beta^3 - \dfrac{bs}{E^*(s)I\beta} & -2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{-T(s)}{E^* I\beta^2} \end{bmatrix}$$

(20)

where $S_\beta = \sin\beta L$, $Sh_\beta = \sinh\beta L$, $C_\beta = \cos\beta L$ and $Ch_\beta = \cosh\beta L$. Following the standard Cramer's rule notation we designate $A_i = \Delta_i/\Delta$ where Δ is the determinant of the 3x3 matrix in 20. Observe that $\Delta = 0$ is the frequency equation for the system of interest. Using the foregoing notation the expression for U(x,s) becomes:

$$U(x,s) = \frac{1}{\Delta}\left[\Delta_1 \sin\beta x + \Delta_2 \sinh\beta x + \Delta_3(\cos\beta x - \cosh\beta x)\right]$$

(21)

where the Δ's are given by:

$$\Delta = 2\{(1-\underline{m}\underline{J}\beta^4+\underline{m}bs/E^*(s)I)(C_\beta Sh_\beta - S_\beta Ch_\beta)- +(bs/E^*(s)I\beta-\underline{J}\beta^3)(1+C_\beta Ch_\beta)-2\underline{m}\beta S_\beta Sh_\beta\} \tag{22}$$

$$\Delta_1 = \frac{T(s)}{EI^*\beta^2}\{1 + S_\beta Sh_\beta + C_\beta Ch_\beta + 2\underline{m}\beta C_\beta Sh_\beta\} \tag{23}$$

$$\Delta_2 = \frac{T(s)}{E^*I\beta^2}\{1 + C_\beta Ch_\beta - S_\beta Sh_\beta - 2\underline{m}\beta S_\beta Ch_\beta\} \tag{24}$$

$$\Delta_3 = \frac{T(s)}{E^*I\beta^2}\{C_\beta Sh_\beta - S_\beta Ch_\beta - 2\underline{m}\beta S_\beta Sh_\beta\} \tag{25}$$

Transfer functions for the variables of interest can be derived directly from equation 21. Those that are of present interest are listed below.

<u>Hub Angle Transfer Function</u>

$$\frac{\Theta(s)}{T(s)} = \frac{U_x(0,s)}{T(s)} = \frac{2}{E^*I\beta\Delta}\{1 + C_\beta Ch_\beta + \underline{m}\beta(C_\beta Sh_\beta - S_\beta Ch_\beta)\} \tag{26}$$

<u>Tip Position Transfer Function</u>

$$\frac{U(L,s)}{T(s)} = \frac{2}{E^*I\beta^2\Delta}\{S_\beta + Sh_\beta\} \tag{27}$$

<u>Strain at x=0</u>

$$\frac{\underline{\sigma}(0,s)}{T(s)} = \frac{h}{2}\frac{U_{xx}(0,s)}{T(s)} = -\frac{h}{\Delta E^*I}\{S_\beta Ch_\beta - C_\beta Sh_\beta + 2\bar{m}S_\beta Sh_\beta\} \tag{28}$$

<u>Strain at an arbitrary position $x=x_i$</u>

$$\frac{\underline{\sigma}(x_i,s)}{T(s)} = \frac{h}{2}\frac{U_{xx}(x_i,s)}{T(s)} = \tag{29}$$

$$\frac{h\beta^2}{2\Delta T(s)} \{-\Delta_1 \sin\beta x_i + \Delta_2 \sinh\beta x_i - \Delta_3(\cos\beta x_i + \cosh\beta x_i)\}$$

Note that the T(s) and β^2 appearing in equation 29 cancel because they also appear in the Δ_i's.

Boundary Condition Feedback

Equations 26 through 29 as expressed above are exact in the sense that they are exact solutions of the model chosen. Approximations to the transfer functions can be obtained by expressing them as ratios of polynomials in s where the polynomials are determined by the zeros of the analytical functions that make up their numerators and denominators. The IMSL subroutine ZANLYT, which finds the roots of a complex analytic function, can be used for this purpose. In the interest of simplifying the expressions and the task of finding the roots it is desirable to express them strictly in terms of β, (recall that β contains s). On examining the equations 26 through 29 we find that if the joint damping coefficient b is non-zero there is no convenient way to eliminate s. There is, however, the alternative of considering the effect of joint damping as a feedback term constituting part of the control torque input. We call this procedure "boundary condition feedback". According to this concept we would proceed as follows. Let $T'(s) = T(s) - bs\Theta(s)$ in boundary condition 13 This modification removes the bs terms that appear in Δ and hence the transfer functions may be expressed in terms of β only. Polynomial ratio approximations for the transfer functions can be constructed in the form:

