Dynamics of Flexible Manipulator Arms: Alternative Derivation, Verification, and Characteristics for Control

This work seeks to provide an effective way for developing the dynamics of a multi-link flexible manipulator consisting of rotary joints connecting two links. Kinematics of both the rotary joint motion and the link deformation are described by 4×4 transformation matrices as proposed in previous works (Book, 1984). The link deflection is assumed small so that the link transformation can be composed of summations of assumed link shapes. To determine the appropriate choice of component mode shapes, two essential techniques employed here are experimental and finite element methods. The resulting equations of motion allow the complete nonlinear model to be recursively derived from the Jacobian matrix and the mass properties via symbolic manipulation. Two prototype models of flexible manipulators are used to verify the dynamics with frequency and time responses. This paper contributes several new results: (1) the velocity terms (Coriolis and centrifugal forces) are related to variations in the mass matrix, (2) the skew symmetry of certain useful terms are shown, (3) the system is theoretically demonstrated to be stable with joint P.D. controllers in addition to an experimental approach, (4) practical and effective incorporation of actuator dynamics (hydraulic cylinder) and structural complexity (non-uniform cross section) is achieved through selection of mode shapes, (5) geometric constraints are incorporated through simplified coordinate transformations and (6) the results are verified on two physical cases.

1 Introduction

The lightweight manipulator arm is a challenging research topic with the potential to improve the performance of robots and other high performance motion systems (Book, 1984; Bayo, 1987). The main problem with lightweight structures is in the resulting flexible vibrations naturally excited as the manipulator is commanded to move or is disturbed (Cannon and Schmitt, 1984). A suitable dynamic model is an important prerequisite for designing the flexible structure because system behavior must be predicted while improved performance is sought. Many proposed control algorithms also require dynamical models for real-time control calculation. Both the control system and mechanical system require correct models for simulation. However, the dynamical model and conventional kinematics (Asada and Slotine, 1985; Paul, 1981) that are widely used for rigid manipulators, are no longer adequate for high performance demands. Consideration of flexibility of the lightweight manipulator arm is necessary. Accuracy in the model is acquired at some cost, and the application of manipulator arms to practical endeavors gives incentive to improve the efficiency of the simulation and formulation of the dynamical models. The recursive nature of the dynamics utilized heavily in numerical formulations of models must be incorporated into symbolic manipulation via programs such as MACSYMA or SMP to reduce simulation time and eliminate the errors of manual manipulation.

In formulation of the dynamic equations of motion for the rigid-link manipulator, much work has been done (Asada and Slotine, 1985; Paul, 1981; Hollerbach, 1980) with both the Newton-Euler method and the Lagrangian method. The Newton-Euler formulation, based on the Newton's Second Law, is greatly complicated by link deflection (Greenwood, 1965). By contrast, the Lagrangian is described in terms of work and energy with generalized coordinates to develop equations of motion so that workless forces and constraint forces are not considered. The resultant equations are generally compact and provide a closed-form expression in joint torques and displacements (Asada and Slotine, 1985). By yet another alternative, Kane's method, the equations are obtained from constructing the generalized active and inertia forces with appropriate selection of the generalized speeds (Kane and Levinson, 1985).

In modeling and controlling a manipulator with a single flexible link, analysis and experimental verifications have been...
II Flexible Arm Kinematics and Kinetic Energy

2.1 Kinematics. A robot positioning task is naturally specified in Cartesian coordinates by a position vector \( P \) and a matrix of direction cosines \( R \). Thus, the position of an arbitrary point attached to the rigid body can be represented as

\[
X_i = A_i E_i X_i
\]

where

\[
X_{i-1} = \{ P_{i-1}, 1 \}^T = \text{the position of the point in system } i - 1,
\]

\( A_i \) = transformation for joint \( i - 1 \),

\( E_i \) = transformation due to link length and deflection.

Considering the \( i \)th consecutive coordinate transformation along a serial linkage, we can derive the location \( r_i \) of a point along the \( i \)th coordinate viewed from the base frame.

\[
r_i = T_i r_i'
\]

where

\[
T_i = A_i E_i A_{i-1} E_{i-1} \cdots A_1 E_1 A_i
\]

and \( r_i' \) is the position vector related to the \( i \)th coordinate before the transformation due to the link \( E_i \) (also see (Book, 1984)).

