TOPICS IN GROUP METHODS FOR INTEGER PROGRAMMING

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Kenneth Chen

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TOPICS IN GROUP METHODS FOR INTEGER PROGRAMMING

Approved by:

William J. Cook, Advisor
School of Industrial & Systems Engineering
Georgia Institute of Technology

Ellis L. Johnson
School of Industrial & Systems Engineering
Georgia Institute of Technology

George L. Nemhauser
School of Industrial & Systems Engineering
Georgia Institute of Technology

R. Gary Parker
School of Industrial & Systems Engineering
Georgia Institute of Technology

Prasad Tetali
School of Mathematics
Georgia Institute of Technology

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SUMMARY

In 2003, Gomory and Johnson gave two different three-slope T-space facet constructions, both of which shared a slope with the corresponding Gomory mixed-integer cut. We give a new three-slope facet which is independent of the GMIC and also give a four-slope T-space facet construction, which to our knowledge, is the first four-slope construction. We describe an enumerative framework for the discovery of T-space facets.

Using an algorithm by Harvey for computing integer hulls in the plane, we give a heuristic for quickly computing lattice-free triangles. Given two rows of the tableau, we derive how to exactly calculate lattice-free triangles and quadrilaterals in the plane which can be used to derive facet-defining inequalities of the integer hull. We then present computational results using these derivations where non-basic integer variables are strengthened using Balas-Jeroslow lifting.

CHAPTER I

INTRODUCTION

1.1 Development of the Corner Polyhedron

In mathematics, computer science and operations research, many well-known and frequently encountered problems can be formulated as an integer program

$$
\begin{aligned}
\text{(P1)} \quad \max \quad & c'x' \\
\text{s.t.} \quad & A'x' \leq b \\
& x' \geq 0 \\
& x' \ \text{integer}
\end{aligned}
$$

where

$$
\begin{aligned}
A' &= m \times n \text{ integer matrix} \\
x' &= \text{integer } n\text{-vector} \\
b &= \text{integer } m\text{-vector} \\
c' &= \text{integer } n\text{-vector}
\end{aligned}
$$

This problem is NP-hard, even when the inputs are restricted to be in $\{0, 1\}$. Without the integer restriction on x', the problem can be solved in polynomial-time by the ellipsoid method.

If we add slack variables to (P1), then an equivalent formulation is

$$
\begin{aligned}
\text{(P2)} \quad \max \quad & cx \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0 \\
& x \ \text{integer}
\end{aligned}
$$

where $A = (A', I)$ and

$$\begin{aligned} A &= m \times (m+n) \text{ integer matrix} \\ x &= \text{integer } (m+n)\text{-vector} \\ b &= \text{integer } m\text{-vector} \\ c &= \text{integer } (m+n)\text{-vector} \end{aligned}$$

Observe that A contains an $m \times m$ identity matrix corresponding to the slack variables that were added. In the sequel, it will be explained why this is desirable.

Now let B be a basis of A, i.e. a non-singular submatrix consisting of m column vectors of A. Then without any loss of generality, we may assume that the columns of A have been rearranged such that $x_1, \ldots, x_m$ are the basic variables and $x_{m+1}, \ldots, x_{m+n}$ are the non-basic variables and the above formulation can be expressed as

$$\begin{array}{llllll} \text{(P3)} & \max & c_B x_B & + & c_N x_N & \\ & \text{s.t.} & B x_B & + & N x_N = b & \\ & & x_B & \geq & 0 & \\ & & x_B & \text{integer} & & \\ & & x_N & \geq & 0 & \\ & & x_N & \text{integer} & & \end{array}$$

Now by the invertibility of B, we may solve

$$B x_B + N x_N = b$$

for x_B to get

$$x_B = B^{-1} b - B^{-1} N x_N \tag{1}$$

Now substituting and dropping the constant term, we get

$$\begin{array}{llrcl}
\text{(P4)} & \max & c_N x_N - c_B B^{-1} N x_N & & \\
& \text{s.t.} & x_B & = & B^{-1}b - B^{-1}Nx_N \\
& & x_B & \geq & 0 \\
& & x_B & & \text{integer} \\
& & x_N & \geq & 0 \\
& & x_N & & \text{integer}
\end{array}$$

Recall that the notation $a \equiv b \bmod n$ means that n divides $a - b$. We may use this notation to express that a number a is integer by writing $a \equiv 0 \bmod 1$. Extending this notation to vectors, we say that x is integral if $x \equiv 0 \bmod 1$. Hence we have the following chain of equivalences

$$\begin{array}{lcl}
x_B \text{ integer} & \Leftrightarrow & B^{-1}b - B^{-1}Nx_N \text{ integer} \\
& \Leftrightarrow & B^{-1}b - B^{-1}Nx_N \equiv 0 \bmod 1 \\
& \Leftrightarrow & B^{-1}b \equiv B^{-1}Nx_N \bmod 1 \\
& \Leftrightarrow & B^{-1}Nx_N \equiv B^{-1}b \bmod 1
\end{array}$$

and substituting above, we get

$$\begin{array}{llrcl}
\text{(P5)} & \max & c_N x_N - c_B B^{-1} N x_N & & \\
& \text{s.t.} & x_B & = & B^{-1}b - B^{-1}Nx_N \\
& & x_B & \geq & 0 \\
& & B^{-1}Nx_N & \equiv & B^{-1}b \bmod 1 \\
& & x_N & \geq & 0 \\
& & x_N & & \text{integer}
\end{array}$$

Now if we convert the problem to a minimization problem and consider x_N to be independent variables and x_B dependent variables, we may then assume that x_B is *defined* to be

$$x_B = B^{-1}b - B^{-1}Nx_N$$

and hence drop it from the formulation. So we now have

$$\begin{array}{llrcl} \text{(P6)} & \min & (c_B B^{-1} N - c_N) x_N & & \\ & \text{s.t.} & x_B & \geq & 0 \\ & & B^{-1} N x_N & \equiv & B^{-1} b \bmod 1 \\ & & x_N & \geq & 0 \\ & & x_N & & \text{integer} \end{array}$$

Now if we relax the non-negativity of the basic variables, we obtain the Corner Polyhedron associated with the basis B

$$\begin{array}{llrcl} \text{(P7)} & \min & (c_B B^{-1} N - c_N) x_N & & \\ & \text{s.t.} & B^{-1} N x_N & \equiv & B^{-1} b \bmod 1 \\ & & x_N & \geq & 0 \\ & & x_N & & \text{integer} \end{array}$$

Let $\overline{c} = c_B B^{-1} N - c_N$ be the reduced costs. If we let g_j denote the jth column of $B^{-1}N$ and g_0 denote $B^{-1}b$, then we may express the above as

$$\begin{array}{llrcl} \text{(P8)} & \min & \sum_{j=1}^{j=n} \overline{c}_j x_{m+j} & & \\ & \text{s.t.} & \sum_{j=1}^{j=n} g_j x_{m+j} & \equiv & g_0 \bmod 1 \\ & & x_{m+j} & \geq & 0 \\ & & x_{m+j} & & \text{integer} \end{array}$$

By the integrality of the x_{m+j}, we have that $x_{m+j} \equiv 0 \bmod 1$ and we may repeatedly add this to or subtract this from any of the m congruences above. By adding the appropriate integral multiples to each congruence, we may obtain $\overline{g}_j$ where

$$\overline{g}_j \equiv g_j \bmod 1, \text{ and } 0 \leq \overline{g}_j < 1$$

Similarly, we may add the appropriate integral multiple of $1 \equiv 0 \bmod 1$ to each congruence to obtain $\overline{g}_0$ where

$$\overline{g}_0 \equiv g_0 \bmod 1, \text{ and } 0 \leq \overline{g}_0 < 1$$

The final system is

$$
\begin{aligned}
\text{(P9)} \quad \min \quad & \sum_{j=1}^{j=n} \overline{c}_j x_{m+j} \\
\text{s.t.} \quad & \sum_{j=1}^{j=n} \overline{g}_j x_{m+j} \equiv \overline{g}_0 \bmod 1 \\
& x_{m+j} \geq 0 \\
& x_{m+j} \ \text{integer}
\end{aligned}
$$

and is known as the Group Minimization Problem.

At this point, we would like to emphasize the point that solving Problem (P9) does not necessarily mean that Problem (P1) has been solved, as (P9) is a relaxation of (P1). The optimal integer solution x_N^* to Problem (P9) must be plugged into Equation 1 and if $x_B^* \geq 0$, then Problem (P1) is solved by $x^* = (x_B^*, x_N^*)$. A formal proof of this fact is given in Theorem 3 of [35].

1.2 A sufficient condition for $x_B \geq 0$

First, we consider the following lemma

Lemma 1.2.1 *If $(x_{m+i}^*)_{i=1}^n$ is a solution to (P9), then $\sum_{i=1}^n x_{m+i}^* \leq |\det(B)| - 1$.*

We omit a proof as the validity of this result will be immediate when the shortest-path problem is introduced in the sequel. Now, consider the following definition

$$K_B = \{y \in \Re^m \mid y = Bx \text{ for some } x \in \Re^n \text{ where } x \geq 0\}$$

In words, K_B is the cone consisting of the non-negative linear combinations of the columns of B, or equivalently, the points in $\Re^m$ for which B is a feasible basis. Now define

$$K_B(d) = \{y \in K_B \mid \|y - \partial K_B\| \geq d\}$$

which is a set consisting of the points in K_B whose Euclidean distance from the boundary of K_B is at least d. Observe that $K_B = K_B(0)$. Figure 1 illustrates these definitions where $B = [b_1 \ b_2]$.

We now present the theorem from [35] which gives a condition under which it is guaranteed that $x_B \geq 0$.

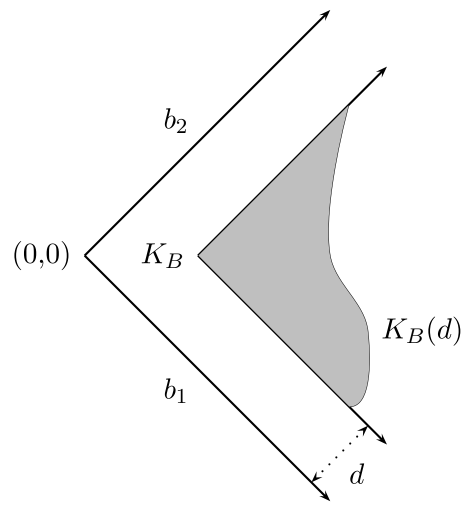

Figure 1: An example of K_B and $K_B(d)$.

Theorem 1.2.2 *Gomory [1969] If* $b \in K_B(l_{\max}(|\det(B)|-1))$ *where* $l_{\max}$ *is the (Euclidean) length of the longest non-basic column, then* $x_B^* = B^{-1}b - B^{-1}Nx_N^* \geq 0$ *for every optimal solution* x_N^* *to (P9).*

Proof

$$\begin{array}{rcll} \|Nx_N^*\| & = & \|\sum_{i=1}^n N_i x_{m+i}^*\| & \\ & \leq & \sum_{i=1}^n \|N_i\| |x_{m+i}^*| & \\ & = & \sum_{i=1}^n \|N_i\| x_{m+i}^* & \text{since } x_{m+i}^* \geq 0 \\ & \leq & l_{\max} \sum_{i=1}^n x_{m+i}^* & \text{by definition of } l_{\max} \\ & \leq & l_{\max}(|\det(B)| - 1) & \text{by Lemma 1.2.1} \end{array}$$

Now observe that if $b \in K_B(l_{\max}(|\det(B)| - 1))$, then $b - Nx_N^* \in K_B$ and so $B^{-1}b - B^{-1}Nx_N^* \geq 0$. ∎

Observe that if the solution to the LP relaxation is degenerate, then x_B lies on the boundary of the cone and the condition given in Theorem 1.2.2 cannot be satisfied, unless $|\det(B)| = 1$. It has been shown by Balas that when the variables in (P1) are binary, then the condition is never satisfied.

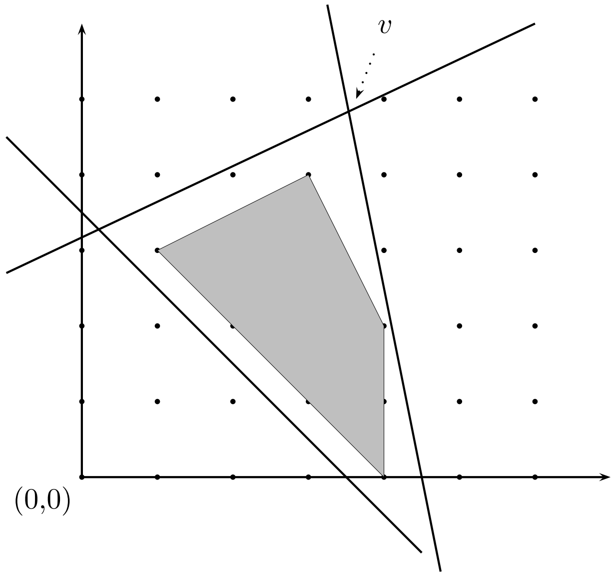

Figure 2: An example of K_B and $K_B(d)$.

1.3 Geometry

Let $\mathcal{N}$ denote the set of non-zero columns in (P9), that is

$$\mathcal{N} = \{\overline{g}_j \mid \overline{g}_j \neq 0 \text{ for } j = 1, \ldots, n\}$$

and let $n' = |\mathcal{N}|$. We must have that $n' \leq n$. The set $\{\overline{g}_1, \ldots, \overline{g}_n\}$ in general may contain zero columns and duplicate columns and it is clear that a zero column serves no purpose in solving (P9). If $\overline{g}_r = \overline{g}_s$ for $r \neq s$, then both of these columns are not necessary and it is desirable to only keep the one with smaller reduced cost. The set $\mathcal{N}$ consists of distinct, non-zero columns.

Let us introduce the variable $t(g)$ corresponding to $g \in \mathcal{N}$ and let T be the n'-vector whose components are $t(g)$. Observe that

$$t(g) = \sum_{\{j \mid \overline{g}_j = g\}} x_{m+j}$$

1.4 Numerical example

We now consider a simple numerical example from Appendix 1 of Gomory's original paper

$$\begin{array}{llllllllllll}
\max & 2x_1 & + & x_2 & + & x_3 & + & 3x_4 & + & x_5 & & \\
\text{s.t.} & & & 2x_2 & + & x_3 & + & 4x_4 & + & 2x_5 & \leq & 41 \\
& 3x_1 & - & 4x_2 & + & 4x_3 & + & x_4 & - & x_5 & \leq & 47 \\
& x_i & \geq & 0 & i = 1, \ldots, 5 & & & & & & & \\
& x_i & \text{integer} & & i = 1, \ldots, 5 & & & & & & &
\end{array}$$

If we solve the linear programming relaxation, then the basis consisting of the first two columns

$$B = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$$

is optimal. Observe that

$$N = \begin{bmatrix} 1 & 4 & 2 & 1 & 0 \\ 4 & 1 & -1 & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned}
B^{-1}b &= \begin{bmatrix} 4/6 & 2/6 \\ 3/6 & 0 \end{bmatrix} \begin{bmatrix} 41 \\ 47 \end{bmatrix} \\
&= \begin{bmatrix} 43 \\ 123/6^{1} \end{bmatrix} \\
B^{-1}N &= \begin{bmatrix} 4/6 & 2/6 \\ 3/6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 & 1 & 0 \\ 4 & 1 & -1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 3 & 1 & 4/6 & 2/6 \\ 3/6 & 2 & 1 & 3/6 & 0 \end{bmatrix}
\end{aligned}$$

As expected, we have that the reduced costs

$$\bar{c} = c_B B^{-1} N - c_N = \begin{bmatrix} 21/6 & 5 & 2 & 11/6 & 4/6 \end{bmatrix}$$

[1] There is a typo in Appendix 4 of [35].

are non-negative. Now problem (P8) is

$$\begin{array}{lccccccccccc}
\max & \frac{21}{6}x_3 & + & 5x_4 & + & 2x_5 & + & \frac{11}{6}x_6 & + & \frac{4}{6}x_7 & & \\
\text{s.t.} & 2x_3 & + & 3x_4 & + & x_5 & + & \frac{4}{6}x_6 & + & \frac{2}{6}x_7 & \equiv & 43 \bmod 1 \\
 & \frac{3}{6}x_3 & + & 2x_4 & + & x_5 & + & \frac{3}{6}x_6 & + & 0x_7 & \equiv & \frac{123}{6} \bmod 1 \\
 & x_{2+i} & \geq & 0 & & i = 1, \ldots, 5 \\
 & x_{2+i} & \text{integer} & & & i = 1, \ldots, 5
\end{array}$$

and problem (P9) is

$$\begin{array}{lccccccccccc}
\max & \frac{21}{6}x_3 & + & 5x_4 & + & 2x_5 & + & \frac{11}{6}x_6 & + & \frac{4}{6}x_7 & & \\
\text{s.t.} & 0x_3 & + & 0x_4 & + & 0x_5 & + & \frac{4}{6}x_6 & + & \frac{2}{6}x_7 & \equiv & 0 \bmod 1 \\
 & \frac{3}{6}x_3 & + & 0x_4 & + & 0x_5 & + & \frac{3}{6}x_6 & + & 0x_7 & \equiv & \frac{3}{6} \bmod 1 \\
 & x_{2+i} & \geq & 0 & & i = 1, \ldots, 5 \\
 & x_{2+i} & \text{integer} & & & i = 1, \ldots, 5
\end{array}$$

The columns in the group minimization problem are

$$g_1 = \begin{bmatrix} 0 \\ \frac{3}{6} \end{bmatrix}, \; g_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; g_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; g_4 = \begin{bmatrix} \frac{4}{6} \\ \frac{3}{6} \end{bmatrix}, \; g_5 = \begin{bmatrix} \frac{2}{6} \\ 0 \end{bmatrix}$$

Let G denote the set of vectors in $\Re^2$ generated by $g_1, \ldots, g_5$ under addition modulo 1. It is not difficult to see that G is an Abelian group. In fact, considering multiples of g_4, we get

k	0	1	2	3	4	5
kg_4	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 4/6 \\ 3/6 \end{bmatrix}$	$\begin{bmatrix} 2/6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3/6 \end{bmatrix}$	$\begin{bmatrix} 4/6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2/6 \\ 3/6 \end{bmatrix}$

and so G is cyclic of order 6 with g_4 as a generator. Since $\phi(6) = 2$ where ϕ is the Euler phi function, there is another generator of G which turns out to be

$$\begin{bmatrix} 2/6 \\ 3/6 \end{bmatrix}$$

although it is not present as a column in the problem.

For a general integer program, the group generated by the non-basic columns transformed by B^{-1} under addition modulo 1 will always be a finite Abelian group, although not necessarily cyclic like the example above.

Let $M(I)$ denote the set of all integer m-vectors and $M(B)$ denote the $\aleph$-module generated by the columns of the basis matrix B. If f denotes the homomorphism from $M(I)$ onto $G = M(I)/M(B)$, then the previous discussion can be rewritten

$$f(Bx_b) + f(Nx_N) = f(b)$$

and by the integrality of x_B,

$$f(Nx_N) = f(b).$$

The module $M(B, N)$ is isomorphic to the module $M(I, B^{-1}N)$ by the mapping induced by B^{-1}.

The order will always be equal to $|\det(B)|$, as long as the original constraint matrix contains the $m \times m$ identity matrix. Otherwise, we can only say that the order of the group will be a divisor of $|\det(B)|$.

Now we consider Theorem 1.2.2 for this problem. Observe that

$$\begin{aligned} l_{\max} &= \max\left\{ \left\| \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\|, \left\| \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\|, \left\| \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\|, \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|, \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| \right\} \\ &= \max\left\{ \sqrt{17}, \sqrt{17}, \sqrt{5}, 1, 1 \right\} \\ &= \sqrt{17} \end{aligned}$$

and so $l_{\max}(|\det(B)| - 1) = 5\sqrt{17}$.

1.5 Solution by Dynamic Programming

In [35], the group minimization problem is cast as a dynamic programming problem as follows. For any set $S \subseteq N$ and $h \in G$, $\phi(S, h)$ is defined to be

$$\begin{aligned} \phi(S,h) \quad = \quad \min \quad & \textstyle\sum_{g\in S} c(g)t(g) \\ \text{s.t.} \quad & \textstyle\sum_{g\in S} t(g)g = h \\ & t(g) \geq 0 \\ & t(g) \quad \text{integer} \end{aligned}$$

This is the same as the original group minimization problem except that only the subset S of columns is allowed and the right-hand size is changed to $h \in G$. For the recursion, we must decide for each $g' \in S$ whether to use the column ($t(g') \geq 1$) or not ($t(g') = 0$). The choice is dictated by whichever choice results in a lower objective value and so we have

$$\phi(S,h) = \min_{g'} \left\{ \phi(S - g', h), c(g') + \phi(S, h - g') \right\}$$

Then the optimal objective value for the problem with right-hand side g can be determined by evaluating $\phi(N, g)$. By maintaining appropriate bookkeeping during the recursion, the optimal solution can be determined.