$$G_\theta(s) = \frac{\Theta(s)}{T'(s)} = \frac{K_\theta}{s^2} \frac{\Pi(\beta^4 - \beta_i^4)}{\Pi(\beta^4 - \beta_j^4)} \quad (30)$$

$$G_{UL}(s) = \frac{U(L,s)}{T'(s)} = \frac{K_{UL}}{s^2} \frac{\Pi(\beta^4 - \beta_i^4)}{\Pi(\beta^4 - \beta_j^4)} \quad (31)$$

$$G_{\sigma 0}(s) = \frac{\underline{\sigma}(0,s)}{T'(s)} = \frac{K_{\sigma 0}}{E^*(s)} \frac{\Pi(\beta^4 - \beta_i^4)}{\Pi(\beta^4 - \beta_j^4)} \quad (32)$$

$$G_{\sigma i}(s) = \frac{\underline{\sigma}(x_i,s)}{T'(s)} = \frac{K_{\sigma i}}{E^*(s)} \frac{\Pi(\beta^4 - \beta_i^4)}{\Pi(\beta^4 - \beta_j^4)} \quad (33)$$

where the K's are constants and β_i and β_j are the zeros of the numerator and denominator transcendental functions, respectively. Note that denominator roots for each of the transfer functions are the same (the roots of $\Delta = 0$) while each of the numerators have a unique set of roots. Given that an expression for the modulus operator $E^*(s)$ is known, the values of s corresponding with a root β_k are obtained by solving for the (complex) roots of the polynomial:

$$s^2(1 + a_1 s + a_2 s^2 + \ldots + a_n s^n) + \beta_k^4 (EI/\rho A)(1 + b_1 s + b_2 s^2 + \ldots + b_m s^m) = 0 \quad (34)$$

to find the s-plane poles and zeros of the transfer functions. Assuming that $m \geq n-2$ (satisfied by the systems of interest) equation 34 has m+2 roots for each β_k. Typically[2] two of these (a complex conjugate pair) will be highly dominant (small negative real parts). This pair describes damped oscillatory motion of the beam. The remaining m "fast" roots describe the motion of the beam during viscoelastic creep.

Now in order to account for the joint damping we consider the feedback system illustrated in Figure 11

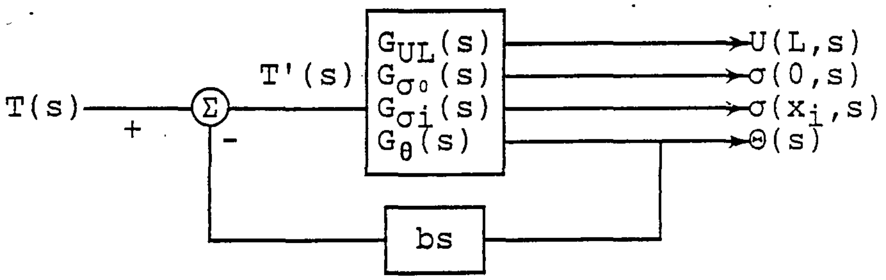

Figure 10. Block diagram illustrating feedback of joint damping

The effect of the joint damping, is to alter the values of the s-plane poles of the transfer functions 26 through 29 The variation of these poles as the damping coefficient b is increased is illustrated graphically by the root locus of Figure 11 The system illustrated has a payload mass to beam mass ratio (m/L) of 0.136 and hub inertia to beam inertia

2/ The s-plane zeros (numerator) of 27 and 29 are exceptions.

ratio[3] ($\underline{J}/L^3$) of 0.0275.

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Figure 11. Locus of open loop transfer function poles as joint damping coefficient b is varied (s-plane).

The example illustrated allows for some observations on the effect of joint damping upon open loop[4] system poles. We note that increasing b effectively shifts the lower frequency eigenvalues to the left while having comparatively little effect on the higher frequency eigenvalues. This type of behavior has been observed in experimental measurements [1]. The fact that the higher modes are relatively unaffected is attributed to the fact that the system represented in Figure 11 has non-zero hub inertia. Recall that the poles of G_θ correspond to the pinned modes of the system while the zeros correspond to the clamped modes. An inertia at the pinned end causes the pinned mode frequencies to approach the clamped mode frequencies at the higher modes. Because the poles and zeros for the higher modes are very closely spaced, boundary condition feedback has relatively little effect on the corresponding loci. This suggests that while damping applied at the joint (active or passive) can quite effectively damp the higher modes of systems with little or no hub inertia [10], the presence of a hub inertia negates this effect somewhat.

The boundary condition feedback strategy can be extended to eliminate the effects of other "non-classical" boundary conditions (i.e. masses and inertias) and accordingly provides a convenient way to parameterize systems which may have varying payloads (e.g. robotic manipulators). This may also provide a convenient scheme for representing systems which may have time varying forces and moments at both ends of the flexible links, such as a multi-link manipulator. Further consideration of this method will be given in [9].