The transformation \( A_i \) is a function of the joint displacement \( q_i \). Only rotational joints are allowed here. The flexible deflection is assumed to be a finite series of separable modes which are the product of admissible shape functions and time-dependent generalized coordinates. Higher modes are comparatively small in amplitude. With small deflections at the end of the link, the matrix \( E_i \) can then be expressed as

\[
E_i = \sum_{j=1}^{m_j} \delta_j \begin{bmatrix} 0 & -\theta_{ij} & \theta_{ij} & u_{ij} \\ \theta_{ij} & 0 & -\theta_{ij} & v_{ij} \\ -\theta_{ij} & \theta_{ij} & 0 & w_{ij} \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

where

\( \delta_j \) is the time-dependent amplitude of mode \( j \) of link \( i \),

\( \theta_{ij}, \theta_{ij}, \) and \( \theta_{ij} \) are the angles of rotation about \( x, y, \) and \( z \), respectively, of mode \( i \) of link \( j \),

\( u_{ij}, v_{ij}, \) and \( w_{ij} \) are the deflection components of mode \( j \) of link \( i \), in \( x, y, \) and \( z \) directions, respectively.

\( m_i \) is the number of modes used to describe the deflection of link \( i \),

\( l_i \) is the length of link \( i \).

Note, if the point in question is not at the end of the link, replace \( l_i \) with \( z_i \) measured to that point.

2.2 Kinetic Energy. In this section, the expression for the system kinetic energy is developed for use in Lagrange's equations. First, consider the kinetic energy of a point on the \( i \)th link:

\[
KE_i = \int_{r_{i-1}}^{r_i} \frac{1}{2} \text{Tr} \left( \frac{dr_i}{dt} \frac{dr_i'}{dt} \right) dm
\]

where

\[
\frac{dr_i}{dt} \text{ is called the velocity vector, and from (2.3a)}
\]

\[
\frac{dr_i}{dt} = \left( \frac{d}{dt} \right)_{r_i'} \left( \frac{d}{dt} \right)_{r_i'}
\]

Equation (2.5) becomes

\[
KE_i = \int_{r_{i-1}}^{r_i} \left( \frac{d}{dt} \right)_{r_i'} \left( \frac{d}{dt} \right)_{r_i'} \text{Tr} \left( \frac{dr_i}{dt} \frac{dr_i'}{dt} \right) + \frac{1}{2} \text{Tr} \left( \frac{dr_i}{dt} \frac{dr_i'}{dt} \right) dm
\]

where

\[
E_i = A_i E_i A_{i-1} E_{i-1} \cdots A_1 E_1 A_i
\]

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\[ KE = \sum_{i=1}^{n} \text{Trace} \left( \dot{T}_i B_i T_i^T + 2 \dot{T}_i B_i T_i^T \dot{T}_i \right) \]  
\[ \text{where} \]
\[ B_i = \sum_{j=1}^{m_i} \sum_{k=1}^{m_i} \delta_{ij} \delta_{ik} C_{ijk} \]  
\[ C_{ijk} = \frac{1}{2} \int_{\text{link}_k} \left[ u_{ijkl} v_{ijkl} w_{ijkl} \right]^T \left[ u_{ijkl} v_{ijkl} w_{ijkl} \right] dm \]  
\[ B_3 = \sum_{j=1}^{m_i} \delta_{ij} C_{ij} + \sum_{k=1}^{m_i} \delta_{ik} \delta_{ij} C_{ijk} \]  
\[ C_{ij} = \frac{1}{2} \int_{\text{link}_k} [0,0,z_{i0}]^T [0,0,z_{i0}] dm \]  
\[ B_3 = C_i + \sum_{j=1}^{m_i} \delta_{ij} \left( C_{ij} + C_{ij'} \right) + \sum_{k=1}^{m_i} \delta_{ik} \delta_{ij} C_{ijk} \]  
\[ C_i = \frac{1}{2} \int_{\text{link}_k} [0,0,z_{i0}]^T [0,0,z_{i0}] dm \]  

Note that \( dm = Mdz_0 \) for a slender beam \( i \), \( M \) is density/area.  

The three terms of the right-hand side of Eq. (2.8b) which include \( \delta_{ij} \delta_{ij} \) are expressed as follows:  
\[ \frac{1}{2} \sum_{j=1}^{m_i} \left( \sum_{k=1}^{m_i} \delta_{ij} \delta_{ik} \right) \int_{\text{link}_k} \frac{1}{2} \int_{\text{link}_k} \left[ u_{ijkl} v_{ijkl} w_{ijkl} \right]^T \left[ u_{ijkl} v_{ijkl} w_{ijkl} \right] dm \]

Those terms (2.12) also appear in (Book, 1984).

### III System Potential Energy

In addition to the computation of the kinetic energy, we need to find the potential energy in order to derive the equations of motion for the dynamic system. The potential energy of the system includes three sources: joint elasticity, gravity and link deformation. The first term is associated with joint coordinates \( q \), the second term is a function of position, and the last term, called the strain energy, results from the energy stored in the link due to deformation. The potential energy related to the gravity and link deflection can be obtained from integrating over the length of the individual link, and then summing over all links.

