NORMALLY ELLIPTIC SINGULAR PERTURBATION PROBLEMS:

LOCAL INVARIANT MANIFOLDS AND APPLICATIONS

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To my parents, my girlfriend and myself.

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TABLE OF CONTENTS

SUMMARY

In this thesis, we study the normally elliptic singular perturbation problems including both finite and infinite dimensional cases, which could also be nonautonomous. In particular, we establish the existence and smoothness of $O(1)$ local invariant manifolds and provide various estimates which are independent of small singular parameters. We also use our results on local invariant manifolds to study the persistence of homoclinic solutions under weakly dissipative and conservative perturbations.

We apply Semi-group Theory and Lyapunov-Perron Integral Equations with some careful estimates to handle the $O(1)$ driving force in the system so that we can approximate the full system through some simpler limiting system. In the investigation of homoclinics, a diagonalization procedure and some normal form transformation should be first carried out. Such diagonalization procedure is not trivial at all. We discuss this issue in the appendix. We use Melnikov type analysis to study the weakly dissipative case, while the conservative case is based on some energy methods.

As a concrete example, we have shown rigrously the persistence of homoclinic solutions of an elastic pendulum model which may be affected by damping, external forcing and other potential fields.

CHAPTER I

INTRODUCTION

A singular perturbation system usually involves different temporal or spatial scales. Here we focus on multiple time scales in which case the system takes the abstract form of

$$\dot{x} = F(x, y, \epsilon)\ ,\ \epsilon\dot{y} = G(x, y, \epsilon). \tag{1.1}$$

The fast motions in the y direction are often some noise or transient behaviors and the slow motions in the x direction are more of the focus of the problem. In the singular limit as $\epsilon \to 0$, we obtain $G(x, y, 0) = 0$. Suppose $y = \phi(x)$ (without loss of generality, assuming $\phi \equiv 0$) solves this equation, the limit motion of x is given by

$$\dot{x} = F(x, 0, 0). \tag{1.2}$$

Let $\tilde{y} = \frac{y}{\epsilon}$, the y equation in (1.1) takes the form

$$\dot{\tilde{y}} = \frac{G_y(x, 0, 0)}{\epsilon}\tilde{y} + g(x, \tilde{y}, \epsilon). \tag{1.3}$$

The singular perturbation system (1.1) is called normally hyperbolic if, for each x, the linear flow $e^{tG_y(x,0,0)}$ on the y space is hyperbolic, i.e. it is exponentially contracting on one closed subspace and exponentially expanding in a complementary subspace. In this case, the standard normally hyperbolic invariant manifold theory [F1, HPS, He, BLZ1, BLZ2] applies to yield a persistent normally hyperbolic invariant slow manifold M_ϵ given by a graph $y = \epsilon\phi(x, \epsilon)$. In the fast (and hyperbolic in nature) motions outside M_ϵ, solutions usually approach a neighborhood of M_ϵ exponentially along its stable direction. After some time moving along the slow manifold, the solutions leave the neighborhood exponentially along the unstable directions. These motions of multiple scales can be connected by tools such as invariant foliations

[F2, F3, HPS, CLL, BLZ3] and this geometric approach has led to a huge success in the study of the dynamics of singular perturbation system (1.1). See for example [F4, Jo, JK, BLZ4].

In the normally elliptic case, i.e. $e^{tG_y(x,0,0)}$ is oscillatory instead of hyperbolic, the persistence of the slow manifold is not always guaranteed [GL1, GL2]. Moreover, solutions starting near $\{y = 0\}$ should stay there at least for some $O(1)$ time period due to the lack of strong exponential instability in the y direction. One typical situation of this type is when $G_y(x, 0, 0)$ is anti-self-adjoint.

In this thesis, with applications to both ODEs and PDEs in mind, we study these normally elliptic singular perturbation problems in an infinite dimensional dynamical system and possibly non-autonomous framework. Assuming $G_y(x, 0, 0) = J$, a constant anti-self-adjoint operator, we first justify the limit equation (1.2) of the slow variable x through a careful averaging.

A more important question is how much of the dynamical structure of the limit slow system (1.2) remains in the singularly perturbed system (1.1). Elliptic type motions in the slow directions such as periodic or quasi-periodic solutions may be resonant with the oscillatory fast motions in the y direction. Some results have been obtained on the persistence of periodic orbits for nonresonant $\epsilon << 1$ [GL1, GL2, Lo, Ma, SZ2]. Here instead we focus on the basic hyperbolic structure – the local invariant manifolds near a steady state. Suppose $(0, 0)$ persists as a steady state of (1.1) for $\epsilon << 1$. Assume the linearization of the limit slow system (1.2) has invariant stable, unstable, and center subspaces $X^{u,s,c}$. For the expanded system (1.1), the normal directions – the Y space – with the oscillatory linearized flow e^{tJ} should obviously be considered as additional center directions. The first observation is, even though system (1.1) is singular, the existence of local invariant manifolds of $(0, 0)$ is guaranteed by the standard theory (see, for example, [BJ, Ha, CL]) after a rescaling of the time by a factor of ϵ. However, since the exponential growth/decay rates in the unstable/stable direction are $O(\epsilon)$ after the rescaling, this approach would only yield local invariant manifolds of the size of $O(\epsilon)$, which is far from being useful in most applications, such as studying the persistence of homoclinic orbits.

Our main result in the thesis is to prove the existence and smoothness and the leading order approximation of invariant manifold of the steady state of size $O(1)$ based on a combination of the averaging and Lyapunov-Perron integral equation methods.

As an application which is also an fundamental problem itself, suppose there exists a homoclinic orbits in the limit slow system (1.2) and we study its persistence in the singular perturbation system (1.1) which can be either weakly dissipative or conservative. In the former, we derive the Melnikov function, which include an additional term coming from the fast directions, whose simple zero indicates a persistent homoclinic orbit to $(0,0)$. In the latter, when the system is analytic in reasonably low dimensions, along with some other structures such as the Hamiltonian setting or the reversibility, it has been shown that the stable and unstable manifold miss each other by an error like $O(e^{-\frac{C}{\epsilon}})$ [Sun, Ge, Lo, To]. Without these assumptions, we prove that there always exist orbits homoclinic to the center-manifold, forming a tube homoclinic to the center manifold. While we follow the well-developed geometric ideas in the finite or infinite dimensional regular perturbation problems [GH, HM, LMSW, SZ3], the proof heavily depends on the invariant manifolds we studied.

Before finishing the introduction, we would like to give two simple examples which partially motivated us to study this subject, while it is also easy to come up with examples in infinite dimensions. One is an elastic pendulum with fast and slow frequencies itself and the other one is a bifurcation problem which does not appear to have any singular parameter.

A pendulum of unit length with a fixed end is described by the Duffing equation. In a more careful model, the pendulum, usually considered as rigid, may have some small elasticity – meaning large elastic constant $\frac{1}{\epsilon^2}$ – allowing the pendulum to be stretched or contracted slightly in the radial direction. Let x be the angular and $1 + y$ be the radial coordinates, respectively, then the system takes of the form of a normally elliptic singular perturbation problem

$$\begin{cases} (1+y)\ddot{x} + 2\dot{x}\dot{y} + g\sin x + 2\epsilon\gamma(1+y)\dot{x} - \dfrac{\epsilon}{1+y}F_1(x, y, \epsilon, t) = 0, \\ \ddot{y} - (1+y)\dot{x}^2 + \dfrac{1}{\epsilon^2}y - g\cos x + 2\epsilon\gamma\dot{y} - \epsilon F_2(x, y, \epsilon, t) = 0, \end{cases} \tag{1.4}$$

where we also included the small damping and forcing. Formally, as $\epsilon \to 0$, i. e. the pendulum converges to be rigid, the corresponding singular limit (1.2) for the above system becomes

$$y \equiv 0, \qquad \ddot{x} + g\sin x = 0. \tag{1.5}$$

When there is no damping and the force is conservative, the problem is in the Lagrangian setting and the limit equation is justified in [RU, Ar, Ta]. In the dynamics, the state $(\pi, 0)$ is a hyperbolic steady state of (1.5) with a homoclinic orbit which often leads to chaos even under small regular perturbation [GH]. One may easily change the variables in the singular equation of y and make it anti-self-adjoint. Our general results apply to (1.4) and give the criterion when the homolcinics persist under either dissipative or conservative perturbation. This example will be revisited in Chapter 5.

The singular perturbation theory also applies to problems which may not be explicitly in the form of (1.1). Consider an autonomous 4-dim ODE system with a parameter ϵ which has the origin O as a fixed point for all $\epsilon << 1$. Assume, at $\epsilon = 0$, the linearized systems has simple eigenvalues $\pm i$ and a double eigenvalue 0. While the unfolding of the focal point has been studied thoroughly (see, for example [CLW]), we note that the oscillatory motions are essentially at a much faster scale in the directions of the pair of elliptic eigenvalues. Under these assumptions, some simple normal forms transformations and near identity time rescaling, the generic form of the system looks like

$$\dot{x} = \begin{pmatrix} a_{11}(\epsilon) & 1 + a_{12}(\epsilon) \\ a_{21}(\epsilon) & a_{22}(\epsilon) \end{pmatrix} x + O(|x|^2 + |y|^2) \qquad \dot{y} = \begin{pmatrix} b(\epsilon) & 1 \\ -1 & b(\epsilon) \end{pmatrix} y + O(|x|^2 + |y|^2)$$

where x, $y \in \mathbb{R}^2$ and $a_{lm}(0) = b(0) = 0, l, m = 1, 2$. Rescaling the system again by

$$x_1 = \epsilon\tilde{x}_1, \; x_2 = \epsilon^{\frac{3}{2}}\tilde{x}_2, \; y = \epsilon\tilde{y}, \; t = \epsilon^{-\frac{1}{2}}\tau,$$

we obtain a singularly perturbed system in the form of (1.1) of normally elliptic type with the singular parameter $\mu = \epsilon^{\frac{1}{2}}$. If $\frac{da_{21}}{d\epsilon}(0) > 0$, the origin becomes hyperbolic in the x directions and we obtain the local center manifolds of order $O(1)$ size in the rescaled variables. If $\frac{db}{d\epsilon}(0) \neq 0$ in addition, we are in the right position to study the Hopf bifurcation from the eigenvalues $\pm i$ in this rather degenerate case. (See [F5] for an approach essentially different from the Hopf bifurcation.)

The rest of the thesis is organized as the following. In Chapter 2 we present the general framework in Section 2.1 and give the justification of the limit slow equations and its linearization as well as the outline of the main results on invariant manifolds and foliations in Section 2.2. In Chapter 3 and 4 we study invariant manifolds and foliations and focus on their leading order approximations. Finally, the homoclinic orbits are considered in Chapter 5. In the Appendix, we outline a process to block-diagonalize the linearized system of (1.1) at a steady state.

CHAPTER II

SET UP AND MAIN RESULTS

2.1 Set Up

Let Z_1, Z_2 be Banach spaces and Z be an open subset of Z_1. We use $|\cdot|_{Z_1}$ to denote the norm of an element in a Banach space which is prescribed by the subscript. For simplicity, we will sometimes just use $|\cdot|$ provided this will not cause confusion.

For any integer $k \geq 1$, let

$$C^k(Z, Z_2) = \big\{ h \big| h : Z \to Z_2 \text{ } k\text{-times differentiable and } \sup_{z\in Z} |D^i f(z)|_{L_i(Z_1,Z_2)} < \infty \text{ for } 0 \leq i \leq k \big\},$$

with norm

$$|h|_{C^k} = \sum_{i=0}^{k} \sup_{z\in Z} |D^i h(z)|_{L_i(Z_1,Z_2)},$$

where D^i is the ith differentiation operator with respect to variables in the phase space and we will use ∂_t and ∂_ϵ for derivatives with respect to t and ϵ. Let $L_i(Z_1, Z_2)$ be the Banach space of all bounded multi-linear operators from Z_1 into Z_2 of order i, where $L_0(Z_1, Z_2) = Z_2$ and $L_1(Z_1, Z_2) = L(Z_1, Z_2)$.

We use $|\cdot|_{C^0}$ to the denote the C_0 norm of some quantity with respect to all its variables.

Let X be a Banach space and Y be a Hilbert space, for $x \in X, y \in Y$, we consider the equations

$$\begin{cases} \dot{x} = Ax + f(x, y, t, \epsilon) \\ \dot{y} = \dfrac{J}{\epsilon} y + g(x, y, t, \epsilon) \end{cases} \tag{2.1}$$

and

$$\dot{x}_0 = Ax_0 + f(x_0, 0, t, 0). \tag{2.2}$$

We assume for some constants C_0:

(A1) $A : X_1 \to X$, where X_1 is a Banach space endowed with graph norm $|\cdot|_{X_1}$, which is continuously embedded into X and X_1 is dense in X.

(A2) A generates a C_0-semigroup e^{tA} on X such that $|e^{tA}| \le Me^{\omega t}$ for $t \ge 0$.

(A3) J is an anti-selfadjoint operator on Y with domain $D(J) = Y_1$, which generates a unitary group $e^{t\frac{J}{\epsilon}}$. Assume $0 \in \rho(J)$ and $|J^{-1}|_{L(Y,Y_1)} \le C_0$. We denote the graph norm on Y_1 by $|\cdot|_{Y_1}$.

(A4) For $k \ge 1$,

$$
\begin{aligned}
&(D^i f, D^i g) \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, L_i(X_1 \times Y_1, X_1 \times Y_1)),\ 0 \le i \le k,\\
&(D^i f, D^i g) \in C^0(X_1 \times Y_1 \times \mathbb{R}^2,\\
&\qquad\qquad\qquad L((X \times Y) \otimes^{i-1} (X_1 \times Y_1), X \times Y)),\ 1 \le i \le k.
\end{aligned}
$$

All of the quantities have a uniform bound C_0.

(A5) $|\cdot|_X \in C^k(X\backslash\{0\}, \mathbb{R}^+)$, where $\mathbb{R}^+$ denotes all positive real numbers.

(A6) $\partial_\epsilon f \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, X_1)$, $D_x\partial_\epsilon f \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, L(X_1, X_1))$, $D_x\partial_t f \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, L(Y_1, X))$, $\partial_t g \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, Y)$, $D\partial_t g \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, L(X_1 \times Y_1, X \times Y))$. Moreover, their norms are bounded by C_0.

Remark 2.1. *In fact we can replace J by $J(\epsilon)$ for each small ϵ, then all results in this manuscript still hold except Theorem 3.21 and 3.23.*

Remark 2.2. *The above assumptions may look complicated which is only due to our intention to make the result applicable to PDEs where unbounded operators and different function spaces are involved. For ODE systems, these assumption would simply be*

- *J is an anti-symmetric matrix and (f, g) are smooth functions.*

Let C be a generic constant, possibly with subscripts, which may have different values as it appears in different places, and it only depends on the quantities involved in (A1)-(A6). Let C' be another generic constant, possibly with subscripts, and the dependence will be specified in the context.

2.2 Main Results

Before we introduce our main result, we first prove the following two theorems which show that (2.2) is a good approximation of (2.1), so we can view (2.1) as a perturbation of (2.2).

Theorem 2.3. *For any $t_0 \in \mathbb{R}$, let $T > 0$ and $(x(t), y(t))$, $x_0(t)$ be solutions of* (2.1) *and* (2.2) *on $[t_0, t_0 + T]$. Suppose $|x(t_0) - x_0(t_0)|_{X_1} + |y(t_0)|_{Y_1} \leq C_1\epsilon$, then for any $t \in [t_0, t_0+T]$, there exists a constant C' which depends on $M, \omega, T, C_0, C_1, |x_0(t_0)|_{X_1}$, such that*

$$|x(t) - x_0(t)|_{X_1} + |y(t)|_{Y_1} \leq C'\epsilon. \tag{2.3}$$

Proof. The idea of the proof is basically averaging in time which appears in the estimate as integration by parts. By (2.1), (2.2) and variation of parameters formula

$$\begin{aligned}(x - x_0)(t) &= e^{(t-t_0)A}(x - x_0)(t_0) + \int_{t_0}^{t} e^{(t-\tau)A}\big(f(x, y, \tau, \epsilon) - f(x_0, 0, \tau, 0)\big)\, d\tau,\\ y(t) &= e^{(t-t_0)\frac{J}{\epsilon}}y(t_0) + \int_{t_0}^{t} e^{(t-\tau)\frac{J}{\epsilon}}\big(g(x, y, \tau, \epsilon) - g(x_0, 0, \tau, 0)\big)\, d\tau\\ &\quad + \int_{t_0}^{t} e^{(t-\tau)\frac{J}{\epsilon}} g(x_0, 0, \tau, 0)\, d\tau.\end{aligned}$$

Due to the oscillatory nature of $e^{t\frac{J}{\epsilon}}$, we can write

$$\begin{aligned}&\int_{t_0}^{t} e^{(t-\tau)\frac{J}{\epsilon}} g(x_0, 0, \tau, 0)\, d\tau\\ &= -\epsilon J^{-1}g(x_0, 0, t, 0) + e^{(t-t_0)\frac{J}{\epsilon}}\epsilon J^{-1}g(x_0, 0, t_0, 0)\\ &\quad + \int_{t_0}^{t} e^{(t-\tau)\frac{J}{\epsilon}}\epsilon J^{-1}\partial_t g(x_0, 0, \tau, 0)\, d\tau\\ &\quad + \int_{t_0}^{t} e^{(t-\tau)\frac{J}{\epsilon}}\epsilon J^{-1}D_x g(x_0, 0, \tau, 0)\big(Ax_0 + f(x_0, 0, \tau, 0)\big)\, d\tau,\end{aligned}$$

where we also use assumption (A4) to assure the last term on the right hand side is well defined. Therefore,

$$\Big| \int_{t_0}^{t} e^{(t-\tau)\frac{J}{\epsilon}} g(x_0, 0, \tau, 0)\ d\tau \Big|_{Y_1} \leq C'\epsilon,$$

where C' depends on $M, \omega, T, C_0, |x_0(t_0)|_{X_1}$. Consequently,

$$\begin{aligned}(|x - x_0|_{X_1} + |y|_{Y_1})(t) \leq C(|x - x_0|_{X_1} + |y|_{Y_1})(t_0) + C'\epsilon \\ + \int_{t_0}^{t} C(|x - x_0|_{X_1}(\tau) + |y - y_0|_{Y_1}(\tau) + \epsilon)\ d\tau.\end{aligned}$$

Then, by applying Gronwall's inequality, we can get the desired estimate. □

In addition to the convergence of solution on finite time interval, we will also need the convergence of solution of the linearized equations. We linearize (2.1) and (2.2) to have

$$\begin{cases} \dot{\delta x} = A\delta x + D_x f(x, y, t, \epsilon)\delta x + D_y f(x, y, t, \epsilon)\delta y \\ \dot{\delta y} = \dfrac{J}{\epsilon}\delta y + D_x g(x, y, t, \epsilon)\delta x + D_y g(x, y, t, \epsilon)\delta y, \end{cases} \tag{2.4}$$

and

$$\begin{cases} \dot{\delta x_0} = A\delta x_0 + D_x f(x_0, 0, t, 0)\delta x_0 \\ \dot{\delta y_0} = \dfrac{J}{\epsilon}\delta y_0 + D_y g(x_0, 0, t, 0)\delta y_0. \end{cases} \tag{2.5}$$

Theorem 2.4. *Assume $k = 2$ in (A4). Let $(\delta x(t), \delta y(t))$ and $(\delta x_0(t), \delta y_0(t))$ be solutions of* (2.4) *and* (2.5)*, respectively. Suppose that*

$$|\delta x(t_0)|_{X_1} + |\delta y(t_0)|_{Y_1} \leq 1\ ,\ \ |\delta x_0(t_0)|_{X_1} + |\delta y_0(t_0)|_{Y_1} \leq 1, \tag{2.6}$$

and

$$|x(t_0) - x_0(t_0)|_{X_1} + |y(t_0)|_{Y_1} \leq C_1\epsilon, \tag{2.7}$$

$$|\delta x(t_0) - \delta x_0(t_0)|_X + |\delta y(t_0) - \delta y_0(t_0)|_{Y_1} \leq C_1\epsilon. \tag{2.8}$$

Then there exists a constant C' which depends on $M, \omega, T, C_0, C_1, |x(t_0)|_{X_1}$, such that

$$|\delta x(t) - \delta x_0(t)|_X + |\delta y(t) - \delta y_0(t)|_{Y_1} \leq C'\epsilon$$

for all $t \in [t_0, t_0 + T]$.

Proof. By standard semigroup theory in Banach space and (A3), we have

$$\begin{aligned}&\big(|\delta x(\cdot)|+|\delta x_0(\cdot)|\big)_{C^0([t_0,t_0+T],X_1)}\le C,\\&\big(|\delta y(\cdot)|+|\delta y_0(\cdot)|\big)_{C^0([t_0,t_0+T],Y_1)}\le C.\end{aligned}\tag{2.9}$$

First we use (2.4) and (2.5) to obtain

$$\begin{cases}\dot{\delta x}-\dot{\delta x_0}=&A(\delta x-\delta x_0)+\big(D_xf(x,y,t,\epsilon)-D_xf(x_0,0,t,0)\big)\delta x_0\\&+D_xf(x,y,t,\epsilon)(\delta x-\delta x_0)+D_yf(x,y,t,\epsilon)(\delta y-\delta y_0)\\&+D_yf(x,y,t,\epsilon)\delta y_0\\\dot{\delta y}-\dot{\delta y_0}=&(\dfrac{J}{\epsilon}+D_yg(x_0(t),0,t,0))(\delta y-\delta y_0)+D_xg(x_0,0,t,0)\delta x\\&+\big(D_xg(x,y,t,\epsilon)-D_xg(x_0,0,t,0)\big)\delta x\\&+(D_yg(x,y,t,\epsilon)-D_yg(x_0(t),0,t,0))\delta y.\end{cases}$$

By using (2.4), we have

$$\begin{aligned}\dot{\delta x}-\dot{\delta x_0}&=A(\delta x-\delta x_0)+h_1(t,\epsilon)+D_yf(x,y,t,\epsilon)\delta y_0,\\\dot{\delta y}-\dot{\delta y_0}&=(\frac{J}{\epsilon}+D_yg(x_0(t),0,t,0))(\delta y-\delta y_0)+h_2(t,\epsilon)\\&\quad+D_xg(x_0,0,t,0)\delta x,\end{aligned}$$

where

$$|h_1|_X\le C'(\epsilon+|\delta x-\delta x_0|_X+|\delta y-\delta y_0|_Y)\ ,\ |h_2|_{Y_1}\le C'\epsilon.$$

By assumptions (A3) and (A4), $\frac{J}{\epsilon}+D_yg(x_0(t),0,t,0)$ generates an evolution operator $E(t,s;x_0(s),\epsilon)$ which satisfies for $t\ge s$,

$$|E(t,s;x_0(s),\epsilon)|_{L(Y,Y),L(Y_1,Y_1)}\le e^{C_0(t-s)}.$$

For $D_x g(x_0,0,t,0)\delta x$, we use (2.9) and integrate by parts to obtain

$$
\begin{aligned}
&\Big|\int_{t_0}^t E(t,\tau;x_0(\tau),\epsilon)D_x g(x_0,0,\tau,0)\delta x \ \ d\tau\Big|_{Y_1}\\
=&\Big|-(\frac{J}{\epsilon}+D_y g(x_0(t),0,t,0))^{-1}D_x g(x_0(t),0,t,0)\delta x(t)\\
&+E(t,t_0;x_0(t_0),\epsilon)(\frac{J}{\epsilon}+D_y g(x_0(t_0),0,t_0,0))^{-1}D_x g(x_0(t_0),0,t_0,0)\delta x(t_0)\\
&+\int_{t_0}^t E(t,\tau;x_0(\tau),\epsilon)(\frac{J}{\epsilon}+D_y g(x_0,0,\tau,0))^{-1}\\
&\qquad\qquad \big[D_{xy}g(x_0,0,\tau,0)(Ax_0+f(x_0,0,\tau,0))+D_y\partial_t g(x_0,0,t,0)\big]\\
&\qquad\qquad\qquad\qquad (\frac{J}{\epsilon}+D_y g(x_0,0,\tau,0))^{-1}D_x g(x_0,0,\tau,0)\delta x \ d\tau\\
&+\int_{t_0}^t E(t,\tau;x_0(\tau),\epsilon)(\frac{J}{\epsilon}+D_y g(x_0,0,\tau,0))^{-1}D_x g(x_0,0,\tau,0)\\
&\qquad\qquad\qquad \big(A\delta x+D_x f(x,y,\tau,\epsilon)\delta x+D_y f(x,y,\tau,\epsilon)\delta y\big)\ d\tau\\
&+\int_{t_0}^t E(t,\tau;x_0(\tau),\epsilon)(\frac{J}{\epsilon}+D_y g(x_0,0,\tau,0))^{-1}\\
&\qquad\qquad\qquad\qquad D_x^2 g(x_0,0,\tau,0)\big(Ax_0+f(x_0,0,\tau,0),\delta x\big)\ d\tau\\
&+\int_{t_0}^t E(t,\tau;x_0(\tau),\epsilon)(\frac{J}{\epsilon}+D_y g(x_0,0,\tau,0))^{-1}D_x\partial_t g(x_0,0,\tau,0)\delta x \ d\tau\Big|_{Y_1}\\
\le& C'\epsilon.
\end{aligned}
$$

For $D_y f(x,y,t,\epsilon)\delta y_0$, we use (2.5) to obtain

$$
\delta y_0(t)=\epsilon J^{-1}\dot{\delta y}_0(t)-\epsilon J^{-1}D_y g(x_0(t),0,t,0)\delta y_0(t).
$$

Consequently,

$$
\begin{aligned}
&|\int_{t_0}^t e^{(t-\tau)A}D_y f(x,y,\tau,\epsilon)\delta y_0 \ d\tau\big|_X\\
=&\epsilon|\int_{t_0}^t e^{(t-\tau)A}D_y f(x,y,\tau,\epsilon)(J^{-1}\dot{\delta y}_0-J^{-1}D_y g(x_0,0,\tau,0)\delta y_0)\ d\tau\big|_X\\
\le& C'\epsilon+\epsilon|\int_{t_0}^t \frac{d}{d\tau}(e^{(t-\tau)A}D_y f(x,y,\tau,\epsilon))J^{-1}\delta y_0 \ d\tau\big|_X.
\end{aligned}
$$

One can use (2.4) and (A6) to show

$$
|\int_{t_0}^t \frac{d}{d\tau}(e^{(t-\tau)A}D_y f(x,y,\tau,\epsilon))J^{-1}\delta y_0 \ d\tau\big|_X\le C'\epsilon. \tag{2.10}
$$

Therefore,

$$\begin{aligned}&\big(|\delta x-\delta x_0|_X+|\delta y-\delta y_0|_{Y_1}\big)(t)\\ \le& C'\epsilon+\int_{t_0}^t C\big(|\delta x-\delta x_0|_X+|\delta y-\delta y_0|\big)(\tau)\ d\tau.\end{aligned}$$

Then Gronwall's inequality shows the desired result. □

Remark 2.5. *We cannot achieve the estimate on* $|\delta x-\delta x_0|_{X_1}$ *even if we assume* $|\delta x(t_0)-\delta x_0(t_0)|_{X_1}\le C_1\epsilon$, *since the left hand side of* (2.10) *contains a term*

$$\int_{t_0}^t Ae^{(t-\tau)A}D_y f(x,y,\tau,\epsilon)J^{-1}\delta y_0\ d\tau,$$

which is only in X *under current assumptions. In fact, if we assume*

$$|\delta x(t_0)-\delta x_0(t_0)|_{X_1}+|\delta y(t_0)|_{Y_1}\le C_1\epsilon\ ,\ \ |\delta y_0(t_0)|_{Y_1}\le C_1\epsilon,$$

we can prove $|\delta x(t)-\delta x_0(t)|_{X_1}+|\delta y(t)|_{Y_1}\le C_1\epsilon$ *for all* $t\in[t_0,t_0+T]$.

Remark 2.6. *Let* $\Phi(t,t_0,x,y,\epsilon)$ *be the flow of* (2.1). *Higher order derivatives of* Φ *in* x,y *can be obtained in a similar way.*

Corollary 2.7. *Assume the same condition as in Theorem 2.4 and replace* (2.6) *and* (2.8) *by*

$$\begin{aligned}&|\delta x_0(t_0)|_{X_1}+|\delta y_0(t_0)|_{Y_1}\le 1,\\ &|\delta x(t_0)-\delta x_0(t_0)|_X+|\delta y(t_0)-\delta y_0(t_0)|_Y\le C_1\epsilon.\end{aligned}$$

Then there exists C' *such that for all* $t\in[t_0,t_0+T]$,

$$|\delta x(t)-\delta x_0(t)|_X+|\delta y(t)-\delta y_0(t)|_Y\le C'\epsilon.$$

Remark 2.8. *In fact, one can also consider a more careful approximation*

$$\begin{cases}\dot x=Ax+f(x,y,t,\epsilon)\\ \dot y=\dfrac{J}{\epsilon}y+g(x,y,t,\epsilon)\end{cases}\quad \text{and}\quad \begin{cases}\dot x_0=Ax_0+f(x_0,y_0,t,0)\\ \dot y_0=\dfrac{J}{\epsilon}y_0+g(x_0,y_0,t,0)-g(x_0,0,t,0)\end{cases}$$

Similar estimates can also be obtained.

To study the local invariant manifolds, suppose $(0,0)$ is always a steady state and the limit systems is autonomous, i.e.

$$\partial_t f(x,y,t,0) = \partial_t g(x,y,t,0) = 0 \qquad f(0,0,t,\epsilon) = g(0,0,t,\epsilon) = 0.$$

We assume linearized (2.2) at 0 has the exponential trichotomy, i.e. there exist closed subspaces $X^{u,s,c}$ such that there exist constants $a_1 < \min\{a_2, 0\}$ and $a_2' > \max\{0, a_1'\}$ and for $t \geq 0$,

$$|e^{t(A+f_x(0))}|_{X^s} \leq C_1 e^{a_1 t} \quad |e^{-t(A+f_x(0))}|_{X^u} \leq C_1 e^{-a_2' t}$$
$$|e^{t(A+f_x(0))}|_{X^c} \leq C_1 e^{a_1' t} \quad |e^{-t(A+f_x(0))}|_{X^c} \leq C_1 e^{-a_2 t}.$$

Moreover, we assume the linearized flow $e^{t(\frac{J}{\epsilon}+g_y(0))}$ satisfies the same assumption as $e^{-t(A+f_x(0))}|_{X^c}$ and thus the expanded center space of (2.1) should be $X^c \oplus Y$. Along with a few other technical assumptions, rough our main results on invariant manifolds and foliations in the phase space $X_1 \times Y_1$ is

Main Theorem. *For $\epsilon << 1$, in the space $X_1 \times Y_1$,*

1. *There exists smooth invariant stable, unstable, center-stable, center-unstable, and center integral manifolds of $(0,0)$ which can be written as graphs of smooth mappings from a δ-neighborhood of the corresponding subspaces to the complements whose norms and δ are independent of ϵ. Moveover their derivatives in t_0, the time parameter of integral manifolds, is $O(\epsilon)$ when evaluated in the norm $|\cdot|_X + |\cdot|_Y$.*

2. *The center-stable and center-unstable manifolds are foliated into the disjoint union of smooth families of smooth stable and unstable fibers which also written as graphs of mappings whose norms are bounded independent of ϵ.*

3. *The stable and unstable manifolds are $O(\epsilon)$ close to those of (2.2).*

4. *The center-stable, center-unstable, and the center manifolds at* $\{y = 0\}$ *are* $O(\epsilon)$ *close to those of* (2.2) *and their tangent spaces there are* $O(\epsilon)$ *close to the direct sum of the unperturbed ones and* Y, *respectively.*

Here by the term an integral manifold, we mean a family of manifold $M(t)$ parameterized by t so that the solution map of (2.1) starting at initial time t_0 and ending at t_1 maps $M(t_0)$ into $M(t_1)$. They are independent of t if the system is autonomous. The precise statement of these results of the invariant manifolds are given in Chapter 3 and 4.

CHAPTER III

INVARIANT MANIFOLD

In this chapter, we will study the local integral manifold of a stationary solution of (2.1), namely, the center-unstable (stable) manifold, unstable (stable) manifold and ect.

3.1 Preliminary

In this section, we will introduce our main assumptions and some basic quantities that will be used to construct various invariant manifolds. Until Theorem 3.14, we only assume (A1)-(A5). In addition, we assume

(B1) $\partial_t f(x, y, t, 0) = \partial_t g(x, y, t, 0) = 0$,

(B2) $f(0, 0, t, \epsilon) = g(0, 0, t, \epsilon) = 0$,

(B3) (Df, Dg) are equicontinuous functions in x, y and ϵ with respect to t at $x = 0, y = 0, \epsilon = 0$, i.e. for any $s > 0$, there exists $\delta > 0$ such that if $|x|_{X_1} < \delta, |y|_{Y_1} < \delta, |\epsilon| < \delta$, for any $t \in \mathbb{R}$,

$$\big|Df(x, y, t, \epsilon) - Df(0, 0, t, 0)\big|_{L(X_1 \times Y_1, X_1), L(X \times Y, X)} < s,$$
$$\big|Dg(x, y, t, \epsilon) - Dg(0, 0, t, 0)\big|_{L(X_1 \times Y_1, Y_1), L(X \times Y, Y)} < s.$$

We will write $D_{x,y} f(0, 0, t, 0)$ as $f_{x,y}$, and such notation is also applied to g.

For $(x, y) \in X_1 \times Y_1$, let

$$\begin{aligned} F_1(x, y, t, \epsilon) &= f(x, y, t, \epsilon) - f_x x - f_y y \\ G_1(x, y, t, \epsilon) &= g(x, y, t, \epsilon) - g_x x - g_y y \end{aligned} \tag{3.1}$$

and $\lambda(r) \in C_c^\infty(\mathbb{R})$ such that

$$\lambda(r) = \begin{cases} 1, & r < \dfrac{1}{3} \\ 0, & r > 1 \end{cases}, \qquad |\lambda'(r)| \leq 3.$$

Let

$$\begin{aligned} F(x,y,t,\epsilon) &= \lambda(\frac{|x|_{X_1} + |y|_{Y_1}}{r}) F_1(x,y,t,\epsilon), \\ G(x,y,t,\epsilon) &= \lambda(\frac{|x|_{X_1} + |y|_{Y_1}}{r}) G_1(x,y,t,\epsilon), \end{aligned} \tag{3.2}$$

then by assumption (A4), (A6) and (B2), F and G satisfy:

$$\begin{aligned} &F(0,0,t,\epsilon) = G(0,0,t,\epsilon) = 0, \\ &|F(x,y,t,\epsilon)|_{X_1} \leq \overline{r}r, \ \ |G(x,y,t,\epsilon)|_{Y_1} \leq \overline{r}r, \\ &|D_xF(x,y,t,\epsilon)|_{L(X_1,X_1)} \leq \overline{r}, \ \ |D_xG(x,y,t,\epsilon)|_{L(X_1,Y_1)} \leq \overline{r}, \\ &|D_xF(x,y,t,\epsilon)|_{L(X,X)} \leq \overline{r}, \ \ |D_xG(x,y,t,\epsilon)|_{L(X,Y)} \leq \overline{r}, \\ &|D_yF(x,y,t,\epsilon)|_{L(Y_1,X_1)} \leq \overline{r}, \ \ |D_yG(x,y,t,\epsilon)|_{L(Y_1,Y_1)} \leq \overline{r}, \\ &|D_yF(x,y,t,\epsilon)|_{L(Y,X)} \leq \overline{r}, \ \ |D_yG(x,y,t,\epsilon)|_{L(Y,Y)} \leq \overline{r}, \end{aligned} \tag{3.3}$$

where $\overline{r} = \overline{r}(r,\epsilon)$ is given by the product of $\widetilde{r}$ and a constant C depending only on C_0, and

$$\begin{aligned} \widetilde{r} &= \widetilde{r}(r,\epsilon_0) \\ &= \sup_{\substack{|x|_{X_1}+|y|_{Y_1} \leq r, \\ \epsilon' \in [0,\epsilon_0)}} \left\{ \begin{array}{l} |D_xF_1(x,y,t,\epsilon')|_{L(X_1,X_1)}, |D_yF_1(x,y,t,\epsilon')|_{L(X_1,Y_1)}, \\ |D_xG_1(x,y,t,\epsilon')|_{L(Y_1,X_1)}, |D_yG_1(x,y,t,\epsilon')|_{L(Y_1,Y_1)}, \\ |D_xF_1(x,y,t,\epsilon')|_{L(X,X)}, |D_yF_1(x,y,t,\epsilon')|_{L(X,Y)}, \\ |D_xG_1(x,y,t,\epsilon')|_{L(Y,X)}, |D_yG_1(x,y,t,\epsilon')|_{L(Y,Y)} \end{array} \right\}. \end{aligned} \tag{3.4}$$

Clearly, (A4) implies $\lim\limits_{r,\epsilon_0 \to 0} \overline{r} = 0$.

Define $A_f = A + f_x$, we will construct the global invariant manifold for the modified system:

$$\begin{cases} \dot{x}(t) = A_f x + F(x,y,t,\epsilon) + f_y y \\ \dot{y}(t) = (\dfrac{J}{\epsilon} + g_y) y + G(x,y,t,\epsilon) + g_x x \end{cases} \tag{3.5}$$

We further assume

(B4) There exists a pair of continuous projections (P_s, P_{cu}) on X, such that $P_s + P_{cu} = I_X$ [1], clearly $X = P_s X \times P_{cu} X$ and

$$P_{s,cu}X \quad \text{are positively invariant under} \quad e^{tA_f},$$
$$e^{tA_f} \quad \text{can be extended to a group on} \quad P_{cu}X.$$

(B5) There exist constants $a_1 < 0$ and $a_1 < a_2$, such that

$$|e^{tA_f} P_s x|_X \leq K e^{a_1 t}|x|_X \quad \text{for} \quad t \geq 0, \quad x \in X,$$
$$|e^{tA_f} P_{cu} x|_X \leq K e^{a_2 t}|x|_X \quad \text{for} \quad t \leq 0, \quad x \in X,$$
$$|e^{t(\frac{J}{\epsilon}+g_y)} y|_Y \leq K e^{a_2 t}|y|_Y \quad \text{for} \quad t \leq 0, \quad y \in Y.$$

Remark 3.1. *Let $P_s X_1 = X_1^s$ and $P_{cu} X_1 = X_1^{cu}$, (B4) and (B5) imply e^{tA_f} and $e^{t(\frac{J}{\epsilon}+g_y)}$ satisfy the same estimates with all norms replaced by $|\cdot|_{X_1}$ and $|\cdot|_{Y_1}$, respectively.*

Motivated by the exponential dichotomy of the linear part of (3.5), we construct the center-unstable manifold. More precisely, we look for a submanifold,

$$\mathcal{M}_\epsilon^{cu} = \left\{(\xi_{cu}, \xi_y, t_0) + h_s(\xi_{cu}, \xi_y, t_0, \epsilon)\middle|(\xi_{cu}, \xi_y, t_0) \in (X_1^{cu} \times Y_1 \times \mathbb{R})\right\}$$

in $X_1 \times Y_1 \times \mathbb{R}$, which is positively invariant under the augmented flow on $X_1 \times Y_1 \times \mathbb{R}$, where the last dimension correspond to t. Moreover, we expect that

- Every point $p \in \mathcal{M}_\epsilon^{cu}$ has a backward solution $\big(X(t), Y(t)\big) \in \mathcal{M}_\epsilon^{cu}$ for $t \leq t_0$ with $(X(t_0), Y(t_0), t_0) = p$.[2]

[1]Throughout this manuscript, we will use notations I_X, I_Y for identity maps on X, Y, respectively. With slight abuse of notation, we also use the projections P_s and P_{cu} to denote their composition with the projection from $X \times Y$ to X.

[2]To avoid confusion, we remark that for the rest of this manuscript, we will use $x(\cdot)$ and $y(\cdot)$ to denote an element in certain Banach spaces. And we use $X(\cdot)$ and $Y(\cdot)$ to denote the solution of (3.5).

- $\big(|X(t)|_{X_1}+|Y(t)|_{Y_1}\big)\le Ce^{\eta t}$ for some $\eta\in(a_1,a_2)$ and $t\le t_0$.

In general, let $\eta\in\mathbb{R}$, Z be a Banach space, we introduce the following space:

$$C_\eta^-(Z)=\Big\{z(t)\in Z\Big|\sup_{t\le 0}e^{-\eta t}|z(t)|_Z<+\infty, z(t) \text{ is continuous}\Big\}$$

with norm $|\cdot|^-_{\eta,\epsilon_\star,Z}$, where $\epsilon_\star$ is a parameter

$$|z(\cdot)|^-_{\eta,\epsilon_\star,Z}=\sup_{t\le 0}e^{-\eta t}\frac{|z(t)|_Z}{\epsilon_\star}.$$

Let

$$\begin{aligned}B_\eta^-(\rho)=\Big\{\big(x,y\big)\in &C_\eta^-(X_1)\times C_\eta^-(Y_1)\Big|\\ &\big|(x,y)\big|^-_{\eta,\epsilon_\star}=|x|^-_{\eta,1,X_1}+|y|^-_{\eta,\epsilon_\star,Y_1}+|\dot{x}|^-_{\eta,1,X}<\rho\Big\},\end{aligned}\tag{3.6}$$

where we also use $B_\eta^-(\infty)$ to denote the corresponding linear space.

For any $\eta\in(a_1,a_2)$, there exists $\underline{\epsilon}>0$ such that for any $\epsilon_\star\in[0,\underline{\epsilon})$, there exist $r_0,\epsilon_0>0$ satisfying that for any $\epsilon\in[0,\epsilon_0)$ and $r\in(0,r_0)$, it holds

$$\sigma(\eta)=\min\{\sigma_1,\sigma_2(\eta),\sigma_3(\eta)\}>0,\tag{3.7}$$

where

$$\begin{aligned}\sigma_1&=\frac{1}{2}-C_0^2\epsilon,\\ \sigma_2(\eta)&=1-3(\frac{K}{a_2-\eta}+\frac{K}{\eta-a_1}+1)\big(\overline{r}+C_0\epsilon_\star\big),\\ \sigma_3(\eta)&=1-3\frac{(\frac{K}{a_2-\eta}+K+1)\big(\overline{r}+2C_0^2\epsilon\big)}{\epsilon_\star}.\end{aligned}$$

In the rest of this whole chapter, we always assume (3.7).

3.2 Construction of local invariant manifolds

In order to construct the invariant center-unstable manifold, we start with solving the following integral equation by contraction mapping principle. Intuitively, these integral equations are derived by imposing appropriate decay conditions at $t=-\infty$ in the variation of parameter formula and their solutions after certain time translation will be verified to satisfy (3.5).

To simplify notation, we write $F(x(\tau), y(\tau), \tau+t_0, \epsilon) = F(x, y, \tau+t_0, \epsilon)$, $f_y y(\tau) = f_y y$ and such notation also applies to G and g.

