GEORGIA INSTITUTE OF TECHNOLOGY

PROJECT ADMINISTRATION DATA SHEET

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Administrative Data

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Defense Priority Rating: n/a
Military Security Classification: n/a

Restrictions

See Attached NSF Supplemental Information Sheet for Additional Requirements.

Travel: Foreign travel must have prior approval — Contact OCA in each case. Domestic travel requires sponsor approval where total will exceed greater of $500 or 125% of approved proposal budget category.

Equipment: Title vests with GIT.

Comments:
*Includes usual 6 month unfunded flexibility period.

This is a continuing grant.

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GTRI/Ohitlkr School/XXX Math

Date 7/12/84

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Project No. G-37-608

School/MC: Math

Includes Subproject No.(s): N/A

Project Director(s): T.P. Hill

Sponsor: National Science Foundation

Title: "Mathematical Sciences: Optimization and Conditioning Principles for Discrete Parameter Stochastic Processes"

Effective Completion Date: 12/31/86 (Performance) 3/31/87 (Reports)

Grant/Contract Closeout Actions Remaining:

- [x] None
- [ ] Final Invoice or Final Fiscal Report
- [ ] Closing Documents
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- [ ] Classified Material Certificate
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Angela DuBose
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FORM OCA 69.285
MATHEMATICAL SCIENCES: OPTIMIZATION AND CONDITIONING PRINCIPLES FOR DISCRETE PARAMETER STOCHASTIC PROCESSES

Theodore P. Hill
Robert P. Kertz

Award Period
July 1, 1984 to December 31, 1986

Under
NSF Award Number DMS-8401604

March 1987

GEORGIA INSTITUTE OF TECHNOLOGY
A UNIT OF THE UNIVERSITY SYSTEM OF GEORGIA
SCHOOL OF MATHEMATICS
ATLANTA, GEORGIA 30332
Research was carried out in the mathematical theory of probability, focusing chiefly on problems in sequential decision theory and on optimal-partitioning (or "fair-division") problems. The primary objectives were to determine: (i) for various classes of sequences of random variables and information structures, how much better a person with specified information concerning the process can do than one without that information; and (ii) quantitative bounds for the optimal-partitioning of a set or probability space based on various parameters of the underlying distributions.

The project involved two faculty researchers over a two-year period, with input from various collaborators and two beginning graduate students; nearly all of the work was purely theoretical and did not involve computers.

The primary findings included universal bounds for comparisons of various classes of discrete-parameter stochastic processes including averages of independent random variables, powers of martingales, exchangeable random variables and order statistics, and universal bounds for the optimal-partitioning of various classes of value measures such as general nonatomic measures and probability measures with atoms. Applications of those results were given to applied areas such as operations research, control theory and statistics, as well as to theoretical areas such as classical probability theory, function theory and measure theory.
b. Publication Citations

List of papers by T. P. Hill which were partially supported by NSF Grant DMS 84-01604.


List of papers by Robert P. Kertz which were partially supported by NSF Grant DMS 84-01604.


c. **Scientific Collaborators**

Principal Investigators for NSF Grant DMS 84-01604 were Theodore P. Hill and Robert P. Kertz, both associate professors in the School of Mathematics, Georgia Institute of Technology.
e. Technical Summary of Activities and Results

Results of research by T. P. Hill and R. P. Kertz which was supported in part by NSF Grant DMS 84-01604 are summarized below. Abstracts of pertinent papers are given, followed by conferences, workshops, meetings, and universities at which talks were given and names of three professors who were brought in for consultation.
Abstracts of papers by T. P. Hill which were partially supported by NSF Grant DMS 84-01604.


Abstract. If $\mu_1, \mu_2, \ldots, \mu_n$ are non-atomic finite measures on a measure space $(S, \mathcal{F})$, then there is a measurable partition $A_1, \ldots, A_n$ of $S$ satisfying

$$\mu_i(A_i) \geq \frac{1}{n} h\left( \|\mu_1\|, \ldots, \|\mu_n\| \right)$$

for all $i$, where $h(\cdot)$ is the harmonic mean, and $\|\mu\|$ is the total mass of $\mu$. Moreover, this bound is sharp and attained and strict inequality is possible if not all the measures are proportional.