#### 3.1 Elastic Joint Potential Energy

We consider an \( n \)-link manipulator with revolute joints, and model the elasticity of the \( i \)th joint as an equivalent torsional spring with stiffness \( K \) and equivalent stiffness \( K_e \). Since each kinematic joint is actuated directly with some sort of actuator, however, for a linear actuator used to rotate a revolute joint through the use of a four-bar linkage, the equivalent stiffness can be found by the corresponding transformation between joint and actuator spaces (Spong, 1987).

The coordinate \( \theta_i \) in the joint transformation \( A_i \) along with the equivalent stiffness \( K_e \) constitutes the elastic joint potential energy which does not involve the coordinates associated with link deflections. The formula for this potential energy is described as:

\[ PE_e = \sum_{i=1}^{n} \frac{1}{2} K_e \theta_i^2 \]  

The coordinate \( \theta_i \) is measured from the unstretched position.
to $q_i$. In other words, the elastic joint potential energy has the positive value relative to the "basic energy" which is a function of the unstretched position.

3.2 Gravity Potential Energy. In robotic arms with elasticity, the gravity potential energy for a differential element on the $i$th link is

$$dPE_{di} = -g^T T_i' r dm$$

(3.2a)

where the gravity vector $g$ has the form

$$g^T = [g_x, g_y, g_z, 0]$$

(3.2b)

Integrating over the link and summing over all links, the gravity potential energy becomes

$$PE_{g} = -g^T \sum_{i=1}^{n} T_i h_i$$

(3.3a)

where

$$h_i = M_i h_{mi} + \sum_{k=1}^{m_i} \delta_{ik} q_{ik}$$

$M_i$ is the total mass of link $i$, $h_{mi} = [0, 0, h_{mi}]^T$ is a vector to the center of gravity from joint $i$ (undeformed).

$$\epsilon_{ik} = \int_{v_{ink}} [u_{ik}, v_{ik}, w_{ik}, 0]^T dm$$

(3.3c)

From the above equations, we know that if the link is homogeneous, the total distance to the center of gravity is the vectoral sum of the deformed and undeformed parts. However, the gravity potential energy is a function of generalized coordinates $q_i$ and $\delta_{ij}$.

3.3 Link Strain Potential Energy. The link deflection for a slender beam is assumed to be a linear combination of the generalized coordinates $\delta_{ij}(t)$ and mode shapes $u_{ik}$, $v_{ik}$, and $w_{ik}$ in $x$, $y$, and $z$ axes respectively, while the rotational components of the link deflection are taken into account in the $z$ axis. Compression $w_{ik}$ is not initially included as it is generally much smaller. With a truncated modal approximation for the $i$th link deformation, the equation in the $x$-direction is shown as

$$u_{ix} = \sum_{k=1}^{m_i} \delta_{ik} u_{ik}$$

(3.4a)

$u_{ix}$ is similarly represented in the $y$-direction and

$$\theta_{iz} = \sum_{k=1}^{m_i} \delta_{ik} \theta_{ik}$$

(3.4b)

The strain potential energy related to the link deformation which is integrated along the $z$-axis coincident with the link is described as

$$PE_{di} = \frac{1}{2} \sum_{i=1}^{n} \left[ \int_{v_{ink}} \left( EI_x \left( \frac{\partial^2 u_{ix}}{\partial z^2} \right)^2 + EI_y \left( \frac{\partial^2 v_{ix}}{\partial z^2} \right)^2 + E_G J_z \left( \frac{\partial^2 \theta_{iz}}{\partial z^2} \right)^2 \right) dz_i \right]$$

(3.5)

where $E$ is Young's modulus of elasticity, and $I_x$ and $I_y$ are the area moments of inertia of the link about an axis parallel to the $x$ and $y$ axes, respectively, and through the centroid of the cross section. $E_G$ is the shear modulus and $J_z$ is the polar area moment of inertia of the link about the centroid.

Substituting (3.4) into (3.5), $PE$ then becomes

$$PE_{di} = \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{m_i} \delta_{ik} \delta_{ik} (K_{iiik} + K_{iiik} + K_{iiik})$$

(3.6a)

where $K_{iiik}$, $K_{iiik}$, and $K_{iiik}$ are stiffness coefficients,

$$K_{iiik} = \int_{v_{ink}} EI_x \frac{\partial^2 u_{ix}}{\partial z^2} \frac{\partial^2 u_{ix}}{\partial z^2} dz_i$$

(3.6b)

Note that the stiffness coefficient matrix is symmetric, for example, $K_{iiik} = K_{iiik}$. The link potential energy for the total system $PE$ can therefore be written as

$$PE_{di} = \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{m_i} \delta_{ik} \delta_{ik} K_{iiik}$$

where $K_{ijik} = K_{iiik} + K_{iiik} + K_{iiik}$.