Consider the following integral equations with parameters $(\xi_{cu}, \xi_y, t_0) \in (X_1^{cu} \times Y_1 \times \mathbb{R})$ for $x \in C_\eta^-(X_1)$ and $y \in C_\eta^-(Y_1)$:

$$\begin{cases} x(t) = e^{tA_f}\xi_{cu} + \int_0^t e^{(t-\tau)A_f} P_{cu}\big(F(x, y, \tau+t_0, \epsilon) + f_y y\big)\, d\tau \\ \qquad + \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(F(x, y, \tau+t_0, \epsilon) + f_y y\big)\, d\tau, \\ y(t) = e^{t(\frac{J}{\epsilon}+g_y)}\xi_y + \int_0^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} G(x, y, \tau+t_0, \epsilon)\, d\tau \\ \qquad + \int_0^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} g_x x\, d\tau. \end{cases} \tag{3.8}$$

If (x, y) satisfy (3.8), one can compute

$$\begin{aligned} &\frac{d}{dt}\int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(F(x, y, \tau+t_0, \epsilon) + f_y y\big)\, d\tau \\ =& \lim_{h\to 0}\frac{1}{h}\Big(\int_{-\infty}^{t+h} e^{(t+h-\tau)A_f} P_s\big(F(x, y, \tau+t_0, \epsilon) + f_y y\big)\, d\tau \\ &\qquad - \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(F(x, y, \tau+t_0, \epsilon) + f_y y\big)\, d\tau\Big) \\ =& \lim_{h\to 0}\frac{1}{h}\int_{-\infty}^t e^{(t-\tau)A_f}(e^{hA_f} - I) P_s\big(F(x, y, \tau+t_0, \epsilon) + f_y y\big)\, d\tau \\ &+ \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} e^{(t+h-\tau)A_f} P_s\big(F(x, y, \tau+t_0, \epsilon) + f_y y\big)\, d\tau \\ =& A_f \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(F(x, y, \tau+t_0, \epsilon) + f_y y\big)\, d\tau \\ &+ P_s\big(F(x(t), y(t), t+t_0, \epsilon) + f_y y(t)\big). \end{aligned}$$

The last equality follows from the facts that $e^{tA_f}x$ is a continuous function in t and $(x(t), y(t)) \in X_1 \times Y_1$, which implies the integrand is in X_1 by (A4). Here we also use the assumption $x(t) \in C_\eta^-(X_1)$ and $y(t) \in C_\eta^-(Y_1)$ to guarantee the convergence of the improper integrals.

Similarly,

$$\begin{aligned}
&\frac{d}{dt}\int_0^t e^{(t-\tau)A_f}P_{cu}\big(F(x,y,\tau+t_0,\epsilon)+f_y y\big)\ d\tau\\
&= A_f\int_0^t e^{(t-\tau)A_f}P_{cu}\big(F(x,y,\tau+t_0,\epsilon)+f_y y\big)\ d\tau\\
&\quad +P_{cu}\big(F(x(t),y(t),t+t_0,\epsilon)+f_y y(t)\big),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int_0^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}\big(G(x,y,\tau+t_0,\epsilon)+g_x x\big)\ d\tau\\
&= (\frac{J}{\epsilon}+g_y)\int_0^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}\big(G(x,y,\tau+t_0,\epsilon)+g_x x\big)\ d\tau\\
&\quad +G(x(t),y(t),t+t_0,\epsilon)+g_x x(t).
\end{aligned}$$

Now, let $(X(\cdot+t_0),Y(\cdot+t_0))=(x(\cdot),y(\cdot))$, then from the above computation, we have $(X(\cdot+t_0),Y(\cdot+t_0))$ satisfy (3.5) with $(P_{cu}X(t_0),Y(t_0))=(\xi_{cu},\xi_y)$. Therefore, we prove that after a time translation a solution of (3.8) in $C_\eta^-(X_1)\times C_\eta^-(Y_1)$ is a solution of (3.5).

On the other hand, if $(X(t+t_0),Y(t+t_0))=(x(t),y(t))\in C_\eta^-(X_1)\times C_\eta^-(Y_1)$ satisfy (3.5) with $(x(0),y(0))=\xi_0=(\xi_{cu},\xi_y,\xi_s)$, by variation of parameters formula, we get

$$\begin{aligned}
P_{cu}x(t) &= e^{tA_f}\xi_{cu}+\int_0^t e^{(t-\tau)A_f}P_{cu}\big(F(x,y,\tau+t_0,\epsilon)\\
&\quad +f_y y\big)\ d\tau,\\
P_s x(t) &= e^{(t-t_1)A_f}P_s x(t_1)+\int_{t_1}^t e^{(t-\tau)A_f}P_s\big(F(x,y,\tau+t_0,\epsilon)\\
&\quad +f_y y\big)\ d\tau,\\
y(t) &= e^{t(\frac{J}{\epsilon}+g_y)}\xi_y+\int_0^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}\big(G(x,y,\tau+t_0,\epsilon)\\
&\quad +g_x x\big)\ d\tau.
\end{aligned}$$

for any $t_1<0$.

Since $P_s x(\cdot) \in C_\eta^-(X_1)$ and by the exponential dichotomy,

$$\lim_{t_1 \to -\infty} \left| e^{(t-t_1)A_f} P_s x(t_1) \right|_{X_1} = 0.$$

Consequently,

$$\begin{aligned} P_s x(t) &= \lim_{t_1 \to -\infty} \Big[e^{(t-t_1)A_f} P_s x(t_1) + \int_{t_1}^t e^{(t-\tau)A_f} P_s \big(F(x, y, \tau + t_0, \epsilon) \\ &\quad + f_y y \big) \, d\tau \Big] \\ &= \int_{-\infty}^t e^{(t-\tau)A_f} P_s \big(F(x, y, \tau + t_0, \epsilon) + f_y y \big) \, d\tau. \end{aligned}$$

Therefore, (x, y) satisfy (3.8) with parameters (ξ_{cu}, ξ_y).

Let $U(t, \epsilon) = \begin{pmatrix} e^{tA_f} & 0 \\ 0 & e^{t(\frac{J}{\epsilon} + g_y)} \end{pmatrix}$, and for any $(x, y) \in B_\eta^-(\rho)$ and $\xi = (\xi_{cu}, \xi_y) \in X_1^{cu} \times Y_1$,

$$\begin{aligned} \mathscr{T}_{cu}(x, y, \xi, t_0, \epsilon)(t) &= U(t, \epsilon)\xi \\ &\quad + \int_0^t U(t-\tau, \epsilon) \begin{pmatrix} P_{cu} \Big(F(x, y, \tau + t_0, \epsilon) + f_y y \Big) \\ G(x, y, \tau + t_0, \epsilon) + g_x x \end{pmatrix} \, d\tau \\ &\quad + \int_{-\infty}^t U(t-\tau, \epsilon) \begin{pmatrix} P_s \Big(F(x, y, \tau + t_0, \epsilon) + f_y y \Big) \\ 0 \end{pmatrix} \, d\tau. \end{aligned}$$

Lemma 3.2. *For any η with $a_1 < \eta < a_2$ and $\epsilon_\star, r, \epsilon_0$ satisfy* (3.7)*, there exists ρ_0 depending on $|\xi_{cu}|_{X_1}, |\xi_y|_{Y_1}, K, \epsilon_\star, \sigma$, such that for any $\epsilon \in [0, \epsilon_0)$ and $\rho \in [\rho_0, \infty]$, $\mathscr{T}_{cu}$ defines a contraction mapping on $B_\eta^-(\rho)$ under the norm $|\cdot|_{\eta,\epsilon_\star}^-$. Here σ is defined in* (3.7)*.*

Proof. For any $(\xi_{cu}, \xi_y, t_0, \epsilon) \in (X_1^{cu} \times Y_1 \times \mathbb{R}^2)$, since they are fixed, we will skip them in $\mathscr{T}_{cu}$. It is easy to obtain from the definition,

$$\begin{aligned} &\sup_{t \le 0} e^{-\eta t} \Big| (P_s + P_{cu}) \mathscr{T}_{cu}(x, y)(t) \Big|_{X_1} \\ &\le K |\xi_{cu}|_{X_1} + \big(\frac{K}{a_2 - \eta} + \frac{K}{\eta - a_1} \big) \big(|DF|_{C^0} + \epsilon_\star |f_y| \big) \big| (x, y) \big|_{\eta, \epsilon_\star}^-. \end{aligned}$$

Since $(x, y) \in C_\eta^-(X_1) \times C_\eta^-(Y_1)$ and A_f is a closed operator, one can verify $(P_s + P_{cu})\mathscr{T}_{cu}(x, y, \xi, t_0, \epsilon) \in C^1\big((-\infty, 0), X\big)$ and

$$
\begin{aligned}
&(P_s + P_{cu})\frac{d}{dt}\mathscr{T}_{cu}(x, y)(t) \\
=&A_f e^{tA_f}\xi_{cu} + F(x(t), y(t), t + t_0, \epsilon) + f_y y(t) \\
&+ A_f \int_0^t e^{(t-\tau)A_f} P_{cu}\big(F(x, y, \tau + t_0, \epsilon) + f_y y\big)\, d\tau \\
&+ A_f \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(F(x, y, \tau + t_0, \epsilon) + f_y y\big)\, d\tau,
\end{aligned}
\tag{3.9}
$$

Therefore,

$$
\begin{aligned}
&\sup_{t\le 0} e^{-\eta t}\Big|(P_s + P_{cu})\frac{d}{dt}\mathscr{T}_{cu}(x, y)(t)\Big|_X \\
\le & K|\xi_{cu}|_{X_1} + \big(\frac{K}{a_2 - \eta} + \frac{K}{\eta - a_1} + 1\big)\big(|DF|_{C^0} + \epsilon_\star|f_y|\big)\big|(x, y)\big|^-_{\eta,\epsilon_\star}.
\end{aligned}
$$

Again, by the definition of $\mathscr{T}_{cu}$,

$$
\begin{aligned}
&(I - P_s - P_{cu})\mathscr{T}_{cu}(x, y)(t) \\
=& e^{t(\frac{J}{\epsilon}+g_y)}\xi_y + \int_0^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}\Big(G(x, y, \tau + t_0, \epsilon) + g_x x\Big)\, d\tau \\
=& e^{t(\frac{J}{\epsilon}+g_y)}\xi_y + \int_0^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} G(x, y, \tau + t_0, \epsilon)\, d\tau \\
&- e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}(\frac{J}{\epsilon} + g_y)^{-1} g_x x\Big|_0^t + \int_0^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}(\frac{J}{\epsilon} + g_y)^{-1} g_x \dot{x}\, d\tau.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
&\sup_{t\le 0}\frac{1}{\epsilon_\star} e^{-\eta t}\Big|(I - P_s - P_{cu})\mathscr{T}_{cu}(x, y)(t)\Big|_{Y_1} \\
\le & \frac{K|\xi_y|_{Y_1}}{\epsilon_\star} + (\frac{K}{a_2 - \eta} + K + 1)\frac{|DG|_{C^0} + 2\epsilon|J^{-1}||g_x|}{\epsilon_\star}\big|(x, y)\big|^-_{\eta,\epsilon_\star}.
\end{aligned}
$$

Using (3.7), clearly, there exists $\rho_0 > 0$ determined by $|\xi_y|_{Y_1}, |\xi_{cu}|_{X_1}, K, \epsilon_\star$ and σ such that for any $\rho \in (\rho_0, +\infty]$, the above three inequalities imply that $\mathscr{T}_{cu}$ maps $B_\eta^-(\rho)$

to $B_\eta^-(\rho)$. To prove it is a contraction, we can estimate in a similar fashion

$$
\begin{aligned}
&\sup_{t\le 0} e^{-\eta t}\Big|(P_s+P_{cu})\big(\mathscr{T}_{cu}(x_1,y_1)-\mathscr{T}_{cu}(x_2,y_2)\big)(t)\Big|_{X_1}\\
\le&(\frac{K}{a_2-\eta}+\frac{K}{\eta-a_1})(|DF|_{C^0}+\epsilon_\star|f_y|)\times\\
&(|x_1-x_2|^-_{\eta,1,X_1}+|y_1-y_2|^-_{\eta,\epsilon_\star,Y_1}),\\
&\sup_{t\le 0} e^{-\eta t}\Big|(P_s+P_{cu})\frac{d}{dt}\big(\mathscr{T}_{cu}(x_1,y_1)-\mathscr{T}_{cu}(x_2,y_2)\big)(t)\Big|_{X}\\
\le&\big(\frac{K}{a_2-\eta}+\frac{K}{\eta-a_1}+1\big)\big(|DF|_{C^0}+\epsilon_\star|f_y|\big)\times\\
&\big(|x_1-x_2|^-_{\eta,1,X_1}+|y_1-y_2|^-_{\eta,\epsilon_\star,Y_1}\big),\\
&\sup_{t\le 0} e^{-\eta t}\Big|(I-P_s-P_{cu})\big(\mathscr{T}_{cu}(x_1,y_1)-\mathscr{T}_{cu}(x_2,y_2)\big)(t)\Big|_{Y_1}\\
\le&\big(\frac{K}{a_2-\eta}+K+1\big)\frac{|DG|_{C^0}+2\epsilon|J^{-1}||g_x|}{\epsilon_\star}\times\\
&\big(|x_1-x_2|^-_{\eta,1,X_1}+|\dot{x}_1-\dot{x}_2|^-_{\eta,1,X}+|y_1-y_2|^-_{\eta,\epsilon_\star,Y_1}\big).
\end{aligned}
\tag{3.10}
$$

Therefore, $\mathscr{T}_{cu}$ defines a contraction mapping from $B_\eta^-(\rho)$ to itself under the norm $|\cdot|^-_{\eta,\epsilon_\star}$. □

For any $(\xi,t_0,\epsilon)\in X_1^{cu}\times Y_1\times\mathbb{R}^2$, let $\big(x(t),y(t)\big)$ be the fixed point of $\mathscr{T}_{cu}$, and

$$
h_s(\xi,t_0,\epsilon)=P_sx(0)=\int_{-\infty}^0 e^{-\tau A_f}P_s\Big(F\left(x,y,\tau+t_0,\epsilon\right)+f_yy\Big)\,d\tau, \tag{3.11}
$$

$$
\mathcal{M}_\epsilon^{cu}=\big\{(\xi,t_0)+h_s(\xi,t_0,\epsilon)\big|(\xi,t_0)\in X_1^{cu}\times Y_1\times\mathbb{R}\big\},
$$

where $\mathcal{M}_\epsilon^{cu}$ is independent of $\eta\in(a_1,a_2)$. We will prove $\mathcal{M}_\epsilon^{cu}$ is invariant under the augmented flow defined by (3.5).

Theorem 3.3. *For any $(\xi,t_0)\in\mathcal{M}_\epsilon^{cu}$, its unique solution $(X(t),Y(t)),t\ge t_0$ of (3.5) belongs to $\mathcal{M}_\epsilon^{cu}$. Moreover, $(X(t),Y(t))\in\mathcal{M}_\epsilon^{cu}$ is also a backward solution for $t\le t_0$.*

Proof. Let $(\xi_0,t_0)=(\xi_{cu},\xi_y,h_s(\xi_{cu},\xi_y,t_0,\epsilon),t_0)\in\mathcal{M}_\epsilon^{cu}$, and (x,y) satisfy

$$
(x(t),y(t))=\mathscr{T}_{cu}(x,y,\xi,t_0,\epsilon)(t)
$$

with parameter (ξ_{cu}, ξ_y). By definition, $(X(t), Y(t)) = (x(t-t_0), y(t-t_0))$ is a solution of (3.5).

If $t_1 \le t_0$, we need to show that $(x(t_1 - t_0), y(t_1 - t_0), t_1) \in \mathcal{M}_\epsilon^{cu}$. Let $\xi' = (\xi'_{cu}, \xi'_y) = (P_{cu}x(t_1 - t_0), y(t_1 - t_0))$, we have

$$\begin{aligned} x(t) &= e^{tA_f}\xi_{cu} + \int_0^t e^{(t-\tau)A_f} P_{cu}\big(F(x, y, \tau + t_0, \epsilon) + f_y y\big)\, d\tau \\ &\quad + \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(F(x, y, \tau + t_0, \epsilon) + f_y y\big)\, d\tau \\ &= e^{(t-(t_1-t_0))A_f}\Big(e^{(t_1-t_0)A_f}\xi_{cu} \\ &\quad + \int_0^{t_1-t_0} e^{(t_1-t_0-\tau)A_f} P_{cu}\big(F(x, y, \tau + t_0, \epsilon) + f_y y\big)\, d\tau\Big) \\ &\quad + \int_{t_1-t_0}^t e^{(t-\tau)A_f} P_{cu}\big(F(x, y, \tau + t_0, \epsilon) + f_y y\big)\, d\tau \\ &\quad + \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(F(x, y, \tau + t_0, \epsilon) + f_y y\big)\, d\tau \\ &= e^{(t-(t_1-t_0))A_f}\xi'_{cu} + \int_{t_1-t_0}^t e^{(t-\tau)A_f} P_{cu}\big(F(x, y, \tau + t_0, \epsilon) + f_y y\big)\, d\tau \\ &\quad + \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(F(x, y, \tau + t_0, \epsilon) + f_y y\big)\, d\tau, \end{aligned}$$

and similarly,

$$\begin{aligned} y(t) &= e^{(t-(t_1-t_0))(\frac{J}{\epsilon}+g_y)}\xi'_y + \int_{t_1-t_0}^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} G(x, y, \tau + t_0, \epsilon)\, d\tau \\ &\quad + \int_{t_1-t_0}^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} g_x x\, d\tau. \end{aligned}$$

Let $t' = t - (t_1 - t_0)$ and $\big(\widetilde{x}(t'), \widetilde{y}(t')\big) = \big(x(t' + t_1 - t_0), y(t' + t_1 - t_0)\big)$. Plugging $(\widetilde{x}, \widetilde{y})$ into the above equalities and make some change of variables, we find that $(\widetilde{x}, \widetilde{y}) = \mathscr{T}_{cu}(\widetilde{x}, \widetilde{y}, \xi', t_1, \epsilon)$. By the definition of $\mathcal{M}_\epsilon^{cu}$, we have $(x(t_1 - t_0), y(t_1 - t_0)) \in \mathcal{M}_\epsilon^{cu}$.

For $t_1 \ge t_0$, since $\begin{pmatrix} F(x, y, t, \epsilon) + f_y y \\ G(x, y, t, \epsilon) + g_x x \end{pmatrix}$ as a mapping from $X_1 \times Y_1$ to itself has a globally Lipschitz constant independent of t, the solution $(X(t), Y(t))$ of (3.5) is defined for all $t \ge t_0$. Moreover, let

$$W(t) = e^{-\eta t}\Big(\big|X(t)\big|_{X_1} + \frac{\big|Y(t)\big|_{Y_1}}{\epsilon_\star} + \big|X(t)\big|_X\Big),$$

we have

$$\sup_{t\le t_1} W(t) \le \max\big\{\sup_{t_0\le t\le t_1} W(t), \sup_{t\le t_0} W(t)\big\} < +\infty,$$

which follows that $(\widetilde{x},\widetilde{y}) \in C_\eta^-(X_1)\times C_\eta^-(Y_1)$. Clearly,

$$(\widetilde{x}(0),\widetilde{y}(0)) = (X(t_1),Y(t_1)) \in \mathcal{M}_\epsilon^{cu}.$$

□

Remark 3.4. *Note that $X_1^{cu}\times Y_1 \neq T_0\mathcal{M}_\epsilon^{cu}$, we cannot prove h_s is bounded.*

By the exponential dichotomy, we can also construct the stable integral manifold. For a Banach space Z, we use $C_\eta^+(Z)$ to denote

$$C_\eta^+(Z) = \Big\{z(t)\in Z \big| \sup_{t\ge 0} e^{-\eta t}|z(t)|_Z < \infty, z(t) \text{ is continuous}\Big\},$$

We use $|\cdot|^+_{\eta,\epsilon_\star,Z}$ to denote the norm in $C_\eta^+(Z)$, where $\epsilon_\star$ is a parameter and

$$|z|^+_{\eta,\epsilon_\star,Z} = \sup_{t\ge 0} e^{-\eta t}\frac{|z(t)|_Z}{\epsilon_\star}.$$

Similar to $B_\eta^-(\rho)$ introduced in (3.6), let

$$\begin{aligned} B_\eta^+(\rho) = \Big\{\big(x,y\big) \in & C_\eta^+(X_1)\times C_\eta^+(Y_1)\Big| \\ & \big|(x,y)\big|^+_{\eta,\epsilon_\star} = |x|^+_{\eta,1,X_1} + |y|^+_{\eta,\epsilon_\star,Y_1} + |\dot{x}|^+_{\eta,1,X} < \rho\Big\}, \end{aligned} \tag{3.12}$$

where we also use $B_\eta^+(\infty)$ to denote the corresponding linear space.

For $\xi_s \in X_1^s$ and $(x,y)\in B_\eta^+(\infty)$, we define

$$\begin{aligned} &\mathscr{T}_s(x,y,\xi_s,t_0,\epsilon)(t) \\ =&U(t,\epsilon)\xi_s + \int_0^t U(t-\tau,\epsilon)\begin{pmatrix} P_s\big(F_1(x,y,\tau+t_0,\epsilon)+f_y y\big) \\ 0 \end{pmatrix} d\tau \\ &+\int_{+\infty}^t U(t-\tau,\epsilon)\begin{pmatrix} P_{cu}\big(F_1(x,y,\tau+t_0,\epsilon)+f_y y\big) \\ G_1(x,y,\tau+t_0,\epsilon)+g_x x \end{pmatrix} d\tau, \end{aligned} \tag{3.13}$$

where F_1, G_1 are introduced in (3.1). One may note that we do not cut off F_1, G_1 in $\mathscr{T}_s$. It is due to the following fact. To construct the stable integral manifold, we need $\eta < 0$, which makes the functions in $B^+_\eta(\infty)$ exponentially decay as $t \to +\infty$. If we choose $|\xi_s|_{X_1}$ to be sufficiently small, we can work on sufficiently small ball in $B^+_\eta(\infty)$. Consequently, DF_1, DG_1 will be automatically small by continuity. This subtle difference would imply the local stable integral manifold for (2.1) is unique, which is not true for the local center-unstable integral manifold.

Following the same proof as in Lemma 3.2, one can see that if (3.7) is satisfied, $\mathscr{T}_s$ defines a contraction on $B^+_\eta(\rho)$, where ρ is sufficiently small, under the norm $|\cdot|^+_{\eta,\epsilon_\star}$ which is given in (3.12). Let (x_s, y_s) be the fixed point of $\mathscr{T}_s$ with parameters (ξ_s, t_0, ϵ). Define

$$\mathcal{M}^s_\epsilon = \big\{(\xi_s, t_0) + h_{cu}(\xi_s, t_0, \epsilon)\big|(\xi_s, t_0, \epsilon) \in (X^s_1 \times \mathbb{R}^2)\big\},$$

where

$$\begin{aligned} h_{cu}(\xi_s, t_0, \epsilon) =& (I - P_s)(x_s(0), y_s(0)) \\ =& \int_{+\infty}^0 U(-\tau, \epsilon) \begin{pmatrix} P_{cu}\big(F_1(x_s, y_s, \tau + t_0, \epsilon) + f_y y_s\big) \\ G_1(x_s, y_s, \tau + t_0, \epsilon) + g_x x_s \end{pmatrix} d\tau. \end{aligned} \tag{3.14}$$

In fact, all the previous estimates apply to $\mathscr{T}_s$. Hence one may go through the same procedure to prove the existence of the stable integral manifold.

Theorem 3.5. *Let r_s be a positive real number and assume $\epsilon, \epsilon_\star$ and r_s are sufficiently small. There exists ρ depending only on $\epsilon, \epsilon_\star, r_s$ such that for each ξ_s with $|\xi_s|_{X_1} \le r_s$, $\mathscr{T}_s$ which is defined in* (3.13) *has a unique solution on $B^+_\eta(\rho)$. Therefore, we have the unique stable integral manifold $\mathcal{M}^s_\epsilon$ in a neighborhood of the origin.*

Now, we will state the results on center-stable integral manifold. To do that, we need the following assumption and notation.

(C1) There exists a pair of continuous projections (P_{cs}, P_u) on X, such that $P_{cs} + P_u = I_X$ and $P_{cs,u}X$ are positively invariant under e^{tA_f}.

(C2) There exist constants $a_2' > 0$, and $a_1' < a_2'$,

$$|e^{tA_f}P_{cs}x|_X \le Ke^{a_1't}|x|_X \quad \text{for} \quad t \ge 0, \quad x \in X,$$
$$|e^{tA_f}P_u x|_X \le Ke^{a_2't}|x|_X \quad \text{for} \quad t \le 0, \quad x \in X,$$
$$|e^{t(\frac{J}{\epsilon}+g_y)}y|_Y \le Ke^{a_1't}|y|_Y \quad \text{for} \quad t \le 0, \quad y \in Y.$$

Compared with (B4) and (B5), we know that e^{tA_f} and $e^{t(\frac{J}{\epsilon}+g_y)}$ satisfy the same estimates with norms replaced by $|\cdot|_{X_1}$ and $|\cdot|_{Y_1}$.

Let $P_{cs}X_1 = X_1^{cs}$ and $P_u X_1 = X_1^u$. For any $\xi = (\xi_{cs}, \xi_y) \in X_1^{cs} \times Y_1$ and $(x, y) \in B_\eta^+(\rho)$ define

$$\begin{aligned}
&\mathscr{T}_{cs}(x, y, \xi, t_0, \epsilon)(t) \\
=&U(t,\epsilon)\xi + \int_0^t U(t-\tau,\epsilon) \begin{pmatrix} P_{cs}\Big(F(x,y,\tau+t_0,\epsilon) + f_y y\Big) \\ G(x,y,\tau+t_0,\epsilon) + g_x x \end{pmatrix} d\tau \\
&+ \int_{+\infty}^t U(t-\tau,\epsilon) \begin{pmatrix} P_u\Big(F(x,y,\tau+t_0,\epsilon) + f_y y\Big) \\ 0 \end{pmatrix} d\tau.
\end{aligned} \tag{3.15}$$

It's not hard to see that $\mathscr{T}_{cs}$ satisfies estimates similar to $\mathscr{T}_{cu}$. Let $(x(t), y(t))$ be the fixed point of $\mathscr{T}_{cs}(\cdot, \cdot, \xi, t_0, \epsilon)(t)$, and define

$$\begin{aligned}
&h_u(\xi, t_0, \epsilon) = P_u x(0) = \int_{+\infty}^0 e^{-\tau A_f} P_u\big(F(x,y,\tau+t_0,\epsilon) + f_y y\big)\, d\tau, \\
&\mathcal{M}_\epsilon^{cs} = \Big\{(\xi, t_0) + h_u(\xi, t_0, \epsilon)\big|(\xi, t_0) \in (X_1^{cs} \times Y_1 \times \mathbb{R})\Big\}.
\end{aligned}$$

Theorem 3.6. *Assume (A1)—(A5) for $k = 1$, (B1)—(B3) and (C1)—(C2). For sufficiently small $\overline{r}, \epsilon_\star, \epsilon$, there exists a center-stable integral manifold $\mathcal{M}_\epsilon^{cs}$.*

By taking the intersection of $\mathcal{M}_\epsilon^{cu}$ and $\mathcal{M}_\epsilon^{cs}$, we obtain a center manifold. Let $X_1^c = X_1^{cu} \cap X_1^{cs} = (I_X - P_s - P_u)X_1 \triangleq P_c X_1$.

Theorem 3.7. *Assume (A1)—(A5) for $k = 1$, (B1)—(B5) and (C1)—(C2). For sufficiently small $\overline{r}, \epsilon_\star, \epsilon$, there exists a center manifold $\mathcal{M}_\epsilon^\epsilon$ which is parameterized by*

(ξ_c, ξ_y, t_0) *such that*

$$\mathcal{M}^c_\epsilon = \left\{ (\xi_c, \xi_y, t_0) + (\Psi_s + \Psi_u)(\xi_c, \xi_y, t_0, \epsilon) \middle| (\xi_c, \xi_y, t_0) \in X^c_1 \times Y_1 \times \mathbb{R} \right\}.$$

Moreover, Ψ^s *and* Ψ^u *satisfy*

$$\begin{aligned} h_s(\Psi_u(\xi_c, \xi_y, t_0, \epsilon), \xi_c, \xi_y, t_0, \epsilon) =& \Psi_s(\xi_c, \xi_y, t_0, \epsilon), \\ h_u(\Psi_s(\xi_c, \xi_y, t_0, \epsilon), \xi_c, \xi_y, t_0, \epsilon) =& \Psi_u(\xi_c, \xi_y, t_0, \epsilon). \end{aligned} \tag{3.16}$$

Proof. Since we fix (t_0, ϵ), we will suppress the arguments in h_s and h_u. To construct the center manifold, it's equivalent to solve the following system:

$$\begin{aligned} \xi_u =& \quad h_u(\xi_s, \xi_c, \xi_y), \\ \xi_s =& \quad h_s(\xi_u, \xi_c, \xi_y). \end{aligned} \tag{3.17}$$

By the definition of h_s and h_u, we have

$$\begin{aligned} & \left| h_u(\xi_s^1, \xi_c, \xi_y) - h_u(\xi_s^2, \xi_c, \xi_y) \right|_{X_1} \\ \leq & \Big(\int_0^{+\infty} K e^{-a_2' \tau} (\overline{r} + C_0 \epsilon_\star) \frac{3K}{(1-\sigma)} \, d\tau \Big) |\xi_s^1 - \xi_s^2|_{X_1}, \\ & \left| h_s(\xi_u^1, \xi_c, \xi_y) - h_s(\xi_u^2, \xi_c, \xi_y) \right|_{X_1} \\ \leq & \Big(\int_{-\infty}^0 K e^{-a_1 \tau} (\overline{r} + C_0 \epsilon_\star) \frac{3K}{(1-\sigma)} \, d\tau \Big) |\xi_u^1 - \xi_u^2|_{X_1}, \end{aligned}$$

where $\dfrac{3K}{(1-\sigma)}$ is the supremum of Lipschitz constants of fixed points of $\mathscr{T}_{cs}$ and $\mathscr{T}_{cu}$ with respect to ξ_{cs} and ξ_{cu}, respectively. Thus, for each $(\xi_c, \xi_y) \in X^c_1 \times Y_1$, $\big(h_s(\cdot, \xi_c, \xi_y), h_u(\cdot, \xi_c, \xi_y)\big)$ defines a contraction on $(P_s + P_u)X_1$. Thus, there exists a unique pair (Ψ_s, Ψ_u) that satisfy (3.16). □

3.3 Smoothness of invariant manifolds

In this section, we will prove some regularity results of invariant manifolds constructed in the previous chapter. More precisely, we will prove there smoothness with respect to spatial variables, t_0 and other external parameters. And we will only give detailed proof for $\mathcal{M}^{cu}_\epsilon$, while other results can be proved in a similar way. We start with the following lemmas.

Lemma 3.8. *Let X, Y_1, Y_2, Z be Banach spaces such that X is continuously embedded into Y_1 and Y_1 is continuously embedded into Y_2. Let $0 < \theta < 1$ and $\mathcal{F} : X \times Z \to X$, such that for any $x_1, x_2 \in X$ and $z \in Z$,*

$$\big|\mathcal{F}(x_1, z) - \mathcal{F}(x_2, z)\big|_X \le \theta\big|x_1 - x_2\big|_X.$$

Moreover, $D_z\mathcal{F} \in C^0(X \times Z, L(Z, X))$ with $|D_z\mathcal{F}|_{C^0} < \infty$. For every $n \ge 0$,

$$(D_x\mathcal{F})^n \in C^0(X \times Z, L(X, Y_1)), (D_xF)^n \in C^0(X \times Z, L(Y_1, Y_2)),$$

satisfying $|(D_x\mathcal{F})^n|_{C^0(X\times Z, L(X,Y_1))} \le \theta^n$ and $|(D_x\mathcal{F})^n|_{C^0(X\times Z, L(Y_1,Y_2))} \le \theta^n$. Then, there exists a unique mapping $x : Z \to X$, such that $\mathcal{F}(x(z), z) = x(z)$ and

$$x \in Lip(Z, X) \quad , \quad x \in C^1(Z, L(Z, Y_2)).$$

Proof. Since for each $z \in Z$, $\mathcal{F}$ is a contraction mapping on X, there exists a unique $x(z)$, such that $\mathcal{F}(x(z), z) = x(z)$. Note that

$$\begin{aligned}
&\big|x(z_1) - x(z_0)\big|_X \\
=&\big|\mathcal{F}(x(z_1), z_1) - \mathcal{F}(x(z_0), z_0)\big|_X \\
\le&\big|\mathcal{F}(x(z_1), z_1) - \mathcal{F}(x(z_0), z_1)\big|_X + \big|\mathcal{F}(x(z_0), z_1) - \mathcal{F}(x(z_0), z_0)\big|_X \\
\le&\theta\big|x(z_1) - x(z_0)\big|_X + \big|D\mathcal{F}\big|_{C^0}\big|z_1 - z_0\big|_Z,
\end{aligned}$$

which implies

$$|x(z_1) - x(z_0)|_X \le \frac{|D\mathcal{F}|_{C^0}}{1-\theta}|z_1 - z_0|_Z. \tag{3.18}$$

Therefore, we have $x(\cdot)$ is Lipschitz in x, in particular,

$$x \in C^0(Z, X) \subset C^0(Z, Y_2).$$

Let I_{X,Y_1} be the inclusion map from X to Y_1 and I_{Y_1,Y_2} be the inclusion map from Y_1 to Y_2. To prove the second part, first we have

$$\begin{aligned}
&\big(I_{X,Y_1} - D_xF(x(z_0), z_0)\big)\big(x(z_1) - x(z_0)\big) \in Y_1 \\
=&D_zF(x(z_1), z_0)(z_1 - z_0) + F_1(z_1) + F_2(z_1) \in Y_1.
\end{aligned} \tag{3.19}$$

where

$$F_1(z_1) = F(x(z_1), z_0) - F(x(z_0), z_0) - D_x F(x(z_0), z_0)(x(z_1) - x(z_0)),$$

$$F_2(z_1) = F(x(z_1), z_1) - F(x(z_1), z_0) - DF(x(z_1), z_0)(z_1 - z_0).$$

Since $x(\cdot)$ is continuous in z and $F \in C^1(X \times Z, Y_1)$,

$$|F_1(z_1)|_{Y_1} / |x(z_1) - x(z_0)|_X \to 0 \quad \text{as} \quad z_1 \to z_0.$$

By (3.18), we have

$$|F_1(z_1)|_{Y_1} / |z_1 - z_0|_Z \to 0 \quad \text{as} \quad z_1 \to z_0. \tag{3.20}$$

Since $DF \in C^0(X \times Z, L(Z, X))$,

$$|F_2(z_1)|_X / |z_1 - z_0|_Z \to 0 \quad \text{as} \quad z_1 \to z_0. \tag{3.21}$$

Define

$$T^k(z_0) = I_{Y_1,Y_2} + \sum_{n=1}^{k} \big(D_x F(x(z_0), z_0)\big)^n, \quad T(z_0) = \lim_{k\to\infty} T^k(z_0).$$

Since $x(\cdot)$ is continuous in z, $T^k \in C^0(Z, L(Y_1, Y_2))$. For any n and k and z_0, $|T^{n+k}(z_0) - T^n(z_0)| \leq \dfrac{\theta^{n+1}}{1-\theta}$, which implies T^k converges to T uniformly in z_0. So $T \in C^0(Z, L(Y_1, Y_2))$ and $|T|_{C^0} \leq \dfrac{1}{1-\theta}$. Multiply (3.19) by $T(z_0)$, we get

$$\begin{aligned}
&x(z_1) - x(z_0) \\
=&T(z_0) DF(x(z_1), z_0)(z_1 - z_0) + T(z_0)(F_1(z_1) + F_2(z_1)) \\
=&T(z_0) DF(x(z_0), z_0)(z_1 - z_0) + T(z_0)(F_1(z_1) + F_2(z_1)) \\
&+ T(z_0)(DF(x(z_1), z_0) - DF(x(z_0), z_0))(z_1 - z_0) \\
=&T(z_0) D_z F(x(z_0), z_0)(z_1 - z_0) + R(z_1, z_0).
\end{aligned} \tag{3.22}$$

By (3.20), (3.21) and continuity of DF, x, $|R(z_1, z_0)|_{L(Z,Y_2)} = o(|z_1 - z_0|_Z)$. Therefore, x is *Fréchet* differentiable, and

$$D_z x(z_0) = T(z_0) D_z F(x(z_0), z_0) \in C^0(Z, L(Z, Y_2)). \tag{3.23}$$

□

Lemma 3.9. *Let Z, Λ be Banach spaces and $\widetilde{\Lambda} = C^0_{loc}((-\infty, 0), \Lambda)$, which is a topological vector space. For $l \geq 1$, $p_1 + p_2 + \cdots + p_m = l$, where $p_i \geq 0$, suppose*

$$B \in C^0\big(\Lambda \times \mathbb{R}, L_m(X_1 \times Y_1, X_1 \times Y_1)\big),$$

with $|B|_{C^0} < \infty$. For $\widetilde{\lambda} \in \widetilde{\Lambda}$, define

$$\begin{aligned}
&\overline{B}_1(\widetilde{\lambda}, t_0)(z_1, z_2, \cdots, z_l)(t) \\
= &\int_0^t U(t-\tau, \epsilon)(I - P_s)B(\widetilde{\lambda}(\tau), \tau + t_0) \\
&\big(\Psi_1(z_1, \cdots, z_{p_1})(\tau), \cdots, \Psi_m(z_{p_1 + \cdots + p_{m-1}+1}, \cdots, z_l)(\tau)\big)\ d\tau, \\
&\overline{B}_2(\widetilde{\lambda}, t_0)(z_1, z_2, \cdots, z_l)(t) \\
= &\int_{-\infty}^t U(t-\tau, \epsilon)P_s B(\widetilde{\lambda}(\tau), \tau + t_0) \\
&\big(\Psi_1(z_1, \cdots, z_{p_1})(\tau), \cdots, \Psi_m(z_{p_1 + \cdots + p_{m-1}+1}, \cdots, z_l)(\tau)\big)\ d\tau,
\end{aligned}$$

where $z_i \in Z, \Psi_j \in L_{p_j}(Z, B^-_{p_j\eta}(\infty))$, $i = 1, 2, \cdots, l, j = 1, 2, \cdots, m$. Then,

$$\overline{B}_1(\widetilde{\lambda}, t_0), \overline{B}_2(\widetilde{\lambda}, t_0) \in L_l(Z, B^-_{l\eta}(\infty)), \tag{3.24}$$

$$\overline{B}_1, \overline{B}_2 \in C^0\big(\widetilde{\Lambda} \times \mathbb{R}, L_l(Z, B^-_{l\eta'}(\infty))\big), \tag{3.25}$$

for any $\eta' < \eta$ with $a_1 < l\eta' < l\eta < a_2$.

Proof. Obviously, (3.24) follows from the definition of $B^-_{l\eta}(\infty)$ in (3.6) and the assumptions of U in (B5).

To prove (3.25), we fix $\widetilde{\lambda}_0 \in \widetilde{\Lambda}$ and $t_0 \in \mathbb{R}$ and take any sequence $\widetilde{\lambda}_n \to \widetilde{\lambda}_0$, $t_n \to t_0$. Given any $s > 0$, $T_1 < 0$, we claim that there exists $N > 0$, such that for any $t \in [T_1, 0]$, if $n \geq N$

$$\big|B(\widetilde{\lambda}_n(t), t + t_n) - B(\widetilde{\lambda}_0(t), t + t_0)\big|_{L_m(X_1 \times Y_1, X_1 \times Y_1)} < s. \tag{3.26}$$

By the continuity of B, for any $t \in [T_1, 0]$, there exists $S_t > 0$ such that

$$\big|B(\lambda, t') - B(\widetilde{\lambda}_0(t), t + t_0)\big| < \frac{s}{2},$$

if $|t'-(t+t_0)|, |\lambda-\widetilde{\lambda}_0(t)| < S_t$. Since $[T_1,0]$ is compact and $\widetilde{\lambda}_n \to \widetilde{\lambda}_0$, there exists $S'_t \le S_t$ and N_t such that $|\widetilde{\lambda}_n(t')-\widetilde{\lambda}_0(t)| < S_t$ for $n > N_t$ and $|t'-t| < S'_t$. Again, by the compactness of $[T_1,0]$, there exist $t^1,\cdots,t^r$ such that $[T_1,0] \subset \bigcup_{i=1}^{r}(t^i-\frac{S_i}{2}, t^i+\frac{S_i}{2})$, where $S_i = S'_{t^i}$. Let $s' = \min\{\frac{S_1}{2},\cdots,\frac{S_r}{2}\}$. Since $t_n \to t_0$, $\widetilde{\lambda}_n(t) \to \widetilde{\lambda}_0(t)$ uniformly on $[T_1,0]$, there exists $N' > 0$, such that for every $n > N'$, $|t_n - t_0| < s'$ and $|\widetilde{\lambda}_n(t)-\widetilde{\lambda}_0(t)| < s'$ for $t \in [T_1,0]$. Now, let $N = \max\{N', N_{t^1},\cdots,N_{t^r}\}$, for every $n > N$ and $t \in [T_1,0]$, there exists some t^i such that $t \in (t^i-\frac{S_i}{2}, t^i+\frac{S_i}{2})$, thus

$$\begin{aligned}
&\big|B(\widetilde{\lambda}_n(t),t+t_n)-B(\widetilde{\lambda}_0(t),t+t_0)\big|\\
\le&\big|B(\widetilde{\lambda}_n(t),t+t_n)-B(\widetilde{\lambda}_0(t^i),t^i+t_0)\big|\\
&+\big|B(\widetilde{\lambda}_0(t^i),t^i+t_0)-B(\widetilde{\lambda}_0(t),t+t_0)\big|.
\end{aligned}$$

Note that $|t-t^i| < \frac{S_i}{2} < S_i$ and $|t+t_n-(t^i+t_0)| < \frac{S_i}{2}+s' < S_i$,

$$\big|B(\widetilde{\lambda}_n(t),t+t_n)-B(\widetilde{\lambda}_0(t),t+t_0)\big| < \frac{s}{2}+\frac{s}{2} = s.$$

Therefore, (3.26) is proved.

Let $\Psi = (\Psi_1,\cdots,\Psi_m)$ and $\|\Psi\|_{l\eta,\epsilon_\star} = \prod_{i=1}^{m}|\Psi_i|_{L_{p_i}(Z,B^-_{p_i\eta}(\infty))}$. Without loss of generality in the following proof, we choose $\|\Psi\|_{l\eta,\epsilon_\star} = 1$. For any $\eta' < \eta$, clearly we have $\|\Psi\|_{l\eta',\epsilon_\star} \le 1$. Define $b^1_n(t) = \overline{B}_1(\widetilde{\lambda}_n,t_n)(t) - \overline{B}_1(\widetilde{\lambda}_0,t_0)(t)$ and we choose $T_1 = \dfrac{\log s}{l(\eta-\eta')}$. For $n > N$, if $T_1 \le t \le 0$, using (3.26)

$$e^{-l\eta' t}\big|b^1_n(t)\big| \le \Big(\int_t^0 Ke^{(a_2-l\eta')(t-\tau)}s\ d\tau\Big)\|\Psi\|_{l\eta',\epsilon_\star} \le s\frac{K}{a_2-l\eta'}.$$

Since $\frac{d}{dt}b^1_n(t)$ has a similar form to (3.9),

$$e^{-l\eta' t}\Big|P_{cu}\frac{d}{dt}b^1_n(t)\Big| \le s(\frac{K}{a_2-l\eta'}+K).$$

For $t \le T_1$, since $\|\Psi\|_{l\eta,\epsilon_\star} = 1$,

$$\begin{aligned}
e^{-l\eta' t}\big|b^1_n(t)\big| &\le e^{l(\eta-\eta')T_1}\Big(\int_t^0 Ke^{(a_2-l\eta)(t-\tau)}2|B|_{C^0}\ d\tau\Big)\|\Psi\|_{l\eta,\epsilon_\star}\\
&\le s\Big(\frac{K}{a_2-l\eta}2|B|_{C^0}\Big),\\
e^{-l\eta' t}\Big|P_{cu}\frac{d}{dt}b^1_n(t)\Big| &\le s(\frac{K}{a_2-l\eta}+K)2|B|_{C^0}.
\end{aligned}$$

Since s is arbitrary, we have proved $\overline{B}_1 \in C^0(\widetilde{\Lambda} \times \mathbb{R}, L_l(Z, B^-_{l\eta'}(\infty))$.