A probabilistic corollary is the following: If $X_1, \ldots, X_n$ are continuous non-negative $L_1$ random variables on $(\Omega, \mathcal{F}, P)$, then there is a measurable partition $A_1, \ldots, A_n$ of $\Omega$ such that

$$f_{A_i} X_i dP \leq \left( \frac{1}{(EX_1)^{-1} + \ldots + (EX_n)^{-1}} \right)^{-1}$$

for all $i$ and this bound is best possible.


Abstract. Suppose $\mu_1, \ldots, \mu_n$ are nonatomic probability measures on the same measurable space $(S, \mathcal{B})$. Then there exists a measurable partition $\{S_i\}_{i=1}^n$ of $S$ such that $\mu_i(S_i) \geq (n+1-M)^{-1}$ for all $i = 1, \ldots, n$, where $M$ is the total mass of $v = \mu_1 + \ldots + \mu_n$ (the smallest measure majorizing each $\mu_i$). This inequality is the best possible for the functional $M$, and sharpens and quantifies a well-known cake cutting theorem of Urbanik and of Dubins and Spanier. Applications are made to $L_1$-functions, discrete allocation problems, statistical decision theory, and a dual problem.


Abstract. If $X_1, \ldots, X_n$ are independent non-negative random variables, then $E\{\max_{1 \leq j \leq n} \left( \frac{X_1 + \ldots + X_j}{j} \right) \} \leq \left( 2 - n^{-1} \right) \sup \{ E[(X_1 + \ldots + X_t) / t] : t$ is a stop rule for $X_1, \ldots, X_n \}$, and this bound is best possible for all $n \geq 1$. This result sharpens results of Krengel and Sucheston, in which the weaker bound $2(1 + \sqrt{3})$ was obtained, and yields the $k = 2$ prophet inequality for independent non-negative random variables.


Abstract. The distance from the convex hull of the range of an $n$-dimensional vector-valued measure to the range of that measure is no more than $\alpha n / 2$, where $\alpha$ is the largest (one-dimensional) mass of the atoms of the measure. The case $\alpha = 0$ yields Lyapounov's Convexity Theorem; applications are given to the bisection problem and to the bang-bang principle of optimal control theory.

Abstract. In every finite state leavable gambling problem and in every finite-state Markov decision process with discounted, negative or positive reward criteria there exists a Markov strategy which is monotonically improving and optimal in the limit along every history. An example is given to show that for the positive and gambling cases such strategies cannot be constructed by simply switching to a "better" action or gamble at each successive return to a state.


Abstract. Suppose $\mu_1, \ldots, \mu_n$ are probability measures on the same measurable space $(\Omega, F)$. Then if all atoms of each $\mu_i$ have mass $a$ or less, there is a measurable partition $A_1, \ldots, A_n$ of $\Omega$ so that $\mu_i(A_i) \geq V_n(a)$ for all $i = 1, \ldots, n$, where $V_n(\cdot) : [0,1] \to [0, n^{-1}]$ is the unique non-increasing function satisfying $V_n(a) = 1-k(n-1)a$ for $a \in [(k+1)n-1, (kn-1) -1], (kn-1)^{-1}$ for all $k \geq 1$. Moreover, the bound $V_n(a)$ is attained for all $n$ and all $a$.

Applications are given in $L_1$ spaces, to statistical decision theory, and to the classical non-atomic case.


Abstract. For countable-state decision processes (dynamic programming problems), a general class of objective functions is identified for which it is shown that good Markov strategies always exist. This class includes product and lim inf rewards, as well as practically all the classical dynamic programming expected payoff functions.


Abstract. Every collection of $m \leq n$ random $n$-dimensional vectors has a common hyperplane median, that is, there is a single hyperplane which simultaneously bisects each of the distributions in the sense that at least half the mass of each distribution lies on one side of the hyperplane (including the hyperplane) and at least half lies on the other side (again including the hyperplane). This result generalizes the well-known Ham Sandwich Theorem; the proof is based on first principles using an application of the Borsuk-Ulam Theorem to a "midpoint median" function.
9. "A sharp partitioning-inequality for non-atomic probability measures based on the mass of the infimum of the measures," accepted for publication in Prob. Theory (Z. Wahrscheinlichkeitsth.).