$PE_{di}$ is independent of $q_i$, the joint coordinate. In fact, Eq. (3.7) can be made much more general than the initial assumptions regarding the link strain energy. Compression strain energy and link forms other than beam, for example, can also be represented in this formulation. The values of coefficients $K_{iiik}$ are determined analytically or numerically, e.g., by finite element methods. Combining elastic joint and link strain energies leads to

$$PE_{di} + PE_{di} = \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{m_i} \sum_{l=1}^{m_i} K_{iiik} \xi_{ik} \xi_{ik}$$

(3.8)

where

$$\xi_{ik} = \{ \delta_{ik} \} \quad j = 0, 1, 2, \ldots, m_i$$

IV Equations of Motion

The Lagrangian formulation shown as below results in a compact system of equations which is appealing from both the dynamic modeling and control engineering points of view.

$$\frac{d}{dt} \left( \frac{\partial KE}{\partial x} \right) - \frac{\partial KE}{\partial x} = \frac{\partial PE}{\partial x} = Q$$

(4.1)

where $Q$ is the generalized force corresponding to joint variable $q$. For the deflection variables, the corresponding generalized force will be zero if the corresponding modal deflection or rotations have no displacement at those locations where external forces or moments are applied. Thus, it is assumed for the present development that the modal functions are selected so that is the case. This is convenient for utilizing the result as well since joint angle sensors measure the variable $q$.

Since $m_{ijab}$ is a function of $x_{ij}$ or $x_{ijab}$ in (2.10a), the first term in (4.1) is computed as

$$\frac{d}{dt} \left( \frac{\partial KE}{\partial x} \right) = \frac{d}{dt} \left( \sum_{i=1}^{n} \sum_{j=0}^{m_i} m_{ijab} x_{ijab} \right)$$

(4.2)

The second term in (4.1) includes the partial derivative of the kinetic energy given by

$$\frac{\partial KE}{\partial x_{ijab}} = \frac{\partial}{\partial x_{ijab}} \left( \frac{1}{2} \sum_{i=1}^{n} \sum_{j=0}^{m_i} \sum_{a=0}^{m_i} m_{ijab} x_{ijab} x_{ijab} \right)$$

(4.3)
Taking the partial derivative of the potential energies of the elastic joint and the link deflection leads to

$$\frac{\partial (PE_e + PE_s)}{\partial x_{pq}} = \sum_{i=0}^{n} K_{pq,i} \delta x_i$$  \hspace{1cm} (4.4)

And the gravity term comes from (3.3)

$$\frac{\partial PE_g}{\partial x_{pq}} = \begin{cases} -g \sum_{i=p}^{n} \frac{\partial T_i}{\partial x_{pq}} h_i - g TT_{pq} & \text{when } q \neq 0 \\ -g \sum_{i=p}^{n} \frac{\partial T_i}{\partial x_{pq}} h_i & \text{when } q = 0 \end{cases}$$  \hspace{1cm} (4.5)

Notice that

$$\sum_{i=p}^{n} \frac{\partial T_i}{\partial x_{pq}} = \frac{\partial A_n}{\partial q} = 0$$

when $p = n$ and $q = 0$. Hence, the gravity term is a function of $x$ and we define $G(X) = [G_{pq}]$ with elements (4.5).

Finally, substituting (4.2), (4.3), (4.4), and (4.5) into (4.1), we can obtain the equations of motion affected by $Q_{pq}$

$$\sum_{i=1}^{n} \sum_{j=0}^{m} m_i \omega_{ij} \dot{x}_j + \sum_{i=1}^{n} \sum_{j=0}^{m} \sum_{l=0}^{m} C_{ijklpq} \dot{x}_i \dot{x}_j + \sum_{i=0}^{n} K_{ijklpq} \dot{x}_i + G_{pq} = Q_{pq}$$  \hspace{1cm} (4.6a)

where

$$C_{ijklpq} = \frac{\partial m_{ijpq}}{\partial x_{ijpq}} \frac{1}{2} \frac{\partial m_{ijpq}}{\partial x_{pq}}$$  \hspace{1cm} (4.6b)

The dynamic Eq. (4.6a) can also be written in matrix-vector form as

$$M(X) \ddot{X} + H(X, \dot{X}) \dot{X} + K \dot{X} + G(X) = Q$$  \hspace{1cm} (4.7)

where $H(X, \dot{X}) = C(X, \dot{X})$ and $C_{ijklpq}$ is the element's coefficient of $C$ (Appendix A). More detail in $C$ can be found in (Book, 1984). Equation (4.6b) is claimed to be an effective equation to generate $C$ using a symbolic program when the inertia matrix is known.

Conclusively, the element of the inertia matrix $M(X)$ arises from Eq. (2.12). The coupling elements are from (4.6b). The stiffness matrix $K$ is determined by the elastic joint and the link deflection energies (3.8) or (4.4). Appendix A illustrates that the inertia matrix is positive definite as well as symmetric, and (M-2H) is skew-symmetric. For further work of controlling the dynamic system of flexible manipulators, the joint PD controllers, which have exhibited adequate control performance for rigid robotic manipulators (Asada and Slotine, 1985) also theoretically demonstrate stability for flexible manipulators (Appendix B).