Some Observations on Transfer Function Poles and Zeros. In the interest of more general applicability it is convenient to consider the roots β_k in the nondimensional form $\alpha_k = \beta_k L$. These roots are fully symmetric in the (complex) α-plane so that for each root α_k there are corresponding roots $-\alpha_k$, $j\alpha_k$ and $-j\alpha_k$. Because β_k is raised

3/ These dimensionless units where suggested by Schmitz [7].

4/ Open loop in the sense that the system with joint damping is viewed as an open loop system from the standpoint of applying active control.

to the fourth power in equation 34 each of these results in the same s-plane values. Two types of α-plane roots occur, the first (type 1) fall directly on the real and imaginary axes, the second type (type 2) lie on two lines ±45° from the axes.

In undamped systems [7], each type 1 root in the α-plane maps into a complex conjugate s-plane pair on the imaginary axis while each type 2 root maps to one negative and one positive (non-minimum phase) root in the s-plane.

In systems with distributed damping following the three parameter model, each type 1 root maps to a left half s-plane complex conjugate pair and one (comparatively) very "fast" negative real axis root. Each type 2 root in the α-plane produces a positive real root and two negative roots (the number depending on the viscoelastic model order) in the s-plane. Moreover the negative real axis s-plane roots that are a consequence of the viscoelastic model, tend to fall in a tightly grouped clump, whether they be poles or zeros, so that pole-zero cancellation, renders their effect upon the the system negligible in the root locus sense. The authors postulate that roots for higher order modulus operators will follow a similar pattern[5] however this issue remains to be studied in detail.

All of the denominator roots are Type 1 as are the numerator roots of the colocated transfer functions G_θ and $G_{\sigma 0}$. The numerator of the tip transfer function G_{UL} has only Type 2 roots. The spacing of these roots is such that the separation between the real parts of consecutive roots (and imaginary parts, which are equal to the real parts in magnitude) in the α-plane, asymptotically approaches π for the higher modes[6] . The numerator of the non-colocated strain transfer function $G_{\sigma i}$ has both Type 1 and Type 2 roots, the separation of which depend upon the dimensionless distance $\xi_i = x_i/L$ of the strain sensor from the pinned end of the beam. When $\xi_i = 0$, the separation of the type 1 roots approaches π and the separation of the Type 2 roots is ∞. For $\xi_i > 0$ the asymptotic spacing of the Type 1 roots approaches $\pi/(1-\xi_i)$ while the Type 2 roots have real parts (and imaginary) that are separated by π/ξ_i. These relationships, which are illustrated graphically in Figure 12 indicate that the transfer function $G_{\sigma i}$ acquires more (undesirable) [7] non-minimum phase character and the

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5/ Providing that the modulus operator's parameters are chosen such that the dominant complex conjugate s-plane eigenvalues are located in a physically reasonable fashion.

6/ The spacing between the third and fourth roots satisfies this relationship to seven decimal places.

desirable property [11] of alternating (imaginary parts of the) complex conjugate poles and zeros is lost when the dimensionless distance ξ_i is greater than zero.

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Figure 12. Root spacing (α-plane) for the numerator of $G_{\sigma i}$.

Conclusions

A method for modeling a structure with distributed damping has been presented whereby a Laplace domain elastic modulus operator $E^*(s)$ replaces the now more familiar frequency domain complex modulus. The modulus operator is mathematically permissible for systems with transient excitation, giving it a clear advantage over the complex modulus approach where feedback control is concerned. The authors find that a simple modulus operator incorporating only three parameters can be fit to experimental results for a beam treated with a constrained viscoelastic damping treatment with reasonable accuracy. Better fits are possible by increasing the order of $E(s)$. In this paper the beam is modeled as if it were constructed of a homogeneous viscoelastic material while in reality it is a composite structure composed of both elastic and viscoelastic components. The authors maintain that, given a modulus operator representation of the viscoelastic damping layer material, it is possible to derive the the modulus operator for an equivalent homogeneous material representative of the composite structure. Using the three parameter representation, it was found that the system transfer functions where somewhat similar in character to those for the corresponding purely elastic system.