Abstract. If \( u_1, \ldots, u_n \) are non-atomic probability measures on the same measurable space \((S,F)\), then there is an \( F \)-measurable partition \( \{A_i\}_{i=1}^n \) of \( S \) so that \( u_i(A_i) \geq (n-1+m)^{-1} \), where \( m = \| \bigwedge_{i=1}^n u_i \| \) is the total mass of the largest measure dominated by each of the \( u_i \)'s; moreover, this bound is attained for all \( n \geq 1 \) and all \( m \) in \([0,1]\). This result is an analog of the bound \((n+1-M)^{-1}\) of Elton, Hill, and Kertz based on the mass \( M \) of the supremum of the measures; each gives a quantitative generalization of a well-known cake cutting inequality of Urbanik and of Dubins and Spanier.


Abstract. Applications of the original prophet inequalities of Krengel and Sucheston are made to problems of order selection, non-measurable stop rules, look-ahead stop rules, and iterated maps of random variables. Also, proofs are given of two results of Hill and Hordijk concerning optimal orderings of uniform and exponential distributions.


Abstract. Deterministic and randomized solutions are developed for the problem of equitably distributing \( m \) indivisible objects to \( n \) people (whose values may differ), without the use of outside judges or side-payments. Several general bounds for the minimal share are found; a practical method is given for determining an optimal lottery and the largest minimal share; and the case of repeated allocations is analyzed.


Abstract. A widely-accepted empirical observation called the First Digit Phenomenon or Benford's Law says that in collections of random tables of data (such as physical constants, almanacs, newspaper articles, etc.), the first significant digit is more likely to be a low number than a high number. In this study, an analysis of the frequencies of the first and second digits of "random" six-digit numbers guessed by people suggests that people's responses share some of the properties of Benford's Law: first digit 1 occurs much more frequently than expected; first digit 8 or 9 occurs much less frequently; and the second digits are much more uniformly distributed than the first.

Abstract. Generalizations of prophet inequalities for single sequences are obtained for optimal stopping of several parallel sequences of independent random variables. For example, if \( \{X_{i,j}\}_{j=1}^{n} \) are independent non-negative random variables, then

\[
E\left( \sup_{i,j} X_{i,j} \right) \leq (n+1) \max_{i} \sup_{t} \left( E\left( X_{i,t} \right) : t \text{ is a stop rule} \right)
\]

for \( X_{1,1}, X_{1,2}, \ldots \), and this bound is best possible. Applications are made to comparisons of the optimal expected returns of various alternative methods of stopping of parallel processes.

Abstracts of papers by Robert P. Kertz which were partially supported by NSF Grant DMS 84-01604.


Abstract. A procedure is given for proving strictness of some sharp, infinite-sequence martingale inequalities, which arise from sharp, finite-sequence martingale inequalities attained by degenerating extremal distributions. The procedure is applied to obtain strictness of the sharp inequalities of Cox and Kemperman \( P(|X_i| \geq 1 \text{ for some } i = 1, 2, \ldots) \leq \)

\[ (\ln 2)^{-1} \sup \mathbb{E} \left| \sum_{i=0}^{n} X_i \right| \text{ and of Cox (sharp form of Burkholder's inequality)} \]

\[ P(\sum_{i=0}^{n} X_i^2 \geq 1) \leq e^2 \sup \mathbb{E} \left| \sum_{i=0}^{n} X_i \right| \text{ for all nontrivial martingale difference sequences } X_0, X_1, \ldots. \]


Abstract. Exact comparisons are made relating \( \mathbb{E}|Y_0|^p, \mathbb{E}|Y_{n-1}|^p, \) and \( \mathbb{E}(\max_{j \leq n-1} |Y_j|^p), \) valid for all martingales \( Y_0^*, \ldots, Y_{n-1}^* \) for each \( p \geq 1. \)