V Experimental Setups

Two prototype models of flexible robotic arms at the Flexible Automation Laboratory at Georgia Tech are used to verify the dynamic equations derived in the previous sections. The frequency and time responses are two approaches that can be used to demonstrate agreement between analytical and experimental results. The actuator dynamics will be considered because it is presented in the experimental systems. However, a linear case has been adapted for comparing analytical and experimental results, using sufficiently slow and small motion of the links.

The first experimental apparatus (Fig. 5.1) is a one-link flexible manipulator driven by an electric torque motor. The arm is a four foot aluminum beam with the section oriented in the horizontal plane. Two strain gages mounted at the base and at mid-length of the beam measure the link deflection. Table 5.1 lists the physical properties (Hastings, 1986).

The other apparatus is a two link manipulator, RALF (Robotic Arm, Large, and Flexible), with a parallel mechanism (Fig. 5.2). Each link is a cylindrical hollow beam, ten feet long. The parallel mechanism's function is force transmission for actuating the upper link. The weight of the robotic structure is about seventy pounds. More details are given in the paper by Huggins (1988). The analytical work involved is more complicated than the first case.

VI The Case of a One-Link Flexible Manipulator

The process of forming the dynamic model for flexible manipulators has been discussed. One difference from the rigid manipulator is the existence of the stiffness term in (4.7) which determines the system vibration due to the flexible link deflection. Since the one-link beam moves only in the horizontal plane, the flexible deflection is simply described by an infinite series of separable modes, i.e., the deflection in $x$ and $z$ di-

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Table 5.1 Physical properties of one-link flexible manipulator

<table>
<thead>
<tr>
<th>Aspect</th>
<th>System value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>Aluminum 6061-T6</td>
</tr>
<tr>
<td>Form</td>
<td>Rectangular 3/4 x 3/16 in.</td>
</tr>
<tr>
<td>Length</td>
<td>-48 in.</td>
</tr>
<tr>
<td>Moment of Inertia</td>
<td>4.12E-4 in.</td>
</tr>
<tr>
<td>EI Product</td>
<td>4120 lbf-in.</td>
</tr>
</tbody>
</table>

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Fig. 5.1 Single Link flexible Manipulator (SLM)

Fig. 5.2 RALF (Robotic Arm, Large, and Flexible)
Table 6.1 Comparison of modal frequencies (Hz) of one-link case

<table>
<thead>
<tr>
<th>Mode</th>
<th>Measured</th>
<th>Analytical</th>
<th>Finite element</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.08</td>
<td>2.096</td>
<td>2.088</td>
</tr>
<tr>
<td>2</td>
<td>13.92</td>
<td>13.989</td>
<td>13.955</td>
</tr>
<tr>
<td>3</td>
<td>41.38</td>
<td>41.524</td>
<td>41.452</td>
</tr>
<tr>
<td>4</td>
<td>81.18</td>
<td>81.225</td>
<td>81.203</td>
</tr>
<tr>
<td>5</td>
<td>136.352</td>
<td>136.345</td>
<td></td>
</tr>
</tbody>
</table>

rect of $E$ in (2.2) has been ignored. However, the first few modes will be accurate enough to describe the flexible deflection because the amplitudes of higher modes are small compared to the amplitudes of the lower modes. Here, $n$ is selected to be 2 (Hastings and Book, 1987). The transformation of a rigid-body motion has been expressed as $A$ in (2.2). Thus, the equation of motion can be derived as presented in reference (Yuan et al., 1989).

The beam, directly driven by the torque motor (which is here considered as a high bandwidth torque source), is controlled by feedback signals from the joint in the case of a one-link manipulator. Therefore, the clamped-mass boundary conditions are imposed such that the mode shapes can be derived from the Bernoulli-Euler beam formulation. Because it is a simple structure, the solution can be obtained analytically (Yuan et al., 1989). Table 6.1 compares the measured modal frequencies to those computed from the linear dynamical equations with the mode shapes using the analytical method.

When a small amount of proportional damping is employed (Meirovitch, 1967), the simulations of the dynamics motion with two modes result in the plot shown in Fig. 6.1(a), (b) for a step change in the desired joint angle. Note that joint feedback has been implemented in this case. The strain measurement at the base is shown in reference (Hastings and Book, 1987). It can be seen in Table 6.1 that the model implemented with only the first few modes produces system natural frequencies that agree to less than 1 percent with the experimental data (Hastings, 1986). Obviously, the clamped-mass shape is acceptable in representing the link deflection in this case.