Let $b_n^2(t) = \overline{B}_2(\widetilde{\lambda}_n, t_n) - \overline{B}_2(\widetilde{\lambda}_0, t_0)$, if $t \le T_1$, again we use $\|\Psi\|_{l\eta,\epsilon_\star} = 1$ to have

$$\begin{aligned}
e^{-l\eta' t}\big|b_n^2(t)\big| &\le e^{l(\eta-\eta')T_1}\big(\int_{-\infty}^t K e^{(a_1-l\eta)(t-\tau)} 2|B|_{C^0}\, d\tau\big)\|\Psi\|_{l\eta,\epsilon_\star} \\
&\le s\big(\frac{K}{l\eta - a_1} 2|B|_{C^0}\big), \\
e^{-l\eta' t}\Big|\frac{d}{dt} b_n^2(t)\Big| &\le s\big(\frac{K}{l\eta - a_1} + K\big) 2|B|_{C^0}.
\end{aligned}$$

If $T_1 \le t \le 0$, we split the integral into $\displaystyle\int_{-\infty}^{T_1}$ and $\displaystyle\int_{T_1}^{t}$. First we have

$$\begin{aligned}
& e^{-l\eta' t}\Big|\int_{-\infty}^{T_1} U(t-\tau,\epsilon) P_s(B(\widetilde{\lambda}_n, \tau + t_n) - B(\widetilde{\lambda}_0, \tau + t_0))\Psi\, d\tau\Big| \\
\le\, & e^{l(\eta-\eta')t}\int_{-\infty}^{T_1} K e^{(a_1-l\eta)(t-\tau)} 2|B|_{C^0}\|\Psi\|_{l\eta,\epsilon_\star}\, d\tau \\
\le\, & \frac{K}{l\eta - a_1} 2|B|_{C^0} e^{l(\eta-\eta')T_1} \le C's, \\
& e^{-l\eta' t}\Big|\frac{d}{dt}\int_{-\infty}^{T_1} U(t-\tau,\epsilon) P_s(B(\widetilde{\lambda}_n, \tau + t_n) - B(\widetilde{\lambda}_0, \tau + t_0))\Psi\, d\tau\Big| \le C's.
\end{aligned}$$

where C' depends on $K, l, \eta, a_1, |B|_{C^0}$. The estimate of $\displaystyle\int_{T_1}^{t}$ part is similar to the estimate of $b_n^1(t)$ for $T_1 \le t \le 0$. Therefore, we obtain

$$\begin{aligned}
e^{-l\eta' t}\big|b_n^2(t)\big| &\le s\Big(\frac{K}{l\eta' - a_1} + C'\Big), \\
e^{-l\eta' t}\Big|\frac{d}{dt} b_n^2(t)\Big| &\le s\Big(\frac{K}{l\eta' - a_1} + K + C'\Big),
\end{aligned}$$

Therefore, $\overline{B}_2 \in C^0\big(\widetilde{\Lambda} \times \mathbb{R}, L_l(Z, B^-_{l\eta'}(\infty))\big)$. □

Theorem 3.10. *Suppose f, g are C^k functions and there exists some η_0 with $a_1 < \eta_0, k\eta_0 < a_2$. Choose $r_0, \epsilon_0, \underline{\epsilon}$ and δ to be small enough such that $\sigma(\eta') > \delta$ for any $\eta' \in [a_1 + \delta, a_2 - \delta]$, where $\sigma(\eta')$ is given in (3.7). Then, the center-unstable integral manifold $\mathcal{M}^{cu}_\epsilon$ of (3.5) is C^k in (ξ_{cu}, ξ_y). Moreover, the norm of all derivatives of h_s for $1 \le i \le k$ are bounded in $X_1^{cu} \times Y_1$ with an upper bound $\widetilde{\rho}$ independent of $\epsilon \in [0, \epsilon_0)$ and t_0.*

Proof. From assumptions, if such η_0 exists, then $i\eta_0 \in (a_1, a_2)$ for $i = 1, 2, \cdots, k$. For $1 \le i \le k$, define

$$\Omega_j = \left\{\eta \middle| i\eta \in (a_1 + \delta, a_2 - \delta),\ \sigma(i\eta) > \delta \text{ in } \ (3.7), i = 1, 2, \cdots, j\right\},$$

where δ is a positive but sufficiently small quantity. Clearly, each Ω_j is open and $\eta_0 \in \Omega_k \subset \Omega_{k-1} \subset \cdots \subset \Omega_1$.

We will first prove the case for $k = 1$, and the higher order case can be proved inductively.

For $\eta \in \Omega_1$, $\mathscr{T}_{cu}$ is a contraction mapping on $B_\eta^-(\infty)$ under the norm $|\cdot|_{\eta,\epsilon_\star}^-$ according to the proof of Lemma 3.2. Since we do not change t_0, ϵ in this proof, to simply notations, we use $\mathscr{T}_{cu}(x, y, \xi)$ to denote $\mathscr{T}_{cu}(x, y, \xi, t_0, \epsilon)$. We also write

$$\begin{aligned} F(x, y) &= F(x(t), y(t), t + t_0, \epsilon), \\ DF(x, y) &= (D_x F(x(t), y(t), t + t_0, \epsilon), D_y F(x(t), y(t), t + t_0, \epsilon)). \end{aligned}$$

and such notations also apply to G.

For any (x, y) and $(\phi, \psi) \in B_\eta^-(\infty)$, define

$$\begin{aligned} \mathscr{T}_{cu}^1(x, y)(\phi, \psi) = &\int_0^t U(t-\tau, \epsilon) \begin{pmatrix} P_{cu}\big(DF(x, y)(\phi, \psi) + f_y\psi\big) \\ DG(x, y)(\phi, \psi) + g_x\phi \end{pmatrix} d\tau \\ &+ \int_{-\infty}^t U(t-\tau, \epsilon) \begin{pmatrix} P_s\big(DF(x, y)(\phi, \psi) + f_y\psi\big) \\ 0 \end{pmatrix} d\tau. \end{aligned}$$

A basic estimate on $\mathscr{T}_{cu}^1$ are obtained below,

$$\begin{aligned}
&\sup_{t\le 0} e^{-\eta t}\Big|(P_s+P_{cu})\mathscr{T}_{cu}^1(x,y)(\phi,\psi)\Big|_{X_1}\\
\le&\big(\frac{K}{a_2-\eta}+\frac{K}{\eta-a_1}\big)\big(|DF|_{C^0}+\epsilon_\star|f_y|\big)\big|(\phi,\psi)\big|^-_{\eta,\epsilon_\star},\\
&\sup_{t\le 0}\frac{1}{\epsilon_\star}e^{-\eta t}\Big|(I-P_s-P_{cu})\mathscr{T}_{cu}^1(x,y)(\phi,\psi)\Big|_{Y_1}\\
\le&\frac{1}{\epsilon_\star}\big(\frac{K}{a_2-\eta}+K+1\big)\big(|DG|_{C^0}+|(\frac{J}{\epsilon}+g_y)^{-1}||g_x|\big)\big|(\phi,\psi)\big|^-_{\eta,\epsilon_\star},\\
&\sup_{t\le 0} e^{-\eta t}\Big|(P_s+P_{cu})\frac{d}{dt}\mathscr{T}_{cu}^1(x,y)(\phi,\psi)\Big|_{X}\\
\le&\big(\frac{K}{a_2-\eta}+\frac{K}{\eta-a_1}+1\big)\big(|DF|_{C^0}+\epsilon_\star|f_y|\big)\big|(\phi,\psi)\big|^-_{\eta,\epsilon_\star},
\end{aligned}\tag{3.27}$$

where we recall $|\cdot|_{C^0}$ is the C^0 norm with respect to x, y, t, ϵ. From (3.3) and (3.7), we have $|\mathscr{T}_{cu}^1|_{L(B_\eta^-(\infty),B_\eta^-(\infty))}<1$.

From (3.7) and the proof of Lemma 3.2, the Lipschitz constant of $\mathscr{T}_{cu}$ on $B_\eta^-(\infty)$ is uniform in η. By (3.10) and (3.27), there exists $0<\theta<1$ such that for any $\eta\in\Omega_1$, $\xi\in X_1^{cu}\times Y_1$ and $(x,y)\in B_\eta^-(\infty)$,

$$\max\big\{Lip_{(x,y)}\mathscr{T}_{cu}(\cdot,\cdot,\xi),\big|\mathscr{T}_{cu}^1(x,y)\big|\big\}\le\theta<1.\tag{3.28}$$

By Lemma 3.9 and (3.28), for any $\eta<\eta'$, where $\eta',\eta\in\Omega_1$,

$$\big|\mathscr{T}_{cu}^1\big|_{C^0(B_{\eta'}^-(\infty),L(B_{\eta'}^-(\infty),B_\eta^-(\infty)))}\le\theta<1,\tag{3.29}$$

which implies

$$\mathscr{T}_{cu}^1\in C^0(B_\eta^-(\infty)\times B_\eta^-(\infty),B_{\eta'}^-(\infty)).\tag{3.30}$$

In fact, if we choose a positive decreasing sequence $\{\delta_i\}_{i\ge1}$ such that $\eta+\delta_i\in\Omega_1$ for all i, (3.30) shows for every n,

$$(\mathscr{T}_{cu}^1)^n\in C^0(B_{\eta'}^-(\infty)\times B_{\eta'}^-(\infty),B_{\eta+\delta_n}^-(\infty)),$$

which implies

$$(\mathscr{T}_{cu}^1)^n\in C^0(B_{\eta'}^-(\infty)\times B_{\eta'}^-(\infty),B_\eta^-(\infty)).$$

Clearly, $D\mathscr{T}_{cu} = \mathscr{T}_{cu}^1$ and $D_\xi \mathscr{T}_{cu} = U(t,\epsilon)$. By (3.29),

$$\big|(\mathscr{T}_{cu}^1)^n\big|_{C^0(B_\eta^-(\infty)\times B_\eta^-(\infty), B_{\eta'}^-(\infty))} \le \theta^n. \tag{3.31}$$

In the rest of this proof, we let $Z = X_1^{cu} \times Y_1$. For any $\eta \in \Omega_1$ choose $\eta_1, \eta_2 \in \Omega_1$ such that $\eta < \eta_2 < \eta_1$, let $X = B_{\eta_1}^-(\infty), Y_1 = B_{\eta_2}^-(\infty), Y_2 = B_\eta^-(\infty)$ and $F(z,\xi) = \mathscr{T}_{cu}(z,\xi)$. We have

$$\begin{aligned}
&|DF|_{C^0(X\times Z, L(Z,X))} \le \frac{2K}{a_2-\eta_0},\\
&|(D_xF)^n|_{C^0(X\times Z, L(X,Y_1))} \le \theta^n \quad , \quad |(D_xF)^n|_{C^0(X\times Z, L(Y_1,Y_2))} \le \theta^n.
\end{aligned}$$

By Lemma 3.8, we obtain a unique mapping z, such that

$$\mathscr{T}_{cu}(z(\xi),\xi) = z(\xi),$$

and $z \in C^1(Z, B_\eta^-(\infty))$. From (3.20), we also have

$$|D_\xi z|_{C^0} \le \frac{2K}{(1-\theta)(a_2-\eta_0)} \le \frac{2K}{(1-\theta)\delta}.$$

By the integral equation which defines h_s in (3.11),

$$|D_\xi h_s|_{C^0} \le \frac{K}{\eta - a_1}(\overline{r} + C_0\epsilon_\star)\frac{2K}{(1-\theta)\delta} \le \frac{2K^2(\overline{r}+C_0\epsilon_\star)}{(1-\theta)\delta^2} \triangleq \widetilde{\rho}_1. \tag{3.32}$$

For $l \ge 2$, let z be the fixed point of $\mathscr{T}_{cu}$, by differentiating (3.5) formally, we have

$$\begin{aligned}
D_\xi^l z(t) = &\int_0^t U(t-\tau,\epsilon)\begin{pmatrix} P_{cu}\big(DF(z(\xi))D_\xi^l z + f_y D_\xi^l y + \cdots\big) \\ D_xG(z(\xi))D_\xi^l z + g_x D_\xi^l x + \cdots \end{pmatrix} d\tau \qquad (3.33)\\
&+ \int_{-\infty}^t U(t-\tau,\epsilon)\begin{pmatrix} P_s\big(DF(z(\xi))D_\xi^l z + f_y D_\xi^l y + \cdots\big) \\ 0 \end{pmatrix} d\tau.
\end{aligned}$$

Here the skipped terms are in the form of

$$\sum_{i_1+\cdots+i_m=l} \big(H_{i_1,\cdots,i_m}(z)\big)\big(D_\xi^{i_1}z, \cdots, D_\xi^{i_m}z\big), \tag{3.34}$$

where each $H_{i_1,\cdots,i_m}(z) = D^m F(z)$ or $D^m G(z)$ is a multi-linear operator and C^{k-m} in z. Moreover, we have here $m > 1$ and $1 \le i_j < l$ for $j = 1, 2, \cdots, m$.

We will prove by induction that for any $l \le k$ and $\eta \in \Omega_l$,

$$z \in C^l\big(Z, B^-_{l\eta}(\infty)\big),$$

and there exists $\widetilde{\rho}_l$ which depends on $K, a_1, a_2, \sigma, l, \delta$, such that

$$|D^l_\xi z|_{C^0} \le \widetilde{\rho}_l.$$

For $l = 1$, this has been proved in the above. For $l_0 \le k$, we assume the result holds for $l < l_0$. Let

$$\begin{aligned}
\mathscr{F}_1^{l_0-1}(\xi)(\alpha,\beta) = &\int_0^t U(t-\tau,\epsilon)\begin{pmatrix} P_{cu}\big(DF(z(\xi))(\alpha,\beta) + f_y\beta\big) \\ DG(z(\xi))(\alpha,\beta) + g_x\alpha \end{pmatrix} d\tau \\
&+ \int_{-\infty}^t U(t-\tau,\epsilon)\begin{pmatrix} P_s\big(DF(z(\xi))(\alpha,\beta) + f_y\beta\big) \\ 0 \end{pmatrix} d\tau, \\
\mathscr{F}^{l_0-1}(\xi) = &I + \sum_{n=1}^{\infty}\big(\mathscr{F}_1^{l_0-1}(\xi)\big)^n,
\end{aligned}$$

and (3.33) shows

$$D_\xi^{l_0-1} z(\xi) = \mathscr{F}_1^{l_0-1}(\xi) D_\xi^{l_0-1} z(\xi) + R(\xi),$$

where $R(\xi)$ includes all skipped terms in (3.34) for $l = l_0 - 1$ which contain $D^j_\xi z$ for $j \le l_0 - 2$.

By Lemma 3.9 with $m = 1$, for any $n, 1 \le j \le k$, and $\eta < \eta'$ such that $\eta', \eta \in \Omega_j$, we have

$$\big(\mathscr{F}_1^{l_0-1}\big)^n \in C^0\Big(Z \times L_j(Z, B^-_{j\eta'}(\infty)), L_j(Z, B^-_{j\eta}(\infty))\Big).$$

Obviously, $|\big(\mathscr{F}_1^{l_0-1}\big)^n|_{C^0} \le \theta^n$. It implies

$$\mathscr{F}^{l_0-1} \in C^0\Big(Z \times L_j(Z, B^-_{j\eta'}(\infty)), L_j(Z, B^-_{j\eta}(\infty))\Big). \tag{3.35}$$

For $2 \le l_0 \le k$, denote $S_j = \big\{j\eta \big| \eta \in \Omega_j\big\}$, we claim that $S_j \subset S_{j-1}$. We recall that $a_1 < 0$ by assumption, so we need to prove two cases.

If $a_1 < a_2 \leq 0$, then

$$\begin{aligned} S_{j-1} &= \big\{(j-1)\eta \big| \frac{a_1+\delta}{j-1} < \eta < a_2 - \delta, \eta \in \Omega_{j-1}\big\} \\ &= \big\{(j-1)\eta \big| a_1+\delta < (j-1)\eta < (j-1)(a_2-\delta), \eta \in \Omega_{j-1}\big\}. \end{aligned}$$

On the other hand,

$$S_j = \big\{j\eta \big| \frac{a_1+\delta}{j} < \eta < a_2 - \delta, \eta \in \Omega_j\big\} = \big\{j\eta \big| a_1+\delta < j\eta < j(a_2-\delta), \eta \in \Omega_j\big\}.$$

By our assumption, $a_1 < ka_2$, so S_j and S_{j-1} are not empty. Clearly, $S_j \subset S_{j-1}$.

If $a_1 < 0 < a_2$, then

$$\begin{aligned} S_{j-1} &= \big\{(j-1)\eta \big| a_1+\delta < (j-1)\eta < a_2 - \delta, \eta \in \Omega_{j-1}\big\} \\ &\subset \big\{j\eta \big| a_1+\delta < j\eta < a_2-\delta, \eta \in \Omega_j\big\} = S_j. \end{aligned}$$

This fact implies for any $l_0\eta \in S_{l_0}$, there exists $(l_0-1)\overline{\eta} \subset S_{l_0-1}$ such that $l_0\eta = (l_0-1)\overline{\eta}$. Therefore, with slight abuse of notation

$$D_\xi^{l_0-1} z(\xi) = \mathscr{F}_1^{l_0-1}(\xi) D_\xi^{l_0-1} z(\xi) + R(\xi) \in L_{l_0-1}\big(Z, B^-_{l_0\eta}(\infty)\big). \tag{3.36}$$

So we have

$$\begin{aligned} & D_\xi^{l_0-1} z(\xi_1) - D_\xi^{l_0-1} z(\xi_0) \\ =& \mathscr{F}_1^{l_0-1}(\xi_1) D_\xi^{l_0-1} z(\xi_1) - \mathscr{F}_1^{l_0-1}(\xi_0) D_\xi^{l_0-1} z(\xi_0) + R(\xi_1) - R(\xi_0) \\ =& \mathscr{F}_1^{l_0-1}(\xi_0)\big(D_\xi^{l_0-1} z(\xi_1) - D_\xi^{l_0-1} z(\xi_0)\big) \\ & + \big(\mathscr{F}_1^{l_0-1}(\xi_1) - \mathscr{F}_1^{l_0-1}(\xi_0)\big) D_\xi^{l_0-1} z(\xi_1) + R(\xi_1) - R(\xi_0) \\ =& \mathscr{F}_1^{l_0-1}(\xi_0)\big(D_\xi^{l_0-1} z(\xi_1) - D_\xi^{l_0-1} z(\xi_0)\big) \\ & + D_\xi \mathscr{F}_1^{l_0-1}(\xi_0)(\xi_1-\xi_0) D_\xi^{l_0-1} z(\xi_0) + D_\xi R(\xi_0)(\xi_1-\xi_0) \\ & + R_1(\xi_1,\xi_0) + R_2(\xi_1,\xi_0), \end{aligned} \tag{3.37}$$

where

$$\begin{aligned}
&R_1(\xi_1,\xi_0)\\
=&\big(\mathscr{F}_1^{l_0-1}(\xi_1)-\mathscr{F}_1^{l_0-1}(\xi_0)\big)D_\xi^{l_0-1}z(\xi_1)\\
&-D_\xi\mathscr{F}_1^{l_0-1}(\xi_0)(\xi_1-\xi_0)D_\xi^{l_0-1}z(\xi_0)\\
=&\big(\mathscr{F}_1^{l_0-1}(\xi_1)-\mathscr{F}_1^{l_0-1}(\xi_0)-D_\xi\mathscr{F}_1^{l_0-1}(\xi_0)(\xi_1-\xi_0)\big)D_\xi^{l_0-1}z(\xi_0)\\
&+\big(\mathscr{F}_1^{l_0-1}(\xi_1)-\mathscr{F}_1^{l_0-1}(\xi_0)\big)\big(D_\xi^{l_0-1}z(\xi_1)-D_\xi^{l_0-1}z(\xi_0)\big).
\end{aligned} \tag{3.38}$$

and

$$R_2(\xi_1,\xi_0)=R(\xi_1)-R(\xi_0)-D_\xi R(\xi_0)(\xi_1-\xi_0). \tag{3.39}$$

For any $\eta\in\Omega_{l_0}$, there exits $\eta'>\eta$ with $\eta'\in\Omega_{l_0}$, such that

$$D_\xi^{l_0-1}z(\xi)=\mathscr{F}_1^{l_0-1}(\xi)D_\xi^{l_0-1}z(\xi)+R(\xi)\in L_{l_0-1}\big(Z,B^-_{l_0\eta'}(\infty)\big).$$

For $1\le j\le l_0-1$, since $\eta'\in\Omega_{l_0}\subset\Omega_j$, $D_\xi^j z\in C^0\big(Z,L_j(Z,B^-_{j\eta'}(\infty))\big)$. let $\eta''=\dfrac{\eta+\eta'}{2}$, by Lemma 3.9 and the fact that R consists of terms given in (3.34), we have

$$R\in C^1\big(Z,L_{l_0-1}(Z,B^-_{l_0\eta''}(\infty))\big), \tag{3.40}$$

which implies

$$\lim_{\xi_1\to\xi_0}\frac{\big|R_2(\xi,\xi_0)\big|_{L_{l_0-1}(Z,B^-_{l_0\eta''}(\infty))}}{|\xi_1-\xi_0|_Z}\longrightarrow 0.$$

By differentiating $\mathscr{F}_1^{l_0-1}$, Lemma 3.9 also shows,

$$\begin{aligned}
&\mathscr{F}_1^{l_0-1}\in C^1\Big(Z\times L_{l_0-1}(Z,B^-_{(l_0-1)\eta'}(\infty)),L_{l_0-1}(Z,B^-_{(l_0-1)\eta''}(\infty))\Big),\\
&\mathscr{F}_1^{l_0-1}\in C^1\Big(Z\times L_{l_0-1}(Z,B^-_{(l_0-1)\eta''}(\infty)),L_{l_0-1}(Z,B^-_{(l_0-1)\eta}(\infty))\Big).
\end{aligned}$$

which follows

$$\begin{aligned}
&\Big|\big(\mathscr{F}_1^{l_0-1}(\xi_1)-\mathscr{F}_1^{l_0-1}(\xi_0)-D_\xi\mathscr{F}_1^{l_0-1}(\xi_0)(\xi_1-\xi_0)\big)D_\xi^{l_0-1}(\xi_0)\Big|_{L_{l_0-1}(Z,B^-_{l_0\eta''}(\infty))}\\
=&\,o(|\xi_1-\xi_0|_Z),
\end{aligned}$$

and along with the continuity of $D_\xi^{l_0-1}z$,

$$
\begin{aligned}
&\left|\left(\mathscr{F}_1^{l_0-1}(\xi_1)-\mathscr{F}_1^{l_0-1}(\xi_0)\right)\left(D_\xi^{l_0-1}z(\xi_1)-D_\xi^{l_0-1}z(\xi_0)\right)\right|_{L_{l_0-1}(Z,B^-_{l_0\eta''}(\infty))} \\
=&\, O(|\xi_1-\xi_0|_Z)\times O(|\xi_1-\xi_0|_Z) \\
=&\, o(|\xi_1-\xi_0|_Z).
\end{aligned}
$$

Therefore,

$$
\lim_{\xi_1\to\xi_0}\frac{\left|R_1(\xi,\xi_0)\right|_{L_{l_0-1}(Z,B^-_{l_0\eta''}(\infty))}}{|\xi_1-\xi_0|_Z}\longrightarrow 0.
$$

Then, by using (3.37), we have

$$
\begin{aligned}
&D_\xi^{l_0-1}z(\xi_1)-D_\xi^{l_0-1}z(\xi_0) \\
=&\mathscr{F}^{l_0-1}(\xi_0)\Big(D_\xi\mathscr{F}_1^{l_0-1}(\xi_0)D_\xi^{l_0-1}z(\xi_0)+D_\xi R(\xi_0)\Big)(\xi_1-\xi_0) \\
&+o(|\xi_1-\xi_0|_Z),
\end{aligned}
$$

where $\mathscr{F}^{l_0-1}\Big(D_\xi\mathscr{F}_1^{l_0-1}D_\xi^{l_0-1}z+D_\xi R\Big)=D_\xi^{l_0}z\in C^0\big(Z,L_{l_0}(Z,B^-_{l_0\eta}(\infty))\big)$.

Moreover, from (3.33), there exists ρ_l which depends on $K,a_1,a_2,\sigma,l,\delta$ such that $|D_\xi^{l_0}z|_{C^0}\le\rho_l$ which follows

$$
\left|D_\xi^l h_s\right|_{C^0}\le\frac{K}{l_0\eta-a_1}\big((\overline{r}+C_0\epsilon_\star)\rho_l+Q(\rho_1,\rho_2,\cdots,\rho_{l-1})\big)\triangleq\widetilde{\rho}_l,
$$

where $Q(\rho_1,\rho_2,\cdots,\rho_{l-1})$ is a polynomial of degree l.

Finally, let $\widetilde{\rho}=\max\limits_{1\le i\le k}\widetilde{\rho}_i$, we have $\left|D_\xi^i h_s\right|_{C^0}\le C'(\overline{r}+\epsilon_\star)\widetilde{\rho}$ for $1\le i\le k$, where C' depends on K,a_1,η,l,C_0. By the definition of $\mathcal{M}_\epsilon^{cu}$, $\mathcal{M}_\epsilon^{cu}$ is C^k in ξ. □

Remark 3.11. *The C^k center-unstable integral manifold we obtained is unique for* (3.5). *However, it's not unique for the original system* (2.1)*, which is due to different cut-off functions λ.*

If f,g depend on some other parameter $\alpha\in\Lambda$, which is a Banach space, we replace (A4) and (B1)-(B3) by

(A4$'$) For $k \geq 1$ and $0 \leq i \leq k$,

$$(D^i f, D^i g) \in C^0(X_1 \times Y_1 \times \Lambda \times \mathbb{R}^2, L_i(X_1 \times Y_1 \times \Lambda, X_1 \times Y_1)),$$
$$(D^i f, D^i g) \in C^0(X_1 \times Y_1 \times \Lambda \times \mathbb{R}^2, L_i(X \times Y \times \Lambda, X \times Y)).$$

where D is the differentiation in $X_1 \times Y_1 \times \Lambda$ space. And all of the quantities have a uniform bound C_0 for $0 \leq i \leq k$.

(B1$'$) $\partial_t f(x, y, \alpha, t, 0) = \partial_t g(x, y, \alpha, t, 0) = 0$,

(B2$'$) $f(0, 0, \alpha, t, \epsilon) = g(0, 0, \alpha, t, \epsilon) = 0$,

(B3$'$) (Df, Dg) are equicontinuous functions in x, y, α, ϵ with respect to t at $x = 0, y = 0, \alpha = \alpha_\star, \epsilon = 0$.

Then, we have the following corollary

Corollary 3.12. *Assume (B4)-(B5) for some $\alpha_\star \in \Lambda$ and the same conditions as in Theorem 3.10 except (A4), (B1)-(B3) replaced by (A4$'$), (B1$'$)-(B3$'$), then*

i) There exists a neighborhood $U \subset \Lambda$ of $\alpha_\star$ such that $\mathcal{M}_\epsilon^{cu}$ is C^k in $\alpha \in U$ and there exists ρ such that $|D^i h_s| \leq \rho$ on $X_1 \times Y_1 \times U$.

ii) If in addition $(\partial_t^i f, \partial_t^i g) \in C^{k-i}(X_1 \times Y_1 \times \Lambda \times \mathbb{R}^2, X_1 \times Y_1)$ with uniform bound C_0, then $h_s \in C^k(X_1^{cu} \times Y_1 \times U \times \mathbb{R}, X_1^s)$ for sufficiently small $\epsilon > 0$.

Proof. We consider the augmented system

$$\begin{cases} x(t) = e^{tA_f}\xi_{cu} + \int_0^t e^{(t-\tau)A_f} P_{cu}\big(F(x, y, \tau + t_0, \alpha, \epsilon) + f_y y\big)\, d\tau \\ \qquad + \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(F(x, y, \tau + t_0, \alpha, \epsilon) + f_y y\big)\, d\tau, \\ \alpha(t) = \alpha_0, \\ y(t) = e^{t(\frac{J}{\epsilon} + g_y)}\xi_y + \int_0^t e^{(t-\tau)(\frac{J}{\epsilon} + g_y)} G(x, y, \tau + t_0, \alpha, \epsilon)\, d\tau \\ \qquad + \int_0^t e^{(t-\tau)(\frac{J}{\epsilon} + g_y)} g_x x\, d\tau. \end{cases}$$

where $f_y = f_y(0, 0, \alpha_\star, t, 0), D_x g = D_x g(0, 0, \alpha_\star, t, 0)$ and F, G are defined in a similar way as in (3.2)-(3.4). Then, we can apply Theorem 3.10 to obtain i). For ii), we define

$$\widetilde{f}(x, y, \alpha, t, t_0, \epsilon) = f(x, y, \alpha, t + t_0, \epsilon),$$

$$\widetilde{g}(x, y, \alpha, t, t_0, \epsilon) = g(x, y, \alpha, t + t_0, \epsilon).$$

Then, we can apply i) to obtain ii). □

Next we will look at the dependence of $\mathcal{M}_\epsilon^{cu}$ on t_0. We still assume (A1)-(A5), (B1)-(B3) and replace (A6) by

(A6′) $(\partial_t f, \partial_t g) \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, X \times Y)$, $(D\partial_t f, D\partial_t g) \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, L(X_1 \times Y_1, X \times Y))$, $(D\partial_\epsilon \partial_t f, D\partial_\epsilon \partial_t g) \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, L(X_1 \times Y_1, X \times Y))$ Moreover, their norms are bounded by C_0.

Remark 3.13. *In fact, the assumptions in (A6′) on $D\partial_t f$ and $D\partial_t g$ are only needed when one has to work with $\eta > 0$ and here our following theorem still holds without the second part in (A6′). However, we still assume it and the proof of the theorem also works for the case when $\eta < 0$.*

Before we state next theorem, we will first introduce another transformation and relating properties which will be used in the proofs of the following theorems.

For fixed$z=(x,y)$, we define $\widetilde{\mathscr{T}_{cu}}$ on $C_\eta^-(X_1)\times C_\eta^-(Y_1)$ as

$$
\begin{aligned}
&\widetilde{\mathscr{T}_{cu}}(z,t_0)(t)\\
=&U(t,\epsilon)\xi+\int_0^t U(t-\tau,\epsilon)\begin{pmatrix} P_{cu}\Big(F(z,\tau+t_0,\epsilon)+f_y y\Big)\\ G(z,\tau+t_0,\epsilon)\end{pmatrix}\,d\tau\\
&+\int_{-\infty}^t U(t-\tau,\epsilon)\begin{pmatrix} P_s\Big(F(z,\tau+t_0,\epsilon)+f_y y\Big)\\ 0\end{pmatrix}\,d\tau\\
&+\begin{pmatrix} 0\\ -(\frac{J}{\epsilon}+g_y)^{-1}g_x x(t)+(\frac{J}{\epsilon}+g_y)^{-1}e^{t(\frac{J}{\epsilon}+g_y)}g_x x(0)\end{pmatrix}\\
&+\begin{pmatrix} 0\\ \int_0^t(\dfrac{J}{\epsilon}+g_y)^{-1}e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}g_x(A_f x+F(z,\tau+t_0,\epsilon)+f_y y)\,d\tau\end{pmatrix},
\end{aligned}
\tag{3.41}
$$

where the norm on $C_\eta^-(X_1)\times C_\eta^-(Y_1)$ is given by

$$
\|z\|_\eta^1=\sup_{t\le 0}e^{-\eta t}\big(|x(t)|_{X_1}+\frac{|y(t)|_{Y_1}}{\epsilon_\star}\big),
\tag{3.42}
$$

and we will also use $\|\cdot\|_\eta$ to denote

$$
\|z\|_\eta=\sup_{t\le 0}e^{-\eta t}\big(|x(t)|_X+\frac{|y(t)|_Y}{\epsilon_\star}\big).
$$

Much as in Lemma 3.2, $\widetilde{\mathscr{T}_{cu}}$ is still a contraction on $C_\eta^-(X_1)\times C_\eta^-(Y_1)$ under the norm in (3.42). Clearly, $\widetilde{\mathscr{T}_{cu}}$ and $\mathscr{T}_{cu}$ have the same unique fixed point. Moreover, by the same proof as in Lemma 3.2, one has

$$
\Big\|\widetilde{\mathscr{T}_{cu}}(z,t_0)-\widetilde{\mathscr{T}_{cu}}(z',t_0)\Big\|_\eta\le(1-\sigma')\|z-z'\|_\eta,
\tag{3.43}
$$

where $0<\sigma'<1$ and has a similar form as σ.

Theorem 3.14. *Assume the conditions in Theorem 3.10 for $k=1$, then*

$$
h_s(\cdot,\cdot,\epsilon)\in C^0(X_1^{cu}\times Y_1\times\mathbb{R},X_1^s).
$$

If we further assume (A6′),

$$
\partial_{t_0}h_s(\cdot,\cdot,\epsilon)\in C^0(X_1^{cu}\times Y_1\times\mathbb{R},X^s).
$$

Proof. To simplify notation, we will also ignore ϵ argument in F, G. We first claim that

$$\lim_{t_1\to t_0}\left|\widetilde{\mathscr{T}_{cu}}(z,t_1)-\widetilde{\mathscr{T}_{cu}}(z,t_0)\right|_{\eta}^{1}=0, \tag{3.44}$$

for any $a_1<\eta<\eta'<a_2$ and $z\in \widetilde{B}_{\eta'}^{-}(\infty)$, where $\widetilde{B}_{\eta'}^{-}(\infty)=\left\{z\middle|\|z\|_{\eta'}^{1}<\infty\right\}$.

For any $s>0$, we choose $T_2=\dfrac{\log s}{\eta'-\eta}$. By (A4), $DF(pz(t),t'), DG(pz(t),t')$ are uniformly continuous on $(p,t,t')\in[0,1]\times[T_2,0]\times[T_2+t_0-1,t_0+1]$. Therefore, there exists $s'>0$ such that if $|t_1-t_0|<s'$,

$$\begin{aligned}
&\left|DF(pz(t),t+t_1)-DF(pz(t),t+t_0)\right|_{L(X_1\times Y_1,X_1)}<s,\\
&\left|DG(pz(t),t+t_1)-DG(pz(t),t+t_0)\right|_{L(X_1\times Y_1,Y_1)}<s,
\end{aligned}$$

for $(p,t)\in[0,1]\times[T_2,0]$.

We write $F(z(t),t+t_1)-F(z(t),t+t_0)$ as

$$\begin{aligned}
&F(z(t),t+t_1)-F(z(t),t+t_0)\\
=&F(z(t),t+t_1)-F(0,t+t_1)-F(z(t),t+t_0)-F(0,t+t_0)\\
=&\left(\int_0^1 DF(pz(t),t+t_1)-DF(pz(t),t+t_0)dp\right)z(t),
\end{aligned} \tag{3.45}$$

and such operation also applies to G. It follows that for $|t_1-t_0|<s'$,

$$\begin{aligned}
&\left|F(z(t),t+t_1)-F(z(t),t+t_0)\right|_{X_1}\le s|z(t)|_{X_1\times Y_1},\\
&\left|G(z(t),t+t_1)-G(z(t),t+t_0)\right|_{Y_1}\le s|z(t)|_{X_1\times Y_1}.
\end{aligned} \tag{3.46}$$

Following the same procedure in the proof of Lemma 3.9 and using (3.46), we obtain (3.44). Moreover, we can prove the following stronger statements,

$$\widetilde{\mathscr{T}_{cu}}\in C^0(\widetilde{B}_{\eta'}^{-}(\infty)\times\mathbb{R},\widetilde{B}_{\eta}^{-}(\infty)), \tag{3.47}$$

$$D\widetilde{\mathscr{T}_{cu}}\in C^0(\widetilde{B}_{\eta'}^{-}(\infty)\times\mathbb{R},L(\widetilde{B}_{\eta'}^{-}(\infty),\widetilde{B}_{\eta}^{-}(\infty)), \tag{3.48}$$

where $a_1<\eta<\eta'<a_2$ and D is the differentiation with respect to z.

Remark 3.15. *In the follow, when we use* (3.44), (3.47) *and* (3.48) *at fixed points, since they belong to* $\widetilde{B}_{\eta}^{-}(\infty)$ *for any* $\eta\in(a_1,a_2)$, *the loss of* η *decay is harmless.*

Let z_i be the fixed point of $\widetilde{\mathscr{T}_{cu}}(\cdot, t_i)$ for $i = 0, 1$. Since $\mathscr{T}_{cu}(\cdot, t_0)$ is a contraction in z,

$$
\begin{aligned}
|z_1 - z_0|_\eta^1 &\le \left|\widetilde{\mathscr{T}_{cu}}(z_1, t_1) - \widetilde{\mathscr{T}_{cu}}(z_0, t_1)\right|_\eta^1 + \left|\widetilde{\mathscr{T}_{cu}}(z_0, t_1) - \widetilde{\mathscr{T}_{cu}}(z_0, t_0)\right|_\eta^1 \\
&\le (1 - \sigma')|z_1 - z_0|_\eta^1 + \left|\widetilde{\mathscr{T}_{cu}}(z_0, t_1) - \widetilde{\mathscr{T}_{cu}}(z_0, t_0)\right|_\eta^1,
\end{aligned}
$$

together with (3.44), it implies

$$
\lim_{t_1 \to t_0} |z(t_1) - z(t_0)|_{\eta,\epsilon_\star}^- \le \frac{1}{\sigma} \lim_{t_1 \to t_0} \left|\mathscr{T}_{cu}(z_0, t_1) - \mathscr{T}_{cu}(z_0, t_0)\right|_{\eta,\epsilon_\star}^- = 0, \tag{3.49}
$$

Since $|h_s(\xi, t_1, \epsilon) - h_s(\xi, t_0, \epsilon)|_{X_1} = |P_s(z(t_1) - z(t_0))(0)|_{X_1}$ and h_s is Lipschitz in ξ, we have

$$
h_s(\cdot, \cdot, \epsilon) \in C^0(X_1^{cu} \times Y_1 \times \mathbb{R}, X_1^s).
$$

To prove the second part, by our assumptions in (A4) and (A6′) involving X and Y, one may also prove

$$
D\widetilde{\mathscr{T}_{cu}} \in C^0(\widetilde{B}_{\eta'}^-(\infty) \times \mathbb{R}, L(\overline{B}_{\eta'}^-(\infty), \overline{B}_\eta^-(\infty)), \tag{3.50}
$$

$$
\partial_{t_0}\widetilde{\mathscr{T}_{cu}} \in C^0(\widetilde{B}_{\eta'}^-(\infty) \times \mathbb{R}, \overline{B}_\eta^-(\infty)), \tag{3.51}
$$

where $\overline{B}_\eta^-(\infty) = \left\{z \Big| \|z\|_\eta < \infty\right\}$. Since (B2) implies $F(0, t) = G(0, t) = 0$, we obtain

$$
\begin{aligned}
&F(z_0, t + t_1) - F(z_0, t + t_0) - \partial_{t_0}F(z_0, t + t_0)(t_1 - t_0) \\
=&(t_1 - t_0)\Big(\int_0^1 \partial_t F(z_0, t + pt_1 + (1 - p)t_0) - \partial_t F(z_0, t + t_0)dp\Big) \\
=&(t_1 - t_0)\Big(\int_0^1\int_0^1 D\partial_{t_0}F(qz_0, t + pt_1 + (1 - p)t_0) \\
&- D\partial_t F(qz_0, t + t_0)dqdp\Big)\big(z_0(t)\big).
\end{aligned} \tag{3.52}
$$

Assumptions (A6′) and (3.52), (3.51) yield that

$$
\|\partial_{t_0}\widetilde{\mathscr{T}_{cu}}(z)\|_\eta \le C'\|z\|_\eta^1,
$$

which implies

$$
\left\|\widetilde{\mathscr{T}_{cu}}(z, t_1) - \widetilde{\mathscr{T}_{cu}}(z, t_0)\right\|_\eta \le C'|t_1 - t_0|\|z\|_\eta^1, \tag{3.53}
$$

where C' depends on C, a_1, a_2, η.

Since

$$z_1 - z_0 = \widetilde{\mathscr{T}_{cu}}(z_1, t_0) - \widetilde{\mathscr{T}_{cu}}(z_0, t_0) + \widetilde{\mathscr{T}_{cu}}(z_1, t_1) - \widetilde{\mathscr{T}_{cu}}(z_1, t_0), \tag{3.54}$$

using (3.50) and (3.53), we obtain

$$\|z_1 - z_0\|_\eta \leq \frac{C'}{\sigma'} \|z_1\|_\eta^1 |t_1 - t_0|. \tag{3.55}$$

We continue to write $z_1 - z_0$ as

$$z_1 - z_0 = \partial_{t_0} \widetilde{\mathscr{T}_{cu}}(z_0, t_0)(t_1 - t_0) + D\widetilde{\mathscr{T}_{cu}}(z_0, t_0)(z_1 - z_0) + R_1 + R_2, \tag{3.56}$$

where

$$R_1 = \widetilde{\mathscr{T}_{cu}}(z_1, t_1) - \widetilde{\mathscr{T}_{cu}}(z_1, t_0) - \partial_{t_0} \widetilde{\mathscr{T}_{cu}}(z_0, t_0)(t_1 - t_0),$$
$$R_2 = \widetilde{\mathscr{T}_{cu}}(z_1, t_0) - \widetilde{\mathscr{T}_{cu}}(z_0, t_0) - D\widetilde{\mathscr{T}_{cu}}(z_0, t_0)(z_1 - z_0).$$

By (3.49) and (3.51),

$$\begin{aligned} \|R_1\|_\eta =& |t_1 - t_0| \Big\| \int_0^1 \partial_{t_0} \widetilde{\mathscr{T}_{cu}}(z_1, pt_1 + (1-p)t_0) - \partial_{t_0} \widetilde{\mathscr{T}_{cu}}(z_0, t_0) dp \Big\|_\eta \\ \leq& |t_1 - t_0| o(1) \leq o(|t_1 - t_0|). \end{aligned}$$

Using (3.50), we have

$$\|R_2\|_\eta = o(\|z_1 - z_0\|_\eta) = o(|t_1 - t_0|).$$

Finally, by (3.56),

$$\partial_{t_0} z_0 = \Big(I - D\widetilde{\mathscr{T}_{cu}}(z_0, t_0)\Big)^{-1} \partial_{t_0} \widetilde{\mathscr{T}_{cu}}(z_0, t_0). \tag{3.57}$$

Therefore, one has $\partial_{t_0} h_s(\cdot, \cdot, \epsilon) \in C^0(X_1^{cu} \times Y_1 \times \mathbb{R}, X^s)$. □

For $\mathcal{M}_\epsilon^s, \mathcal{M}_\epsilon^{cu}$ and $\mathcal{M}_\epsilon^c$, we have the following results.

