SHARP WEIGHTED ESTIMATES FOR SINGULAR INTEGRAL OPERATORS

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SHARP WEIGHTED ESTIMATES FOR SINGULAR INTEGRAL OPERATORS

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To my loving parents.

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SUMMARY

The thesis provides answers to two conjectures (in one case partial and in the other final) in the area of weighted inequalities for Singular Integral Operators. It is interesting to understand the mapping properties of these operators. In particular, if the kernel satisfies certain size and smoothness conditions, it is well established that Singular Integral Operators map Lebesgue spaces $L^p(dx)$ into $L^p(dx)$ for $1 < p < \infty$. If we want to replace Lebesgue measure by a general weight wdx, where w is a nonnegative locally integrable function, $L^p(w)$ bounds can be obtained if w belongs to the so called Muckenhoupt A_p class. The latest results were established in the early seventies. The novelty of this thesis resides in proving sharp dependence of the operator norm on the A_p constant associated to the weight w. The question was known as the A_2 conjecture. In joint work with my advisor, M. Lacey, and one of his collaborators, S. Petermichl, we were able to prove the conjecture for the special case of dyadic Singular Integral Operators. The full conjecture has been proved by T. Hytönen. Another interesting question considers the end point $p = 1$. The open problem was known as the Muckenhoupt-Wheeden conjecture. The thesis provides a counterexample to this conjecture in the dyadic setting. The full conjecture has been answered in the negative in a later result with my coauthor C. Thiele, hence closing a problem that has been open since the early seventies.

CHAPTER I

INTRODUCTION

1.1 Main Concepts

The thesis addresses two conjectures formulated in the area of weighted inequalities for Singular Integral Operators. In the first conjecture, known as the A_2 conjecture, we provide a positive result for a smaller but highly relevant class of operators. This and some recent developments will be discussed in Chapter 2, which is based on the paper [21]. In the second one, known as the Muckenhoupt-Wheeden conjecture, we completely solve the question in the negative by providing counterexamples, first in the dyadic setting and later in the continuous one. The dyadic counterexample will be explained in Chapter 3, and it is based on the paper [38]. The continuous counterexample will be covered in Chapter 4, and it is part of the paper [39].

Singular integral operators have been vastly studied in Harmonic Analysis, as they appeared regularly in Partial Differential Equations. The class of operators we will be interested in are called Calderón-Zygmund operators.

Definition 1.1.1. Let K be a function defined on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) \,:\, x \in \mathbb{R}^d\}$. We say that K is a standard kernel if there exist $\delta > 0$ and constant A such that the following size and smoothness conditions are satisfied:

$$|K(x,y)| \leq \frac{A}{|x-y|^d},$$

$$|K(x,y) - K(x',y)| \leq \frac{A|x-x'|^\delta}{(|x-y| + |x'-y|)^{d+\delta}},$$

when $2|x-x'| \leq max(|x-y|, |x'-y|)$,

$$|K(x,y) - K(x,y')| \leq \frac{A|y-y'|^{\delta}}{(|x-y|+|x-y'|)^{d+\delta}}$$

when $2|y-y'| \leq max(|x-y|, |x-y'|)$.

We are now ready to define Calderón-Zygmund operators.

Definition 1.1.2. An operator T is Calderón-Zygmund operator if

1. T is bounded from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$;

2. there exists a standard kernel K such that for $f \in L^2(\mathbb{R}^d)$ with compact support,

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy, \quad x \notin \mathrm{supp}(f).$$

We define the Calderón-Zygmund constant $\|T\|_{CZO} = \sup(A, \|T\|_{L^2 \mapsto L^2})$, where A is the constant that appears in Definition 1.1.1.

Boundedness of these operators from the Lebesgue spaces $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, $1 < p < \infty$, and $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ is well understood. Results have been present in the literature since the 1960s. We refer the curious reader to Chapter 5 of [13].

The question that arises next is: What happens if we replace Lebesgue measure by a general weight w? A weight is a locally integrable, non-negative function. The new measure is that whose Radon-Nikodym derivative is precisely w. To answer this question we need to define the A_p class, first introduced by B. Muckenhoupt in [26].

Definition 1.1.3. For w a weight on $\mathbb{R}^d$, we define the A_p characteristic of w to be

$$\|w\|_{A_p} := \sup_Q |Q|^{-1} \int_Q w \, dx \cdot \left[|Q|^{-1} \int_Q w^{-1/(p-1)} \, dx \right]^{p-1}, \qquad 1 < p < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^d$. We say that a weight w is in the A_p class if and only if $\|w\|_{A_p} < \infty$. For $p = 1$, we say that w is in the A_1 class if there exists a constant $c > 0$ such that $Mw(x) \leq cw(x)$ a.e. The smallest of such constants c is called the A_1 characteristic of w and is denoted by $\|w\|_{A_1}$

For a weight w, belonging to the A_p class is equivalent to boundedness of the Hardy-Littlewood operator in $L^p(w)$, [26]. A precise definition of Hardy-Littlewood operator can be found in Section 1.3 of this chapter, Definition 1.3.1. The answer to the singular integral case was given by R. Hunt, B. Muckenhoupt and R. Wheeden in the one dimensional case, [17], and R. Coifman and C. Fefferman in higher dimensions, [6]. Belonging to the A_p class turns out to be sufficient for boundedness, although not always necessary.

Theorem 1.1.4. *If T is a Calderón-Zygmund operator, then for any $w \in A_p$, $1 < p < \infty$, T maps $L^p(w)$ to $L^p(w)$ and for any $w \in A_1$, T maps $L^1(w)$ to $L^{1,\infty}(w)$.*

1.2 A_2 conjecture

The question that concerns us in this part of the thesis is the dependence of the operator norm $\|T\|_{L^p(w)\mapsto L^p(w)}$ on $\|w\|_{A_p}$ when $1 < p < \infty$. The problem has received special attention after applications of these sharp estimates were found when studying the regularity of solutions to the Beltrami equation; see the work of S. Petermichl and S. Volberg in [37]. We formulate the conjecture below (after the recent developments, the conjecture should be called theorem, discussion of recent developments is postponed until the end of Chapter 2).

Conjecture 1.2.1. *Let $w \in A_p$ and let* T *be a Calderón-Zygmund operator as defined in Definition 1.1.2, we have the estimate*

$$\|\mathrm{T}\, f\|_{L^p(w)} \lesssim \|w\|_{A_p}^{\alpha(p)} \|f\|_{L^p(w)}\,, \qquad \alpha(p) = \max\{1, 1/(p-1)\}\,.$$

The main contribution in relation to this question of this thesis is formulated in the next theorem.

Theorem 1.2.2. *Let* T *be a Haar shift operator of index τ, and let w be an A_2 weight. We have the inequality*

$$\|\mathrm{T}\|_{L^2(w)\mapsto L^2(w)} \lesssim \|w\|_{A_2}$$

The implied constant depends only dimension d and the index τ of the operator.

For a complete description of Haar shift operators of a fixed index τ we refer the reader to Chapter 2, Definition 2.1.5.

1.3 Muckenhoupt-Wheeden Conjecture

Another problem related to these sharp estimates at the end point $p = 1$ is Muckenhoupt-Wheeden Conjecture. This question is a natural extension of a classical inequality by C. Fefferman and E. Stein, see [14]. Before we state it, let us recall the definition of Hardy-Littlewood maximal operator.

Definition 1.3.1. Let $f \in L^1_{loc}(\mathbb{R}^d)$, we define the Hardy-Littlewood maximal operators M as

$$Mf(x) = \sup_Q \frac{1_Q}{|Q|} \int_Q |f(y)|\, dy,$$

where 1_Q denotes the characteristic function at cube Q and the supremum is taken over all cubes in $\mathbb{R}^d$. When the supremum is taken over dyadic cubes, the operator is known as the dyadic Hardy-Littlewood maximal operator.

The conjecture is as follows,

Conjecture 1.3.2. *(Muckenhoupt-Wheeden) Let w be a weight and M be the Hardy-Littlewood maximal operator. Let T be a Calderón-Zygmund operator with $\|T\|_{CZO} \leq 1$. Then*

$$\sup_{t>0} tw\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq C \int_{\mathbb{R}} |f| Mw(x) dx$$

The contribution to this question answers it completely: the Conjecture is false. Let us state the main two theorems,

Theorem 1.3.3. *The dyadic version of Muckenhoupt-Wheeden Conjecture is false, i.e., there exist a weight w and a Haar shift operator T for which the weak L^1 inequality fails when M is replaced by the dyadic Hardy-Littlewood maximal operator.*

Theorem 1.3.4. *For each constant $C > 0$ there is a weight function w on the real line and an integrable compactly supported function f and a $t > 0$ such that*

$$t\, w\left(\{x \in \mathbb{R} : |Hf(x)| > t\}\right) \geq C \int |f(x)| Mw(x)\, dx \quad .$$

CHAPTER II

SHARP A_2 INEQUALITY FOR HAAR SHIFT OPERATORS

2.1 Background

We are interested in weighted estimates for singular integral operators, and cognate operators, with a focus on sharp estimates in terms of the A_p characteristic of the weight. In particular we give a new proof of the estimate of Petermichl [33]

$$\|\mathrm{H}\, f\|_{L^2(w)} \lesssim \|w\|_{A_2} \|f\|_{L^2(w)} \,,$$

where $\mathrm{H}\, f(x) = \text{p.v.} \int f(x-y)\ dy/y$ is the Hilbert transform. Petermichl's proof, as well as corresponding inequalities for the Beurling operator [37] and the Riesz transforms [36] have relied upon a Bellman function approach to the estimate for the corresponding Haar shift. We also analyze the Haar shifts, but instead use a deep two-weight inequality of Nazarov-Treil-Volberg [30] as a way to quickly reduce the question to certain Carleson measure estimates. The latter estimates are proved by using the usual Haar functions together with appropriate corona decomposition. The linear growth in terms of the A_2 characteristic is neatly explained by this decomposition.

Let us precede to the definitions.

Definition 2.1.1. For w a positive function (a weight) on $\mathbb{R}^d$ we define the A_p characteristic of w to be

$$\|w\|_{A_p} \coloneqq \sup_Q |Q|^{-1} \int_Q w\ dx \cdot \left[|Q|^{-1} \int_Q w^{-1/(p-1)}\ dx \right]^{p-1} , \qquad 1 < p < \infty \,,$$

where the supremum is over all cubes in $\mathbb{R}^d$.

The relevant conjecture concerning the behavior of singular integral operators on the spaces $L^p(w)$ is

Conjecture 2.1.2. *For a smooth singular integral operator* T *which is bounded on* $L^2(dx)$ *we have the estimate*

$$\|\mathrm{T}\, f\|_{L^p(w)} \lesssim \|w\|_{A_p}^{\alpha(p)} \|f\|_{L^p(w)} \,, \qquad \alpha(p) = \max\{1, 1/(p-1)\} \,. \tag{2.1.3}$$

An extrapolation estimate [37], [11] shows that it suffices to prove this estimate for $p = 2$, which is the case we consider in the remainder of this chapter. Currently this estimate is known for the Hilbert transform, Riesz transforms and the Beurling operator, with the proof using in an essential way the so-called Haar shift operators. This proof will do so as well, but handle all Haar shifts at the same time.

Definition 2.1.4. By a *Haar function* h_Q on a cube $Q \subset \mathbb{R}^d$, we mean any function which satisfies

1. h_Q is a function supported on Q, and is constant on dyadic subcubes of Q. (That is, h_Q is in the linear span of the indicators of the 'children' of Q.)
2. $\|h_Q\|_\infty \le |Q|^{-1/2}$. (So $\|h_Q\|_2 \le 1$.)
3. $\int_Q h_Q(x)\, dx = 0$.

Definition 2.1.5. We say that T is a *Haar shift operator of index* τ iff

$$\mathrm{T}\, f = \sum_{Q \in \mathcal{Q}} \sum_{\substack{Q', Q'' \subset Q \\ 2^{-\tau d}|Q| \le |Q'|, |Q''|}} a_{Q',Q''} \langle f, h_{Q'} \rangle h_{Q''} \,,$$

$$|a_{Q',Q''}| \le \left[\frac{|Q'|}{|Q|} \cdot \frac{|Q''|}{|Q|} \right]^{1/2} .$$

The point of the conditions in the definition is that T be not only an $L^2(dx)$ bounded operator, but that it also be a Calderón-Zygmund operator. In particular, it should admit a weak-$L^1(dx)$ bound that depends only on the index τ. See Proposition 2.3.12.

Theorem 2.1.6. *Let* T *be a Haar shift operator of index* τ*, and let* w *be an* A_2 *weight. We have the inequality*

$$\|\mathrm{T}\|_{L^2(w)\mapsto L^2(w)} \lesssim \|w\|_{A_2} \tag{2.1.7}$$

The implied constant depends only dimension d *and the index* τ *of the operator.*

We have this Corollary:

Corollary 2.1.8. *The inequalities* (2.1.3) *holds for the Hilbert transform, the Riesz transforms in any dimension* d*, and the Beurling operator on the plane.*

As is well-known, these singular integral operators are obtained by appropriate averaging of the Haar shifts, an argument invented in [35], to address the Hilbert transform. For the Riesz transforms, see [34], and the Beurling transform, see [12]. We also derive, as a corollary, the sharp A_2 bound for Haar square functions. We leave the details of this to the reader.

The starting point of our proof is a beautiful 'two weight $T1$ Theorem for Haar shifts' due to Nazarov-Treil-Volberg [30]. We recall a version of this Theorem in Section 2.2. This Theorem supplies necessary and sufficient conditions for an individual Haar shift to satisfy a two-weight L^2 inequality, with the conditions being expressed in the language of the T 1 Theorem. In particular, it neatly identifies three estimates that need to be proved, with two related to paraproduct estimates. In fact, this step is well-known, and is taken up immediately in e. g. [33]. We then check the paraproduct bounds for A_2 weights in Section 2.3 and Section 2.4, which is the main new step in this chapter.