VII The Case of RALF

The total system of RALF should include the actuator dynamics in addition to the two-link manipulator with a parallel mechanism. Hydraulic actuators are employed to drive the structure. Since the actuator has an equivalent stiffness as its dynamical characteristic, natural frequencies of the total system may differ from the original static system.

For this experiment, the hydraulic actuator including the servovalve and amplifier is simplified as a third order transfer function from the input voltage of the valve motor $x_i$ to the piston displacement $x_p$ of the hydraulic cylinder (Merritt, 1967). The hydraulic natural frequency $\omega_n$ is, therefore, obtained from the transfer function, while the hydraulic spring rate $k_h$ is simply a useful concept in computing hydraulic natural frequencies and interpreting dynamic response. By imposing a swept sine input and using an LVDT (linear variable-differential transformer) attached to the piston rod to measure the displacement data, the frequency response test for the joint 1 and 2 actuators were performed. No load was placed on the actuators. Curve fitting the measured frequency response data, the transfer functions are:

For actuator 1: $\frac{x_p}{x_i} = \frac{5.217E3}{S(S^2 + 3.836E2S + 7.509E4)}$ (7.1a)

For actuator 2: $\frac{x_p}{x_i} = \frac{3.374E3}{S(S^2 + 4.838E2S + 9.869E4)}$ (7.1b)

The hydraulic natural frequencies for the actuators at 1 and 2 are approximately 43.6 Hz and 50.0 Hz. Thus, the hydraulic spring rates are calculated to be 1.54 $E3$ lb/in for actuator 1 and 6.03 $E3$ lb/in for actuator 2.

Experimental mode shapes are observed from the frequency functions. The excitation consisted of a swept sine wave. Measurements were taken by accelerometers sequentially placed along the link at 10 points on each link. The deflection amplitude is then obtained by integrating the acceleration spectrum twice. A cubic spline is used to connect the amplitude values at discrete points to obtain the curves in Fig. 7.1 through 7.4. Note that a third order polynomial is the lowest order that can satisfy the Bernoulli-Euler equation for continuity of bending moments.

7.1 Finite Element Method for Modeling RALF. From the previous sections, we recognize that the link deflection is the main component of system oscillation for the flexible manipulator. However, the clamped-mass mode shapes of simple beams, which are conveniently derived analytically, may not be suitable for complicated structures like RALF. It is easily observed in Fig. 5.2 that the major difference between the RALF and two serial-link arm is a drive link parallel to the lower link (link 1) used to drive the upper link (link 2) in RALF. The parallel drive link and link 1 form a closed kinematic chain. Finite element methods are used to analyze the system and comparisons are made between the numerical and experimental results. Table 7.1 shows comparison of the results from experiments and finite element methods with the joint angle between link 1 and link 2 equal 90 degrees. Measurements are taken by a force transducer on the shaker and an accelerometer on the link.
When the linear hydraulic actuators are attached to the structure, the clamped boundary condition used previously must be modified. However, the hydraulic spring rate can be thought of as a "dynamic" spring in some sense so that the boundary condition for the driving joint can be modeled as a concentrated spring with an equivalent stiffness. The results for natural frequencies are shown in Table 7.2.

The parallel linkage driving joint 2 can be treated precisely as a geometric constraint to the dynamic behavior. An approximate method is used here which yields accurate results and exhibits the versatility of the modeling approach. The simplifications are motivated conceptually and experimentally. Conceptually, one expects flexure of the first and parallel driving links to have minimal effect on the distance between their pinned ends. Deflection of these links can thus cause no rotation of link 2, as would happen in the serial link case. This decouples the deflection rotations of the two links. Deflection translations show simplified coupling when joint two is near a right angle. By choosing the transformation matrix that describes link 2 to be independent of link 1 rotations the effect of the parallel linkage is thus incorporated. The symbolic manipulation of the equations easily handles this non-standard form.

Table 7.1 Comparison of modal frequencies (Hz) of RALF without actuators attached

<table>
<thead>
<tr>
<th>Mode</th>
<th>Experiment</th>
<th>Finite element</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.37</td>
<td>5.95</td>
</tr>
<tr>
<td>2</td>
<td>12.00</td>
<td>12.78</td>
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<td>3</td>
<td>37.87</td>
<td>30.19</td>
</tr>
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<td>4</td>
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<td>60.60</td>
</tr>
<tr>
<td>5</td>
<td>94.02</td>
<td>95.05</td>
</tr>
</tbody>
</table>

Table 7.2 Comparison of structural frequencies (Hz, with actuators attached)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Experiment</th>
<th>Finite element</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.70</td>
<td>6.08</td>
</tr>
<tr>
<td>3</td>
<td>30.00</td>
<td>29.70</td>
</tr>
<tr>
<td>4</td>
<td>49.50</td>
<td>49.50</td>
</tr>
</tbody>
</table>