Specifically, for \( p > 1, \) the set of ordered triples \( \{(x,y,z): x = \mathbb{E}|Y_0|^p, y = \mathbb{E}|Y_{n-1}|^p, \text{ and } z = \mathbb{E}(\max_{j \leq n-1} |Y_j|^p) \} \) for some martingale \( Y_0^*, \ldots, Y_{n-1}^* \) is precisely the set \( \{(x,y,z): 0 \leq x \leq y \leq z \leq \psi_{n,p}(x,y) \}, \) where \( \psi_{n,p}(x,y) = x \psi_{n,p}(y/x) \text{ if } x > 0, \) and \( = a_{n-1,p} y \text{ if } x = 0; \) here \( \psi_{n,p} \) is a specific recursively defined function. The result yields families of sharp inequalities, such as \( \mathbb{E}(\max_{j \leq n-1} |Y_j|^p) + \psi_{n,p}(a)\mathbb{E}|Y_0|^p \leq a\mathbb{E}|Y_{n-1}|^p, \) valid for all martingales \( Y_0^*, \ldots, Y_{n-1}^*, \) where \( \psi_{n,p} \) is the concave conjugate function of \( \psi_{n,p}. \) Both the finite sequence and infinite sequence cases are developed.

Proofs utilize moment theory, induction, conjugate function theory, and functional equation analysis.


Abstract. For all uniformly bounded sequences of independent random variables \( X_1, X_2, \ldots, \) a complete comparison is made between the optimal value \( V(X_1, X_2, \ldots) = \sup \{ \mathbb{E}X_t: t \text{ is an (a.e.) finite stop rule for } X_1, X_2, \ldots \} \)

and \( \mathbb{E}(k^{-1} \sum_{i=1}^{k} M_i(X_1, X_2, \ldots)), \) where \( M_i(X_1, X_2, \ldots) \) is the ith largest order statistic for \( X_1, X_2, \ldots. \) In particular, for \( k > 1, \) the set of ordered pairs \( \{(x,y): x = V(X_1, X_2, \ldots) \text{ and } y = \mathbb{E}(k^{-1} \sum_{i=1}^{k} M_i(X_1, X_2, \ldots)) \} \) for some independent
random variables $X_1, X_2, \ldots$ taking values in $[0,1]$ is precisely the set
\[
\{(x,y): k^{-1}x \leq y \leq B_k(x); 0 < x \leq 1\} \cup \{(0,0),(1,1)\},
\]
where $B_k(0) = 0$, $B_k(1) = 1$, and for $0 < x < 1$, $B_k(x) = 1 - k^{-1} \sum_{i=0}^{k-1} \binom{1}{j} (-ln(1-x))^j / j!$.

The result yields sharp, universal inequalities for independent random variables comparing two choice mechanisms, the mortal’s value of the game $V(X_1, X_2, \ldots )$ and the prophet’s constrained maxima expectation of the game $E(k^{-1} \sum_{i=1}^{k} M_i(X_1, X_2, \ldots ))$. Techniques of proof include probability- and convexity-based reductions; calculus-based, multivariate, extremal problem analysis; and limit theorems of Poisson-approximation type. Precise results are also given for finite sequences of independent random variables.


Abstract. Suppose $\mu_1, \ldots, \mu_n$ are nonatomic probability measures on the same measurable space $(S, \mathcal{S})$. Then there exists a measurable partition $\{S_i\}_{i=1}^{n}$ of $S$ such that $\mu_i(S_i) \geq (n+1-M)^{-1}$ for all $i = 1, \ldots, n$, where $M$ is the total mass of $\nu_{i=1}^{n} \mu_i$ (the smallest measure majorizing each $\mu_i$). This inequality is the best possible for the functional $M$, and sharpens and quantifies a well-known cake-cutting theorem of Urbanik and of Dubins and Spanier. Applications are made to $L_1$-functions, discrete allocation problems, statistical decision theory, and a dual problem.


