Applications of Nonstandard Analysis to Mathematical Physics

PART II—SUMMARY OF COMPLETED PROJECT (FOR PUBLIC USE)

The mathematical theory of infinitesimals is developed and used to analyze mathematical problems arising in quantum physics.

1. In "An Application of the Nonstandard Trotter Product Formula" a new formulation of the Feynman path integral is given.

2. In "A Note on Exponentials of Distributions" it is proposed that instead of attempting to treat nonlinear functions of distributions within distribution theory on should go beyond to nonstandard analysis. It was noted that the exponential of a negative Dirac delta function could be defined within the context of distributions to be identically one whereas it is not one in the nonstandard framework and this leads to the correct perturbation formula. This paper also contains the nonstandard analysis of strong resolvent convergence.

3. In "The Strong Convergence of Schrödinger Propagators" time dependent perturbation formulas are developed for highly singular potentials in quantum mechanics using techniques of the infinitesimal approach.

PART III—TECHNICAL INFORMATION (FOR PROGRAM MANAGEMENT USES)

1. ITEM (Check appropriate blocks)
   a. Abstracts of Theses
   b. Publication Citations
   c. Data on Scientific Collaborators
   d. Information on Inventions
   e. Technical Description of Project and Results
   f. Other (specify)

2. Principal Investigator/Project Director Name (Typed)
   Alan D. Sloan

3. Principal Investigator/Project Director Signature

4. Date
   3/14/81
An application of the nonstandard Trotter product formula

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The nonstandard Trotter product formula is used to extend the Feynman integral interpretation of solutions to the Schrödinger equation in the presence of a highly singular potential.

We seek an answer to the question of when a quantum mechanical Hamiltonian \( H \) determines a system which evolves according to a Feynman path integral. Specifically, we want to know when there is a potential \( V \) so that, upon setting \( H = 1 \), and fixing \( t > 0 \)

\[
\exp(-itH)u = \lim_{n \to \infty} T_{n, V}(u), \quad u \in L^2(\mathbb{R}^3), 
\]

where \( T_{n, V}(u) \) is defined as the integral

\[
(T_{n, V}(u))(x) = c_n \int_{\mathbb{R}^3} \exp[S(x, x, x; n, t, V)] \times u(x) \, dx 
\]

and

\[
c_n = (2\pi\hbar/n)^{3/2}, \quad m > 0, 
\]

\[
x = (x_1, x_2, \ldots, x_m), 
\]

\[
S(x, x, x; n, t, V) = \sum_{j=1}^{n} \left( \frac{m(x_j - x_{j-1})^2}{2(t/n)^2} - V(x_j) \right) t/n. 
\]

An explicit computation shows that

\[
T_{n, V} = \exp(-itH_0/n) \exp(-itV/n), 
\]

where

\[
H_0 = -\Delta/2m \quad \text{on } L^2(\mathbb{R}^3), 
\]

for \( V \) any real measurable function. Consequently, Nelson was able to employ the Trotter product formula to verify (1) in case \( H \) is the closure of the operator sum, \( H_0 + V \), and \( V \) is chosen so that \( H_0 + V \) is essentially self-adjoint. Nelson and Faris extended this type of formula to more singular potentials by including an additional limiting operation after analytic continuation in the mass parameter \( m \). Although there has been progress in extending Trotter's formula for \( \exp(-itH_0 + V) \), when \( V \) is singular relative to \( H_0 \), analogous results are not known for the unitary groups \( \exp[itH_0 + V] \). Through the use of elementary nonstandard techniques we shall find a formula of the type (1) valid for singular potentials.

For details on nonstandard analysis and notation we refer the reader to Refs. 4 and 8–10. Here we adopt the convention that \( X \subset X \) for any set \( X \). If \( X \) is a topological space, in \( X \), \( b \in X \), we write \( a = b \) if and only if \( b \) is in the monad of \( a \). If \( a = b \) we say \( a \) is the standard part of \( b \) and that \( a \) and \( b \) are infinitely close. We use the Euclidean topology on \( \mathbb{R}^n \) and the norm topology on \( L^2(\mathbb{R}^3) \).

Suppose \( H \) is a self-adjoint operator on \( L^2(\mathbb{R}^3) \) which can be approximated in the generalized strong sense by bounded perturbations of \( H_0 \); i.e., there is a sequence \( V_k \) of bounded self-adjoint operators on \( L^2(\mathbb{R}^3) \) such that the resolvents of \( H_0 + V_k \) converge strongly to those of \( H \). Then, (Ref. 11, 502), providing \( [H_0 + V_k] \) is uniformly bounded below, it follows that \( \exp(-itH) \) is the strong limit of \( \exp[it(H + V_k)] \). Thus, for all \( K \) in \( \mathbb{N} \), \( N \), and for all \( u \) in \( L^2 \) we have

\[
\exp(-itH)u = \lim_{K \to \infty} \exp[-it(H + V_K)]u. 
\]

By the Trotter product formula and the transfer principle, for each \( K \) in \( \mathbb{N} \)

\[
\exp(-t(H_0 + V_K)) = \lim_{n \to \infty} \exp[-itH_0/n] \exp[-itV_K/n].
\]

Combining (6) and (7) we find

\[
\exp(-itH)u = \lim_{n \to \infty} \exp[-itH_0/n] \exp[-itV_K/n].
\]

For all finite \( n \) and \( K \), (4) shows that

\[
T_{n, V_K} = \exp(-itH_0/n) \exp(-itV_K/n), 
\]

so by the transfer principle (9) holds for all \( n, K \) in \( \mathbb{N} \). Thus, for each fixed infinite \( K \), (8) yields

\[
\exp(-itH)u = \lim_{n \to \infty} T_{n, V_K}(u). 
\]

For each \( n, K \) in \( \mathbb{N} \), (2) shows that \( T_{n, V_K}(u) \) is given by an action integral. The transfer principle then implies that for all \( n, K \) in \( \mathbb{N} \), (2) and (3) remain valid with \( V_K \) replacing \( V \). Consequently, we may conclude that

\[
\exp(-itH)u = \lim_{n \to \infty} c_n \int_{\mathbb{R}^3} \exp[iS(\cdot, x, x; n, tV_K)] \times u(x) \, dx, 
\]

Lemma: Let \( X \) be a separable normed linear space. Let \( f : X \to X \) be bounded linear operators which are uniformly bounded in the sense that there is an \( M \in \mathbb{N} \) such that \( \|f_n\| \leq M \) for all \( n \in \mathbb{N} \). If \( f_n \to f_0 \) pointwise on \( X \) as \( n \to \infty \), there is an \( N \in \mathbb{N} \) such that \( n > N \) implies \( f_n(x) = f_0(x) \) for all \( x \in X \).

Proof: Let \( C \) be a countable dense subset of \( X \). Let \( \lambda \) be a positive infinitesimal. Choose \( N \in \mathbb{N} \) so that \( n > N \) implies \( \|f_n(x) - f_0(x)\| < \lambda \). Let \( N \) be an upper bound for \( \{N \in \mathbb{N} : \|f_n(x) - f_0(x)\| < \lambda \} \); \( N \) exists by Ref. 5, p. 59. Let \( x \in X \) be arbitrary. Fix \( \delta > 0 \) in \( \mathbb{R} \). There is a \( c \in C \) so that

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\[ \|c - x\| < \delta. \text{ Then for } n > N \]
\[ \|f_n(x) - f_0(x)\| \leq \|f_n(x) - f_n(c)\| + \|f_n(c) - f_0(c)\| + \|f_0(c) - f_0(x)\| \]
\[ < M \|x - c\| + \lambda + M \|c - x\| < 3M\delta. \]

Because \(\delta\) is arbitrary we conclude that \(f_n(x) = f_0(x)\) whenever \(n > N\).