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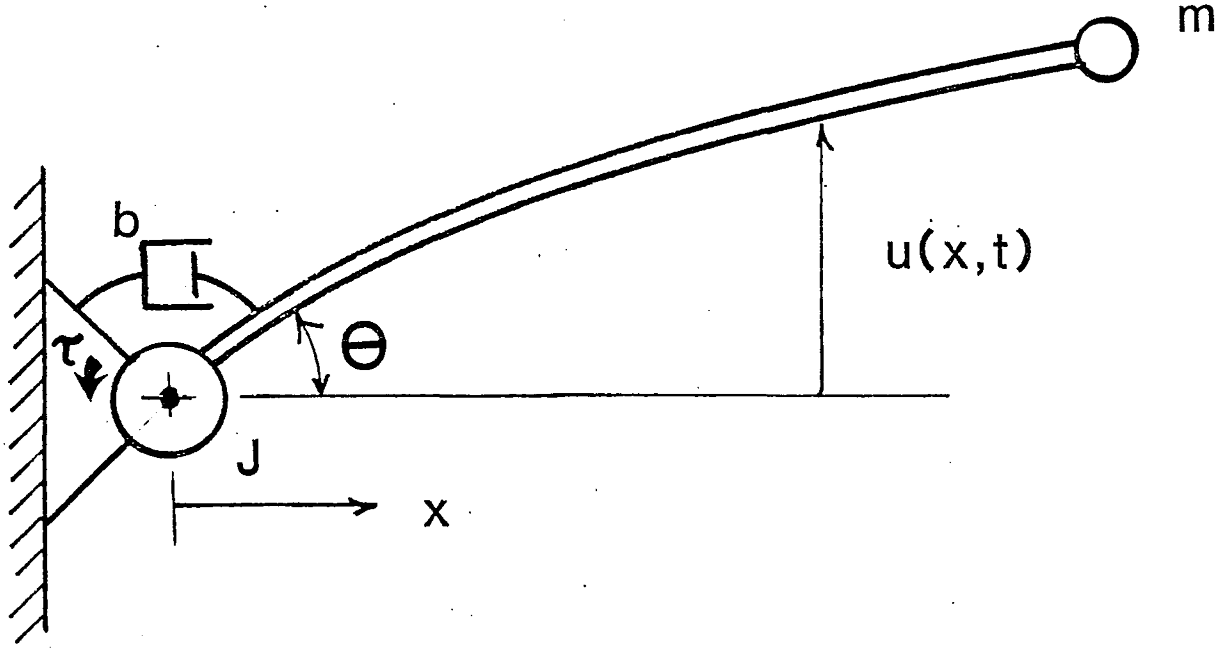