Table 7.3 Comparison of structural frequencies (Hz, with actuators attached)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Experiment</th>
<th>Finite element</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.70</td>
<td>5.82</td>
</tr>
<tr>
<td>2</td>
<td>9.12</td>
<td>9.18</td>
</tr>
<tr>
<td>3</td>
<td>30.00</td>
<td>55.70</td>
</tr>
</tbody>
</table>
Fig. 7.7 Simulated strain response at the mid-point of lower link

Fig. 7.8 Simulated strain response at the mid-point of upper link

Fig. 7.9 Simulated strain response of lower link (sine wave)

Fig. 7.10 Simulated strain response of upper link (sine wave)

each link arising from an impulsive force when the actuators are controlled. It is obvious that the structural damping should be included in the dynamics. From Figs. 7.5 and 7.6, the proportional damping ratio of about 0.2 is selected for use in the simulations. The Runge-Kutta method using sampling time 0.1 ms is utilized to solve the nonlinear dynamics. The results are shown in Figs. 7.7-7.8.

The responses from experiments and simulations show similar characteristics. A frequency of about 5.7 Hz for experiment and 6.1 Hz for simulation is most apparent in the lower link and a frequency of about 9.12 Hz for experiment and 9.18 Hz for simulation is most apparent in the upper link. Furthermore, the sine wave response can also be used to illustrate a property of the dynamic system. Figures 7.9 and 7.10 show the strain responses of simulations for the lower and upper links, while Figs. 7.11 and 7.12 show the experimental results. Further tuning of the model might improve the damping ratio of higher frequency modes. However, the fact that the simplified model used in the simulation may cause small deviation from measured experimental data is expected and acceptable from the engineering point of view.

VIII Summary and Conclusion

A transformation between two coordinates which includes rigid body motion and link deformation has been established in the form of a 4 x 4 matrix. Any point on the robotic arm can be described from the base coordinate in terms of those transformations. Due to the recursive nature of the transformation chain, the Lagrangian formulation of the dynamics is derived as efficiently as has been done in the rigid link case. The inertia matrix is shown to be positive and symmetric and a condition of skew-symmetry exists in the equations of motion that is useful in Lyapunov stability proofs (Yuan et al., 1989). There also exists a stiffness term in the equations, which is not present in the case of rigid-link manipulators. However, the approach requires that the link deflection is assumed to be small compared to joint motion and only rotational joints are allowed.

The system frequency deduced from the analytical formulation is highly dependent on the mode shapes of the link deflection, while the mode shapes are determined by the boundary conditions present. The application of feedback control to flexible manipulators also impacts the resultant flexible vibration modes. With the correct dynamical equation obtained in symbolic form, the choice of reasonable modes will result in the correct prediction of dynamic response. In the case of a one-link flexible manipulator, clamped-mass modes are selected when control action constrains the motion of the joint. The manipulator may have pinned-mass modes with no feedback control on the joint actuator (Cannon and Schmitz, 1984). RALF provides a more thorough and complicated case to verify the analysis. The dynamics of the actuator needs to be considered if the bandwidth of the actuator is not large enough to be ignored, i.e., the dynamics of the actuator with low bandwidth contributes to the boundary conditions. To elim-


Huggins, J. D., 1988, "Experimental Verification of a Model of a Two-Link Flexible Lightweight Manipulator," M.S. thesis, Georgia Institute of Technology, Atlanta, GA.


Paul, R. P., Robot Manipulators, MIT Press.


APPENDIX A

From Eq. (2.6), $r_i$ can also be represented as (Yuan et al., 1989)

$$r_i = J_i \dot{x}_i$$  \hspace{1cm} (A.1)

where

$L_i$ is the total number of generalized coordinates
$J_i$ is $4 \times L_i$ of Jacobian matrix
$x_i$ is the time derivative vector including $i$ joints, say $q_1, q_2, \ldots, q_i$ and ($L_i - i$) time dependent flexible coordinates.
The kinetic energy of the $n$-link flexible arm is then
\[ KE = \frac{1}{2} \sum_{i=1}^{n} \int_{\text{link}_i} r_i \dot{r}_i \, dm \]
\[ = \frac{1}{2} \dot{x}^T M \dot{x} \quad \text{(A.2a)} \]
where
\[ M = \sum_{i=1}^{n} \int_{\text{link}_i} [\mathbf{J}^T \mathbf{J}] \, dm \quad \text{(A.2b)} \]

Obviously, the inertia matrix $M$ is positive and symmetric in (A.2b) and in (2.10a). The kinetic energy in (2.10a) can be expressed by physical reasoning. The necessary and sufficient conditions for this are that the inertia matrix satisfies positive definiteness, unless the system is at rest.