Q. E. D.

Combining this lemma with the previous conclusion, noting that \(\exp(-itH)\) and the \(T_n\) are uniformly bounded by 1, yields the following:

**Theorem:** If \(V_k\) are bounded self-adjoint operators such that \(H_0 + V_k\) converge to \(H\) in the generalized strong sense as \(n \to \infty\) in \(N\), then for each fixed positive infinite integer \(K\) there is an \(N\) such that for all \(n > N\) and for all \(u\) in \(L^2\)

\[ \exp(itH)u(x) = c_n \int_{R^m} \exp(is\cdot x + x_n; n, t, V_k) \]
\[ \times u(x_n) \, dx \, dx_n, \quad (11) \]

in \(*L^2*\).

To compare the types of potentials covered by formulas (1) and (11) we note that\(^2\) (1) holds if \(V\) is in \(L^p + L^\infty, p > 2\), whereas (11) holds whenever \(H\) is the generalized strong limit of bounded self-adjoint perturbations of \(H_0\). Examples of such \(H\)'s can be found by defining \(H\) to be the form sum of \(H_0\) and \(V\) when either

(a) \(V\) is in \(L^p + L^\infty, p > \frac{3}{2}\);

(b) \(V > 0\) is locally in \(L^1\) outside a closed set of measure zero;

(c) \(V\) is a delta function distribution concentrated on the surface of a compact \(C^1\) hypersurface in \(R^3\).

For the necessary approximation theorem in case (b) see Ref. 12 where the \(V_k\)'s are defined by truncation. In cases (a) and (c) see Ref. 13 where the \(V_k\)'s are given by regularization of \(V\). See Ref. 13 for additional examples.

Similar techniques apply in dimensions other than 3 and for other \(H_0\)'s.

\(^8\)A. Robinson, *Nonstandard Analysis* (North-Holland, Amsterdam, 1974).
The Strong Convergence of Schrodinger Propagators

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Abstract

Time dependent versions of the Trotter-Kato theorem are discussed using nonstandard analysis. Both standard and non-standard results are obtained. In particular, it is shown that if a sequence of generators converges in the strong resolvent topology at each time to a limiting generator and if the sequence of generators and limiting generator uniformly satisfy Kisynski type hypotheses then the corresponding Schrodinger propagators converge strongly. The results are used to analyze time dependent, form bounded perturbations of the Laplacian.
I. Introduction.

The time dependent Schrödinger equation

\[ \frac{dx}{dt} (t) = -i A(t)x(t) \]

may be advantageously discussed in terms of unitary propagators. Equation (1) is set in a complex Hilbert space, \( H \), and the time variables, \( t, s \), are to range in some closed and possibly infinite interval, \( J \), of real numbers. For each such \( t \), \( A(t) \) is a selfadjoint operator on \( H \) while \( x(t) \) is an element of \( H \) as is \( x_s \). \( A(\cdot) \) is called the generator of (1).

**Definition:** A jointly strongly continuous map \((t,s) \rightarrow U(t,s)\) from \( J \times J \) into the unitary operators on \( H \) is a unitary propagator providing

(a) \( U(t,t) = I \)

(b) \( U(t,s)U(s,r) = U(t,s) \)

hold for all \( r, s, t \) in \( J \).

Given a generator \( A \) the relevant propagator is expected to have the property that \( x(t) = U(t,s)x_s \) is the "solution" to
equation (1). The precise sense in which $x(\cdot)$ is a solution and the exact collection of initial states $x_s$ for which such a solution exists, remain to be specified later. At times, to avoid ambiguity, we will denote the propagator related to equation (1) by $U_A$.

The most satisfactory results are known in case $A(t) = A$ for each $t$ in $J$ where $A$ is some fixed selfadjoint operator. This is the time independent case. The related propagator may be explicitly given as $U_A(t,s) = e^{i(t-s)A}$. Then $x(t) = U(t,s)x_s$ is differentiable in norm and satisfies equation (1) for all $x_s$ in $D(A)$, the domain of $A$.

Once a solution is known to exist, an analysis of stability properties of the equation may begin; this is perturbation theory. The basic question in this theory is: If two generators are close, are the corresponding solutions close? In the time independent case, this question is affirmatively answered by the Trotter-Kato Theorem 1 [5, page 502]:

If $A_n$ is a sequence of selfadjoint operators whose resolvents converge strongly to the resolvent of a selfadjoint operator $A$, then $U_{A_n}(t,s)$ converges strongly to $U_A(t,s)$ for each $t,s$ in $J$.

The situation is more complicated in the time dependent case. For example, for $t$ in $\mathbb{R}$, let $f_n(t) = 1 + 2^n X_n(t)$, where $X_n$ is the characteristic function of the interval $[n^{-1} - 2^{-(n+1)}, n^{-1} + 2^{-(n+1)}]$ for $n = 1,2,\ldots$. Let $A_n(t) = f_n(t)I$ and $A = I$. Then $A_n(t)$ converges in the strong resolvent topology to $A$ but $U_{A_n}(t,0)$ does not converge strongly to $U_A(t,0)$ for any $t > 0$, even though the $A_n$'s and $A$ are bounded.
In II we present a Trotter-Kato type theorem for the case of bounded generators. In III a similar result is presented for a certain class of unbounded generators. For unbounded and time dependent generators, $A$, there is a general technique for constructing $U_A$. First one approximates $A$ by bounded generators, $A_n$, next one constructs $U_{A_n}$ explicitly, and finally one proves that the $U_{A_n}$ converge to a propagator which is then defined to be $U_A$.

Yoshida, [15], developed an approximation procedure which has proved to be very durable. He defined $A_n(t)$ to be $A(t)[1+A(t)/n]^{-1}$. This method has recently been applied in [16] to study the "hyperbolic case" of evolution equations. In addition to its continued applicability, another indication of the strength of the Yoshida approximation may be described in the framework of nonstandard analysis. In the next paragraph we informally discuss the ideas involved.

Given some set $S$ one can form $^*S$, a new set containing not only a copy of $S$ but also ideal elements, whenever $S$ is infinite. For example if $\mathbb{N}$ is the set of positive integers then $^*\mathbb{N}$ contains infinite integers while for $\mathbb{R}$ the reals, $^*\mathbb{R}$ contains both infinite and infinitesimal numbers. The map $V \mapsto ^*V$ of subsets of $S$ to subsets of $^*S$ is a Boolean algebra isomorphism into. Consequently, all the concepts of standard analysis on $S$ may be transferred to $^*S$. In particular there is a natural order on $^*\mathbb{R}$ and an induced definition of convergence: for $a_n, a$ in $^*\mathbb{R}$, $n \in \mathbb{N}, a_n \rightarrow a$ means for every $\varepsilon > 0$ in $^*\mathbb{R}$ there is an $N$ in $^*\mathbb{N}$ such that $n > N$ implies $|a_n - a| < \varepsilon$. Given such a sequence $a_n$ one may also ask if $a_n \rightarrow a$
in the standard sense that for every \( \epsilon > 0 \) in \( \mathbb{R} \) \( \exists N \) in \( \mathbb{N} \) such that \( n > N \) implies \( |a_n - a| < \epsilon \). These two types of convergence are not equivalent. For example, if \( m \) is an infinite integer then \( a_n = m/n + a = 0 \) in the transferred sense, but not the standard sense.