Figure # 1

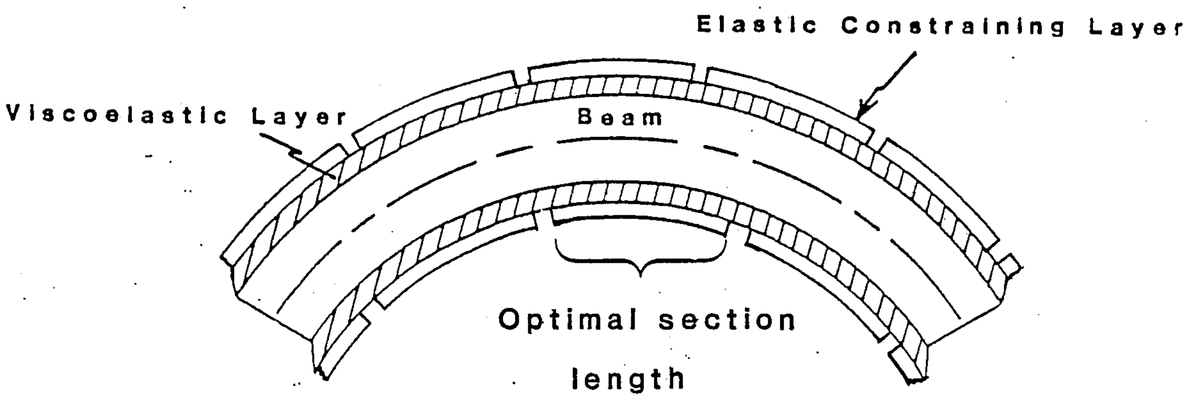

Figure #2

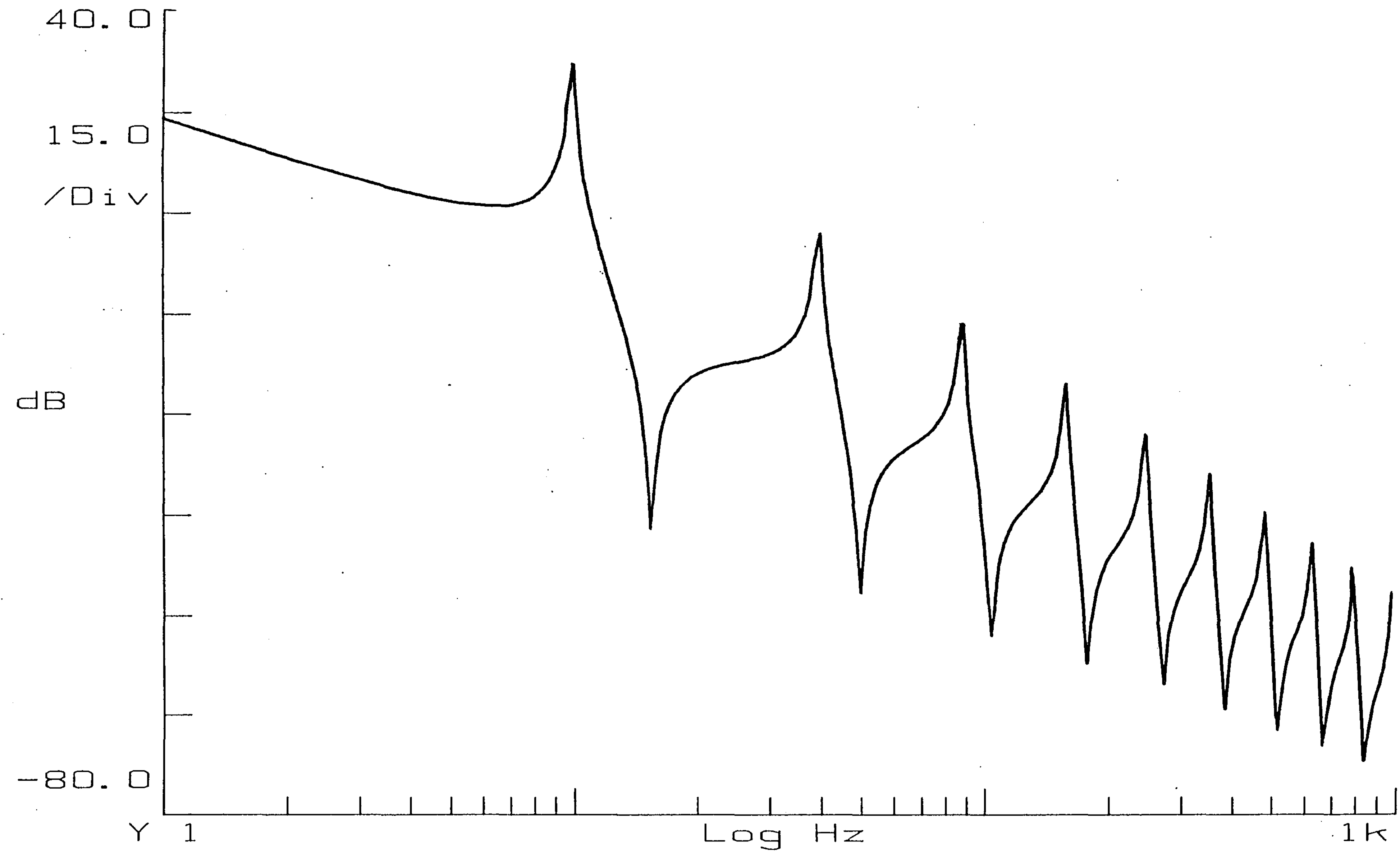