In order to solve the group minimization problem by dynamic programming, it suffices to consider each group element one by one. So we define

$$\begin{array}{rcl} \phi'(k,h) & = & \min \quad \sum_{i=1}^{k} c(g_i)t(g_i) \\ & & \text{s.t.} \quad \sum_{i=1}^{k} t(g_i)g_i = h \\ & & \quad t(g_i) \quad \geq \quad 0 \\ & & \quad t(g_i) \quad \text{integer} \end{array}$$

$\phi'(k, h)$ is the optimal objective value for the group minimization problem using the first k group elements and with right-hand side h. The optimal objective value for the problem with right-hand side g is then $\phi'(n, g)$. To make the recursion work, we set $\phi'(0, h) = M$ where M represents an arbitrarily large value. Since B is assumed to have been an optimal basis, we have that $c(g_i) \geq 0$ for all i and hence, $\phi'(k, 0) = 0$ for all k.

The difference between $\phi'(k, h)$ and $\phi'(k - 1, h)$ is that we are allowed to use the group element g_k in the former.

$$\phi'(k,h) = \min \left\{ \phi'(k-1,h), c(g_k) + \phi'(k, h - g_k) \right\}$$

Observe that in the second term, we use $\phi'(k, h - g_k)$ instead of $\phi'(k - 1, h - g_k)$. This allows for the element g_k to be used more than once.

Unfortunately, we cannot directly apply this framework to most problems and in fact, we will quickly run into difficulty with the earlier numerical example. The difficulty is that not every non-basic column generates the entire group, and so the recursive procedure gets "stuck." When computing $\phi'(k, h)$ for some $h \in G \setminus < g_k >$, then $\phi'(k, h - g_k)$ is not available.

In the numerical example, if we drop the columns corresponding to the identity element and rearrange and rename the columns, we get

$$\begin{array}{lccccccc}
\max & \frac{11}{6}x_1 & + & \frac{4}{6}x_2 & + & \frac{21}{6}x_3 & & \\
\text{s.t.} & \frac{4}{6}x_1 & + & \frac{2}{6}x_2 & + & 0x_3 & \equiv & 0 \bmod 1 \\
 & \frac{3}{6}x_1 & + & 0x_2 & + & \frac{3}{6}x_3 & \equiv & \frac{3}{6} \bmod 1 \\
 & x_i & \geq & 0 & i = 1,2,3 & & & \\
 & x_i & \text{integer} & & i = 1,2,3 & & &
\end{array}$$

From the following table, observe that the first column generates the entire group, the second column generates a subgroup of order 3 and the third column generates a subgroup of order 2.

g	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 4/6 \\ 3/6 \end{bmatrix}$	$\begin{bmatrix} 2/6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3/6 \end{bmatrix}$	$\begin{bmatrix} 4/6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2/6 \\ 3/6 \end{bmatrix}$
ig_1	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
jg_2	$j = 0,3$	$-$	$j = 1,4$	$-$	$j = 2,5$	$-$
kg_3	$k = 0,2,4$	$-$	$-$	$k = 1,3,5$	$-$	$-$

For $k = 1$, we have

$$\begin{array}{rcccccl}
\phi'(1, g_1) & = & c(g_1) + \phi'(1, 0) & = & c(g_1) & = & \frac{11}{6} \\
\phi'(1, 2g_1) & = & c(g_1) + \phi'(1, 2g_1 - g_1) & = & 2c(g_1) & = & \frac{22}{6} \\
\phi'(1, 3g_1) & = & c(g_1) + \phi'(1, 3g_1 - g_1) & = & 3c(g_1) & = & \frac{33}{6} \\
\phi'(1, 4g_1) & = & c(g_1) + \phi'(1, 4g_1 - g_1) & = & 4c(g_1) & = & \frac{44}{6} \\
\phi'(1, 5g_1) & = & c(g_1) + \phi'(1, 5g_1 - g_1) & = & 5c(g_1) & = & \frac{55}{6}
\end{array}$$

The first row of the table below shows the elements of the group. The next two rows follow by definition and we have just derived the fourth and fifth rows. In the sequel, we derive the remainder of the table.

g	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 4/6 \\ 3/6 \end{bmatrix}$	$\begin{bmatrix} 2/6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3/6 \end{bmatrix}$	$\begin{bmatrix} 4/6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2/6 \\ 3/6 \end{bmatrix}$
$\phi'(0,g)$	M	M	M	M	M	M
$\phi'(1,g)$	0	$\frac{11}{6}$	$\frac{22}{6}$	$\frac{33}{6}$	$\frac{44}{6}$	$\frac{55}{6}$
$\phi'(2,g)$	0	$\frac{11}{6}$	$\frac{4}{6}$	$\frac{15}{6}$	$\frac{8}{6}$	$\frac{19}{6}$
$\phi'(3,g)$	0	$\frac{11}{6}$	$\frac{4}{6}$	$\frac{15}{6}$	$\frac{8}{6}$	$\frac{19}{6}$

For $k = 2$, we have

$$\begin{aligned}
\phi'(2,g_2) &= \min\{\phi'(1,g_2), c(g_2) + \phi'(2,0)\} &&= \min\left\{\tfrac{22}{6}, \tfrac{4}{6} + 0\right\} &&= \tfrac{4}{6} \\
\phi'(2,2g_2) &= \min\{\phi'(1,2g_2), c(g_2) + \phi'(2,g_2)\} &&= \min\left\{\tfrac{44}{6}, \tfrac{4}{6} + \tfrac{4}{6}\right\} &&= \tfrac{8}{6} \\
\phi'(2,3g_2) &= \min\{\phi'(1,3g_2), c(g_2) + \phi'(2,2g_2)\} &&= \min\left\{0, \tfrac{4}{6} + \tfrac{19}{6}\right\} &&= 0
\end{aligned}$$

Now we are stuck. However, for $\phi'(2,g_1)$, the trick is to assume that $\phi'(2,g_1) = \phi'(1,g_1) = \frac{11}{6}$. This may be an overestimate of the true value. Proceeding,

$$\begin{aligned}
\phi'(2,g_1+g_2) &= \min\{\phi'(1,g_1+g_2), c(g_2) + \phi'(2,g_1)\} = \min\{\tfrac{33}{6}, \tfrac{15}{6}\} = \tfrac{15}{6} \\
\phi'(2,g_1+2g_2) &= \min\{\phi'(1,g_1+2g_2), c(g_2) + \phi'(2,g_1+g_2)\} = \min\{\tfrac{55}{6}, \tfrac{19}{6}\} = \tfrac{19}{6} \\
\phi'(2,g_1+3g_2) &= \min\{\phi'(1,g_1+3g_2), c(g_2) + \phi'(2,g_1+2g_2)\} = \min\{\tfrac{11}{6}, \tfrac{23}{6}\} = \tfrac{11}{6}
\end{aligned}$$

Now observe that $\phi'(2,g_1) = \phi'(2,g_1+3g_2)$. By a theorem of T. C. Hu, this justifies the earlier estimate and so in fact, $\phi'(2,g_1) = \frac{11}{6}$. Now for $\phi'(2,g_1+g_2)$, we assume that $\phi'(2,g_1+g_2) = \phi'(1,g_1+g_2) = \frac{33}{6}$. Proceeding,

$$\begin{aligned}
\phi'(2,g_1+2g_2) &= \min\{\phi'(1,g_1+2g_2), c(g_2) + \phi'(2,g_1+g_2)\} = \min\{\tfrac{55}{6}, \tfrac{37}{6}\} = \tfrac{37}{6} \\
\phi'(2,g_1+3g_2) &= \min\{\phi'(1,g_1+3g_2), c(g_2) + \phi'(2,g_1+2g_2)\} = \min\{\tfrac{11}{6}, \tfrac{41}{6}\} = \tfrac{11}{6} \\
\phi'(2,g_1+4g_2) &= \min\{\phi'(1,g_1+4g_2), c(g_2) + \phi'(2,g_1+3g_2)\} = \min\{\tfrac{33}{6}, \tfrac{15}{6}\} = \tfrac{15}{6}
\end{aligned}$$

Now this does not agree with our earlier estimate, but now we estimate that $\phi'(2,g_1+g_2) = \frac{15}{6}$. Proceeding,

$$\begin{aligned}
\phi'(2,g_1+2g_2) &= \min\{\phi'(1,g_1+2g_2), c(g_2) + \phi'(2,g_1+g_2)\} = \min\{\tfrac{55}{6}, \tfrac{19}{6}\} = \tfrac{37}{6} \\
\phi'(2,g_1+3g_2) &= \min\{\phi'(1,g_1+3g_2), c(g_2) + \phi'(2,g_1+2g_2)\} = \min\{\tfrac{11}{6}, \tfrac{23}{6}\} = \tfrac{11}{6} \\
\phi'(2,g_1+4g_2) &= \min\{\phi'(1,g_1+4g_2), c(g_2) + \phi'(2,g_1+3g_2)\} = \min\{\tfrac{33}{6}, \tfrac{15}{6}\} = \tfrac{15}{6}
\end{aligned}$$

Now we have $\phi'(2, g_1+4g_2)$ agreeing with our estimate for $\phi'(2, g_1+g_2)$ and so $\phi'(2, g_1+g_2) = \frac{15}{6}$. For $\phi'(2, g_1+2g_2)$, we assume that $\phi'(2, g_1+2g_2) = \phi'(1, g_1+2g_2) = \frac{55}{6}$. Proceeding,

$$\begin{aligned}
\phi'(2, g_1+3g_2) &= \min\{\phi'(1, g_1+3g_2), c(g_2)+\phi'(2, g_1+2g_2)\} = \min\{\tfrac{11}{6}, \tfrac{59}{6}\} = \tfrac{11}{6}\\
\phi'(2, g_1+4g_2) &= \min\{\phi'(1, g_1+4g_2), c(g_2)+\phi'(2, g_1+3g_2)\} = \min\{\tfrac{33}{6}, \tfrac{63}{6}\} = \tfrac{33}{6}\\
\phi'(2, g_1+5g_2) &= \min\{\phi'(1, g_1+5g_2), c(g_2)+\phi'(2, g_1+4g_2)\} = \min\{\tfrac{55}{6}, \tfrac{37}{6}\} = \tfrac{37}{6}
\end{aligned}$$

Again, this does not agree with our earlier estimate, so we now must estimate that $\phi'(2, g_1+2g_2) = \frac{37}{6}$. Proceeding,

$$\begin{aligned}
\phi'(2, g_1+3g_2) &= \min\{\phi'(1, g_1+3g_2), c(g_2)+\phi'(2, g_1+2g_2)\} = \min\{\tfrac{11}{6}, \tfrac{41}{6}\} = \tfrac{11}{6}\\
\phi'(2, g_1+4g_2) &= \min\{\phi'(1, g_1+4g_2), c(g_2)+\phi'(2, g_1+3g_2)\} = \min\{\tfrac{33}{6}, \tfrac{15}{6}\} = \tfrac{15}{6}\\
\phi'(2, g_1+5g_2) &= \min\{\phi'(1, g_1+5g_2), c(g_2)+\phi'(2, g_1+4g_2)\} = \min\{\tfrac{55}{6}, \tfrac{19}{6}\} = \tfrac{19}{6}
\end{aligned}$$

Again, this does not agree with the revised estimate, so we now must estimate that $\phi'(2, g_1+2g_2) = \frac{19}{6}$. Proceeding,

$$\begin{aligned}
\phi'(2, g_1+3g_2) &= \min\{\phi'(1, g_1+3g_2), c(g_2)+\phi'(2, g_1+2g_2)\} = \min\{\tfrac{11}{6}, \tfrac{23}{6}\} = \tfrac{11}{6}\\
\phi'(2, g_1+4g_2) &= \min\{\phi'(1, g_1+4g_2), c(g_2)+\phi'(2, g_1+3g_2)\} = \min\{\tfrac{33}{6}, \tfrac{15}{6}\} = \tfrac{15}{6}\\
\phi'(2, g_1+5g_2) &= \min\{\phi'(1, g_1+5g_2), c(g_2)+\phi'(2, g_1+4g_2)\} = \min\{\tfrac{55}{6}, \tfrac{19}{6}\} = \tfrac{19}{6}
\end{aligned}$$

We now have agreement and so $\phi'(2, g_1+2g_2) = \frac{19}{6}$. For $k = 3$, we have

$$\begin{aligned}
\phi'(3, g_3) &= \min\{\phi'(2, g_3), c(g_3)+\phi'(3, 0)\} &= \min\{\tfrac{15}{6}, \tfrac{21}{6}+0\} &= \tfrac{15}{6}\\
\phi'(3, 2g_3) &= \min\{\phi'(2, 2g_3), c(g_3)+\phi'(3, g_3)\} &= \min\{0, \tfrac{21}{6}+\tfrac{15}{6}\} &= 0
\end{aligned}$$

We assume that $\phi'(3, g_1) = \phi'(2, g_1) = \frac{11}{6}$. Proceeding,

$$\begin{aligned}
\phi'(3, g_1+g_3) &= \min\{\phi'(2, g_1+g_3), c(g_3)+\phi'(3, g_1)\} = \min\{\tfrac{8}{6}, \tfrac{32}{6}\} = \tfrac{8}{6}\\
\phi'(3, g_1+2g_3) &= \min\{\phi'(2, g_1+2g_3), c(g_3)+\phi'(3, g_1+g_3)\} = \min\{\tfrac{11}{6}, \tfrac{29}{6}\} = \tfrac{11}{6}
\end{aligned}$$

This agrees with the estimate and so we have that $\phi'(3, g_1) = \frac{11}{6}$. We now assume that $\phi'(3, g_1+g_3) = \phi'(2, g_1+g_3) = \frac{11}{6}$. Proceeding,

$$\begin{aligned}
\phi'(3, g_1+2g_3) &= \min\{\phi'(2, g_1+2g_3), c(g_3)+\phi'(3, g_1+g_3)\} = \min\{\tfrac{11}{6}, \tfrac{32}{6}\} = \tfrac{11}{6}\\
\phi'(3, g_1+3g_3) &= \min\{\phi'(2, g_1+3g_3), c(g_3)+\phi'(3, g_1+2g_3)\} = \min\{\tfrac{8}{6}, \tfrac{32}{6}\} = \tfrac{8}{6}
\end{aligned}$$

This does not agree with our estimate and so we revise it to $\phi'(3, g_1 + g_3) = \frac{8}{6}$. Proceeding,

$$\begin{aligned}\phi'(3, g_1 + 2g_3) &= \min\{\phi'(2, g_1 + 2g_3), c(g_3) + \phi'(3, g_1 + g_3)\} = \min\{\tfrac{11}{6}, \tfrac{32}{6}\} = \tfrac{11}{6} \\ \phi'(3, g_1 + 3g_3) &= \min\{\phi'(2, g_1 + 3g_3), c(g_3) + \phi'(3, g_1 + 2g_3)\} = \min\{\tfrac{8}{6}, \tfrac{32}{6}\} = \tfrac{8}{6}\end{aligned}$$

This agrees with the estimate and so we have that $\phi'(3, g_1 + g_3) = \frac{8}{6}$. We assume that $\phi'(3, g_2) = \phi'(2, g_2) = \frac{4}{6}$. Proceeding,

$$\begin{aligned}\phi'(3, g_2 + g_3) &= \min\{\phi'(2, g_2 + g_3), c(g_3) + \phi'(3, g_2)\} = \min\{\tfrac{19}{6}, \tfrac{25}{6}\} = \tfrac{19}{6} \\ \phi'(3, g_2 + 2g_3) &= \min\{\phi'(2, g_2 + 2g_3), c(g_3) + \phi'(3, g_2 + g_3)\} = \min\{\tfrac{4}{6}, \tfrac{40}{6}\} = \tfrac{4}{6}\end{aligned}$$

This agrees with the estimate and so we have that $\phi'(3, g_2) = \frac{4}{6}$. We now assume that $\phi'(3, g_2 + g_3) = \phi'(2, g_2 + g_3) = \frac{19}{6}$. Proceeding,

$$\begin{aligned}\phi'(3, g_2 + 2g_3) &= \min\{\phi'(2, g_2 + 2g_3), c(g_3) + \phi'(3, g_2)\} = \min\{\tfrac{4}{6}, \tfrac{40}{6}\} = \tfrac{4}{6} \\ \phi'(3, g_2 + 3g_3) &= \min\{\phi'(2, g_2 + 3g_3), c(g_3) + \phi'(3, g_2 + g_3)\} = \min\{\tfrac{19}{6}, \tfrac{25}{6}\} = \tfrac{19}{6}\end{aligned}$$

This agrees with the estimate and so we have that $\phi'(3, g_2 + g_3) = \frac{19}{6}$. We are now ready to solve the integer program. Observe that $\phi'(3, 3g_1) = \frac{15}{6}$ and so $x_2 = 1$. Now

$$\phi'(3, 3g_1 - g_2) = \phi'(3, g_1) = \frac{11}{6}$$

and so $x_1 = 1$. So we have that $x_1 = x_2 = 1$. In the original variables, this corresponds to the solution

$$x_3 = x_4 = x_5 = 0, x_6 = x_7 = 1$$

Now solving for the basic variables, we get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B^{-1}b - B^{-1}Nx_N = \begin{bmatrix} 43 \\ 123/6 \end{bmatrix} - \begin{bmatrix} 1 \\ 3/6 \end{bmatrix} = \begin{bmatrix} 42 \\ 20 \end{bmatrix}$$

which are non-negative. Hence, we have solved the integer program. Observe that it is relatively simple for us to solve the same problem with a different RHS which is one advantage of the dynamic programming approach.

1.6 Shortest path formulation

Consider the directed graph $H(\mathcal{G}, \mathcal{N}, \overline{c}) = (N, A)$ where

$$\begin{aligned} N &= \text{elements of the group generated by } \{\overline{g}_1, \ldots, \overline{g}_n\} \\ A &= \{(r, s) \mid s - r \equiv \overline{g}_j \bmod 1 \text{ for some } j\}\end{aligned}$$

The graph has a node for each group element and an arc (r, s) from node r to node s whenever $s - r$ is equal to a column in (P9) modulo 1, say $\overline{g}_j$. The traversal of the arc (r, s) corresponds to incrementing x_{m+j} and so we naturally assign the cost of arc (r, s) to be the value $\overline{c}_j$.

This construction is best illustrated by an example. Continuing the numerical example, we first have six nodes with one node for each element of the group. Since the group in the example is cyclic and generated by g_4, the group element kg_4 is labeled by k in Figure 3. We first add the arcs corresponding to g_4. Since the reduced cost of g_4 is 11/6, the arcs are labeled with 11/6.

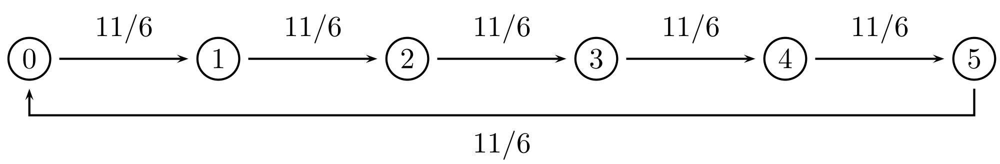

Figure 3: Network with arcs corresponding to g_4 added.

Now we consider the group element

$$g_1 = \begin{bmatrix} 0 \\ 3/6 \end{bmatrix} = 3g_4$$

which has a reduced cost of 21/6. Since

$$0g_4 + g_1 = 0g_4 + 3g_4 = 3g_4,$$

we add an arc from node 0 to node 3 with a cost of 21/6. Since

$$1g_4 + g_1 = 1g_4 + 3g_4 = 4g_4,$$

we add an arc from node 1 to node 4 with a cost of 21/6 and similarly for the remaining 4 arcs. The resulting network is shown in Figure 4.

Now we consider the group element

$$g_5 = \begin{bmatrix} 2/6 \\ 0 \end{bmatrix} = 2g_4$$

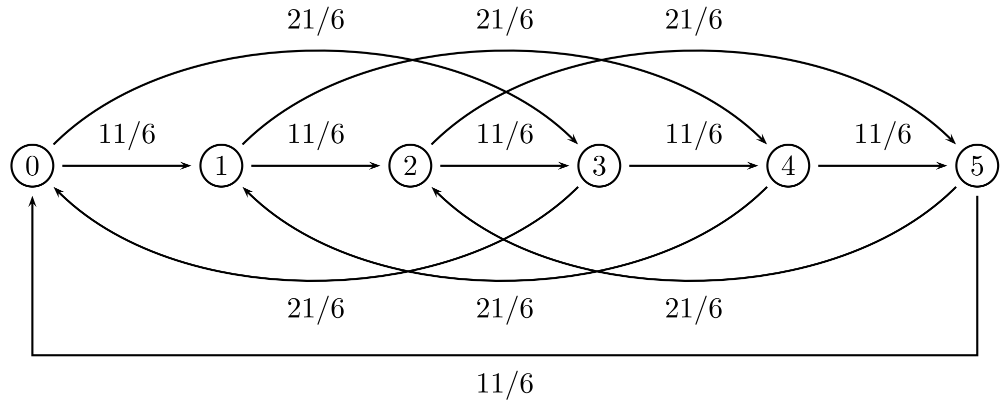

Figure 4: Network with arcs corresponding to g_1 and g_4 added.

which has a reduced cost of 4/6. Adding six arcs of cost 4/6 appropriately to the network in Figure 4, we obtain the network shown in Figure 5.