The question of bounds for singular integral operators on $L^p(w)$ that are sharp with respect to the A_p characteristic was identified in an influential paper of Buckley, [3]. It took many years to find the first proofs of such estimates. We refer the reader to [33] for some of this history, and point to the central role of the work of Nazarov-Treil-Volberg [29] in shaping much of the work cited here. The prior proofs of Corollary 2.1.8 have all relied upon Bellman function techniques. And indeed, this technique will supply a proof of the results in this chapter. The Beurling operator is the most easily available, since this operator can be seen as the average of the simplest of Haar shifts, namely martingale transforms, see [12]. The A_2 bound was derived for Martingale transforms by J. Wittwer [44]. The paraproduct structure is much more central to the problem if one works with Haar shifts that pair a 'parent' Haar with a 'child' Haar. If one considers Square Functions, sharp results were obtained in L^2 by Wittwer [45], and Hukovic-Treil-Volberg [16]. Recently, Beznosova [2], has proved the linear bound for discrete paraproduct operators, again using the Bellman function method. It would be of interest to prove her Theorem with techniques closer to those of this chapter.

2.2 The Characterization of Nazarov-Treil-Volberg

The success of this approach is based upon a beautiful characterization of two weight inequalities. Indeed, this characterization is true for *individual* two-weight inequalities. This Theorem can be thought of as a 'Two Weight $T1$ Theorem.' We are stating only a sub-case of their Theorem, which does not assume that the operators satisfy an $L^2(dx)$ bound.

Theorem 2.2.1. *[Nazarov-Treil-Volberg [30]] Let* T *be a Haar shift operator of index* τ*, as in Definition 2.1.5, and* σ, μ *two positive measures. The* L^2 *inequality*

$$\|\mathrm{T}(\sigma f)\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\sigma)}$$

holds iff the following three conditions hold. For all cubes Q, Q', Q'' *with* $Q', Q'' \subset Q$ *and* $2^{-(\tau-1)d}|Q| \le |Q'|, |Q''|$*,*

$$\Big|\int_{Q''} \mathrm{T}(\sigma \mathbf{1}_{Q'})\ \mu(dx)\Big| \le C_{\mathrm{WB}}\sqrt{\sigma(Q')\mu(Q'')} \qquad \text{(Weak Bnded)} \qquad (2.2.2)$$

$$\|\mathrm{T}(\sigma \mathbf{1}_Q)\|_{L^2(Q,\mu)} \le C_{T1}\sqrt{\sigma(Q)} \qquad (T1 \in BMO) \qquad (2.2.3)$$

$$\|\mathrm{T}^*(\mu \mathbf{1}_Q)\|_{L^2(Q,\sigma)} \le C_{T^*1}\sqrt{\mu(Q)} \qquad\qquad (T^*1 \in BMO)$$

Moreover, we have the inequality

$$\|\mathrm{T}(\sigma\cdot)\|_{L^2(\sigma)\to L^2(\mu)} \lesssim C_{\mathrm{WB}} + C_{T1} + C_{T^*1}\,. \tag{2.2.4}$$

This Theorem is contained in [30], Theorem 1.4, aside from the claim (2.2.4). But this inequality can be seen from the proof in their paper. Indeed, their proof is in close analogy to the $T1$ Theorem. Briefly, the proof is as follows. The operator $\mathrm{T}(\sigma\cdot)$ is expanded in 'Haar basis', but the Haar bases are adapted to the two measures σ and μ. This technique appeared in [29], and has been used subsequently in [12], [44], [33]. Expressing the bilinear form $\int \mathrm{T}(\sigma f)\cdot g\ \mu$ as a matrix in these two bases, the matrix is split into three parts. Those terms 'close to the diagonal' are controlled by the 'weak boundedness' condition (2.2.2). Those terms below and above the diagonal are recognized as paraproducts. One of these is of the form

$$\mathrm{P}(f) \coloneqq \sum_Q \sigma(Q)^{-1}\int_Q f\sigma\ dy \cdot \Delta_Q^w(\mathrm{T}(\sigma 1)) \tag{2.2.5}$$

Here the first term is an average of f with respect to the measure σ, and the second is a martingale difference of $\mathrm{T}(\sigma 1)$ with respect to the measure w. In particular, $\Delta_Q^w(\mathrm{T}(\sigma 1))$ are w-orthogonal functions in Q. Thus, one has the equality

$$\|\mathrm{P}(f)\|_{L^2(w)}^2 = \sum_Q \Big|\sigma(Q)^{-1}\int_Q f\sigma\ dy\Big|^2 \cdot \|\Delta_Q^w(\mathrm{T}(\sigma 1))\|_{L^2(w)}^2\,.$$

The inequality $\|\mathrm{P}(f)\|_{L^2(w)} \lesssim \|f\|_{L^2(\sigma)}$ is a weighted Carleson embedding inequality that is implied by the '$\mathrm{T}\,1 \in BMO$' condition (2.2.3). The other paraproduct is dual to the one in (2.2.5).

2.3 Initial Considerations

We collect together a potpourri of facts that will be useful to us, and are of somewhat general nature. We begin with a somewhat complicated definition that we will use in order to organize the proof of our main estimate.

Definition 2.3.1. Let $\mathcal{Q}' \subset \mathcal{Q}$ be any collection of dyadic cubes, and μ a positive measure. Call $(\mathcal{L} \,:\, \mathcal{Q}'(L))$ a *μ-corona decomposition of $\mathcal{Q}'$* if these conditions hold.

1. For each $Q \in \mathcal{Q}'$ there is a member of $\mathcal{L}$ that contains Q, and letting $\lambda(Q) \in \mathcal{L}$ denote the minimal cube which contains Q we have

$$4\frac{\mu(\lambda(Q))}{|\lambda(Q)|} \geq \frac{\mu(Q)}{|Q|} .$$

2. For all $L \subsetneq L' \in \mathcal{L}$

$$\frac{\mu(L)}{|L|} > 4\frac{\mu(L')}{|L'|} . \tag{2.3.2}$$

We set $\mathcal{Q}'(L) \coloneqq \{Q \in \mathcal{Q}' \,:\, \lambda(Q) = L\}$. The collections $\mathcal{Q}'(L)$ partition $\mathcal{Q}'$.

Decompositions of this type appear in a variety of questions. We are using terminology which goes back to (at least) David and Semmes [10], [9], though the same type of construction appears as early as 1977 in [27], where it is called the 'principle cube' construction. A subtle corona decomposition is central to [42], and the paper [1] includes several examples in the context of dyadic analysis.

A basic fact is this.

$$\Big| \bigcup_{\substack{L' \in \mathcal{L} \\ L' \subsetneq L}} L' \Big| \leq \tfrac{1}{4}|L| , \qquad L \in \mathcal{L} . \tag{2.3.3}$$

This follows from (2.3.2), which says that the intervals $L' \subset L$ have much more than their fair share of the mass of μ, hence the L' have to be smaller intervals. And this easily implies

$$\Big\| \sum_{\substack{L' \in \mathcal{L} \\ L' \subset L}} \mathbf{1}_{L'} \Big\|_2 \lesssim |L|^{1/2} . \tag{2.3.4}$$

We have the following (known) Lemma, but we detail it as it is one way that the A_2 condition enters in the proof.

Lemma 2.3.5. *Let $\mathcal{L}$ be associated with corona decomposition for an A_2 weight w. For any cube Q we have*

$$\sum_{\substack{L\in\mathcal{L}\\ L\subset Q}} w(L) \le \tfrac{16}{9}\|w\|_{A_2} w(Q)\,. \tag{2.3.6}$$

Proof. It suffices to show this: For $L \in \mathcal{L}$

$$w\big(\bigcup\{L' \in \mathcal{L} \;:\; L' \subsetneq L\}\big) \le (1 - c\|w\|_{A_2}^{-1})w(L)\,, \qquad c = \tfrac{9}{16}\,. \tag{2.3.7}$$

We begin with a calculation related to A_∞. Let E be a measurable subset of L. Then,

$$\begin{aligned}\frac{|E|}{|L|} &= |L|^{-1}\int_E w^{1/2}\cdot w^{-1/2}\,dx\\ &\le \left[\frac{w(E)}{|L|}\cdot\frac{w^{-1}(L)}{|L|}\right]^{1/2}\\ &\le \left[\|w\|_{A_2}\frac{w(E)}{w(L)}\right]^{1/2}.\end{aligned} \tag{2.3.8}$$

Apply this with $L - E = \bigcup\{L' \in \mathcal{L} \;:\; L' \subsetneq L\}$. Then, by (2.3.3), $|L - E| < \frac{1}{4}|L|$, so that $|E| \ge \frac{3}{4}|L|$. It follows that we then have

$$\frac{9}{16\|w\|_{A_2}}\cdot w(L) \le w(E)\,.$$

Whence, we see that (2.3.7) holds. Our proof is complete. □

Concerning the Haar shift operators T, we make the following definition.

Definition 2.3.9. We say that T is a *simple Haar shift operator of index τ* iff

$$\mathrm{T}\, f = \sum_{Q\in\mathcal{Q}} \langle f, g_Q\rangle \gamma_Q\,,$$

$$g_Q, \gamma_Q \in \operatorname{span}(h_{Q'} \;:\; Q' \subset Q\,, 2^{-\tau d}|Q| \le |Q'|)\,, \tag{2.3.10}$$

$$\|g_Q\|_\infty\,,\ \|\gamma_Q\|_\infty \le |Q|^{-1/2}\,. \tag{2.3.11}$$

Below, we will only consider simple Haar shift operators. The important property they satisfy is

Proposition 2.3.12. *A simple Haar shift operator* T *with index* τ *maps* $L^2(dx)$ *into itself with norm at most* $\lesssim \tau$*. It maps* $L^1(dx)$ *into* $L^{1,\infty}(dx)$ *with norm* $\lesssim 2^{\tau d}$.

The point is that these bounds only depend upon the index τ.

Proof. The proof is well-known, but we present it as some similar difficulties appear later in the proof; see the discussion following (2.5.5). Set

$$\mathrm{T}_s f := \sum_{\substack{Q\in\mathcal{Q} \\ |Q|=2^{sd}}} \langle f, g_Q\rangle \gamma_Q \,,$$

which is the operator at scale 2^s. The 'size condition' (2.3.11) implies that $\|\mathrm{T}_s\|_{L^2(dx)} \le 1$. The 'cancellation condition' (2.3.10) then implies that

$$\mathrm{T}_s \, \mathrm{T}_{s'}^* = \mathrm{T}_s^* \, \mathrm{T}_{s'} = 0 \,, \qquad |s-s'| > \tau \,.$$

So we see that $\|\mathrm{T}\|_{L^2(dx)} \le \tau + 1$.

Concerning the weak $L^1(dx)$ inequality, we use the usual proof. Fix $f \in L^1(dx)$. Apply the dyadic Calderón-Zygmund Decomposition to f at height λ. Thus, $f = g+b$ where $\|g\|_2 \lesssim \sqrt{\lambda}\|f\|_{L^1(dx)}^{1/2}$, and b is supported on a union of disjoint dyadic cubes $Q \in \mathcal{B}$ with

$$\int_Q b \; dx = 0 \,, \qquad Q \in \mathcal{B} \,, \tag{2.3.13}$$

$$\sum_{Q\in\mathcal{B}} |Q| \lesssim \lambda^{-1}\|f\|_1 \,. \tag{2.3.14}$$

For the 'good' function g, using the $L^2(dx)$ estimate we have

$$\begin{aligned} |\{\mathrm{T}\, g > \tau\lambda\}| &\le (\tau\lambda)^{-2}\|\mathrm{T}\, g\|_{L^2(dx)}^2 \\ &\lesssim \lambda^{-2}\|g\|_2^2 \lesssim \lambda^{-1}\|f\|_{L^1(dx)} \,. \end{aligned}$$

For the ‘bad’ function, we modify the usual argument. For a dyadic cube Q, and integer t, let $Q^{(t)}$ denote it's t-fold parent. Thus, $Q^{(1)}$ is the minimal dyadic cube that strictly contains Q, and inductively, $Q^{(t+1)} = (Q^{(t)})^{(1)}$. Observe that (2.3.14) implies

$$\left|\bigcup \{Q^{(\tau)} \ : \ Q \in \mathcal{B}\}\right| \lesssim 2^{\tau d} \lambda^{-1} \|f\|_1 \,.$$

And, the 'cancellation condition' (2.3.10), with (2.3.13), imply that for $Q \in \mathcal{B}$, and $x \notin Q^{(\tau)}$, we have

$$\mathrm{T}(\mathbf{1}_Q b)(x) = \sum_{Q' \,:\, Q^{(\tau)} \subsetneq Q'} \langle \mathbf{1}_Q b, g_{Q'} \rangle \gamma_{Q'}(x) = 0$$

since $g_{Q'}$ will be constant on the cube Q.

Hence, we have

$$|\{\mathrm{T}(b) > \lambda\}| \leq \left|\bigcup \{Q^{(\tau)} \ : \ Q \in \mathcal{B}\}\right| \lesssim 2^{\tau d} \lambda^{-1} \|f\|_1 \,.$$

This completes the proof.

□

We need a version of the John-Nirenberg inequality, which says that a ‘uniform L^0 condition implies exponential integrability.’