Figure #5. Hook's Law + Structural

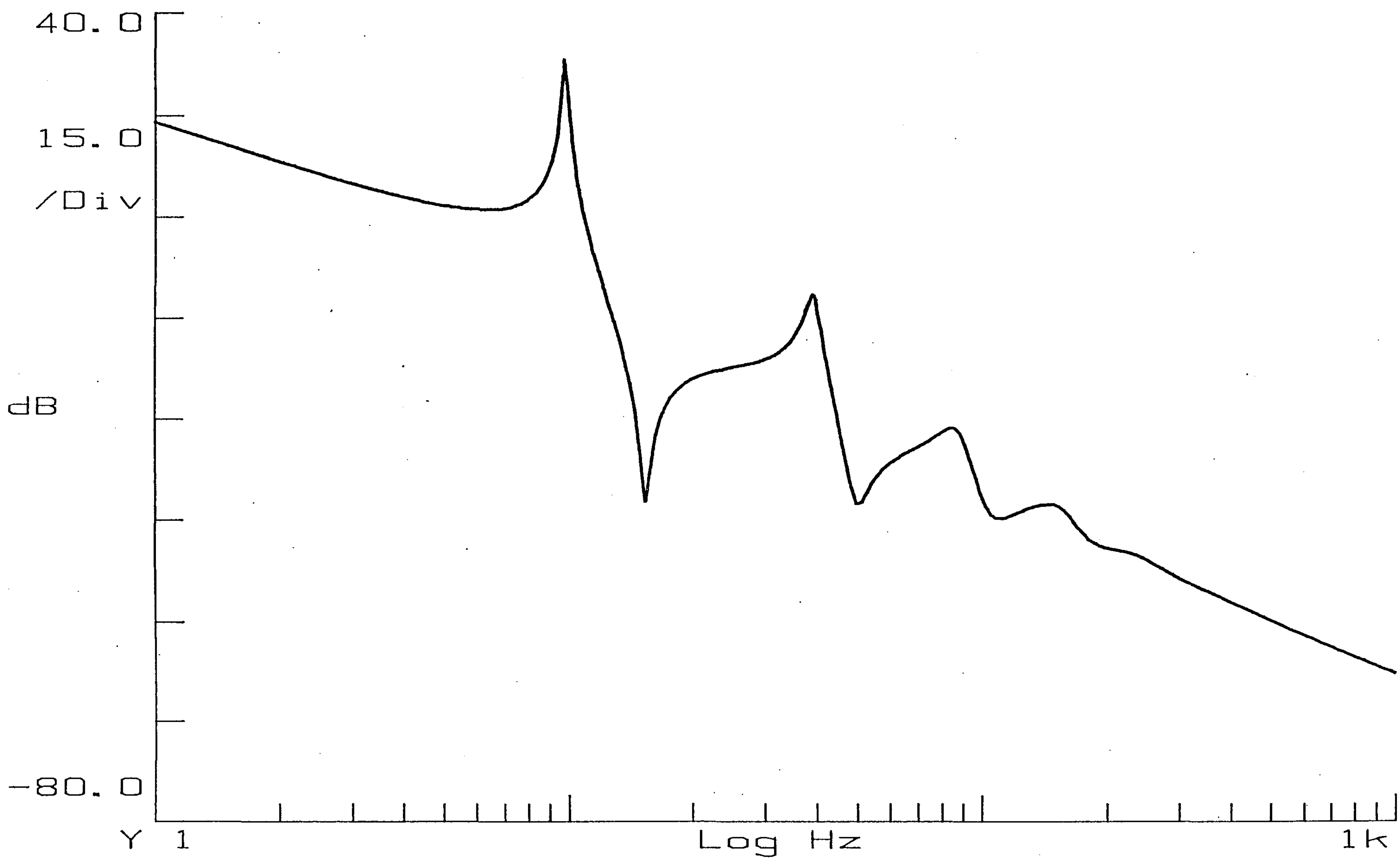