Now observe that g_2 and g_3 are both the identity element and have non-negative reduced costs. Hence, adding them to the network would be nothing more than just adding self-loops at each node, which serves no purpose.

Now to solve the Group Minimization Problem, it suffices to compute the shortest path from the node representing the identity element of $\mathcal{G}$ to the node representing the right-hand side of the problem. For the numerical example, we want the shortest path from 0 to 3 in Figure 5. It can be computed using Dijkstra's algorithm that the paths $(0, 2, 3)$ and $(0, 1, 3)$ are both shortest paths from 0 to 3 with cost 15/6. Both paths correspond to the solution

$$x_N^* = (x_3^*, x_4^*, x_5^*, x_6^*, x_7^*) = (0, 0, 0, 1, 1)$$

which is the same solution that we found via dynamic programming.

1.7 Basic properties

The group problem on the unit interval can be considered when the group is finite or infinite, and with or without continuous variables. There are only a handful of known constructions which give facets of the infinite group polyhedron. When the group is finite, the extreme inequalities are completely known. Let $T(U, u_0)$ denote the set of functions t which satisfy

$$\sum_{u \in U} ut(u) = u_0$$

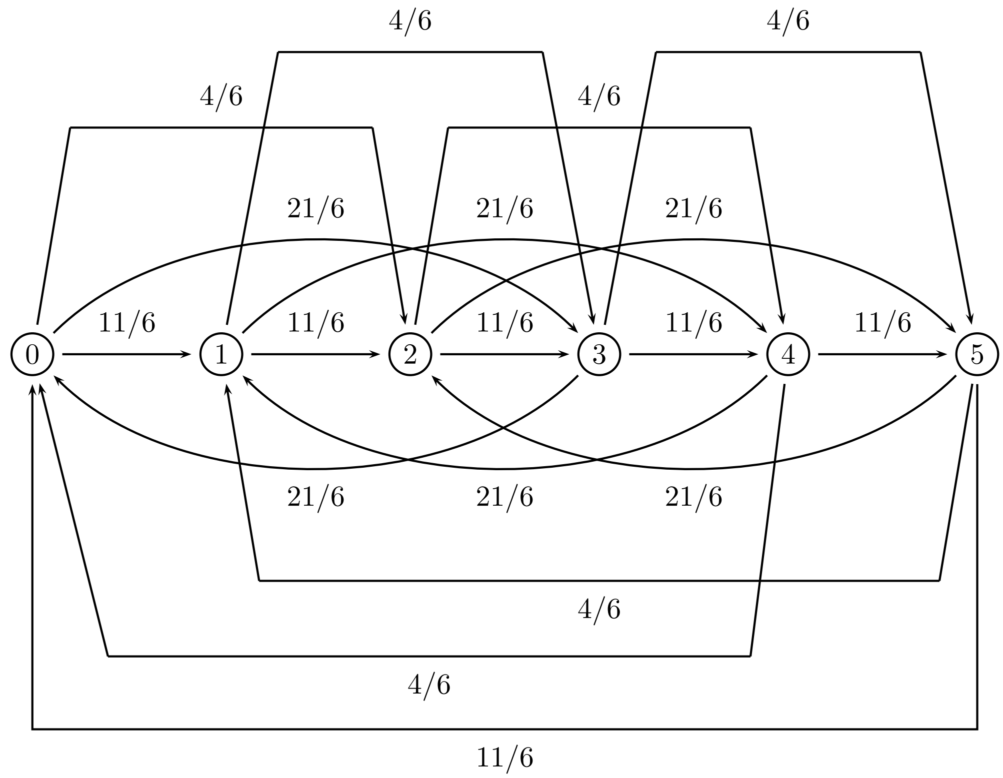

Figure 5: Network with arcs corresponding to g_1, g_4 and g_5 added.

where the operations are addition and multiplication modulo 1. Here, U is a subset of $[0, 1]$. Let $T_-^+(U, u_0)$ denote the set of solutions $t' = (t, s^+, s^-)$ which satisfy

$$\sum_{u \in U} ut(u) + \widehat{s^+} - \widehat{s^-} = u_0$$

where the operations are again addition and multiplication modulo 1.

Definition For $P(U, u_0)$, a *valid inequality* is a function $\pi : U \to \Re$ such that

$$\pi(u) \geq 0 \text{ for all } u \in I,\ \pi(0) = 0$$

and

$$\sum_{u \in U} \pi(u)t(u) \geq 1 \text{ for all } t \in T(U, u_0).$$

Definition For $P_-^+(U, u_0)$, a *valid inequality* is a function $\pi' = (\pi, \pi^+, \pi^-)$ where π is as above and $\pi^+, \pi^- \in \Re$ such that

$$\sum_{u \in U} \pi(u)t(u) + \pi^+ s^+ + \pi^- s^- \geq 1 \text{ for all } t' \in T_-^+(U, u_0).$$

Inequalities vary in their usefulness and a desirable property of a valid inequality is minimality.

Definition A valid inequality π for $P(U, u_0)$ is a *minimal valid inequality* if there does not exist a valid inequality ρ for $P(U, u_0)$ with $\rho(u) \leq \pi(u)$ for all $u \in U$ with $\rho(u) < \pi(u)$ for at least one $u \in U$.

In order to show a valid inequality is minimal, the definition cannot be applied directly. We will see later a theorem which gives a simple characterization of minimal valid inequalities. A property that is even more desirable than minimality is extremality.

Definition A valid inequality π for $P(U, u_0)$ is an *extreme valid inequality* if there does not exist valid inequalities ρ and σ for $P(U, u_0)$ such that $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$.

Theorem 1.1 of [36] says that the extreme valid inequalities are a subset of the (strictly larger) set of minimal valid inequalities.

Theorem 1.7.1 *The extreme valid inequalities are minimal valid inequalities.*

Theorem 1.2 of [36] says that the minimal valid inequalities are a subset of the (strictly larger) set of subadditive valid inequalities.

Theorem 1.7.2 *The minimal valid inequalities are subadditive valid inequalities.*

The set of valid inequalities is a convex set which contains the strictly smaller convex subset of subadditive valid inequalities. The extreme points of the set of subadditive valid inequalities contain all the extreme valid inequalities. Theorem 1.3[2] from [36] allows us to actually extract the extreme valid inequalities:

Theorem 1.7.3 *If π (or π') is extreme among the subadditive valid inequalities for $P(U, u_0)$ (or $P_-^+(U, u_0)$), that is, π (or π') is not the midpoint of any two different subadditive valid inequalities, and if π (or π') is also a minimal valid inequality, then it is an extreme valid inequality.*

[2]There is a typo on p. 33 in [36] and the theorem is mistakenly labeled as Theorem 1.1.

When this theorem is specialized to G_n, the cyclic group on n elements, we get Theorem 2.2 of [36].

Theorem 1.7.4 *The extreme valid inequalities for $P(G_n, u_0), u_0 \in G_n$, are the extreme points of the solutions to*

$$\begin{aligned} \pi(g_i) &\geq 0, \pi(0) = 0 \\ \pi(g_i) + \pi(g_j) &\geq \pi(g_i + g_j) \\ \pi(u_0) &\geq 1 \end{aligned}$$

which satisfy the additional equations

$$\pi(g_i) + \pi(u_0 - g_i) = 1, g_i \in G_n$$

1.8 2-row theory

Previous work in this area has focused on essentially applying integrality arguments to a linear combination of the rows of $Ax = b$. Currently, there is substantial interest in applying integrality arguments to two rows simultaneously in the hopes of generating cutting planes that cannot be obtained from arguments involving a single row.

The initial results in this area were obtained by Dey and Richard [26], Andersen, Louveaux, Weismantel and Wolsey [1], Borozan and Cornuéjols [14], Cornuéjols and Margot [21], and Dey and Wolsey [27]. Computational results were obtained by Espinoza [30].

Suppose we have a mixed-integer programming problem of the form:

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & x_j \geq 0 \text{ for } j = 1, \ldots, n \\ & x_j \in \mathbb{Z} \text{ for } j = 1, \ldots, p \end{aligned}$$

for $p \leq n$, where A is a rational $m \times n$ matrix, c is a rational $1 \times n$-vector and b is a rational $m \times 1$-vector. Without loss of generality, A is assumed to have full row rank.

If B and J are the basic and non-basic variables respectively of a solution of the LP relaxation, then the solution can be represented as

$$x_i = f_i + \sum_{j \in J} r^j x_j \text{ for } i \in B.$$

and the system can be rewritten as

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & x_i = f_i + \sum_{j \in J} r^j x_j \text{ for } i \in B \cap \{1, \ldots, p\} \\ & x_i = f_i + \sum_{j \in J} r^j x_j \text{ for } i \in B \cap \{p+1, \ldots, n\} \\ & x_j \in \mathbb{Z} \text{ for } j \in \{1, \ldots, p\} \cap B \\ & x_j \in \mathbb{Z} \text{ for } j \in \{1, \ldots, p\} \cap J \\ & x_j \geq 0 \text{ for } j \in B \\ & x_j \geq 0 \text{ for } j \in J \end{aligned}$$

By feasibility, we have $f_i \geq 0$ for all $i = 1, \ldots, n$. If for all $i \in B \cap \{1, \ldots, p\}$, we have that $f_i \in \mathbb{Z}$, then the basic solution is an optimal solution of the mixed-integer linear program. Otherwise, we will want to generate one or more cutting planes that are violated by this solution, but are satisfied by all feasible solutions of the mixed-integer LP.

Recall that in Gomory's corner polyhedron, we relax the non-negativity constraints on the x_i for $i \in B$ and so the constraints

$$x_j \geq 0 \text{ for } j \in B$$

get dropped. However, the constraints

$$x_i = f_i + \sum_{j \in J} r^j x_j \text{ for } i \in B \cap \{p+1, \ldots, n\}$$

can also be dropped since these variables are not otherwise constrained.

If we further relax the integrality constraints on the non-basic variables x_j, i.e. we drop the constraints

$$x_j \in \mathbb{Z} \text{ for } j \in \{1, \ldots, p\} \cap J,$$

then our problem becomes

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & x_i = f_i + \sum_{j \in J} r^j x_j \text{ for } i \in B \cap \{1, \ldots, p\} \\ & x_j \in \mathbb{Z} \text{ for } j \in \{1, \ldots, p\} \cap B \\ & x_j \geq 0 \text{ for } j \in J. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} x &= f + \sum_{j=1}^{k} r^j s_j \\ x &\in \mathbb{Z}_q \\ s &\geq 0 \end{aligned}$$

where s is now the set of non-basic variables, and $q = |\{1, \ldots, p\} \cap B|$. $R_f(r^1, \ldots, r^k)$ is used to denote the convex hull of all vectors s satisfying the above constraints, where $f, r^1, \ldots, r^k$ are all $q \times 1$ rational vectors.

A further relaxation first suggested by Gomory and Johnson is to relax the finite dimensional space of variables to an infinite dimensional space. Instead of only considering the particular $r^1, \ldots, r^k$, consider any q-dimensional rational vector r. The problem then becomes

$$\begin{aligned} x &= f + \sum_{r \in \mathbb{Q}^q} r s_r \\ x &\in \mathbb{Z}_q \\ s &\geq 0 \text{ with finite support} \end{aligned}$$

The convex hull of all vectors $s \geq 0$ satisfying the above constraints is denoted by R_f. Recall that the vector $s \geq 0$ has finite support if $|\{r : s_r > 0\}| < \infty$. In order to avoid issues such as convergence, only vectors s with finite support are considered. By setting $s_r = 0$ for $r \in \mathbb{Q}^q \backslash \{r^1, \ldots, r^k\}$, $R_f(r^1, \ldots, r^k)$ is observed to be a face of R_f. R_f is simpler than $R_f(r^1, \ldots, r^k)$, but is not a closed set. By a theorem of Meyer [44], $R_f(r^1, \ldots, r^k)$ is a polyhedral set. As an aside, the model where the integer variables are required to be non-negative has been studied and results for this model have been obtained by Fukasawa and Günlük [32].

The $q = 2$ case where just two rows of the tableau are simultaneously considered was studied by Andersen, Louveaux, Weismantel and Wolsey [1] where they showed that all the non-trivial facets of $R_f(r^1, \ldots, r^k)$ are the intersections cuts of Balas [3]. In our own results in the sequel, we only consider the $q = 2$ case.

We will assume that $f \in \mathbb{Q}^q \backslash \mathbb{Z}^q$, so the basic solution $s = 0$ is not a feasible solution. A linear inequality $\alpha s \geq \beta$ is *valid* for R_f (respectively $R_f(r^1, \ldots, r^k)$) if it is satisfied by all the feasible solutions of R_f (respectively $R_f(r^1, \ldots, r^k)$). A valid inequality of the form $s_i \geq 0$ is considered *trivial*. Since $s = 0$ is not a feasible solution for R_f, we are interested in valid inequalities that cut it off and they are of the form

$$\sum \psi(r) s_r \geq 1$$

where $\psi : \mathbb{Q}^2 \to \mathbb{R} \cup \{+\infty\}$ and s has finite support. In general, ψ need not be finite or continuous. In the event that for some r we have $s_r = 0$ and $\psi(r) = +\infty$, then the product $\psi(r)s_r$ is defined to be 0. Observe that the restriction of a valid inequality for R_f to the space $r^1, \ldots, r^k$ results in a valid inequality for $R_f(r^1, \ldots, r^k)$.

Not all valid inequalities are equal, however. For example, a function ψ that is $+\infty$ everywhere is valid, but basically useless. A valid inequality $\sum \psi(r)s_r \geq 1$ is *minimal* if there does not exist another valid inequality $\sum \psi'(r)s_r \geq 1$ such that $\psi'(r) \leq \psi(r)$ for all $r \in \mathbb{Q}^2$ and $\psi'(r) < \psi(r)$ for at least one $r \in \mathbb{Q}^2$. In the event that $\psi(r) = +\infty$, then the convention is that $\psi'(r) < \psi(r)$ if and only if $\psi'(r) < \infty$.

Minimal inequalities are of interest because they are the (non-trivial) inequalities that characterize R_f. In [14], Borozan and Cornuéjols showed that for a minimal valid inequality $\sum \psi(r)s_r \geq 1$, ψ has a number of important properties.

Theorem 1.8.1 *If $\psi : \mathbb{Q}^q \to \mathbb{R} \cup \{+\infty\}$ is a minimal valid function, then ψ is zero at the origin, subadditive and positively homogeneous.*

Recall that ψ being positively homogeneous means that $\psi(\lambda r) = \lambda\psi(r)$ for any $r \in \mathbb{Q}^2$ and $\lambda \in \mathbb{Q}$ where $\lambda > 0$. The proofs of these properties are fairly straightforward. The basic idea is that if ψ is minimal and valid, a slightly different ψ' can be defined and then shown to be valid.

This is done by considering a feasible $(\overline{x}, \overline{s}) \in R_f$ and then defining a slightly different $(\overline{x}, \tilde{s})$. The ψ' and $\tilde{s} \geq 0$ are specially chosen so that

$$\sum_r \psi'(r)\overline{s}_r = \sum_r \psi(r)\tilde{s}_r$$

and

$$\overline{x} = f + \sum r\overline{s}_r = f + \sum r\tilde{s}_r$$

both hold. Then $(\overline{x}, \tilde{s})$ is a feasible point of R_f and by ψ being valid, we have

$$\sum \psi'(r)\overline{s}_r \geq 1$$

and hence ψ' is valid. But ψ is minimal and we get that $\psi(r) \leq \psi'(r)$ for an r which shows the desired property.

Now if ψ is a function that is valid but not necessarily minimal, then we know at least that it is non-negative everywhere by the following theorem.

Theorem 1.8.2 *If $\psi : \mathbb{Q}^q \to \mathbb{R} \cup \{+\infty\}$ is a valid function, then $\psi(r) \geq 0$ for all r.*

For minimal valid functions, we know that they are subadditive and positive homogeneous and from this, convexity immediately follows.

Theorem 1.8.3 *If $\psi : \mathbb{Q}^q \to \mathbb{R} \cup \{+\infty\}$ is a minimal valid function, then ψ is convex.*

Borozan and Cornuéjols also show that for general q, a minimal valid function ψ for R_f that is finite has at most 2^q pieces and that such a ψ can be extended to a continuous function of $\mathbb{R}^q$. Borozan and Cornuéjols found a very nice and simple characterization of validity in terms of lattice-points in the following theorem.

Theorem 1.8.4 *If ψ is a non-negative, positively homogeneous and subadditive function, then $\psi(x - f) \geq 1$ for all $x \in \mathbb{Z}^q$ is necessary and sufficient for ψ to be valid for R_f.*

We discuss the argument behind this theorem as it nicely illustrates and employs the properties of minimal valid functions. Suppose ψ is a non-negative, positively homogeneous and subadditive function. Suppose further that for all $x \in \mathbb{Z}^q$, we have $\psi(x-f) \geq 1$. If $(x,s) \in \mathbb{R}_f$, then

$$x = f + \sum r s_r$$

and by re-arranging and applying ψ to both sides, we have

$$\psi\left(\sum r s_r\right) = \psi(x-f).$$

So we get

$$\sum \psi(r) s_r = \sum \psi(r s_r) \geq \psi\left(\sum r s_r\right) = \psi(x-f) \geq 1$$

where the first equality follows from positive homogeneity and the second inequality follows from subadditivity. Since (x,s) was an arbitrary element of R_f, this shows that ψ is valid.

On the other hand, if there exists $\overline{x} \in \mathbb{Z}^q$ such that $\psi(\overline{x}-f) < 1$, then for

$$\overline{s}_r = \begin{cases} 1 & \text{if } r = \overline{x} - f \\ 0 & \text{otherwise} \end{cases}$$

we have $(\overline{x}, \overline{s}) \in \mathbb{R}_f$ and

$$\sum \psi(r) \overline{s}_r = \psi(\overline{x} - f) < 1$$

contradicting the validity of ψ.

Now suppose we have a function ψ that is minimal. The necessary and sufficient condition for validity leads naturally to the following definition. Define

$$B_\psi = \{x \in \mathbb{Q}^2 : \psi(x-f) \leq 1\}.$$

Since ψ is a convex function, B_ψ is a convex set (in $\mathbb{Q}^2$). In a sense, B_ψ is another representation or "view" of ψ and there is a close connection between them. When ψ is a minimal valid function of ψ, B_ψ has the very important property of being lattice-free which means that it does not contain an integral point in its interior. Integral points are allowed to exist on the boundary of B_ψ however.

If ψ is minimal valid and $\overline{x} \in \mathbb{Z}^q$, then we have that $\psi(\overline{x}-f) \geq 1$. If $\overline{x} \in B_\psi$, then by definition $\psi(\overline{x}-f) \leq 1$ and so we must have $\psi(\overline{x}-f) = 1$. If $\overline{x}$ were in the interior of B_ψ, then consider any point $\overline{\overline{x}}$ where

$$\overline{\overline{x}} \in \{f + \lambda(\overline{x} - f) \in B_\psi : \lambda > 1\}.$$

By positive homogeneity, it must be the case that $\psi(\overline{\overline{x}} - f) > 1$ contradicting $\overline{\overline{x}}$ belonging to B_ψ. This is the basic argument behind the first part of the following theorem:

Theorem 1.8.5 *If ψ is a minimal valid for R_f, then $cl(B_\psi)$ is a lattice-free convex set in $\mathbb{R}^q$. In addition, $f \in B_\psi$ and if $\psi(r) < +\infty$ for all $r \in \mathbb{Q}^q$, then f is in the interior of $cl(B_\psi)$.*

It is of course desirable to find functions ψ with the lowest possible coefficients and the following result shows that in terms of the B_ψ, larger lattice-free sets are better.

Theorem 1.8.6 *If $\psi, \psi' : \mathbb{Q}^q \to \mathbb{R} \cup \{+\infty\}$ are convex functions, then $\psi \leq \psi'$ if and only if $B_{\psi'} \subseteq B_\psi$.*

Naturally with these results, maximal lattice-free convex sets are of interest. In 1989, Lovász showed the following Minkowski-Weyl-style theorem concerning maximal lattice-free convex sets.

Theorem 1.8.7 *A maximal lattice-free convex subset of $\mathbb{R}^n$ is an irrational hyperplane or a full-dimensional polyhedron which is the sum of a polytope and a rational linear space.*

First recall that for a set S and a point $x \in S$, a vector r is called a *recession direction* of S if

$$\{x + \lambda r : \lambda \geq 0\} \subseteq S.$$

The *recession cone* of S is simply the set of all recession directions. Now given a maximal lattice-free convex set B, a corresponding function $\psi_B : \mathbb{Q}^q \to \mathbb{R}$ can be defined that is non-negative and positively homogeneous satisfying

$$B_{\psi_B} = B \cap \mathbb{Q}^q.$$

If $r \in \mathbb{Q}^q$ is in the recession cone of B, then $\psi_B(r)$ is defined to be zero. Otherwise, if $r \in \mathbb{Q}^q$ is not a recession direction, then if $\lambda > 0$ is such that $f + \lambda r$ is a boundary point of B, $\psi_B(r)$ is defined to be $1/\lambda$. Borozan and Cornuéjols show that this construction results in a minimal valid function for R_f.