Lemma 2.3.15. *Let $\{\phi_Q \ : \ Q \in \mathcal{Q}\}$ be functions so that for all dyadic cubes Q we have*

1. *ϕ_Q is supported on Q and is constant on each sub-cube $Q' \subset Q$ with $|Q'| = 2^{-\tau d}|Q|$;*

2. *$\|\phi_Q\|_\infty \leq 1$;*

3. *there is a constant K so that for all dyadic cubes Q, and any collection $\mathcal{P} \subset \mathcal{Q}$ of dyadic subcubes of Q, we have*

$$\left|\left\{\Big|\sum_{Q' \in \mathcal{P}} \phi_{Q'}\Big| > K\right\}\right| \leq \frac{1}{2}|Q| \,.$$

It then follows that we have the estimate

$$\Big|\Big\{\Big|\sum_{Q'\,:\,Q'\subset Q}\phi'_Q\Big|>2\tau Kt\Big\}\Big|\le \tau 2^{-t+1}|Q|\,,\qquad t>1\,. \tag{2.3.16}$$

Proof. For the purposes of this proof, we are free to assume that if $Q'\subset Q$, and neither of $\phi_{Q'}$ and ϕ_Q are identically zero, then $\ell(Q')\le 2^\tau\ell(Q)$. We can then prove the conclusion, without the additional terms τ that appear in (2.3.16).

It is enough to prove that for any cube Q,

$$\Big|\Big\{\Big|\sum_{Q'\,:\,Q'\subset Q}\phi'_Q\Big|>(K+1)t\Big\}\Big|\le 2^{-t+1}|Q|\,,\qquad t\ge 1\,.$$

Set

$$G(t)=\sup_Q|Q|^{-1}\Big|\Big\{\Big|\sum_{Q'\,:\,Q'\subset Q}\phi'_Q\Big|>(K+1)t\Big\}\Big|$$

We have by assumption that $G(1)\le 2^{-1}$. We argue that $G(t+1)\le\frac12 G(t)$, which, by induction, will prove the Lemma.

Fix a cube Q. Each $\phi_{Q'}$ is bounded by one in L^∞ norm, hence if the sum over Q' contained in Q exceeds $(K+1)(t+1)$, then it must first exceed $(K+1)t$. Accordingly, we let $\mathcal{S}$ be the maximal subcubes S of Q such that for some $x\in S$ we have

$$|\Phi_S|:=\Big|\sum_{\substack{Q'\,:\,S\subset Q'\subset Q\\ \ell(Q')\ge 2^\tau\ell(S)}}\phi_{Q'}(x)\Big|>(K+1)t$$

Note that the sum above is in fact constant on S, and it is at most $(K+1)t+1$. Moreover, if we set $\mathcal{P}$ to be those $Q'\subset Q$ such that Q' contains some $S\in\mathcal{S}$, with $2^\tau\ell(S)\le\ell(Q')$. We then have

$$\Big\{\Big|\sum_{Q'\in\mathcal{P}}\phi_{Q'}\Big|>(K+1)t\Big\}=\bigcup_{S\in\mathcal{S}}S\,. \tag{2.3.17}$$

And then, the definition of $\mathcal{S}$ proves the inequality.

The measure of the set on the left in (2.3.17) is controlled by $G(t)|Q|$, by assumption. It also follows that we have the inclusion

$$\Big\{\Big|\sum_{Q'\,:\,Q'\subset Q}\phi_{Q'}\Big|>(K+1)(t+1)\Big\}\subset\bigcup_{S\in\mathcal{S}}\Big\{\Big|\sum_{Q'\,:\,Q'\subset S}\phi_{Q'}\Big|>K\Big\}.$$

The sum on the left is over all $Q'\subset Q$, but the terms on the right are localized to the cubes in $\mathcal{S}$. And so we can complete the proof by estimating

$$\begin{aligned}\Big|\Big\{\Big|\sum_{Q'\,:\,Q'\subset Q}\phi_{Q'}\Big|>(K+1)(t+1)\Big\}\Big| &\le \sum_{S\in\mathcal{S}}\Big|\Big\{\Big|\sum_{Q'\,:\,Q'\subset S}\phi_{Q'}\Big|>K\Big\}\Big|\\ &\le \frac{1}{2}\sum_{S\in\mathcal{S}}|S|\\ &\le \frac{1}{2}G(t)|Q|\,.\end{aligned}$$

This completes the induction.

□

2.4 The Main Argument

We begin the main line of argument to prove (2.1.7). We no longer try to keep track of the dependence on τ in our estimates. (It is, in any case, exponential in τ.) Accordingly, we assume that we work with a subset $\mathcal{Q}_\tau$ of dyadic cubes with 'scales separated by τ.' That is, we assume that for $Q'\subsetneq Q$ and $Q',Q\in\mathcal{Q}_\tau$ we have $|Q'|\le 2^{-d\tau}|Q|$, where d is dimension.

It is well-known that (2.1.7) is equivalent to showing that

$$\|\mathrm{T}(fw)\|_{L^2(w^{-1})}\lesssim\|w\|_{A_2}\|f\|_{L^2(w)}\,.$$

Here we are using the dual-measure formulation, so that the measure w appears on both sides of the inequality, as in Theorem 2.2.1.

By Theorem 2.2.1, and the symmetry of the A_2 condition, it is sufficient to check that the two inequalities below hold for all simple Haar shift operators T of index τ:

$$|\langle\mathrm{T}(w\mathbf{1}_Q),w^{-1}\mathbf{1}_R\rangle|\lesssim\|w\|_{A_2}\sqrt{w(Q)w^{-1}(R)}\,,\qquad(2.4.1)$$

$$\int_Q |\mathrm{T}(w\mathbf{1}_Q)|^2 \, w^{-1}dx \lesssim \|w\|_{A_2}^2 w(Q)\,. \tag{2.4.2}$$

These should hold for all dyadic cubes Q, and in (2.4.1), we have $2^{-(\tau+1)d}|Q| \le |R| \le 2^{(\tau+1)d}|Q|$.

In the present circumstance, the 'weak boundedness' inequality (2.4.1) can be derived from the '$T1$' inequality (2.4.2). We can assume that $|Q| \le |R|$ by passing to the dual operator and replacing w by w^{-1}. If $|Q| = |R|$, the inner product is zero unless $Q = R$. But then we just appeal to (2.4.2),

$$\begin{aligned} |\langle \mathrm{T}(w\mathbf{1}_Q), w^{-1}\mathbf{1}_Q\rangle| &\le \sqrt{w^{-1}(Q)} \cdot \|\mathbf{1}_Q \, \mathrm{T}(w\mathbf{1}_Q)\|_{L^2(w^{-1})} \\ &\lesssim \|w\|_{A_2} \sqrt{w(Q)\cdot w^{-1}(Q)}\,. \end{aligned}$$

If $|Q| < |R|$, let assume that $Q \subset R$, and write

$$|\langle \mathrm{T}(w\mathbf{1}_Q), w^{-1}\mathbf{1}_R\rangle| \le |\langle \mathrm{T}(w\mathbf{1}_Q), w^{-1}\mathbf{1}_Q\rangle| + |\langle \mathrm{T}(w\mathbf{1}_Q), w^{-1}\mathbf{1}_{R-Q}\rangle|\,.$$

The first term on the right is handled just as in the previous case. In the second case, we use the fact that $2^{-\tau d}|R| \le |Q| < |R|$, so that there is a difference in scales between the two cubes of only at most τ scales. That, with the size conditions on T lead to

$$\begin{aligned} |\langle \mathrm{T}(w\mathbf{1}_Q), w^{-1}\mathbf{1}_{R-Q}\rangle| &\lesssim \frac{w(Q)w^{-1}(R)}{|R|} \\ &\lesssim \|w\|_{A_2}\sqrt{w(Q)\cdot w^{-1}(R)}\,. \end{aligned}$$

The last inequality follows since

$$\sqrt{\frac{w(Q)w^{-1}(R)}{|R|^2}} \le \sqrt{\frac{w(R)w^{-1}(R)}{|R|^2}} \le \sqrt{\|w\|_{A_2}} \le \|w\|_{A_2}\,.$$

Indeed, we always have $1 \le \|w\|_{A_2}$. The case of $Q \cap R = \emptyset$ is handled in a similar fashion.

To verify (2.4.2), we first treat the 'large scales',

$$\Big\| \mathbf{1}_{Q_0} \sum_{Q\,:\,Q \supsetneq Q_0} \langle w\mathbf{1}_{Q_0}, g_Q\rangle \gamma_Q \Big\|_{L^2(w^{-1})} \lesssim \frac{w(Q_0)w^{-1}(Q_0)^{1/2}}{|Q_0|}$$

$$\lesssim \sqrt{w(Q_0)} \cdot \|w\|_{A_2}.$$

Therefore, it suffices to prove

$$\Big\| \sum_{Q\,:\,Q \subset Q_0} \langle w, g_Q \rangle \gamma_Q \Big\|_{L^2(w^{-1})} \lesssim \|w\|_{A_2} \sqrt{w(Q_0)}\,. \tag{2.4.3}$$

Let us define for dyadic cubes Q_0 and collections of dyadic cubes $\mathcal{Q}'$,

$$\begin{aligned} H(Q_0, \mathcal{Q}') &:= \sum_{\substack{Q \subset Q_0 \\ Q \in \mathcal{Q}'}} \langle w, g_Q \rangle \gamma_Q\,, \\ \mathbf{H}(\mathcal{Q}') &:= \sup_{Q_0} \frac{\|H(Q_0, \mathcal{Q}')\|_{L^2(w^{-1})}}{\sqrt{w(Q_0)}}\,. \end{aligned} \tag{2.4.4}$$

It is a useful remark that in estimating $\mathbf{H}(\mathcal{Q}')$ we can restrict the supremum to cubes $Q_0 \in \mathcal{Q}'$. Of course, we are seeking to prove $\mathbf{H}(\mathcal{Q}') \lesssim \|w\|_{A_2}$.

The first important definition here is

$$\mathcal{Q}_n := \left\{ Q \in \mathcal{Q}' \;:\; 2^{n-1} < \frac{w(Q)}{|Q|} \cdot \frac{w^{-1}(Q)}{|Q|} \leq 2^n \right\}. \tag{2.4.5}$$

We show that

$$\mathbf{H}(\mathcal{Q}_n) \lesssim 2^{n/2} \|w\|_{A_2}^{1/2}\,. \tag{2.4.6}$$

Since $2^n \leq \|w\|_{A_2}$, this estimate is summable in n to prove (2.4.3).

Now fix a $Q_0 \in \mathcal{Q}_n$ for which we are to test the supremum in (2.4.4). Let $\mathcal{P}_n = \{Q \in \mathcal{Q}_n \;:\; Q \subset Q_0\}$. Let $(\mathcal{L}_n \;:\; \mathcal{P}_n(L))$ be a corona decomposition of $\mathcal{P}_n$ relative to measure w. (The reader is advised to recall the Definition 2.3.1.)

The essence of the matter is contained in the following Lemma.

Lemma 2.4.7. *We have these distributional estimates, uniform over $L \in \mathcal{L}_n$:*

$$\big|\big\{x \in L \;:\; |H(L, \mathcal{P}_n(L))(x)| > Kt\tfrac{w(L)}{|L|}\big\}\big| \lesssim \mathrm{e}^{-t}|L|\,, \tag{2.4.8}$$

$$w^{-1}\big(\big\{x \in L \;:\; |H(L, \mathcal{P}_n(L))(x)| > Kt\tfrac{w(L)}{|L|}\big\}\big) \lesssim \mathrm{e}^{-t}\, w^{-1}(L)\,. \tag{2.4.9}$$

Let us complete the proof of our Theorem based upon this Lemma. Set $H_n(L) := |H(L, \mathcal{P}_n(L))|$, and estimate

$$\|H(Q_0, \mathcal{Q}_n)\|^2_{L^2(w^{-1})} \le \Big\| \sum_{L \in \mathcal{L}_n} H_n(L) \Big\|^2_{L^2(w^{-1})}$$

$$= A + 2B = A + 2 \sum_{L \in \mathcal{L}_n} B(L)\,, \tag{2.4.10}$$

$$A := \sum_{L \in \mathcal{L}_n} \|H_n(L)\|^2_{L^2(w^{-1})}$$

$$B(L) := \sum_{\substack{L' \in \mathcal{L}_n \\ L' \subsetneq L}} \int H_n(L) \cdot H_n(L')\, w^{-1}\,. \tag{2.4.11}$$

Note that these estimates show that all cancellation necessary for the truth of the theorem is already captured in the corona decomposition.

The estimate of A is straight forward. By (2.4.9), we see that the A_2 estimate reveals itself.

$$\|H_n(L)\|^2_{L^2(w^{-1})} \lesssim \Big[\frac{w(L)}{|L|}\Big]^2 w^{-1}(L)$$

$$\lesssim w(L)\frac{w(L)}{|L|} \cdot \frac{w^{-1}(L)}{|L|}$$

$$\lesssim 2^n w(L)\,.$$

Therefore, by (2.3.6)

$$A \lesssim 2^n \sum_{L \in \mathcal{L}_n} w(L) \lesssim 2^n \|w\|_{A_2} w(Q_0)\,. \tag{2.4.12}$$

In the expression (2.4.11), the integral is not as complicated as it immediately appears. We have assumed that 'scales are separated by τ' at the beginning of this section, so that as L' is strictly contained in L, we have for any $Q \in \mathcal{P}_n(L)$, that $(L')^{(\tau)}$ is either contained in Q or disjoint from it. It follows that $H_n(L)$ takes a single value on all of L', which we denote by $H_n(L; L')$. This observation simplifies our task of estimating the integral.

For $L' \subsetneq L$ we use (2.4.9) and (2.4.5) to see that

$$\int H_n(L) \cdot H_n(L')\, w^{-1} \lesssim H_n(L;L') \frac{w(L')}{|L'|} \cdot w^{-1}(L') \lesssim 2^n H_n(L;L') \cdot |L'| \,. \tag{2.4.13}$$

Note that the A_2 characteristic has entered in. And the presence of $|L'|$ indicates that there is an integral against Lebesgue measure here.