In III we consider the class, \( K \), of generators introduced by Kisynski, [8], and studied also by Simon [11]. The aforementioned authors showed that whenever \( A \) is in \( K \), then \( U_{A_n} \) converges strongly to \( U_A \) where \( A_n \) is the Yoshida approximation to \( A \). Transferring this result we obtain the fact that whenever \( B \) is in \( K \) then \( U_{B_n} \) converges strongly to \( U_B \). A priori, we may not assert that \( U_{B_n} \) converges to \( U_B \) in the standard sense. Nevertheless we are able to prove that this additional convergence takes place, giving an additional indication of the strength of the Yoshida approximation. Moreover, the general techniques of nonstandard analysis may be used to show that in order to obtain a standard Trotter-Kato type theorem in \( K \) (i.e., whenever \( C_n, C \) are in \( K \) and \( C_n(t) + C(t) \) in the strong resolvent topology for each \( t \), then \( U_{C_n}(t,s) \) converges strongly to \( U_C(t,s) \)) the standard convergence of \( U_{B_n}(t,s) \) to \( U_B(t,s) \) for \( B \) in \( K \) is sufficient. Thus we are able to present a new standard result obtained first using nonstandard techniques. Our methods also revealed a standard proof which we present. The nonstandard results do have interesting consequences in IV:

In IV we are particularly interested in generators of the form \( A(t) = -\Delta + V(t) \), selfadjoint on \( L^2(\mathbb{R}^n) \) for each \( t \). The corresponding propagator describes the evolution of the quantum system
with Hamiltonian $A(t)$ at time $t$. If $V(t)$ is a quadratic form then the usual formulas of quantum physics do not have an obvious meaning; see for example the Dyson expansion, Theorem 3, which is typically the starting point for time dependent perturbation theory. Nonetheless, if one regularizes the $V(t)$ it is possible to obtain a nonstandard bounded $W(t)$ so that $U_A$ and $U_B$ are infinitely close, where $B(t) = -A + W(t)$. Thus, the usual perturbation formulas, which are valid for $U_B$, may be used for $U_A$ with only infinitesimal errors. Consequently, we obtain a nonstandard Dyson expansion for propagators of quantum mechanical systems of particles in potentials given as distributions. These and other applications are discussed further in IV.

Finally, we remark that the reader interested in perturbation theory but not nonstandard analysis may read the remark following Theorem 7, the introduction to III through the discussion of Theorem 8, Corollary 10, Theorems 11, 12, Corollary 13 and Examples 14 and 15 as these standard results are self contained.

II. Stability of Solutions for Bounded Generators.

Let $\star X$ be an adequate ultrapower of a structure $X$ containing the real numbers, $\mathbb{R}^1$.

For $A$ selfadjoint and $B$ an internal selfadjoint operator we write $A \bowtie B$ if $B$ is in the monad of $A$ relative to the strong resolvent topology. In this case we say that $B$ is near standard and that $A$ is the standard part of $B$. If $P$ is a bounded linear operator and $Q$ an internal finitely bounded linear operator write $P \bowtie Q$ in case $Ph = Qh$ for all $h$ in $H$. In case $P$ and $Q$ are additionally selfadjoint the two definitions of $\bowtie$ coincide. Here we have written $h \bowtie k$ to mean $\|h-k\| \bowtie 0$ for $h,k$ in $\star H$. This coincides with $h$ being in the monad of $k$, in case $k \in H$, relative to the norm topology on $H$. The weak topology relation will be denoted by $h \bowtie_\omega k$. This means $(h,v) \bowtie (k,v)$ for all $v$ in $H$. On $\mathbb{R}$ $\bowtie$ is with respect to the ordinary Euclidean topology.
In [13] it was shown that a standard technique of proving the Trotter-Kato theorem, (see [5]), could be used to prove a nonstandard version:

\textbf{Theorem 2:} If $A$ is selfadjoint and $B$ is internal selfadjoint then $A \subset B$ implies $U_A(t,s) \subset U_B(t,s)$ for all finite $t,s$ in $\mathbb{R}^1$.

See [13] for a discussion of the strong resolvent topology from a nonstandard point of view. For an introduction to nonstandard analysis the reader is referred to [9] and [14].

\textbf{Definition:} A D-generator is a strongly continuous map of $\mathcal{J}$ into the bounded selfadjoint operators.

An existence theorem for solutions of the Schrödinger equation defined by a D-generator is proved by explicit construction via the Dyson expansion.

\textbf{Theorem 3:} [10] Let $A$ be a D-generator. Then defining $U(t,s)$ by $I + \sum_{n=1}^\infty (-i)^n T_n(A,t,s)$ for $t \geq s$ in $\mathcal{J}$, where

$$T_n(A,t,s) = \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} A(t_1) \cdots A(t_n) \, dt_n \cdots dt_1,$$

and by $U^*(s,t)$ for $s \geq t$ in $\mathcal{J}$, gives a unitary propagator. Here the integrals are strong while the sum converges in operator norm. $x(t) = U(t,s)x_s$ gives a norm differentiable function which is a solution to equation (1) for all $x_s$ in $\mathcal{H}$. 
The main ingredient in the proof of theorem 3 is an application of the uniform boundedness principle to prove Lemma 4 [10]: Let $A$ be a $D$-generator and $K$ a compact interval in $J$. Then $\sup_{t \in K} ||A(t)|| < \infty$.

**Lemma 5:** Let $A$ be a bounded linear operator and $B$ an internal finitely bounded linear operator. If $g, h \in H, g \preceq h$ and $Ag \preceq Bg$ then $Ah \preceq Bh$.

**Proof:** $||Ah-Bh|| \leq
||A(h-g)|| + ||(A-B)g|| + ||B(g-h)||
\leq ||A|| ||h-g|| + ||(A-B)g|| + ||B|| ||g-h|| \preceq 0.
Q.E.D.

**Lemma 6:** Let $A$ be a $D$-generator. Then for all finite $t, s$ in $^*J$, $t \preceq s$ implies $A(t) \preceq A(s)$. In particular $A(t)$ is near standard for each finite $t$ in $^*J$.

**Proof:** If $t, s$ in $^*J$ are finite there is an $N$ in $\mathbb{N}$, the non-negative integers, such that $|t| \leq N$ and $|s| \leq N$. By lemma 4 $\sup_{t \in \mathbb{R}} ||A(t)|| = c$ is finite. By transfer, $||A(t)|| \leq c$ for all $t \in ^*\mathbb{R}$, $|t| \leq N$. If $s \preceq t$, then $s \preceq r \preceq t$, $r \in \mathbb{R}$ and $|r| \leq N$. By strong continuity, $A(s)h \preceq A(r)h \preceq A(t)h$. Since $A(s), A(r), A(t)$ are all finitely bounded we obtain $A(s) \preceq A(r) \preceq A(t)$.

Q.E.D.
Theorem 7: Let $A$ be a $D$-generator and $B$ an internal $D$-generator. Suppose additionally that $\|B(t)\|$ is finite for all finite $t$ in $^*J$.

Then $A(t) \preceq B(t)$ for all finite $t$ in $^*J$ implies $U_A(t,s) \preceq U_B(t,s)$ for all finite $t,s$ in $^*J$.

Proof: Fix $t,s$ finite in $^*J$. Let $K$ be a compact interval in $J$ so that $t,s$ are in $^*K$. It suffices to consider the case $t > s$.

$B$ is internal so $\{N \in ^*\mathbb{N} : \|B(r)\| \leq N \forall r \in ^*K\}$ is internal and contains $^*\mathbb{N}$ -- $N$. As $^*\mathbb{N}$ -- $N$ is external we conclude that there is a $c$ in $\mathbb{N}$ such that $\|B(r)\| \leq c$ for all $r \in ^*K$.