Theorem 1.8.8 *If B is a full-dimensional maximal lattice-free convex subset of $\mathbb{R}^q$ and $f \in \mathbb{Q}^q$ is in the interior of B, then ψ_B is minimal valid for R_f with cl$(B_{\psi_B}) = B$.*

Observe that by construction, ψ_B necessarily satisfies

$$\psi_B(x - f) \geq 1 \text{ for all } x \in \mathbb{Z}^q$$

since B is lattice-free. By its definition, ψ_B is non-negative and positively homogeneous, and so to show validity, it suffices to show subaddivity by applying Theorem 1.8.4. This is a pretty straightforward case-analysis depending upon whether the points are in the recession cone or not. Showing that ψ_B is minimal is also not too difficult.

The interesting thing about this construction is that Borozan and Cornuéjols show that any minimal valid function ψ for R_f that is finite everywhere must arise from some B, where cl(B_ψ) is a maximal lattice-free convex set.

By Lovász's Theorem 1.8.7, cl(B_ψ) is a polyhedral set and by results due to Doignon [28], Bell [10], and Scarf [47], this polyhedron can have at most 2^q facets. The argument essentially just uses the pigeonhole principle. Each facet must have an integral point in its relative interior and if there were more than 2^q facets, then there exists distinct $x_1, x_2 \in \mathbb{Z}^q$ which are congruent modulo 2 and their midpoint is also integral and would be in the interior of the polyhedron. This contradicts its choice as being lattice-free. From this, it can be argued that ψ is piecewise-linear with no more than 2^q pieces. The theorem that Borozan and Cornuéjols showed is stated below.

Theorem 1.8.9 *If $f \in \mathbb{Q}^q \backslash \mathbb{Z}^q$ and ψ is a minimal valid function for R_f with $\psi(r) < \infty$ for all $r \in \mathbb{Q}^q$, then ψ is a non-negative, positively homogeneous and piecewise-linear convex function with at most 2^q pieces. ψ can also be extended from $\mathbb{Q}^q$ to $\mathbb{R}^q$ in a continuous fashion.*

Borozan and Cornuéjols also consider the difficult case where f lies on the boundary of cl(B_ψ). This case is difficult because when r points away from cl(B_ψ), we must define $\psi(r)$ to be $+\infty$. This is the *degenerate* case. In order to even define ψ in this case, we have to concern ourselves with each face of cl(B_ψ) that contains f.

In the case of $q = 2$, Cornuéjols and Margot showed by case analysis of all the possible two-dimensional maximal lattice-free convex sets and degeneracies occurring at edges and vertices that degenerate cases are not needed for $R_f(r_1, \ldots, r_k)$. They did this by showing that if ψ was a minimal valid function that was degenerate, another minimal ψ' that is non-degenerate could be constructed that is identical for $r^1, \ldots, r^k$. In [50], Zambelli later gave a short argument to show that this is true for general q.

CHAPTER II

NEW FACETS OF T-SPACE

In 2003, Gomory and Johnson gave two different three-slope T-space facet constructions, both of which shared a slope with the corresponding Gomory mixed-integer cut. In this chapter, we give a new three-slope facet which is independent of the GMIC. We also give a four-slope T-space facet construction, which to our knowledge, is the first four-slope construction.

Let G denote the interval $[0, 1)$ under addition mod 1. For each u in the Abelian group G, we assign a non-negative integer $t(u)$. If $\sum t(u)u = u_0$, then $\{t(u)\}$ is a *path* to u_0. To avoid issues about convergence, t is assumed to have finite support. Typically, u_0 is the fractional part of the value of an integer-constrained variable in a tableau row from an integer or mixed-integer program. *T-space* is the vector space with a dimension for each non-zero element of G.

A function π defined on G is a *valid* function with rhs element u_0 if π is continuous, non-negative, $\pi(0) = 0$, $\pi(u_0) = 1$ and

$$\sum t(u)u = u_0 \text{ implies } \sum \pi(u)t(u) \geq 1$$

A function π is *subadditive* if $\pi(u_1 + u_2) \leq \pi(u_1) + \pi(u_2)$ for all $u_1, u_2 \in G$. A valid function need not be subadditive, but can always be improved to be subadditive and so we may restrict our attention to only subadditive functions.

A valid function π is *minimal* if there does not exist a π' such that $\pi'(u) \leq \pi(u)$ for all $u \in G$ and $\pi'(v) < \pi(v)$ for some $v \in G$. The following theorem from [36, 38] gives a simple necessary and sufficient condition for a valid function to be minimal.

Theorem 2.0.10 *(Minimality Theorem [36, 38]) A valid function π is minimal if and only if π is subadditive and the symmetry condition $\pi(u) + \pi(u_0 - u) = \pi(u_0) = 1$ holds for all $u \in G$.*

By choosing $u = u_0/2$ and $u = (1+u_0)/2$, observe that any minimal function π is forced to pass through the halfway points $P_1 = (u_0/2, 1/2)$ and $P_2 = ((1+u_0)/2, 1/2)$. Once the symmetry of a piecewise linear function π has been established, a useful theorem to establish its subadditivity is the following theorem:

Theorem 2.0.11 *(Subadditivity Checking Theorem [38]) If π is piecewise linear, minimal and $\pi(u_1 + u_2) \leq \pi(u_1) + \pi(u_2)$ whenever u_1 and u_2 are convex endpoints of π, then π is subadditive.*

A path $\{t(u)\}$ *lies on an inequality* π if $\sum \pi(u)t(u) = 1$. Let $P(\pi)$ denote the set of paths which lie on the inequality π. Then π is a *facet* if $P(\pi^*) \supseteq P(\pi)$ implies $\pi^* = \pi$. Let $E(\pi)$ denote the set of equalities satisfied by π. Then,

Theorem 2.0.12 *(Facet Theorem [38]) If π is subadditive and minimal, and if the set $E(\pi)$ of all equalities has no solution other than π itself, then π is a facet.*

An important tool that will be used repeatedly is the following lemma.

Lemma 2.0.13 *(Interval Lemma [38]) Let $U = [u_1, u_2], V = [v_1, v_2]$ and $U + V = [u_1 + v_1, u_2 + v_2]$ be three closed intervals in G. If, whenever $u \in U$ and $v \in V$, we have $\pi(u) + \pi(v) = \pi(u+v)$, then $\pi(u)$ must be a straight line with some constant slope s for all $u \in U, V$, and $U + V$.*

The *cylindrical space* S is the set of all points (u, h) where $u \in G$ and h is any non-negative real number. The u values are plotted horizontally and the h values are plotted vertically. In S, the origin is represented twice, once by $O_1 = (0, 0)$ and also by $O_2 = (1, 0)$. Given $u \in G$, the corresponding real number in $[0, 1)$ is denoted $\eta(u)$.

The cylindrical topology of S gives the property that a non-origin point can be connected by a straight line to the origin by countably-infinite different lines. In S, multiplying a vector by a non-integer scalar is not well-defined. So given a vector (u, h), an *s-vector* is (u, h) with one of the slopes $s = h/(\eta(u) + n)$ for some integer n.

A major result with a remarkably simple proof is the following:

Theorem 2.0.14 *(Gomory Johnson Two-Slope Theorem [36, 38]) If $\pi(u)$ is subadditive, minimal and has only two slopes, then it is a facet.*

The Two-Slope theorem was first proved in [36] and a different proof using the Interval Lemma and the Facet Theorem is given in [38].

A technical result on subadditive functions which will be needed is the following lemma from [37]:

Lemma 2.0.15 *If π is a subadditive function on $[0,1]$ and if $\pi(u) \to 0$ as $u \downarrow 0$ and $\pi(u) \to 0$ as $u \uparrow 1$, then π is continuous at every $u \in [0,1]$.*

The following result can be helpful when showing subaddivity:

Lemma 2.0.16 *(Separation Lemma [38]) If π is a piecewise linear function with the slopes s of all of its segments satisfying $s^- \leq s \leq s^+$, w^+ is an upward pointing s-vector with slope s^+, and w^- is an upward pointing s-vector with slope s^-, then if p lies on π, $p + w^+ + w^-$ cannot lie below π.*

2.1 Construction 1

The shooting experiments conducted in [39] suggest that a relatively small number of facets of the corner polyhedra are important. Instead of looking at the most frequently hit facets, we observed the unusual structure in some of the less frequently hit facets. In Figure 6, the first two facets had vertices only at the heights $0, 1/3, 2/3$, and 1 and the last two facets had vertices only at the heights $0, 1/4, 1/2, 3/4$ and 1.

We will construct a piecewise-linear function π whose vertices have only four possible heights: $0, 1/3, 2/3$, and 1. We require that $u_0 \leq 0.5$. We first define π on the interval $[0, \eta(u_0)]$ by constructing the vertices of its line segments. The line segments will start from the origin O_1 and end at $R = (u_0, 1)$.

Choose λ such that $\lambda > 3\max\{1/\eta(u_0), 1/(1-\eta(u_0))\}$ and define

$$\alpha = \eta(u_0)/2 - 3/2\lambda \text{ and } \beta = 1/2 - \eta(u_0)/2 - 3/2\lambda.$$

Observe that $6/\lambda + 2\alpha + 2\beta = 1$ and $\alpha, \beta > 0$.

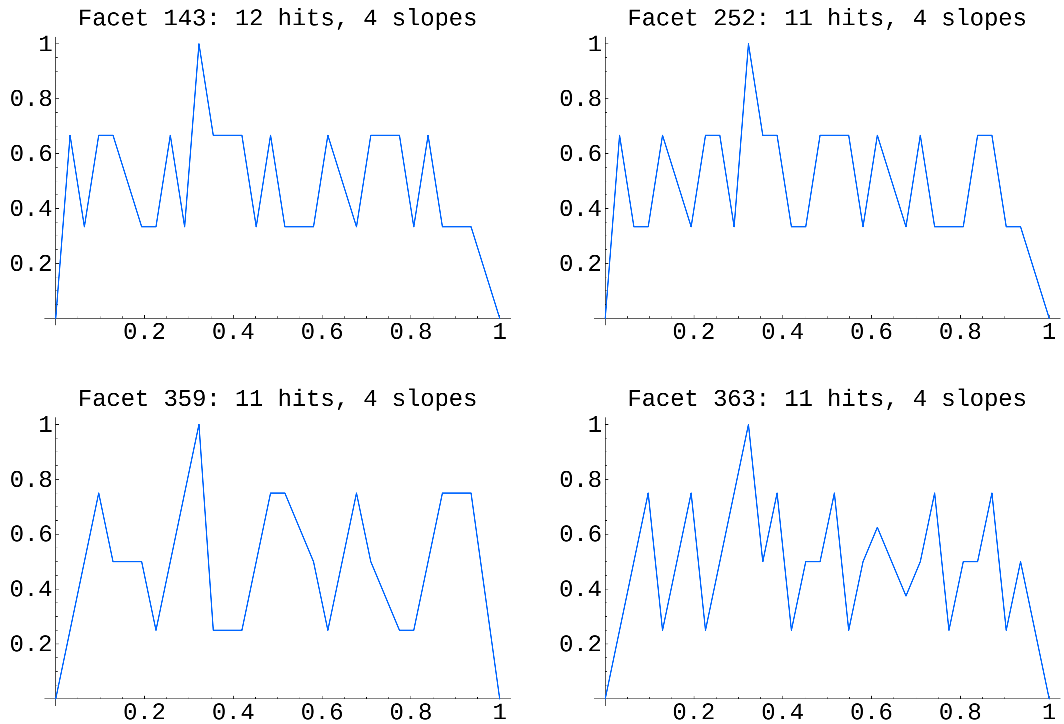

Figure 6: Some facets from a shooting experiment which motivated Construction 1.

Let v_i be the s-vector from O_1 to $(1/\lambda, 2/3)$ with slope $2\lambda/3$ and let v_h be the s-vector from O_1 to O_2 with slope 0 and length 1. Then we define

$$A = O_1 + v_i \text{ and } AA = R - v_i.$$

We then add horizontal line segments to these two points to define

$$B = A + \alpha v_h \text{ and } BB = AA - \alpha v_h.$$

Observe that $h(A) = h(B) = 2/3$ and that $h(AA) = h(BB) = 1/3$. By the choice of α, we have that $u(BB) = u(B) + 1/\lambda$.

Now we define π on the interval $[\eta(u_0), 1]$. Let v_d be the s-vector from O_2 to $(1-1/\lambda, 1/3)$ with slope $-\lambda/3$. Then we define

$$C = O_2 + v_d \text{ and } CC = R - v_d.$$

We then add horizontal line segments to these two points to define

$$D = C - \beta v_h \text{ and } DD = CC + \beta v_h.$$

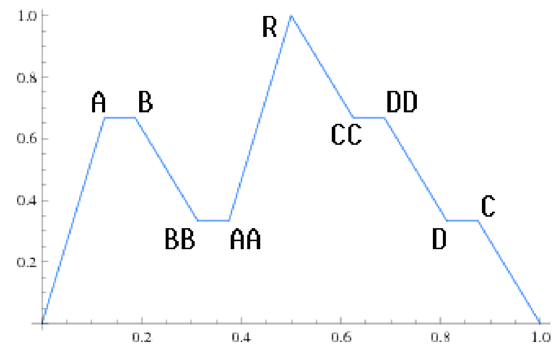

Figure 7: Example of Facet Construction 1 with $u_0 = 0.5$ ($\alpha = \beta$).

Observe that $h(C) = h(D) = 1/3$ and $h(CC) = h(DD) = 2/3$. By the choice of β, we have that $u(D) = u(DD) + 1/\lambda$.

Theorem 2.1.1 *The $\pi(u)$ formed by the direct segments connecting the successive pairs of points in the sequence $O_1, A, B, BB, AA, R, CC, DD, D, C, O_2$ is a facet.*

2.1.1 Minimality and Subadditivity

For a piecewise-linear function, it suffices to check the symmetry condition for the vertices of each line segment. Observe that $A + AA = R$ and $B + BB = R$ and so $\pi(u)$ for $u \in [0, \eta(u_0)/2]$ is symmetric to $\pi(u)$ for $u \in [\eta(u_0)/2, \eta(u_0)]$. On $[\eta(u_0), 1]$, we have that $C + CC = R$ and $D + DD = R$ and so π satisfies the symmetry condition.

We now prove that π is subadditive by showing that $\pi(u_1+u_2) \leq \pi(u_1)+\pi(u_2)$ whenever u_1 and u_2 are convex endpoints of π. Observe that

$$\begin{aligned}
O &= (0,0) = (1,0) \\
BB &= (2/\lambda + \alpha, 1/3) \\
AA &= (2/\lambda + 2\alpha, 1/3) \\
CC &= (4/\lambda + 2\alpha, 2/3) \\
DD &= (4/\lambda + 2\alpha + \beta, 2/3) \\
D &= (5/\lambda + 2\alpha + \beta, 1/3) \\
C &= (5/\lambda + 2\alpha + 2\beta, 1/3)
\end{aligned}$$

Table 1: Subadditivity of Construction 1.

u	v	$\pi(u)+\pi(v)$	$u+v$	$\pi(u+v)$
BB	BB	$2/3$	CC	$2/3$
BB	AA	$2/3$	$4/\lambda+3\alpha$	$2/3$
BB	CC	1		
BB	DD	1		
BB	D	$2/3$	$1/\lambda+\alpha-\beta$	$\leq 2/3$
BB	C	$2/3$	B	$2/3$
AA	AA	$2/3$	$4/\lambda+4\alpha$	$\leq 2/3$
AA	CC	1		
AA	DD	1		
AA	D	$2/3$	$1/\lambda+2\alpha-\beta$	$\leq 2/3$
AA	C	$2/3$	$1/\lambda+2\alpha$	$\leq 2/3$
CC	CC	$4/3$		
CC	DD	$4/3$		
CC	D	1		
CC	C	1		
DD	DD	$4/3$		
DD	D	1		
DD	C	1		
D	D	$2/3$	CC	$2/3$
D	C	$2/3$	DD	$2/3$
C	C	$2/3$	$4/\lambda+2\alpha+2\beta$	$< 2/3$

are the convex endpoints of π. We enumerate all of the 28 possible cases, of which 7 cases involve O and are omitted in Table 1, due to the subadditivity condition being trivially satisfied. In 11 of the remaining cases, we find that $\pi(u)+\pi(v) \geq 1$ and we do not compute either $u+v$ or $\pi(u+v)$ since $\pi(w) \leq 1$ for all w and the subadditivity condition is trivially satisfied.

2.1.2 Uniqueness

Now that we have shown π to be minimal and subadditive, it remains to show that $\pi(u)$ is the only solution to all the equalities $E(\pi)$ and then invoke the Facet Theorem. Consider a function π^* that satisfies all the equations $E(\pi)$.

We first consider the line segments of the graph of π with slope $2\lambda/3$. We choose both U_1 and V_1 in the Interval Lemma to be $[0, 1/2\lambda]$. Then the interval $U_1 + V_1$ is $[0, 1/\lambda] =$

$[u(O_1), u(A)]$. Now for any $u \in U_1$ and $v \in V_1$, we have that $(u, \pi(u)), (v, \pi(v))$ and $(u + v, \pi(u + v)) \in [O_1, A]$ and so $\pi(u) + \pi(v) = \pi(u + v)$. By the choice of π^*, we must also have $\pi^*(u) + \pi^*(v) = \pi^*(u + v)$. By the Interval Lemma, π^* must be a straight line segment on $U_1 \cup V_1 \cup (U_1 + V_1) = [u(O_1), u(A)]$ with some slope s_1.

Now consider

$$\begin{aligned} U_2 &= [0, 1/2\lambda] \subset [u(O_1), u(A)], \text{ and} \\ V_2 &= [2/\lambda + 2\alpha, 2/\lambda + 2\alpha + 1/2\lambda] \subset [u(AA), u(R)] \end{aligned}$$

Then

$$U_2 + V_2 = [2/n + 2\alpha, 3/n + 2\alpha] = [u(AA), u(R)].$$

Since $\pi(u) + \pi(v) = \pi(u + v)$ for $u \in U_2$ and $v \in V_2$, we must also have that $\pi^*(u) + \pi^*(v) = \pi^*(u + v)$ for $u \in U_2$ and $v \in V_2$, By the Interval Lemma, π^* must be a straight line segment with some constant slope on U_2, V_2 and $U_2 + V_2 = [u(AA), u(R)]$. Since $U_1 = U_2$, the slope must be s_1.

We now consider the line segments with slope $-\lambda/3$. Let

$$\begin{aligned} U_3 &= [3/\lambda + 2\alpha + 1/2\lambda, 4/n + 2\alpha] \subset [u(R), u(CC)], \text{ and} \\ V_3 &= [5/\lambda + 2\alpha + 2\beta + 1/2\lambda, 1] \subset [u(C), u(O_2)] \end{aligned}$$

Then

$$U_3 + V_3 = [3/\lambda + 2\alpha, 4/\lambda + 2\alpha] = [u(R), u(CC)].$$

By the Interval Lemma, π^* must be a straight line segment with some constant slope s_2 on U_3, V_3 and $U_3 + V_3$. Let

$$\begin{aligned} U_4 &= [4/\lambda + 2\alpha + \beta + 1/2\lambda, 5/\lambda + 2\alpha + \beta] \subset [u(DD), u(D)], \text{ and} \\ V_4 &= [5/\lambda + 2\alpha + 2\beta + 1/2\lambda, 1] \subset [u(C), u(O_2)] \end{aligned}$$

Then

$$U_4 + V_4 = [4/\lambda + 2\alpha + \beta, 5/\lambda + 2\alpha + \beta] = [u(DD), u(D)].$$

By the Interval Lemma, π^* must be a straight line segment with some constant slope on U_4, V_4 and $U_4 + V_4$. Since $V_4 = V_3$, the slope must be s_2. Let

$$\begin{aligned} U_5 &= [1/\lambda + \alpha + 1/2\lambda, 2/\lambda + \alpha] \subset [u(B), u(BB)], \text{ and} \\ V_5 &= [5/\lambda + 2\alpha + 2\beta + 1/2\lambda, 1] \subset [u(C), u(O_2)] \end{aligned}$$

Then

$$U_5 + V_5 = [1/\lambda + 1\alpha, 2/\lambda + 1\alpha] = [u(B), u(BB)].$$

By the Interval Lemma, π^* must be a straight line segment with some constant slope on U_5, V_5 and $U_5 + V_5$. Since $V_5 = V_4 = V_3$, the slope must be s_2.

Let $U_6 = V_6 = [5/\lambda + 2\alpha + 2\beta + 1/2\lambda, 1] \subset [u(C), u(O_2)]$. Then

$$U_6 + V_6 = [5/\lambda + 2\alpha + 2\beta, 1] = [u(C), u(O_2)].$$

By the Interval Lemma, π^* must be a straight line segment with some constant slope on $U_6 = V_6$ and $U_6 + V_6$. Since $V_6 = V_5 = V_4 = V_3$, the slope must be s_2.