Theorem 3.16. *Let k be a positive integer and assume $\epsilon_\star, \epsilon$ are sufficiently small. If f, g are C^k functions and there exists $\eta_0 < 0$ with $a_1 < k\eta_0 < \eta_0 < a_2$, then we have the unique stable integral manifold $\mathcal{M}_\epsilon^s$ defined in a small neighborhood of the origin, which is C^k in ξ_s. Moreover, all the derivatives of h_{cu} defined in* (3.14) *with respect to ξ_s are uniformly bounded in (t_0, ϵ).*

Theorem 3.17. *Let k be a positive integer and assume $\overline{r}, \epsilon_\star, \epsilon$ are sufficiently small. If f, g are C^k functions and there exists η with $a_1' < i\eta < a_2'$ for $i = 1, 2, \cdots, k$. Then there exists a center-stable integral manifold $\mathcal{M}_\epsilon^{cs}$ which is C^k in $(\xi_s, \xi_c, \xi_y) \in X_1^s \times X_1^c \times Y_1$.*

Theorem 3.18. *Assume the conditions in Theorems 3.10 and 3.17 are both satisfied. Then the center integral manifold $\mathcal{M}_\epsilon^c$ is C^k in $(\xi_c, \xi_y) \in X_1^c \times Y_1$.*

Proof. From Theorem 3.10 and Theorem 3.16, we know that h_s and h_u are C^k mappings in (ξ_c, ξ_y). Uniform contraction principle shows (Ψ_s, Ψ_u) are C^k in (ξ_c, ξ_y). Therefore, by the definition of $\mathcal{M}_\epsilon^c$, which is given in Theorem 3.7, we can finish the proof. □

3.4 Asymptotics of Invariant Manifolds

In Theorems 2.4 and 2.4, we have demonstrated that (2.2) can be viewed as the singular limit of (2.1) as $\epsilon \to 0$. Therefore, one may expect the perturbed invariant manifolds should be close to the unperturbed ones. In this section, we will prove their closeness. Moreover, we will give asymptotic expansions of some invariant manifolds.

As in Theorem 2.4, equation

$$\dot{x}_0(t) = A_f x_0(t) + F(x_0(t), 0, t, 0), \tag{3.58}$$

can be viewed as the singular limit of (3.5). For $\xi_{cu} \in X_1^{cu}$ and $x \in C_\eta^-(X_1)$, let

$$\begin{aligned}\widetilde{\mathscr{T}_0}(x)(t) =& e^{tA_f}\xi_{cu} + \int_0^t e^{(t-\tau)A_f} P_{cu} F(x(\tau), 0, \tau + t_0, 0)\, d\tau \\ &+ \int_{-\infty}^t e^{(t-\tau)A_f} P_s F(x(\tau), 0, \tau + t_0, 0)\, d\tau.\end{aligned} \tag{3.59}$$

By the exponential dichotomy property in (B5), one may construct the center-unstable manifold $\mathcal{M}_0^{cu}$ of (3.58) from the fixed point of $\widetilde{\mathscr{T}_0}$, which is a contraction on $C_\eta^-(X_1)$. For $\xi_{cu} \in X_1^{cu}$, let $x_0(t)$ be the fixed point of $\widetilde{\mathscr{T}_0}$. Define

$$h_s^0(\xi_{cu}) = P_s x_0(0) = \int_{-\infty}^0 e^{-\tau A_f} P_s F(x_0, 0, \tau + t_0, 0)\, d\tau,$$

$$\mathcal{M}_0^{cu} = \Big\{\xi_{cu} + h_s^0(\xi_{cu}) \big| \xi_{cu} \in X_1^{cu}\Big\},$$

which is the center-unstable manifold of the modified unperturbed system. By assumption (B1), the system (3.58) is autonomous, so $\mathcal{M}_0^{cu}$ is independent of t_0. A natural question is that if $\mathcal{M}_\epsilon^{cu}$ converges to $\mathcal{M}_0^{cu}$ as $\epsilon \to 0$.

Theorem 3.19. *Assume the conditions in Theorem 3.10 for $k = 1$ and (A6), (A6′). Then*

$$\big|h_s(\xi_{cu}, 0, t_0, \epsilon) - h_s^0(\xi_{cu})\big|_{X_1} \le C'\epsilon.$$

If in addition to the conditions in Theorem 3.10 hold for $k = 2$, then

$$\big|D_{\xi_{cu}} h_s(\xi_{cu}, 0, t_0, \epsilon) - D_{\xi_{cu}} h_s^0(\xi_{cu})\big|_{L(X_1^{cu}, X_1^s)} \le C'\epsilon,$$

$$\big|D_{\xi_y} h_s(\xi_{cu}, 0, t_0, \epsilon)\big|_{L(Y_1, X^s)} \le C'\epsilon,$$

where C' depends on $K, a_1, a_2, \eta, |\xi_{cu}|_{X_1}, \epsilon_\star, \overline{r}$.

Proof. Let (x, y) is the fixed point of $\widetilde{\mathscr{T}_{cu}}$ with parameter $\xi_{cu}, \xi_y = 0, t_0, \epsilon$ and x_0 be the fixed point of $\widetilde{\mathscr{T}_0}$ with parameter ξ_{cu}. In the rest of the proof, we will use $\widetilde{\mathscr{T}_{cu}}(x, y, \epsilon)$ to denote $\widetilde{\mathscr{T}_{cu}}(z, t_0)$, which is introduced in (3.41). Note that

$$\begin{aligned}\|(x - x_0, y)\|_\eta^1 &\le \|\widetilde{\mathscr{T}_{cu}}(x, y, \epsilon) - \widetilde{\mathscr{T}_{cu}}(x_0, 0, \epsilon)\|_\eta^1 + \|\widetilde{\mathscr{T}_{cu}}(x_0, 0, \epsilon) - \widetilde{\mathscr{T}_0}(x_0)\|_\eta^1 \\ &\le (1 - \sigma')\|(x - x_0, y)\|_\eta^1 + \|\widetilde{\mathscr{T}_{cu}}(x_0, 0, \epsilon) - \widetilde{\mathscr{T}_0}(x_0)\|_\eta^1,\end{aligned}$$

which implies

$$\|(x - x_0, y)\|_\eta^1 \le \frac{1}{\sigma'}\|\widetilde{\mathscr{T}_{cu}}(x_0, 0, \epsilon) - \widetilde{\mathscr{T}_0}(x_0)\|_\eta^1.$$

By assumptions (A6) and (B2), $\partial_\epsilon F(0,0,\tau+t_0,\epsilon)=0$ for any ϵ. It implies

$$\begin{aligned}&\big|F(x_0,0,\tau+t_0,\epsilon)-F(x_0,0,\tau+t_0,0)\big|_{X_1}\\ =&\epsilon\big|\int_0^1 \partial_\epsilon F(x_0,0,\tau+t_0,p\epsilon)dp\big|_{X_1}\le C_0\epsilon|x_0(\tau)|_{X_1}.\end{aligned} \tag{3.60}$$

Then, using (3.41), (3.59) and (3.60),

$$\begin{aligned}&\|\widetilde{\mathscr{T}_{cu}}(x_0,0,\epsilon)-\widetilde{\mathscr{T}_0}(x_0)\|_\eta^1\\ =&\sup_{t\le 0}e^{-\eta t}\Bigg(\frac{1}{\epsilon_\star}\Big|-(\frac{J}{\epsilon}+g_y)^{-1}g_x x_0(t)+(\frac{J}{\epsilon}+g_y)^{-1}e^{t(\frac{J}{\epsilon}+g_y)}g_x x_0(0)\\ &+\int_0^t(\frac{J}{\epsilon}+g_y)^{-1}e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}g_x(A_f x_0+F(x_0,0,t+t_0,\epsilon))\,d\tau\Big|_{Y_1}\\ &+\Big|\int_0^t e^{(t-\tau)A_f}P_{cu}\big(F(x_0,0,\tau+t_0,\epsilon)-F(x_0,0,\tau+t_0,0)\big)\,d\tau\\ &+\int_{-\infty}^t e^{(t-\tau)A_f}P_s\big(F(x_0,0,\tau+t_0,\epsilon)-F(x_0,0,\tau+t_0,0)\big)\,d\tau\Big|_{X_1}\Bigg)\\ \le&\epsilon\left(\frac{1}{\epsilon_\star}\big(1+K+\frac{K}{a_2-\eta}(1+\overline{r})\big)2C_0+(\frac{K}{\eta-a_1}+\frac{K}{a_2-\eta})\right)C_0|x_0|_{C_\eta^-(X_1)},\end{aligned}$$

which implies

$$\|(x-x_0,y)\|_\eta^1\le\frac{C_1'}{\sigma'\epsilon_\star}\epsilon. \tag{3.61}$$

Therefore, $\big|h_s(\xi_{cu},0,t_0,\epsilon)-h_s^0(\xi_{cu})\big|_{X_1}=|(x-x_0,y)(0)|_{X_1}\le\frac{C_1'}{\sigma'\epsilon_\star}\epsilon$.

To prove the second part, choose η such that $a_1<\eta, 2\eta<s_2$. Let $(\phi^\epsilon(t),\psi^\epsilon(t))$ be the derivative of $(x(t),y(t))$ with respect to ξ_{cu} at $(\xi_{cu},0)$ and $\phi^0(t)$ be the derivative of $x_0(t)$ with respect to ξ_{cu}, so we have

$$(\phi^\epsilon,\psi^\epsilon)=D\widetilde{\mathscr{T}_{cu}}(x,y,\epsilon)(\phi^\epsilon,\psi^\epsilon)+e^{tA_f}\quad,\quad\phi^0=D\widetilde{\mathscr{T}_0}(x_0)(\phi^0)+e^{tA_f}. \tag{3.62}$$

As in Theorem 3.10, we can show ϕ^0 is uniformly bounded in X_1^{cu}. And we choose ρ_1

which is independent of x_0 so that $|\phi^0|_{L(X_1^{cu},C_{i\eta}^-(X_1))} \le \rho_1$ for $i = 1, 2$. By (3.62),

$$\begin{aligned}
&\Big|(\phi^\epsilon - \phi^0, \psi^\epsilon)\Big|_{L(X_1^{cu},\widetilde{B}_{2\eta}^-(\infty))} \\
\le&\Big|D\widetilde{\mathscr{T}_{cu}}(x, y, \epsilon)(\phi^\epsilon - \phi^0, \psi^\epsilon)\Big|_{L(X_1^{cu},\widetilde{B}_{2\eta}^-(\infty))} \\
&+ \Big|(D\widetilde{\mathscr{T}_{cu}}(x, y, \epsilon) - D\widetilde{\mathscr{T}_{cu}}(x_0, 0, \epsilon))(\phi^0, 0)\Big|_{L(X_1^{cu},\widetilde{B}_{2\eta}^-(\infty))} \\
&+ \Big|D\widetilde{\mathscr{T}_{cu}}(x_0, 0, \epsilon)(\phi^0, 0) - D\widetilde{\mathscr{T}_0}(x_0)\phi^0\Big|_{L(X_1^{cu},\widetilde{B}_{2\eta}^-(\infty))}.
\end{aligned} \tag{3.63}$$

Since (F, G) are C^2 with respect to (x, y) and by (3.61),

$$\begin{aligned}
&\Big|(D\widetilde{\mathscr{T}_{cu}}(x, y, \epsilon) - D\widetilde{\mathscr{T}_{cu}}(x_0, 0, \epsilon))(\phi^0, 0)\Big|_{L(X_1^{cu},\widetilde{B}_{2\eta}^-(\infty))} \\
\le&(\frac{K}{2\eta - a_1} + \frac{K}{a_2 - 2\eta} + 2C_0^2\epsilon\frac{K}{a_2 - 2\eta})C_0\rho_1 C_1'\epsilon.
\end{aligned} \tag{3.64}$$

For $D\widetilde{\mathscr{T}_{cu}}(x_0, 0)(\phi^0, 0) - D\widetilde{\mathscr{T}_0}(x_0)\phi^0$, we have

$$\begin{aligned}
&D\widetilde{\mathscr{T}_{cu}}(x_0, 0, \epsilon)(\phi^0, 0) - D\widetilde{\mathscr{T}_0}(x_0)\phi^0 \\
=&\int_0^t e^{(t-\tau)A_f} P_{cu}\big(D_xF(x_0, 0, \tau + t_0, \epsilon) - D_xF(x_0, 0, \tau + t_0, 0)\big)\phi^0 \, d\tau \\
&+ \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(D_xF(x_0, 0, \tau + t_0, \epsilon) - D_xF(x_0, 0, \tau + t_0, 0)\big)\phi^0 \, d\tau \\
&+ \int_0^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} D_xG(x_0, 0, \tau + t_0, \epsilon)\phi^0 \, d\tau \\
&- (\frac{J}{\epsilon} + g_y)^{-1} g_x\phi^0(t) + (\frac{J}{\epsilon} + g_y)^{-1} e^{t(\frac{J}{\epsilon}+g_y)} g_x\phi^0(0) \\
&+ \int_0^t (\frac{J}{\epsilon} + g_y)^{-1} e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} g_x(A_f\phi^0 + D_xF(x_0, 0, \tau + t_0, \epsilon)\phi^0) \, d\tau.
\end{aligned}$$

Except the third integral, the estimates of other integrals are straightforward. we integrate by parts for the third one to obtain

$$
\begin{aligned}
&\int_0^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} D_x G(x_0, 0, t+t_0, \epsilon)\phi^0 \; d\tau \\
= & -(\frac{J}{\epsilon}+g_y)^{-1} D_x G(x_0(t), 0, t+t_0, \epsilon)\phi^0(t) \\
& + (\frac{J}{\epsilon}+g_y)^{-1} e^{t(\frac{J}{\epsilon}+g_y)} D_x G(x_0(0), 0, t_0, \epsilon)\phi^0(0) \\
& + \int_0^t (\frac{J}{\epsilon}+g_y)^{-1} e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} D_x^2 G(x_0, 0, \tau+t_0, \epsilon) \\
& (A_f x_0 + F(x_0, 0, \tau+t_0, 0), \phi^0) \; d\tau \\
& + \int_0^t (\frac{J}{\epsilon}+g_y)^{-1} e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} D_x \partial_t G(x_0, 0, \tau+t_0, \epsilon)\phi^0 \; d\tau \\
& + \int_0^t (\frac{J}{\epsilon}+g_y)^{-1} e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} D_x G(x_0, 0, \tau+t_0, \epsilon) \\
& (A_f \phi^0 + D_x F(x_0, 0, \tau+t_0, 0)\phi^0) \; d\tau.
\end{aligned}
$$

Therefore,

$$
\sup_{t\le 0} \frac{1}{\epsilon_\star} e^{-2\eta t} \Big| \int_0^t e^{(t-\tau)(\frac{J}{\epsilon}+g_y)} D_x G(x_0, 0, t+t_0, \epsilon)\phi^0 \; d\tau \Big| \le \frac{C_2'}{\epsilon_\star}\epsilon.
$$

Consequently,

$$
\Big| D\widetilde{\mathscr{T}_{cu}}(x_0, 0)(\phi^0, 0) - D\widetilde{\mathscr{T}_0}(x_0)\phi^0 \Big|_{L(X_1^{cu}, \widetilde{B}_{2\eta}^-(\infty))} \le \frac{C_2'}{\epsilon_\star}\epsilon, \tag{3.65}
$$

where C_2' depends on $K, a_2, \eta, C_0, |\xi_{cu}|_{X_1}, \rho_1$. The first term of (3.63) can be estimated by the fact that $D\widetilde{\mathscr{T}_{cu}}(x, y)$ is a linear contraction with Lipschitz constant $(1-\sigma')$. Combining (3.63), (3.64) and (3.65), we obtain

$$
\Big| (\phi^\epsilon - \phi^0, \psi^\epsilon) \Big|_{L(X_1^{cu}, \widetilde{B}_{2\eta}^-(\infty))} \le \frac{1}{\sigma' \epsilon_\star} C_2' \epsilon, \tag{3.66}
$$

where C_2' depends on $K, a_1, a_2, \eta, C_0, |\xi_{cu}|_{X_1}$.

With slight abuse of notation, we still use $(\phi^\epsilon, \psi^\epsilon)$ to denote the derivative of (x, y)

with respect to ξ_y at $\xi_y = 0$. We have $(\phi^\epsilon, \psi^\epsilon)$ satisfy

$$\begin{aligned}\phi^\epsilon(t) = &\int_0^t e^{(t-\tau)A_f} P_{cu} D_x F(x, y, \tau + t_0, \epsilon)\phi^\epsilon(\tau)\ d\tau \\ &+ \int_{-\infty}^t e^{(t-\tau)A_f} P_s D_x F(x, y, \tau + t_0, \epsilon)\phi^\epsilon(\tau)\ d\tau \\ &+ \int_0^t e^{(t-\tau)A_f} P_{cu}\big(D_y F(x, y, \tau + t_0, \epsilon) + f_y\big)\psi^\epsilon(\tau)\ d\tau \\ &+ \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(D_y F(x, y, \tau + t_0, \epsilon) + f_y\big)\psi^\epsilon(\tau)\ d\tau,\end{aligned} \tag{3.67}$$

$$\begin{aligned}\dot{\psi}^\epsilon(t) = &(\frac{J}{\epsilon} + g_y + D_y G(x, y, t + t_0, \epsilon))\psi^\epsilon(t) \\ &+\big(D_x G(x, y, t + t_0, \epsilon) + g_x\big)\phi^\epsilon(t).\end{aligned} \tag{3.68}$$

By rewriting (3.68), we have

$$\psi^\epsilon = \epsilon J^{-1}\dot{\psi}^\epsilon - \epsilon J^{-1}\big((g_y + D_y G)\psi^\epsilon + (D_x G + g_x)\phi^\epsilon\big). \tag{3.69}$$

Substituting (3.69) into (3.67). Since the first two integrals in (3.67) define a contraction mapping for ϕ^ϵ, once we prove

$$\begin{aligned}&\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t e^{(t-\tau)A_f} P_{cu}\Big(D_y F(x, y, \tau + t_0, \epsilon) + f_y\Big) \\ &\qquad\qquad \epsilon J^{-1}\dot{\psi}^\epsilon\ d\tau\Big|_{L(Y_1, X^{cu})} \le C'\epsilon, \\ &\sup_{t\le 0} e^{-2\eta t}\Big|\int_{-\infty}^t e^{(t-\tau)A_f} P_s\Big(D_y F(x, y, \tau + t_0, \epsilon) + f_y\Big) \\ &\qquad\qquad \epsilon J^{-1}\dot{\psi}^\epsilon\ d\tau\Big|_{L(Y_1, X^s)} \le C'\epsilon,\end{aligned}$$

we obtain the estimate on ϕ^ϵ and thus complete the proof. Here we only prove the first inequality. Integrating by parts, we have

$$
\begin{aligned}
&\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t e^{(t-\tau)A_f}P_{cu}\Big(D_yF(x,y,\tau+t_0,\epsilon)+f_y\Big)\epsilon J^{-1}\dot{\psi}^\epsilon\, d\tau\Big| \\
\le&\sup_{t\le 0} e^{-2\eta t}\Big|e^{(t-\tau)A_f}P_{cu}\Big(D_yF(x,y,\tau+t_0,\epsilon)+f_y\Big)\epsilon J^{-1}\psi^\epsilon(\tau)\Big|_0^t\Big| \\
&+\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t \frac{d}{d\tau}\Big(e^{(t-\tau)A_f}P_{cu}\Big(D_yF(x,y,\tau+t_0,\epsilon)+f_y\Big)\Big)\epsilon J^{-1}\psi^\epsilon\, d\tau\Big| \\
\le& C_0\epsilon(\frac{K}{a_2-2\eta}+K)(\overline{r}+C_0)|\psi^\epsilon|_{L(Y_1,\widetilde{B}^-_{2\eta}(\infty))} \\
&+\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t \frac{d}{d\tau}\Big(e^{(t-\tau)A_f}P_{cu}\Big(D_yF(x,y,\tau+t_0,\epsilon)+f_y\Big)\Big)\epsilon J^{-1}\psi^\epsilon\, d\tau\Big|,
\end{aligned}
$$

where we recall that in Theorem 3.10 we proved $|\phi^\epsilon|+|\psi^\epsilon|$ are uniformly bounded. Since $\xi_y=0$, by (3.61), $|y(t)|_{Y_1}\le C_3' e^{\eta t}\epsilon$, where C_3' depends on constants in assumptions and σ. Using this fact, assumption (A6$'$) and a straight forward computation, we obtain

$$
\begin{aligned}
&\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t \frac{d}{d\tau}\Big(e^{(t-\tau)A_f}P_{cu}\Big(D_yF(x,y,\tau+t_0,\epsilon)+f_y\Big)\Big)\epsilon J^{-1}\psi\, d\tau\Big| \\
\le& C_0\epsilon\frac{K}{a_2-2\eta}(\overline{r}+C_0)|\psi^\epsilon|_{L(Y_1,\widetilde{B}^-_{2\eta}(\infty))}+C_0\epsilon\frac{K}{a_2-2\eta}|\psi^\epsilon|_{L(Y_1,\widetilde{B}^-_{2\eta}(\infty))} \\
&+C_0^2\epsilon\frac{K}{a_2-2\eta}(\overline{r}+C_0+1)(|x|_{C^-_\eta(X_1)}+|y|_{C^-_\eta(Y_1)})|\psi^\epsilon|_{L(Y_1,\widetilde{B}^-_\eta(\infty))} \\
&+C_0^2\epsilon\frac{K}{a_2-2\eta}\big(C_3'+(\overline{r}+C_0)(|x|_{C^-_\eta(X_1)}+|y|_{C^-_\eta(Y_1)})\big)|\psi^\epsilon|_{L(Y_1,\widetilde{B}^-_\eta(\infty))} \\
\le& C_4'\epsilon.
\end{aligned}
$$

Let $C'=\max\{\frac{C_1'}{\sigma'\epsilon_\star},\frac{C_2'}{\sigma'\epsilon_\star},C_3',C_4'\}$, we complete the proof. □

Let $\Phi(T,t_0,\xi,\epsilon)$ and $\Phi^0(T,t_0,\xi')$ be solutions of (2.1) and (2.2) at terminal time T with $\Phi(t_0,t_0,\xi,\epsilon)=\xi$ and $\Phi(t_0,t_0,\xi')=\xi'$. Combining with Theorem 2.4, Theorem 2.4 and Corollary 2.7, we have

Theorem 3.20. *If the conditions in Theorem 3.10 hold for $k=2$, then there exists*

C' *which depends on* $C, K, \eta, a_1, a_2, \overline{r}, \epsilon_\star, T - t_0, |\xi_{cu}|_{X_1}$ *such that*

$$\begin{aligned}
&\left|\Phi(T, t_0, \xi_{cu} + h_s(\xi_{cu}, 0, t_0, \epsilon), \epsilon) - \Phi^0(T, t_0, \xi_{cu} + h_s^0(\xi_{cu}))\right|_{X_1 \times Y_1} \leq C'\epsilon, \\
&\Big| D_{\xi_{cu}} \left(\Phi(T, t_0, \xi_{cu} + h_s(\xi_{cu}, 0, t_0, \epsilon), \epsilon)\right) \\
&\qquad\qquad - D_{\xi_{cu}} \left(\Phi^0(T, t_0, \xi_{cu} + h_s^0(\xi_{cu}))\right) \Big|_{L(X_1^{cu}, X_1 \times Y_1)} \leq C'\epsilon, \\
&\left| P_X \left(D_{\xi_y} \left(\Phi(T, t_0, \xi_{cu} + \xi_y + h_s(\xi_{cu}, \xi_y, t_0, \epsilon), \epsilon)\right) \right)\Big|_{\xi_y = 0} \right|_{L(Y_1, X)} \leq C'\epsilon, \\
&\Big| P_Y \Big(D_{\xi_y} \left(\Phi(T, t_0, \xi_{cu} + \xi_y + h_s(\xi_{cu}, \xi_y, t_0, \epsilon), \epsilon)\right) \\
&\qquad\qquad - E(T, t_0; \xi_{cu} + h_s^0(\xi_{cu}), \epsilon) \Big)\Big|_{\xi_y = 0} \Big|_{L(Y_1, Y_1)} \leq C'\epsilon.
\end{aligned}$$

where $E(T, t_0; \xi_{cu} + h_s^0(\xi_{cu}), \epsilon)$ *is the evolution operator generated by* $\frac{J}{\epsilon} + D_y g(\Phi^0(t, t_0, \xi_{cu} + h_s^0(\xi_{cu})), 0, t, 0)$ *and* P_X, P_Y *denote the projection map from* $X \times Y$ *to* X *and* Y*, respectively.*

Proof. From Theorem 3.19, since we have

$$\begin{aligned}
&\left|\Phi(t_0, t_0, \xi_{cu} + h_s(\xi_{cu}, 0, t_0, \epsilon), \epsilon) - \Phi^0(t_0, t_0, \xi_{cu} + h_s^0(\xi_{cu}))\right|_{X_1 \times Y_1} \\
= &\left|\xi_{cu} + h_s(\xi_{cu}, 0, t_0, \epsilon) - \xi_{cu} - h_s^0(\xi_{cu})\right|_{X_1 \times Y_1} \leq C'\epsilon.
\end{aligned}$$

And thus we obtain the first part which follows from Theorem 2.4. To prove the second inequality, first we have

$$\begin{aligned}
&\left| D_{\xi_{cu}} \left(\xi_{cu} + h_s(\xi_{cu}, 0, t_0, \epsilon)\right) - D_{\xi_{cu}} \left(\xi_{cu} + h_s^0(\xi_{cu})\right) \right|_{L(X_1^{cu}, X_1^s)} \\
= &\left| D_{\xi_{cu}} h_s(\xi_{cu}, 0, t_0, \epsilon) - D_{\xi_{cu}} h_s^0(\xi_{cu}) \right|_{L(X_1^{cu}, X_1^s)} \leq C'\epsilon.
\end{aligned}$$

Since the range of $I + D_{\xi_{cu}} h_s$ is in X_1, by (2.6) and the remark after Theorem 2.4, we obtain the second part. Finally, from Theorem 3.19 we have

$$\left| D_{\xi_y} h_s(\xi_{cu}, \xi_y, t_0, \epsilon)\Big|_{\xi_y = 0} \right|_{L(Y_1, X^s)} \leq C'\epsilon,$$

we can apply Theorem 2.4 to finish the proof. □

In fact, we can calculate the leading order of $h_s(\xi_{cu}, \xi_y, t_0, \epsilon) - h_s^0(\xi_{cu})$ in X^s explicitly. Let $x_0(t)$ be the fixed point of $\widetilde{\mathscr{T}_0}$ defined in (3.59),

$$
\begin{aligned}
(\mathcal{L}\Delta x)(t) &\triangleq D\widetilde{\mathscr{T}_0}(x_0)(\Delta x)(t) \\
&= \int_0^t e^{(t-\tau)A_f} P_{cu} D_x F(x_0, 0, \tau + t_0, 0)\Delta x(\tau)\, d\tau \\
&\quad + \int_{-\infty}^t e^{(t-\tau)A_f} P_s D_x F(x_0, 0, \tau + t_0, 0)\Delta x(\tau)\, d\tau.
\end{aligned}
$$

Clearly, $\mathcal{L}$ defines a linear operator from $C_\eta^-(X)$ to $C_\eta^-(X)$ with norm strictly less than 1.

Theorem 3.21. *Assume (A4) for $k = 1$, (A6), and*

$$\sup_{t \le 0} |D_{yx} F(x_0, 0, t + t_0, 0)|_{L(X \otimes Y_1, X)} < C_0.$$

Let (x, y) be the fixed point of $\widetilde{\mathscr{T}_{cu}}$ with parameter (ξ_{cu}, ξ_y) and $|\xi_y|_{Y_1} \le C_1 \epsilon$. If there exists η with $a_1 < \eta, 2\eta < a_2$, then

$$
\begin{aligned}
&\sup_{t \le 0} e^{-2\eta t} \Bigg| (x - x_0)(t) + \epsilon (I - \mathcal{L})^{-1} \\
&\Big\{ \int_0^t e^{(t-\tau)A_f} P_{cu} \Big(\partial_\epsilon F - (D_y F + f_y) J^{-1}(G + g_x x_0) \Big)\, d\tau \\
&+ \int_{-\infty}^t e^{(t-\tau)A_f} P_s \Big(\partial_\epsilon F + (D_y F + f_y) J^{-1}(G + g_x x_0) \Big)\, d\tau \Big\} \Bigg|_X = o(\epsilon),
\end{aligned}
$$

where $\partial_\epsilon F, D_y F$ and G are evaluated at $(x_0(\tau), 0, \tau + t_0, 0)$.

Remark 3.22. *Theorem 3.16 shows the difference between $h_s(\xi_{cu}, \xi_y, t_0, \epsilon)$ and $h_s^0(\xi_{cu})$ is small in X_1^s, but we can only calculate the leading order of the difference in X^s. If we further assume $x_0 \in C^1(\mathbb{R}^-, X_1)$, the above theorem can be proved in X_1. If $g = 0$, then x equation and y equation are decoupled. So we recover the formula for $\partial_\epsilon h_s$ in the regular perturbation case. The term $(D_y F + f_y) J^{-1}(G + g_x x_0)$ is the contribution from y equation after averaging.*

Proof. First, we note

$$
\begin{aligned}
&x(t) - x_0(t) \\
&= \int_0^t e^{(t-\tau)A_f} P_{cu}\big(F(x,y,\tau+t_0,\epsilon) - F(x_0,0,\tau+t_0,0) + f_y y\big)\, d\tau \\
&\quad + \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(F(x,y,\tau+t_0,\epsilon) - F(x_0,0,\tau+t_0,0) + f_y y\big)\, d\tau \\
&= \mathcal{L}(x - x_0) + \epsilon\left(\left(\int_0^t e^{(t-\tau)A_f} P_{cu} + \int_{-\infty}^t e^{(t-\tau)A_f} P_s\right) \partial_\epsilon F\, d\tau\right) \\
&\quad + \left(\int_0^t e^{(t-\tau)A_f} P_{cu} + \int_{-\infty}^t e^{(t-\tau)A_f} P_s\right)\big((D_y F + f_y)y\big)\, d\tau + R
\end{aligned}
$$

where $\partial_\epsilon F, D_y F, G$ are evaluated at $(x_0(\tau), 0, \tau + t_0, 0)$ and R is the remainder. The explicit form of R is

$$
\begin{aligned}
R = &\left(\int_0^t e^{(t-\tau)A_f} P_{cu} + \int_{-\infty}^t e^{(t-\tau)A_f} P_s\right)\left[\left(\int_0^1 (DF(p) - DF(0))\, dp\right)\right. \\
&\left.(x - x_0, y) + \epsilon\left(\int_0^1 (\partial_\epsilon F(p) - \partial_\epsilon F(0))\, dp\right)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
DF(p) &= DF(px + (1-p)x_0, py, \tau + t_0, p\epsilon), \\
\partial_\epsilon F(p) &= \partial_\epsilon F(px + (1-p)x_0, py, \tau + t_0, p\epsilon).
\end{aligned}
$$

When we estimate R, we will lose some exponential weight η. Since we only work on fixed points, as in the remark in the proof of Theorem 3.10, the loss of η does not cause any problem. From the C^1 assumption on F, (3.61) and the proof of Theorem 3.10, for any $\eta < \eta'$, by choosing (x, y) and x_0 in $C^-_{2\eta'}(X_1) \times C^-_{2\eta'}(Y_1)$ and $C^-_{2\eta'}(X_1)$, respectively, one can show

$$
\sup_{t \le 0} e^{-2\eta t} |R|_X = o(\epsilon).
$$

Moreover, by assumptions (A6) and (B2),

$$
\begin{aligned}
&\sup_{t\le 0} e^{-2\eta t}\left|\left(\int_0^t e^{(t-\tau)A_f}P_{cu}+\int_{-\infty}^t e^{(t-\tau)A_f}P_s\right)\partial_\epsilon F\ d\tau\right|_X\\
\le&\sup_{t\le 0} e^{-2\eta t}\Bigg|\left(\int_0^t e^{(t-\tau)A_f}P_{cu}\partial_\epsilon+\int_{-\infty}^t e^{(t-\tau)A_f}P_s\right)\\
&\Big(\partial_\epsilon F(x_0,0,\tau+t_0,0)-\partial_\epsilon F(0,0,\tau+t_0,0)\Big)\ d\tau\Bigg|_X\\
\le&\left(\frac{K}{a_2-2\eta}+\frac{K}{2\eta-a_1}\right)C_0|x_0|^-_{2\eta,1,X_1}<\infty.
\end{aligned}
$$

For simplicity of notation, we let $w(t)=D_yF(x_0(t),0,\tau+t_0,0)+f_y$, which satisfies $w\in C^0(\mathbb{R},L(Y_1,X_1))$ with $|w|_{C^0}<\infty$ and $\dot w\in C^-_\eta(L(Y_1,X))$. We claim that

$$
\begin{aligned}
\sup_{t\le 0} e^{-2\eta t}\Bigg|&\int_0^t e^{(t-\tau)A_f}P_{cu}w(\tau)y(\tau)\ d\tau+\epsilon\Big(\int_0^t e^{(t-\tau)A_f}P_{cu}\\
&w(\tau)J^{-1}\big(G(x_0,0,\tau+t_0,0)+g_xx_0\big)\ d\tau\Big)\Bigg|_X=O(\epsilon^2),
\end{aligned}
\tag{3.70}
$$

and

$$
\begin{aligned}
\sup_{t\le 0} e^{-2\eta t}\Bigg|&\int_{-\infty}^t e^{(t-\tau)A_f}P_s w(\tau)y(\tau)\ d\tau-\epsilon\Big(\int_{-\infty}^t e^{(t-\tau)A_f}P_s\\
&w(\tau)J^{-1}\big(G(x_0,0,\tau+t_0,0)+g_xx_0\big)\ d\tau\Big)\Bigg|_X=O(\epsilon^2).
\end{aligned}
\tag{3.71}
$$

Clearly, $|\xi_y|_{Y_1}\le C_1\epsilon$ along with integration by parts implies

$$
\sup_{t\le 0} e^{-2\eta t}\left|\left(\int_0^t e^{(t-\tau)A_f}P_{cu}+\int_{-\infty}^t e^{(t-\tau)A_f}\right)w(\tau)e^{\tau(\frac{J}{\epsilon}+g_y)}\xi_y\ d\tau\right|_X=O(\epsilon^2).
$$

It implies that when $|\xi_y|_{Y_1}\le C_1\epsilon$, it doesn't contribute anything to the ϵ order of $x-x_0$. In the rest of the proof, we will take $\xi_y=0$.

To prove (3.70), we use (3.8) and interchange of the order of integrals to deduce

$$
\begin{aligned}
&\int_0^t e^{(t-\tau)A_f}P_{cu}w(\tau)y(\tau)\ d\tau\\
=&\int_0^t \Big(\int_s^t e^{(t-\tau)A_f}P_{cu}w(\tau)e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau\Big)\big(G(x,y,s+t_0,\epsilon)+g_xx\big)ds.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_s^t e^{(t-\tau)A_f} P_{cu} w(\tau) e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau \\
&= P_{cu} w(t)(\frac{J}{\epsilon}+g_y)^{-1} e^{(t-s)(\frac{J}{\epsilon}+g_y)} - e^{(t-s)A_f} P_{cu} w(s)(\frac{J}{\epsilon}+g_y)^{-1} \\
&\quad - \int_s^t \frac{d}{d\tau}(e^{(t-\tau)A_f} P_{cu} w(\tau))(\frac{J}{\epsilon}+g_y)^{-1} e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau,
\end{aligned}
$$

by (3.61) and assumption (A6),

$$
\begin{aligned}
&\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t P_{cu} w(t)(\frac{J}{\epsilon}+g_y)^{-1} e^{(t-s)(\frac{J}{\epsilon}+g_y)} \\
&\qquad \big(G(x,y,s+t_0,\epsilon)+g_x x\big)ds\Big|_X \\
&\le \sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t P_{cu} w(t)(\frac{J}{\epsilon}+g_y)^{-1} e^{(t-s)(\frac{J}{\epsilon}+g_y)} \\
&\qquad (G(x_0,0,s+t_0,\epsilon)+g_x x_0)ds\Big|_X + O(\epsilon^2) \le O(\epsilon^2),
\end{aligned}
\tag{3.72}
$$

where we integrated by parts to get the second inequality. Moreover, by assumptions (A6) and (B2)

$$
\begin{aligned}
&\sup_{t\le 0} e^{-2\eta t}\Bigg|\int_0^t -e^{(t-s)A_f} P_{cu} w(s)(\frac{J}{\epsilon}+g_y)^{-1}(G(x,y,s+t_0,\epsilon)+g_x x)ds \\
&+\epsilon\int_0^t e^{(t-s)A_f} P_{cu} w(s) J^{-1}(G(x_0,0,s+t_0,\epsilon)+g_x x_0)ds\Bigg|_X = O(\epsilon^2).
\end{aligned}
$$

We note that

$$
\begin{aligned}
&\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t\int_s^t e^{a_2(t-\tau)+a_2(\tau-s)+2\eta s}\ d\tau ds\Big| < \infty, \\
&\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t\int_s^t e^{a_2(t-\tau)+\eta\tau+a_2(\tau-s)+\eta s}\ d\tau ds\Big| < \infty.
\end{aligned}
$$

Consequently, by (3.61)

$$\begin{aligned}
&\sup_{t\le 0} e^{-2\eta t}\Big| -\int_0^t \Big(\int_s^t \frac{d}{d\tau}(e^{(t-\tau)A_f}P_{cu}w(\tau))(\frac{J}{\epsilon}+g_y)^{-1} \\
&\qquad e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau\Big)\Big(G(x,y,s+t_0,\epsilon)+g_x x\Big)ds\Big|_X \\
&\le \sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t \frac{d}{d\tau}(e^{(t-\tau)A_f}P_{cu}w(\tau))\Big(\int_0^\tau (\frac{J}{\epsilon}+g_y)^{-1}e^{(\tau-s)(\frac{J}{\epsilon}+g_y)} \\
&\qquad \Big(G(x_0,0,s+t_0,\epsilon)+g_x x_0\Big)ds\Big)\ d\tau\Big|_X + O(\epsilon^2)\le O(\epsilon^2).
\end{aligned} \tag{3.73}$$

where we integrate by parts with respect to s to obtain the last inequality. Also, in the above estimate when $\dfrac{d}{d\tau}$ applies to $w(\tau)$, we use $|\cdot|^{-}_{\eta,1,X_1}$ norm for x_0, and while $\dfrac{d}{d\tau}$ applies to $e^{(t-\tau)A_f}$, we use $|\cdot|_{2\eta,1,X_1}$ norm for x_0. Therefore, (3.70) is proved.

For $\displaystyle\int_{-\infty}^t e^{(t-\tau)A_f}P_s w(\tau)y(\tau)\ d\tau$, we recall that $\xi_y=0$ and write

$$\begin{aligned}
&\int_{-\infty}^t e^{(t-\tau)A_f}P_s w(\tau)\left(\int_0^\tau e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}(G(x,y,s+t_0,\epsilon)+g_x x)ds\right)\ d\tau \\
&=\int_{-\infty}^t e^{(t-\tau)A_f}P_s w(\tau)\left(\int_0^t e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\,(G(x,y,s+t_0,\epsilon)+g_x x)\,ds\right)\ d\tau \\
&\quad+\int_{-\infty}^t e^{(t-\tau)A_f}P_s w(\tau)\left(\int_t^\tau e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\,(G(x,y,s+t_0,\epsilon)+g_x x)\,ds\right)\ d\tau \\
&=\int_0^t\Big(\int_{-\infty}^t e^{(t-\tau)A_f}P_s w(\tau)e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau\Big)\,(G(x,y,s+t_0,\epsilon)+g_x x)\,ds \\
&\quad+\int_{-\infty}^t\Big(\int_{-\infty}^s e^{(t-\tau)A_f}P_s w(\tau)e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau\Big)\,(G(x,y,s+t_0,\epsilon)+g_x x)\,ds.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{-\infty}^t e^{(t-\tau)A_f}P_s w(\tau)e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau \\
&=P_s w(t)(\frac{J}{\epsilon}+g_y)^{-1}e^{(t-s)(\frac{J}{\epsilon}+g_y)} \\
&\quad-\int_{-\infty}^t \frac{d}{d\tau}(e^{(t-\tau)A_f}P_s w(\tau))(\frac{J}{\epsilon}+g_y)^{-1}e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau,
\end{aligned}$$

we can use the same argument as in (3.77) to obtain

$$\begin{aligned}
&\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t P_s w(t)(\frac{J}{\epsilon}+g_y)^{-1}e^{(t-s)(\frac{J}{\epsilon}+g_y)} \\
&\qquad \Big(G(x,y,s+t_0,\epsilon)+g_x x\Big)ds\Big|_X = O(\epsilon^2).
\end{aligned}$$

Similar to (3.73), we have

$$\sup_{t\le 0} e^{-2\eta t}\Big| -\int_0^t \Big(\int_{-\infty}^t \frac{d}{d\tau}(e^{(t-\tau)A_f}P_s w(\tau))(\frac{J}{\epsilon}+g_y)^{-1}$$

$$e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau\Big)\Big(G(x,y,s+t_0,\epsilon)+g_x x\Big)ds\Big|_X$$

$$\le \sup_{t\le 0} e^{-2\eta t}\Big|\int_{-\infty}^t \frac{d}{d\tau}(e^{(t-\tau)A_f}P_s w(\tau))\Bigg(\int_0^t (\frac{J}{\epsilon}+g_y)^{-1}e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}$$

$$\Big(G(x_0,0,s+t_0,\epsilon)+g_x x_0\Big)ds\Bigg)\ d\tau\Big|_X + O(\epsilon^2) \le O(\epsilon^2),$$

where we use the facts

$$\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t\int_{-\infty}^t e^{a_1(t-\tau)+a_2(\tau-s)+2\eta s}\ d\tau ds\Big| < \infty,$$

$$\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t\int_{-\infty}^t e^{a_1(t-\tau)+\eta\tau+a_2(\tau-s)+\eta s}\ d\tau ds\Big| < \infty.$$

Consequently,

$$\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t \Big(\int_{-\infty}^t e^{(t-\tau)A_f}P_s w(\tau)e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau\Big)$$

$$\Big(G(x,y,s+t_0,\epsilon)+g_x x\Big)ds\Big|_X = O(\epsilon^2).$$

Meanwhile,

$$\int_{-\infty}^s e^{(t-\tau)A_f}P_s w(\tau)e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau$$

$$= e^{(t-s)A_f}P_s w(s)(\frac{J}{\epsilon}+g_y)^{-1}$$

$$-\int_{-\infty}^s \frac{d}{d\tau}(e^{(t-\tau)A_f}P_s w(\tau))(\frac{J}{\epsilon}+g_y)^{-1}e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau.$$

Again similar to (3.73), we have

$$\sup_{t\le 0} e^{-2\eta t}\Big| -\int_{-\infty}^t \Big(\int_{-\infty}^s \frac{d}{d\tau}(e^{(t-\tau)A_f}P_s w(\tau))(\frac{J}{\epsilon}+g_y)^{-1}$$

$$e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}\ d\tau\Big)\Big(G(x,y,s+t_0,\epsilon)+g_x x\Big)ds\Big|_X$$

$$\le \sup_{t\le 0} e^{-2\eta t}\Big|\int_{-\infty}^t \frac{d}{d\tau}(e^{(t-\tau)A_f}P_s w(\tau))\Bigg(\int_\tau^t (\frac{J}{\epsilon}+g_y)^{-1}e^{(\tau-s)(\frac{J}{\epsilon}+g_y)}$$

$$\Big(G(x_0,0,s+t_0,\epsilon)+g_x x_0\Big)ds\Bigg)\ d\tau\Big|_X + O(\epsilon^2) \le O(\epsilon^2),$$

where we also use the facts

$$\sup_{t\leq 0} e^{-2\eta t}\Big|\int_{-\infty}^{t}\int_{-\infty}^{s} e^{a_1(t-\tau)+a_2(\tau-s)+2\eta s}\ d\tau ds\Big| < \infty,$$
$$\sup_{t\leq 0} e^{-2\eta t}\Big|\int_{-\infty}^{t}\int_{-\infty}^{s} e^{a_1(t-\tau)+\eta\tau+a_2(\tau-s)+\eta s}\ d\tau ds\Big| < \infty.$$

Moreover,

$$\sup_{t\leq 0} e^{-2\eta t}\Bigg|\int_{-\infty}^{t} e^{(t-s)A_f}P_s w(s)(\frac{J}{\epsilon}+g_y)^{-1}(G(x,y,s+t_0,\epsilon)+g_x x)ds$$
$$-\epsilon\int_{0}^{t} e^{(t-s)A_f}P_s w(s)J^{-1}(G(x_0,0,s+t_0,\epsilon)+g_x x_0)ds\Bigg|_X = O(\epsilon^2).$$

Therefore, (3.71) is proved.