Write $T_n(o)$ for $T_n(o,t,s)$ and $U_o$ for $U_o(t,s)$.

Choose $\alpha$ in $^*\mathbb{N}$ -- $\mathbb{N}$. Observe that $\|U_B - (I + \sum_{n=1}^{\infty} (-i)^n T_n(B))\| \leq \sum_{n=\alpha+1}^{\infty} \frac{c^\alpha(t-s)^n}{n!} \sim 0$. Similarly,

$\|U_A - (I + \sum_{n=1}^{\infty} (-i)^n T_n(A))\| \sim 0$. Consequently, we obtain

$$\|U_B - U_A\| h \leq 0$$ (2)

$$\sum_{n=1}^{\alpha} \int t \int t_1 \int \cdots \int t_{n-1} |B(t_1) \cdots B(t_n) - A(t_1) \cdots A(t_n)| h |dt_n \cdots dt_1|$$

for all $h$ in $^*H$ and $\alpha$ in $^*\mathbb{N}$ -- $\mathbb{N}$.

$$B(t_1) \cdots B(t_n) - A(t_1) \cdots A(t_n)$$

$$= \sum_{i=1}^{n} B(t_1) \cdots B(t_{i-1}) [B(t_i) - A(t_i)] A(t_{i+1}) \cdots A(t_n).$$ (3)
If the $t_i$'s are all finite and if $h$ is near standard then by lemmas 5 and 6 $A(t_{i+1}) \cdots A(t_n) h$ is near standard. By lemma 5 $[B(t_i) - A(t_i)] A(t_{i+1}) \cdots A(t_n) h$ is some vector $h_{i,n}$ of infinitesimal norm, providing $n$ is finite. Let $\delta_n = \max\{|h_{i,n}|: i=1,2,\cdots,n\} < 0$, for $n$ finite. From (3) we conclude

$$||[B(t_1) \cdots B(t_n) - A(t_1) \cdots A(t_n)] h||$$

$$\leq \sum_{i=1}^{n} c_i^{i-1} ||h_{i,n}|| \leq \delta_n (1-c)^n/(1-c).$$

Then

$$||[B(t_1) \cdots B(t_n) - A(t_1) \cdots A(t_n)] h|| < 0$$

(4)

for all $n \in \mathbb{N}$ and $t_i$ finite.

Fix $h \in H$.

Let $I_0 = \{m \in \mathbb{N} : (n < m, |t_1| < m, \cdots, |t_n| < m) \lor (|m[B(t_1) \cdots B(t_n) - A(t_1) \cdots A(t_n)] h| \leq 1)\}$. $I_0$ is internal and contains, by (4) above, the external set $\mathbb{N}$. Thus $\gamma \in I_0$ for some $\gamma$ in $\mathbb{N}$. By (2) there now follows

$$||U_A - U_B|| \leq \sum_{i=1}^{\gamma} \frac{i(t-s)^n}{i!} \leq \frac{e(t-s)}{\gamma} < 0.$$  

Thus, $U_A h \cong U_B h$ so that $U_A(t,s) \cong U_B(t,s)$ for all finite $t,s$ in $\mathbb{J}$.

Q.E.D.

Remark: If $B_n$ is a sequence of standard generators with

$$\sup \sup_{n,t \in K} |B_n(t)| < \infty$$

for all compact $K$, then the Dyson expansions

for $B_n$ converge to $U_B(t,s)$ uniformly in $n$ for each $t,s$.
Consequently, if \( B_n(r) \) converges strongly to \( A(r) \) then \( T_m(B_n,t,s) \) converges to \( T_m(A,t,s) \) for each \( m \) as \( n \to \infty \) by dominated convergence and one may conclude \( U_{B_n}^m(t,s) \) converges to \( U_A^m(t,s) \) strongly.

The previous proof follows this argument.

III. Stability of Solutions for Unbounded Generators.

**Definition:** A K-generator is a map \( A \) of \( J \) into the selfadjoint operator on \( H \) for which there exists a selfadjoint operator \( H \geq 1 \) and a positive real number \( c \) such that

\[
c^{-1}(H+1) \leq A(t) \leq c(H+1)
\]

(5)

\[
G(t) = \frac{d(A(t)^{-1})}{dt}
\]

exists in norm

(6)

\[
||A(t)^{1/2}G(t)A(t)^{1/2}|| \leq c
\]

(7)

hold for all \( t \) in \( J \). \( H \) and \( c \) are the related constants.

In [11] Simon presents an existence theorem for equations with K-generators. We have included the relevant part of this result as Theorem 8 below. Simon points out that, in fact, this theorem is a special case of a result due to Kisynski, [8]. In this theorem there is no hypothesis concerning the independence of \( D(A(t)) \) from \( t \). Thus it is particularly useful in treating problems of singular perturbations.
Theorem 8: Let $A$ be a $K$-generator. There is a unitary propagator $U$ such that defining $a(t) = U(t,s)x_s$ in $D(H^{1/2})$ gives a weak solution to Schrodinger's equation (1) in the sense that

$$\frac{d}{dt} (f,a(t)) = -i(A^{1/2}(t)f,A^{1/2}(t)a(t))$$

for all $f$ in $D(H^{1/2})$.

Some Remarks on Theorem 8:

Although the theorem only gives a weak solution it follows by a related uniqueness result that if a strong solution exists, it is the one given in theorem 8.

Both Simon and Kisynski use a modification of techniques invented by Yoshida. One defines

$$A_n(t) = A(t)(1+n^{-1}A(t))^{-1}$$

for $n = 1, 2, \cdots$. Observing that $t + A_n(t)$ is a $D$-generator and that

$$||A_n(t)|| \leq n$$

for all $t$ in $J$, one obtains a strong solution $a_n(t)$ to the Schrodinger equation (1) with generator $A_n$, using theorem 3. One then proves

$$\lim_{n \to \infty} (f,a_n(t)) = (f,a(t))$$

for all $f$ in $H$ and $t$ in $J$. 
Defining \( \| g \|_{-1} = \| (H+1)^{-1/2} g \| \)
\( \| g \|_{+1} = \| (H+1)^{1/2} g \| \)
it can be shown that for any compact interval \( T \) containing \( s \),
there is a constant \( K_T \) such that

\[
\left\| \frac{d}{dt} a_n(t) \right\|_{-1} \leq K_T
\]

(11)

and

\[
\left\| a_n(t) \right\|_{+1} \leq K_T
\]

(12)

for all \( t \) in \( T \). Here if \( \tau \) is the largest \( t \) in \( T \), then

\[
K_T \leq (c+1)^2 e^{3c(\tau-s)/2} \| x_s \|_{+1} .
\]

(13)

The proofs of all these numbered facts may be found in [11].

**Theorem 9:** Let \( A \) be a \( K \)-generator with constants \( H,c \). Let \( B \) be an internal \( K \)-generator with constants \( H,c \).

If \( A(t) \prec B(t) \) for all finite \( t \) in \( J \) then \( U_A(t,s) \prec U_B(t,s) \)
for all finite \( t,s \) in \( J \).

**Corollary 10:** Let \( A_n, A \) be \( K \)-generators on \( J \) with constants \( (H,c) \)
for \( n = 1,2,\cdots \). Suppose \( A_n(t) \) converges in the strong resolvent
topology to \( A(t) \) for each \( t \) in \( J \). Then \( U_{A_n}(t,s) \) converges strongly
to \( U_A(t,s) \) for every \( t \) and \( s \) in \( J \).
To obtain corollary 10 from theorem 9 one may proceed as follows. Choose $B(t) = A_n(t)$ for some positive infinite integer $n$.