We now finally consider the horizontal line segments. In this case, we choose $m \geq 2$ and let

$$\begin{aligned} U_7 &= [5/n + 2\alpha + \beta, 5/n + 2\alpha + \beta(m+1)/m] \subset [u(D), u(C)], \text{ and} \\ V_7 &= [5/n + 2\alpha + \beta, 5/n + 2\alpha + \beta(2m-1)/m] \subset [u(D), u(C)] \end{aligned}$$

Then $U_7 + V_7 = [4/\lambda + 2\alpha, 4/\lambda + 2\alpha + \beta] = [u(CC), u(DD)]$. By the Interval Lemma, π^* must be a straight line segment with some constant slope s_3 on $U_7 \cup V_7 \cup (U_7 + V_7)$. Now we may make m as large as we wish, and by the continuity of π^*, we have that π^* is a straight line segment on the entire closed interval $[u(D), u(C)]$ with slope s_3.

For sufficiently large m, let

$$\begin{aligned} U_8 &= [2/\lambda + \alpha + \alpha/m, 2/\lambda + 2\alpha] \subset [u(BB), u(AA)], \text{ and} \\ V_8 &= [5/\lambda + \alpha(m-1)/m + 2\beta, 5/\lambda + \alpha + 2\beta] \subset [u(D), u(C)] \end{aligned}$$

Then $U_8 + V_8 = [1\lambda, 1\lambda + \alpha] = [u(A), u(B)]$. By the Interval Lemma, π^* must be a straight line segment with some constant slope on $U_8 = V_8$ and $U_8 + V_8$. Since V_8 and V_7 have non-empty intersection, the slope must be s_3. Again, we may make m as large as we wish, and by the continuity of π^*, we have that π^* is a straight line segment on the entire closed interval $[u(BB), u(AA)]$ with slope s_3.

Now observe that the following equations belong to $E(\pi)$:

$$\begin{array}{ccccc} \pi(B) & + & \pi(BB) & = & \pi(R) \\ \pi(D) & + & \pi(C) & = & \pi(DD) \\ \pi(D) & + & \pi(D) & = & \pi(CC) \\ \pi(D) & + & \pi(DD) & = & \pi(R) \\ \pi(BB) & + & \pi(C) & = & \pi(B) \end{array}$$

and hence, must also be satisfied by π^*. By first principles, we have that $\pi^*(O_1) = \pi^*(O_2) = 0$ and $\pi^*(R) = 1$. Over the interval $[\eta(u_0), 1]$, a height decrease of 1 occurs from which it follows that

$$-\pi^*(C) + \pi^*(CC) + \pi^*(D) - \pi^*(DD) = 0$$

This equation together with the above five equations yields a system of six equations in six unknowns with the unique solution

$$\pi^*(BB) = \pi^*(D) = \pi^*(C) = 1/3, \pi^*(B) = \pi^*(CC) = \pi^*(DD) = 2/3.$$

Observe that π^* has the same values as π for the points B, BB, C, CC, D and DD. By the slope condition on the intervals corresponding to the line segments with slope s_3, it follows that π and π^* have horizontal segments on those intervals and $\pi^*(A) = 2/3$ and $\pi^*(AA) = 1/3$. Hence, π^* must be equal to π.

2.2 Construction 2

We assume that $u_0 \geq 0.5$. Through O_1, construct a line L^+ with positive slope $s^+ \geq 1/\eta(u_0)$. Similarly, through O_2, construct a line L^- with negative slope $s^- \leq 1/(\eta(u_0) - 1)$. Without any loss of generality, and in the interest of cleaner notation, we will frequently use u_0 hereinafter when we actually mean $\eta(u_0)$. Let v_1 be the direct vector from $O_1 = (0, 0)$ to $P_1 = (u_0/2, 1/2)$ and v_2 be the direct vector from $O_2 = (1, 0)$ to $P_2 = ((1 + u_0)/2, 1/2)$. For

$$0 \leq \lambda_1 < \min\left\{\frac{1}{2}, \frac{s^+ - s^-}{s^+(1 - s^- u_0)}\right\},$$

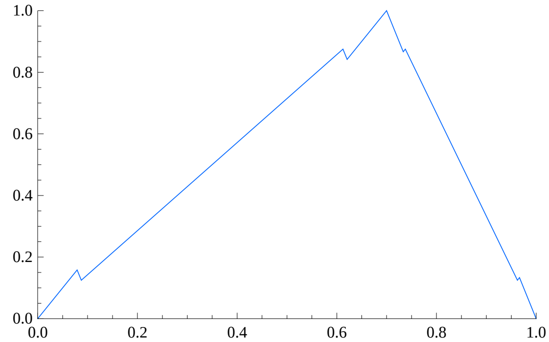

Figure 8: Example of Facet Construction 2 with $u_0 = 0.7$.

let A be the point

$$\lambda_1 v_1 = (\lambda_1 u_0/2, \lambda_1/2)$$

and B be the complementary point $R - A$. For

$$u_0 - \frac{1}{s^+} < \lambda_2 < \min\left\{\frac{1}{2}, \frac{s^+ - s^-}{s^-(s^+(u_0 - 1) - 1)}\right\},$$

let C be the point

$$\lambda_2 v_2 = (1 + \lambda_2(u_0 - 1)/2, \lambda_2/2)$$

and D be the complementary point $R - C$.

Now through A, construct a line with slope s^-. Within the vertical strip $\{(u,h) : 0 \leq \eta(u) \leq u(A), h \geq 0\}$, this line has a unique intersection with the line L^+ at a point, call it AA. Let v_3 denote the direct vector from A to AA with slope s^-. Let $BB = R - AA$.

Through C, construct a line with slope s^+. Within the vertical strip $\{(u,h) : u(C) \leq \eta(u) \leq 1, h \geq 0\}$, this line has a unique intersection with the line L^- at a point, call it CC. Let v_4 denote the direct vector from C to CC with slope s^+. Let $DD = R - CC$.

Theorem 2.2.1 *The $\pi(u)$ formed by the direct segments connecting the successive pairs of points in the sequence $O_1, AA, A, B, BB, R, DD, D, C, CC, O_2$ is a facet.*

2.2.1 Example

Suppose that $u_0 = 7/10$. Then the slopes of the GMIC are $10/7$ and $-10/3$. Now let $s^+ = 2 > 10/7$, $s^- = -4 < -10/3$ and $\lambda_1 = \lambda_2 = 1/4$.

We have that $A = (7/80, 1/8)$ and $C = (77/80, 1/8)$, and by symmetry, $B = (49/80, 7/8)$ and $D = (59/80, 7/8)$. $AA = (19/240, 38/240)$ and $CC = (29/30, 2/15)$, and by symmetry, $BB = (149/240, 202/240)$ and $DD = (11/15, 13/15)$.

2.2.2 Minimality and Subadditivity

We check the symmetry condition for each vertex of π. Observe that $AA + BB = R$, $A + B = R$, $DD + CC = R$, and $D + C = R$. It suffices to check subadditivity for the convex vertices of π: A, BB, DD, and C. The origin is a convex vertex, but we may omit it because of the subadditivity condition being trivially satisfied. A short calculation shows that

$$\begin{aligned}
u(BB) &= u_0 - \frac{\lambda_1(1 - s^- u_0)}{2(s^+ - s^-)} \\
h(BB) &= 1 - s^+ \frac{\lambda_1(1 - s^- u_0)}{2(s^+ - s^-)} \\
u(CC) &= 1 + \frac{s^+\lambda_2(u_0 - 1) - \lambda_2}{2(s^+ - s^-)} \\
h(CC) &= s^- \left(\frac{s^+\lambda_2(u_0 - 1) - \lambda_2}{2(s^+ - s^-)}\right) \\
u(DD) &= u_0 - \frac{s^+\lambda_2(u_0 - 1) - \lambda_2}{2(s^+ - s^-)} \\
h(DD) &= 1 - s^- \left(\frac{s^+\lambda_2(u_0 - 1) - \lambda_2}{2(s^+ - s^-)}\right)
\end{aligned}$$

We first observe that by the choice of λ_1, we have that

$$\begin{aligned}
\lambda_1 < \frac{s^+ - s^-}{s^+(1 - s^- u_0)} \quad &\Leftrightarrow \quad s^+ \frac{\lambda_1}{2} \frac{1 - s^- u_0}{s^+ - s^-} < \frac{1}{2} \\
&\Leftrightarrow \quad 1 - s^+ \frac{\lambda_1}{2} \frac{1 - s^- u_0}{s^+ - s^-} > \frac{1}{2} \\
&\Leftrightarrow \quad h(BB) > \frac{1}{2}
\end{aligned}$$

and by the choice of λ_2, we have that

$$\begin{aligned}
\lambda_2 < \frac{s^+ - s^-}{s^-(s^+(u_0 - 1) - 1)} \quad &\Leftrightarrow \quad s^- \left(\frac{s^+\lambda_2(u_0 - 1) - \lambda_2}{2(s^+ - s^-)}\right) < \frac{1}{2} \\
&\Leftrightarrow \quad 1 - s^- \left(\frac{s^+\lambda_2(u_0 - 1) - \lambda_2}{2(s^+ - s^-)}\right) > \frac{1}{2} \\
&\Leftrightarrow \quad h(DD) > \frac{1}{2}
\end{aligned}$$

By the Subadditivity Checking Theorem, it suffices to consider the following cases:

Case 1: $p_1 = A, p_2 = A$. In this case, observe that $A + A = (\lambda_1 u_0, \lambda_1) = \lambda_1 R$ and so $A + A$ lies on the line segment of π connecting A and B and so the subadditivity condition is satisfied.

Case 2: $p_1 = A, p_2 = BB$. Observe that $A + BB = A + (R - AA) = R - v_3$. Now $DD = R - CC = R - \sigma v_3$ for $\sigma \geq 1$. So $A + BB$ lies on the line segment of π between R and DD and the subadditivity condition is satisfied.

Case 3: $p_1 = A, p_2 = C$. Observe that $A + C = (AA - v_3) + (CC - v_4) = (\tau - 1)v_4 + (\sigma - 1)v_3$. By the Separation Lemma, $A + C$ cannot lie below π.

Case 4: $p_1 = A, p_2 = DD$. If $u(A) + u(DD) \leq u(C)$, then we have

$$
\begin{aligned}
& \pi(A) + \pi(DD) \\
= \; & \frac{\lambda_1}{2} + 1 - s^- \left(\frac{s^+ \lambda_2 (u_0 - 1) - \lambda_2}{2(s^+ - s^-)} \right) \\
> \; & \frac{\lambda_1}{2} + 1 + \frac{1}{1 - u_0} \left(\frac{s^+ \lambda_2 (u_0 - 1) - \lambda_2}{2(s^+ - s^-)} \right) \\
> \; & \frac{\lambda_1}{2} + 1 + \frac{1}{1 - u_0} \left(\frac{s^+ \lambda_2 (u_0 - 1) - \lambda_2}{2(s^+ - s^-)} - \frac{\lambda_1}{2} \right) \\
= \; & \frac{\lambda_1}{2} + 1 + \frac{1}{u_0 - 1} \left(\frac{\lambda_1}{2} - \frac{s^+ \lambda_2 (u_0 - 1) - \lambda_2}{2(s^+ - s^-)} \right) \\
= \; & \frac{1}{u_0 - 1} \left(\frac{\lambda_1}{2} u_0 - \frac{\lambda_1}{2} + \frac{\lambda_1}{2} + u_0 - 1 - \frac{s^+ \lambda_2 (u_0 - 1) - \lambda_2}{2(s^+ - s^-)} \right) \\
= \; & \frac{1}{u_0 - 1} \left(\frac{\lambda_1}{2} u_0 + u_0 - 1 - \frac{s^+ \lambda_2 (u_0 - 1) - \lambda_2}{2(s^+ - s^-)} \right) \\
= \; & \pi \left(\frac{\lambda_1}{2} u_0 + u_0 - \frac{s^+ \lambda_2 (u_0 - 1) - \lambda_2}{2(s^+ - s^-)} \right) \\
= \; & \pi(A + DD)
\end{aligned}
$$

If $u(C) < u(A) + u(DD) \leq 1$ or $\eta(u(A) + u(DD)) \leq u(A)$, we trivially have $\pi(A) +$

$\pi(DD) > \pi(A + DD)$ since $\pi(A + DD) < 1/2$. If $u(A) < \eta(u(A) + u(DD)) \leq u(B)$, then

$$\begin{aligned}
\pi(A+DD) &= \pi\left(\frac{\lambda_1}{2}u_0 + u_0 - \left(\frac{s^+\lambda_2(u_0-1)-\lambda_2}{2(s^+-s^-)}\right)\right) \\
&= \frac{1}{u_0}\left(\frac{\lambda_1}{2}u_0 + u_0 - \left(\frac{s^+\lambda_2(u_0-1)-\lambda_2}{2(s^+-s^-)}\right)\right) \\
&= \frac{\lambda_1}{2} + 1 - \frac{1}{u_0}\left(\frac{s^+\lambda_2(u_0-1)-\lambda_2}{2(s^+-s^-)}\right) \\
&< \frac{\lambda_1}{2} + 1 + \frac{1}{1-u_0}\left(\frac{s^+\lambda_2(u_0-1)-\lambda_2}{2(s^+-s^-)}\right) \\
&< \frac{\lambda_1}{2} + 1 - s^-\left(\frac{s^+\lambda_2(u_0-1)-\lambda_2}{2(s^+-s^-)}\right) \\
&= \pi(A) + \pi(DD)
\end{aligned}$$

Now

$$\begin{aligned}
u(A) + u(DD) &< u(A) + u(D) \\
&= \frac{\lambda_1 u_0}{2} + u_0 + \lambda_2\frac{1-u_0}{2} \\
&< \frac{5}{4}u_0 + \frac{1}{4}(1-u_0) \\
&< 1 + u(B)
\end{aligned}$$

and so we do not have to consider the case where $u(B) < \eta(u(A) + u(DD)) < u(DD)$.

Case 5: $p_1 = BB, p_2 = BB$. Since $\pi(BB) > 1/2$, we have $\pi(BB + BB) \leq 1 < 2\pi(BB)$ and subadditivity is trivially satisfied.

Case 6: $p_1 = BB, p_2 = C$. We first consider the case $u(A) \leq \eta(u(BB) + u(C)) \leq u(B)$.

$$\begin{aligned}
\lambda_2 > u_0 - \frac{1}{s^+} &\Leftrightarrow \frac{s^+u_0 - 1}{\lambda_2 s^+} < 1 \\
&\Leftrightarrow \frac{s^+u_0 - 1}{\lambda_2 s^+}\frac{\lambda_1 s^+(1-s^-u_0)}{s^+-s^-} < 1 \\
&\Leftrightarrow \left(\frac{1}{u_0} - s^+\right)\frac{\lambda_1}{2}\frac{1-s^-u_0}{s^+-s^-} > -\frac{\lambda_2}{2u_0} \\
&\Leftrightarrow 1 - s^+\frac{\lambda_1(1-s^-u_0)}{2(s^+-s^-)} + \frac{\lambda_2}{2} > 1 - \frac{1}{u_0}\frac{\lambda_1(1-s^-u_0)}{2(s^+-s^-)} + \frac{\lambda_2}{2}\frac{u_0-1}{u_0}
\end{aligned}$$

and hence,

$$\begin{aligned}
\pi(BB)+\pi(C) &= 1 - s^{+}\frac{\lambda_1(1-s^{-}u_0)}{2(s^{+}-s^{-})}+\frac{\lambda_2}{2} \\
&> 1-\frac{1}{u_0}\frac{\lambda_1(1-s^{-}u_0)}{2(s^{+}-s^{-})}+\frac{\lambda_2}{2}\frac{u_0-1}{u_0} \\
&= \frac{1}{u_0}\left(u_0-\frac{\lambda_1(1-s^{-}u_0)}{2(s^{+}-s^{-})}+\frac{\lambda_2}{2}(u_0-1)\right) \\
&= \pi\left(u_0-\frac{\lambda_1(1-s^{-}u_0)}{2(s^{+}-s^{-})}+\frac{\lambda_2}{2}(u_0-1)\right) \\
&= \pi\left(u_0-\frac{\lambda_1(1-s^{-}u_0)}{2(s^{+}-s^{-})}+1+\frac{\lambda_2}{2}(u_0-1)\right) \\
&= \pi(BB+C)
\end{aligned}$$

If $0 \leq \eta(u(BB)+u(C)) \leq u(A)$ or $\eta(u(BB)+u(C)) \geq u(C)$, we trivially have $\pi(BB)+\pi(C) > \pi(BB+C)$ since $\pi(BB+C) < 1/2$. By the assumption that $u_0 \geq 0.5$, the case that $u(D) < \eta(u(BB)+u(C)) \leq u(C)$ does not occur.

Case 7: $p_1 = BB, p_2 = DD$. Since $\pi(BB) > 1/2$ and $\pi(DD) > 1/2$, we have $\pi(BB+DD) \leq 1 < \pi(BB)+\pi(DD)$ and subadditivity is trivially satisfied.

Case 8: $p_1 = C, p_2 = C$. This is similar to Case 1. $C+C$ lies on the line segment of π between D and C, so the subadditivity condition is satisfied.

Case 9: $p_1 = C, p_2 = DD$. This is similar to Case 2. Observe that $C+DD = C+(R-CC) = R-v_4$. Now $BB = R-\tau v_4$ for $\tau \geq 1$. So $C+DD$ lies on the line segment of π between BB and R and the subadditivity condition is satisfied.

Case 10: $p_1 = DD, p_2 = DD$. This is similar to Case 5. We have that $\pi(DD+DD) \leq 1 < 2\pi(DD)$.

2.2.3 Uniqueness

Suppose that π^* satisfies all the equations satisfied by π. The segments of π with slope s^+ can be dealt with using the Interval Lemma in a manner similar to the previous construction. Let

$$\epsilon = \frac{1}{2}\min\{u(AA), u(R)-u(BB), u(D)-u(DD), u(CC)-u(C)\}$$

which is half the length of the smallest interval corresponding to a segment of π with slope s^+. Then for any interval $[u_1, u_2]$ in

$$S^+ = \{[0, u(AA)], [u(BB), u(R)], [u(DD), u(D)], [u(C), u(CC)]\}$$

we may apply the Interval Lemma by choosing $U = [0, \epsilon], V = [u_1, u_2 - \epsilon]$ and $U + V = [u_1, u_2]$. Then, since for any $u \in U$ and $v \in V$, we have $\pi(u) + \pi(v) = \pi(u + v)$, it follows that π^* must be linear on $U \cup V \cup (U + V)$ with some slope s'^+. Now all the intervals $[u_1, u_2] \in S^+$ must have the same slope s'^+ since each interval has the same slope as π^* on $U = [0, \epsilon]$. The segments of π with slope s^- can be dealt with in a similar manner.

By the choice of A, we have that $u(A) < u_0/4$ and so $u(P_1 - A) > u_0/4$. So we may choose $U = [A, P_1 - A], V = [P_1 - A, P_1]$ and $U + V = [P_1, 2P_1 - A] = [P_1, B]$ in the Interval Lemma and we get that π^* is linear with a single slope over U, V and $U + V$. Now $U \cup V \cup (U + V) = [A, B]$ and π^* must be continuous by, and so π^* is linear over $[A, B]$ with a single slope, say s_1. Now we argue that on $[u(A), u(B)]$, the slope of π^* is the same as the slope of π. Observe that $\pi(2A) = 2\pi(A)$ and $2\pi(P_1) = \pi(R) = 1$, which are relations that must also be satisfied by π^*. Now $\pi^*(2A) = 2\pi^*(A)$ implies that the line passing through $(u(A), \pi^*(A))$ and $(u(B), \pi^*(B))$ also passes through O_1. $\pi^*(P_1) = 1/2$ implies that the line passes through P_1, and in the vertical strip $\{(u, h) : 0 \leq \eta(u) \leq \eta(u_0), h \geq 0\}$, there is only one line passing through O_1 and P_1. Hence, π and π^* have the same slope on $[u(A), u(B)]$. This exact same line of argument can be used to show that π and π^* have the same slope on $[u(D), u(C)]$.

2.3 Merit Index

In [38], Gomory and Johnson introduced the notion of the *merit index* as a way to compare the quality of different facets. The merit index $MI(\pi)$ of a function π is defined to be twice the area of the set of points (x, y) in the unit square such that $\pi(x) + \pi(y) = \pi(x + y)$. The maximum possible merit index of a function is 1.0. The GMIC has merit $u_0^2 + (1 - u_0)^2$.

The merit index of the example for Construction 1 and the corresponding GMIC is shown in Figure 9.