Employ this observation with Cauchy-Schwartz, *both* distributional estimates (2.4.8) and (2.4.9) as well as (2.3.4) to estimate

$$\begin{aligned}
B(L) &:= \sum_{\substack{L' \in \mathcal{L}_n \\ L' \subsetneq L}} \int H_n(L) \cdot H_n(L')\, w^{-1} \\
&\lesssim 2^n H_n(L;L') \sum_{\substack{L' \in \mathcal{L}_n \\ L' \subsetneq L}} |L'| && \text{(by (2.4.13))} \\
&= 2^n \int H_n(L;L') \cdot \sum_{\substack{L' \in \mathcal{L}_n \\ L' \subset L}} \mathbf{1}_{L'}\, dx && \text{(by defn.)} \\
&\le 2^n \|H_n(L)\|_{L^2(dx)} \Big\| \sum_{\substack{L' \in \mathcal{L}_n \\ L' \subset L}} \mathbf{1}_{L'} \Big\|_{L^2(dx)} && \text{(Cauchy-Schwartz)} \\
&\lesssim 2^n w(L) \,. && \text{(by (2.4.8) and (2.3.4))}
\end{aligned}$$

Therefore, by (2.3.6) again,

$$B \lesssim 2^n \sum_{L \in \mathcal{L}_n} w(L) \lesssim 2^n \|w\|_{A_2} w(Q_0) \,.$$

Combining this estimate with (2.4.10) and (2.4.12) completes the proof of (2.4.6), and so our Theorem, assuming Lemma 2.4.7.

2.5 *The essence of the matter.*

We prove Lemma 2.4.7. In this situation, both a cube Q_0 and cube $L \in \mathcal{L}_n$ are given. It is an important point that all the relevant cubes that we sum over are in the collection $\mathcal{Q}_n$, as defined in (2.4.5).

One more class of dyadic cubes are needed. For integers $\alpha \geq 0$ define $\mathcal{P}_{n,\alpha}(L)$ to be those $Q \in \mathcal{P}_n(L)$ such that

$$2^{-\alpha+1} \frac{w(L)}{|L|} \leq \frac{w(Q)}{|Q|} < 2^{-\alpha+2} \frac{w(L)}{|L|} \,. \tag{2.5.1}$$

The essential observation is this: By Proposition 2.3.12, T maps $L^1(dx)$ into weak-$L^1(dx)$, with norm depending only on the index τ of the operator. Hence,

$$\Big\| \sum_{\substack{Q \subset Q_1 \\ Q \in \mathcal{P}_{n,\alpha}(L)}} \langle w, g_Q \rangle \gamma_Q \Big\|_{L^{1,\infty}(dx)} \lesssim w(Q_1) \,.$$

This is a uniform statement in Q_1. If in addition $Q_1 \in \mathcal{P}_{n,\alpha}(L)$, we have

$$\Big\| \sum_{\substack{Q \subset Q_1 \\ Q \in \mathcal{P}_{n,\alpha}(L)}} \langle w, g_Q \rangle \gamma_Q \Big\|_{L^{1,\infty}(dx)} \lesssim 2^{-\alpha} \frac{w(L)}{|L|} \cdot |Q_1| \,. \tag{2.5.2}$$

Due to the functions g_Q and γ_Q are supported on Q, we see that this estimate also holds uniformly in Q_1.

Note that we have by the definition of Haar functions Definition 2.1.4, and a simple Haar shift, Definition 2.3.9,

$$|\langle w, g_Q \rangle \gamma_Q(x)| \leq \frac{w(Q)}{|Q|} \lesssim 2^{-\alpha} \frac{w(L)}{|L|} \,. \tag{2.5.3}$$

The point of these observations is that Lemma 2.3.15 applies. Define

$$E_\alpha(t) \coloneqq \left\{ x \in L \ : \ \Big| \sum_{Q \in \mathcal{P}_{n,\alpha}(L)} \langle w, g_Q \rangle \gamma_Q(x) \Big| > K t 2^{-\alpha} \frac{w(L)}{|L|} \right\}, \qquad t \geq 1 \,. \tag{2.5.4}$$

We have the exponential inequality $|E_\alpha(t)| \lesssim \mathrm{e}^{-t}|L|$ for an appropriate choice of constant K in (2.5.4). (The choice of K is dictated only by the exact constants that enter into (2.5.2) and (2.5.3) as well as the parameter τ associated with the simple Haar shift.)

This is one of our two claims, the distributional estimate in Lebesgue measure (2.4.8), for the collection $\mathcal{P}_{n,\alpha}(L)$, not the collection $\mathcal{P}_n(L)$. But with the term $2^{-\alpha}$ appearing in (2.5.4), it is easy to supply (2.4.8) as written. Indeed, for $K' = K \sum_\alpha 2^{-\alpha/2}$, and K as in (2.5.4), we can estimate

$$\left|\left\{x \in L \;:\; \left|\sum_{Q\in\mathcal{P}_n(L)} \langle w, g_Q\rangle \gamma_Q(x)\right| > K' t \frac{w(L)}{|L|}\right\}\right| \le \sum_{\alpha=0}^{\infty} |E_\alpha(t2^{\alpha/2})| \lesssim e^{-t}|L|.$$

We want the corresponding inequality in w^{-1}-measure. But note that $E_\alpha(t)$ is a union of disjoint dyadic cubes in a collection $\mathcal{E}_\alpha(t)$, where for each $Q \in \mathcal{E}_\alpha(t)$, we can choose dyadic $\phi(Q) \in \mathcal{P}_{n,\alpha}(L)$ with $Q \subset \phi(Q)$, and $|Q| \ge 2^{-\tau d}|\phi(Q)|$. This follows from the definition of a simple Haar shift. It follows that we have

$$\left|\bigcup\{\phi(Q) \;:\; Q \in \mathcal{E}_\alpha(t)\}\right| \lesssim \mathrm{e}^{-t}|L|\,. \tag{2.5.5}$$

(Recall that there is a similar difficulty in Proposition 2.3.12.) The point of these considerations is this: For each $Q' \in \mathcal{P}_{n,\alpha}(L)$, we have both the equivalences (2.4.5) and (2.5.1). Hence, $w^{-1}(Q') \simeq \rho|Q'|$ where ρ is a fixed quantity. (It depends upon L, and we can we can compute it, but as it appears on both sides of the distributional inequality, its value is irrelevant to our conclusion.) We can conclude from (2.5.5) the same inequality in w^{-1}-measure by the following argument. Let $\mathcal{E}^*_\alpha(t)$

$$\begin{aligned} w^{-1}\left\{\left|\sum_{Q\in\mathcal{P}_{n,\alpha}(L)} \langle w, g_Q\rangle \gamma_Q\right| > Kt2^{-\alpha}\frac{w(L)}{|L|}\right\} &\le w^{-1}\big(\bigcup\{\phi(Q) \;:\; Q \in \mathcal{E}_\alpha(t)\}\big) \\ &= \sum_{Q\in\mathcal{E}^*_\alpha(t)} w^{-1}(\phi(Q)) \\ &\simeq \rho \sum_{Q\in\mathcal{E}^*_\alpha(t)} |\phi(Q)| \\ &\lesssim \rho\left|\bigcup\{\phi(Q) \;:\; Q\in\mathcal{E}_\alpha(t)\}\right| \\ &\lesssim \rho\,\mathrm{e}^{-t}|L| \simeq e^{-t} w^{-1}(L)\,. \end{aligned}$$

This (2.4.9), except for the occurence of the $2^{-\alpha}$ on the right, and so the proof is complete.

2.6 Sufficient Conditions for a Two Weight Inequality

There are a great many sufficient conditions for a two-weight inequality. To these results, let us add this statement, for it's elegance. (It is probably already known.)

Theorem 2.6.1. *Let α, β be positive functions on $\mathbb{R}^d$. For the inequality below to hold for all Haar shift operators* T

$$\|\mathrm{T}(f\alpha)\|_{L^2(\beta)} \lesssim \|f\|_{L^2(\alpha)}$$

It is sufficient that $\alpha, \beta \in A_\infty$ and the following 'two-weight A_2' hold:

$$\sup_Q \frac{\alpha(Q)}{|Q|} \cdot \frac{\beta(Q)}{|Q|} < \infty\,.$$

Of course these conditions are not necessary, for example one can take $\alpha = \beta = \mathbf{1}_E$, for any measurable subset E of $\mathbb{R}^d$. By $\alpha \in A_\infty$ we mean the measures α and β satisfy a variant of the estimate in (2.3.8).

Definition 2.6.2. We say that measure $\alpha \in A_\infty$ if this condition holds. For all $0 < \epsilon < 1$ there is a $0 < \eta < 1$ so that for all cubes Q and sets $E \subset Q$ with $|E| < \epsilon|Q|$, then $\alpha(E) < \eta\alpha(Q)$.

The proof is a modification of what we have already presented, so we do not give the details. The resulting estimate is however sharp in the dependence upon the two weight A_2 constant, and the A_∞ constants.

2.7 Recent developments

The result on recovering singular integrals with sufficiently smooth kernels from Haar shift operators in dimension 1 by A. Vagharshakyan, [43], allow us to extend corollary 2.1.8 to the later class of singular integrals.

It is important to mention that there is another line of investigation, one that completely avoids the use of two weighted results, due to Cruz-Uribe, Martell and Pérez [8]. Instead they use a very powerful inequality by A. Lerner, [22]. Their argument only works for dyadic Haar shift operators though, and an extension to the continuous setting has not been found yet.

A striking result by Pérez-Treil-Volberg [32] allow to deduce the strong L^2 estimate from the weak one. The techniques used in that paper, together with a careful examination of the ones presented in this chapter, resulted in the solution of the full conjecture by T. Hytönen, see [18]. At the end of his paper, T. Hytönen was calling for a simplification of his argument, that was long and repeatedly used powerful techniques from the two weighted setting. During October 2010 a simplified proof was provided by T. Hytönen, C. Pérez, S. Treil and A. Volberg, see [20].

CHAPTER III

ON MUCKENHOUPT-WHEEDEN CONJECTURE

3.1 Background and main results

The starting point of this work goes back to 1971 [14], when C. Fefferman and E. Stein, in order to establish vector-valued estimates for the maximal function, proved that if w is a weight, namely a non-negative locally integrable function, and M denotes the Hardy-Littlewood maximal operator then

$$\sup_{t>0} tw\left(\left\{x \in \mathbb{R}^d : Mf(x) > t\right\}\right) \leq c \int_{\mathbb{R}^d} |f| Mw(x).$$

A very natural question was then raised by B. Muckenhoupt and R. Wheeden (see [23]): could we replace the Hardy-Littlewood maximal operator M by a Calderón-Zygmund operator? Their conjecture, known as the Muckenhoupt-Wheeden Conjecture, is stated below.

Conjecture 3.1.1. *(Muckenhoupt-Wheeden) Let w be a weight and M be the Hardy-Littlewood maximal operator. Let T be a Calderón-Zygmund operator with $\|T\|_{CZO} \leq 1$. Then*

$$\sup_{t>0} tw\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq C \int_{\mathbb{R}} |f| Mw(x) dx \tag{3.1.2}$$

The exact definition of Calderón-Zygmund operator need not concern us here, though it certainly includes the non-positive Hilbert transform (see chapter VII of [41] for precise definitions). The hope was that the Conjecture identified a somewhat robust principle. We herein disprove the *dyadic version* of this conjecture. So, M is replaced by the (smaller) dyadic maximal function, and T will be a Haar multiplier, which are the simplest possible dyadic Calderón-Zygmund operators.

Endpoint estimates are known to be the most delicate ones, and very frequently they are also the most powerful. That is the case of Muckenhoupt-Wheeden Conjecture. For instance, an extrapolation result due to D. Cruz-Uribe and C. Pérez [7] shows this: If w is a weight and (3.1.2) holds with T a sublinear operator then

$$\int_{\mathbb{R}} |T(f)|^p w(x)dx \leq \int_{\mathbb{R}} |f|^p \left(\frac{Mw}{w}\right)^p w(x)dx. \tag{3.1.3}$$

The dyadic version of this result is also true (see [7], Remark 1.5).

With a few partial results that we shall discuss later in the introduction, the Muckenhoupt-Wheeden Conjecture has been open up to today's date. In this chapter, we answer the dyadic version of (3.1.2) in the negative by disproving (3.1.3). We are ready to state our main theorem.

Theorem 3.1.4. *There exist a weight w and a Haar multiplier T for which T is unbounded as map from $L^2\left((\frac{Mw}{w})^2 w\right)$ to $L^2(w)$.*

As a corollary we solve a long-standing conjecture,

Corollary 3.1.5. *The Muckenhoupt-Wheeden conjecture in its dyadic version is false.*

For the proof we construct a measure w and a Haar multiplier T that avoids all cancellations. The tool behind this construction is the corona decomposition, that has proven to be very useful in finding sharp estimates when the weight is in the A_p class [21], [19], [18], [20].

Throughout the literature, there has been evidence for a positive answer to the conjecture as well as for a negative one. S. Chanillo and R. Wheeden [5] showed that a square function satisfied the Muckenhoupt-Wheeden Conjecture. We also mention the work of Buckley [3], who in dimension n proved that (3.1.2) holds for weights $w_\delta(x) = |x|^{-n(1-\delta)}$ for $0 < \delta < 1$.