Equation (7) implies that $|\frac{dB^{-1}(t)}{dt}| \leq c^2$ for all $t$ in $J$. The norm mean value theorem then yields $|B^{-1}(t) - B^{-1}(s)| \approx 0$ whenever $t \approx s$ similarly for $A^{-1}$. Since we are assuming $A(s) \approx B(s)$ for each $s$ in $J$ we arrive at the conclusion $B^{-1}(t) \approx B^{-1}(s) \approx A^{-1}(s) \approx A^{-1}(t)$ whenever $s$ is in $J$ and $t \approx s$, using the fact that $J$ is closed. Since, however, corollary 10 is an interesting standard result, such a proof is presented. Both the proof of theorem 9 and the corollary are based on the following standard computation.

Technical Lemma: Let $T$ be any compact interval of $\mathbb{R}$ and let $K_T$ be as in equations (10), (11), (13). Then for all $A$ in $K(H,c)$, $t > s$ in $T$, and $x$ in $D(H^{1/2})$

$$||A^{-1/2}(t)(U_{A_n}(s,t) - U_{A_m}(s,t))x|| \leq 8K_T^2(e^{c(t-s)}-1)(\frac{1}{m} + \frac{1}{n})/c.$$  

Here $A_n$ is defined as in (8).

Proof: Fix $s$ in $T$ and for $t > s$ in $T$ define $a_n(t) = U_{A_n}(s,t)x$.

Fix positive integers $m$ and $n$. Define

$$p(t) = a_n(t) - a_m(t)$$
$$w(t) = (p(t), A^{-1}(t)p(t))$$

and

$$\frac{dA^{-1}}{dt}(t) = G(t).$$

In general we denote $\frac{df}{dt}$ by $\dot{f}$ so that $\dot{p} = -iA_n a_n + iA_m a_m$ and

$$\dot{w} = (\dot{p}, A^{-1}p) + (p, Gp) + (p, A^{-1}Gp).$$

Since $A^{-1} = A_n^{-1} - \frac{1}{n}$ we may write
\[ w = (-iA_n a_n + iA_m a_m A^{-1} p) + (p, Gp) + (p, A^{-1} (-iA_n a_n + iA_m a_m)) \]

\[ = i(A_n a_n, (A_n^{-1} - \frac{1}{n}) p) - i(A_m a_m, (A_m^{-1} - \frac{1}{m}) p) + (p, Gp) \]

\[ + i(p, (A_n^{-1} - \frac{1}{n}) A_n a_n) - i(p, (A_m^{-1} - \frac{1}{m}) A_m a_m) \]

\[ w = i(a_n, p) - \frac{i}{n} (A_n a_n, p) - i(a_m, p) + \frac{i}{m} (A_m a_m, p) + (p, Gp) \]

\[ - i(p, a_n) + \frac{i}{n} (p, A_n a_n) + i(p, a_m) - \frac{i}{m} (p, A_m a_m) \]

\[ = i(a_n - a_m, p) - i(p, a_n - a_m) + (p, Gp) + \frac{1}{n} (p, -iA_n a_n) \]

\[ - \frac{1}{n} (-iA_n a_n, p) + \frac{1}{m} (-iA_m a_m, p) - \frac{1}{n} (p, -iA_n a_n) \]

\[ = i(p, p) - i(p, p) + (p, Gp) - \frac{1}{n} (a_n, p) + \frac{1}{m} (a_m, p) - \frac{1}{n} (p, a_n) \]

\[ + \frac{1}{m} (p, a_m) \]

Thus

\[ \dot{w} = (p, Gp) + \frac{2}{m} \text{Re}(a_m, p) - \frac{2}{n} \text{Re}(a_n, p) \quad (14) \]

Next observe that since

\[ |(p, Gp)| = |(A^{-1/2}_p, A^{1/2} G A^{1/2}_p, A^{-1/2}_p)| \]

\[ \leq ||A^{1/2} G A^{1/2}|| ||A^{-1/2}_p||^2 \]

\[ = ||A^{1/2} G A^{1/2}|| (p, A^{-1}_p) \]

we obtain

\[ |(p, Gp)| \leq cw \quad (15) \]
Also, for \( r \) in \( \mathbb{N} \),

\[
|\langle \dot{a}_r, p \rangle| = |\langle (H+1)^{-1/2} \dot{a}_r, (H+1)^{1/2} p \rangle| 
\leq ||\dot{a}_r||_{-1} ||p||_{+1} \leq ||\dot{a}_r||_{-1} (||a_n||_{+1} + ||a_m||_{+1}).
\]

so that from (11), (12) and (13) there follows

\[
|\langle \dot{a}_r(t), p(t) \rangle| \leq 2K_T^2
\] (16)

Substituting (15) and (16) into (14) we find

\[
\frac{d\omega}{dt}(t) \leq c\omega(t) + K \left( \frac{1}{n} + \frac{1}{m} \right)
\] (17)

for all \( t > s \) in \( T \). Here \( K \) is the finite constant \( 8K_T^2 \).

To finish the proof we observe that \( w(s) = (p(s), A^{-1}(s)p(s)) = (a_n(s) - a_m(s), A^{-1}(s)p(s)) = (x - x, A^{-1}(s)p(s)) = 0 \). Since \( w \geq 0 \), (17) implies

\[
0 \leq w(t) \leq K c \left( \frac{1}{n} + \frac{1}{m} \right) e^{c(t-s)} - K c \left( \frac{1}{n} + \frac{1}{m} \right).
\]

Q.E.D.

Proof of Theorem 9: Fix a finite \( s \) in \( \ast J \) and \( x \) in \( D(H^{1/2}) \). Define \( a_n \) as in the proof of the technical lemma and \( a \) as in theorem 8. Similarly, define \( B_n(t) = B(t)(1 + n^{-1}B(t))^{-1} \) so that \( B \) is an internal D-generator with

\[
||B_n(t)|| \leq n.
\] (18)
Let \( b_n \) be the solution to the Schrödinger equation (1) with generator \( B_n \) and initial state \( x \). Since

\[
A_n(t)^{-1} = A(t)^{-1} + \frac{1}{n} \hat{\psi} B(t)^{-1} + \frac{1}{n} = B_n(t)^{-1}
\]

it follows from bounded stability, theorem 7, that

\[
a_n(t) \preceq b_n(t), \forall n \in \mathbb{N}, \forall \text{ finite } t \text{ in } \mathbb{J}.
\]  

(19)

Let \( I_0 = \{ m \in \mathbb{N} : n \leq m + 1 \} \) 

\[
(\forall t \in \mathbb{J}, |t| \leq n + ||n(a_n(t) - b_n(t))|| \leq 1)\}
\]

\( I_0 \) is internal and by (15) \( I_0 \) contains \( \mathbb{N} \). Since \( \mathbb{N} \) is external there is an \( \omega \) in \( I_0 \cap (\mathbb{N} - \mathbb{N}) \) such that

\[
a_n(t) \preceq b_n(t), \forall n \leq \omega, \forall \text{ finite } t \text{ in } \mathbb{J}.
\]  

(20)

Let \( p(t) = b_n(t) - b_m(t) \) and

\[
w(t) = (p(t), B^{-1}(t)p(t)),
\]

for \( n, m \) infinite. Since every finite \( t \) in \( \mathbb{J} \) is in \( \mathbb{T} \) for some compact interval \( \mathbb{T} \) with \( s \) in \( \mathbb{T} \) it follows transferring the technical lemma that

\[
w(t) \preceq 0 \text{ for all finite } t \text{ in } \mathbb{J}.
\]  

(21)
Now suppose \( g \) is in \( D(H^{1/2}) \). Then

\[
\|(p(t),g)\| = \|(B^{-1/2}(t)p(t),B^{1/2}(t)g)\|
\]

\[
\leq \|B^{-1/2}(t)p(t)\| \|B^{1/2}(t)g\|
\]

\[
\leq w(t)^{1/2} c((H+1)g,g)^{1/2}.
\]

Therefore from (21) there follows

\[
(p(t),g) \overset{\omega}{\rightarrow} 0
\]  
(22)

for all finite \( t \) in \( {}^*J \) and \( g \) in \( D(H^{1/2}) \).