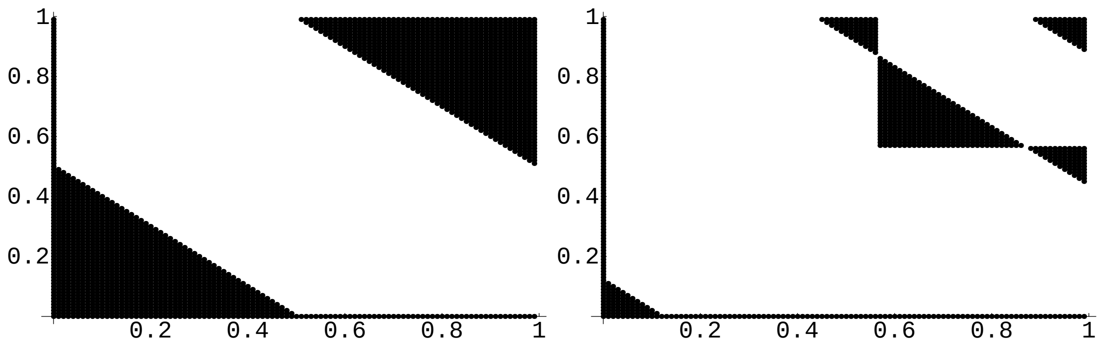

Figure 9: Merit index for GMIC and Construction 1.

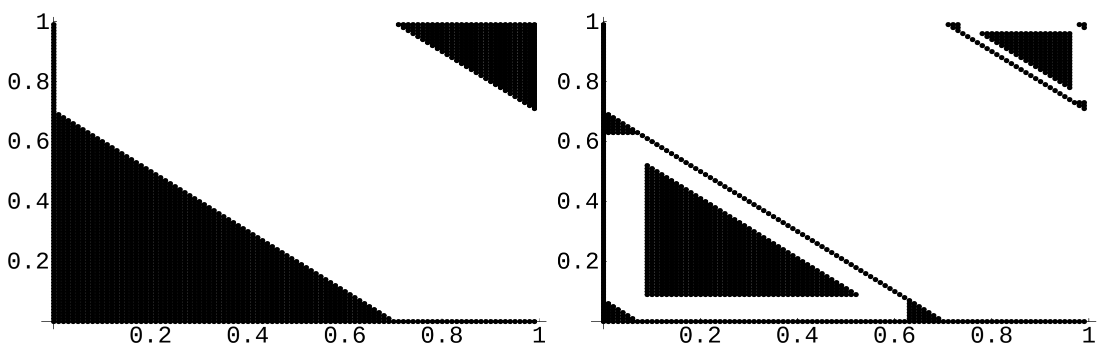

Figure 10: Merit index for GMIC and Construction 2.

CHAPTER III

FINDING NEW FACETS OF T-SPACE

3.1 Other cuts and the T-space framework

In classical derivations of the Gomory mixed-integer cut, the following simple two-variable mixed-integer set is usually first considered

$$X^{\geq} = \{(x, y) \in \mathbb{R} \times \mathbb{Z} : x + y \geq b, x \geq 0\}.$$

By a simple case analysis, it can be shown that

$$\frac{x}{\hat{b}} + y \geq \lceil b \rceil$$

is the only non-trivial facet of X. The notation $\hat{b}$ represents the fractional part of b, i.e. $\hat{b} = b - \lfloor b \rfloor$.

This simple result applies more generally because given an equation defining a mixed-integer set with more than two variables, one can extract an integral part out of the inequality and a continuous part, and then apply the above inequality to derive a valid inequality. If we take a row of the tableau corresponding to a basic integer variable that is fractional, then the set

$$X^G = \{(y, x, v) \in \mathbb{Z} \times \mathbb{Z}^{|N|} \times \mathbb{R}^2 : y + \sum_{j \in N} a_j x_j + v_1 - v_2 = b, x \geq 0, v \geq 0\}$$

is of interest. If we take the floors of the coefficients of the integer variables with indices in $S \subseteq N$ and take the ceilings for $N \backslash S$, then the equation can be rewritten as

$$y + \sum_{j \in S} \lfloor a_j \rfloor x_j + \sum_{j \in N \backslash S} \lceil a_j \rceil x_j + \sum_{j \in S} \hat{a}_j x_j - \sum_{j \in N \backslash S} (1 - \hat{a}_j) x_j + v_1 - v_2 = b.$$

Since $\sum_{j \in N \backslash S} (1 - \hat{a}_j) x_j$ and v_2 are non-negative, it follows that

$$y + \sum_{j \in S} \lfloor a_j \rfloor x_j + \sum_{j \in N \backslash S} \lceil a_j \rceil x_j + \sum_{j \in S} \hat{a}_j x_j + v_1 \geq b,$$

and by the integrality of the first three terms, we can apply the basic mixed-integer inequality to obtain

$$\sum_{j\in S} \hat{a}_j x_j + v_1 \geq \hat{b}\left(\lceil b\rceil - \left(y + \sum_{j\in S}\lfloor a_j\rfloor x_j + \sum_{j\in N\setminus S}\lceil a_j\rceil x_j\right)\right)$$

which is equivalent to

$$\sum_{j\in S} \hat{a}_j x_j + v_1 \geq \hat{b}\left(\lceil b\rceil - b + \sum_{j\in S}\hat{a}_j x_j - \sum_{j\in N\setminus S}(1-\hat{a}_j)x_j + v_1 - v_2\right)$$

and can be rewritten

$$\sum_{j\in S}\frac{\hat{a}_j}{\hat{b}}x_j + \sum_{j\in N\setminus S}\frac{1-\hat{a}_j}{1-\hat{b}}x_j + \frac{v_1}{\hat{b}} + \frac{v_2}{1-\hat{b}} \geq 1.$$

Now since

$$\frac{\hat{a}_j}{\hat{b}} \leq \frac{1-\hat{a}_j}{1-\hat{b}} \text{ if and only if } \hat{a}_j \leq \hat{b},$$

one should choose

$$S = \{j \in N : \hat{a}_j \leq \hat{b}\}$$

to get the best possible inequality. This is of course not the only possible derivation. For example, Gomory gave a disjunctive proof of his inequality in 1963.

In 2006, Dash and Günlük [23], considered a slightly more general mixed-integer set

$$\{(v, y, z) \in \mathbb{R} \times \mathbb{Z}^2 : v + \alpha y + z \geq \beta, v, y \geq 0\}$$

with one continuous and two integer variables, where $0 < \alpha < \beta < 1$. They found valid inequalities that are facets when some conditions are satisfied and called them two-step MIR inequalities. Dash and Günlük's work has been generalized even further by Kianfar and Fathi's [42] n-step MIR facets.

In 2003, Cornuéjols, Li and Vandenbussche [20] found that by scaling the equation by a positive integer k before applying the mixed-integer inequality results in different inequalities. They called the resulting inequalities k-cuts, and so the Gomory mixed-integer cut can be viewed as just a 1-cut. They only considered positive integers k as the $(-k)$-cut is just a scalar multiple of the corresponding k-cut. The idea of first multiplying the tableau row by a non-zero scalar to obtain different cuts was discussed even earlier in 1972 by Garfinkel and Nemhauser [33].

Cornuéjols, Li and Vandenbussche did computational experiments on randomly generated 0-1 and bounded knapsack problems and also on integer programs with multiple rows. The k-cut over various values of k had roughly the same performance as the GMIC for both the 0-1 and bounded knapsack problems, and adding multiple k-cuts simultaneously closed a signifcant percentage of the gap in most cases. However, for integer programs with multiple constraints, the performance was far poorer and the additional improvement on top of the GMIC was minimal.

All of these inequalities and more can be viewed within the T-space framework. The importance of subadditivity and the connection between generating cutting planes and the theory of T-space was discussed in 1972 by Gomory and Johnson [36, 37]. The theory leads to a far simpler and "graphical" derivation of Gomory's mixed-integer cut which we describe now. The process can be used to derive valid inequalities for both pure integer and mixed-integer programs. Given a function $\pi : [0,1] \rightarrow \mathbb{R}^+$ from the T-space theory, π can be directly applied to a tableau row corresponding to an integer basic variable that is fractional and give a valid inequality that is violated by the current basic feasible solution.

For a non-basic integer variable, the coefficient of the variable in the inequality is simply the value of π at the fractional value of its coefficient. For a non-basic continuous variable with positive coefficient, the coefficient of the variable in the inequality is the slope of π to the right of the origin. For a non-basic continuous variable with negative coefficient, the coefficient of the variable in the inequality is the slope of π to the left of 1 (or equivalently, the slope to the left of the origin). The right-hand size of the inequality is the value of π at the fractional value of the basic integer variable, and is typically 1. The basic variable itself has a coefficient of 0 in the inequality. For the equation defining X^G, we get

$$\sum_{j \in N} \pi(a_j)x_j + \pi^+ v_1 - \pi^- v_2 \geq \pi(\hat{b})$$

where

$$\pi^+ = \lim_{x \rightarrow 0^+} \frac{\pi(x)}{x}$$

and

$$\pi^- = -\lim_{x\to 0^+} \frac{\pi(1-x)}{x}.$$

If $\pi(\hat{b}) = 1$, then the right-hand size of the inequality is simply 1. Since $\pi^- < 0$, the derived inequality only has non-negative coefficients.

Consider the following numerical example from [38][1]

$$x_1 + 4.72t_1 - 2.93t_2 + 0.51t_3 + 0.14t_4 + 1.1t^+ - 1.4t^- = 2.79$$

which is a tableau row where $x_1, t_1, \ldots, t_4$ are integer-constrained variables and t^+ and t^- are continuous variables. x_1 is basic and the rest are non-basic variables. The coefficients

$$4.72, -2.93, 0.51, 0.14$$

of the non-basic integer variables $t_1, \ldots, t_4$ have respective fractional parts

$$0.72, 0.07, 0.51, 0.14.$$

Consider the two-sloped, piecewise-linear function π which passes through the points $(0, 0)$, $(0.79, 1)$ and $(1, 0)$. This π in fact yields the Gomory mixed-integer cut for this problem. Evaluating π at the fractional parts of the coefficients of the non-basic integer variables, we get

$$0.911, 0.089, 0.646, 0.177$$

respectively. The coefficients of the continuous variables t^+ and t^- are $1.1/.79$ and $1.4/.21$ respectively. The inequality that is derived is

$$0.911t_1 + 0.089t_2 + 0.646t_3 + 0.177t_4 + 1.392t^+ + 6.667t^- \geq 1.$$

Using this procedure, any facet of the infinite group polyhedron can be directly used to generate cutting planes for practical problems, with perhaps, the Merit Index and the closely-related Intersection Index giving some guidance regarding the quality of facets. The theory does not require the computation of the determinant of the basis matrix or even

[1] We correct the typos in [38].

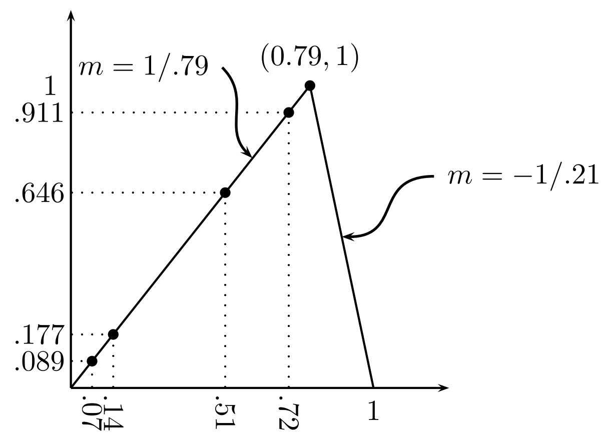

Figure 11: π function illustrating the cutting plane construction process.

knowledge of which group is actually present in a given problem. Because of this, it is desirable to find as many families of facets of the infinite group polyhedron as possible as the more facets that are known, the more variety of cutting planes that can be produced. Ultimately, it would be desirable to find results that allow us to take a given facet of the infinite group polyhedron, and make local changes to it which preserve facetness. This way, facets can be specially tailored for the specific problem at hand. Knowledge of more infinite group facets lead us in that direction.

Recall that the master cyclic group polyhedron $P(n, r)$ is the convex hull of

$$\left\{ u \in \mathbb{Z}^{n-1} : \sum_{i=1}^{n-1} \left(\frac{i}{n}\right) u_i \equiv \frac{r}{n} (\text{mod} 1), u \geq 0 \right\}$$

where $n, r \in \mathbb{Z}$ and $0 < r < n$. Suppose

$$\sum_{i=1}^{n-1} \pi_i u_i \geq 1$$

is a facet of $P(n, r)$. Recall that such a non-trivial facet is necessarily an extreme point of the system

$$\pi_r = 1 \quad (2)$$

$$\pi_i + \pi_j = \pi_r \text{ where } r = (i + j)(\text{mod} n) \quad (3)$$

$$\pi_i + \pi_j \geq \pi_{i+j (\bmod n)} \quad (4)$$

$$\pi_i \geq 0. \quad (5)$$

For a direction vector $d \geq 0$, recall that *shooting* is the procedure of determining the last facet hit by the ray

$$\{\lambda d : \lambda \geq 0\}$$

before entering the polyhedron. It is determined by solving the LP with objective function

$$\begin{aligned} \min \quad & \pi d \\ \text{s.t.} \quad & (2) - (5) \end{aligned}$$

since

$$\begin{aligned} & \max \quad \lambda \\ & \text{s.t.} \quad \pi(\lambda d) = 1 \text{ for facet } \pi \\ \Leftrightarrow \; & \max \quad 1/\pi d \\ & \text{s.t.} \quad \pi \text{ is a facet} \\ \Leftrightarrow \; & \min \quad \pi d \\ & \text{s.t.} \quad \pi \text{ is a facet.} \end{aligned}$$

Now the straight-line interpolation of π is the piecewise linear function

$$h(x) = \begin{cases} 0 & \text{if } \hat{x} = 0 \\ \pi_i & \text{if } \hat{x} = \frac{i}{n} \text{ for } i \in \{1, \ldots, n-1\} \\ \delta h(\frac{i}{n}) + (1-\delta) h(\frac{i+1}{n}) & \text{if } \hat{x} = \frac{i+\delta}{n} \text{ for } i \in \{1, \ldots, n-1\} \end{cases}$$

These functions h derived from facets of $P(n, r)$ can be used to derive cutting planes in exactly the same manner as described above for T-space facets. If the tableau row of a pure integer program has right-hand side b and if $\overline{n}$ is the smallest positive integer such that the tableau row multiplied by $\overline{n}$ becomes integral, then $P(\overline{n}, \overline{n}\hat{b})$ is the *canonical* master polyhedron. Generally, choosing $\overline{n}$ to be the absolute value of the determinant of the basis matrix suffices. However, $\overline{n}$ can be extremely large in practice and it becomes infeasible to work with $P(\overline{n}, \overline{n}\hat{b})$ directly.

In 2003, Gomory, Johnson and Evans conducted shooting experiments to find "important" facets of $P(n, r)$ for $n \leq 30$. They fired $10,000$ shots at each polyhedron they studied. Their computational results showed that generally the GMIC and 2-slope facets are important due to being hit frequently. In the shooting framework, facets with large solid angle subtended at the origin are considered important. The shooting approach of course only applies to non-trivial facets, as the non-negativity constraints have zero probability of being hit and hence, would be judged unimportant. Despite this shortcoming of shooting, this empirical approach is generally accepted.

In [23], Dash and Günlük extended the shooting experiments of Gomory, Johnson and Evans by considering $P(n, r)$ for $n \leq 200$. They fired $100,000$ shots at each polyhedron they studied. For $n \leq 90$, they kept track of all facets that were hit along with the number of times they were hit. For greater n up to 200, they only kept track of hits on MIR-based facets. This was likely due to the number of MIR-based facets growing quadratically in n whereas the number of facets of $P(n, r)$ grows exponentially in n.

In their empirical experiments, Dash and Günlük did again find that a small number of facets were hit a non-negligible fraction of the time, and most of them were MIR-based. They found that MIR and two-step MIR facets were frequently hit facets of $P(n, r)$.

A number of facets for finite master polyhedra in [2] called *seeds* were shown to be facets for the infinite group polyhedra in [38]. Recall that we previously mentioned Theorem 3.2 of [37] which allows us to take a facet π of the infinite group problem $P(I, u_0)$ and obtain a facet of a finite group problem $P(G_m, u_0)$ as long as the vertices of π belong to G_m. Hence, the group problem in the finite group case and the infinite unit interval group are closely related.

Patterns in the structure of facets for corner polyhedra and master equality knapsack polyhedra and mapping relationships between them are discussed in [2].

3.2 Enumerative algorithm

Because of the lack of a shooting theorem for the infinite group polyhedron, we describe in this chapter an enumerative and heuristic process which can be used to identify candidate facets of the continuous interval problem. The idea is that we choose positive integers m and n with m even, and divide the unit square into a checkerboard with m equally-spaced rows and n equally-spaced columns. We then have $(m+1) \times (n+1)$ grid points at

$$v_{i,j} = (i/m, j/n) \text{ for } i \in \{0, 1, \ldots, m\}, j \in \{0, 1, \ldots, n\}$$

There are m^{n+1} functions on the unit interval with vertices occurring at the grid points. We are interested in enumerating those functions which give candidate facets of the infinite group polyhedron.

Recall that any minimal function π must pass through the halfway points $P_1 = (u_0/2, 1/2)$ and $P_2 = ((1+u_0)/2, 1/2)$. We also have that $\pi(0) = 0, \pi(1) = 0$, and $\pi(u_0) = 1$. By minimality, once π is determined on $[0, u_0/2]$, it is determined on $[u_0/2, u_0]$. Likewise, once π is determined on $[u_0, u_0/2 + 1/2]$, it is determined on $[u_0/2 + 1/2, 1]$.

We assume that u_0 is rational and that the even positive integer m has been chosen such that $u_0/2 = i/m$ for some integer $i \in \{0, 1, \ldots, m-1\}$. The enumeration process entails the selection of $n+1$ values to determine a function π. For validity, the value of the function at 0 and 1 must be 0 and the value at u_0 must be 1. By the minimality condition, our only degrees of freedom are the grid points which fall in the intervals $[0, u_0/2]$ and $[u_0, u_0/2 + 1/2]$.

If we were to combinatorially enumerate all such possible functions, we would get a number of functions that are not even subadditive. The following lemma on subadditive functions is Lemma 2.4 from [37].

Lemma 3.2.1 *If π is a subadditive function on $[0,1]$ and if*

$$\lim_{u \downarrow 0} \frac{\pi(u)}{|u|} = \beta,$$

then

$$\limsup_{u \downarrow v} \frac{\pi(u) - \pi(v)}{|u| - |v|} \leq \beta$$

for any $v \in [0,1]$.

A similar result can be shown for

$$\lim_{u \uparrow 1} \frac{\pi(u)}{1 - |u|}$$

and

$$\limsup_{u \uparrow v} \frac{\pi(u) - \pi(v)}{|v| - |u|}.$$

We use these results in the enumeration procedure to cut-off candidate functions that are not subadditive. In addition, recall that for a piecewise-linear subadditive function that it suffices to check the convex vertices.

Theorem 3.2.2 *(Subadditivity Checking Theorem [38]) If π is piecewise linear, minimal and $\pi(u_1 + u_2) \le \pi(u_1) + \pi(u_2)$ whenever u_1 and u_2 are convex endpoints of π, then π is subadditive.*

Algorithm 1 Enumerative algorithm for finding candidate infinite group facets

---

m $\leftarrow$ even positive integer such that $u_0/2 = i/m$ for some integer $i \in \{0, 1, \ldots, m-1\}$
n $\leftarrow$ positive integer
for assignment σ_1 of values in $\{0, 1/n, 2/n, \ldots, 1\}$ to grid points in $[0, u_0/2]$ **do**
  for assignment σ_2 of values in $\{0, 1/n, 2/n, \ldots, 1\}$ to grid points in $[u_0/2 + 1/2, 1]$ **do**
    $\pi(0) \leftarrow 0$
    $\pi(1) \leftarrow 0$
    $\pi(u_0) \leftarrow 1$
    assign values to π on grid points in $[0, u_0/2]$ according to σ_1
    assign values to π on grid points in $[u_0/2, u_0]$ using σ_1 to maintain minimality
    assign values to π on grid points in $[u_0/2 + 1/2, 1]$ according to σ_2
    assign values to π on grid points in $[u_0, u_0/2 + 1/2]$ using σ_2 to maintain minimality
    if π can't be cut-off using subaddivity properties, etc. **then**
      print π
    end if
  end for
end for

In the algorithm, the choice for $\pi(\frac{1}{m})$ and $\pi(\frac{m-1}{m})$ leads immediately to the coefficients multiplying the continuous variables. These can be chosen as desired depending upon the importance of the continuous variables relative to the integer-constrained variables. In addition, we may also assign the values to grid points such that we only consider functions π with a fixed number of slopes. Functions with two slopes are already taken care of by the Gomory-Johnson two-slope theorem, but there is currently no known similar theorem for three or more slopes.

In Figures 12- 16, we show the output of our algorithm with $m = 10$ and $n = 9$. Even for these relatively small values, almost 500 functions were found and for the sake of brevity, we show just a subset of them. Observe that Plot 400 in Figure 16 is exactly the $u_0 = 0.5$ case of Construction 1 which we showed to be a facet of the infinite group polyhedron in the previous chapter. Plot 74 in Figure 13, Plot 81 in Figure 14 and Plot 408 in Figure 16 are already known to be facets by theorems of Gomory and Johnson.