The sharpest results in this direction are due to C. Pérez [31]: If T is a Calderón-Zygmund operator and M^2 is the Hardy-Littlewood maximal operator iterated 2 times,

$$\sup_{t>0} tw\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq C \int_{\mathbb{R}} |f| M^2 w(x) dx.$$

He actually proved something better, M^2 can be replaced by the smaller operator $M_{L(logL)^\epsilon}$. In an attempt to understand these endpoint estimates, A. Lerner, S. Ombrosy and C. Pérez considered a somehow 'dual' problem of Muckenhoupt-Wheeden, we refer the reader to [25]. A negative answer to (3.1.2) was provided by M.J. Carro, C. Pérez, F. Soria and J. Soria when T is a fractional integral, [4]. There are two points that distinguish this example from the singular integral one: 1) the lack of cancellation when treating positive operators and 2) the construction depends upon T being a true fractional integral and does not allow an immediate extension to the singular integral case.

By imposing an extra condition on the weight w, a weaker version of Muckenhoupt-Wheeden can be formulated. This is known as the Weak Muckenhoupt-Wheeden Conjecture and appears in work of A. Lerner, S. Ombrosy and C. Pérez [23], [24].

Conjecture 3.1.6. *(Weak Muckenhoupt-Wheeden) Let w be an A_1 weight and let $\|w\|_{A_1}$ be the A_1 constant associated to it. Let T be a Calderón-Zygmund operator with $\|T\|_{CZO} \leq 1$. Then*

$$\sup_{t>0} tw\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq C\|w\|_{A_1} \int_{\mathbb{R}} |f| w(x) dx \qquad (3.1.7)$$

Recall that w is an A_1 weight if there exists a constant $c > 0$ such that $Mw(x) \leq cw(x)$ a.e. The smallest of such constants c is denoted by $\|w\|_{A_1}$. Thus, the Weak Muckenhoupt-Wheeden Conjecture would be an immediate consequence of (3.1.2), were it true. The continuity of Calderón-Zygmund operators in $L^1 \mapsto L^{1,\infty}$ when w is an A_1 weight is well known and goes back to the origins of the weighted theory with R. Hunt, B. Muckenhoupt and R. Wheeden [17] in dimension 1 and R. Coifman and C. Fefferman in higher dimensions [6]. The novelty of (3.1.7) resides with the linear dependence on $\|w\|_{A_1}$. Linear growth of the A_1 constant has been proven in the strong case for $p > 1$, that is

$$\|T\|_{L^p(w) \mapsto L^p(w)} \leq C(p,n)\|w\|_{A_1},$$

and this is sharp. The result was first proven by R. Fefferman and J. Pipher for $p \geq 2$ and T a classical singular integral operator in [15]. Later on it was extended to $p > 1$ and general Calderón-Zygmund operators by A. Lerner, S. Ombrosy and C. Pérez in [23]. The proof of A. Lerner *et al.* provides not only linear dependence on the A_1 constant, but also explicit dependence of the operator norm on p. The explicit dependence on p allows one to get the weak endpoint below,

$$\sup_{t>0} tw\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq C\|w\|_{A_1}(1 + \log\|w\|_{A_1}) \int_{\mathbb{R}} |f|w(x)dx.$$

We want to point out that even though this estimate is far from proving (3.1.7), it is the best result known up to this date. The Weak Muckenhoupt-Wheeden Conjecture remains open and we will not make any new contribution to it in this chapter, but we are hoping to shed some light in the understanding of these endpoint estimates.

3.2 Basic concepts

The space we will be working on is $\mathbb{R}$. Throughout the chapter $|\cdot|$ will stand for the Lebesgue measure in $\mathbb{R}$, 1_E will be the characteristic function associated to the set $E \subset \mathbb{R}$, and for $x \geq 0$, $[x]$ denotes the integer part. The letters i, j, l, k will stand for positive integers. C will denote a universal constant, not necessarily the same in each case.

In the sequel when referring to M we will understand the dyadic maximal function, i.e., for $f \in L^1_{loc}$

$$Mf(x) = \sup_{Q \text{ dyadic}} \frac{1_Q}{|Q|} \int_Q f(x)dx.$$

We will use a different formulation of the two weight inequality (3.1.3). This characterization was first introduced by E. Sawyer in [40] and has been used since then, becoming one of the standard approaches. The proof is a well known exercise that we are not including in this chapter.

Proposition 3.2.1. *Let w, v be two positive Borel measures, continuous with respect to Lebesgue measure and let T be a sublinear operator. Let C be a universal constant and $1 < p < \infty$, the statements below are equivalent,*

$$\|Tf\|_{L^p(w)} \leq C\|f\|_{L^p(v)},$$
$$\|T(f\sigma)\|_{L^p(w)} \leq C\|f\|_{L^p(\sigma)}, \quad \sigma = v^{1-p'}1_{\text{supp}(v)}. \tag{3.2.2}$$

Remark 3.2.3. This new formulation provides a more symmetric estimate for T^*, the dual operator with respect to Lebesgue measure, i.e., (3.2.2) is equivalent to

$$\|T^*(fw)\|_{L^{p'}(\sigma)} \leq C\|f\|_{L^{p'}(w)}, \tag{3.2.4}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

The estimate we want to disprove is a particular case of a 2-weight inequality. We will work with weights $w \geq 0$, $w \in L^1_{\text{loc}}$ and $\sigma = \frac{w}{(Mw)^2}$, which is the dual measure of $v = \left(\frac{Mw}{w}\right)^2 w$. Throughout the chapter σ will take the above form. The operators we consider are discrete dyadic operators. Let us recall some of the basic concepts associated to them before getting to the precise definition.

Definition 3.2.5. Let $\mathcal{D}$ be the dyadic grid in $\mathbb{R}$ $\mathcal{D} = \{\left[2^k m, 2^k(m+1)\right), \ m, k \in \mathbb{Z}\}$. Let $I = [a, b]$ be an interval in $\mathcal{D}$, then $I^- = [a, \frac{a+b}{2})$ is the *left child of I* and $I^+ = [\frac{a+b}{2}, b)$ is the *right child of I.* We define the L^2-normalized Haar function associated to I, h_I as

$$h_I = \frac{1_{I^+} - 1_{I^-}}{|I|^{1/2}}$$

Our interest will lay on particular examples of dyadic operators, the Haar multipliers.

Definition 3.2.6. Let $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be a bounded sequence. The operator T_ϵ is a *Haar multiplier* associated to ϵ iff

$$T_\epsilon f = \sum_{I\in\mathcal{D}} \epsilon_I \langle f, h_I\rangle h_I.$$

3.3 The Corona Decomposition

This Section provides the tool to, given w, decompose the measure σ. At the same time, the corona decomposition allows to group the dyadic intervals into families and consequently decompose any dyadic operator into a sum of operators. This was previously done in [21].

Definition 3.3.1. Let $\mathcal{D}' \subset \mathcal{D}$ be any collection of dyadic cubes. Call $(\mathcal{L} : \mathcal{D}'(L))$ a *corona decomposition of* $\mathcal{D}'$ *relative to measure* w if these conditions are met. Let $L, L' \in \mathcal{L}$, we have

$$\frac{w(L')}{|L'|} \geq 4\frac{w(L)}{|L|}, \qquad L' \subsetneq L.$$

Define $\Gamma : \mathcal{D}' \to \mathcal{L}$ by requiring that $\Gamma(I)$ be the minimal element of $\mathcal{L}$ that contains I. We set $\mathcal{D}'(L) := \{I \in \mathcal{D}' : \Gamma(I) = L\}$. Then for all $I \in \mathcal{D}'(L)$ we require

$$4\frac{w(L)}{|L|} > \frac{w(I)}{|I|}.$$

Remark 3.3.2. The corona decomposition is obtained by a stopping time argument. It is for this reason that we will refer to $\mathcal{L}$ as the *stopping collection* in the corona decomposition.

The collections $\mathcal{D}'(L)$ partition $\mathcal{D}'$. Since decompositions of dyadic intervals correspond directly to decompositions of dyadic operators, we can write any Haar multiplier as

$$T_\epsilon = \sum_{L\in\mathcal{L}} T_L \quad \text{where} \quad T_L = \sum_{I\in\mathcal{D}(L)} \epsilon_I \langle f, h_I\rangle h_I. \tag{3.3.3}$$

We now focus on the structure of the measure σ. We will denote $\mathcal{L}_0$ to be the set of maximal intervals in $\mathcal{L}$. In general, we denote $\mathcal{L}_j$ to be the maximal intervals on $\mathcal{L}\setminus\bigcup_{i=0}^{j-1}\mathcal{L}_i$.

Definition 3.3.4. Let $E \in \mathcal{D}(L)$, $L \in \mathcal{L}_j$, we define the *take away the children* operator on sets of $\mathbb{R}$ as $\Delta_1 E = E\setminus \bigcup_{L'\in\mathcal{L}_{j+1},\, L'\subset E} L'$. In general, we define

$$\Delta_l E = \cup_{\tilde{L}\in\mathcal{L}_{j+l-1},\, \tilde{L}\subset E}\tilde{L}\setminus \cup_{L'\in\mathcal{L}_{j+l},\, L'\subset E} L' \quad \text{for } l > 1.$$

Remark 3.3.5. This last definition helps us track the value of Mw. Notice that for every $x \in \Delta_l E$, $E \in \mathcal{D}(L)$, and $L \in \mathcal{L}$, we have $8^l\frac{w(L)}{|L|} \geq Mw(x) \geq 4^l\frac{w(L)}{|L|}$. Since $\{\Delta_l E\}_{l\geq 1}$ forms a partition of E, we can estimate

$$\sigma(E) = \int_E \frac{w}{Mw^2}(x)dx \geq \sum_{l=1}^{\infty} 8^{-2l}\left(\frac{|L|}{w(L)}\right)^2 w(\Delta_l E) \tag{3.3.6}$$

3.4 The inductive construction

In this Section, we describe the inductive procedure that will provide measures w_k and operators T_k, the key elements in proving Theorem 3.1.4. We start with a few definitions associated to the base case.

Definition 3.4.1. Let $J = [a, a+\alpha)$ be a dyadic interval, we define the *jumping point* of J, and we denote it by $\mathrm{jp}(J)$, as $\mathrm{jp}(J) := a + \frac{\alpha}{3}$. We also denote the right end point of J by $\mathrm{rep}(J) := a + \alpha$.

Notice that the "jumping point" divides the interval into two intervals of lengths one-third and two-thirds the length of the original interval, thus the "jumping point" has a periodic binary expansion, which fact is important to the construction. We now define the measure that gives name to the jumping point.

Definition 3.4.2. Let J be a dyadic interval and $\lambda > 0$ be a height, we define the measure associated to J with height λ as

$$\mu_J^\lambda = \lambda 1_{[\mathrm{jp}(J),\mathrm{rep}(J))}.$$

Having listed the key ingredients to construct our measure w_k inductively, we now focus on those associated to the construction of the operator T^k.

Definition 3.4.3. Let $k \geq 1$ be a fixed integer. Let J be a dyadic interval, we define Ξ_J as the following collection of intervals associated to J,

$$\Xi_J := \left\{ J = I_0 \supseteq \ldots \supseteq I_{2k} \,:\, \mathrm{jp}(J) \in I_i^-, \, |I_i| = 4|I_{i+1}| \right\}. \tag{3.4.4}$$

We denote $I(J) := I_{2k}$, the minimal interval in the collection Ξ_J. And we define the collection of the right children of the intervals in Ξ_J as

$$\Xi_J^+ := \left\{ I_i^+ \,:\, I_i \in \Xi_J \setminus I(J) \right\}.$$

We are now ready to define the Haar multiplier associated to J with sign r_J, $S_{J,r(J)}$, as

$$S_{J,r(J)}(f) := r_J \sum_{I \in \Xi_J} \langle f, h_I \rangle h_I. \tag{3.4.5}$$

where $r_J \in \{+1, -1\}$.

Remark 3.4.6. There are two ideas about the jumping point that should be clarified. (1) If $\mathrm{jp}(J) \in I$, then either $\mathrm{jp}(J) \in I^-$ or $\mathrm{jp}(J) \in I^+$. Moreover, these two events alternate, i.e., let $I \subset I' \subset J$ with $|I| = \frac{1}{2}|I'|$ and $\mathrm{jp}(J) \in I, I'$. We have that if $\mathrm{jp}(J) \in I'^-$ (respectively I'^+) then $\mathrm{jp}(J) \in I^+$ (respectively I^-). This phenomenon explains that the chosen intervals in (3.4.4) satisfy $|I_i| = 4|I_{i+1}|$. (This is the consequence of the binary expansion of 1/3.) (2) We take advantage of the localization of the jumping point in another way. The intervals $I \in \Xi_J^+$ "almost" form a partition of the support of μ_J^λ. If $k \to \infty$, they will actually form a partition, since we consider only a fixed number of them, we can only get

$$\mathrm{supp}\mu_J^\lambda = [\mathrm{jp}(J), \mathrm{rep}(I(J))) \cup \bigcup_{I \in \Xi_J^+} I. \tag{3.4.7}$$

Remark 3.4.8. Notice that Ξ_J and consequently Ξ_J^+ and $S_{J,r(J)}$ depend on the parameter k, that will play the role of the induction index in the proof of Proposition 3.4.13. For the sake of simplicity, we omit the parameter k in the notation of those objects.

The following lemma takes advantage of the lack of cancellation in $S_{J,r(J)}(\mu_J^\lambda)$ to compare the μ_J^λ measure of a level set associated to $S_{J,r(J)}$ with that of J.

Lemma 3.4.9. *Let $k \geq 1$ be a fixed integer, J be a dyadic interval and μ_J^λ and $S_{J,r(J)}$ as above. Then, first, the inner products $\langle \mu_J^\lambda, h_{I_i} \rangle$, for $I_i \in \Xi_J$ depend only on the numbers $\mu_J^\lambda(I(J))$ and $\{\mu_J^\lambda(I) \ : \ I \in \Xi_J^+\}$. Second,*

$$\mu_J^\lambda\left(\left\{x \ : \ |S_{J,r(J)}\mu_J^\lambda(x)| > k\frac{\mu_J^\lambda(J)}{|J|}\right\}\right) \geq \frac{1}{4}2^{-4k}\mu_J^\lambda(J). \tag{3.4.10}$$

Proof. For the first claim, $\langle \mu_J^\lambda, h_{I_i} \rangle$ depends only on the measure μ_J^λ assigned to the two children for I_i. And, these two children are unions of the sets in (3.4.7).