Let \( h \) in \( H \) be given. Let \( \varepsilon \) be an arbitrary positive real number. \( D(H^{1/2}) \) is dense in \( H \), so there is a \( g \) in \( D(H^{1/2}) \) such that \( \|g-h\| < \varepsilon \). Then

\[
\|(p(t),h)\| \leq \|(p(t),g)\| + \|(p(t),h-g)\|
\]

\[
\overset{\omega}{\leq} \|(p(t),(h-g))\| \ \text{by (22)}
\]

\[
\leq \|p(t)\|_+ \|h-g\|_+ \leq \|p(t)\|_+ \|h-g\|.
\]

Since \( \|p(t)\|_+ \) is finite for all finite \( t \) in \( {}^*J \) we conclude that \( \|(p(t),h)\| \leq 2K_T\varepsilon \) where \( T \) is any compact interval chosen so that \( {}^*T \) contains \( t \) and \( s \). Since \( \varepsilon \) is an arbitrary positive real number \( (p(t),h) \overset{\omega}{\rightarrow} 0 \). That is

\[
b_n(t) \overset{\omega}{\rightarrow} b_m(t)
\]

(23)

for all infinite \( n \) and \( m \) and finite \( t \) in \( {}^*J \). By transfer of theorem 8 and its proof we know that for each \( t \) in \( {}^*J \)

\[
\text{weak limit } \lim_{n \to \infty} b_n(t) = b(t)
\]

for \( n \) in \( {}^*N \).
In particular if $\varepsilon \gtrsim 0$, $\varepsilon > 0$ in $^*\mathbb{R}$, $t$ is finite in $^*J$, $h$ is in $H$ there is an $n(t,h,\varepsilon)$ in $^*\mathbb{N}$ such that for $n \geq n(t,h,\varepsilon)$ we have

$$|(b(t),h)-(b_n(t),h)| < \varepsilon.$$ 

If $m$ in $^*\mathbb{N}$ is infinite we argue, using (23) that

$$|(b_m(t)-b(t),h)|$$

$$\leq |(b_m(t)-b_n(t),h)| + |(b_n(t)-b(t),h)|$$

$$\gtrsim 0.$$

The conclusion is

$$b_m(t) \sim b(t)$$

for all infinite $m$ and finite $t$ in $^*J$. An analogous argument shows

$$a_m(t) \sim a(t)$$

for all finite $m$ and finite $t$ in $^*J$.

Combining (16), (24) and (25) we may write

$$a(t) \sim a_n(t) \sim b_n(t) \sim b(t)$$

for all finite $t$ in $^*J$ and infinite, though sufficiently small $n$. In other words
for all finite \( t, s \) in \( \mathcal{J} \) and \( x_s \) in \( \mathcal{D}(H^{1/2}) \).

If \( y \) is arbitrary in \( \mathcal{H} \), choose \( \epsilon > 0 \) arbitrary in \( \mathbb{R} \) and \( x \) in \( \mathcal{D}(H^{1/2}) \) such that \( ||x - y|| < \epsilon \). Then for \( g \) in \( \mathcal{H} \), using (26)

\[
|\langle U_A y - U_B y, g \rangle| \\
\leq |\langle U_A (y-x), g \rangle| + |\langle U_A - U_B \rangle x, g \rangle| \\
+ |\langle U_B (x-y), g \rangle| \\
\leq 2||y-x|| ||g|| \leq 2\epsilon ||g||
\]

since \( U_A \) and \( U_B \) are unitary. As \( \epsilon \) is arbitrary we conclude

\[
U_A(t,s)y \sim U_B(t,s)y
\]

(27)

for all finite \( t, s \) in \( \mathcal{J} \) and \( y \) in \( \mathcal{H} \). Again, since the \( U_A \)'s and \( U_B \)'s are unitary we may argue that

\[
||U_A y - U_B y||^2 = \langle U_A y - U_B y, U_A y - U_B y \rangle \\
= ||U_A y||^2 + ||U_B y||^2 - \langle U_B y, U_A y \rangle - \langle U_A y, U_B y \rangle \\
= 2||y||^2 - \langle U_B y, U_A y \rangle - \langle U_A y, U_B y \rangle \\
\sim 2||y||^2 - \langle U_A y, U_A y \rangle - \langle U_A y, U_A y \rangle \\
= 2||y||^2 - 2||y||^2 = 0
\]

using (27).

Thus

\[
U_A(t,s)y \sim U_B(t,s)y
\]
for all finite t, s in J and y in H. Since $||U_A|| = 1 = ||U_B||$
we conclude $U_A(t, s) \sim U_B(t, s)$ for all finite t, s in J.

Q.E.D.

Proof of Corollary 10: Fix $\varepsilon > 0$, t > s and x in $D(H^{1/2})$. Let T
be a compact interval containing t and s. Choose $K_T$ as in (11),
(12) and (13). Choose N to be an integer larger than
$8k_T^2(e^c(t-s)-1)/2\varepsilon c$. Let $m > N$. For any n, let $B_m(n)$ be the $m^{th}$
Yoshida approximation to $A_n$ and C the $m^{th}$ Yoshida approximation
to A. Then by the Technical Lemma and the triangle inequality
$||A^{1/2}(t)(U_{A_n}(s,t)-U_A(s,t))x|| \leq 2\varepsilon + ||A^{-1/2}(t)(U_{B_m(n)}-U_C)x||$.

Since for any fixed m, $||B_m(n)|| \leq m$, $||C|| \leq m$ and $B_m(n)$ con-
verges strongly to C as $n \rightarrow \infty$ it follows from theorem 7 and the
remark which follows it that for n sufficiently large
$||A^{-1/2}(t)(U_{A_n}(s,t)-U_A(s,t))x|| < 3\varepsilon$. Since $D(A^{1/2}(\varepsilon)) = D(H^{1/2})$
is dense it follows that $U_{A_n}(s,t)x \rightarrow U_A(s,t)x$ weakly as $n \rightarrow \infty$ for
all $x \in D(H^{1/2})$. Since, in general, weak convergence on a dense
set implies strong convergence of unitary operators the proof is
completed.

Q.E.D.

Remarks. 1. In the last section of this paper we apply the above
results to the analysis of form bounded perturbations. Proof simi-
lar to those just given lead to results useful in studying opera-
tor bounded perturbations. Here we just state these results.
Definition: A Y-generator on $J$ is a function $A(\cdot)$ from $J$ into the selfadjoint operators on $H$ such that there are positive numbers $E$ and $M$ satisfying

(i) $A(t) \geq E + 1$;

(ii) $t + (A(t)+E)^{-1}$ is strongly differentiable with derivative $G(t)$;

(iii) $G$ is strongly continuous; and

(iv) $\| (A(t)+E)B(t) \| \leq M$.

Then according to a theorem of Yoshida [15, 429] there is a unitary propagator $U_A$ such that for all $x$ in $D(H(s))$, defining $a(t) = U(t,s)x$ gives a solution to the Schrödinger equation in the sense that

$$\frac{da(t)}{dt} = -iA(t)a(t).$$

The techniques given above lead to the following theorem and corollary.