Multiplying both sides of the above equality by $(I-\mathcal{L})^{-1}$, we complete the proof.

□

For $\xi_s \in X_1^s$, let $x_0(t)$ satisfy

$$x_0(t) = e^{tA_f}\xi_s + \int_0^t e^{(t-\tau)A_f}P_s F_1(x_0,0,\tau+t_0,0)\ d\tau$$
$$+\int_{+\infty}^{t} e^{(t-\tau)A_f}P_{cu}F_1(x_0,0,\tau+t_0,0)\ d\tau.$$

and let

$$h_{cu}^0(\xi_s) = \int_{+\infty}^{0} e^{-\tau A_f}P_{cu}F_1(x_0,0,\tau+t_0,0)\ d\tau.$$

Similarly, we can compute the leading order of $h_{cu}(\xi_s,t_0,\epsilon)-h_{cu}^0(\xi_s)$. Let

$$(\mathcal{L}'\Delta x)(t) = \int_0^t e^{(t-\tau)A_f}P_s D_x F_1(x_0,0,\tau+t_0,0)\Delta x(\tau)\ d\tau$$
$$+\int_{+\infty}^{t} e^{(t-\tau)A_f}P_{cu}D_x F_1(x_0,0,\tau+t_0,0)\Delta x(\tau)\ d\tau.$$

Theorem 3.23. *Assume the conditions in Theorem 3.16 for $k=2$, where the differentiation D also includes ϵ. There exists a constant C' which depends on K,η,a_1',σ,C_0, such that*

$$\Big|h_{cu}(\xi_s,t_0,\epsilon)-h_{cu}^0(\xi_s)\Big|_{X_1\times Y_1} \leq C'\epsilon, \tag{3.74}$$

$$\Big|D_{\xi_s}h_{cu}(\xi_s,t_0,\epsilon)-D_{\xi_s}h_{cu}^0(\xi_s)\Big|_{L(X_1^s,X_1^{cu}\times Y_1)} \leq C'\epsilon. \tag{3.75}$$

Moreover, if we further assume

$$\partial_\epsilon \partial_t Dg \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, L(X_1 \times Y_1, Y))$$

and for every $t \in \mathbb{R}, x \in X_1$,

$$J^{-1} D_x g(x, 0, t, 0) A \in L(X, Y) \ , \ J^{-1} D_x^2 g(x, 0, t, 0) \in L_2(X, Y).$$

We have

$$\begin{aligned} & h_{cu}(\xi_s, t_0, \epsilon) - h^0_{cu}(\xi_s) \\ = \epsilon & \begin{pmatrix} (I - \mathcal{L}')^{-1} \displaystyle\int_{+\infty}^{0} e^{-\tau A_f} P_{cu}\Big(\partial_\epsilon F_1 + (D_y F_1 + f_y) J^{-1}(G_1 + g_x x_0)\Big) \, d\tau \\ -J^{-1}\Big(G_1(\xi_s + h^0_{cu}(\xi_s), 0, t_0, 0) + g_x\big(\xi_s + h^0_{cu}(\xi_s)\big)\Big) \end{pmatrix} \\ & + O(\epsilon^2), \end{aligned}$$

where $D_y F_1, \partial_\epsilon F_1, G_1$ *in the integral are evaluated at* $(x_0(\tau), 0, \tau + t_0, 0)$*. And* $O(\epsilon^2)$ *is measured in* $C^+_{2\eta}(X_1) \times C^+_{2\eta}(Y_1)$*.*

Proof. Let (x, y) be the fixed point of $\mathscr{T}_s$ with parameter (ξ_s, t_0, ϵ). Since

$$\begin{aligned} & \begin{pmatrix} x - x_0 \\ y \end{pmatrix}(t) \\ = & \int_0^t U(t - \tau, \epsilon) \begin{pmatrix} P_s\big(F_1(x, y, \tau + t_0, \epsilon) - F_1(x_0, 0, \tau + t_0, 0) + f_y y\big) \\ 0 \end{pmatrix} d\tau \\ & + \int_{+\infty}^t U(t - \tau, \epsilon) \begin{pmatrix} P_{cu}\big(F_1(x, y, \tau + t_0, \epsilon) - F_1(x_0, 0, \tau + t_0, 0) + f_y y\big) \\ G_1(x, y, \tau + t_0, \epsilon) - G_1(x_0, 0, \tau + t_0, \epsilon) \end{pmatrix} d\tau \\ & + \int_{+\infty}^t U(t - \tau, \epsilon) \begin{pmatrix} 0 \\ G_1(x_0, 0, \tau + t_0, \epsilon) + g_x x \end{pmatrix} d\tau. \end{aligned}$$

Following a similar argument in the proof of Theorem 3.19, we have

$$\big\|(x - x_0, y)\big\|^1_\eta \leq C'\epsilon, \tag{3.76}$$

where C' depends on $K, a_1, a_2, \eta, |\xi_s|_{X_1}, \epsilon_\star, |DF_1|_{C^0}, |DG_1|_{C^0}, |\partial_\epsilon F_1|_{C^0}$. This proves (3.74). Let $(\phi^\epsilon, \psi^\epsilon)$ be the derivative of (x, y) with respect to ξ_s and ϕ^0 be the derivative of x_0 with respect to ξ_s. We have

$$
\begin{aligned}
\phi^\epsilon - \phi^0 = &\int_0^t e^{(t-\tau)A_f} P_s\big(DF_1(x, y, \tau + t_0, \epsilon)(\phi^\epsilon, \psi^\epsilon) \\
&\qquad\qquad - D_x F_1(x_0, 0, \tau + t_0, 0)\phi^0 + f_y \psi^\epsilon\big)\, d\tau \\
&+ \int_{+\infty}^t e^{(t-\tau)A_f} P_{cu}\big(DF_1(x, y, \tau + t_0, \epsilon)(\phi^\epsilon, \psi^\epsilon) \\
&\qquad\qquad - D_x F_1(x_0, 0, \tau + t_0, 0)\phi^0 + f_y \psi^\epsilon\big)\, d\tau, \\
\psi^\epsilon = &\int_{+\infty}^t U(t-\tau) e^{(t-\tau)(\frac{J}{\epsilon} + g_y)} \Big(D_y G_1(x, y, \tau + t_0, \epsilon)\psi^\epsilon \\
&+ \big(D_x G_1(x, y, \tau + t_0, \epsilon) - D_x G_1(x_0, 0, \tau + t_0, 0)\big)\phi^\epsilon \\
&\qquad + (D_x G_1(x_0, 0, \tau + t_0, 0) + g_x)\phi^\epsilon\Big)\, d\tau.
\end{aligned}
$$

Integration by parts and boundedness of ϕ^ϵ from Theorem 3.16 shows

$$
\Big| \int_{+\infty}^t U(t-\tau, \epsilon)(D_x G_1(x_0, 0, \tau + t_0, 0) + g_x)\phi^\epsilon\Big)\, d\tau \Big|_{L(X_1^s, Y_1)} \le C'\epsilon,
$$

while (3.76) implies

$$
\begin{aligned}
\Big| \int_{+\infty}^t & U(t-\tau, \epsilon)\big(D_x G_1(x, y, \tau + t_0, \epsilon) \\
&- D_x G_1(x_0, 0, \tau + t_0, 0)\big)\phi^\epsilon\, d\tau \Big|_{L(X_1^s, X_1^{cu} \times Y_1)} \le C'\epsilon.
\end{aligned}
$$

By using the smallness DG_1, we obtain

$$
\sup_{t \ge 0} e^{-2\eta t} |\psi^\epsilon|_{L(X_1^s, Y_1)} \le C'\epsilon.
$$

One can substitute this estimate into the integral equation for $\phi^\epsilon - \phi^0$ and use (3.76) to obtain

$$
\sup_{t \ge 0} e^{-2\eta t} |\phi^\epsilon - \phi^0|_{L(X_1^s, X_1^{cu})} \le C'\epsilon,
$$

which finishes the proof of (3.75).

To find the exact asymptotic expansion of h_{cu}, we use the equation of $y(t)$ and integrate by parts

$$
\begin{aligned}
y(t) =& -(\frac{J}{\epsilon} + g_y)^{-1}\Big(G_1(x, y, t + t_0, \epsilon) + g_x x(t)\Big) \\
&+ \int_{+\infty}^{t} e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}(\frac{J}{\epsilon} + g_y)^{-1}\Big((D_x G_1 + g_x)\dot{x} + D_y G_1(G_1 + g_x x)\Big)\ d\tau \\
&+ \int_{+\infty}^{t} e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}(J + \epsilon g_y)^{-1} D_y G_1 (J + \epsilon g_y) y\ d\tau \\
&+ \int_{+\infty}^{t} e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}(\frac{J}{\epsilon} + g_y)^{-1}\partial_t G_1\ d\tau \triangleq I_0 + I_1 + I_2 + I_3.
\end{aligned}
$$

We start to show that

$$
\sup_{t\geq 0} \frac{1}{\epsilon_\star} e^{-2\eta t}\Big|y(t) + \epsilon J^{-1}\Big(G_1(x_0(t), 0, t + t_0, 0) + g_x x_0(t)\Big)\Big|_{Y_1} \leq C'\epsilon^2, \tag{3.77}
$$

where C' depends on $K, a_2, \eta, |\xi_s|_{X_1}, \epsilon_\star, |DG_1|_{C^0}$.

From (3.76), it's clear that

$$
\begin{aligned}
\sup_{t\geq 0} \frac{1}{\epsilon_\star} e^{-2\eta t}\Big| &- (\frac{J}{\epsilon} + g_y)^{-1}\Big(G_1(x, y, t + t_0, \epsilon) + g_x x(t)\Big) \\
&+ \epsilon J^{-1}\Big(G_1(x_0(t), 0, t + t_0, 0) + g_x x_0(t)\Big)\Big|_{Y_1} \leq C'\epsilon^2,
\end{aligned}
$$

where we use the fact that $\partial_\epsilon DG_1$ exists and (3.76).

Next, we observe that the rest of the terms are oscillatory. The key ingredients in the proof of (3.77) are integration by parts and (3.76). In general, to improve their smallness, we first replace every function evaluated at $(x, y, t+t_0, \epsilon)$ by $(x_0, 0, t+t_0, 0)$ and then apply integration by parts. The $O(\epsilon^2)$ smallness of the difference is given by $(\frac{J}{\epsilon} + g_y)^{-1}$ and (3.76). More precisely, by assumptions

$$
J^{-1} D_x G_1(x_0(t), 0, t + t_0, 0)A \in L(X, Y),
$$

$$
J^{-1} D_x^2 G_1(x_0(t), 0, t + t_0, 0) \in L_2(X, Y),
$$

which ensures

$$
\begin{aligned}
\sup_{t\geq 0} e^{-2\eta t}\frac{1}{\epsilon_\star}\Big| &\int_{+\infty}^{t} e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}(\frac{J}{\epsilon} + g_y)^{-1} \\
&D_x G_1(x_0(\tau), 0, \tau + t_0, 0) A_f x_0(\tau)\ d\tau\Big|_{Y_1} \leq C'\epsilon^2.
\end{aligned}
$$

Consequently, I_1 is of order ϵ^2 in $C^+_{2\eta}(Y_1)$ and I_3 can be proved in a similar way. For I_2, we write y in the integral as

$$y(\tau) = y(\tau) + \epsilon J^{-1}\Big(G_1(x_0(\tau), 0, \tau + t_0, \epsilon) + g_x x_0(\tau)\Big)$$
$$-\epsilon J^{-1}\Big(G_1(x_0(\tau), 0, \tau + t_0, \epsilon) + g_x x_0(\tau)\Big).$$

Consequently, one can show that

$$\sup_{t\geq 0}\frac{1}{\epsilon_\star}e^{-2\eta t}\Big|y(t) + \epsilon J^{-1}\Big(G_1(x_0(t), 0, t, \epsilon) + g_x x_0(t)\Big)\Big|_{Y_1}$$
$$\leq C'\epsilon^2 + C'|D_y G_1|_{C^0}$$
$$\left(\sup_{t\geq 0}\frac{1}{\epsilon_\star}e^{-2\eta t}\Big|y(t) + \epsilon J^{-1}\Big(G_1(x_0(t), 0, t, \epsilon) + g_x x_0(t)\Big)\Big|_{Y_1}\right).$$

By taking advantage of the smallness of DG_1, one can move the second term on the right hand side to the left to obtain

$$\sup_{t\geq 0}\frac{1}{\epsilon_\star}e^{-2\eta t}\Big|y(t) + \epsilon J^{-1}\Big(G_1(x_0(t), 0, t, \epsilon) + g_x x_0(t)\Big)\Big|_{Y_1} \leq C'\epsilon^2.$$

Plugging this asymptotic expansion into $(x - x_0)(t)$ equation and using the smallness of DF_1, we can prove the result. □

Remark 3.24. *If one assume $J^{-k}D_x g(x, 0, t, 0)A^k \in L(X, Y)$ for every $t \in \mathbb{R}$ and $x \in X_1$, then one can further compute higher order expansions.*

Theorem 3.25. *Assume the conditions in Theorem 3.16 for $k = 2$, where the differentiation D also includes ϵ. There exists C' which depends on $C, K, \eta, a_1, a_2, \overline{r}, \epsilon_\star, T - t_0, |\xi_s|_{X_1}$ such that*

$$\Big|\Phi(T, t_0, \xi_s + h_{cu}(\xi_s, 0, t_0, \epsilon), \epsilon) - \Phi^0(T, t_0, \xi_s + h^0_{cu}(\xi_s))\Big|_{X_1\times Y_1} \leq C'\epsilon,$$
$$\Big|D_{\xi_s}\left(\Phi(T, t_0, \xi_s + h_{cu}(\xi_s, 0, t_0, \epsilon), \epsilon)\right)$$
$$-D_{\xi_s}\left(\Phi^0(T, t_0, \xi_s + h^0_{cu}(\xi_s))\right)\Big|_{L(X_1^s, X_1^{cu}\times Y_1)} \leq C'\epsilon.$$

Proof. We apply Theorem 2.4 and use the remark after Theorem 2.4 to complete the proof. □

Theorem 3.26. *Assume the same condition as in Theorem 3.14 for $k = 2$ and $|\xi_y| \le C_1\epsilon$, then*

$$|\partial_{t_0} h_s(\xi_{cu}, \xi_y, \cdot, \epsilon)|_{C^0(\mathbb{R},X^s)} \le C'\epsilon,$$

where C' depends on $C_1, |\xi_{cu}|_{X_1}$ and constants in assumptions.

Proof. Let (x_0, y_0) be the fixed point of $\widetilde{\mathscr{T}_{cu}}(\cdot, t_0)$ and $(\phi, \psi) = (\partial_{t_0}x_0, \partial_{t_0}y_0)$. Differentiate (3.8) with respect to t_0, we have

$$\begin{aligned}
\phi(t) =& \int_0^t e^{(t-\tau)A_f} P_{cu}\partial_t F(x_0, y_0, \tau + t_0, \epsilon)\ d\tau \qquad (3.78)\\
&+ \int_{-\infty}^t e^{(t-\tau)A_f} P_s\partial_t F(x_0, y_0, \tau + t_0, \epsilon)\ d\tau\\
&+ \int_0^t e^{(t-\tau)A_f} P_{cu}D_x F(x_0, y_0, \tau + t_0, \epsilon)\phi(\tau)\ d\tau\\
&+ \int_{-\infty}^t e^{(t-\tau)A_f} P_s D_x F(x_0, y_0, \tau + t_0, \epsilon)\phi(\tau)\ d\tau\\
&+ \int_0^t e^{(t-\tau)A_f} P_{cu}\big(D_y F(x_0, y_0, \tau + t_0, \epsilon) + f_y\big)\psi(\tau)\ d\tau\\
&+ \int_{-\infty}^t e^{(t-\tau)A_f} P_s\big(D_y F(x_0, y_0, \tau + t_0, \epsilon) + f_y\big)\psi(\tau)\ d\tau,\\
\dot{\psi}(t) =& (\frac{J}{\epsilon} + g_y + D_y G(x_0, y_0, t + t_0, \epsilon))\psi(t) \qquad (3.79)\\
&+\big(D_x G(x_0, y_0, t + t_0, \epsilon) + g_x\big)\phi(t) + \partial_t G(x_0, y_0, t + t_0, \epsilon).
\end{aligned}$$

By using $\partial_\epsilon\partial_t F(0, 0, t, \epsilon) = 0$ and $(A6')$, one can prove

$$\begin{aligned}
\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t e^{(t-\tau)A_f} P_{cu}\partial_t F(x_0, y_0, \tau + t_0, \epsilon)\ d\tau\Big|_X &\le \frac{K}{a_2 - 2\eta} C_0\epsilon\|z_0\|_{2\eta}^1,\\
\sup_{t\le 0} e^{-2\eta t}\Big|\int_{-\infty}^t e^{(t-\tau)A_f} P_s\partial_t F(x_0, y_0, \tau + t_0, \epsilon)\ d\tau\Big|_X &\le \frac{K}{2\eta - a_1} C_0\epsilon\|z_0\|_{2\eta}^1.
\end{aligned}$$

Since the third and fourth integral in (3.78) define a contraction mapping acting on ϕ, by rewriting (3.79),

$$\psi = \epsilon J^{-1}\dot{\psi} - \epsilon J^{-1}\big((g_y + D_y G)\psi + (D_x G + g_x)\phi + \partial_t G\big), \qquad (3.80)$$

and substituting (3.80) into (3.78), once we prove

$$\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t e^{(t-\tau)A_f}P_{cu}\Big(D_yF(x_0,y_0,\tau+t_0,\epsilon)+f_y\Big)$$
$$\epsilon J^{-1}\dot{\psi}^\epsilon\, d\tau\Big|_X\le C'\epsilon,$$
$$\sup_{t\le 0} e^{-2\eta t}\Big|\int_{-\infty}^t e^{(t-\tau)A_f}P_{s}\Big(D_yF(x_0,y_0,\tau+t_0,\epsilon)+f_y\Big)$$
$$\epsilon J^{-1}\dot{\psi}^\epsilon\, d\tau\Big|_X\le C'\epsilon,$$

we can complete the proof. Here we only prove the first inequality. First we suppose $\psi(t)\in Y_1$, then (3.80) implies $\dot{\psi}(t)\in Y$.

$$\begin{aligned}
&\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t e^{(t-\tau)A_f}P_{cu}\Big(D_yF(x_0,y_0,\tau+t_0,\epsilon)+f_y\Big)\epsilon J^{-1}\dot{\psi}\, d\tau\Big|\\
\le&\sup_{t\le 0} e^{-2\eta t}\Big|e^{(t-\tau)A_f}P_{cu}\Big(D_yF(x_0,y_0,\tau+t_0,\epsilon)+f_y\Big)\epsilon J^{-1}\psi(\tau)\Big|_0^t\Big|\\
&+\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t \frac{d}{d\tau}\Big(e^{(t-\tau)A_f}P_{cu}\Big(D_yF(x_0,y_0,\tau+t_0,\epsilon)+f_y\Big)\Big)\epsilon J^{-1}\psi\, d\tau\Big|\\
\le& C_0\epsilon(\frac{K}{a_2-2\eta}+K)(\overline{r}+C_0)|\psi|_{\overline{B}^-_{2\eta}(\infty)}\\
&+\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t \frac{d}{d\tau}\Big(e^{(t-\tau)A_f}P_{cu}\Big(D_yF(x_0,y_0,\tau+t_0,\epsilon)+f_y\Big)\Big)\epsilon J^{-1}\psi\, d\tau\Big|.
\end{aligned}$$

Since $|\xi_y|\le C_1\epsilon$, by (3.61), $|y_0(t)|^-_{2\eta,\epsilon_\star,Y_1}\le C_3'\epsilon$, where C_3' depends on C_1 and constants in assumptions and σ'. Using this fact, assumption (A6$'$) and a straight forward computation, we obtain

$$\begin{aligned}
&\sup_{t\le 0} e^{-2\eta t}\Big|\int_0^t \frac{d}{d\tau}\Big(e^{(t-\tau)A_f}P_{cu}\Big(D_yF(x,y,\tau+t_0,\epsilon)+f_y\Big)\Big)\epsilon J^{-1}\psi\, d\tau\Big|\\
\le& C_0\epsilon\frac{K}{a_2-2\eta}(\overline{r}+C_0)|\psi|_{\overline{B}^-_{2\eta}(\infty)}+C_0\epsilon\frac{K}{a_2-2\eta}|\psi|_{\overline{B}^-_{2\eta}(\infty)}\\
&+C_0^2\epsilon\frac{K}{a_2-2\eta}(\overline{r}+C_0+1)(|x|_{C^-_\eta(X_1)}+|y|_{C^-_\eta(Y_1)})|\psi|_{\overline{B}^-_{2\eta}(\infty)}\\
&+C_0^2\epsilon\frac{K}{a_2-2\eta}\big(C_3'+(\overline{r}+C_0)(|x|_{C^-_\eta(X_1)}+|y|_{C^-_\eta(Y_1)})\big)|\psi|_{\overline{B}^-_{2\eta}(\infty)}.
\end{aligned}$$

Now, if $\psi(t)\in Y$, for $\delta>0$ and $\psi\in Y$, define

$$\psi_\delta=B_\delta\psi=\frac{1}{\delta}\int_0^\delta e^{sJ}\psi ds.$$

One can verify that $B_\delta \in L(Y, Y_1)$ and $B_\delta \to I$ strongly as $\delta \to 0$. Therefore, for $\psi(t) \in Y$, $\psi_\delta(t) \to \psi(t)$ locally uniformly in t. Then we can use $\dot{\psi}_\delta(t)$ on finite time interval and take η' norm on the complement of that time interval to prove the result. $\square$

For stable integral manifold, we have the following result.

Theorem 3.27. *Assume (A4) for $k = 2$, (A6′),*

$$\partial_\epsilon \partial_t Dg \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, L(X_1 \times Y_1, Y)),$$

and for $x \in X_1$, $t \in \mathbb{R}$

$$J^{-1} D_x g(x, 0, t, 0) A \in L(X, Y) \ , \ J^{-1} D_x^2 g(x, 0, t, 0) \in L_2(X, Y).$$

Moreover, if there exists $\eta < 0$ such that $a_1 < 2\eta < \eta < a_2$, then we have

$$h_{cu}(\cdot, \cdot, \epsilon) \in C^0(X_1^s \times \mathbb{R}, X_1^{cu} \times Y_1),$$
$$\partial_{t_0} h_{cu}(\cdot, \cdot, \epsilon) \in C^0(X_1^s \times \mathbb{R}, X^{cu} \times Y),$$
$$|\partial_{t_0} h_{cu}(\xi_s, \cdot, \epsilon)|_{X^{cu} \times Y} \le C' \epsilon,$$

where C' depends on constants in assumptions and $|\xi_s|_{X_1}$.

Proof. The proof of the first two parts are similar to Theorem 3.14 and thus we only prove the last one. Let (x_0, y_0) satisfy (3.8) for fixed parameters (ξ_s, ϵ) and denote $(\partial_{t_0} x_0, \partial_{t_0} y_0)$ as (ϕ, ψ). Since the construction of the stable integral manifold doesn't need cut-off function, we replace F, G in (3.8) by F_1, G_1 and differentiate with respect to t_0 which yields

$$\phi(t) = \int_0^t e^{(t-\tau)A_f} P_s \partial_t F_1(x_0, y_0, \tau + t_0, \epsilon)\, d\tau \tag{3.81}$$
$$+ \int_{+\infty}^t e^{(t-\tau)A_f} P_{cu} \partial_t F_1(x_0, y_0, \tau + t_0, \epsilon)\, d\tau$$
$$+ \int_0^t e^{(t-\tau)A_f} P_s \big(DF_1(x_0, y_0, \tau + t_0, \epsilon)(\phi, \psi) + f_y \psi\big)\, d\tau$$
$$+ \int_{+\infty}^t e^{(t-\tau)A_f} P_{cu} \big(DF_1(x_0, y_0, \tau + t_0, \epsilon)(\phi, \psi) + f_y \psi\big)\, d\tau,$$
$$\psi(t) = \int_{+\infty}^t e^{(t-\tau)(\frac{J}{\epsilon} + g_y)} \big(DG_1(x_0, y_0, \tau + t_0, \epsilon)(\phi, \psi) \tag{3.82}$$
$$+ g_x \phi + \partial_t G_1\big)\, d\tau.$$

By using $\partial_\epsilon \partial_t F(0, 0, t, \epsilon) = 0$ and $(A6')$, one can prove

$$\sup_{t \geq 0} e^{-2\eta t} \Big| \int_0^t e^{(t-\tau)A_f} P_s \partial_t F_1(x_0, y_0, \tau + t_0, \epsilon)\, d\tau \Big|_X \leq \frac{C_0 K \epsilon}{2\eta - a_1} \|(x_0, y_0)\|^1_{2\eta},$$
$$\sup_{t \geq 0} e^{-2\eta t} \Big| \int_{+\infty}^t e^{(t-\tau)A_f} P_{cu} \partial_t F_1(x_0, y_0, \tau + t_0, \epsilon)\, d\tau \Big|_X \leq \frac{C_0 K \epsilon}{a_2 - 2\eta} \|(x_0, y_0)\|^1_{2\eta}.$$

Since the right hand side of (3.81) defines a linear contraction for ϕ, once we have

$$\sup_{t \geq 0} e^{-2\eta} |\psi|_Y \leq C' \epsilon,$$

we can finish the proof. By (3.76), we have $|y_0(\cdot)|^+_{2\eta, \epsilon_\star, Y_1} \leq C' \epsilon$. Combining with integration by parts and the assumption $D_x^2 g(x, 0, t, 0) \in L_2(X, Y)$, one can show

$$\sup_{t \geq 0} e^{-2\eta t} \Big| \int_{+\infty}^t e^{(t-\tau)(\frac{J}{\epsilon} + g_y)} \big(D_x G_1 \phi + g_x \phi + \partial_t G_1\big)\, d\tau \Big|_Y \leq C' \epsilon. \tag{3.83}$$

Since

$$\sup_{t \geq 0} e^{-2\eta t} \Big| \int_{+\infty}^t e^{(t-\tau)(\frac{J}{\epsilon} + g_y)} D_y G_1(x_0, y_0, \tau + t_0, \epsilon) \psi\, d\tau \Big|_Y$$
$$\leq \sigma \sup_{t \geq 0} e^{-2\eta} |\psi|_Y, \tag{3.84}$$

Therefore, $\sup_{t \geq 0} e^{-2\eta} |\psi|_Y \leq C' \epsilon$, where C' depends on $|\xi_s|_{X_1}$ and constants in assumptions. □

Finally, we will present a theorem for the closeness between the perturbed and unperturbed center manifold. We introduce the following notation. For any $\xi_c \in X_1^c$, let Ψ_s^0 and Ψ_u^0 satisfy

$$h_s^0(\Psi_u^0(\xi_c), \xi_c) = \Psi_s^0(\xi_c),$$
$$h_u^0(\Psi_s^0(\xi_c), \xi_c) = \Psi_u^0(\xi_c),$$

where h_u^0 is defined similarly to h_s^0. Define

$$\mathcal{M}_0^c = \left\{ \xi_c + \Psi_s^0(\xi_c) + \Psi_u^0(\xi_c) \middle| \xi_c \in X_1^c \right\}.$$

One can easily see $\mathcal{M}_0^c$ represents the center manifold for the unperturbed system (3.58).

Theorem 3.28. *Assume the conditions in (A4) for $k = 1$ and $\overline{r}, \epsilon_\star, \epsilon$ are sufficiently small. Then we have*

$$\left| \Psi_s(\xi_c, 0, t_0, \epsilon) - \Psi_s^0(\xi_c) \right|_{X_1^s} \leq C'\epsilon,$$
$$\left| \Psi_u(\xi_c, 0, t_0, \epsilon) - \Psi_u^0(\xi_c) \right|_{X_1^s} \leq C'\epsilon,$$

where C' depends on $C_0, C_1, K, a_1, a_2, \eta, |\xi_c|_{X_1}, \overline{r}, \epsilon_\star$.

Proof.

$$\begin{aligned}
&\Psi_s(\xi_c, 0, t_0, \epsilon) - \Psi_s^0(\xi_c) + \Psi_u(\xi_c, 0, t_0, \epsilon) - \Psi_u^0(\xi_c) \\
=& h_s(\Psi_u(\xi_c, 0, t_0, \epsilon), \xi_c, 0, t_0, \epsilon) - h_s(\Psi_u^0(\xi_c), \xi_c, 0, t_0, \epsilon) \\
&+ h_s(\Psi_u^0(\xi_c), \xi_c, 0, t_0, \epsilon) - h_s^0(\Psi_u^0(\xi_c), \xi_c) \\
&+ h_u(\Psi_s(\xi_c, 0, t_0, \epsilon), \xi_c, 0, t_0, \epsilon) - h_u(\Psi_s^0(\xi_c), \xi_c, 0, t_0, \epsilon) \\
&+ h_u(\Psi_s^0(\xi_c), \xi_c, 0, t_0, \epsilon) - h_u^0(\Psi_s^0(\xi_c), \xi_c).
\end{aligned}$$

By Theorem 3.19 and similar estimates for h_u, we obtain

$$\begin{aligned}
&\left| \Psi_s - \Psi_s^0 \right|_{X_1} + \left| \Psi_u - \Psi_u^0 \right|_{X_1} \\
\leq& C'\epsilon + |Dh_s|_{C^0} |\Psi_u - \Psi_u^0|_{X_1} + |Dh_u|_{C^0} |\Psi_s - \Psi_s^0|_{X_1}.
\end{aligned}$$

Consequently, using (3.32) and similar estimate for $|Dh_u|_{C^0}$ we finish the proof. □

CHAPTER IV

INVARIANT FOLIATION

In the previous chapter, we obtain a local C^k center manifold. In this chapter, we will construct stable (unstable) fibres on center-stable (center-unstable) manifold. We start with stable fibres, and unstable fibres can be obtained in a similar manner.

For $\xi_{cy} = (\xi_c, \xi_y) \in X_1^c \times Y_1$, let $(x(\xi_{cy})(t-t_0), y(\xi_{cy})(t-t_0))$ be the solution of (3.5) with the initial value

$$\xi = \xi_{cy} + \Psi_s(\xi_{cy}, t_0, \epsilon) + \Psi_u(\xi_{cy}, t_0, \epsilon). \tag{4.1}$$

The solution is on $\mathcal{M}_\epsilon^c$ and satisfies

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = U(t,\epsilon)\xi + \int_0^t U(t-\tau,\epsilon) \begin{pmatrix} F(x,y,\tau+t_0,\epsilon) + f_y y \\ G(x,y,\tau+t_0,\epsilon) + g_x x \end{pmatrix} d\tau \tag{4.2}$$

To simplify our notation, for $(\widetilde{x}, \widetilde{y}) \in X_1 \times Y_1$, let

$$\begin{aligned} \widetilde{F}(\widetilde{x}, \widetilde{y}, \xi_{cy}, t, \epsilon) &= F(x(\xi_{cy})(t-t_0) + \widetilde{x}, y(\xi_{cy})(t-t_0) + \widetilde{y}, t, \epsilon) \\ &\quad - F(x(\xi_{cy})(t-t_0), y(\xi_{cy})(t-t_0), t, \epsilon). \end{aligned}$$

We often write it in short as $\widetilde{F}(\widetilde{x}, \widetilde{y}, \xi_{cy}, \epsilon)$. Such notation also applies to G.

For each triple $(\xi_s, \xi_c, \xi_y) \in X_1^s \times X_1^c \times Y_1$ and $a_1 < \eta < a_2$, we look for a solution

$(\widetilde{x}, \widetilde{y}) \in B_{\eta}^{+}(\infty)$ for the following integral equation,

$$\begin{aligned}
\begin{pmatrix} \widetilde{x}(t) \\ \widetilde{y}(t) \end{pmatrix} &= U(t,\epsilon)\xi_s \\
&+ \int_0^t U(t-\tau,\epsilon) \begin{pmatrix} P_s\big(\widetilde{F}(\widetilde{x}(\tau),\widetilde{y}(\tau),\xi_{cy},\tau+t_0,\epsilon) + f_y\widetilde{y}(\tau)\big) \\ 0 \end{pmatrix} d\tau \qquad (4.3) \\
&+ \int_{+\infty}^t U(t-\tau,\epsilon) \begin{pmatrix} P_{cu}\big(\widetilde{F}(\widetilde{x}(\tau),\widetilde{y}(\tau),\xi_{cy},\tau+t_0,\epsilon) + f_y\widetilde{y}(\tau)\big) \\ \widetilde{G}(\widetilde{x}(\tau),\widetilde{y}(\tau),\xi_{cy},\tau+t_0,\epsilon) + g_x\widetilde{x}(\tau) \end{pmatrix} d\tau.
\end{aligned}$$

Next, we will prove the existence of the solution of (4.2) and its smooth dependence with respect to ξ_s and ξ_{cy}, respectively. Before we proceed, we first prove a technical lemma, which will be used in the proof of our next theorem. It is not hard but less obvious.

Lemma 4.1. *Let X be a measurable space with finite measure and $\{f_n\}_{n\geq 1}, f : X \to Y$ such that*

$$\lim_{n\to+\infty} f_x(x) = f(x) \quad a.e. \ ,$$

where Y is a Banach space. Moreover, $\sup_{x\in X, n\geq 1} |f_n(x)| \leq M < +\infty$, then

$$\lim_{n\to+\infty} \int_X \big(f_n(x) - f(x)\big) d\mu(x) \to 0,$$

where μ is the measure on X.

Proof. Define $A_{N,\epsilon} = \big\{x \in X \big| |f_n(x) - f(x)| < \epsilon, n \geq N\big\}$, clearly,

$$A_{N,\epsilon} \subseteq A_{N+1,\epsilon} \quad , \quad \bigcup_{N=1}^{+\infty} A_{N,\epsilon} = X.$$

For any $\epsilon > 0$, there exists $N(\epsilon)$ such that $\mu(X \backslash A_{N(\epsilon),\epsilon}) < \epsilon$. Now, for $n \geq N(\epsilon)$

$$\begin{aligned}
&|\int_X f_n(x) - f(x) d\mu(x)| \\
&\leq \int_{A_{N(\epsilon),\epsilon}} |f_n(x) - f(x)| d\mu(x) + \int_{X\backslash A_{N(\epsilon),\epsilon}} |f_n(x) - f(x)| d\mu(x) \\
&\leq (\mu(X) + 2M)\epsilon.
\end{aligned}$$

Let $\epsilon \to 0$, we finish the proof. $\square$

Theorem 4.2. *Assume (A1)-(A5) for k which is a positive integer and (B4), (B5), (C1), (C2). If there exists $\eta_0 < 0$ with $a_1 < k\eta_0 < \eta_0 < a_2$ and $\overline{r}, \epsilon, \epsilon_\star$ are sufficiently small, then for each triple $(\xi_s, \xi_c, \xi_y) \in X_1^s \times X_1^c \times Y_1$, (4.3) has a unique solution $(\widetilde{x}, \widetilde{y}) \in B_{\eta_0}^+(\infty)$ such that*

i) If $\xi_s = 0$, $(\widetilde{x}, \widetilde{y}) \equiv (0,0)$.

ii) $(D_{\xi_s}^j \widetilde{x}, D_{\xi_s}^j \widetilde{y}) \in C^0\left(X_1^s \times X_1^c \times Y_1, B_{j\eta_0}^+(\infty)\right)$, where $j = 1, \cdots, k$.

Remark 4.3. *Clearly, $(\widetilde{x} + x, \widetilde{y} + y)\mathcal{M}_\epsilon^{cs}$, where (x, y) is the solution of (4.2) with parameters (ξ_c, ξ_y, t_0).*

Proof. Assumptions (C1) and (C2) imply

$$(x(\xi_{cy}), y(\xi_{cy})) \in C^1\left(X_1^c \times Y_1, B_{\eta_0'}^+(\infty)\right),$$

where $a_1' < \eta' < a_2'$.

For any $\eta \in (a_1, a_2)$ and $(\widetilde{x}, \widetilde{y}) \in B_\eta^+(\infty)$, let

$$\begin{aligned}
&\mathscr{G}_s(\xi_s, \xi_{cy}, t_0, \epsilon)(\widetilde{x}, \widetilde{y})(t) \\
=&U(t,\epsilon)\xi_s + \int_0^t U(t-\tau,\epsilon) \begin{pmatrix} P_s\big(\widetilde{F}(\widetilde{x}, \widetilde{y}, \xi_{cy}, \epsilon) + f_y\widetilde{y}\big) \\ 0 \end{pmatrix} d\tau \\
&+ \int_{+\infty}^t U(t-\tau,\epsilon) \begin{pmatrix} P_{cu}\big(\widetilde{F}(\widetilde{x}, \widetilde{y}, \xi_{cy}, \epsilon) + f_y\widetilde{y}\big) \\ \widetilde{G}(\widetilde{x}, \widetilde{y}, \xi_{cy}, \epsilon) + g_x\widetilde{x} \end{pmatrix} d\tau.
\end{aligned} \tag{4.4}$$

One can see that $\mathscr{G}_s$ has the same form as $\mathscr{T}_s$ with an additional parameter ξ_{cy}. Moreover, we note that

$$\widetilde{F}(0,0,\xi_{cy},\epsilon) = \widetilde{G}(0,0,\xi_{cy},\epsilon) = 0,$$

and by (3.3)

$$|D\widetilde{F}|_{C^0} = |DF|_{C^0} \le \overline{r}, |D\widetilde{G}|_{C^0} = |DF|_{C^0} \le \overline{r},$$

where D is the differentiation with respect to $(\widetilde{x},\widetilde{y})$ or (x,y). Same as Lemma 3.2, $\mathscr{G}_s$ defines a contraction on $B^+_{\eta_0}(\infty)$ under the norm $|\cdot|^+_{\eta,\epsilon_\star}$. By the same procedure as in Theorem 3.14, we obtain $(\widetilde{x},\widetilde{y})$ is C^k from X^s_1 to $B^+_{\eta_0}(\infty)$.

Clearly, if $\xi_s=0$, $(\widetilde{x},\widetilde{y})=(0,0)$ is the unique solution of (4.3).

Therefore, for each $(\xi,t_0)\in\mathcal{M}^\epsilon_c$, its stable fibre is given by

$$\begin{aligned}\mathcal{W}^s_\epsilon(\xi_{cy},t_0)=\Big\{\sigma_{cu}(\xi_s,\xi_{cy},t_0)\Big|\\ \sigma_{cu}(\xi_s,\xi_{cy},t_0)=\xi+\left(\widetilde{x}(0),\widetilde{y}(0)\right),\xi_s\in X^s_1\Big\},\end{aligned}\tag{4.5}$$

where ξ is given in (4.1) and $(\widetilde{x},\widetilde{y})$ is the unique solution of (4.3) with parameters (ξ_s,ξ_{cy},t_0).

For $i=0,1$, let $(x_i,y_i)=\big(x(\xi^i_{cy}),y(\xi^i_{cy})\big)\,,(\widetilde{x}_i,\widetilde{y}_i)=\big(\widetilde{x}(\xi^i_s,\xi^i_{cy}),\widetilde{y}(\xi^i_s,\xi^i_{cy})\big)$ be the solutions of (4.2) and (4.3), respectively, with parameters $(\xi^i_s,\xi^i_{cy})\in X^s_1\times X^c_1\times Y_1$. In the rest of the proof, we will suppress (t,ϵ) in all functions that contain them. We will also write $\widetilde{x}_1-\widetilde{x}_0$ as $\overline{x}$, $\widetilde{y}_1-\widetilde{y}_0$ as $\overline{y}$. We further simply our notation to write $z_i=(x_i,y_i),\widetilde{z}_i=(\widetilde{x}_i,\widetilde{y}_i)$ and $\overline{z}=(\overline{x},\overline{y})$, where $i=0,1$. Fix (ξ^0_s,ξ^0_{cy}), we first note that

$$\begin{aligned}&\widetilde{F}(\widetilde{z}_1,\xi^1_{cy})-\widetilde{F}(\widetilde{z}_0,\xi^0_{cy})\\ =&F(z_1+\widetilde{z}_1)-F(z_1)-F(z_0+\widetilde{z}_0)+F(z_0)\\ =&\int_0^1 D_zF\big(z_1+\widetilde{z}_0+p\overline{z}\big)dp\;\overline{z}+F(z_1+\widetilde{z}_0)-F(z_1)-F(z_0+\widetilde{z}_0)+F(z_0)\\ =&\int_0^1 D_zF\big(z_1+\widetilde{z}_0+p\overline{z}\big)dp\;\overline{z}+\int_0^1\Big(D_zF(z_1+p\widetilde{z}_0)-D_zF(z_0+p\widetilde{z}_0)\Big)dp\;\widetilde{z}_0.\end{aligned}$$

We write

$$\widetilde{F}(\widetilde{z}_1,\xi^1_{cy})-\widetilde{F}(\widetilde{z}_0,\xi^0_{cy})\triangleq\int_0^1 D_zF\big(z_1+\widetilde{z}_0+p\overline{z}\big)dp\;\overline{z}+Q_0(\xi^1_s,\xi^1_{cy}).\tag{4.6}$$

Similarly,

$$\widetilde{G}(\widetilde{z}_1,\xi^1_{cy})-\widetilde{G}(\widetilde{z}_0,\xi^0_{cy})\triangleq\int_0^1 D_zG\big(z_1+\widetilde{z}_0+p\overline{z}\big)dp\;\overline{z}+R_0(\xi^1_s,\xi^1_{cy}).\tag{4.7}$$

Let $C(\cdot) = \int_0^1 D_z F(\cdot + p\widetilde{z}_0)dp$. By Lemma 4.1, one can verify that C satisfies the same property as B which is defined in Lemma 3.9. Apply a similar proof as in Lemma 3.9, one can prove

$$\int_0^t U(t-\tau,\epsilon)\begin{pmatrix} P_s Q_0(\cdot) \\ 0 \end{pmatrix} d\tau, \int_{+\infty}^t U(t-\tau,\epsilon)\begin{pmatrix} P_{cu} Q_0(\cdot) \\ R_0(\cdot) \end{pmatrix} d\tau = o(1).$$

And $o(1) \to 0$ as $(\xi_s^1, \xi_{cy}^1) \to (\xi_s^0, \xi_{cy}^0)$. Consequently, by fixed point property of $\widetilde{z}_i$ for $i = 0, 1$,

$$|\overline{z}|^+_{\eta_0,\epsilon_\star} \leq \frac{2K}{\eta - a_1}|\xi_s^1 - \xi_s^0| + (1-\sigma(\eta))|\overline{z}|^+_{\eta_0,\epsilon_\star} + o(1),$$

where $\sigma(\eta)$ is defined in (3.7) and $o(1) \to 0$ as $(\xi_s^1, \xi_{cy}^1) \to (\xi_s^0, \xi_{cy}^0)$. Since $\overline{r}, \epsilon, \epsilon_\star$ are sufficiently small, $\sigma(\eta) > 0$. Therefore,

$$\widetilde{z} \in C^0(X_1^s \times X_1^c \times Y_1, B_\eta^+(\infty),$$

where $\widetilde{z}$ is the unique solution of (4.3).