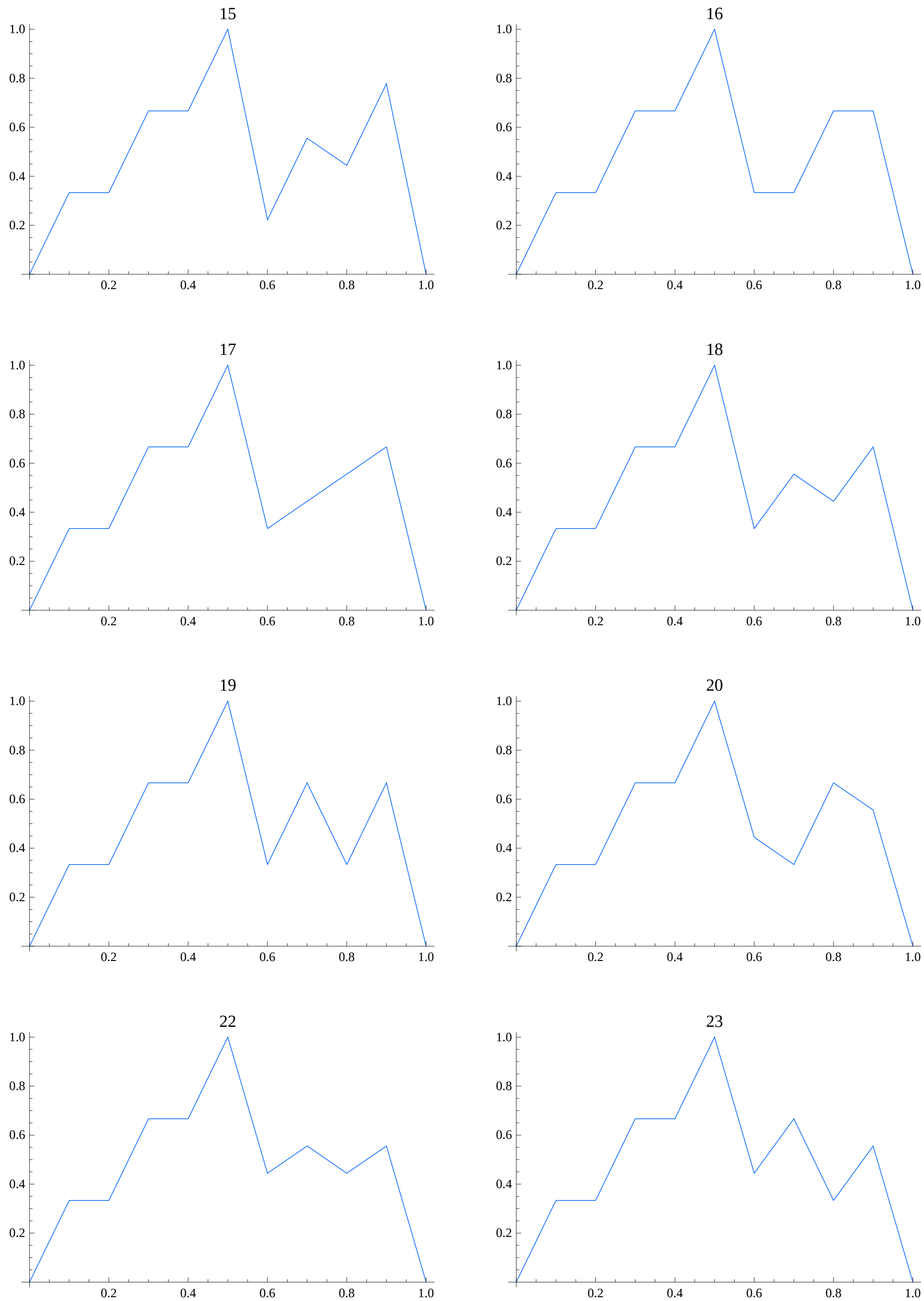

Figure 12: Output of enumerative algorithm.

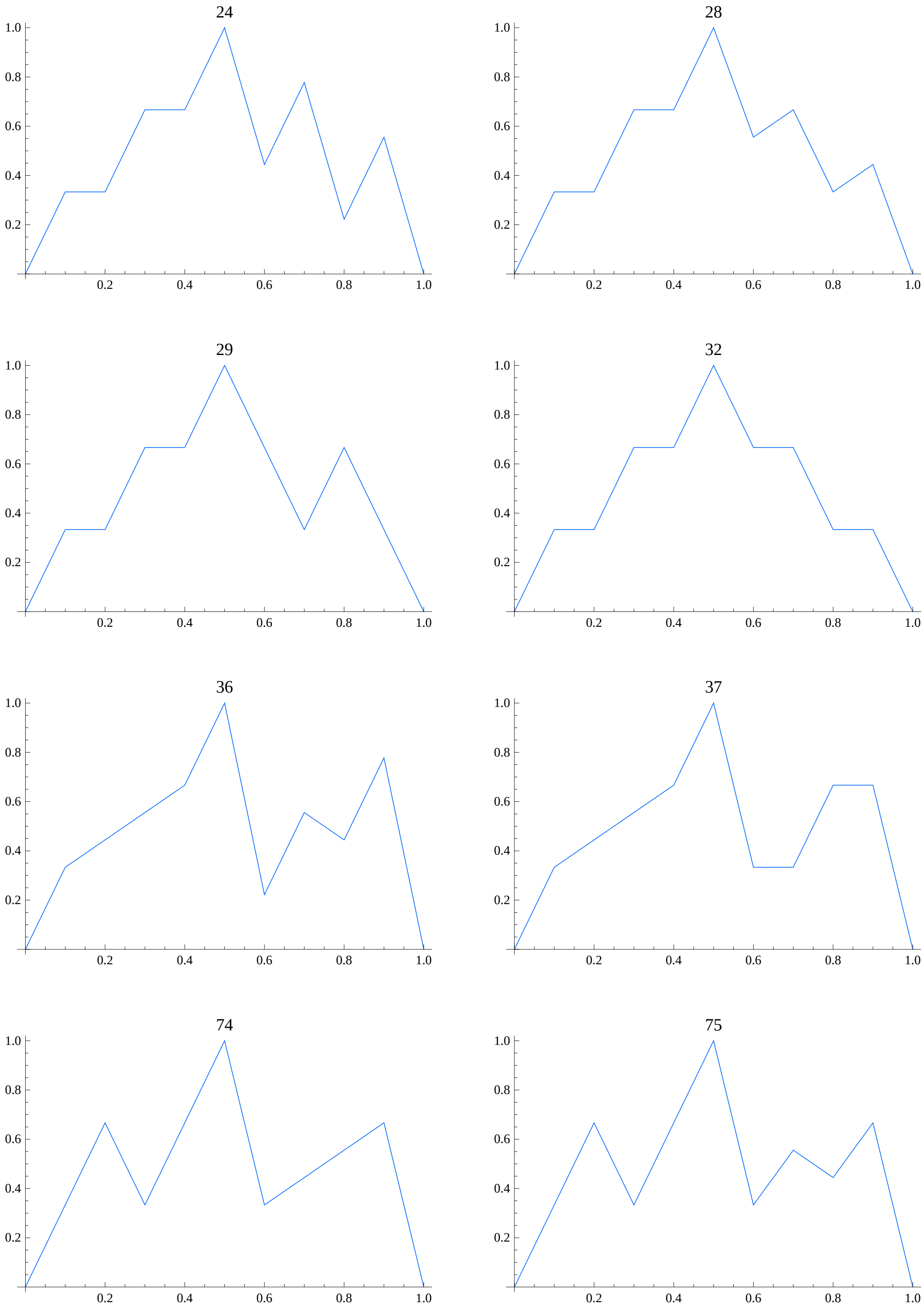

Figure 13: Output of enumerative algorithm, continued.

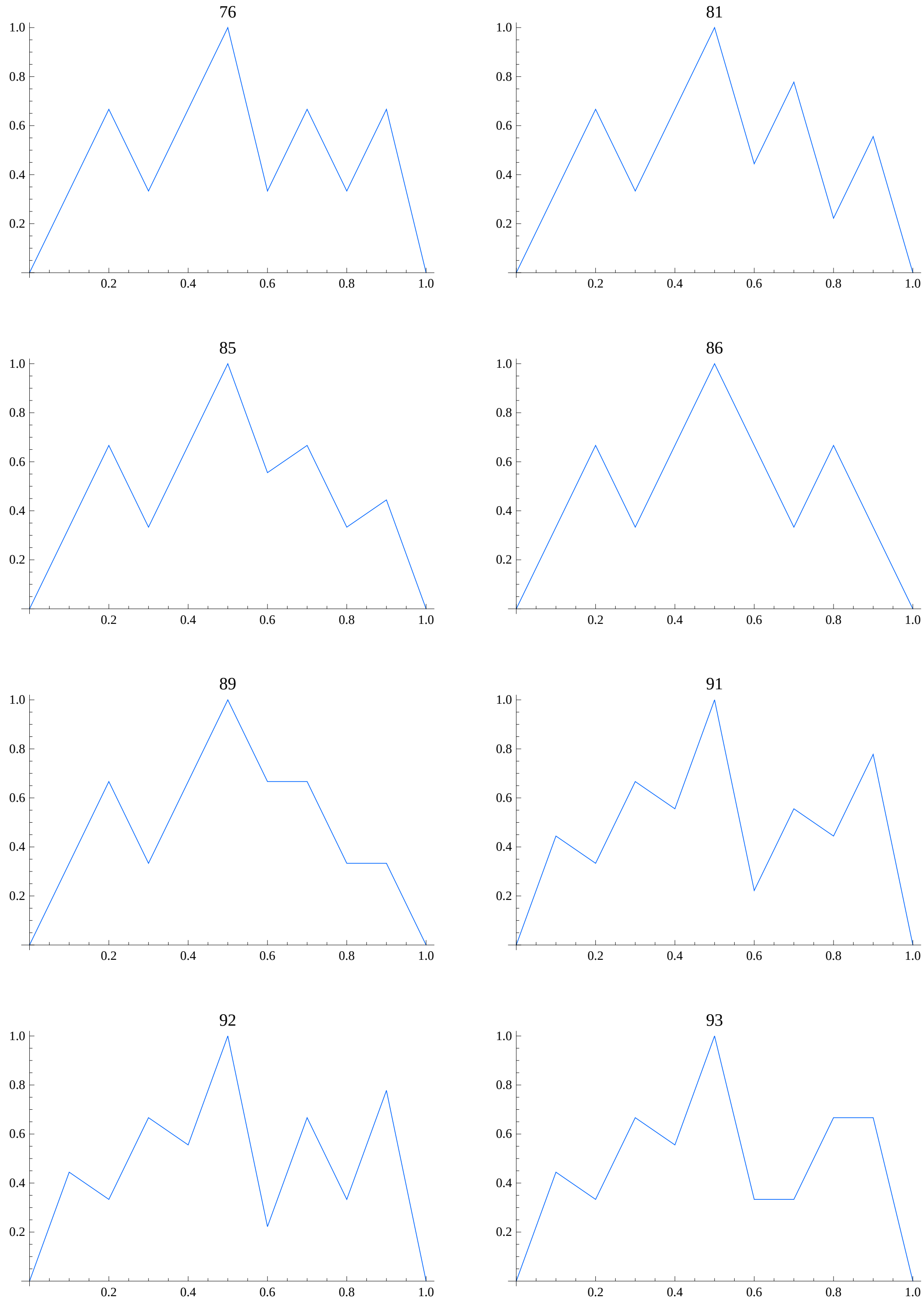

Figure 14: Output of enumerative algorithm, continued.

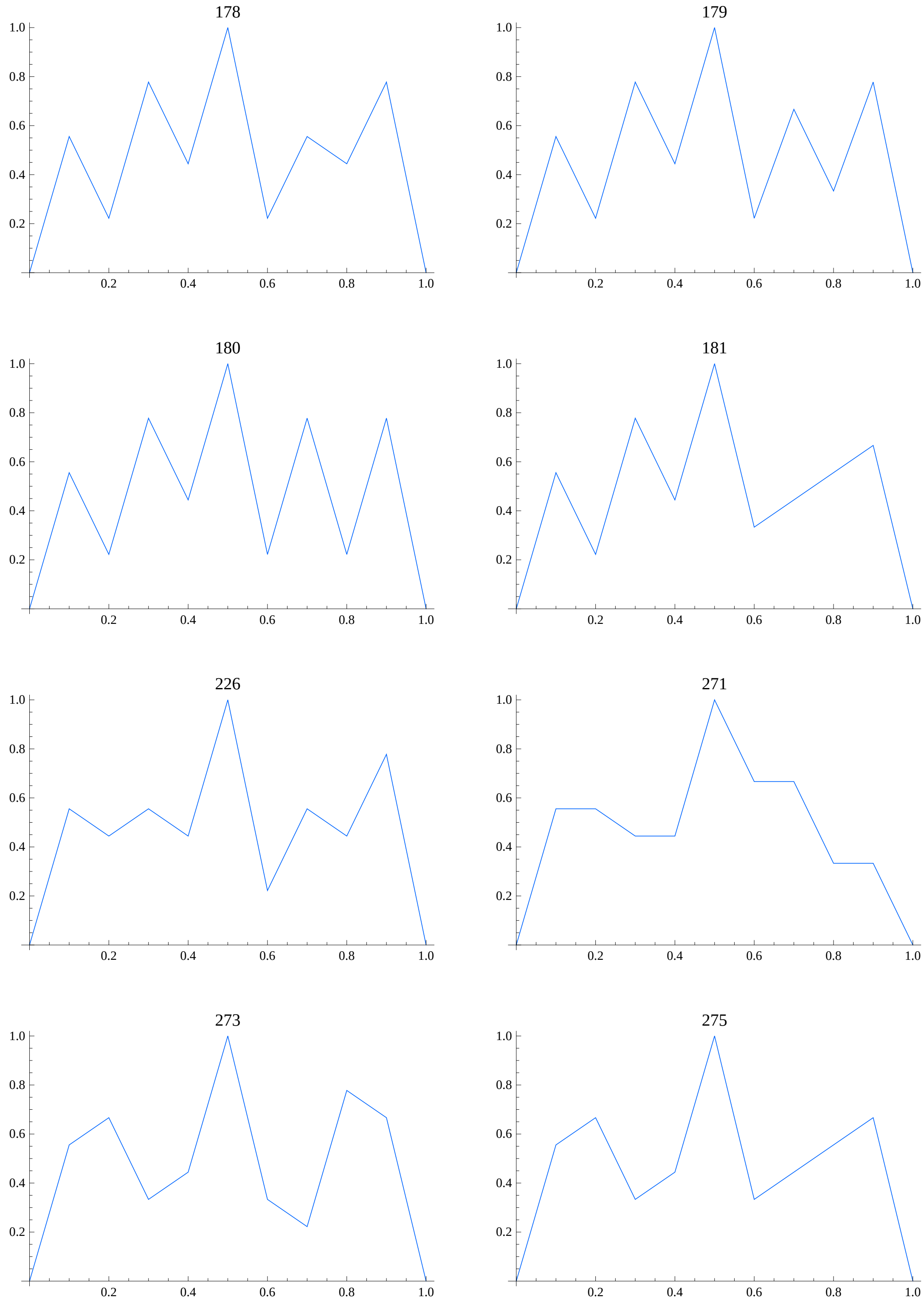

Figure 15: Output of enumerative algorithm, continued.

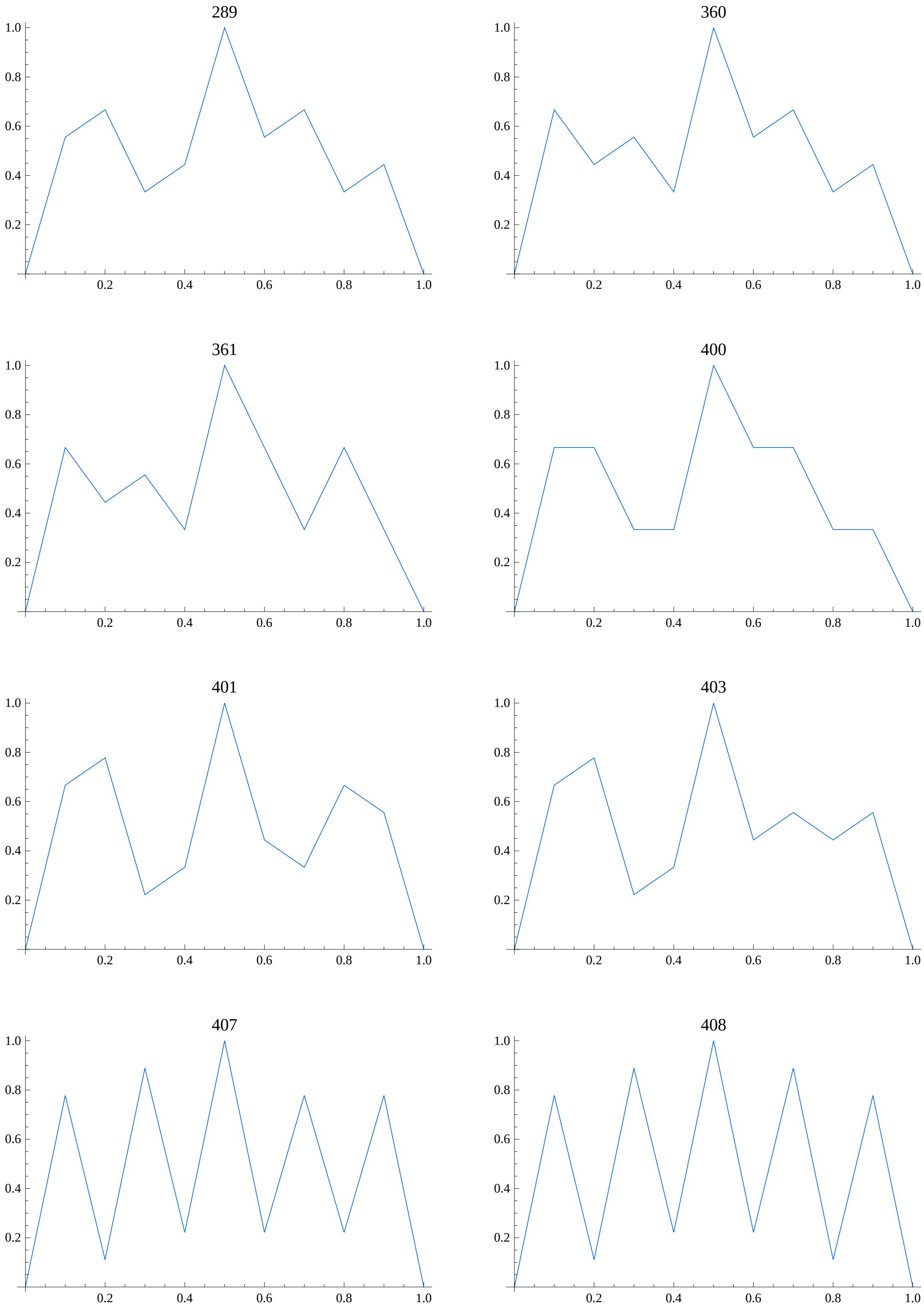

Figure 16: Output of enumerative algorithm, continued.

CHAPTER IV

HEURISTIC LATTICE-FREE TRIANGLES

4.1 Integer hulls in two-dimensional space

The integer hull of a polyhedron is the convex hull of the integral vectors inside of it. In the course of doing research in integer solvers for Constraint Logic Programming, W. Harvey found in 1999 [41] an optimal algorithm for computing the integer hull of a two-dimensional convex region defined by a set of linear inequalities. In computational geometry, a number of algorithms for computing the convex hull of a finite set of points are well-known, but the algorithms cannot handle an infinite set of points. In Harvey's algorithm, the region defined by the inequalities can be unbounded.

Given a pair of rational inequalities, they can be rewritten as

$$\begin{aligned} a_1x + b_1y &\leq c_1 \\ a_2x + b_2y &\leq c_2 \end{aligned}$$

where all the coefficients are integer, and we may assume without loss of generality that $\gcd(a_i, b_i) = 1$ for $i = 1, 2$. If we let

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix},$$

we may assume that $\det(A) = a_1b_2 - b_1a_2 > 0$ since the inequalities can be swapped. If the supporting lines happen to intersect in an integral point, then that integral point is the integer hull and we are done. Observe that the set of integral points that satisfy the pair of inequalities does not change when rewriting the inequalities.