Abstract. For sequences of exchangeable random variables $X_1, \ldots, X_n$ taking values in $[0,1]$, results are given which compare $E(\max_{1 \leq i \leq n} X_i)$ and $V(X_1, \ldots, X_n) = \sup\{EX_t: t$ is a stop rule for $X_1, \ldots, X_n\}$. Complete results are given for the $n = 2$ and $n = \infty$ cases; results are also given for the $2 < n < \infty$ case. A constructive approach is taken, with extremal distributions given where applicable. For the $n = 2$ case, conjugate function theory is used in an essential way. The transformations and constructions of finite sequences of exchangeable random variables which are given should be of independent interest to those working with this class of processes.


Abstract. A problem of optimization under constraints is stated for general nonstationary stochastic decision models, and studied using convexity theory. For constrained multi-armed bandit processes with geometric
discounting, optimal plans are characterized through dynamic allocation indices, and value functions are expressed in terms of single-arm value functions. Techniques of proof include randomizations, conjugate duality, and characterizations for the unconstrained problem. Explicit closed-form representations are given for some Bernoulli bandit processes.


Abstract. A player starts at $x$ in $(-G,G)$ and attempts to leave the interval in a limited playing time. In the discrete time problem, $G$ is a positive integer and the position is described by a random walk starting at integer $x$, with mean increments zero, and variance increment chosen by the player from $[0,1]$ at each integer playing time. In the continuous time problem, the player's position is described by an Itô diffusion process with infinitesimal mean parameter zero and infinitesimal diffusion parameter chosen by the player from $[0,1]$ at each time instant of play. To maximize the probability of leaving the interval $(-G,G)$ in a limited playing time, the player should play boldly by always choosing largest possible variance increment in the discrete-time setting and largest possible diffusion parameter in the continuous-time setting, until the player leaves the interval. In the discrete time setting, this result affirms a conjecture of Spencer. In the continuous time setting, the value function of play is also identified.


Abstract. This paper presents background, statements, techniques, and variations of prophet problems in optimal stopping. The main formulation of the prophet problem and equivalent formulations are given in sections one and two. Techniques of proof are given in sections three and four; these include reduction procedures, use of verification lemmas, use of moment theory, and other techniques. Variations of the prophet problem in optimal stopping are given in section five. Prophet-type stochastic control problems for transforms of processes are discussed in section six. Examples of prophet inequalities and prophet regions for many classes of processes are given. Proofs in section 5b and in the appendix illustrate the techniques used to prove these results.


Talks were presented at the following meetings, conferences, and universities by T. P. Hill.

1. Amer. Math. Society Annual Meeting, Anaheim  
   Jan 1985
2. Mathematics Dept., University of Miami  
   Apr 1986
3. Mathematics and Statistics Departments, Ohio State  
   Apr 1986
4. Conference on optimal stopping and gambling theory, Oberwolfach  
   Jun 1986
5. Mathematical Stochastics Department, Univ. of Göttingen  
   Jun 1986
6. Applied Mathematics Department, Univ. of Bonn  
   Jun 1986
7. International Congress of Mathematicians, Berkeley  
   Aug 1986
8. Mathematics Dept., U.S. Military Academy  
   Oct 1986

Talks were presented at the following meetings, conference, and universities by Robert P. Kertz.

1. Amer. Math. Society 91st Annual Meeting, Anaheim  
   Jan 1985
2. Southeastern Meeting on Diff. Eqns., Atlanta  
   Oct 1985
3. ORSA/TIMS 20th Joint National Meeting, Atlanta  
   Nov 1985
4. Inst. of Math. Statist. 195th Meeting, Atlanta  
   Mar 1986
5. Conference on Optimal Stopping and Gambling, Oberwolfach  
   Jun 1986
6. Workshop on Probabilistic Problems Concerning Strategies, Göttingen  
   Jun 1986
7. Statistical Laboratory, Cambridge Univ., Cambridge  
   Jul 1986
The following professors were brought to Georgia Tech for research consultations with T. P. Hill and R. P. Kertz with partial assistance from NSF Grant DMS 84-01604.

1. Prof. Dr. Ulrich Krengel, Director, Institut für Mathematische Stochastik, Georg-August-Universität, Göttingen (August 1985).

2. Prof. Donald Geman, Department of Mathematics and Statistics, University of Massachusetts, Amherst (March 1986).

3. Prof. Douglas Kennedy, Statistical Laboratory and Trinity College, University of Cambridge (September 1986).