In the following, we will prove the continuity of $D_{\xi_s}^j \widetilde{z}$ for $j = 1, 2, \cdots, k$. By induction, we assume the continuity of derivatives of $\widetilde{z}$ with respect to ξ_s of order lower than j. We differentiate (4.3) j times with respect to ξ_s to obtain

$$\begin{aligned}
&D_{\xi_s}^j \widetilde{z} - \delta_{1j} U(t,\epsilon) \\
&= \int_0^t U(t-\tau,\epsilon)\begin{pmatrix} P_s\big(D_z\widetilde{F}(z+\widetilde{z})D_{\xi_s}^j\widetilde{z} + f_y D_{\xi_s}^j \widetilde{y} + M_j(z,\widetilde{z})\big) \\ 0 \end{pmatrix} d\tau \\
&\quad + \int_{+\infty}^t U(t-\tau,\epsilon)\begin{pmatrix} P_{cu}\big(D_z\widetilde{F}(z+\widetilde{z})D_{\xi_s}^j\widetilde{z} + f_y D_{\xi_s}^j \widetilde{y} + M_j(z,\widetilde{z})\big) \\ D_z\widetilde{G}(z+\widetilde{z})D_{\xi_s}^j\widetilde{z} + g_x\widetilde{x} + N_j(z,\widetilde{z}) \end{pmatrix} d\tau,
\end{aligned}$$

where δ_{1j} is the Kronecker delta function and $M(z,\widetilde{z}), N(z,\widetilde{z})$ are in the form of

$$\sum_{\substack{i_1+\cdots+i_m=j,\\ m>1}} \big(H_{i_1,\cdots,i_m}(z+\widetilde{z})\big)\big(D_{\xi_{cy}}^{i_1}\widetilde{z},\cdots,D_{\xi_{cy}}^{i_m}\widetilde{z}\big).$$

Here each $H_{i_1,\cdots,i_m}(z+\widetilde{z})$ is a multi-linear operator and C^{j-m} in z.

From the continuity of lower order derivatives and a similar proof as in Lemma 3.9 again, we have

$$\left| \int_0^t U(t-\tau,\epsilon) \begin{pmatrix} P_s(M_j(z_1,\widetilde{z}_1) - M_j(z_0,\widetilde{z}_0)) \\ 0 \end{pmatrix} d\tau \right.$$
$$\left. + \int_{+\infty}^t U(t-\tau,\epsilon) \begin{pmatrix} P_{cu}(M_j(z_1,\widetilde{z}_1) - M_j(z_0,\widetilde{z}_0)) \\ N_j(z_1,\widetilde{z}_1) - N_j(z_0,\widetilde{z}_0) \end{pmatrix} d\tau \right|_{j\eta_0,\epsilon_\star}^+ \longrightarrow 0,$$

as $(\xi_s^1, \xi_{cy}^1) \to (\xi_s^0, \xi_{cy}^0)$. Therefore, following the same procedure as $j = 0$, we have

$$D_{\xi_s}^j \widetilde{z} \in C^0\left(X_1^s \times X_1^c \times Y_1, B_{j\eta_0}^+(\infty)\right).$$

□

For a positive integer $k \geq 2$, we define

$$\Lambda_k = \Big\{(\eta,\eta') \in \mathbb{R}^2 \big| a_1 < j\eta < \min\{0, a_2\}, a_1' < j\eta' < a_2',$$
$$a_1 < \eta + j\eta' < a_2,\ j = 1, 2, \cdots, k-1, \Big\}. \tag{4.8}$$

Theorem 4.4. *For $k \geq 2$, assume (A1)-(A5), (B4), (B5), (C1), (C2) and Λ_k is nonempty. For any compact subset Σ of Λ_k, when the Lipschitz constants of F and G are sufficiently small, then for any $(\eta,\eta') \in \Sigma$,* (4.3) *has a unique solution $(\widetilde{x},\widetilde{y}) \in B_\eta^+(\infty)$ such that*

i) $(D_{\xi_{cy}}^j \widetilde{x}, D_{\xi_{cy}}^j \widetilde{y}) \in C^0\left(X_1^s \times X_1^c \times Y_1, B_{\eta+j\eta'}^+(\infty)\right)$,

ii) $(D_{\xi_s}^{m-j} D_{\xi_{cy}}^j \widetilde{x}, D_{\xi_s}^{m-j} D_{\xi_{cy}}^j \widetilde{y}) \in C^0\left(X_1^s \times X_1^c \times Y_1, B_{\eta+j\eta'}^+(\infty)\right)$,

where $m = 2, \cdots, k,\ j = 1, \cdots, m-1$.

Proof. In this proof, we often use the notation introduced in the previous proof. We will prove the result inductively. For $k = 2$, since $a_1 < \eta + \eta' < a_2$, from Theorem 4.2 one has

$$\overline{z} = (\overline{x}, \overline{y}) \in B_{\eta+\eta'}^+(\infty). \tag{4.9}$$

From Q_0, R_0 defined in (4.6) and (4.7) as functions of (ξ_s^1, ξ_{cy}^1), one can use the Lipschitz property of DF, DG and C^2 and C^2 smoothness of the center manifold given in Theorem 3.18 to obtain

$$\int_0^t U(t-\tau,\epsilon)\begin{pmatrix} P_s Q_0(\cdot) \\ 0 \end{pmatrix} d\tau \ , \ \int_{+\infty}^t U(t-\tau,\epsilon)\begin{pmatrix} P_{cu} Q_0(\cdot) \\ R_0(\cdot) \end{pmatrix} d\tau$$
$$= O(|\xi_s^1 - \xi_s^0| + |\xi_{cy}^1 - \xi_{cy}^0|),$$

where $O(|\xi_s^1 - \xi_s^0| + |\xi_{cy}^1 - \xi_{cy}^0|)$ is measured in $B^+_{\eta+\eta'}(\infty)$. It implies

$$\widetilde{z} = (\widetilde{x}, \widetilde{y}) \in Lip\left(X_1^s \times X_1^c \times Y_1, B^+_{\eta+\eta'}(\infty)\right). \tag{4.10}$$

Now we fix $\xi_s^1 = \xi_s^0$. From Taylor expansion we write

$$\begin{aligned} Q_1(\xi_{cy}^1) =& \widetilde{F}(\widetilde{z}_1, \xi_{cy}^1) - \widetilde{F}(\widetilde{z}_0, \xi_{cy}^0) - D_z F(z_0 + \widetilde{z}_0)\overline{z} \\ & - \big(D_z F(z_0 + \widetilde{z}_0) - D_z F(z_0)\big) D_{\xi_{cy}} z(\xi_{cy}^0)(\xi_{cy}^1 - \xi_{cy}^0), \\ R_1(\xi_{cy}^1) =& \widetilde{G}(\widetilde{z}_1, \xi_{cy}^1) - \widetilde{G}(\widetilde{z}_0, \xi_{cy}^0) - D_z G(z_0 + \widetilde{z}_0)\overline{z} \\ & - \big(D_z G(z_0 + \widetilde{z}_0) - D_z G(z_0)\big) D_{\xi_{cy}} z(\xi_{cy}^0)(\xi_{cy}^1 - \xi_{cy}^0), \end{aligned}$$

where we recall $\widetilde{z}_i = (\widetilde{x}_i, \widetilde{y}_i), z_i = (x_i, y_i)$ for $i = 0, 1$.

By C^2 smoothness of F, G and (4.10), we obtain

$$\left| \int_0^t U(t-\tau,\epsilon)\begin{pmatrix} P_s Q_1(\xi_{cy}^1) \\ 0 \end{pmatrix} d\tau + \int_{+\infty}^t U(t-\tau,\epsilon)\begin{pmatrix} P_{cu} Q_1(\xi_{cy}^1) \\ R_1(\xi_{cy}^1) \end{pmatrix} d\tau \right|^+_{\eta+\eta',\epsilon_\star}$$
$$= o(|\xi_{cy}^1 - \xi_{cy}^0|).$$

Define

$$\begin{aligned} \mathscr{G}(\xi_s^0, \xi_{cy}^0, t_0, \epsilon)(\overline{z})(t) =& \int_0^t U(t-\tau,\epsilon)\begin{pmatrix} P_s\big(D_z F(z_0 + \widetilde{z}_0)\overline{z} + f_y \overline{y}\big) \\ 0 \end{pmatrix} d\tau \\ & + \int_{+\infty}^t U(t-\tau,\epsilon)\begin{pmatrix} P_{cu}\big(D_z F(z_0 + \widetilde{z}_0)\overline{z} + f_y \overline{y}\big) \\ D_z G(z_0 + \widetilde{z}_0)\overline{z} + g_x \overline{x} \end{pmatrix} d\tau, \end{aligned}$$

and

$$\mathscr{H}(\xi_s^0, \xi_{cy}^0, t_0, \epsilon)(\xi_{cy}^1 - \xi_{cy}^0)$$

$$= \int_0^t U(t-\tau,\epsilon) \begin{pmatrix} P_s\big(D_zF(z_0+\widetilde{z}_0) - D_zF(z_0)\big)D_{\xi_{cy}}z(\xi_{cy}^0)(\xi_{cy}^1 - \xi_{cy}^0) \\ 0 \end{pmatrix} d\tau$$

$$+ \int_{+\infty}^t U(t-\tau,\epsilon) \begin{pmatrix} P_{cu}\big(D_zF(z_0+\widetilde{z}_0) - D_zF(z_0)\big)D_{\xi_{cy}}z(\xi_{cy}^0)(\xi_{cy}^1 - \xi_{cy}^0) \\ \big(D_zG(z_0+\widetilde{z}_0) - D_zG(z_0)\big)D_{\xi_{cy}}z(\xi_{cy}^0)(\xi_{cy}^1 - \xi_{cy}^0) \end{pmatrix} d\tau.$$

Integrating $g_x\overline{x}$ by parts, one can prove that $\mathscr{G}$ defines a linear contraction on $B_\eta^+(\infty)$ for any η in any compact subset of (a_1, a_2). And $\mathscr{H}$ has to be estimated on $B_{\eta+\eta'}^+(\infty)$. From (4.3), one has

$$\overline{z} = \mathscr{G}(\xi_s^0, \xi_{cy}^0, t_0, \epsilon)\overline{z} + \mathscr{H}(\xi_s^0, \xi_{cy}^0, t_0, \epsilon)(\xi_{cy}^1 - \xi_{cy}^0) + o(|\xi_{cy}^1 - \xi_{cy}^0|).$$

Therefore, $\widetilde{z}$ is *Fréchet* differentiable with respect to ξ_{cy}. Furthermore,

$$D_{\xi_{cy}}\widetilde{z}(\xi_{cy}^0) = (I - \mathscr{G})^{-1}\mathscr{H}(\xi_s^0, \xi_{cy}^0, t_0, \epsilon).$$

Based on the above formula and Theorem 3.18, Theorem 4.2 and a similar proof as in Lemma 3.9, $D_{\xi_{cy}}\widetilde{z}$ is continuous in (ξ_s, ξ_{cy}).

Assume the result holds for $k' = 1, 2, \cdots, k_0 - 1 < k$. We will prove it for

$k' = k_0 \geq 3$. We differentiate (4.2) $k_0 - 2$ times with respect to ξ_{cy} to obtain

$$D_{\xi_{cy}}^{k_0-2}\widetilde{z}(t) \tag{4.11}$$
$$= \int_0^t U(t-\tau,\epsilon)\begin{pmatrix} P_s\Big(D_zF(z+\widetilde{z})D_{\xi_{cy}}^{k_0-2}\widetilde{z} + f_yD_{\xi_{cy}}^{k_0-2}\widetilde{y}\Big) \\ 0 \end{pmatrix} d\tau$$
$$+ \int_0^t U(t-\tau,\epsilon)\begin{pmatrix} P_s\Big(\big(D_zF(z+\widetilde{z}) - D_zF(z)\big)D_{\xi_{cy}}^{k_0-2}z + Q_{k_0-1}(z,\widetilde{z})\Big) \\ 0 \end{pmatrix} d\tau$$
$$+ \int_{+\infty}^t U(t-\tau,\epsilon)\begin{pmatrix} P_{cu}\Big(D_zF(z+\widetilde{z})D_{\xi_{cy}}^{k_0-2}\widetilde{z} + f_yD_{\xi_{cy}}^{k_0-2}\widetilde{y}\Big) \\ D_zG(z+\widetilde{z})D_{\xi_{cy}}^{k_0-2}\widetilde{z} + g_xD_{\xi_{cy}}^{k_0-2}\widetilde{x} \end{pmatrix} d\tau$$
$$+ \int_{+\infty}^t U(t-\tau,\epsilon)\begin{pmatrix} P_{cu}\Big(\big(D_zF(z+\widetilde{z}) - D_zF(z)\big)D_{\xi_{cy}}^{k_0-2}z + Q_{k_0-2}(z,\widetilde{z})\Big) \\ \big(D_zF(z+\widetilde{z}) - D_zF(z)\big)D_{\xi_{cy}}^{k_0-2}z + R_{k_0-2}(z,\widetilde{z}) \end{pmatrix} d\tau.$$

Here Q_{k_0-2} and R_{k_0-2} are in the form of

$$\begin{aligned} &\sum_{\substack{i_1+\cdots+i_m=k_0-2,\\ m>1}} \big(H_{i_1,\cdots,i_m}(z+\widetilde{z})\big)\big(D_{\xi_{cy}}^{i_1}(z+\widetilde{z}),\cdots,D_{\xi_{cy}}^{i_m}(z+\widetilde{z})\big) \\ -&\sum_{\substack{i_1+\cdots+i_m=k_0-2,\\ m>1}} \big(H_{i_1,\cdots,i_m}(z)\big)\big(D_{\xi_{cy}}^{i_1}z,\cdots,D_{\xi_{cy}}^{i_m}z\big), \end{aligned} \tag{4.12}$$

where each $H_{i_1,\cdots,i_m}(z)$ is a multi-linear operator and C^{k_0-m} in z.

We need to prove $D_{\xi_{cy}}^{k_0-1}\widetilde{z}$ exists under the assumptions F, G are C^{k_0} and Λ_{k_0} is nonempty. By substituting $D_{\xi_{cy}}\widetilde{z}_1, D_{\xi_{cy}}\widetilde{z}_0$ into (4.11), respectively, and taking difference to follow a similar procedure as in the case of $k' = 2$, one can show

$$D_{\xi_{cy}}^{k_0-1}\widetilde{z} \in C^0\left(X_1^s \times X_1^c \times Y_1, B_{\eta+(k_0-1)\eta'}^{+}(\infty)\right).$$

Thus, we have finished part i).

For part ii), one can still it by induction. For any fixed (m, j), we can derive the formula from (4.3) that $D_{\xi_s}^{m-j-1}D_{\xi_{cy}}^{j}\widetilde{z}$ should satisfy. Again, we substitute $D_{\xi_s}^{m-j-1}D_{\xi_{cy}}^{j}\widetilde{z}_1$ and $D_{\xi_s}^{m-j-1}D_{\xi_{cy}}^{j}\widetilde{z}_0$ into that formula and take the difference to prove the result. □

Given any $\xi_c \in X_1^c$, let $\xi_0 = \xi_c + (\Psi_s^0 + \Psi_u^0)(\xi_c) \in \mathcal{M}_0^c$, where we recall that Ψ_s^0, Ψ_u^0 are independent of t_0, and $x_0(\xi_c)(t)$ be the solution on $\mathcal{M}_0^c$ such that $x_0(\xi_c)(0) = \xi_0$. Let $\widetilde{x}_0(t)$ satisfy

$$\begin{aligned}\widetilde{x}_0(t) =& e^{tA_f}\xi_s + \Big(\int_0^t e^{(t-\tau)A_f} P_s + \int_{+\infty}^t e^{(t-\tau)A_f} P_{cu}\Big)\\ &\big(F(\widetilde{x}_0 + x_0(\xi_c), 0, \tau + t_0, 0) - F(x_0(\xi_c), 0, \tau + t_0, 0)\big)\, d\tau.\end{aligned}$$

Therefore, $(\widetilde{x}_0 + x_0(\xi_c))(t)$ is the solution of the unperturbed fibre starting at the based point ξ_0 with height ξ_s such that $(\widetilde{x}_0 + x_0)(0) = \xi_s + (I - P_u)\xi_0 + h_u(\xi_s, (I - P_u)\xi_0)$. Define

$$\mathcal{W}_0^s(\xi_c) = \Big\{\sigma_{cu}^0(\xi_s, \xi_c)\Big|\sigma_{cu}^0(\xi_s, \xi_c) = \xi_0 + \widetilde{x}_0(0), \xi_s \in X_1^s\Big\}.$$

A natural question is that what happens to each stable fibre when ϵ tends to 0.

Theorem 4.5. *For $k = 2$, assume (A1)-(A6), (B4), (B5), (C1), (C2) and Λ_2 in* (4.8) *is nonempty. For $\xi_y = 0$ and the above given ξ_c, let (x, y) be the solution of* (4.2) *and $(\widetilde{x}, \widetilde{y})$ be the solution of* (4.3). *Then, we have*

$$\big|\sigma_{cu}(\xi_s, \xi) - \sigma_{cu}^0(\xi_s, \xi_0)\big|_{X_1 \times Y_1} \leq C'\epsilon,$$

where C' depends on $K, a_1, a_1', a_2', \eta, \eta', \overline{r}, \epsilon_\star, |\xi_c|_{X_1}$.

Proof. By (4.3),

$$\begin{aligned}\widetilde{y}(t) = &\int_{+\infty}^t e^{(t-\tau)(\frac{J}{\epsilon} + g_y)}\Big(\widetilde{G}(\widetilde{x}, \widetilde{y}, \xi_c, \epsilon) - \widetilde{G}(\widetilde{x}, 0, \xi_c, \epsilon)\Big)\, d\tau\\ &+ \int_{+\infty}^t e^{(t-\tau)(\frac{J}{\epsilon} + g_y)}\Big(\widetilde{G}(\widetilde{x}, 0, \xi_c, \epsilon) + g_x\widetilde{x}\Big)\, d\tau.\end{aligned} \tag{4.13}$$

Since $|D_{(\widetilde{x},\widetilde{y})}\widetilde{G}|_{C^0} = |D_{(x,y)}G|_{C^0} \leq \overline{r}$ and $a_1 < \eta < \eta + \eta' < a_2$,

$$\begin{aligned}\sup_{t\geq 0} \frac{1}{\epsilon_\star} e^{-(\eta+\eta')t}\Big|&\int_{+\infty}^t e^{(t-\tau)(\frac{J}{\epsilon} + g_y)}\Big(\widetilde{G}(\widetilde{x}, \widetilde{y}, \xi_c, \epsilon)\\ &- \widetilde{G}(\widetilde{x}, 0, \xi_c, \epsilon)\Big)\, d\tau\Big|_{Y_1} \leq \frac{K\overline{r}}{\eta + \eta' - a_1}|\widetilde{y}|_{\eta+\eta',\epsilon_\star,Y_1}.\end{aligned} \tag{4.14}$$

For the second term on the right hand side of (4.13), we integrate by parts to obtain

$$
\begin{aligned}
&\int_{+\infty}^{t} e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}\Big(\widetilde{G}(\widetilde{x},0,\xi_c,\epsilon)+g_x\widetilde{x}\Big)\,d\tau \\
=&-(\frac{J}{\epsilon}+g_y)^{-1}\Big(\widetilde{G}(\widetilde{x}(t),0,\xi_c,\epsilon)+g_x\widetilde{x}(t)\Big) \\
&+\int_{+\infty}^{t}(\frac{J}{\epsilon}+g_y)^{-1}e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}\Big[\big(D_xG(x(\xi_c)+\widetilde{x},y(\xi_c),\tau+t_0,\epsilon)+g_x\big)\dot{\widetilde{x}} \\
&+\big(D_xG(x(\xi_c)+\widetilde{x},y(\xi_c),\tau+t_0,\epsilon)-D_xG(x(\xi_c),y(\xi_c),\tau+t_0,\epsilon)\big)\dot{x}(\xi_c) \\
&+\big(D_yG(x(\xi_c)+\widetilde{x},y(\xi_c),\tau+t_0,\epsilon)-D_yG(x(\xi_c),y(\xi_c),\tau+t_0,\epsilon)\big)\dot{y}(\xi_c) \\
&+\big(\partial_tG(x(\xi_c)+\widetilde{x},y(\xi_c),\tau+t_0,\epsilon)-\partial_tG(x(\xi_c),y(\xi_c),\tau+t_0,\epsilon)\big)\Big]\,d\tau,
\end{aligned}
$$

where $(x(\xi_c),y(\xi_c))=(x(\xi_c)(\tau),y(\xi_c)(\tau))$ and $\widetilde{x}=\widetilde{x}(\tau)$.

Since $\xi_y=0$, (3.61) shows $|y(\xi_c)|^+_{\eta',\epsilon_\star,Y_1}\le C'\epsilon$, which implies

$$
\begin{aligned}
|\dot{y}(\xi_c)(\tau)|_{Y_1}\le&(\frac{1}{\epsilon}+C_0)C'\epsilon_\star\epsilon e^{\eta'\tau}+C_0|x(\xi_c)|^+_{\eta',1,X_1}e^{\eta'\tau} \\
&+\overline{r}\big(|x(\xi_c)|^+_{\eta',1,X_1}e^{\eta'\tau}+C'\epsilon_\star\epsilon e^{\eta'\tau}\big),
\end{aligned}
$$

where C' depends on $K,a_1',\eta',\overline{r},\epsilon_\star,|\xi_c|_{X_1}$. It follows that

$$
\begin{aligned}
&\sup_{t\ge0}\frac{1}{\epsilon_\star}e^{-(\eta+\eta')t}\Big|\int_{+\infty}^{t}(\frac{J}{\epsilon}+g_y)^{-1}e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}\big(D_yG(x(\xi_c)+\widetilde{x},y(\xi_c),\tau+t_0,\epsilon) \\
&\qquad-D_yG(x(\xi_c),y(\xi_c),\tau+t_0,\epsilon)\big)\dot{y}(\xi_c)\,d\tau\Big|_{Y_1} \\
\le&\frac{2C_0\epsilon}{\epsilon_\star}\frac{K}{\eta+\eta'-a_1}C_0|\widetilde{x}|^+_{\eta,1,X_1}\Big[(\frac{1}{\epsilon}+C_0)\epsilon_\star|y(\xi_c)|^+_{\eta',\epsilon_\star,Y_1} \\
&+\overline{r}\big(|x(\xi_c)|^+_{\eta',1,X_1}+\epsilon_\star|y(\xi_c)|^+_{\eta',\epsilon_\star,Y_1}\big)+C_0|x(\xi_c)|^+_{\eta',1,X_1}\Big]\le C'\epsilon.
\end{aligned}
$$

And other terms can be estimated as before to be bounded by $C'\epsilon$. Consequently, by (4.3)

$$
\begin{aligned}
|\widetilde{y}|_{\eta+\eta',\epsilon_\star,Y_1}=&\sup_{t\ge0}\frac{1}{\epsilon_\star}e^{-(\eta+\eta')t}\Big|\int_{+\infty}^{t}e^{(t-\tau)(\frac{J}{\epsilon}+g_y)}\Big(\widetilde{G}(\widetilde{x},0,\xi_c,\epsilon)+g_x\widetilde{x}\Big)\,d\tau\Big|_{Y_1} \\
\le&C'\epsilon,
\end{aligned}
$$

where C' depends on $K,a_1,a_1',a_2',\eta,\eta',\overline{r},\epsilon_\star,|\xi_c|_{X_1}$.

Using integral equations of $\widetilde{x}(t)$ and $\widetilde{x}_0(t)$, we have

$$\begin{aligned}
&\widetilde{x}(t) - \widetilde{x}_0(t)\\
=& \int_0^t e^{(t-\tau)A_f} P_s\Big(\widetilde{F}(\widetilde{x}, \widetilde{y}, \xi_c, \epsilon) - \widetilde{F}(\widetilde{x}_0, 0, \xi_c, 0) + f_y \widetilde{y}\Big)\, d\tau\\
&+ \int_{+\infty}^t e^{(t-\tau)A_f} P_{cu}\Big(\widetilde{F}(\widetilde{x}, \widetilde{y}, \xi_c, \epsilon) - \widetilde{F}(\widetilde{x}_0, 0, \xi_c, 0) + f_y \widetilde{y}\Big)\, d\tau.
\end{aligned}$$

By the facts $|\widetilde{y}|_{\eta+\eta',\epsilon_\star,Y_1}$ and $|D\widetilde{F}|_{C^0} \leq \overline{r}$, one can easily deduce

$$|\widetilde{x} - \widetilde{x}_0|_{\eta+\eta',1,X_1} \leq C'\epsilon.$$

In particular, let $t = 0$, we finish the proof. □

CHAPTER V

NORMALLY ELLIPTIC SINGULAR PERTURBATION TO HOMOCLINIC ORBITS AND PRELIMINARIES

In this chapter, we will discuss the persistence of a homoclinic orbit under the normally elliptic singular perturbation. We assume (A1)-(A5) for $k = 2$, (A6′), (B1)-(B5) in Chapter 4 and (C1)-(C2) after Theorem 3.5 . As we proved in chapter 3, (2.2) can be viewed as the singular limit of the singularly perturbed system (2.1). In this whole chapter, we assume

(D1) A generates a strongly continuous group on X and X_1^u has finite dimension.

(D2) For any $(x, t) \in X_1 \times \mathbb{R}$,

$$J^{-1}D_x g(x, 0, t, 0)A \in L(X, Y)\ ,\ J^{-1}D_x^2 g(x, 0, t, 0) \in L_2(X, Y).$$

Moreover,

$$\partial_\epsilon \partial_t Dg \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, L(X_1 \times Y_1, Y)).$$

(D3) There exist η and η' such that

$$a_1 < \eta < \min\{0, a_2\}\ ,\ \max\{0, a_1'\} < \eta' < 2\eta' < a_2',$$
$$a_1 < \eta + \eta' < a_2\ ,\ a_1 + \eta' < 0.$$

(D4) When $\epsilon = 0$, (2.2) has a homoclinic orbit $x_h(t)$ such that

$$Ax_h(t) \subset X_1,$$

and

$$\sup_{t \geq 0} e^{-a_1 t}|x_h(t)|_{X_1} < \infty\ ,\ \sup_{t \leq 0} e^{-a_2' t}|x_h(t)|_{X_1} < \infty,$$

where a_1, a_2, a_1', a_2 are defined in (B5) and (C2).

(D5) There exists a C^2 invariant quantity $H : X_1 \to \mathbb{R}$ with $DH : X \to \mathbb{R}$ such that

$$H(0) = 0 \ , \ DH(0) = 0.$$

(D6) There exists $x_0 = x_h(0)$ such that

$$DH(x_0) \neq 0 \ , \ \dim(T_{x_0}\mathcal{M}_0^u \bigcap T_{x_0}\mathcal{M}_0^{cs}) = 1.$$

Our question is if (2.1) has a homoclinic solution to $(0, 0)$ when $0 < \epsilon \ll 1$.

This chapter is organized as follows. In the first section, we will establish a coordinate system around the unperturbed homoclinic orbit. The second section is devoted to study the persistence of the homoclinic orbit under weakly dissipative perturbations and the last section is to study the conservative case. In section 6.3, we also assume f, g are independent of t for all $\epsilon \geq 0$.

5.1 Coordinates around the unperturbed homoclinic orbit

With slight abuse of notation, we extend H from X_1 to $X_1 \times Y_1$ such that the extension is independent of y variable. Clearly,

$$H(0, 0) = 0 \ , \ DH(0, 0) = 0. \tag{5.1}$$

Let $v = Ax_0 + f(x_0, 0, t, 0)$. Since $v \in X_1 \subset X$, there exists a hyperplane $\Sigma' \subset X$ that is transverse to v. Let $\Sigma = (\Sigma' \bigcap X_1) \times Y_1$, by using $v \in X_1$, one can prove v and Σ are transverse in $X_1 \times Y_1$. Let Q_v, Q_v' be the projections from $X_1 \times Y_1$ and $X \times Y$ onto $\mathbb{R}v$ with kernel Σ and $\Sigma' \times Y$, respectively. We will identify the range of Q_v and Q_v', i.e., $\mathbb{R}v$ with $\mathbb{R}$. Locally, we cut off the nonlinearity as in chapter 4 to obtain h_{cs}, h_u and thus the local invariant integral manifolds. Let r be the cut-off radius defined in chapter 4, there exist $t_1 > 0, t_2 < 0$ such that

$$|x_{1,2}|_{X_1} < \frac{r}{2\big(1 + |P_{cs}|(1 + |Dh_u|_{C^0}) + |P_u|(1 + |Dh_{cs}|_{C^0})\big)}, \tag{5.2}$$

where $x_{1,2} = x_h(t_{1,2})$. With slight abuse of notation, we will also use $\mathcal{M}^{\alpha}_{\beta}(t_0)$ to denote various invariant integral manifolds extended by the flow from the local ones with initial time $t_0 \in \mathbb{R}$ for systems (2.1) and (2.2), where $\alpha = cs, u, cu, s, c$, $\beta = 0, \epsilon$. Next, we will prove $\mathcal{M}^{cs}_{\epsilon}(t_0)$ (or $\mathcal{M}^{u}_{\epsilon}(t_0)$) are C^1 close to $\mathcal{M}^{cs}_{0}(t_0) \times Y_1$ (or $\mathcal{M}^{u}_{0}(t_0)$) so that $\mathcal{M}^{cs}_{\epsilon}(t_0)$ (or $\mathcal{M}^{u}_{\epsilon}(t_0)$) and Σ are transverse. Then we will work on their intersections. We recall that Φ and Φ^0 denote the flow maps of (2.1) and (2.2), respectively.

We first show that for any $t_0 \in \mathbb{R}$, $\mathcal{M}^{cs}_{\epsilon}(t_0)$ does intersect Σ near x_0 for $\epsilon \ll 1$.

Lemma 5.1. *For any $t_0 \in \mathbb{R}$, there exists a unique $t' = t'(t_0)$ such that*

$$\Phi(t_0, t_0 + t', x_1', \epsilon) \in x_0 + \Sigma,$$

$$\left|\Phi(t_0, t_0 + t', x_1', \epsilon) - x_0\right|_{X_1 \times Y_1} \le C'\epsilon,$$

$$|t' - t_1| < C'\epsilon \ , \ |\partial_{t_0} t'| \le C'\epsilon,$$

where $x_1' = P_{cs}x_1 + h_u(P_{cs}x_1, t_0 + t', \epsilon)$ and C' depends on constants in assumptions.

Proof. We will use $\partial_1\Phi, \partial_2\Phi$ to denote the differentiation with respect to terminal and initial time, respectively. Such notations also apply to Φ^0. For any $t_0 \in \mathbb{R}$, since (2.2) is autonomous, we have

$$Q_v'(\Phi^0(t_0, t_0 + t_1, x_1) - x_0) = 0 \ , \ Q_v'\partial_1\Phi^0(t_0, t_0 + t_1, x_1) = 1.$$

Clearly, h_u satisfies similar properties as h_s in Theorem 3.19 and 3.20. Let

$$\gamma(t', \epsilon) = Q_v'(\Phi(t_0, t_0 + t', x_1', \epsilon) - x_0) \ , \ \gamma(t', 0) = Q_v'(\Phi^0(t_0, t_0 + t', x_1) - x_0).$$

Theorem (2.4) shows for t' on any bounded interval

$$|\gamma(t', \epsilon) - \gamma(t', 0)|_{X_1 \times Y_1} \le C'\epsilon. \tag{5.3}$$

From (A4), one can easily prove

$$D\Phi \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, L(X \times Y, X \times Y).$$

By using chain rule, one can show

$$\begin{aligned}
&\Big|\partial_{t'}\gamma(t',\epsilon)-\partial_{t'}\gamma(t',0)\Big| \\
\leq&\Big|Q'_v\big(D\Phi(t_0,t_0+t',x'_1,\epsilon)V_\epsilon(t_0+t',x'_1)-D\Phi^0(t_0,t_0+t',x_1)V_0(t_0+t',x_1)\big)\Big| \\
&+\Big|Q'_vD\Phi(t_0,t_0+t',x'_1,\epsilon)\partial_{t_0}h_u(x'_1,t_0+t',\epsilon)\Big|,
\end{aligned}$$

where $V_\epsilon(t,x), V_0(t,x)$ represent the velocity field of (2.1) and (2.2) at (t,x), respectively. By a similar proof as in Theorem 3.19, we have $|x'_1-x_1|_{X_1\times Y_1}\leq C'\epsilon$ which along with the fact $x'_1\in X_1$ implies $|P_X(V_\epsilon(t_0+t',x'_1)-V_0(t_0+t',x_1))|_X\leq C'\epsilon$. Applying Theorem 2.4, we obtain

$$\Big|Q'_v\big(D\Phi(t_0,t_0+t',x'_1,\epsilon)V_\epsilon(t_0+t',x'_1)-D\Phi^0(t_0,t_0+t',x_1)V_0(t_0+t',x_1)\big)\Big|\leq C'\epsilon.$$

Similar to Theorem 3.26 which was stated for h_s, we have $|\partial_{t_0}h_u(x'_1,t_0+t',\epsilon)|_X\leq C'\epsilon$, which implies

$$\Big|Q'_vD\Phi(t_0,t_0+t',x'_1,\epsilon)\partial_{t_0}h_u(x'_1,t_0+t',\epsilon)\Big|\leq C'\epsilon. \tag{5.4}$$

Therefore, we have proved $\gamma(t',\epsilon)$ and $\gamma(t',0)$ are C^1 close for t' on any bounded intervals. Since the system (2.2) is autonomous when $\epsilon=0$,

$$\partial_{t'}\gamma(t_1,0)=-Q'_v\partial_1\Phi^0(t_0,t_0+t_1,x_1)=-Q'_vv=-1.$$

By implicit function theorem, there exists a unique $t'=t'(t_0)$ such that

$$Q'_v(\Phi(t_0,t_0+t',x'_1,\epsilon)-x_0)=0\ ,\ |t'(t_0)-t_1|\leq C'\epsilon.$$

Since $\Phi(t_0,t_0+t',x'_1,\epsilon)\in X_1\times Y_1$, we have

$$\Phi(t_0,t_0+t',x'_1,\epsilon)\in x_0+\Sigma.$$

Since $Ax_h\subset X_1$ and (2.2) is autonomous, one may obtain

$$\big|\Phi^0(t_0,t_0+t',x_1)-\Phi^0(t_0,t_0+t_1,x_1)\big|_{X_1}\leq C'\epsilon.$$

Together with Theorem 2.1, we have

$$\left|\Phi(t_0, t_0 + t', x_1', \epsilon) - x_0\right|_{X_1 \times Y_1} \leq C'\epsilon. \tag{5.5}$$

Note that

$$\begin{aligned}\partial_{t_0}\gamma(t', \epsilon) =& Q_v'\big(V_\epsilon(t_0, \Phi(t_0, t_0 + t', x_1', \epsilon)) - D\Phi(t_0, t_0 + t', x_1', \epsilon)V_\epsilon(t_0 + t', x_1') \\ &+ D\Phi(t_0, t_0 + t', x_1', \epsilon)\partial_{t_0} h_u\big).\end{aligned}$$

By Theorem 2.4 and (5.4), (5.5), we have $|\partial_{t_0}\gamma(t', \epsilon)| \leq C'\epsilon$, which implies

$$|\partial_{t_0} t'| \leq C'\epsilon. \tag{5.6}$$

□

Next we study the tangent space $T(\Sigma \bigcap \mathcal{M}_\epsilon^{cs}(t_0))$ at $\Phi(t_0, t_0 + t'(t_0), x_1', \epsilon)$. We use $B_\rho(p, S)$ to denote the ball in a space S centered at p with radius ρ. In the following lemma, we need to use the evolutionary operator $E(t, t_0; x, \epsilon)$ defined as the solution operator of the linear equation

$$\dot{E} = (\frac{J}{\epsilon} + D_y g(\Phi^0(t, t_0, x), 0, t, 0))E(t, t_0; \Phi^0(t, t_0, x)),$$

where $x \in X_1$.

Lemma 5.2. *Let C be positive constants and $t' = t'(t_0)$ be the one found in Lemma 5.1. When $0 < \delta \ll 1$, for any $(\xi_{cs}, \xi_y) \in B_\delta(P_{cs}x_1, X_1^{cs}) \times B_{C\epsilon}(0, Y_1)$,*

$$\left|\Phi(t_0, t_0 + t', \xi_{cs} + \xi_y + h_u(\xi_{cs}, \xi_y, t_0 + t', \epsilon), \epsilon) - x_0\right|_{X_1 \times Y_1} \leq C'\delta. \tag{5.7}$$

Moreover, if $(\delta x, \delta y) \in X_1^{cs} \times Y_1$ with $|\delta x|_{X_1} + |\delta y|_{Y_1} \leq 1$,

$$\begin{aligned}\Big|D_{\xi_{cs}}&\big(\Phi(t_0, t_0 + t', \xi_{cs} + \xi_y + h_u(\xi_{cs}, \xi_y, t_0 + t', \epsilon), \epsilon)\big)\delta x \\ &- D_{\xi_{cs}}\big(\Phi^0(t_0, t_0 + t', \xi_{cs} + h_u^0(\xi_{cs}))\big)\delta x\Big|_{X_1} \leq C'\epsilon,\end{aligned} \tag{5.8}$$

and

$$
\begin{aligned}
&\Big|P_X D_{\xi_y}\big(\Phi(t_0, t_0+t', \xi_{cs}+\xi_y+h_u(\xi_{cs},\xi_y,t_0+t',\epsilon),\epsilon)\big)\delta y\Big|_{X_1} \le C'\epsilon,\\
&\Big|\Big(P_Y D_{\xi_y}\big(\Phi(t_0, t_0+t', \xi_{cs}+\xi_y+h_u(\xi_{cs},\xi_y,t_0+t',\epsilon),\epsilon)\big)\\
&\qquad\qquad - E(t_0, t_0+t'; \xi_{cu}+h_u^0(\xi_{cu}),\epsilon)\Big)\delta y\Big|_{Y_1} \le C'\epsilon.
\end{aligned}
\tag{5.9}
$$

where P_X, P_Y denote the projection from $X \times Y$ onto X and Y, respectively. And C' depends on C and constants in assumptions.

Proof. By assumption (D3), Φ is C^2 with respect to phase space variables. Inequality (5.7) follows from Lemma 5.1 and the C^2 smoothness of Φ. Using the fact $|\xi_y|_{Y_1} \le C\epsilon$ and the same proof as in Theorem 3.20, one can obtain (5.8). Finally, the proof of (5.9) follows from Theorem 3.20 and the assumptions $|\xi_y|_{Y_1} \le C'\epsilon$ and $\dim X_1^u < \infty$. Therefore, we have

$$
\big|D_{\xi_y} h_u(\xi_{cs},\xi_y,t_0+t',\epsilon)\big|_{L(Y_1,X_1^u)} = \big|D_{\xi_y} h_u(\xi_{cs},\xi_y,t_0+t',\epsilon)\big|_{L(Y_1,X^u)} \le C'\epsilon.
$$

From (5.2), we have

$$
|\xi_{cs}+\xi_y+h_u(\xi_{cs},\xi_y,t_0+t',\epsilon)|_{X_1\times Y_1} < r.
$$

By Theorem 3.20, we can finish the proof. □

We notice from assumptions (A3) and (A4) that $\frac{J}{\epsilon} + D_y g$ generates an evolution operator $E(t, t_0;\epsilon)$ bounded on any finite time interval uniformly in $0 < \epsilon \ll 1, (\xi_{cs}, \xi_y) \in X_1^{cs} \times Y_1$ and so its inverse. Therefore, Lemma 5.1 and 5.2 imply $\mathcal{M}_\epsilon^{cs}(t_0)$ is C^1 close to $\mathcal{M}_0^{cs} \times Y_1$. Since $|t' - t_1| \le C'\epsilon$, C^2 smoothness of Φ and $Ax_h \subset X_1$ imply

$$
\begin{aligned}
&\Big|\big(D_{\xi_{cs}}\Phi^0(t_0, t_0+t', \xi_{cs}+h_u^0(\xi_{cs}))\big)\delta x\\
&\qquad - \big(D_{\xi_{cs}}\Phi^0(t_0, t_0+t_1, P_{cs}x_1+h_u^0(P_{cs}x_1))\big)\delta x\Big|_{X_1} \le C'\delta.
\end{aligned}
$$

And thus for any $(\xi_{cs}, \xi_y) \in B_\delta(P_{cs}x_1, X_1^{cs}) \times B_{C\epsilon}(0, Y_1)$,

$$\begin{aligned} &Q_v D\Phi(t_0, t_0 + t', \xi_{cs} + \xi_y + h_u(\xi_{cs}, \xi_y, t_0 + t', \epsilon), \epsilon) \\ &\qquad\qquad (Ax_1 + f(x_1, 0, t, 0)) \in (\frac{1}{2}, \frac{3}{2}), \end{aligned} \tag{5.10}$$

where $t' = t'(t_0)$ is the one found in Lemma 5.1. Therefore, $Q_v^{-1}(0) = \mathcal{M}_\epsilon^{cs}(t_0) \bigcap \Sigma$ is a submanifold of $\mathcal{M}_\epsilon^{cs}(t_0)$ and Σ with codimension 1 in $\mathcal{M}_\epsilon^{cs}(t_0)$. Obviously, $\mathcal{M}_0^{cs} \bigcap \Sigma$ is a submanifold of $\mathcal{M}_0^{cs}$ with codimension 1 in $\mathcal{M}_0^{cs}$. Similarly, for any $t_0 \in \mathbb{R}$, there exists a unique $t'' = t''(t_0)$ with $|t'' - t_2| \leq C'\epsilon$ such that for $\xi_u \in B_{\delta'}(P_u x_2, X_1^u)$, $\Phi(t_0, t_0 + t'', \xi_u + h_{cs}(\xi_u, t_0 + t'', \epsilon), \epsilon) \bigcap \Sigma$ is a submanifold of $\mathcal{M}_\epsilon^u(t_0)$ with codimension 1, where like h_s defined in (3.11), the graph of h_{cs} gives the local unstable integral manifold of (2.1) near 0.