Theorem: Let $A$ be a Y-generator with constants $E$ and $M$. Let $B$ be an internal Y-generator with constants $E$ and $M$. If $(A(t)+E)^{-1} \preceq (B(t)+E)^{-1}$ for all finite $t$ in $J$ then $U_A(t,s) \preceq U_B(t,s)$ for all finite $t,s$ in $J$.

Corollary: Let $A_{n}$ be Y-generators with constants $E,M$ for $n = 1,2,\ldots$. Let $A_n(t)$ converge to $A(t)$ in the strong resolvent topology for each $t$ in $J$. Then $U_{A_n}(t,s)$ converges strongly $U_A(t,s)$ for each $t,s$ in $J$.

2. In corollary 10 and the last corollary, the convergence of propagators is uniform in any finite interval of $J$. 
IV. Perturbations of a Time Independent Generator.

In this section $H_0$ will be a non-negative selfadjoint operator on $H$. If $F$ is a Hermitian form $Q(F)$ will denote the form domain of $F$. In particular $Q(H_0) = \mathcal{D}(H_0^{1/2})$.

**Definition:** A $K$-perturbation of $H_0$ is a Hermitian form valued function, $V$, defined on $J$ with the properties

(i) $Q(H_0) \subset Q(V(t))$.

(ii) There are constants $0 < a < 1$, $0 < b < \infty$ such that

$$|V(t)(f,f)| \leq a(H_0^{1/2} f, H_0^{1/2} f) + b||f||^2$$

for all $f$ in $Q(H_0)$.

(iii) There is a Hermitian form, $\dot{V}(t)$, which is the derivative of $V$ in the sense that $(H_0 + 1)^{-1/2} V(t)(H_0 + 1)^{-1/2}$ is norm differentiable and its derivative is $(H_0 + 1)^{-1/2} \dot{V}(t)(H_0 + 1)^{-1/2}$

and

(iv) $|\dot{V}(t)(f,f)| \leq a(H_0^{1/2} f, H_0^{1/2} f) + b||f||^2$

for all $f$ in $Q(H_0)$.

$(a, b)$ are the $H_0$-bounds of $V$.

The following theorem is due to Simon. He states it for the case where each $V(t)$ is a selfadjoint operator but the proof he gives works just as well for each $V(t)$ being a Hermitian form.

**Theorem 11** [11, pg. 66]:

Let $V$ be a $K$-perturbation of $H_0$. There is a constant $E$ such that $A(t) = H_0 + V(t) + E$ is a $K$-generator with constants $(H_0, c)$ for some real number $c$. 
Theorem 12: Let $V, \tilde{W}_n$ be $K$-perturbations of $H_0$, for $n = 1, 2, \ldots$.

Suppose the $H_0$-bounds of $\tilde{W}_n, (a_n, b_n)$ satisfy $a_n + b_n \to 0$ as $n \to \infty$. Then there is a constant $E$ such that for $n$ sufficiently large $A_n(t) = H_0 + V(t) + \tilde{W}_n(t) + E$ is a $K$ generator and $U_{A_n}(t, s)$ converges strongly to $U_{H_0 + V + E}(t, s)$ for all $t, s$ in $J$.

Proof: First note that

$Q(H_0) \subset Q(V(t)) \cap Q(\tilde{W}_n(t)) = Q(V(t) + \tilde{W}_n(t))$. Next observe that

$| (V(t) + \tilde{W}_n(t))(f, f) | \leq | V(t)(f, f) | + | \tilde{W}_n(t)(f, f) |
\leq a(H_0^{1/2}f, H_0^{1/2}f) + b(f, f) + a_n(H_0^{1/2}f, H_0^{1/2}f) + b_n ||f||^2
\leq \frac{(a+1)}{2} (H_0^{1/2}f, H_0^{1/2}f) + 2b ||f||^2$

for all $f$ in $Q(H_0)$ and $n$ sufficiently large. Also, letting

$K = (H_0 + 1)^{-1/2}$, it is true that $K(V(t) + \tilde{W}_n(t))K = KV(t)K + \tilde{K}W_n(t)K$ and so $\frac{d}{dt} K(V(t)) + \tilde{W}_n(t)K$ exists in norm and equals $KV(t)K + \tilde{K}W_n(t)K$. As above,

$| (\dot{V}(t) + \dot{W}_n(t))(f, f) | \leq \frac{a+1}{2} (H_0^{1/2}f, H_0^{1/2}f) + 2b ||f||^2$.

By theorem 11, there is a constant $E$ such that for $n$ sufficiently large $A_n(t)$ is a $K$-generator.

$|(K\tilde{W}_n(t)Kf, f)| \leq (a_n + b_n) ||f||^2$

so $||K\tilde{W}_n(t)K|| \to 0$ as $n \to \infty$, uniformly in $t$. Consequently, by the Neumann series for the resolvent, one obtains that

$(1 + K(V(t) + E)K + K\tilde{W}_n(t)K)^{-1}$
converges in norm to \((1+K(V(t)+E)K)^{-1}\) as \(n \to \infty\). We conclude that \((A_n(t)+1)^{-1} = K(I+K(V(t)+EK) + KW_nK)^{-1}\) converges in norm, and hence strongly, to \((H_0+V(t)+E+1)^{-1}\). We are done by corollary 10.

Q.E.D.

As a special case of the previous theorem we note

Corollary 13: Let \(V,W\) be \(K\)-perturbations of \(H_0\). There is a constant \(E\) so that \(A_\epsilon(t) = H_0 + V(t) + \epsilon W(t) + E\) is a \(K\)-generator for all \(\epsilon\) sufficiently small and \(U_{A_\epsilon}(t,s)\) converges strongly to \(U_{A}(t,s)\) for all \(t,s\) in \(J\) where \(A = H_0 + V(t) + E\) as \(\epsilon \to 0\).

Example 14: Let \(V\) be a small Hermitian form perturbation of \(H_0\), i.e., \(Q(H_0) \subset Q(V)\) and \(|V(f,f)| \leq a(H_0^{1/2}f, H_0^{1/2}f) + b||f||^2\) for all \(f \in Q(H_0)\) where \(0 \leq a < 1\) and \(0 \leq b < \infty\). Let \(f\) be a bounded \(C^1\) function with bounded derivative. Let \(A_\epsilon(t) = H_0 + \epsilon f(t)V + E\). Then \(U_{A_\epsilon}(t,s)\) converges strongly to \(U_{H_0+E}(t,s)\) as \(\epsilon \to 0\).

Example 15: Let \(H_0\) here denote \(-\Delta\) on \(L^2(\mathbb{R}^n)\). Let \(\tau\) be a real distribution on \(\mathbb{R}^n\). \(\tau\) is called an \(H_0\)-small distribution if there are numbers \(0 \leq a < 1, 0 \leq b < \infty\) such that

\[ \pm \tau(f^2) \leq a(H_0^{1/2}f, H_0^{1/2}f) + b||f||^2 \quad (36) \]
for all \( f \) in \( C_c^\infty(\mathbb{R}^n) \). \((a, b)\) are the \( H_0\)-bounds of \( \tau \). Given such a distribution one can define a Hermitian form \( \tau(f, g) = \tau(\bar{f}g) \) which can be extended to a Hermitian form \( \tau \) on \( Q(H_0) \) satisfying

\[
|\tau(f, f)| \leq a(H_0^{1/2}f, H_0^{1/2}f) + b(f, f)
\]  

(37)

for all \( f \) in \( Q(H_0) \).