Turning to the second claim, let J be a dyadic interval, for any $I_i \in \Xi_J$ we have the following equality

$$\frac{\langle \mu_J^\lambda, h_{I_i} \rangle}{\sqrt{|I_i|}} = \frac{1/2\lambda|I_i| - (1/2 - 1/3)\,\lambda|I_i|}{|I_i|} = \frac{\lambda}{3}. \tag{3.4.11}$$

Since $I_{i+1} \subset I_i^-$ for all i, $\langle \mu_J^\lambda, h_{I_i} \rangle h_{I_i}$ is constant on I_{i+1}. Therefore, using (3.4.11) for every $x \in I(J)^-$

$$|S_{J,r(J)}\mu_J^\lambda(x)| = \sum_{i=0}^{2k} \frac{\langle \mu_J^\lambda, h_{I_i} \rangle}{\sqrt{|I_i|}} 1_{I(J)^-} = \frac{(2k+1)\lambda}{3}.$$

On the other hand,

$$\frac{\mu_J^\lambda(J)}{|J|} = \frac{\frac{2}{3}\lambda|J|}{|J|} = \tfrac{2}{3}\lambda,$$

and $\dfrac{(2k+1)\lambda}{3} > \dfrac{2}{3}k\lambda$ trivially. This added to the fact that $|S_{J,r(J)}\mu_J^\lambda(x)| \leq \frac{2}{3}k\lambda$ for all $x \in J$ and $x \notin I(L)^-$ gives,

$$I(L)^- = \left\{x \ : \ |S_{J,r(J)}\mu_J^\lambda(x)| > k\frac{\mu_J^\lambda(J)}{|J|}\right\}, \tag{3.4.12}$$

and

$$\mu_J^\lambda\left(\left\{x\,:\,|S_{J,r(J)}\mu_J^\lambda(x)|>k\frac{\mu_J^\lambda(J)}{|J|}\right\}\right)\geq\mu_J^\lambda(I(J)^-)=\tfrac{1}{6}\lambda 2^{-4k}|J|=\tfrac{1}{4}2^{-4k}\mu_J^\lambda(J),$$

as desired. □

Actually this estimate (3.4.10) is unimprovable, as follows from the John-Nirenberg inequality. (Our point of view in this construction is informed by the extension of the John-Nirenberg inequality in the weighted setting, as established in the work of the author with M. Lacey and S. Petermichl [21], page 137).

We have shown that we can construct a particular Haar multiplier, that with respect to Lebesgue measure has no cancellation, and we have reversed the John-Nirenberg inequality. The success of this proof is based upon the observation that we can iterate this construction on the elements of the partition in (3.4.7). Namely, we are free to change the measure μ_J^λ provided we *do not change the numbers* $\mu_J^\lambda(I)$, for $I\in\Xi_J^+$. And so, we will change the definition of μ_J^λ, without changing its total measure, in such a way that we carefully track the corona, so that we have (3.3.6). This means that at a different threshold, and a different part of our Haar multiplier, we will have a second reversal of the John-Nirenberg inequality. This construction will then have to be iterated many times, to overcome the exponential nature of the John-Nirenberg inequality. All of these considerations are incorporated into this proposition.

Proposition 3.4.13. *Let $k\geq 1$ be a fixed integer. There exist a family of random Haar multipliers T^k and a weight $w_k\not\equiv 0$, $w_k\in L^1_{loc}$ such that*

$$\sum_{L\in\mathcal{L},L\subset[0,1]}w_k\left(\left\{x\,:\,|T_L^k w_k(x)|>k\frac{w_k(L)}{|L|}\right\}\right)\geq\frac{1}{6}w_k([0,1]),$$

where $\mathcal{L}$ is the stopping collection in the corona decomposition associated to measure w_k as defined in (3.3.1) *and T_L^k as in in* (3.3.3).

For the proof we need this definition.

Definition 3.4.14. Let $k \geq 1$ be an integer, J be a dyadic interval and Ξ_J be as above. We define the set of intervals $\mathcal{L}(J)$ as

$$\mathcal{L}(J) := \left\{ L'(I) := I^{--} \ : \ I \in \Xi_J^+ \right\}$$

Notice that the map

$$\begin{aligned} \Psi \ : \ \Xi_J^+ &\longmapsto \mathcal{L}(J) \\ I \quad &\longmapsto L'(I) \end{aligned} \tag{3.4.15}$$

is a bijection and $|L'(I)| = \frac{1}{4}|I|$. Moreover, given J dyadic interval,

$$I(J) \cap L' = \emptyset \quad \text{ for all } \ L' \in \mathcal{L}(J). \tag{3.4.16}$$

Remark 3.4.17. The passage to the 'left-left child' above helps us keep track of the corona. We will rescale all of the measure assigned to $I \in \Xi_J^+$ to 'right two-thirds' of I^{--}. Now, $\frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6}$, so to preserve measure, we will need to multiply by 6. This explains the powers of 6 that appear below.

Proof. The proof follows from an inductive procedure. Let $\mathcal{L}_0$, μ_0 and S^0 be as follows, $\mathcal{L}_0 = \{[0,1)\}$, $\mu_0 := \mu^1_{[0,1)}$ and $S^0 := S_{[0,1),r[0,1)}$. For a picture of the first stage see Figure 1 below. Notice that for easy of presentation we have sketched only three intervals I_i, that will correspond to the case $k = 1$. In general, for every $j \geq 1$ we define

$$\begin{aligned} \mathcal{L}_j &= \bigcup_{L \in \mathcal{L}_{j-1}} \mathcal{L}(L) \\ \mu_j &= \sum_{i=0}^{j-1} \sum_{L \in \mathcal{L}_i} \mu^{6^i}_{I(L)} + \sum_{L' \in \mathcal{L}_j} \mu^{6^j}_{L'} \\ S^j &= S^{j-1} + \sum_{L' \in \mathcal{L}_j} S_{L',r(L')}. \end{aligned}$$

For the proof of the proposition the selection of signs $r(L)$ is irrelevant. See Figure 2 for a descriptive drawing of the second stage of the construction.

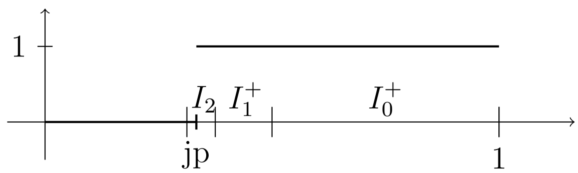

Figure 1: The first stage of the construction.

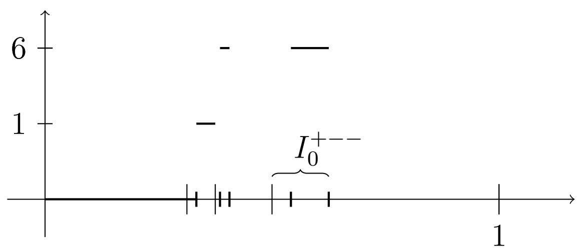

Figure 2: The second stage of the construction.

The next lemma states the main properties of the construction, namely, that the support of the measure built at each stage has shrunk with respect to the previous one. At the same time the new measure preserves the total measure and the measure of the intervals playing a role in previous stages. It is for this reason that we will refer to it as the 'Measure Preserving Lemma'. Its proof is postponed until Section 6.

Lemma 3.4.18. *(Measure Preserving Lemma) Let $\mathcal{L}_j$, μ_j and S^j as above for any $j \geq 1$. Let $L \in \mathcal{L}_j$, we have the following estimates*

1. $\sum_{L \in \mathcal{L}_j} |L| \leq \frac{(1-2^{-4k})^j}{6^j}$,

2. $I(L) \cap L' = \emptyset$ *for all* $L' \in \mathcal{L}_i$, $L' \neq L$, $i \geq 1$. *In particular* $I(L) \cap I(L') = \emptyset$,

3. $\mu_{j+1}|_{[0,1] \setminus \cup_{L \in \mathcal{L}_j} L} = \mu_j|_{[0,1] \setminus \cup_{L \in \mathcal{L}_j} L}$,

4. $\mu_{j+1}(I) = \mu_{j+1}(L'(I)) = \mu_j(I)$ *for all* $I \in \Xi_L^+$,

5. $\mu_i(I(L)) = \mu_j(I(L))$ *for all* $i \geq j$,

6. $\mu_i(L) = \mu_j(L)$ *for all* $i \geq j$,

7. $\mu_i\left(\left\{|S_{L,r(L)}\mu_i(x)| > k\frac{\mu_i(L)}{|L|}\right\}\right) = \mu_j\left(\left\{|S_{L,r(L)}\mu_j(x)| > k\frac{\mu_j(L)}{|L|}\right\}\right)$ *for all* $i \geq j$.

where in the last line $S_{L,r(L)}$ *is as in* (3.4.5).

We are now ready to prove the proposition. Let $M = M(k) > 0$ be an integer that will depend on k and will be chosen later. Let $T^k := S^M$ and $w_k := \mu_M$. We claim that the corona decomposition of $\mathcal{D}'$ associated to μ_l is $(\mathcal{L} : \mathcal{D}'(L))$ where $\mathcal{L} = \bigcup_{i=0}^{l} \mathcal{L}_i$ and $\mathcal{D}'$ is the set of dyadic intervals contained in $[0, 1)$. The first stopping interval in the corona decomposition associated to μ_0 is $[0, 1)$. It is easy to see that this is actually the only stopping interval, therefore $\mathcal{L}_0$ is the stopping collection in the corona decomposition associated to μ_0 and the claim is true in this case. The two facts that allow us to conclude the claim in general are: (a) parts (3) and (6) of Lemma 3.4.18, which let us keep the corona of the measure μ_j when we move to stage $j+1$. Using backwards induction we have proved $\bigcup_{i=0}^{l-1} \mathcal{L}_i$ forms part of the stopping intervals associated to μ_l.

(b) For the next stage of the corona: due to parts (4) and (5) of Lemma 3.4.18 and (3.4.7), it is enough to consider the children of $I \in \Xi_L^+$ for all $L \in \mathcal{L}_{l-1}$. We have these relations:

$$\begin{aligned}
\mu_l(I^+) &= 0 \\
\frac{\mu_l(I^-)}{|I^-|} &< 4\frac{\mu_l(L)}{|L|} \\
\mu_l(I^{-+}) &= 0 \\
\frac{\mu_l(I^{--})}{|I^{--}|} &\geq 4\frac{\mu_l(L)}{|L|} \\
\frac{\mu_l(J)}{|J|} &< 4\frac{\mu_l(I^{--})}{|I^{--}|}, \quad \text{for all } J \subset I^{--}
\end{aligned}$$

which describe the new level of stopping intervals as $\left\{L'(I) : I \in \Xi_L^+,\ L \in \mathcal{L}_{l-1}\right\}$. It is easy to check that this is the last stage. Herein we have proved that the corona decomposition associated to w_k is $(\mathcal{L} : \mathcal{D}'(L))$ with $\mathcal{L} = \bigcup_{i=0}^M \mathcal{L}_i$.

We now decompose the operator as described in (3.3.3), $T^k = \sum_{L\in\mathcal{L}} T_L^k$. Notice that $T_L^k = S_{L,r(L)}$ as defined in (3.4.5). We finish the proof using parts (4), (6) and (7) of Lemma 3.4.18 together with Lemma 3.4.9.

$$\sum_{\substack{L\in\mathcal{L}\\ L\subset[0,1]}} w_k\left(\left\{x : |T_L^k w_k(x)| > k\frac{w(L)}{|L|}\right\}\right) =$$

$$\sum_{i=0}^{M}\sum_{L\in\mathcal{L}_i} w_k\left(\left\{|S_{L,r(L)}w_k(x)| > k\frac{w_k(L)}{|L|}\right\}\right) =$$

$$\sum_{i=0}^{M}\sum_{L\in\mathcal{L}_i} \mu_i\left(\left\{|S_{L,r(L)}\mu_i(x)| > k\frac{\mu_i(L)}{|L|}\right\}\right) =$$

$$\sum_{i=0}^{M}\sum_{L\in\mathcal{L}_i} \mu_L^{6^i}\left(\left\{|S_{L,r(L)}\mu_L^{6^i}(x)| > k\frac{\mu_L^{6^i}(L)}{|L|}\right\}\right) \geq$$

$$\sum_{i=0}^{M}\sum_{L\in\mathcal{L}_i} \frac{1}{6}6^i 2^{-4k}|L| =$$

$$\tfrac{1}{6}2^{-4k}\sum_{i=0}^{M}(1-2^{-4k})^i =$$

$$\tfrac{1}{6}\left(1-(1-2^{-4k})^{M+1}\right) > \tfrac{1}{6}w_k([0,1]),$$

for $M = \left[\frac{\log(3)}{\log(1-2^{-4k})^{-1}}\right] + 1$. Note that M behaves exponentially with respect to k, which is exactly what one should expect from (3.4.10).

□

3.5 *Proof of Main Theorem*

Using Proposition 3.2.1 and (3.2.4), we can reduce the proof of the main theorem to for any $C > 0$ finding a weight w, a Haar multiplier T and a function $f \in L^2(w)$ such that

$$\int_{\mathbb{R}} |T(fw)|^2\sigma(x)dx \geq C\int_{\mathbb{R}} |f|^2 w(x)dx. \tag{3.5.1}$$

There is one more reduction, we can use a gliding hump argument to deduce the infinitary inequality above from the following finitary one. We refer the reader to Section 7 for a detailed explanation of the gliding hump argument.