Figure # 7 Kelvin – Voight

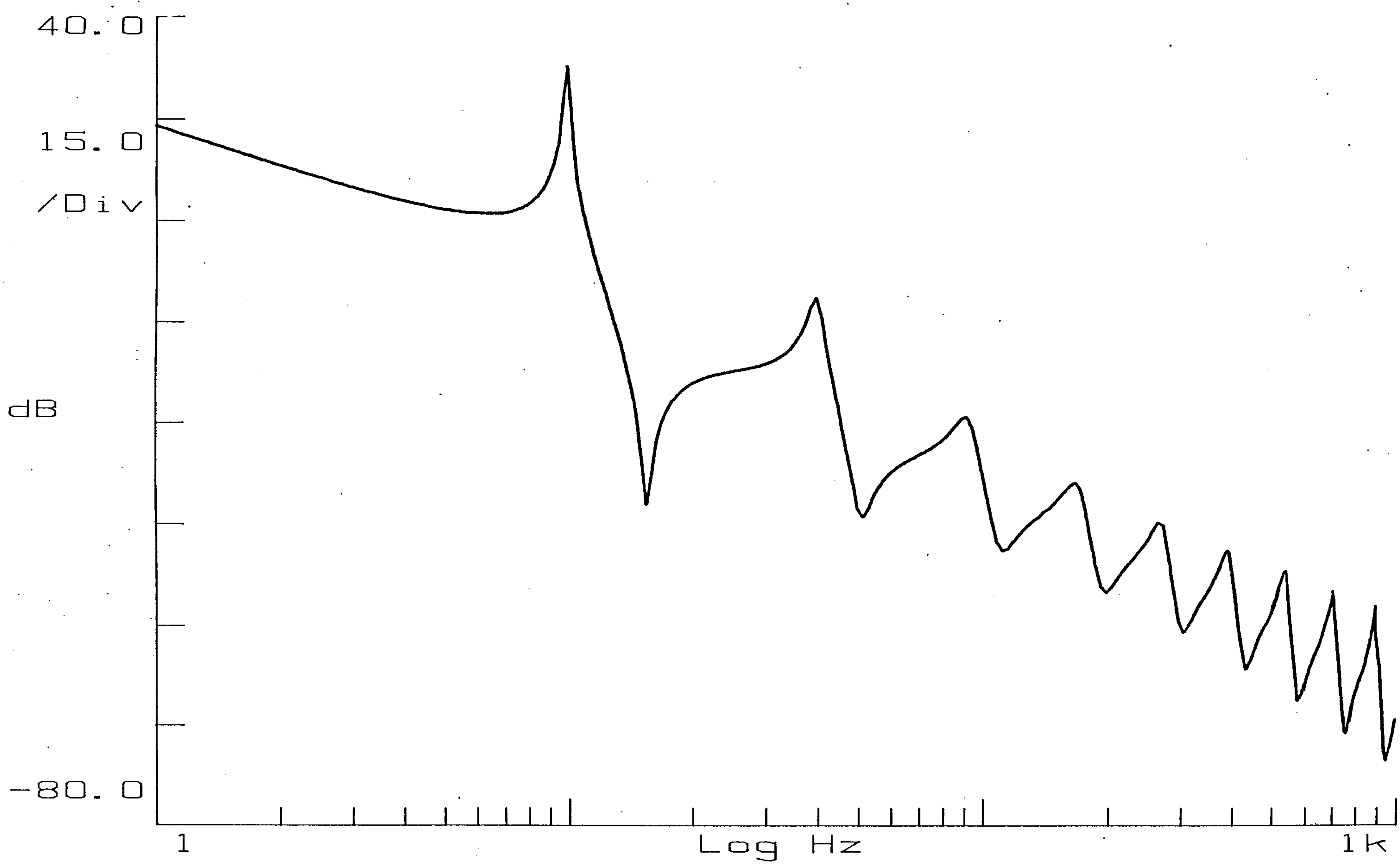

Figure #9 3-parameter