Let \( \rho \) be a \( C_c^\infty(\mathbb{R}^n) \) function which satisfies \( \rho(x) = 1 \) for \(||x|| \leq 1 \) and \( 0 \leq \rho \leq 1 \). Set \( \beta_k(x) = \rho(x/k) \) and \( \gamma_n(x) = \rho(kx)(\int \rho(kx)dx)^{-1} \). Define \( V_k(f, g) = \tau(\gamma_n(\beta_k \bar{f}g)) \), so that each \( V_k \) is a \( C_c^\infty(\mathbb{R}^n) \) function. The \( V_k \)'s are the regularizations of \( \tau \) and converge to \( \tau \) in the sense of distributions. Moreover \( H_0 + V_k \) converges in the strong resolvent topology to the form sum \( H_0 + \tau \). See [3] for details.

Suppose \( f \) is \( C^1 \) on \( J \). Define

\[
|||f||| = \max\{||f||, |\frac{df}{dt}|\}.
\]

Suppose \( |||f||| < \frac{1}{a} \).

Let \( A(t) = H_0 + f(t)\tau \) and \( A_k(t) = H_0 + f(t)V_k \). Then \( f(t)V \) and \( f(t)V_k \) are \( K \)-perturbations of \( H_0 \) and by the preceding remarks \( A_k(t) \) converges in the strong resolvent topology to \( A(t) \) for each \( t \) on \( J \). By theorems 11, 9 and corollary 10 \( U_{A_k}(t, s) \) converges strongly to \( U_A(t, s) \) for each \( t, s \) in \( J \) while

\( U_{A_k}(t, s) \ngtr U_A(t, s) \) for each infinite \( k \) in \( \mathbb{N} \) and every finite \( t, s \) in \( J \).
For specific examples of $\tau$'s which may be chosen we refer the reader to [3]. Here we note that $\tau$ may be in $L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ where $p > 1$ if $n = 1$, $p > 1$ if $n = 2$ and $p > n/2$ if $n > 3$. Also $\tau$ may be a delta function concentrated on a $C^1$ compact hypersurface or the distributional directional derivative of a bounded function.

If $\tau_1$ and $\tau_2$ are two $H_0$-small distributions then for suitable $f$'s $V(t) = f(t) \tau_1 + (1-f(t))\tau_2$ is a K-perturbation of $H_0$.

Choosing $f$ so that $f = 0$ for $t \leq 0$ and $f = 1$ for $t > 1$ one can treat the problem of switching from one potential to another. By choosing $f(t) = \sin wt$ one obtains periodically varying potentials.

One obtains considerable technical advantage in continuing the study of time dependent singular perturbations of $-\Delta$ by introducing infinite regularizations and by accepting infinitesimal errors. This naturally occurs because $U_{H_0 + V_k + E}$ has the generator $H_0 + V_k + E$ defined as an operator sum with $V_k$ in $C_0^\infty$ while $U_{H_0 + V + E}$ has the generator $H_0 + V + E$ defined as a form sum.

For example, since $V_k \in L^p(\mathbb{R}^n)$ for all $p \geq 1$, the product formula of Faris [2] is valid and one can express the solution $u$ of the Schrodinger equation (1) as

$$u(t) \approx \sum_{j=0}^{m-1} \left( \frac{it}{m} \Delta - \frac{it}{m} V_k \frac{jt}{m} \right) u_s$$

(38)
with \( u(0) = u_s \) for \( u_s \) in \( Q(H_0) \), for \( m \) sufficiently large in \( \mathbb{N} \).

Since \( L^2(\mathbb{R}^n) \) is separable one \( m \) can be chosen so that (38) holds for all \( u_s \) in \( Q(H_0) \) and all finite \( t \) (see [13] for details of this argument). Following Faris, (38) leads to a Feynman path integral:

For each pair of points \( (x_0, x_m) \) in \( \mathbb{R}^{2n} \) we may view \( \mathbb{R}^{n \cdot (m-2)} \) as a space of "polygonal paths" connecting \( x_0 \) and \( x_m \).

These paths are not standard, but if \( x_0 \) and \( x_m \) are finite and if \( \omega \) is any standard path connecting \( x_0 \) and \( x_m \); i.e.

\[
\omega : [0,t] \to \mathbb{R}^n \quad \text{and} \quad \omega(0) = st(x_0), \omega(t) = st(x_m),
\]

then there are polygonal paths \( \omega_1 \) so that \( \omega(s) \overset{\sim}{\approx} \omega_1(s) \) for all \( s \) in \( [0,t] \).

There are many more polygonal paths than standard paths. Since for each \( t \), \( V_k(t) \) is a \( C^\infty_c \) function we let \( (V_k(t))(y) = V_k(y,t) \).

For each polygonal path \( x = (x_1, \ldots, x_{m-1}) \) in \( \mathbb{R}^{n \cdot (m-2)} \) let

\[
S(x_0, x, x_m, m, t, V) = \sum_{j=1}^{m} \left( \frac{(x_j - x_{j-1})^2}{2(t/m)^2} - V_k(x_j, jt/m) \right) 
\]

Let \( u(\cdot, t) \in L^2(\mathbb{R}^n) \) be the state at time \( t \) of the quantum mechanical system with time dependent Hamiltonian \( H_0 + V(t) \) with state at time 0 given by \( u(\cdot, 0) \in Q(H_0) \). Thus

\[
(U_{H_0 + V(t)})(u(\cdot, 0)) = u(\cdot, t).
\]

Define the amplitude of the corresponding quantum mechanical particle to go from \( x_m \) at time 0 to \( x_0 \) at time \( t \) by
\[ K((x_0, t), (x_m, 0)) = \]
\[ \int_{\mathbb{R}^{m-2} \cdot n} e^{iS(x_0, x_m, t, V_k)} u(x_m, 0) \, dx . \]

Since \( e^{iA} \) is convolution by \((4\pi is)^{-3n/2} e^{iy^2/4s}\) in \(\mathbb{R}^n\), (38) implies

\[ u(\cdot, t) \cong \int_{\mathbb{R}^n} K((\cdot, t), (x_m, 0)) u(x_n, 0) \, dx_m \]

in \(L^2(\mathbb{R}^n)\).

(39) and (40) are a formulation of the Feynman path integral for particles moving in singular and time dependent potentials.

Since adding a delta function to \(-\Delta = -\frac{d^2}{dy^2}\) in \(L^2(\mathbb{R}^1)\) is equivalent to imposing a boundary condition on the maximally defined \(-\frac{d^2}{dy^2}\), it follows that the problem of smoothly changing boundary conditions in time can be given a Feynman path integral interpretation.

We conclude the discussion of this example by noting that, in general, the solution to equation (1) is shown to exist by proving certain approximate solutions converge weakly. However, by accepting infinitesimal errors one can explicitly construct the propagator. For this purpose we enter the interaction representation defining \( \tilde{V}(t) = e^{itH_0} V_k(t) e^{-itH_0} \), where \(k\) remains an infinite positive integer. Then \( t \mapsto \tilde{V}(t) \) is an internal D-generator so that
\[ \tilde{U}(t,s) = 1 + \sum_{p=1}^{\infty} (-i)^p \int_{s}^{t} \cdots \int_{s}^{t} \tilde{V}(t_1) \cdots \tilde{V}(t_p) dt_p \cdots dt_1 \]

converges in norm to a unitary propagator and

\[ U_a(t,s) = e^{\frac{-iH_0}{\hbar}} \tilde{U}(t,s) e^{\frac{iH_0}{\hbar}} \]

is almost the propagator for equation (1) in the sense that

\[ U(t,s)x_s \approx U_a(t,s)x_s \]

for all \( x_s \) in \( \mathcal{Q}(H_0) \) and \( t,s \) finite. For the standard details, see [10]. Thus we have defined time ordered exponentials of certain quadratic form valued functions.
References