By the standard invariant foliation theory, we can foliate $\mathcal{M}_0^{cs}$ into invariant stable fibres. By assumption (D3), there exists $f^s \in C^1(\mathcal{M}_0^{cs}, \mathcal{M}_0^c)$ such that

$$f^s|_{\mathcal{M}_0^s} = (0,0) \ , \ f^s|_{\mathcal{M}_0^c} = I \ , \ \Phi^0 \circ f^s = f^s \circ \Phi^0 \ , \ \forall t \in \mathbb{R},$$

where Φ^0 is the flow map of (2.2). Since $H(0,0) = 0$, $H(x) = 0$ for any $x \in \mathcal{M}_0^u$, which implies $T_{x_0}\mathcal{M}_0^u \subset \ker(DH(x_0))$. On the other hand, clearly, we have $H(x) = H(f^s(x))$ for any $x \in \mathcal{M}_0^{cs}$. By using the facts $x_0 \in \mathcal{M}_0^s$ and $DH(0,0) = 0$, for any $\delta x_0 \in T_{x_0}\mathcal{M}_0^{cs}$,

$$DH(x_0)\delta x_0 = DH(f^s(x_0))Df^s(x_0)\delta x_0 = DH(0,0)Df^s(x_0)\delta x_0 = 0,$$

which implies $T_{x_0}\mathcal{M}_0^{cs} \subset \ker(DH(x_0))$. Define

$$\widetilde{\mathcal{M}}_0^u = \mathcal{M}_0^u \bigcap \Sigma \ , \ \widetilde{\mathcal{M}}_0^{cs} = \mathcal{M}_0^{cs} \bigcap \Sigma \ , \ \overline{X}_1^u = T_{x_0}\widetilde{\mathcal{M}}_0^u \ , \ \overline{X}_1^{cs} = T_{x_0}\widetilde{\mathcal{M}}_0^{cs}.$$

Clearly,

$$\overline{X}_1^{cs} \ , \ \overline{X}_1^u \ , \ Y_1 \subset \ker(DH(x_0)) \bigcap \Sigma \triangleq \Pi.$$

We use $\mathrm{Codim}_W(Z)$ to represent the codimension of a linear subspace Z in a Banach space W. Clearly, we have $\mathrm{Codim}_\Sigma(\Pi) = 1$. On the other hand, (D4) implies

$\overline{X}_1^{cs} \bigcap \overline{X}_1^{u} = \{0\}$. Moreover, we have $\text{Codim}_\Sigma(\overline{X}_1^{cs} \oplus \overline{X}_1^{u} \oplus Y_1) = 1$, since X_1^u is finite dimensional. Therefore,

$$\Pi = \overline{X}_1^{cs} \oplus \overline{X}_1^{u} \oplus Y_1.$$

Let $\omega \in \Sigma$ be transversal to Π such that $DH(x_0)\omega = 1$ and $Q_\omega, Q_{cs}, Q_u, Q_y$ be projections from Σ onto $\omega, \overline{X}_1^{cs}, \overline{X}_1^{u}$ and Y_1. Thus,

$$\Sigma = \text{span}\{\omega\} \oplus \Pi = \text{span}\{\omega\} \oplus \overline{X}_1^{cs} \oplus \overline{X}_1^{u} \oplus Y_1.$$

We will use the coordinates

$$\begin{aligned} &(d, x^{cs}, y, x^u) \\ =&(Q_\omega(p - x_0), Q_{cs}(p - x_0), Q_y(p - x_0), Q_u(p - x_0)) \\ =&(DH(x_0)(p - x_0), Q_{cs}(p - x_0), Q_y(p - x_0), Q_u(p - x_0)) \end{aligned} \tag{5.11}$$

to represent any $p \in \Sigma + x_0$. Locally, there exist $\delta > 0$ independent of ϵ and

$$\Upsilon_0 : B_\delta(0, \overline{X}_1^{cs}) \longrightarrow \mathbb{R} \times \overline{X}_1^{u} \ , \ \Psi_0 : B_\delta(0, \overline{X}_1^{u}) \longrightarrow \mathbb{R} \times \overline{X}_1^{cs},$$

such that $\widetilde{\mathcal{M}}_0^{cs}, \widetilde{\mathcal{M}}_0^{u}$ contain the graphs of Υ_0, Ψ_0, respectively. We extend Υ_0 to $B_\delta(0, \overline{X}_1^{cs}) \times Y_1$ trivially in y. Since the perturbed and unperturbed invariant manifolds are C^1 close, using this coordinate system we can write integral manifolds $\widetilde{\mathcal{M}}_\beta^\alpha(t_0) = \mathcal{M}_\epsilon^\alpha(t_0) \bigcap (x_0 + \Sigma)$ as graphs, where $\alpha = cs, u$, $\beta = 0, \epsilon$. Before we state our next lemma, we first introduce some notations. For $\epsilon = 0$,

$$\Phi(t, t_0, x + y, 0) \triangleq \Phi^0(t, t_0, x) \ , \ h_u(x_{cs}, y, t_0, 0) \triangleq h_u^0(x_{cs}), \tag{5.12}$$

where $x \in X_1, x_{cs} \in X_1^{cs}, y \in Y_1$.

Lemma 5.3. *For any $b > 0$, there exist $\epsilon_0 > 0$, $b' > 0$ independent of $\epsilon \in [0, \epsilon_0)$ and*

$$\Upsilon = (\Upsilon^d, \Upsilon^u) : B_{b'}(0, \overline{X}_1^{cs}) \times B_b(0, Y_1) \times \mathbb{R} \times [0, \epsilon_0) \to (\mathbb{R}, \overline{X}_1^{u})$$

such that

$$\begin{aligned} \widetilde{\mathcal{M}}_\epsilon^{cs}(t_0) \supset \big\{ &x_0 + \Upsilon^d(x^{cs}, y, t_0, \epsilon) + x^{cs} + \epsilon y \\ &+ \Upsilon^u(x^{cs}, y, t_0, \epsilon) \big| x^{cs} \in B_{b'}(0, \overline{X}_1^{cs}), y \in B_b(0, Y_1) \big\} \\ &\triangleq Grap(\Upsilon(t_0, \epsilon)). \end{aligned}$$

Moreover, Υ are C^2 in x^{cs}, y and satisfy

$$\Upsilon(0,0,t_0,0) = 0\ ,\ D\Upsilon(0,0,t_0,0) = D\Upsilon_0(0) = 0, \tag{5.13}$$

$$\left|D_{x^{cs}}\Upsilon - D_{x^{cs}}\Upsilon_0\right|_{C^0(B_{b'}(0,\overline{X}_1^{cs})\times B_b(0,Y_1))} + |\Upsilon - \Upsilon_0|_{C^0} \le C'\epsilon, \tag{5.14}$$

$$|D_y\Upsilon|_{C^0(B_{b'}(0,\overline{X}_1^{cs})\times B_b(0,Y_1))} \le C'\epsilon^2, \tag{5.15}$$

$$|\partial_{t_0}\Upsilon| \le C'\epsilon. \tag{5.16}$$

where C' only depends on constants in assumptions.

For $\epsilon = 0$, we define $\mathrm{Grap}(\Upsilon_0) \subset \widetilde{\mathcal{M}}_0^{cs} \times \{0 \in Y_1\}$. We notice in the definition of $\mathrm{Grap}(\Upsilon)$ we scale y to ϵy. This is to avoid the dependence on ϵ of the domain where the function is defined.

Proof. Let $w = P_{cs}(Ax_1 + f(x_1, 0, t, 0))$ and $\widetilde{X}_1^{cs} \subset X_1^{cs}$ such that $X_1^{cs} = \mathbb{R}w \oplus \widetilde{X}_1^{cs}$, and define

$$\begin{aligned}\widetilde{\mathcal{F}}(a, \xi'_{cs}, \xi_y, \epsilon) = \Phi\big(t_0, &t_0 + t', P_{cs}x_1 + aw + \xi'_{cs} + \epsilon\xi_y \\ &+ h_u(P_{cs}x_1 + aw + \xi'_{cs}, \epsilon\xi_y, t_0 + t', \epsilon), \epsilon\big) - x_0.\end{aligned}$$

Here $a \in [-\delta, \delta]$, $\xi'_{cs} \in B_\delta(0, \widetilde{X}_1^{cs})$ and $\xi_y \in B_{b_1}(0, Y_1)$, where $\delta > 0$ sufficiently small but independent of ϵ and b_1 is arbitrary. From Theorem 2.4 and Lemma 5.2, we have

$$\left|\widetilde{\mathcal{F}}(\cdot, \epsilon) - \widetilde{\mathcal{F}}(\cdot, 0)\right|_{C^1([-\delta,\delta]\times B_\delta(0,\widetilde{X}_1^{cs})\times B_{b_1}(0,Y_1), X_1\times Y_1)} \le C'\epsilon, \tag{5.17}$$

where C' depends on constants in assumptions. From Lemma 5.1 and (5.12), we have

$$Q_v\widetilde{\mathcal{F}}(0,0,0,\epsilon) = 0\ ,\ \epsilon \in [0, \epsilon_0).$$

Clearly, $\widetilde{\mathcal{F}} \in x_0 + \Sigma$ if and only if $Q_v\widetilde{\mathcal{F}} = 0$. By (5.17), one can use implicit function theorem to obtain for arbitrary $b_1 > 0$ there exists $\delta' > 0$ sufficiently small, independent of ϵ and $a : B_{\delta'}(0, \widetilde{X}_1^{cs}) \times B_{b_1}(0, Y_1) \times [0, \epsilon_0) \to [-\delta', \delta']$ such that

$$\mathcal{F}(\xi'_{cs}, \xi_y, \epsilon) \triangleq \widetilde{\mathcal{F}}(a(\xi'_{cs}, \xi_y, \epsilon), \xi'_{cs}, \xi_y, \epsilon) \in \Sigma,$$

where $(\xi'_{cs}, \xi_y) \in B_{\delta'}(0, \widetilde{X}_1^{cs}) \times B_{b_1}(0, Y_1)$. For $\epsilon = 0$, we identify $a(\xi_{cs'}, \xi_y, 0)$ with $a_0(\xi'_{cs})$, which satisfy

$$\begin{aligned}\Phi^0(t_0, t_0 + t_1, P_{cs}x_1 + & a_0(\xi'_{cs})w + \xi'_{cs} \\ & + h_u^0(P_{cs}x_1 + a_0(\xi'_{cs})w + \xi_{cs'})) \in \Sigma \cap X_1.\end{aligned}$$

Moreover, by assumption (D3) and Theorem 3.10, a is C^2 in $\xi_{cs'}, \xi_y$, a_0 is C^2 in ξ'_{cs} and $a(0, 0, \epsilon) = a_0(0) = 0$. Based on (5.12), we also define

$$\mathcal{F}_0(\xi'_{cs}) = \mathcal{F}(\xi'_{cs}, \xi_y, 0) = \widetilde{\mathcal{F}}(a(\xi'_{cs}, \xi_y, 0), \xi_{cs'}, \xi_y, 0).$$

To estimate $a(\xi'_{cs}, \xi_y, \epsilon) - a_0(\xi'_{cs})$, we have

$$\begin{aligned}0 = & Q_v\mathcal{F}(\xi'_{cs}, \xi_y, \epsilon) - Q_v\mathcal{F}_0(\xi'_{cs}) \\ = & Q_v\mathcal{F}(\xi'_{cs}, \xi_y, \epsilon) - Q_v\widetilde{\mathcal{F}}(a(\xi'_{cs}, \xi_y, \epsilon), \xi'_{cs}, \xi_y, 0) \\ & + Q_v\widetilde{\mathcal{F}}(a(\xi'_{cs}, \xi_y, \epsilon), \xi'_{cs}, \xi_y, 0) - Q_v\mathcal{F}_0(\xi'_{cs}).\end{aligned} \tag{5.18}$$

By (5.18), we have

$$\left|Q_v\mathcal{F}(\xi'_{cs}, \xi_y, \epsilon) - Q_v\widetilde{\mathcal{F}}(a(\xi'_{cs}, \xi_y, \epsilon), \xi'_{cs}, \xi_y, 0)\right| \leq C'\epsilon,$$

We note from (5.10) that

$$\frac{1}{2}|a(\xi'_{cs}, \xi_y, \epsilon) - a(\xi'_{cs}, \xi_y, 0)| \leq \left|Q_v\widetilde{\mathcal{F}}(a(\xi'_{cs}, \xi_y, \epsilon), \xi'_{cs}, \xi_y, 0) - Q_v\mathcal{F}_0(\xi'_{cs})\right|.$$

From (5.12), we conclude

$$|a(\xi'_{cs}, \xi_y, \epsilon) - a_0(\xi'_{cs})| = |a(\xi'_{cs}, \xi_y, \epsilon) - a(\xi'_{cs}, \xi_y, 0)| \leq C'\epsilon. \tag{5.19}$$

Differentiating (5.18) with respect to ξ'_{cs}, we obtain

$$\begin{aligned}0 = & \partial_a\widetilde{F}(a(\epsilon), \xi'_{cs}, \xi_y, \epsilon)D_{\xi'_{cs}}a(\epsilon)w + D_{\xi'_{cs}}\widetilde{F}(a(\epsilon), \xi'_{cs}, \xi_y, \epsilon) \\ & - \partial_a\widetilde{F}(a(0), \xi'_{cs}, \xi_y, 0)D_{\xi'_{cs}}a(0)w - D_{\xi'_{cs}}\widetilde{F}(a(0), \xi'_{cs}, \xi_y, 0)\end{aligned}$$

where $a(\epsilon) = a(\xi'_{cs}, \xi_y, \epsilon), a_0 = a(\xi'_{cs}, \xi_y, 0)$. By (5.18), (5.19) and C^2 smoothness of $\widetilde{F}_0$, we have

$$\left|D_{\xi'_{cs}}\widetilde{F}(a(\epsilon), \xi'_{cs}, \xi_y, \epsilon) - D_{\xi'_{cs}}\widetilde{F}(a(0), \xi'_{cs}, \xi_y, 0)\right| \leq C'\epsilon,$$

$$\left|\partial_a\widetilde{F}(a(\epsilon), \xi'_{cs}, \xi_y, \epsilon) - \partial_a\widetilde{F}(a(0), \xi'_{cs}, \xi_y, 0)\right| \leq C'\epsilon,$$

which implies

$$\left|D_{\xi'_{cs}}(a(\xi_{cs},\xi_y,\epsilon)-a(\xi_{cs},\xi_y,0))\right|\le C'\epsilon.$$

Differentiating (5.18) with respect to ξ_y, we have

$$0=Q_v\partial_a\widetilde{F}(\xi_{cs'},\xi_y,\epsilon)D_{\xi_y}a+Q_vD_{\xi_y}\widetilde{F}.$$

By (5.9),

$$\begin{aligned}\left|Q_vD_{\xi_y}\widetilde{F}\right|=\big|&\epsilon Q_vD\Phi(t_0,t_0+t',\xi+h_u(\xi,t_0+t',\epsilon),\epsilon)\\&(I+Dh_u(\xi,t_0+t',\epsilon))\big|\le C'\epsilon^2,\end{aligned}$$

where $\xi=P_{cs}x_1+a(\xi'_{cs},\xi_y,\epsilon)w+\xi'_{cs}+\epsilon\xi_y$. It follows

$$|D_{\xi_y}a(\xi'_{cs},\xi_y,\epsilon)|\le C'\epsilon^2. \tag{5.20}$$

Therefore,

$$\left|a(\cdot,\cdot,\epsilon)-a_0(\cdot)\right|_{C^1(B_{\delta'}(0,\widetilde{X}_1^{cs})\times B_{b_1}(0,Y_1)\times[0,\epsilon_0),[-\delta',\delta'])}\le C'\epsilon. \tag{5.21}$$

Consequently,

$$|\mathcal{F}(\cdot,\cdot,\epsilon)-\mathcal{F}_0(\cdot)|_{C^1}\le C'\epsilon. \tag{5.22}$$

Let $\xi'_{cs}(a)\triangleq P_{cs}x_1+aw+\xi'_{cs}$, we note that

$$\begin{aligned}&\left|\partial_{t_0}Q_v\widetilde{\mathcal{F}}(a,\xi'_{cs},\xi_y,\epsilon)\right|\\
\le&\Big|Q_v\big(V_\epsilon(t_0,\widetilde{\mathcal{F}}(a,\xi'_{cs},\xi_y,\epsilon)+x_0)-D\Phi(t_0,t_0+t',\xi'_{cs}(a)+\epsilon\xi_y\\
&+h_u(\xi'_{cs}(a),\epsilon\xi_y,t_0+t',\epsilon),\epsilon)V_\epsilon(t_0+t',\xi'_{cs}(a)+\epsilon\xi_y\\
&+h_u(\xi'_{cs}(a),\epsilon\xi_y,t_0+t',\epsilon))(1+\partial_{t_0}t')+D\Phi\partial_{t_0}h_u\big)\Big|\\
\le&\Big|Q_v\big(V_0(t_0,\widetilde{\mathcal{F}}(a,\xi'_{cs},\xi_y,0))-D\Phi^0(t_0,t_0+t_1,\xi'_{cs}(a)\\
&+h_u^0(\xi'_{cs}(a)))V_0(t_0+t_1,\xi'_{cs}(a)+h_u^0(\xi'_{cs}(a)))\big)\Big|+C'\epsilon\le C'\epsilon.\end{aligned}$$

Here we use Lemma 5.2, Theorem 3.26 and (5.6) to obtain the above estimates. It implies

$$|\partial_{t_0}a|\le C'\epsilon\ ,\ |\partial_{t_0}\mathcal{F}|\le C'\epsilon. \tag{5.23}$$

By Lemma 5.2, we have

$$
\begin{aligned}
&Q_{cs}D_{\xi'_{cs}}\mathcal{F} = Q_{cs}D_{\xi'_{cs}}\mathcal{F}_0 + \epsilon O_1 \ , \ Q_{cs}D_{\xi_y}\mathcal{F} = \epsilon^2 O_2, \\
&Q_y D_{\xi'_{cs}}\mathcal{F} = \epsilon O_3 \ , \ Q_y D_{\xi_y}\mathcal{F} = \epsilon E + \epsilon^2 O_4,
\end{aligned}
\tag{5.24}
$$

where O_1—O_4 are linear operators bounded uniformly in ϵ and E is the linear evolutionary operator defined at base point

$$
P_{cs}x_1 + a_0(\xi'_{cs})w + \xi'_{cs} + h_u^0(P_{cs}x_1 + a_0(\xi'_{cs})w + \xi_{cs'}).
$$

Let $(x_\epsilon^{cs}, y_\epsilon) = (Q_{cs} + Q_y)\mathcal{F}(0, 0, \epsilon)$, clearly, $|x_\epsilon^{cs}|_{X_1} + |y_\epsilon|_{Y_1} \le C'\epsilon$, where C' depends on those constants in assumptions and $(x_0^{cs}, y_0) = (0, 0)$. We claim for any $b > 0$ there exists $b' > 0$ depending on b but independent of ϵ such that the map

$$
(Q_{cs}\mathcal{F}, \frac{1}{\epsilon}Q_y\mathcal{F})^{-1} : B_{b'}(0, \overline{X}_1^{cs}) \times B_b(0, Y_1) \longrightarrow B_{\delta'}(0, \widetilde{X}_1^{cs}) \times B_{b_1}(0, Y_1)
$$

is well defined and the norm of the linearized map is independent of ϵ. To prove this, we need solve the equation

$$
Q_{cs}\mathcal{F}(\xi'_{cs}, \xi_y, \epsilon) = x^{cs} \ , \ \frac{1}{\epsilon}Q_y\mathcal{F}(\xi'_{cs}, \xi_y, \epsilon) = y.
$$

We first look at the equation

$$
E\xi_y + \frac{1}{\epsilon}Q_y\mathcal{F}(\xi'_{cs}, 0, \epsilon)) = y.
$$

By (5.24), for fixed ξ'_{cs}, we have

$$
\Big|\frac{1}{\epsilon}Q_y\mathcal{F}(\xi'_{cs}, \xi_y, \epsilon) - \big(E\xi_y + \frac{1}{\epsilon}Q_y\mathcal{F}(\xi'_{cs}, 0, \epsilon)\big)\Big|_{C^1} \le C'\epsilon.
$$

Note E and E^{-1} both have upper bounds independent of ϵ. For any $b > 0$, by implicit function theorem argument, there exist sufficiently small $\delta' > 0$, reasonably large $b_1 > 0$ independent of ϵ and sufficiently small $\epsilon_0 > 0$ such that for any $y \in B_b(0, Y_1)$, $\xi'_{cs} \in B_{\delta'}(0, \widetilde{X}_1^{cs})$ and $\epsilon \in [0, \epsilon_0)$ there exists a unique $\xi_y(\xi'_{cs}, y, \epsilon)$ which is C^2 in ξ'_{cs} and y.satisfying

$$
\frac{1}{\epsilon}Q_y\mathcal{F}(\xi'_{cs}, \xi_y(\xi'_{cs}, y, \epsilon), \epsilon) = y.
$$

By differentiating the above equality with respect to ξ'_{cs} and using (5.24), we obtain

$$|D_{\xi'_{cs}}\xi_y| \leq C'. \tag{5.25}$$

Combining the above estimate with (5.24), we obtain

$$\Big|Q_{cs}\mathcal{F}(\xi'_{cs}, \xi_y(\xi'_{cs}, \xi_y, \epsilon), \epsilon) - Q_{cs}\mathcal{F}_0(\xi'_{cs})\Big|_{C^1} \leq C'\epsilon.$$

Since $Q_{cs}\mathcal{F}_0$ is independent of ϵ and is locally invertible, one can use inverse function theorem argument again to prove there exist sufficiently small $b' > 0$, $\epsilon_0 > 0$ so that for $(x^{cs}, y, \epsilon) \in B_{b'}(0, \overline{X}_1^{cs}) \times B_b(0, Y_1) \times [0, \epsilon_0)$, there exists a unique $\xi'_{cs}(x^{cs}, y, \epsilon)$ which is C^2 in x^{cs} and y satisfying

$$Q_{cs}\mathcal{F}(\xi'_{cs}(x^{cs}, y, \epsilon), \xi_y(\xi'_{cs}(x^{cs}, y, \epsilon), \xi_y, \epsilon), \epsilon) = x^{cs}.$$

For $(x^{cs}, y, t_0, \epsilon) \in B_{b'}(0, \overline{X}_1^{cs}) \times B_b(0, Y_1) \times \mathbb{R} \times [0, \epsilon_0)$, let

$$x^{cs} + \epsilon y + \Upsilon(x^{cs}, y, t_0, \epsilon) = \mathcal{F}\Big(\big(Q_{cs}\mathcal{F}, \frac{1}{\epsilon}Q_y\mathcal{F}\big)^{-1}(x^{cs}, y), \epsilon\Big), \tag{5.26}$$

when $\epsilon = 0$, (5.26) is replaced by

$$x^{cs} + \Upsilon_0(x^{cs}) = \mathcal{F}_0\Big(\big(Q_{cs}\mathcal{F}_0\big)^{-1}(x^{cs})\Big).$$

Based on (5.12) and the definition of Σ, (5.13) is obvious. Since $\mathcal{F}$ is C^2, Υ is also C^2. Let I_{cs}, I_y be the identity maps on $\overline{X}_1^{cs}, Y_1$, respectively. Differentiating (5.26) with respect to (x^{cs}, y) and using (5.24), we obtain

$$\begin{pmatrix} I_{cs} & 0 \\ 0 & \epsilon I_y \end{pmatrix} = \begin{pmatrix} Q_{cs}D_{\xi'_{cs}}\mathcal{F}_0 + \epsilon O_1 & \epsilon^2 O_2 \\ \epsilon O_3 & \epsilon E + \epsilon^2 O_4 \end{pmatrix} \begin{pmatrix} D_{x^{cs}}\xi'_{cs} & D_y\xi'_{cs} \\ D_{x^{cs}}\xi_y & D_y\xi_y \end{pmatrix},$$

which implies

$$D_{x^{cs}}\xi'_{cs} = (Q_{cs}D_{\xi_{cs}'}\mathcal{F}_0)^{-1} + \epsilon O_5 \ , \ D_y\xi'_{cs} = \epsilon^2 O_6,$$

$$D_{x^{cs}}\xi_y = -E^{-1}O_3(Q_{cs}D_{\xi'_{cs}}\mathcal{F}_0)^{-1} + \epsilon O_7 \ , \ D_y\xi_y = E^{-1} + \epsilon O_8,$$

where O_5—O_8 are bounded linear operators with bounds independent of ϵ. Therefore,

$$\begin{aligned}
D_{x^{cs},y}\Upsilon(\cdot,\cdot,t_0,\epsilon) =& Q_{w,u}(D_{\xi'_{cs},\xi_y}\mathcal{F})(D_{x^{cs},y}(\xi'_{cs},\xi_y)) \\
=& Q_{w,u}\begin{pmatrix} D_{\xi'_{cs}}\mathcal{F}_0+\epsilon O_1 & \epsilon^2 O_2 \end{pmatrix} \\
& \begin{pmatrix} (Q_{cs}D_{\xi cs'}\mathcal{F}_0)^{-1}+\epsilon O_5 & \epsilon^2 O_6 \\ -E^{-1}O_3(Q_{cs}D_{\xi'_{cs}}\mathcal{F}_0)^{-1}+\epsilon O_7 & E^{-1}+\epsilon O_8 \end{pmatrix} \\
=& Q_{w,u}\begin{pmatrix} D_{\xi'_{cs}}\mathcal{F}_0(Q_{cs}D_{\xi cs'}\mathcal{F}_0)^{-1}+\epsilon O_9 & \epsilon^2 O_{10} \end{pmatrix},
\end{aligned}$$

which implies (5.14) and (5.15). Finally, one can differentiate (5.26) with respect to t_0 and use (5.23) to prove

$$|\partial_{t_0}(Q_{cs}\mathcal{F})^{-1}| + |\partial_{t_0}(Q_y\mathcal{F})^{-1}| \leq C'\epsilon,$$

which implies (5.16). □

Similarly, there exists $b > 0$ sufficiently small but independent of ϵ and

$$\Psi = (\Psi^d, \Psi^{cs}, \Psi^y) : B_b(0, \overline{X}_1^u) \times \mathbb{R} \times [0, \epsilon_0) \to (\mathbb{R}, \overline{X}_1^{cs}, Y_1)$$

such that

$$\begin{aligned}
\widetilde{\mathcal{M}}_\epsilon^u(t_0) \supset \Big\{ & x_0 + \Psi^d(x^u, t_0, \epsilon) + \Psi^y(x^u, t_0, \epsilon) \\
& + \Psi^{cs}(x^u, t_0, \epsilon) \big| x^u \in \overline{X}^u \Big\} \triangleq \text{Graph}(\Psi),
\end{aligned}$$

where $(\Psi^d, \Psi^{cs}, \Psi^y)$ are C^2 in x^u and satisfy

$$\Psi(0, t_0, 0) = 0\ ,\ D\Psi(0, t_0, 0) = 0\ ,\ \Psi^y(x^u, t_0, 0) \equiv 0. \tag{5.27}$$

Furthermore,

$$\big|\Psi(\cdot, t_0, \epsilon) - \Psi(\cdot, t_0, 0)\big|_{C^1(B_b(0,\overline{X}_1^u), \mathbb{R}\times\overline{X}_1^{cs}\times Y_1)} \leq C'\epsilon\ ,\ |\partial_{t_0}\Psi| \leq C'\epsilon, \tag{5.28}$$

where C' is independent of ϵ.

From the construction of the coordinate system, the intersection of $\widetilde{\mathcal{M}}_\epsilon^{cs}(t_0)$ and $\widetilde{\mathcal{M}}_\epsilon^u(t_0)$ is equivalent to the following system:

$$x^u = \Upsilon^u(x^{cs}, y, t_0, \epsilon)\ ,\ x^{cs} = \Psi^{cs}(x^u, t_0, \epsilon)\ ,\ \epsilon y = \Psi^y(x^u, t_0, \epsilon), \tag{5.29}$$

$$d = \Upsilon^d(x^{cs}, y, t_0, \epsilon) = \Psi^d(x^u, t_0, \epsilon). \tag{5.30}$$

Lemma 5.4. *There exists ϵ_0 such that for every $\epsilon \in [0, \epsilon_0)$ and $t_0 \in \mathbb{R}$, there exist $x^{cs} = x^{cs}(t_0, \epsilon), x^u = x^u(t_0, \epsilon), y = y(t_0, \epsilon)$, which are continuous in t_0 and ϵ, satisfying* (5.29). *Moreover,*

$$|x^{cs}|_{X_1} + |x^u|_{X_1} + \epsilon |y|_{Y_1} \le C' \epsilon, \tag{5.31}$$

where C' depends on constants in assumptions and uniform in t_0 and ϵ.

Proof. The proof is based on (5.14), (5.15), (5.28) and a contraction mapping argument. □

5.2 *Persistence of homoclinic orbits under weakly dissipative perturbation*

In this section, we assume additionally

(A7) For $i = 0, 1, 2$, the following quantities have a uniform bound C_0,

$$(\partial_\epsilon^{2-i} D^i f, \partial_\epsilon^{2-i} D^i g) \in C^0(X_1 \times Y_1 \times \mathbb{R}^2, L_i(X_1 \times Y_1, X_1 \times Y_1)).$$

The goal is to study the persistence of the homoclinic solution of (2.1). Our strategy goes as following. We will first derive the Melnikov integral to measure the distance between two special points on $\widetilde{\mathcal{M}}_\epsilon^{cs}(t_0)$ and $\widetilde{\mathcal{M}}_\epsilon^{u}(t_0)$. Then we study the stable region on the center-stable integral manifold.

Using the notations in Chapter 6.1 and Lemma 5.4, for any $t_0 \in \mathbb{R}$, we let

$$\begin{aligned}
&P^u(t_0, \epsilon) = (\Psi^d(x^u(t_0, \epsilon), t_0, \epsilon), x^u(t_0, \epsilon), x^{cs}(t_0, \epsilon), \epsilon y(t_0, \epsilon)) + x_0, \\
&P^{cs}(t_0, \epsilon) = (\Upsilon^d(x^{cs}(t_0, \epsilon), y(t_0, \epsilon), t_0, \epsilon), x^u(t_0, \epsilon), x^{cs}(t_0, \epsilon), \epsilon y(t_0, \epsilon)) + x_0, \\
&(x_-(t, t_0, \epsilon), y_-(t, t_0, \epsilon)) \triangleq \Phi(t, t_0, P^u(t_0, \epsilon), \epsilon), \\
&(x_+(t, t_0, \epsilon), y_+(t, t_0, \epsilon)) \triangleq \Phi(t, t_0, P^{cs}(t_0, \epsilon), \epsilon).
\end{aligned}$$

From the coordinate system we constructed in the previous section, clearly

$$P^u = P^{cs} \Longleftrightarrow \Psi^d(x_-, t_0, \epsilon) = \Upsilon^d(x_+, y_+, t_0, \epsilon) \Longleftrightarrow H(P^u) = H(P^{cs}),$$

where (5.11) is used.

From (5.1), we have

$$\begin{aligned} H(P^u) =& H(P^u) - H(\Phi(-\infty, t_0, P^u, \epsilon)) \\ =& \int_{-\infty}^{t_0} \frac{d}{dt} H\big(\Phi(t, t_0, P^u, \epsilon)\big) dt \\ =& \int_{-\infty}^{t_0} DH\big(\Phi(t, t_0, P^u, \epsilon)\big) \dot{\Phi}(t, t_0, P^u, \epsilon) dt \\ =& \int_{-\infty}^{t_0} DH\big(x_-(t, t_0, \epsilon)\big)\big(f(x_-(t, t_0, \epsilon), y_-(t, t_0, \epsilon), t, \epsilon) \\ & \qquad - f(x_-(t, t_0, \epsilon), 0, t, 0)\big) dt, \end{aligned} \tag{5.32}$$

where the last equality follows from the fact that H is invariant under (2.2) so that for any $x \in X_1$

$$DH(x)(Ax + f(x, 0, t, 0)) = 0.$$

Next, we will analyze the leading order of (5.32). Since (x_-, y_-) and x_h are on perturbed and unperturbed unstable manifold, for all $t \leq 0$ and $\max\{a_1', 0\} < \eta' < a_2'$,

$$\big|x_-(t, t_0, \epsilon) - x_h(t - t_0)\big|_{X_1} + \big|y_-(t, t_0, \epsilon)\big|_{Y_1} \leq C' e^{\eta'(t - t_0)} \epsilon. \tag{5.33}$$

Substituting (5.33) into the last equality of (5.32), we obtain

$$\begin{aligned} & \int_{-\infty}^{t_0} DH\big(x_-(t, t_0, \epsilon)\big)\big(f(x_-(t, t_0, \epsilon), y_-(t, t_0, \epsilon), t, \epsilon) \\ & \qquad - f(x_-(t, t_0, \epsilon), 0, t, 0)\big) dt \\ =& \int_{-\infty}^{t_0} DH\big(x_h(t - t_0)\big)\big(D_y f(x_h(t - t_0), 0, t, 0) y_-(t, t_0, \epsilon) \\ & \qquad + \epsilon \partial_\epsilon f(x_h(t - t_0), 0, t, 0)\big) dt + O(\epsilon^2), \end{aligned}$$

where C^2 smoothness of H and $DH(0) = 0$ guarantee the convergence of the above integral. By variation of constants formula

$$y_-(t, t_0, \epsilon) = e^{(t - t_0)\frac{J}{\epsilon}} y(t_0, \epsilon) + \int_{t_0}^{t} e^{(t - \tau)\frac{J}{\epsilon}} g(x_-, y_-, \tau, \epsilon)\, d\tau.$$

Integrating by parts and using the fact $|y_-| \le C_1\epsilon$, we obtain

$$\Big|\int_{-\infty}^{t_0} DH\big(x_h(t-t_0)\big)D_yf(x_h(t-t_0),0,t,0)e^{(t-t_0)\frac{J}{\epsilon}}y(t_0,\epsilon)dt\Big| = O(\epsilon^2).$$

Moreover,

$$\begin{aligned}
&\int_{-\infty}^{t_0} DH(x_h(t-t_0))D_yf(x_h(t-t_0),0,t,0)\\
&\qquad\qquad\Big(\int_{t_0}^{t} e^{(t-\tau)\frac{J}{\epsilon}}g(x_-(\tau,t_0,\epsilon),y_-(\tau,t_0,\epsilon),\tau,\epsilon)\ d\tau\Big)dt\\
=&\int_{-\infty}^{t_0}\Big(\int_{-\infty}^{\tau} DH(x_h)D_yf(x_h,0,t,0)e^{(t-\tau)\frac{J}{\epsilon}}dt\Big)g(x_-,y_-,\tau,\epsilon)\ d\tau\\
=&\int_{-\infty}^{t_0}\Big(DH(x_h)D_yf(x_h,0,t,0)\epsilon J^{-1}e^{(t-\tau)\frac{J}{\epsilon}}\Big|_{-\infty}^{\tau}\Big)g(x_-,y_-,\tau,\epsilon)\ d\tau\\
&-\int_{-\infty}^{t_0}\Big(\int_{-\infty}^{\tau}\frac{d}{dt}\big(DH(x_h)D_yf(x_h,0,t,0)\big)\epsilon J^{-1}e^{(t-\tau)\frac{J}{\epsilon}}dt\Big)g(x_-,y_-,\tau,\epsilon)\ d\tau
\end{aligned}$$

It's easy to see from (5.33)

$$\begin{aligned}
&\int_{-\infty}^{t_0}\Big(DH(x_h)D_yf(x_h,0,t,0)\epsilon J^{-1}e^{(t-\tau)\frac{J}{\epsilon}}\Big|_{-\infty}^{\tau}\Big)g(x_-,y_-,\tau,\epsilon)\ d\tau\\
=&-\epsilon\int_{-\infty}^{t_0} DH(x_h(t-t_0))D_yf(x_h(t-t_0),0,t,0)J^{-1}g(x_h(t-t_0),0,t,0))dt\\
&+O(\epsilon^2).
\end{aligned}$$

Integrate by parts again, we have

$$\begin{aligned}
&-\int_{-\infty}^{t_0}\Big(\int_{-\infty}^{\tau}\frac{d}{dt}\big(DH(x_h)D_yf(x_h,0,t,0)\big)\epsilon J^{-1}e^{(t-\tau)\frac{J}{\epsilon}}dt\Big)g(x_-,y_-,\tau,\epsilon)\ d\tau\\
=&O(\epsilon^2).
\end{aligned}$$

Summarizing all the estimates, we obtain

$$\begin{aligned}
&H(P^u)\\
=&\epsilon\int_{-\infty}^{t_0} DH(x_h(t-t_0))\Big(\partial_\epsilon f(x_h(t-t_0),0,t,0)\\
&-D_yf(x_h(t-t_0),0,t,0)J^{-1}g(x_h(t-t_0),0,t,0)\Big)dt+O(\epsilon^2) \qquad (5.34)\\
=&\epsilon\int_{-\infty}^{0} DH(x_h(t))\Big(\partial_\epsilon f(x_h(t),0,t+t_0,0)\\
&-D_yf(x_h(t),0,t+t_0,0)J^{-1}g(x_h(t),0,t+t_0,0)\Big)dt+O(\epsilon^2).
\end{aligned}$$

To approximate $H(P^{cs})$, the difficulty is $\Phi(t, t_0, P^{cs}(t_0, \epsilon), \epsilon)$ doesn't lie necessarily in a small neighborhood of the origin for all $t \geq 0$. Nevertheless, we will give an approximation similar to $H(P^u)$. Let $a = \frac{1}{a_1 - \eta'} < 0$ and $T_1 = a \log \epsilon > 0$, we claim for any $t \in [t_0, T_1 + t_0]$,

$$\left| \Phi(t, t_0, P^{cs}, \epsilon) - x_h(t - t_0) \right|_{X_1} \leq C' e^{\eta'(t-t_0)} \epsilon, \tag{5.35}$$

where C' is independent of t and ϵ. To see this, we first note $(x_+, y_+)(t_1 + t_0, t_0, \epsilon)$ and $x_h(t_1)$ are in a small neighborhood of the origin inside the perturbed and unperturbed center-stable manifold, respectively. To prove our claim, we introduce the following notations. Let

$$x_+ = x_+(t_1 + t_0, t_0, \epsilon) \ , \ y_+ = y_+(t_1 + t_0, t_0, \epsilon) \ , \ x_{0+} = x_h(t_1),$$
$$(x_+(t), y_+(t)) = \Phi(t + t_1 + t_0, t_1 + t_0, x_+ + y_+, \epsilon).$$

Clearly, $|x_+ - x_{0+}| + |y_+| \leq C'\epsilon$. Consider the integral equation

$$\begin{aligned}
&x_+(t) - x_h(t + t_1) \\
=&e^{tA_f}(x_+ - x_{0+}) + \int_0^t e^{(t-\tau)A_f} \big(F(x_+(\tau), y_+(\tau), \tau + t_1 + t_0, \epsilon) \\
&- F(x_h(\tau + t_1), 0, \tau + t_1 + t_0, \epsilon) + f_y y_+(\tau) \big) \, d\tau, \\
&y_+(t) \\
=&e^{t(\frac{J}{\epsilon} + g_y)} y_+ + \int_0^t e^{(t-\tau)(\frac{J}{\epsilon} + g_y)} \big(G(x_h(\tau + t_1), 0, \tau + t_1 + t_0, \epsilon) \\
&+ g_x x_+(\tau) \big) \, d\tau + \int_0^t e^{(t-\tau)(\frac{J}{\epsilon} + g_y)} \big(G(x_+(\tau), y_+(\tau), \tau + t_1 + t_0, \epsilon) \\
&- G(x_h(\tau + t_1), 0, \tau + t_1 + t_0, \epsilon) \big) \, d\tau,
\end{aligned}$$

where F, G and A_f are defined in (3.2) and (3.5), respectively. By (3.3), we have

$$|DF|_{C^0} + |DG|_{C^0} \longrightarrow 0 \ \text{ as } \ r \longrightarrow 0.$$

By (5.2), there exists T such that

$$|x_+(\cdot) - x_h(\cdot + t_1)| + |y_+(\cdot)| < \frac{r}{2} \ \text{ for } \ t \in [0, T].$$

From integration by parts and Gronwall's inequality, we obtain

$$\begin{aligned}&e^{-a_1' t}\big(|x_+(t)-x_h(t+t_1)|_{X_1}+\frac{|y_+(t)|_{Y_1}}{\epsilon_\star}\big)\\ \le& C'(|x_+-x_{0+}|+\frac{|y_+|}{\epsilon_\star}+\epsilon)e^{C'(|DF|_{C^0}+|DG|_{C^0}+\epsilon+\epsilon_\star)t},\end{aligned}$$

where $\epsilon_\star$ is a small parameter and $C', \epsilon_\star$ are independent of r, ϵ and t. By first taking r and $\epsilon_\star$ sufficiently small, when ϵ is small, the above estimate can be used to extend T to $T_1 - t_1$. Therefore,

$$\begin{aligned}&\Big|\Phi(T_1+t_0,t_0,P^{cs}(\epsilon),\epsilon)-x_0(T_1)\Big|_{X_1}\le C'\epsilon^{1+a\eta'},\\ &\Big|x_h(T_1)\Big|_{X_1}\le C'e^{a_1T_1}=C'\epsilon^{aa_1}.\end{aligned} \tag{5.36}$$

Based on the definition of a, we have

$$aa_1 = 1+a\eta' > \frac{1}{2}\ ,\ 1+2a\eta' > 0. \tag{5.37}$$

Let

$$\begin{aligned}\omega(t,t_0)=&DH(x_h(t-t_0))\Big(\partial_\epsilon f(x_h(t-t_0),0,t,0)\\ &-D_yf(x_h(t-t_0),0,t,0)J^{-1}g(x_h(t-t_0),0,t,0)\Big).\end{aligned}$$

Since $H(0)=0$,

$$\begin{aligned}&H(P^{cs})-\epsilon\int_{+\infty}^{t_0}\omega(t,t_0)dt\\ =&H(P^{cs})-H(\Phi(T_1+t_0,t_0,P^{cs},\epsilon))-\epsilon\int_{T_1+t_0}^{t_0}\omega(t,t_0)dt\\ &+H(\Phi(T_1+t_0,t_0,P^{cs},\epsilon))-\epsilon\int_{+\infty}^{T_1+t_0}\omega(t,t_0)dt.\end{aligned}$$

Using (5.36) and a similar procedure as in the approximation of $H(P^u)$, we obtain

$$\begin{aligned}&\Big|H(P^{cs})-H(\Phi(T_1+t_0,t_0,P^{cs},\epsilon))-\epsilon\int_{T_1+t_0}^{0}\omega(t,t_0)dt\Big|\le C'\epsilon^{2+2a\eta'},\\ &\Big|\epsilon\int_{+\infty}^{T_1+t_0}\omega(t,t_0)dt\Big|\le C'\epsilon^{1+aa_1},\\ &\Big|H(\Phi(T_1+t_0,t_0,P^{cs},\epsilon))\Big|\le|D^2H|_{C^0}|\Phi(T_1+t_0,t_0,P^{cs},\epsilon)|^2\le C'\epsilon^{2+2a\eta'}.\end{aligned}$$

Let

$$M(t_0) \triangleq \int_{-\infty}^{+\infty} DH(x_h(t-t_0))\Big(\partial_\epsilon f(x_h(t-t_0),0,t,0)$$
$$- D_y f(x_h(t-t_0),0,t,0)J^{-1}g(x_h(t-t_0),0,t,0)\Big)dt$$
$$= \int_{-\infty}^{+\infty} DH(x_h(t))\Big(\partial_\epsilon f(x_h(t),0,t+t_0,0)$$
$$- D_y f(x_h(t),0,t+t_0,0)J^{-1}g(x_h(t),0,t+t_0,0)\Big)dt.$$

We have

$$H(P^u) - H(P^{cs}(\epsilon)) = \epsilon M(t_0) + \epsilon^{2+2a\eta'}O(1). \tag{5.38}$$

Lemma 5.5. *Suppose $M(t_0)$ has a simple zero at t_0, then there exists ϵ_0 such that for each $\epsilon \in [0,\epsilon_0)$, there exists t^* (possibly not unique) satisfying*

$$H(P^u(t^*,\epsilon)) - H(P^{cs}(t^*,\epsilon)) = 0.$$

Proof. Since P^u, P^{cs} are C^1 in t_0,

$$\widetilde{H}(\cdot,\epsilon) \triangleq \frac{1}{\epsilon}\big(H(P^{cs}(\cdot,\epsilon)) - H(P^{cs}(\cdot,\epsilon))\big) \in C^1(\mathbb{R},\mathbb{R}).$$

By intermediate value theorem, we can finish the proof. □

Lemma 5.5 gives a condition for nonempty intersection of center-stable and unstable manifold. This intersection means the existence of a solution which converges to the steady solution as $t \longrightarrow -\infty$. As t increases and $t \le a\log\epsilon + t_0$, based on the stable foliation in the center-stable manifold, this solution will approach a neighborhood of the steady state inside the center manifold. However, there might be some weak instability in the center manifold such that the solution will exit a small neighborhood of the steady state as $t > a\log\epsilon + t_0$. Suppose in some sense there is no instability in the unperturbed center directions and the perturbation is weakly dissipative, i.e. there is some weak exponential stability on the center manifold of (2.1). In the following, we will study the size of the stable region on the center manifold under the weakly dissipative assumption. We first use Taylor's expansion to expand f and g, namely,

$$\begin{aligned} f(x,y,t,\epsilon) &= f(x,y,t,0) + \epsilon f_1(x,y,t,\epsilon) \\ &= f_x x + f_y y + f_0(x,y) + \epsilon f_1(x,y,t,\epsilon), \\ g(x,y,t,\epsilon) &= g(x,y,t,0) + \epsilon g_1(x,y,t,\epsilon) \\ &= g_x x + g_y y + g_0(x,y) + \epsilon g_1(x,y,t,\epsilon), \end{aligned}$$

where $f_{x,y} = D_{x,y}f(0,0,t,0)$ and $g_{x,y} = D_{x,y}g(0,0,t,0)$ which are independent of t. Let $P_{c,su}$ be linear projections from X_1 onto $X_1^{c,su}$ which are invariant under $e^{t(A+f_x)}$, where $X_1 = X_1^c \oplus X_1^{su}$. For any $x \in X_1$, we denote $x_c = P_c X_1$ and $x_{su} = P_{su}x$. In addition, we assume

(E1) $\dim X_1^s < +\infty$ and (f,g) are C^3 in (x,y) with upper bound uniform in t.