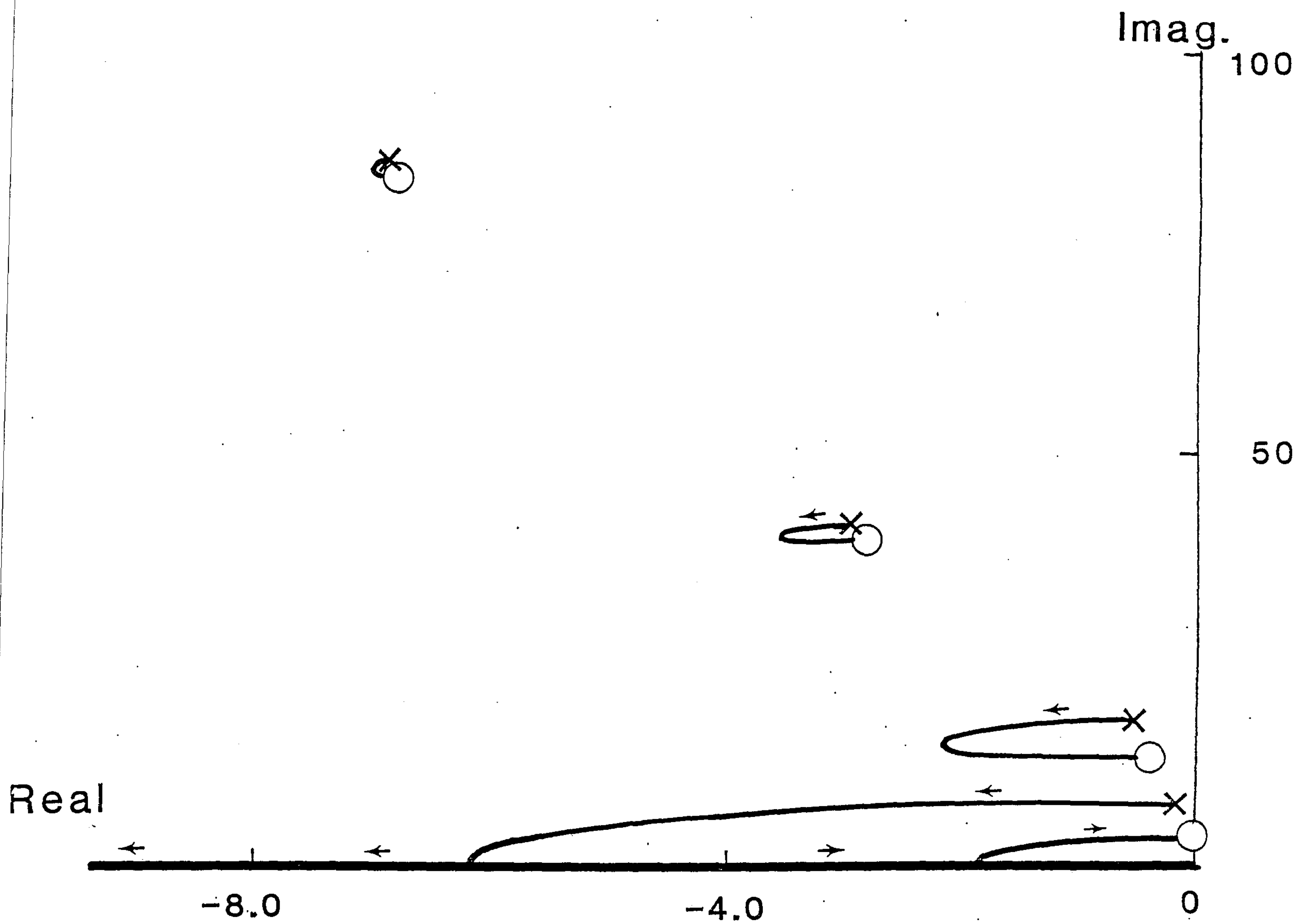

Figure #11

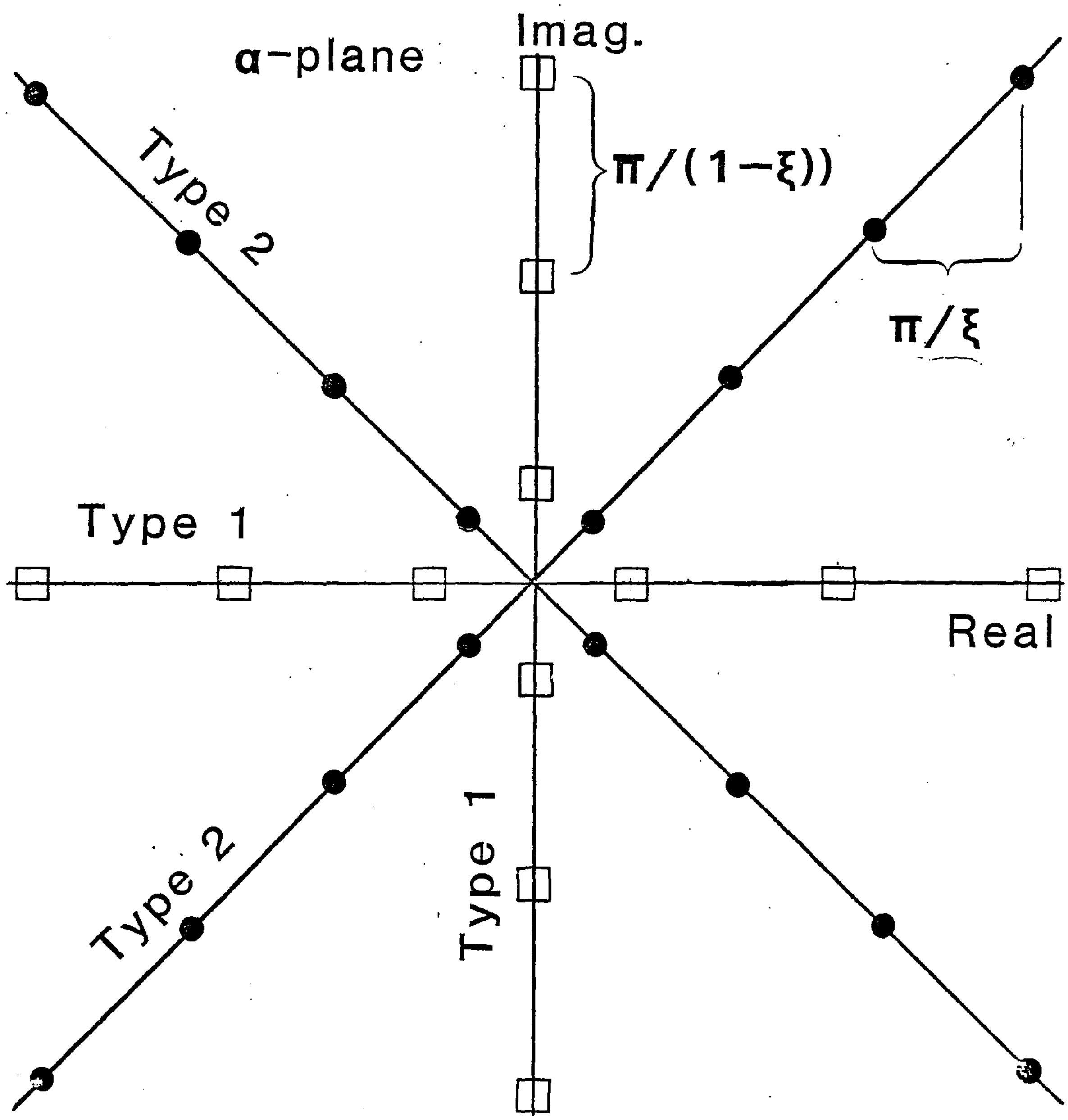

Figure 12