Lemma 3.5.2. *(Main Lemma) For all $k \geq 1$, there exist a Haar multiplier T^k and a weight $w_k \not\equiv 0$, such that*

$$\int_{[0,1)} |T^k \left(w_k 1_{[0,1)}\right)(x)|^2 \sigma_k(x) dx \geq C k^2 w_k([0,1)),$$

where C is a universal constant and $\sigma_k = \frac{w_k}{(M w_k)^2}$.

Proof. Let $k \geq 1$ be a fixed natural number. Define T^k and w_k as in Proposition 3.4.13 and let $(\mathcal{L} : \mathcal{D}(L))$ be the corona decomposition associated to w_k. We now use the decomposition of T^k and σ_k as suggested in (3.3.3) and (3.3.6) respectively. It is now time to determine the more convenient choices of signs r_L that appear in the definition of (3.4.5). By Khintchine's inequalities we can find a sequence of signs $\{r_L\}_{L \in \mathcal{L}}$, $r_L \in \{+1, -1\}$ so that the first inequality below holds. That together with Chebyshev's inequality and Proposition 3.4.13 provide the desired estimate.

$$\begin{aligned}
\int_{[0,1)} |T^k \left(w_k 1_{[0,1)}\right)(x)|^2 \sigma_k(x) dx &\geq \sum_{L \in \mathcal{L}} \int_{[0,1)} |T_L^k \left(w_k 1_{[0,1)}\right)(x)|^2 \sigma_k(x) dx \\
&\geq \sum_{L \in \mathcal{L}} \left(\frac{w_k(L)}{|L|}\right)^2 k^2 \sigma_k \left(\left\{T_L^k w_k > k \frac{w_k(L)}{|L|}\right\}\right) \\
&\geq \frac{k^2}{64} \sum_{L \in \mathcal{L}} w_k \left(\Delta_1 \left(\left\{T_L^k w_k > k \frac{w_k(L)}{|L|}\right\}\right)\right) \\
&\geq \frac{k^2}{64} \sum_{L \in \mathcal{L}} w_k \left(\left\{T_L^k w_k > k \frac{w_k(L)}{|L|}\right\}\right) \\
&\geq C k^2 w_k([0,1)),
\end{aligned}$$

where C is a universal constant. In the last line, we have used (3.4.12) and property (2) of Lemma 3.4.18. □

3.6 Proof of the measure preserving lemma

Proof. Let us start proving (1). A backwards induction argument allows us to reduce the problem to proving that given $L \in L_{j-1}$ and for any $j \geq 1$,

$$\sum_{L' \subset L, L' \in \mathcal{L}_j} |L'| = \frac{1}{6}(1 - 2^{-4k})|L|. \tag{3.6.1}$$

The proof of (3.6.1) uses $|L'(I)| = \frac{1}{4}|I|$, with $L'(I)$ defined in (3.4.15).

$$\begin{aligned} \sum_{L' \subset L, L' \in \mathcal{L}_j} |L'| &= \frac{1}{4}|[\mathrm{rep}(I(L)), \mathrm{rep}(L)]| \\ &= \frac{1}{4}\left[\frac{2}{3}|L| - \frac{2}{3}|I(L)|\right] \\ &= \frac{1}{6}(1 - 2^{-4k})|L|. \end{aligned}$$

Proof of (2). Let $L, L' \in \mathcal{L}$, we need to distinguish two cases.

1. If $L \cap L' = \emptyset$ then $I(L) \cap L' = \emptyset$ trivially.

2. Suppose $L' \subset L$, then there exist i and j, $i > j$ such that $L' \in \mathcal{L}_i$ and $L \in \mathcal{L}_j$. We can find $\tilde{L} \in \mathcal{L}(L)$ such that $L' \subset \tilde{L}$, then $I(L) \cap \tilde{L} = \emptyset$ by (3.4.16). Therefore $I(L) \cap L' = \emptyset$ as desired.

Conclusion (3) follows trivially from the definition of the measures μ_j and μ_{j+1}.

Next we are proving (4). Let $I \in \Xi_L^+$ and let $L'(I)$ be as in (3.4.15), then

$$\mu_{j+1}(L'(I)) = \mu_{L'(I)}^{6^{j+1}}(L'(I)) = 6^{j+1}\frac{2}{3}|L'(I)| = 6^{j+1}\frac{2}{3}\frac{1}{4}|I| = 6^j|I| = \mu_L^{6^j}(I) = \mu_j(I),$$

where we have used the definition of μ_j and the fact that $\mathrm{jp}(L) \notin I$. We want to make one more comment, estimate (4) is the heart of the measure preserving property.

Proof of (5). Let $L \in \mathcal{L}_j$, then $I(L) \cap L' = \emptyset$ for all $L' \neq L$ by property (2). The following estimates conclude the proof,

$$\mu_j(I(L)) \quad = \mu_L^{6^j}(I(L)) \quad = 6^j\frac{2}{3}|I(L)|$$

$$\mu_i(I(L)) \;\; = \mu_{I(L)}^{6^j}(I(L)) \;\; = 6^j \frac{2}{3}|I(L)|.$$

The proof of (6) can be deduced from the following equality and a backwards induction argument,

$$\mu_{j+1}(L) = \mu_j(L) \quad \text{for all } j \geq 0 \text{ and for all } L \in \mathcal{L}_j. \tag{3.6.2}$$

We complete the proof of (3.6.2) using (4) and (3.4.7). Let $L \in \mathcal{L}_j$, then $I(\tilde{L}) \cap L = \emptyset$ for all $\tilde{L} \in \mathcal{L}_i$, $i < j$ by property (2) and we get the following

$$\begin{aligned} \mu_{j+1}(L) &= \mu_{I(L)}^{6^j}(L) + \sum_{L' \in \mathcal{L}_{j+1}, L' \subset L} \mu_{L'}^{6^{j+1}}(L) \\ &= \mu_{I(L)}^{6^j}(L) + \sum_{L' \in \mathcal{L}_{j+1}, L' \subset L} \mu_{L'}^{6^{j+1}}(L') \\ &= \mu_{I(L)}^{6^j}(L) + \sum_{I \in \Xi_L^+} \mu_L^{6^j}(I) \\ &= \mu_L^{6^j}(L) = \mu_j(L). \end{aligned}$$

We now turn to proving (7). Let $L \in \mathcal{L}_j$, again a backwards induction argument reduces (7) to prove

$$\mu_{j+1}\left(\left\{\left|S_{L,r(L)}\mu_{j+1}\right| > k\frac{\mu_{j+1}(L)}{|L|}\right\}\right) = \mu_j\left(\left\{\left|S_{L,r(L)}\mu_j\right| > k\frac{\mu_j(L)}{|L|}\right\}\right).$$

The strategy will be to verify $S_{L,r(L)}\mu_{j+1} = S_{L,r(L)}\mu_j$. The rest of the proof follows from (3.4.12) and properties (5) and (6). This said, we are going to prove

$$\langle \mu_{j+1}, h_I \rangle = \langle \mu_j, h_I \rangle \quad \text{for all } I \in \Xi_L. \tag{3.6.3}$$

Suppose $I = I(L)$, then $\mu_{j+1}|_{I(L)} = \mu_j|_{I(L)}$ proving (3.6.3) for this particular case. Suppose $I \in \Xi_L$ but $I \neq I(L)$, then $I^+ \in \Xi_L^+$ and $I^- = \left(I^- \setminus \bigcup_{J \in \Xi_L^+, J \subset I^-} J\right) \cup \bigcup_{J \in \Xi_L^+, J \subset I^-} J$. The decomposition of I^+ and I^- together with property (4) proves the desired estimate

$$\langle \mu_{j+1}, h_I \rangle \;\; = \;\; \mu_{j+1}(I^+) - \mu_{j+1}(I^-)$$

$$
\begin{aligned}
&= \mu_{j+1}(L'(I^+)) - \mu_{j+1}(I^- \setminus \cup_{J\in\Xi_L^+, J\subset I^-} J) - \sum_{J\in\Xi_L^+, J\subset I^-} \mu_{j+1}(J) \\
&= \mu_j(I^+) - \mu_j(I(L)) - \sum_{J\in\Xi_L^+, J\subset I^-} \mu_j(J) \\
&= \mu_j(I^+) - \mu_j(I^-) = \langle \mu_j, h_I\rangle.
\end{aligned}
$$

□

3.7 A gliding hump argument

The construction of T^k and w_k in Lemma 3.5.2 is completed on the interval $[0, 1)$. As a matter of fact, there is nothing particular about the interval $[0, 1)$, and the construction could be repeated in any interval of the form $[n, n + 1)$ for $n \geq 1$.

Let $k \geq 1$ be a fixed integer, and let us consider T^k and w_k as in Lemma 3.5.2, then

$$
T^k(f) = \sum_{I\in\mathcal{O}_k} \epsilon_I \langle f, h_I\rangle h_I, \tag{3.7.1}
$$

where $\mathcal{O}_k \subset \mathcal{D}$ such that $\epsilon_I \neq 0$.

We can now define the operators associated to the intervals $[k, k + 1)$.

Definition 3.7.2. We define the Haar multiplier $T^k_{[k,k+1)}$ and the weights $w_{[k,k+1)}$ and $\sigma_{[k,k+1)}$ as

$$
T^k_{[k,k+1)} f = \sum_{I\in\mathcal{O}_k} \epsilon_I \langle f, h_{I+k}\rangle h_{I+k},
$$

$$
w_{[k,k+1)} = w_k(\cdot - k),
$$

$$
\sigma_{[k,k+1)} = \frac{w_{[k,k+1)}}{(Mw_{[k,k+1)})^2},
$$

with $\mathcal{O}_k$ as in (3.7.1).

Lemma 3.7.3. *Let $k \geq 1$ be a natural number, $T^k_{[k,k+1)}$, $w_{[k,k+1)}$ and $\sigma_{[k,k+1)}$ as in definition 3.7.2, then*

$$
\int_{[k,k+1)} |T^k_{[k,k+1)} \left(w_{[k,k+1)} 1_{[k,k+1)}\right)(x)|^2 \sigma_{[k,k+1)}(x) dx \geq C k^2 w_{[k,k+1)}([k, k + 1)),
$$

where C is a universal constant.

Proof. Let $w_{[k,k+1)}$ as in definition (3.7.2) and w_k as in Lemma 3.5.2, using a change of variables we can see that

$$\langle w_{[k,k+1)}, h_{I+k}\rangle = \langle w_k, h_I\rangle. \tag{3.7.4}$$

We use (3.7.4) to conclude

$$T^k_{[k,k+1)}\left(w_{[k,k+1)}\right)(y+k) = T^k w_k(y), \tag{3.7.5}$$

with T^k as in (3.7.1).

Also,

$$\sigma_{[k,k+1)}(y+k) = \sigma_k(y), \tag{3.7.6}$$

with σ_k as in Lemma 3.5.2. The proof uses the fact that $Mw_{[k,k+1)}(y+k) = Mw_k(y)$.

Finally, (3.7.5) and (3.7.6) together with Lemma 3.5.2 allow us to finish the proof.

$$\int_{[k,k+1)} |T^k_{[k,k+1)}\left(w_{[k,k+1)}1_{[k,k+1)}\right)(y)|^2\sigma_{[k,k+1)}(y)dy =$$
$$\int_{[0,1)} |T^k_{[k,k+1)}\left(w_{[k,k+1)}1_{[k,k+1)}\right)(x+k)|^2\sigma_{[k,k+1)}(x+k)dx =$$
$$\int_{[0,1)} |T^k\left(w_k 1_{[0,1)}\right)(x)|^2\sigma_k(x)dx \geq Ck^2 w_k([0,1)) = Ck^2 w_{[k,k+1)}([k,k+1))$$

□

Proof of (3.5.1). We will prove something stronger than (3.5.1). We will find T, w and $f \in L^2(w)$ such that the left hand side of (3.5.1) is infinite. Let $T = \sum_{n=1}^{\infty} T^n_{[n,n+1)}$, then T is well defined and is a Haar multiplier. Let $w = \sum_{n=1}^{\infty} w_{[n,n+1)}$, w is in $L^1_{loc}(\mathbb{R})$. And let $f = \sum_{n=1}^{\infty} \frac{1}{n^{1/2+\epsilon}} 1_{[n,n+1)}$ where $0 < \epsilon < 1/2$. It is easy to see that $f \in L^2(w)$.

Let us now consider the left hand side of (3.5.1). We first notice that for all $x \in [n, n+1)$, $\sigma(x) \approx \sum_{n=1}^{\infty} \frac{w_{[n,n+1)}}{(Mw_{[n,n+1)})^2}$. Actually we only need to check that $Mw(x) \approx$

$Mw_{[n,n+1)}$. A similar argument has been used in the next chapter to prove (4.2.2), we skip it here and refer the reader to Chapter 4, page 44. Using this observation, Lemma 3.7.3 and the fact that $\epsilon < 1/2$, we conclude the proof.

$$\begin{aligned}\int_{\mathbb{R}} |T(fw)|^2 \sigma(x)dx &\approx \sum_{n=1}^{\infty} \int_{[n,n+1)} \left| \sum_{k=1}^{\infty} T^k_{[k,k+1)} \left(\sum_{i=1}^{\infty} \frac{1}{i^{1/2+\epsilon}} 1_{[i,i+1)} w_{[i,i+1)} \right) \right|^2 \sigma_{[n,n+1)} dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{1+2\epsilon}} \int_{[n,n+1)} \left| T^n_{[n,n+1)} \left(1_{[n,n+1)} w_{[n,n+1)} \right) \right|^2 \sigma_{[n,n+1)} dx \\ &\geq C \sum_{n=1}^{\infty} n^{1-2\epsilon} = \infty\end{aligned}$$

□

CHAPTER IV

THE HILBERT TRANSFORM DOES NOT MAP $L^1(MW)$ TO $L^{1,\infty}(W)$

4.1 Statement of main result

In [14], C. Fefferman and E. Stein observed the following a priori estimate for the Hardy-Littlewood maximal operator M:

$$\sup_{t>0} t\, w\{x \in \mathbb{R} : |Mf(x)| > t\} \leq C \int |f(x)| Mw(x)\, dx \quad .$$

Here the weight w is a non-negative, locally integrable function, and $w(E)$ denotes the integral of the weight over the set E. We give a negative answer to the question whether such an inequality holds when the Hardy Littlewood maximal operator on the left hand side is replaced by the Hilbert transform. For a discussion of the history of this question we refer to [38] or Section 3.1 in the previous chapter.