(E2) For $(x_c, x_{su}, y, t, \epsilon) \in X_1^c \times X_1^{su} \times Y_1 \times \mathbb{R} \times [0, \epsilon_0)$,

$$P_c f_y = 0 \ , \ D^2_{(x_c,y)} P_c f_0(0,0,0) = 0,$$

$$P_c f_1(x_c, x_{su}, y, t, \epsilon) = -x_c + \epsilon B_0(x_c, x_{su}, y) + B_1(x_c, x_{su}, y, t, \epsilon),$$

B_0 is a bounded linear operator acting on (x_c, x_{su}, y),

$$B_1(0,0,0,t,\epsilon) = 0 \ , \ DB_1(0,0,0,t,\epsilon) = 0.$$

(E3) For $(x_c, x_{su}, y, t, \epsilon) \in X_1^c \times X_1^{su} \times Y_1 \times \mathbb{R} \times [0, \epsilon_0)$,

$$g_x = 0 \ , \ g_0(0,0,0) = 0 \ , \ D^2_{(x_c,y)} g_0(0,0,0) = 0,$$

$$g_1(x_c, x_{su}, y, t, \epsilon) = -y + \epsilon B_2(x_c, x_{su}, y) + B_3(x_c, x_{su}, y, t, \epsilon),$$

B_2 is a bounded linear operator acting on (x_c, x_{su}, y),

$$B_3(0,0,0,t,\epsilon) = 0 \ , \ DB_3(0,0,0,t,\epsilon) = 0.$$

Remark 5.6. *One may think the above assumption is too strong, which makes the system* (2.1) *very restrictive. However, one should first try to 'diagonalize' the linear part to remove f_y and g_x. As a separate topic, we will discuss this transformation in* *the Appendix. With this 'diagonalized' linear part, one is in a position to carry out a normal form transformation to eliminate some quadratic terms. Assumption (E) should be considered for the form after performing a normal form transformation.*

Let $A(\epsilon) = A + f_x - \epsilon$ and $\frac{J(\epsilon)}{\epsilon} = \frac{J}{\epsilon} + g_y - \epsilon$. We further assume $a_1' \leq 0$ in (C2), it follows

$$|e^{tA(\epsilon)} x_c|_{X_1} \leq K e^{-\epsilon t} |x_c|_{X_1} \quad \text{for} \quad t \geq 0, \quad x_c \in X_1^c,$$
$$|e^{t\frac{J(\epsilon)}{\epsilon}} y|_{Y_1} \leq K e^{-\epsilon t} |y|_{Y_1} \quad \text{for} \quad t \geq 0, \quad y \in Y_1.$$

For sufficiently small r, from Theorem 3.7, for $(x_c, y, \epsilon) \in B_r(0, X_1^c) \times B_r(0, Y_1) \times [0, \epsilon_0)$ and any $t_0 \in \mathbb{R}$ there exists a local center manifold $\mathcal{M}_\epsilon^c(t_0)$ and (Ψ_u, Ψ_s) with

$$h_{su}(x_c, y, \epsilon) \triangleq (\Psi_s(x_c, y, \epsilon), \Psi_u(x_c, y, \epsilon)) \subset X_1^s \times X_1^u$$

such that h_{su} is uniformly bounded in C^2 in (x_c, y) and

$$\begin{aligned} & h_{su}(0,0,\epsilon) = 0 \ , \ |Dh_{su}(x_c, y, \epsilon)| \leq C'(\epsilon + |x_c| + |y|) \\ & |h_{su}(x_c, y, \epsilon)| \leq C'(\epsilon + |x_c| + |y|)(|x_c + |y|), \\ & \big\{x_c + y + h_{su}(x_c, y, \epsilon)\big\} = \mathcal{M}_\epsilon^c(t_0) \cap \big(B_r(0, X_1^s) \times B_r(0, X_1^u) \\ & \qquad\qquad \times B_r(0, X_1^c) \times B_r(0, Y_1)\big), \end{aligned} \tag{5.39}$$

where C' depends on constants in assumptions. Here we use the assumption that X^u and X^s are finite dimensional. On the center manifold, the flow is reduced to the x_c and y direction only, where the solutions are given in the form of

$$\begin{aligned} x_c(t) =& e^{(t-t_\star)A(\epsilon)} x_c(t_\star) + \int_{t_\star}^t e^{(t-\tau)A(\epsilon)} \widetilde{f}(x_c, y, \tau, \epsilon) \, d\tau, \\ y(t) =& e^{(t-t_\star)\frac{J(\epsilon)}{\epsilon}} y(t_\star) + \int_{t_\star}^t e^{(t-\tau)\frac{J(\epsilon)}{\epsilon}} \widetilde{g}(x_c, y, \tau, \epsilon) \, d\tau, \end{aligned} \tag{5.40}$$

where

$$\widetilde{f}(x_c, y, t, \epsilon) = P_c(f_0 + \epsilon f_1)(x_c + h_{su}(x_c, y, \epsilon), y, t, \epsilon),$$
$$\widetilde{g}(x_c, y, t, \epsilon) = (g_0 + \epsilon g_1)(x_c + h_{su}(x_c, y, \epsilon), y, t, \epsilon).$$

From assumptions (E1)—(E3) and (5.39), $(\widetilde{f}, \widetilde{g})$ satisfies

$$\widetilde{f}(0,0,t,0) = 0\ ,\ \widetilde{g}(0,0,t,0) = 0\ ,\ |D\widetilde{f}| + |D\widetilde{g}| \leq C'(\epsilon^2 + |x_c|^2 + |y|^2).$$

Suppose we have solution $(x_c(t), y(t))$ such that $|x_c(t_\star)| + |y(t_\star)| \leq \delta\epsilon^{\frac{1}{2}}$ and $|x_c(t)| + |y(t)| \leq (2+K)\delta\epsilon^{\frac{1}{2}}$ for $t \in [t_\star, T']$ for some T'. By using Gronwall's inequality, we obtain

$$e^{\epsilon(t-t_\star)}(|x_c(t)| + |y(t)|) \leq K\delta\epsilon^{\frac{1}{2}} e^{C'\delta^2\epsilon(t-t_\star)}.$$

Since C' only depends on constants in assumptions and independent of t, we can extend T' to $+\infty$. Recall that (x_+, y_+) denote the solution which satisfies

$$(x_+(t_0, t_0, \epsilon), y_+(t_0, t_0, \epsilon)) \in \mathcal{M}_\epsilon^{cs}(t_0) \cap \mathcal{M}_\epsilon^{u}(t_0).$$

In particular, for sufficiently small δ and by choosing $t_\star = T_1 + t_0$, (5.36) implies for $t > T_1 + t_0$

$$|P_c x^+(t, t_0, \epsilon)| + |y^+(t, t_0, \epsilon)| \leq \delta\epsilon^{\frac{1}{2}} e^{(t-T_1-t_0)\epsilon(C'\delta^2-1)},$$

which means there is a $O(\epsilon^{\frac{1}{2}})$ weakly stable neighborhood of the origin inside the center manifold.

Theorem 5.7. *Assume (A1)—(A5), (A6′), (A7) for $k = 2$, (B1)—(B5), (C1)—(C2), (D1)—(D6) and (E1)—(E3), where $a_1' \leq 0$ in (C2). There exists $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0)$,* (2.1) *has a homoclinic solution to the origin.*

Finally, we would like to apply Theorem 5.7 to (1.4). We let $y = \epsilon u$ and $\dot{y} = u_1 - \epsilon^2\gamma u$. We rewrite (1.4) as a first order system.

$$\begin{cases} \dot{x} = \dfrac{x_1}{(1+\epsilon u)^2} \\ \dot{x}_1 = -g(1+\epsilon u)\sin x - 2\epsilon\gamma x_1 + \epsilon F_1(x, \epsilon u, t, \epsilon) \\ \dot{u} = \dfrac{1}{\epsilon}u_1 - \epsilon\gamma u \\ \dot{u}_1 = -\dfrac{1}{\epsilon}u - \epsilon\gamma u_1 + \dfrac{x_1^2}{(1+\epsilon u)^3} + \epsilon^3\gamma^2 u + g\cos x + \epsilon F_2(x, \epsilon u, t, \epsilon). \end{cases} \tag{5.41}$$

We assume

$$\gamma > 0 \ , \ F_1(\pi, y, t, \epsilon) \equiv 0 \ , \ \partial_t F_2(\pi, y, t, \epsilon) \equiv 0.$$

By implicit function theorem, there exists a unique $(u^\epsilon, u_1^\epsilon) = (O(\epsilon), O(\epsilon^3))$ such that

$$u_1^\epsilon - \epsilon^2\gamma u^\epsilon = 0 \ , \ -u^\epsilon - \epsilon^2\gamma u_1^\epsilon + \epsilon^4\gamma^2 u^\epsilon - \epsilon g + \epsilon^2 F_2(\pi, \epsilon u^\epsilon, t, \epsilon) = 0, \tag{5.42}$$

which follows $(\pi, 0, u^\epsilon, u_1^\epsilon)$ is fixed point of (5.41). Let $(\widetilde{x}, v, v_1) = (x - \pi, u - u^\epsilon, u - u_1^\epsilon)$, (5.41) becomes

$$\begin{cases} \dot{\widetilde{x}} = x_1 + (\dfrac{x_1}{(1+\epsilon u^\epsilon + \epsilon v)^2} - 1)x_1 \\ \dot{x}_1 = g\sin\widetilde{x} + \epsilon g(v + u^\epsilon)\sin\widetilde{x} - 2\epsilon\gamma x_1 + \epsilon F_1(\widetilde{x} + \pi, \epsilon u^\epsilon + \epsilon v, t, \epsilon) \\ \dot{v} = \dfrac{1}{\epsilon} v_1 - \epsilon\gamma v \\ \dot{v}_1 = -\dfrac{1}{\epsilon} v - \epsilon\gamma v_1 + \dfrac{x_1^2}{(1+\epsilon u^\epsilon + \epsilon v)^3} + \epsilon^3\gamma^2 v + g(1 - \cos\widetilde{x}) \\ \qquad + \epsilon(F_2(\widetilde{x} + \pi, \epsilon u^\epsilon + \epsilon v, t, \epsilon) - F_2(\pi, \epsilon u^\epsilon, t, \epsilon)). \end{cases} \tag{5.43}$$

When $\epsilon = 0$, $(\pi, 0)$ is a hyperbolic fixed point of the first two equations of (5.43), thus, all assumptions in (E2) for f are automatically satisfied. We rewrite the right hand side of last equation in (5.43) as

$$-\frac{1}{\epsilon} v - \epsilon\gamma v_1 + \epsilon D_x F_2(\pi, \epsilon u^\epsilon, t, \epsilon)\widetilde{x} + \overline{g}(\widetilde{x}, x_1, v, v_1, t, \epsilon),$$

where in view of (5.42)

$$\begin{aligned} &\overline{g}(\widetilde{x}, x_1, v, v_1, t, \epsilon) \\ \triangleq &\frac{x_1^2}{(1+\epsilon u^\epsilon + \epsilon v)^3} + g(1 - \cos\widetilde{x}) + \epsilon^3\gamma^2 v + \epsilon\big(F_2(\widetilde{x} + \pi, \epsilon u^\epsilon \\ &+ \epsilon v, t, \epsilon) - F_2(\pi, \epsilon u^\epsilon, t, \epsilon) - D_x F_2(\pi, \epsilon u^\epsilon, t, \epsilon)\widetilde{x}\big). \end{aligned}$$

Clearly, $\overline{g}$ satisfies all assumptions in (E1)—(E3). Finally, we will perform a linear transformation to block diagonalize the linear part of (5.43). Since the transformation is linear, the nonlinearity in the new system still satisfy (E1)—(E3). To simply our notations, we let

$$M_1 \triangleq \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix}, \; M_2 \triangleq \begin{pmatrix} -\epsilon\gamma & \frac{1}{\epsilon} \\ -\frac{1}{\epsilon} & -\epsilon\gamma \end{pmatrix}, \; M_3 \triangleq \begin{pmatrix} 0 & 0 \\ D_x F_2(\pi, \epsilon u^\epsilon, t, \epsilon) & 0 \end{pmatrix}.$$

By implicit function theorem there exist $L_1 \in L(\mathbb{R}^2, \mathbb{R}^2)$ with $|L_1| \le C'\epsilon^2$, where C' depends on constants in assumptions, such that

$$JL_1 - \epsilon^2\gamma L_1 - \epsilon L_1 M_1 + \epsilon^2 M_3 = 0,$$

which implies

$$\begin{pmatrix} I & 0 \\ L_1 & I \end{pmatrix}^{-1} \begin{pmatrix} M_1 & 0 \\ \epsilon M_3 & M_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ L_1 & I \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ 0 & \begin{pmatrix} 0 & \frac{1}{\epsilon} \\ -\frac{1}{\epsilon} & 0 \end{pmatrix} - \epsilon\gamma I \end{pmatrix}.$$

Therefore, one can apply Theorem 5.7 to (1.4).

5.3 Persistence of homoclinic orbit under conservative perturbation

In this section, our goal is to study the persistence of homoclinic orbits under conservative perturbations. We further assume

(D7) X_1^s is finite dimensional. Moreover,

$$\partial_t f(x, y, t, \epsilon) \equiv 0 \; , \; \partial_t g(x, y, t, \epsilon) \equiv 0,$$
$$a_1 < 2\eta < \eta < \min\{0, a_2\} \; , \; \dim(T_{x_0}\mathcal{M}_0^s \bigcap T_{x_0}\mathcal{M}_0^{cu}) = 1.$$

(D8) There exists a family of invariant quantities $\{H_\epsilon\}$ for (2.1) in terms of Taylor expansion in $u = \frac{y}{\epsilon}$, which takes the following form

$$H_\epsilon(x, u) = H_0(x, \epsilon) + H_1(x, \epsilon)u + H_2(x, \epsilon)(u, u) + H_3(x, u, \epsilon),$$
$$H_0(x, 0) = H(x) \; , \; H_i \in C^3(X_1 \times \mathbb{R}, L_i(Y_1, \mathbb{R})) \text{ for } i = 0, 1, 2,$$
$$H_3 \in C^3(X_1 \times Y_1 \times \mathbb{R}, \mathbb{R}).$$

Moreover, there exist $a_0 > 0, a_1 \geq 0, a_2 > 0, a_3 \geq 0$ such that for any $\xi_c \in X_1^c$ and $(x, u) \in B_r(0, X_1) \times B_b(0, Y_1)$

$$
\begin{aligned}
& H_0(0,\epsilon) = 0\ ,\ DH_0(0,\epsilon) = 0\ ,\ D^2H_0(0,0)(\xi_c,\xi_c) \geq a_0|\xi_c|^2, \\
& H_1(0,\epsilon) = 0\ ,\ |DH_1(0,\epsilon)| \leq a_1\ ,\ |H_2(0,\epsilon)(u,u)| \geq a_2|u|^2, \\
& H_3(x,0,\epsilon) = 0\ ,\ D_uH_3(x,0,\epsilon) = 0\ ,\ |D^2H_3(x,u,\epsilon)| \leq C_0\epsilon, \\
& \overline{a} \triangleq \frac{a_0a_2 - a_1^2}{4a_2} > 0.
\end{aligned}
$$

We note (D8) implies

$$H_\epsilon(0,0) = 0\ ,\ DH_\epsilon(0,0) = 0.$$

Then by using (5.39), for any $z = \xi_c + \epsilon\xi_y + h_{su}(\xi_c, \xi_y, \epsilon) = P_Xz + \epsilon\xi_y$

$$
\begin{aligned}
H_0(P_Xz,\epsilon) \geq & (\frac{a_0}{2} - \epsilon|\partial_\epsilon D^2H_0|_{C^0})|\xi_c|^2 - C'|D^2H_0|_{C^0}|\xi_c|((\overline{r}+\epsilon_\star)|\xi_c| + \epsilon^2|\xi_y|) \\
& - C'|D^2H_0|_{C^0}((\overline{r}+\epsilon_\star)^2|\xi_c|^2 + \epsilon^4|\xi_y|^2) - C'(|\xi_c|^3 + \epsilon^6|\xi_y|^3), \\
\geq & (\frac{a_0}{2} - C'\epsilon)|\xi_c|^2 - C'\epsilon^4|\xi_y|^2, \\
H_2(P_Xz,\epsilon)(\xi_y,\xi_y) \geq & (a_2 - C'\epsilon^2)|\xi_y|^2 - C'|\xi_c||\xi_y|^2, \\
|H_1(P_Xz,\epsilon)\xi_y| \leq & a_1(1 + C'(\overline{r}+\epsilon_\star))|\xi_c||\xi_y| + C'\epsilon^2|\xi_y|^2 + C'(|\xi_c|^2 + \epsilon^4|\xi_y|^2)|\xi_y|, \\
|H_3(P_Xz,\xi_y,\epsilon)| \leq & C_0\epsilon|\xi_y|^2.
\end{aligned}
$$

By the last inequality in (D8), we have by choosing sufficiently small r and $\epsilon_\star$,

$$H_\epsilon(p) \geq \frac{a_0}{4}|\xi_c|^2 + \frac{\overline{a}}{2}|\xi_y|^2.$$

The above shows that for any $p \in \mathcal{M}_\epsilon^c \bigcap \big(B_r(0, X_1^s) \times B_r(0, X_1^u) \times B_r(0, X_1^c) \times B_{b\epsilon}(0, Y_1)\big)$ and $p \neq 0$, $H_\epsilon(p)$ is positive with quadratic lower bound. It implies the origin is stable both in forward and backward time on the center manifold. Consequently, $\mathcal{M}_\epsilon^\alpha$ are unique, where $\alpha = c, cu, cs$.

We choose Σ as in Subsection 6.1. Since $DH(x_0)\big|_\Sigma \neq 0$, by continuity we have

$$DH_\epsilon(x_0, 0) = DH_0(x_0,\epsilon) + DH_3(x_0, 0, \epsilon) = DH_0(x_0,\epsilon),$$

which implies $DH_\epsilon(x_0,0)\big|_\Sigma \neq 0$. We will use similar coordinate system to represent $p \in \Sigma + x_0$ as in Subsection 6.1. We define $\overline{X}_1^c = \overline{X}_1^{cs} \cap \overline{X}_1^{cu}$ and $\overline{X}_1^s = T_{x_0}\mathcal{M}_0^s \cap \Sigma$. Let $Q_{c,s}$ be the projections from Σ onto $\overline{X}_1^{c,s}$, respectively. For any $p \in \Sigma + x_0$, its coordinates can be written as

$$\begin{aligned}
&(d, x^{cs}, y, x^u)\\
=&(Q_\omega(p-x_0), Q_s(p-x_0), Q_c(p-x_0), Q_y(p-x_0), Q_u(p-x_0))\\
=&(DH_0(x_0,0)(p-x_0), Q_s(p-x_0), Q_c(p-x_0), Q_y(p-x_0), Q_u(p-x_0)).
\end{aligned}$$

The center-stable and unstable manifolds in Σ can be written as graphs of Υ and Ψ as given in Lemma 5.3. Moreover precisely, for sufficiently small $r > 0$, arbitrary $b > 0$ and $(x^s, x^c, x^u, y) \in B_r(0, \overline{X}_1^s) \times B_r(0, \overline{X}_1^c) \times B_r(0, \overline{X}_1^u) \times B_b(0, Y_1)$, there exist Υ and Ψ such that

$$\begin{aligned}
&\Big\{(\Upsilon^d + \Upsilon^u)(x^c, x^s, y, \epsilon) + x^c + x^s + \epsilon y\Big\} \subset \mathcal{M}_\epsilon^{cs} \cap \Sigma = \widetilde{\mathcal{M}_\epsilon^{cs}},\\
&\Big\{(\Psi^d + \Psi^y + \Psi^s + \Psi^c)(x^u, \epsilon) + x^u\Big\} \subset \mathcal{M}_\epsilon^{u} \cap \Sigma = \widetilde{\mathcal{M}_\epsilon^{u}},
\end{aligned}$$

Similarly, there exist Υ_1 and Ψ_1 such that

$$\begin{aligned}
&\Big\{(\Upsilon_1^d + \Upsilon_1^s)(x^c, x^u, y, \epsilon) + x^c + x^u + \epsilon y\Big\} \subset \mathcal{M}_\epsilon^{cu} \cap \Sigma \triangleq \widetilde{\mathcal{M}_\epsilon^{cu}},\\
&\Big\{(\Psi_1^d + \Psi_1^y + \Psi_1^c + \Psi_1^u)(x^s, \epsilon) + x^s\Big\} \subset \mathcal{M}_\epsilon^{s} \cap \Sigma \triangleq \widetilde{\mathcal{M}_\epsilon^{s}},
\end{aligned}$$

where Υ_1, Ψ_1 satisfy similar properties as Υ, Ψ in (5.13), (5.14), (5.15) and (5.27), (5.28). To find the intersection of $\widetilde{\mathcal{M}_\epsilon^{cs}}$ and $\widetilde{\mathcal{M}_\epsilon^{cs}}$, by using implicit function theorem, we first obtain $x^{s,u}(x^c, y, \epsilon)$ such that

$$\begin{aligned}
&\Upsilon^u(x^c, x^s(x^c, y, \epsilon), y, \epsilon) + x^c + x^s(x^c, y, \epsilon) + \epsilon y\\
=&x^u(x^c, y, \epsilon) + x^c + \Upsilon_1^s(x^c, x^u(x^c, y, \epsilon), y, \epsilon) + \epsilon y,
\end{aligned} \tag{5.44}$$

where $x^{s,u}(0,0,0) = 0$, $Dx^{s,u}(0,0,0) = 0$, $|D_y x^{s,u}|_{C^0} \leq C'\epsilon^2$. Substituting x^s into Υ, Ψ_1 and using implicit function theorem again, we obtain $(x^c(\epsilon), y(\epsilon))$ such that

$$\begin{aligned}
&(\Upsilon^d + \Upsilon^u)(x^c(\epsilon), x^s(x^c(\epsilon), y(\epsilon), \epsilon), y(\epsilon), \epsilon)\\
&\qquad\qquad + x^c(\epsilon) + x^s(x^c(\epsilon), y(\epsilon), \epsilon) + \epsilon y(\epsilon)\\
=&(\Psi_1^d + \Psi_1^y + \Psi_1^c + \Psi_1^u)(x^s(x^c(\epsilon), y(\epsilon), \epsilon), \epsilon) + x^s(x^c(\epsilon), y(\epsilon), \epsilon) \in \widetilde{\mathcal{M}_\epsilon^{s}}.
\end{aligned}$$

Similarly, by substituting x^u into Υ_1, Ψ, we have $(x_1^c(\epsilon), y_1(\epsilon))$ satisfying

$$
\begin{aligned}
(\Upsilon_1^d + \Upsilon_1^u)(x_1^c(\epsilon), x^u(x_1^c(\epsilon), y_1(\epsilon), \epsilon), y_1(\epsilon), \epsilon) \\
+ x_1^c(\epsilon) + x^u(x_1^c(\epsilon), y_1(\epsilon), \epsilon) + \epsilon y_1(\epsilon) \\
=(\Psi^d + \Psi^y + \Psi^c + \Psi_1^u)(x^u(x_1^c(\epsilon), y_1(\epsilon), \epsilon), \epsilon) + x^u(x_1^c(\epsilon), y_1(\epsilon), \epsilon) \in \widetilde{\mathcal{M}}_\epsilon^u.
\end{aligned}
$$

Let $q(\tau) = (q_c(\tau), q_y(\tau)) \triangleq \big((1-\tau)x^c(\epsilon)+\tau x_1^c(\epsilon), (1-\tau)y(\epsilon)+\tau y_1(\epsilon)\big)$, where $\tau \in [0, 1]$. We define

$$
\begin{aligned}
h(\tau) \triangleq & H_\epsilon\Big(\Upsilon(q(\tau), x^s(q(\tau), \epsilon), \epsilon) + q_c(\tau) + x^s(q(\tau), \epsilon) + \epsilon q_y(\tau)\Big) \\
& - H_\epsilon\Big(\Upsilon_1(q(\tau), x^u(q(\tau), \epsilon), \epsilon) + q_c(\tau) + x^u(q(\tau), \epsilon) + \epsilon q_y(\tau)\Big).
\end{aligned}
$$

Clearly, from (5.44), $\widetilde{\mathcal{M}}_\epsilon^{cs} \cap \widetilde{\mathcal{M}}_\epsilon^{cu} \neq \emptyset$ is equivalent to $h(\tau_0) = 0$ for some $\tau_0 \in [0, 1]$. Since $h(1) \geq 0 \geq h(0)$, by intermediate value theorem, the center-unstable and center-stable manifold have a nonempty intersection.

Theorem 5.8. *Assume (A1)—(A5), (A6′) for $k = 2$, (B1)—(B5), (C1)—(C2) and (D1)—(D8). There exists $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0)$, the center-stable manifold and center-unstable manifold of* (2.1) *has nonempty intersection.*

The intersection of the center-stable and center-unstable manifold is generically transversal and forms a high dimensional tube homoclinic to the center manifold. See [SZ3] for more discussion when there is no singular perturbation. Finally, we verify that (1.5) under the conservative perturbation satisfies all the assumptions in the above theorem. Let $G(x, y, \epsilon)$ to be a smooth function and consider (1.4) which can be viewed as a singular perturbation of (1.5) of conservative type. We can rewrite (1.4) into the following form

$$
\begin{cases}
\dot{x} = \dfrac{x_1}{(1+y)^2} \\
\dot{x}_1 = -g(1+y)\sin x - \epsilon D_x G(x, y, \epsilon) \\
\epsilon \dot{y} = y_1 \\
\epsilon \dot{y}_1 = -y + \dfrac{\epsilon^2 x_1^2}{(1+y)^3} + \epsilon^2 g \cos x - \epsilon^3 D_y G(x, y, \epsilon).
\end{cases}
\tag{5.45}
$$

From implicit function theorem, for each ϵ, there exist a unique $(x^\epsilon, y^\epsilon) = (O(\epsilon), O(\epsilon^3))$ such that

$$\begin{cases} g(1+y^\epsilon)\sin x^\epsilon + \epsilon D_x G(x^\epsilon, y^\epsilon, \epsilon) = 0 \\ y^\epsilon - \epsilon^2 g\cos x + \epsilon^3 D_y G(x^\epsilon, y^\epsilon, \epsilon) = 0, \end{cases} \tag{5.46}$$

which implies $(x^\epsilon, 0, u^\epsilon, 0)$ is a fixed point of (5.45). Let $\widetilde{x} = x - x^\epsilon, \widetilde{y} = y - y^\epsilon$, we can rewrite (5.45) as

$$\begin{cases} \dot{\widetilde{x}} = \dfrac{x_1}{(1+y^\epsilon+\widetilde{y})^2} \\ \dot{x}_1 = -g(1+y^\epsilon+\widetilde{y})\sin(x^\epsilon+\widetilde{x}) - \epsilon D_x G(x^\epsilon+\widetilde{x}, y^\epsilon+\widetilde{y}, \epsilon) \\ \dot{\widetilde{y}} = \dfrac{1}{\epsilon} y_1 \\ \dot{y}_1 = -\dfrac{1}{\epsilon}\widetilde{y} - \dfrac{1}{\epsilon} y^\epsilon + \dfrac{\epsilon x_1^2}{(1+y^\epsilon+\widetilde{y})^3} + \epsilon g\cos(x^\epsilon+\widetilde{x}) \\ \qquad - \epsilon^2 D_y G(x^\epsilon+\widetilde{x}, y^\epsilon+\widetilde{y}, \epsilon), \end{cases}$$

which has an invariant quantity

$$\begin{aligned} H_\epsilon(\widetilde{x}, x_1, v, u_1) = & \frac{x_1^2}{2(1+\epsilon y^\epsilon+\epsilon v)^2} + \frac{u_1^2}{2} + \frac{(\frac{y^\epsilon}{\epsilon}+v)^2}{2} - \frac{(y^\epsilon)^2}{2\epsilon^2} \\ & - g\big((1+y^\epsilon+\epsilon v)\cos(x^\epsilon+\widetilde{x}) - (1+y^\epsilon)\cos x^\epsilon\big) \\ & + \epsilon\big(G(x^\epsilon+\widetilde{x}, y^\epsilon+\epsilon v, \epsilon) - G(x^\epsilon, y^\epsilon, \epsilon)\big), \end{aligned}$$

where $v = \frac{\widetilde{y}}{\epsilon}$, $u_1 = \frac{y_1}{\epsilon}$. One can use Taylor's expansion to compute

$$\begin{aligned} H_0 = & \frac{1}{2(1+y^\epsilon)^2} x_1^2 - g(1+y^\epsilon)\big(\cos(x^\epsilon+\widetilde{x}) - \cos x^\epsilon\big) \\ & + \epsilon\big(G(x^\epsilon+\widetilde{x}, y^\epsilon, \epsilon) - G(x^\epsilon, y^\epsilon, \epsilon)\big), \\ H_1 = & \big(-\frac{x_1^2}{(1+\epsilon u^\epsilon)^3}\epsilon + \frac{y^\epsilon}{\epsilon} - \epsilon g\cos(x^\epsilon+\widetilde{x}) + \epsilon^2 D_y G(x^\epsilon+\widetilde{x}, y^\epsilon, \epsilon), 0\big), \\ H_2 = & \begin{pmatrix} \frac{1}{2} + \frac{\epsilon^3}{2} D_y^2 G(x^\epsilon+\widetilde{x}, y^\epsilon, \epsilon) + \frac{3x_1^2\epsilon^2}{2(1+\epsilon u^\epsilon)^4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \\ H_3 = & H_\epsilon(\widetilde{x}, x_1, v, u_1) - H_1(\widetilde{x}, \epsilon)\cdot(v, u_1) - (v, u_1) H_2(\widetilde{x}, \epsilon)(v, u_1)^T. \end{aligned}$$

One can use (5.46) to verify the above H_i, where $i = 0, 1, 2$, satisfy assumption (D8). Therefore, for $\epsilon \ll 1$, the center stable manifold and center-unstable manifold of

(5.45) intersect near the unperturbed homoclinic orbit $x_h(t)$, which generically form a 2-parameter family of solutions homoclinic to small amplitude fast oscillations.

CHAPTER VI

APPENDIX

In this chapter, we will discuss the normal form transformation related to (2.1). And we will discuss two cases, namely, A is a bounded linear operator on X and A is a closed unbounded operator with dense domain on X. Our strategy is to block-diagonalize the linear part of the vector field of (2.1). We assume

(B) For $(t,\epsilon) \in \mathbb{R} \times [0,\epsilon_0)$,

$$(f,g)(0,0,t,\epsilon) = 0\ ,\ \partial_t(Df,Dg)(0,0,t,\epsilon) = 0.$$

We look for two invariant subspaces $(x, L_1^\epsilon(x))$ and $(L_2^\epsilon(y), y)$ such that they are invariant under $\begin{pmatrix} A + D_x f(0,\epsilon) & D_y f(0,\epsilon) \\ D_x g(0,\epsilon) & \frac{J}{\epsilon} + D_y g(0,\epsilon) \end{pmatrix}$. For simplicity, we write $D_{x,y}(f,g)(0,\epsilon)$ as $D_{x,y}(f,g)$. Therefore, $(L_1^\epsilon, L_2^\epsilon)$ should satisfy the following system:

$$\begin{aligned} &(J + \epsilon D_y g)L_1^\epsilon - \epsilon L_1^\epsilon(A + D_x f + D_y f L_1^\epsilon) + \epsilon D_x g = 0, \\ &L_2^\epsilon(\epsilon D_x g L_2^\epsilon + J + \epsilon D_y g) - \epsilon(A + D_x f)L_2^\epsilon - \epsilon D_y f = 0. \end{aligned} \tag{6.1}$$

Lemma 6.1. *Assume $A \in L(X,X)$, (A3) and (A4) for $k = 1$ and (B). There exists $\epsilon_0 > 0$ such that for any $\epsilon \in [0,\epsilon_0)$ there exists a unique pair of*

$$(L_1^\epsilon, L_2^\epsilon) \in L(X, Y_1) \times L(Y, X),$$

which satisfies (6.1).

Proof. Define

$$\widetilde{G}(L_1, L_2, \epsilon) = \begin{pmatrix} (J + \epsilon D_y g)L_1 - \epsilon L_1(A + D_x f + D_y f L_1) + \epsilon D_x g \\ L_2(\epsilon D_x g L_2 + J + \epsilon D_y g) - \epsilon(A + D_x f)L_2 - \epsilon D_y f \end{pmatrix}.$$

Clearly, $\widetilde{G}$ is a smooth mapping from $L(X, Y_1) \times L(Y, X) \times \mathbb{R}$ to $L(X, Y) \times L(Y_1, X)$. Since $\widetilde{G}(0,0,0) = 0$ and for $(\widetilde{L}_1, \widetilde{L}_2) \in L(X, Y) \times L(Y_1, X)$,

$$D\widetilde{G}(0,0,0)^{-1}(\widetilde{L}_1, \widetilde{L}_2) = (J^{-1}\widetilde{L}_1, \widetilde{L}_2 J^{-1}),$$

which is a bounded isomorphism by assumption (A3). By implicit function theorem, there exists a unique pair of $(L_1^\epsilon, L_2^\epsilon)$ such that (6.1) is satisfied. $\square$

Formal asymptotic expansion shows that

$$L_1^\epsilon = -\epsilon J^{-1} D_x g + \epsilon^2 \Big(J^{-2} D_x g (A + D_x f) - J^{-1} D_y g J^{-1} D_x g \Big) + O(\epsilon^3),$$
$$L_2^\epsilon = \epsilon D_y f J^{-1} + \epsilon^2 \Big((A + D_x f) D_y f J^{-2} - D_y f J^{-1} D_y g J^{-1} \Big) + O(\epsilon^3).$$

Moreover,

$$\begin{pmatrix} I & L_2^\epsilon \\ L_1^\epsilon & I \end{pmatrix}^{-1} = \begin{pmatrix} I + L_2^\epsilon (I - L_1^\epsilon L_2^\epsilon)^{-1} L_1^\epsilon & -L_2^\epsilon (I - L_1^\epsilon L_2^\epsilon)^{-1} \\ -(I - L_1^\epsilon L_2^\epsilon)^{-1} L_1^\epsilon & (I - L_1^\epsilon L_2^\epsilon)^{-1} \end{pmatrix}.$$

Let $\mathcal{L} = \begin{pmatrix} I & L_2^\epsilon \\ L_1^\epsilon & I \end{pmatrix}$, we shall use $\mathcal{L}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$ as new variables which are still denoted by (x, y) to rewrite (2.1). After some computation, we have

$$\begin{cases} \dot{x} = (A + D_x f + D_y f L_1^\epsilon) x + \widetilde{F}(x, y, t, \epsilon) \\ \dot{y} = (\dfrac{J}{\epsilon} + D_y g + D_x g L_2^\epsilon) y + \widetilde{G}(x, y, t, \epsilon). \end{cases} \tag{6.2}$$

To apply our results in previous sections, we need $\widetilde{F}, \widetilde{G}$ to satisfy the same properties in (A4). This can be justified by the following embedding of spaces, namely,

$$L(X, Y_1) \subset L(X, Y)\ ,\ L(Y, Y_1) \subset L(Y, Y)\ ,\ L(Y, Y_1) \subset L(Y_1, Y_1).$$

Therefore, all results in previous sections still hold for the new system (6.2).

In the case of A is unbounded, it's not obvious how to apply implicit function theorem. Instead, we will use an integral equation to resolve the difficulty under the assumption

(F) There exist closed subspaces $X^{s,c,u}$ of X such that $X = X^s \oplus X^c \oplus X^u$ and $A + D_x f(0,\epsilon)$ is invariant on $X^{s,c,u}$. Let $A^{u,c,s} = A + D_x f(0,\epsilon)\big|_{X^{s,c,u}}$. We further assume A^c is a bounded linear operator on X^c and there exist $\omega_s < 0$ and $\omega_u > 0$ such that

$$|e^{tA^s}| \le K e^{\omega_s t} \text{ for } t \ge 0\ ,\ |e^{tA^u}| \le K e^{\omega_u t} \text{ for } t \le 0.$$

To find L_1^ϵ, we consider the following coupled Riccati type equations

$$G^u(L^u) = \int_{+\infty}^{0} e^{tJ}(\epsilon L D_y f L^u - \epsilon D_x g - \epsilon D_y g L^u) e^{-\epsilon t A^u} dt, \tag{6.3}$$

$$G^c(L^c) = \epsilon (J + \epsilon D_y g)^{-1} (L D_y f L^c - D_x g + L^c A^c), \tag{6.4}$$

$$G^s(L^s) = \int_{-\infty}^{0} e^{tJ}(\epsilon L D_y f L^s - \epsilon D_x g - \epsilon D_y g L^s) e^{-\epsilon t A^s} dt. \tag{6.5}$$

where $L^u \in L(X^u, Y), L^s \in L(X^s, Y), L^c \in L(X^c, Y), L = L^u + L^c + L^s$.

Lemma 6.2. *Let $\overline{G}(L^u, L^c, L^s) = \big(G^u(L^u), G^c(L^c), G^s(L^s)\big)$, if $|\omega_{u,s}|$ satisfy* (6.6) *below, then $\overline{G}$ is a contraction from a bounded ball in $L(X,Y)$ to itself. Thus, $\overline{G}$ has a unique fixed point $L = L^u + L^c + L^s$. Moreover, L is also in $L(X_1, Y_1)$ and satisfy the first equation of* (6.1) *with*

$$|L|_{L(X,Y)} \le C'\ ,\ |L|_{L(X_1,Y_1)} \le C'\epsilon.$$

Proof. Let

$$\mathcal{B} = \Big\{(L^u, L^c, L^s)\Big| |L^u|_{L(X^u,Y)} + |L^c|_{L(X^c,Y)} + |L^s|_{L(X^s,Y)} \le 1\Big\},$$

and

$$C_0 = \max\big\{|A^c|, |D_y f(0,\epsilon)|, |D_x g(0,\epsilon)|, |D_y g(0,\epsilon)|\big\}.$$

$\overline{G}$ satisfies the following estimates on $\mathcal{B}(\rho)$

$$\Big|\overline{G}(L^u, L^c, L^s)\Big| \le 3C_0 K\Big(\frac{1}{|\omega_s|} + \frac{1}{|\omega_u|} + 2\epsilon|J^{-1}|\Big),$$

$$\Big|\overline{G}(L_1^u, L_1^c, L_1^s) - \overline{G}(L_2^u, L_2^c, L_2^s)\Big|$$

$$\le 3KC_0\Big(\frac{1}{|\omega_s|} + \frac{1}{|\omega_u|} + 2\epsilon|J^{-1}|\Big)\Big(|L_1^u - L_2^u| + |L_1^c - L_2^c| + |L_1^s - L_2^s|\Big).$$

Now, if $|\omega_{u,s}|$ satisfy

$$\frac{1}{|\omega_s|}+\frac{1}{|\omega_u|}+2\epsilon|J^{-1}|<\frac{1}{3KC_0}, \tag{6.6}$$

one can verify that $\overline{G}$ defines a contraction mapping on $\mathcal{B}$. Therefore, $\overline{G}$ has a fixed point L. Integrating (6.3) by parts

$$\begin{aligned}L^u =&J^{-1}e^{tJ}(\epsilon LD_yfL^u-\epsilon D_xg-\epsilon D_ygL^u)e^{-\epsilon tA^u}\big|_{+\infty}^{0}\\&+\int_{+\infty}^{0}J^{-1}e^{tJ}(\epsilon LD_yfL^u-\epsilon D_xg-\epsilon D_ygL^u)e^{-\epsilon tA^u}\epsilon A^u dt\\=&J^{-1}(\epsilon LD_yfL^u-\epsilon D_xg-\epsilon D_ygL^u)\\&+\int_{+\infty}^{0}J^{-1}e^{tJ}(\epsilon LD_yfL^u-\epsilon D_xg-\epsilon D_ygL^u)e^{-\epsilon tA^u}\epsilon A^u dt\end{aligned}$$

shows $L^u\in L(X_1,Y_1)$. Moreover, we have

$$\begin{aligned}|L|_{L(X_1,Y_1)}\leq&\epsilon(|J^{-1}|+\frac{1}{|\omega_u|}|J^{-1}|)\big(|D_yf|_{L(Y,X)}|L|^2_{L(X,Y)}+|D_xg|_{L(X,Y)}\\&+|D_yg|_{L(X,Y)}|L|_{L(X,Y)}\big)\leq C'\epsilon.\end{aligned}$$

Now we need to verify $L=L^u+L^c+L^s$ satisfies the first equation of (6.1). We will only work out L^u, L^s follows similarly, and L^c is obvious.

$$\begin{aligned}JL^u-\epsilon L^uA^u=&\int_{+\infty}^{0}Je^{tJ}\big(\epsilon LD_yfL^u-\epsilon D_xg-\epsilon D_ygL^u\big)e^{-\epsilon tA^u}dt\\&-\int_{+\infty}^{0}e^{tJ}\big(\epsilon LD_yfL^u-\epsilon D_xg-\epsilon D_ygL^u\big)e^{-\epsilon tA^u}\epsilon A^u dt\\=&\,e^{tJ}\big(\epsilon LD_yL^u-\epsilon D_xg-\epsilon D_ygL^u\big)e^{-\epsilon tA^u}\Big|_{+\infty}^{0}\\=&\,\epsilon LD_yfL^u-\epsilon D_xg-\epsilon D_ygL^u.\end{aligned}$$

□

To solve for L_2^ϵ, we consider $L^u\in L(Y_1,X_1^u), L^c\in L(Y_1,X_1^c), L^s\in L(Y_1,X_1^s)$ and define

$$\widetilde{G}(L^u,L^c,L^s)=\begin{cases}\epsilon\displaystyle\int_{+\infty}^{0}e^{-\epsilon tA^u}\big(D_yf-L^uD_xgL-L^uD_yg\big)e^{tJ}dt\\\epsilon\big(D_yf-L^cD_xgL+A^cL\big)(J+\epsilon D_yg)^{-1}\\\epsilon\displaystyle\int_{-\infty}^{0}e^{-\epsilon tA^s}\big(D_yf-L^sD_xgL-L^sD_yg\big)e^{tJ}dt.\end{cases}$$

Lemma 6.3. *If* $|\omega_{u,s}|$ *satisfy* (6.6), *there exists a unique* $L \in L(Y_1, X_1)$ *such that* $\widetilde{G}(L) = L$. *Moreover,* $L \in L(Y, X)$ *with estimates* $|L|_{L(Y_1,X_1)} \leq C'$, $|L|_{L(Y,X)} \leq C'\epsilon$.

Proof. Define

$$\mathcal{B}_1 = \Big\{(L^u, L^c, L^s)\Big||L^u|_{L(Y_1,X_1^u)} + |L^c|_{L(Y_1,X_1^c)} + |L^s|_{L(Y,X_1^s)} \leq 1\Big\}.$$

Then one can apply the same proof in Lemma 6.2 □

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