The coupling element of $C$ which represents the coefficient in the second term in (4.6a) has the following relation:
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{m} \sum_{b=0}^{m} \left( \frac{\partial m_{ab}}{\partial x_{ai3}} - \frac{1}{2} \frac{\partial m_{ab}}{\partial x_{aj3}} \right) \dot{x}_{ai3} \dot{x}_{aj3} \]
\[ = \frac{1}{2} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\partial m_{ab}}{\partial x_{ij}} - \frac{\partial m_{ab}}{\partial x_{ij}} \right) \dot{x}_{ij} \dot{x}_{ij} \right) \quad \text{(A.3)} \]

In comparison with (4.7) and defining the element of the coupling matrix $H$ and $[H_{ijpq}]$, we have
\[ [H_{ijpq}] = \frac{1}{2} \sum_{a=1}^{n} \sum_{b=0}^{m} \left( \frac{\partial m_{ab}}{\partial x_{ij}} - \frac{\partial m_{ab}}{\partial x_{ij}} \right) \dot{x}_{ij} \dot{x}_{ij} \]
\[ = \frac{1}{2} \sum_{a=1}^{n} \sum_{b=0}^{m} \left( \frac{\partial m_{ab}}{\partial x_{ij}} - \frac{\partial m_{ab}}{\partial x_{ij}} \right) \dot{x}_{ij} \dot{x}_{ij} \quad \text{(A.4)} \]

Defining $W = M - 2H$, the above (A.4) gives
\[ W_{ijpq} = \sum_{a=1}^{n} \sum_{b=0}^{m} \left( \frac{\partial m_{ab}}{\partial x_{ij}} - \frac{\partial m_{ab}}{\partial x_{ij}} \right) \dot{x}_{ij} \dot{x}_{ij} \quad \text{(A.5)} \]

This shows that $(M-2H)$ is skew-symmetric; i.e., $W + W^T = 0$. By setting $m_a = 0$ in (A.3), it becomes the case of rigid robotic arms, which was found in reference (Asada and Slotine, 1985).

**APPENDIX B**

Independent linear controllers at each joint, commonly called joint proportional-derivative (PD) controllers, which are based on the local measurements of joint positions ($q_i$) and velocities ($\dot{q}_i$) are described as follows:
\[ \tau_i = -K_p q_i - K_D \dot{q}_i \quad \text{(B.1)} \]
where $	au_i$ is the torque acting on the $i$th joint.

$K_p$ and $K_D$ are positive constants. $\dot{q}_i = q_i - q_n$ and $\dot{q} = \dot{q}_1 = \cdots = \dot{q}_n$ ($Q = \dot{q}$).

While $q_n$ is the reference state and assumed to be constant. Physically, the feedback system effectively equips each joint with equivalent rotary spring and damper. The frequency domain approach has been taken with the linearized system in previous work (Book, 1974). A Lyapunov approach is applied here to show the resulting stability.

First, the following equality (nonconservative forces) exists,
\[ X^T Q = \dot{q}^T \tau \quad \text{(B.2)} \]
where $X$, $Q$ are given in (4.7), $\dot{q}^T = [q_1, q_2, \ldots, q_n]$ represents the joint coordinates in (4.7). $\tau = [\tau_1, \tau_2, \ldots, \tau_n]^T$ and $\tau_1 = Q_0$.

Consider a Lyapunov candidate $V$ associated with the total mechanical energy of the feedback system:
\[ V = \frac{1}{2} [\dot{X}^T M X + \dot{X}^T K X + q^T K_D q] \quad \text{(B.3)} \]

where $K_D = \text{diag}[K_p]$, and $K$ is the positive stiffness as well as time independent matrix as (4.7). Note that $K$ is derived from link strain potential energy which is composed of the quadratic form of the mode shape (Section 3.3).

Differentiating $V$ with respect to time gives,
\[ \dot{V} = \dot{q}^T K_D \dot{q} + \dot{X}^T M X + \frac{1}{2} \dot{X}^T M X + \dot{X}^T K X \]
\[ = \dot{q}^T K_D \dot{q} + \dot{X}^T (M X + K X) + \frac{1}{2} \dot{X}^T M X \quad \text{(B.4)} \]

Note that $\dot{X} = \ddot{X}$. By substituting (B.1), (B.2), (4.7), and (M-2H) into above
\[ \dot{V} = \dot{q}^T K_D \dot{q} + \dot{X}^T (Q - H \dot{X}) + \frac{1}{2} \dot{X}^T M \dot{X} \]
\[ = \dot{q}^T K_D \dot{q} + \dot{X}^T Q + \frac{1}{2} \dot{X}^T (M - 2H) \dot{X} \]
\[ = \dot{q}^T K_D \dot{q} + \dot{q}^T \tau \]
\[ = -\dot{q}^T K_D \dot{q} \leq 0 \]

where $K_D = \text{diag}[K_p]$ is a positive matrix.

Therefore, the linearized system with joint PD controller is stable.