Theorem 4.1.1. *For each constant $C > 0$ there is a weight function w on the real line and an integrable compactly supported function f and a $t > 0$ such that*

$$t\, w\{x \in \mathbb{R} : |Hf(x)| > t\} \geq C \int |f(x)| Mw(x)\, dx \quad .$$

Similarly as in [38], we prove Theorem 4.1.1 as a consequence of the following:

Proposition 4.1.2. *For each constant $C > 0$ there is an everywhere positive weight function w on the real line and an integrable compactly supported function f and a $t > 0$ such that*

$$t^2\, w\{x \in \mathbb{R} : |Hf(x)| > t\} \geq C \int |f(x)|^2 \left(\frac{Mw(x)}{w(x)}\right)^2 w(x)\, dx \quad .$$

The reduction to Proposition 4.1.2 is taken from [7], we include the argument at the end of this chapter. Following [38] further, we reduce Proposition 4.1.2 to the dual proposition:

Proposition 4.1.3. *For each constant C there is a nontrivial weight w on the real line such that*

$$\|H(w1_{[0,1)})\|_{L^2(w/(Mw)^2)} \geq C\|1_{[0,1)}\|_{L^2(w)} \quad .$$

Our construction of the weight w is a somewhat simpler variant of the construction in [38]. It was discovered during a stimulating summer school on "Weighted estimates for singular integrals" at Lake Arrowhead, Oct 3-8. 2010.

4.2 Proof of Theorem 4.1.2

Recall that a triadic interval I is of the form $[3^j n, 3^j(n+1))$ with integers j, n. Denote by I^m the triadic interval of one third the length of I which contains the center of I.

Fix an integer k which will be chosen large enough depending on the constant C in Proposition 4.1.3. Define $\mathbf{K}_0$ to be $\{[0,1)\}$ and recursively for $i \geq 1$:

$$\mathbf{J}_i := \{K^m : K \in \mathbf{K}_{i-1}\} \quad ,$$

$$\mathbf{K}_i := \{K : K \text{ triadic}, \ |K| = 3^{-ik}, \ K \subset \bigcup_{J \in \mathbf{J}_i} J\} \quad .$$

Proceeding recursively from the larger to the smaller intervals, we choose for each $J \in \mathbf{J} := \bigcup_{i \geq 1} \mathbf{J}_i$ a sign $\epsilon(J) \in \{-1, 1\}$. More precisely, $\epsilon(J)$ depends on the values $\epsilon(J')$ with $|J'| > |J|$. The exact choice will be specified below. Define for each $J \in \mathbf{J}$ the interval $I(J)$ to be the triadic interval of length $3^{1-k}|J|$ whose right endpoint equals the left endpoint of J if $\epsilon(J) = 1$, and whose left endpoint equals the right endpoint of J if $\epsilon(J) = -1$. Note that $I(J)$ has the same length as the intervals in $\mathbf{K}_i$.

Next we define a sequence of absolutely continuous measures on $[0, 1]$. We continue to use the same symbol for a measure and its Lebesgue density. Let w_0 be the uniform

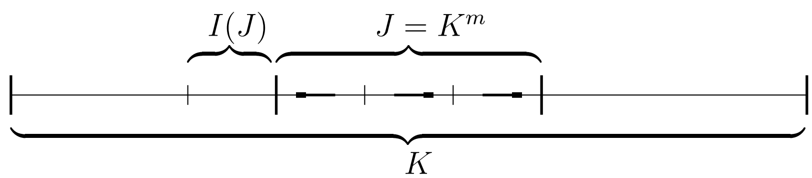

Figure 3: The construction of the measure.

measure on $[0,1)^m \cup I([0,1))^m$ with total mass 1. Recursively we define the measure w_i by the following properties: It coincides with w_{i-1} on the complement of $\bigcup_{K\in\mathbf{K}_i} K$. For $K \in \mathbf{K}_i$ we have $w_i(K) = w_{i-1}(K)$ and the restriction of w_i to K is supported and uniformly distributed on $K^m \cup I(K^m)$.

Let w be the weak limit of the sequence w_i and note that w is supported on $\bigcup_{J\in\mathbf{J}} I(J)$. For $K \in \mathbf{K}_i$, $J \in \mathbf{J}_i$, $x \in I(J)$, and any triadic interval K' with $|K'| \geq |K|$ we have

$$w(x) = \frac{w(I(J))}{|I(J)|} = \frac{w(K)}{|K|} \geq \frac{w(K')}{|K'|} \quad . \tag{4.2.1}$$

We claim that for $J \in \mathbf{J}$ and $x \in I(J)^m$ we have

$$Mw(x) \leq 7w(x) \quad . \tag{4.2.2}$$

To see this, let I be a (not necessarily triadic) interval containing x. If I is contained in $I(J)$, then by the first identity of (4.2.1) the average of w over I equals $w(x)$. If I is not contained in $I(J)$, then $|I| \geq |I(J)|/3$. Let $\mathbf{K}'$ be the collection of triadic intervals of length $|I(J)|$ which intersect I and note that

$$\sum_{K'\in\mathbf{K}'} |K'| \leq |I| + 2|I(J)| \leq 7|I|$$

because at most two intervals in $\mathbf{K}'$ are not entirely covered by I. With (4.2.1) we conclude that the average of w over I is no more than $7w(x)$, which completes the proof of (4.2.2).

Lemma 4.2.3. *For $K \in \mathbf{K}_i$, $J = K^m$, $x \in I(J)^m$, and $k > 3000$ we have*

$$|Hw(x)| \geq (k/3)w(x) \quad .$$

This Lemma proves Proposition 4.1.3, because with (4.2.2) and since w is constant on every $I(J)$ we have

$$49\|Hw\|^2_{L^2(w/(Mw)^2)} \geq (k^2/9)\sum_{J\in\mathbf{J}}\int_{I(J)^m} w(y)\,dy \geq (k^2/27)\|1_{[0,1)}\|^2_{L^2(w)} \quad .$$

Proof of Lemma 4.2.3: We split the principal value integral for $Hw(x)$ into six summands:

$$p.v.\int_{I(J)} \frac{w(y)}{y-x}\,dy \tag{4.2.4}$$

$$+\int_J \frac{w(y)}{y-x}\,dy \tag{4.2.5}$$

$$+\int_{K^c} \left(\frac{w(y)}{y-x} - \frac{w(y)}{y-c(J)}\right)dy \tag{4.2.6}$$

$$+\int_{(\bigcup_{\mathbf{K}_i} K')^c} \frac{w(y)}{y-c(J)}\,dy \tag{4.2.7}$$

$$+\sum_{K'\in\mathbf{K}_i\setminus\{K\}} \int_{K'} \frac{w(y)}{y-c(J)} - \frac{w(y)}{c(K')-c(J)}\,dy \tag{4.2.8}$$

$$+\sum_{K'\in\mathbf{K}_i\setminus\{K\}} \int_{K'} \frac{w(y)}{c(K')-c(J)}\,dy \quad . \tag{4.2.9}$$

The terms (4.2.7) and (4.2.9) remain unchanged if we replace w by w_i and hence depend only on the choices of $\epsilon(J')$ with $|J'| > |J|$. The integrand of (4.2.5) is positive or negative depending on $\epsilon(J)$. Specify the choice of $\epsilon(J)$ so that the sign of (4.2.5) equals the sign of (4.2.7)+(4.2.9). If the latter is zero, we may arbitrarily set $\epsilon(J) = 1$. We estimate

$$|(4.2.5)| \geq \sum_{K'\in\mathbf{K}_{i+1}, K'\subset J} \int_{K'} \frac{w(y)}{|y-x|}\,dy$$

$$\geq \sum_{K' \in \mathbf{K}_{i+1}, K' \subset J} \frac{w(K')}{\sup_{y \in K'} |y-x|}$$

$$\geq \sum_{n=1}^{3^k} \frac{1}{n+1} \frac{w(I(J))}{|I(J)|} \geq (k/2) w(x) \quad .$$

The remaining terms are small error terms, we estimate with $\delta = |I(J)^m|$:

$$|(4.2.4)| = |\int_{I(J) \setminus [x-\delta, x+\delta]} \frac{w(y)}{y-x} \, dy| \leq 3w(x) \quad ,$$

$$|(4.2.6)| \leq 4 \sum_{|K'|=|K|, K' \neq K} \int_{K'} \frac{|x-c(J)|}{|y-c(J)|^2} w(y) \, dy$$

$$\leq 8 \sum_{|K'|=|K|, K' \neq K} \frac{|x-c(J)|}{|c(K')-c(J)|^2} w(K')$$

$$\leq 16 \sum_{n=1}^{\infty} \frac{1}{(n-3/4)^2} \frac{w(I(J))}{|I(J)|} \leq 200 w(x) \quad ,$$

$$|(4.2.8)| \leq 4 \sum_{K' \in \mathbf{K}_i} \int_{K'} \frac{|y-c(K')|}{|c(K')-c(J)|^2} w(y) \, dy \quad ,$$

and the last expression is dominated by the same final bound as (4.2.6). Putting all estimates together, we have

$$|(4.2.4) + (4.2.5) + (4.2.6) + (4.2.7) + (4.2.8) + (4.2.9)|$$

$$\geq |(4.2.5) + (4.2.7) + (4.2.9)| - |(4.2.4)| - |(4.2.6)| - |(4.2.8)|$$

$$\geq |(4.2.5)| - |(4.2.4)| - |(4.2.6)| - |(4.2.8)|$$

$$\geq (k/2 - 403) w(x) \quad .$$

This completes the proof of Lemma 4.2.3 and thus Theorem 4.1.3.

4.3 Remarks

4.3.1 Weights in Theorem 4.1.1

We specify weights satisfying Theorem 4.1.1. Fix a constant C as in Proposition 4.1.3 and consider k and the weight w constructed above. We slightly change w to make it positive by adding ce^{-x^2} for sufficiently small c so as to not change the conclusion of

Proposition 4.1.3. We may normalize the measure to be probability measure and call the remaining measure w again. The conclusion of Proposition 4.1.3 can be written:

$$\left(\int (Hw(x))^2 \frac{w(x)}{(Mw(x))^2}\, dx\right)^{1/2} \geq C \quad . \tag{4.3.1}$$

We now multiply both sides of (4.3.1) by the left hand side of (4.3.1), setting $f = (Hw)w/(Mw)^2$ and using essential self-duality of H we obtain

$$\left|\int w(x) Hf(x)\, dx\right| \geq C \left(\int f(x)^2 \frac{(Mw(x))^2}{(w(x))^2} w(x)\, dx\right)^{1/2} \quad . \tag{4.3.2}$$

Letting f^* be the non-increasing rearrangement of Hf on $[0,1]$, we may estimate the left hand side of (4.3.2)

$$\int_0^1 f^*(y)\, dy \leq 2 \sup_{y\in[0,1]} y^{1/2} f^*(y) = 2 \sup_{t>0} w(\{x : |Hf(x)| \geq t\})^{1/2} t \quad .$$

Hence Proposition 4.1.2 holds for the constant $C/2$ with the weight w and some existentially chosen t. Now let E be the set on the left hand side of Proposition 4.1.2 for the given w, f, and appropriate t, then we have

$$M(w1_E)(x) = \sup_{x\in I} \frac{\int_I w}{\int_I 1} \frac{\int_I 1_E w}{\int_I w} \leq Mw(x) M_w 1_E(x) \quad ,$$

where M_w denotes the Hardy Littlewood maximal function with respect to the weight w. With Hölder's inequality we obtain

$$\int |f(x)| M(w1_E)(x)\, dx \leq \left(\int |f(x)|^2 \frac{Mw(x)^2}{w(x)}\, dx\right)^{1/2} \|M_w 1_E\|_{L^2(w)} \quad .$$

With the Hardy Littlewood maximal theorem with respect to the weight w we can estimate $\|M_w 1_E\|_{L^2(w)}$ by $w(E)^{1/2}$. This shows that Theorem 4.1.1 holds for the weight $w1_E$.

4.3.2 A1 weights

It remains open to date whether the a priori inequality

$$t\, w\{x \in \mathbb{R} : |Hf(x)| > t\} \leq C \|w\|_{A_1} \int |f(x)| w(x)\, dx \tag{4.3.3}$$

Table 1: The three conjectures and its answers

Conjecture	Answer	Authors
Muckenhoupt-Wheeden	False	Reguera-Thiele
Weak Muckenhoupt-Wheeden	False	Nazarov-Reznikov-Vasyunin-Volberg
A_2	True	Hytönen

holds, where the A_1 constant is defined as $\|w\|_{A_1} := \|Mw/w\|_\infty$. Our construction in this paper does not seem to address this question. The recent preprint [28] has announced that the analogue of (4.3.3) for the Hilbert transform is false. In [24], a version of (4.3.3) has been proved with an additional logarithmic factor in the A_1 constant of the weight.

4.3.3 The three conjectures and its answers

In the Table 1 we summarize the answers to the three conjectures we have discussed, some of which have been studied in this thesis.

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