Recall that a non-singular matrix U is called *unimodular* If U is integral and $\det(U) = \pm 1$. The first step is to apply a unimodular transformation from variables x, y to variables X, Y where the second inequality is transformed into a simpler form:

$$\begin{array}{ccccc} tX & + & uY & \leq & c_1 \\ X & + & & \leq & c_2 \end{array}$$

and by unimodularity, the set of feasible integral points is not altered. So we desire an integral matrix

$$U = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

with $\det(U) = \pm 1$ such that

$$AU = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} t & u \\ 1 & 0 \end{bmatrix}.$$

In addition, it is desirable for the first inequality to have a convenient orientation, and so U is chosen to satisfy the additional constraint

$$u > 0 \text{ and } t \leq 0.$$

Since $\gcd(a_2, b_2) = 1$, there exists $\alpha_0, \gamma_0 \in \mathbb{Z}$ such that

$$a_2\alpha_0 + b_2\gamma_0 = 1$$

and if we let

$$\begin{aligned} \alpha &= \alpha_0 + b_2 \left\lfloor -\frac{a_1\alpha_0 + \gamma_0 b_1}{\det(A)} \right\rfloor \\ \beta &= b_2 \\ \gamma &= \gamma_0 - a_2 \left\lfloor -\frac{a_1\alpha_0 + \gamma_0 b_1}{\det(A)} \right\rfloor \\ \delta &= -a_2 \end{aligned}$$

then we have

$$\det(U) = -a_2\alpha_0 - b_2\gamma_0 = -1$$

and

$$u = a_1\beta + b_1\delta = a_1 b_2 - a_2 b_1 > 0.$$

For t, we have

$$\begin{aligned} t &= a_1\alpha + b_1\gamma \\ &= a_1\left(\alpha_0 + b_2 \left\lfloor -\frac{a_1\alpha_0 + \gamma_0 b_1}{\det(A)} \right\rfloor\right) + b_1\left(\gamma_0 - a_2 \left\lfloor -\frac{a_1\alpha_0 + \gamma_0 b_1}{\det(A)} \right\rfloor\right) \\ &= a_1\alpha_0 + b_1\gamma_0 + \det(A) \left\lfloor -\frac{a_1\alpha_0 + \gamma_0 b_1}{\det(A)} \right\rfloor \\ &\leq 0 \end{aligned}$$

Now that we have the desired transformation, let (x_1, y_1) denote the integral point on the supporting line

$$tX + uY = c_1$$

with the largest X coordinate such that $X \leq c_2$. To find (x_1, y_1), we can first find an integral point (x_0, y_0) on the supporting line using the Euclidean algorithm, and then compute

$$(x_1, y_1) = \left(x_0 + \left\lfloor \frac{c_2 - x_0}{u} \right\rfloor u, y_0 - \left\lfloor \frac{c_2 - x_0}{u} \right\rfloor t \right).$$

The coordinate system (X, Y) is then translated to a new system (X', Y') by

$$\begin{aligned} X' &= X - x_1 \\ Y' &= Y - y_1 \end{aligned}$$

so that (x_1, y_1) is the new origin. Applying the translation, the inequalities are now

$$tX' + uY' \leq 0 \quad (6)$$

$$X' + \quad \leq c_2 - x_1. \quad (7)$$

By the translation and the choice of the point (x_1, y_1), there does not exist an integer point between the origin in (X', Y') space and the intersection of the supporting line of (6) and the vertical supporting line of (7).

So the idea is to now rotate the supporting line of (6) clockwise until we hit the first integer point with X' coordinate less than or equal to $c_2 - x_1$. In terms of the slopes of a line through the origin and this first integer piont, the slope p/q of the line should be less than $-t/u$ so that we are rotating clockwise, and to be feasible to the second constraint, q should be at most $c_2 - x_1$. To describe how to find p/q, we first need some definitions from number theory.

Recall that any real number x can be represented as a continued fraction

$$x = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \ddots}}}$$

where $x_0 \in \mathbb{Z}$ and $x_i \in \mathbb{Z}^+$ for all $i > 0$. A standard notation for this expression is

$$[x_0; x_1, x_2, \ldots].$$

The representation is finite if and only if $x \in \mathbb{Q}$. The integers $x_0, x_1, \ldots$ are called *partial quotients* and for any $n/m \in \mathbb{Q}$, they are precisely the quotients computed in the course of the Euclidean algorithm when computing $\gcd(n, m)$.

If we consider only the first k terms of the continued fraction expansion, then the result

$$p_k/q_k = [x_0; x_1, \ldots, x_k]$$

is called a *principal convergent* of x. Observe that the odd convergents decrease and the even convergents increase. The p_k and q_k can be quickly computed using the following second-order recurrence:

$$\begin{aligned} p_k &= x_k p_{k-1} + p_{k-2} \\ q_k &= x_k q_{k-1} + q_{k-2} \end{aligned}$$

for $k \geq 1$ with initial conditions

$$p_0 = x_0, q_0 = 1, p_{-1} = 1, q_{-1} = 0.$$

If $x_k > 1$, then the *intermediate convergents* are defined to be

$$\frac{p_{k-2} + j p_{k-1}}{q_{k-2} + j q_{k-1}}$$

for $j = 1, \ldots, x_k - 1$. We can now state the following theorem from an 1898 algebra book by Chrystal [17]:

Theorem 4.1.1 *The largest fraction p/q with $q \leq D$ and $p/q \leq x$ can be found from the set of even principal convergents of x, and their intermediate convergents when they exist, by taking the fraction with the largest denominator at most D.*

Having found p/q, then the inequality

$$-pX' + qY' \leq 0$$

gives the next segment of the integer hull. If this inequality intersects the supporting line of (7) in an integer point, then we have computed the entire integer hull. If not, then the next segment of the integer hull needs to be computed and that can be done by first doing a translation. We find the integer point with the largest X' value with $X' \leq c_2 - x_1$ on the supporting line

$$-pX' + qY' = 0.$$

This computation is made easier by the fact that we already know that the origin is an integer point on the supporting line. We translate and then repeat. It is not necessary to compute the convergents of p/q as we already computed them in the course of computing the convergents of $-t/u$.

Harvey's algorithm can handle any number of inequalities and is incremental in that an inequality is handled one at a time, but for our purposes, we are only concerned with computing the integer hull of a pair of inequalities. For n inequalities, the running time of the algorithm is $O(n \log A_{\max})$ where $A_{\max}$ is the magnitude of the largest integer in the input. Harvey has shown that his algorithm is optimal by exhibiting instances based on the Fibonacci sequence which results in $\Omega(n \log A_{\max})$ output constraints.

4.1.1 Numerical example

Consider the region spanned by

$$\begin{aligned} 11x + 9y &\leq 10 \\ -x + 5y &\leq 2 \end{aligned}$$

shown in Figure 17.

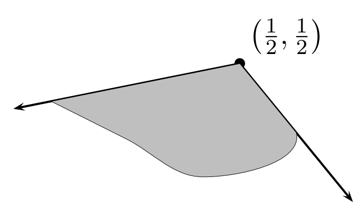

Figure 17: Numerical instance used to illustrate Harvey's algorithm.

Now $a_2(-1) + b_2(0) = 1$, so the unimodular transformation can be given by

$$U = \begin{bmatrix} -1 & 5 \\ 0 & 1 \end{bmatrix}$$

resulting in $t = -11$ and $u = 64$. In (X, Y)-space, we have

$$\begin{aligned} -11X + 64Y &\leq 10 \\ X + \quad &\leq 2. \end{aligned}$$

Now the point $(x_1, y_1) = (-30, -5)$ is the largest integral point on the supporting line

$$-11X + 64Y = 10$$

with $X \leq 2$, and so the transformed inequalities in (X', Y')-space are

$$\begin{aligned} -11X' + 64Y' &\leq 0 \\ X' + \quad &\leq 32 \end{aligned}$$

as shown in Figure 18.

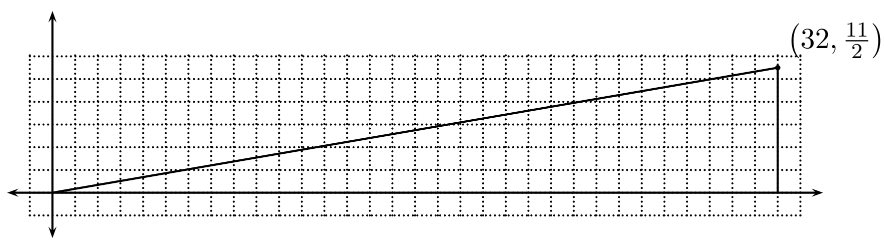

Figure 18: The numerical instance transformed by Harvey's algorithm.

In addition,

$$\begin{bmatrix} -1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -30 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

gives us the first point on the integer hull. The continued fraction representation of 11/64 is

$$x = 0 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{4 + \frac{1}{2}}}}$$

and the convergents are $0, \frac{1}{5}, \frac{1}{6}, \frac{5}{29}$ and $\frac{11}{64}$. The p/q that we desire is $1/6$ and the farthest right integer point on the supporting line

$$-X' + 6Y' = 0$$

satisfying $X' \leq 32$ is $(30, 5)$. This corresponds to $(0, 0)$ in (x, y)-space. If we translate, then the next p/q is simply 0 and in the new translated space, the final integer point is $(2, 0)$ which corresponds to $(-2, 0)$ in (x, y)-space. So we have determined that the vertices on the integer hull are $(-2, 0), (0, 0)$, and $(5, -5)$.

4.2 A heuristic for finding lattice-free triangles

Consider a point

$$f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} \in \mathbb{R}^2,$$

and let

$$r_1 = \begin{bmatrix} r_{1x} \\ r_{1y} \end{bmatrix}, r_2 = \begin{bmatrix} r_{2x} \\ r_{2y} \end{bmatrix}, r_3 = \begin{bmatrix} r_{3x} \\ r_{3y} \end{bmatrix}$$

be three vectors in $\mathbb{R}^2$ whose non-negative cone is all of $\mathbb{R}^2$. We are interested in determining a triangle with vertices

$$\{v_1, v_2, v_3\}$$

such that vertex v_i lies on the open ray

$$\text{ray}_i = \{x \in \mathbb{R}^2 : x = f + \lambda_i r_i \text{ for } \lambda_i > 0\}$$

where the triangle has no integral points in its interior. Convex sets which do not contain integral points in their interior are called *lattice-free*. Integral points are allowed on the boundary. For reasons which will become clear in the sequel, we are interested in triangles that are maximal.

In a result due to Dey and Wolsey [27], the maximal lattice-free triangles in $\mathbb{R}^2$ can be partitioned into three classes:

- *Type 1:* the vertices are integral and there is one integral point in the relative interior of each edge

- *Type 2:* there are multiple integral points in the relative interior of one edge with the opposing vertex being non-integral, and the other two edges have exactly one integral point in their relative interior
- *Type 3:* the vertices are non-integral and there is one integral point in the relative interior of each edge

In Figure 19, we show examples of each of the types of triangles.

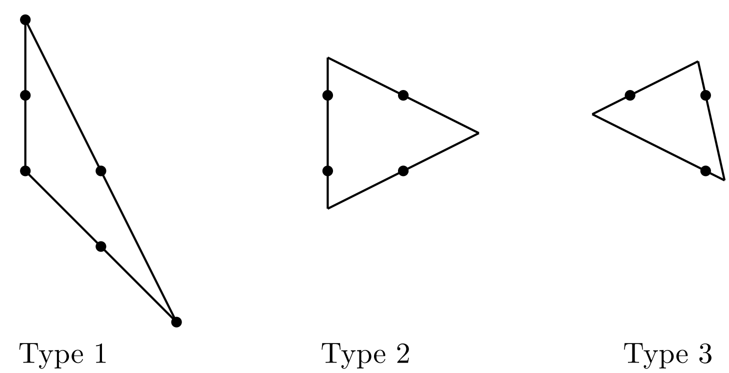

Figure 19: Examples of Type 1, 2, and 3 triangles.

The relative strength of various types of inequalities is naturally of interest. The notion of strength can be made precise. Suppose $Q \subseteq \mathbb{R}^n_+ \backslash \{0\}$ is a polyhedron of the form

$$Q = \{x : Ax \geq b\}$$

where $A \geq 0$ is an $m \times n$ matrix and $b \geq 0$ is an m-vector. For a scalar $\alpha > 0$, the polyhedron αQ is defined

$$\alpha Q = \{x : \alpha Ax \geq b\}.$$

Whenever $\alpha \geq 1$, $Q \subseteq \alpha Q$ and when $\alpha = +\infty$, αQ is defined to be $\mathbb{R}^n_+$. In measuring the strength of inequalities for the Traveling Salesman Problem, Goemans [34] considered how much a polyhedron had to be "blown up" to contain a relaxation. In [9], Basu, Bonami, Cornuéjols and Margot showed the following theorem, which generalizes a theorem of Goemans.

Theorem 4.2.1 *If Q is as above and $P \subseteq \mathbb{R}^n_+$ is a convex set such that $P \supseteq Q$, the smallest $\alpha \geq 1$ such that $P \subseteq \alpha Q$ is*

$$\max_{i=1,\ldots,m} \left\{ \frac{b_i}{\inf\{a^i x : x \in P\}} : b_i > 0 \right\}.$$

If $\inf\{a^i x : x \in P\} = 0$, then $\frac{b_i}{\inf\{a^i x : x \in P\}}$ is defined to be $+\infty$. The theorem allows one to compute the α for any polyhedron Q in the non-negative orthant (where $0 \notin Q$) and corresponding relaxation P by optimizing in the direction of the non-trivial facets of Q over the relaxation.

Suppose that B is a maximal lattice-free triangle and ψ is the corresponding minimal function that defines a facet

$$\sum_{j=1}^{k} \psi(r^j) s_j \geq 1$$

of $R_f(r^1, \ldots, r^k)$. Hence, the set $\{r_1, \ldots, r_k\}$ is assumed to contain rays that point to the vertices of B. By the theorem, the optimization problem

$$\min \left\{ \sum_{j=1}^{k} \psi(r^j) s_j : s \in S_f(r^1, \ldots, r^k) \right\}$$

is of interest.

Without any loss of generality, it can be assumed that for any r_j with $\psi(r_j) > 0$ that the ray r_j is scaled so that $f + r_j$ lies on the boundary of the lattice-free set B_ψ. In addition, Cornuéjols and Margot [21] showed that the triangles and quadrilaterals defining facets of $R_f(r^1, \ldots, r^k)$ are rational and so the scaling can be done using only rationals.

In considering strength, it turns out that not all of the rays r_j are needed to do analysis. Suppose that $B_1, \ldots, B_m$ are lattice-free convex sets containing f in their interior and $R_c \subseteq \{1, \ldots, k\}$ such that if $j \notin R_c$, then there exists $s, t \in R_c$ such that r_j can be expressed as the convex combination of r_s and r_t. Basu et al. showed that the two problems

$$\begin{aligned} \min \quad & \sum_{j=1}^{k} s_j \\ \text{s.t.} \quad & \sum_{j=1}^{k} \psi_{B_p}(r^j) s_j \geq 1 \text{ for } p = 1, \ldots, m \\ & s \geq 0 \end{aligned}$$

and

$$\begin{aligned}\min \quad & \sum_{j\in R_c} s_j \\ \text{s.t.} \quad & \sum_{j\in R_c} \psi_{B_p}(r^j)s_j \geq 1 \text{ for } p = 1,\ldots,m \\ & s \geq 0\end{aligned}$$

have identical optimal objective values. The proof is a fairly straightforward induction.

So we may assume that we have three rays $\{r_1, r_2, r_3\}$ such that the vertices of T are exactly $\{f + r_1, f + r_2, f + r_3\}$. If

$$\sum_{j=1}^{k} \psi(r^j)s_j \geq 1$$

is the inequality generated by T, then Basu et al. showed that

$$\min\left\{\sum_{j=1}^{k} \psi(r^j)s_j : s \in S_f(r^1,\ldots,r^k)\right\}$$

is a piecewise-linear function of f when f lies in the interior of T by computing

$$\begin{aligned}z_{\text{SPLIT}} \quad = \quad \min \quad & \sum_{j=1}^{3} s_j \\ \text{s.t.} \quad & \sum_{j=1}^{3} \psi_B(r^j)s_j \geq 1 \text{ for all splits } B \\ & s \geq 0\end{aligned}$$

Now, a Type 1 triangle T with integral vertices $\{x_1, x_2, x_3\}$ and interior integral points $\{y_1, y_2, y_3\}$ can be transformed via a unimodular transformation into a triangle with vertices $\{(0,0), (2,0), (0,2)\}$ and interior integral points $\{(0,1), (1,0), (1,1)\}$. See Figure 20.

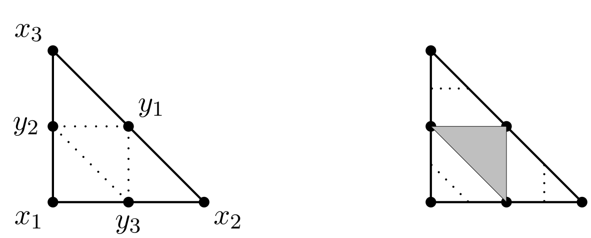

Figure 20: Subdivided Type 1 triangle with some level curves.

When f lies in the interior of the subtriangle with vertices $\{y_1, y_2, y_3\}$ (i.e. the shaded region in Figure 21), then $z_{\text{SPLIT}} = 1/2$. When $f = (f_1, f_2)$ lies in the interior of the corner subtriangle with vertices

$$\{x_1, y_2, y_3\}$$

or on the interior of the line segment connecting y_2 and y_3, then

$$z_{\text{SPLIT}} = 1 - \frac{1}{3 - f_1 - f_2}$$

The two other corner subtriangles with vertices

$$\{y_2, x_3, y_1\} \text{ and } \{y_3, y_1, x_2\}$$

are symmetric.

So z_{SPLIT} ranges from 1/2 in the center subtriangle of T to 2/3 at the vertices of T. It follows that the potential improvement of Type 1 triangles relative to the split closure is limited by a factor of 2.

Basu et al. showed that for any $\alpha > 0$, there exists $f, r_1, \ldots, r_k$ such that

$$S_f(r_1, \ldots, r_k) \not\subseteq \alpha R_f(r_1, \ldots, r_k).$$

In other words, there are problems for which the split closure is an arbitrarily bad approximation of the integer hull. They exhibited a number of integer programs where when optimizing in the direction of a facet from a Type 2 or Type 3 triangle or a quadrilateral, the optimal value over the split closure is arbitrarily close to zero.

For $f = (0, f_2)$ where $f_2 \in (0, 1)$, suppose the three rays are

$$r_1 = \mu_1 \begin{bmatrix} -1 \\ t_1 \end{bmatrix}, r_2 = \mu_2 \begin{bmatrix} 1 \\ t_2 \end{bmatrix}, r_3 = \mu_3 \begin{bmatrix} -1 \\ t_3 \end{bmatrix}$$

for $\mu_i > 0$ and rational t_i satisfying

$$-t_1 < t_2 < -t_3.$$

Then Basu et al. show that

$$z_{\text{SPLIT}} \leq \frac{1}{t_1 - t_3} \left(\frac{1 - f_2}{\mu_1} + \frac{f_2}{\mu_3} \right)$$

If $\mu_1 = \mu_3 = 1$, then the bound on z_{SPLIT} simplifies to $1/(t_1 - t_3)$. As $t_1 - t_3$ grows large, the split closure performs more and more poorly. The proof involves bounding the split closure with pseudo-splits, which in general may contain integral points in their interior and may not be valid for $R_f(r^1, \ldots, r^k)$. We do not get into the notion of a pseudo-split here.

For a concrete example, consider their example of the Type 2 triangle with long vertical edge passing through $(-1, 0)$ and the other two edges passing through the points $(0, 0)$ and $(0, 1)$. Here $f = (0, f_2)$ for $f_2 \in (0, 1)$ so that f lies in the relative interior of the line segment joining $(0, 0)$ and $(0, 1)$. Then the rays from f to the vertices are of the desired form.

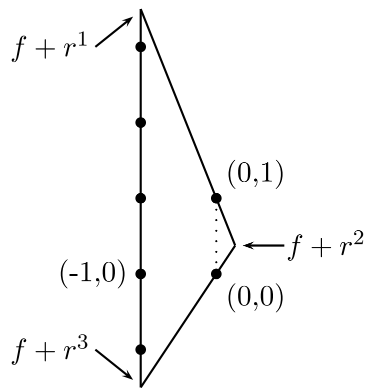

Figure 21: Type 2 triangle with large gap relative to the split closure.

So the theory suggests that Type 2 triangles are interesting. In addition, in the proof by Basu et al. that

$$T_f(r_1, \ldots, r_k) \subseteq S_f(r_1, \ldots, r_k),$$

the split S corresponding to a violated split inequality is shown to contain within it a Type 2 triangle whose corresponding inequality is also violated. This all suggests that Type 2 triangles, especially ones where the edge with multiple integral points is "long," are of interest for computational experiments.

We now describe our heuristic algorithm for finding lattice-free triangles, which finds triangles that are "close" to being Type 2. Our heuristic returns a triangle that is in general a Type 2 triangle or very close to being one, and it may be possible to return a Type 1 triangle, but it will never return a Type 3 triangle.

Algorithm 2 Heuristic algorithm for finding lattice-free triangles

```
Triangles ← {}
IH_1 = Integer-Hull({ray_1, ray_2})
IH_2 = Integer-Hull({ray_2, ray_3})
IH_3 = Integer-Hull({ray_1, ray_3})
for each pair of adjacent points {(x_1, y_1), (x_2, y_2)} in IH_1 do
    p_1 ← point where the line between {(x_1, y_1), (x_2, y_2)} intersects ray_1
    p_2 ← point where the line between {(x_1, y_1), (x_2, y_2)} intersects ray_2
    for each vertex v in IH_2 do
        if v ≠ p_1 then
            find point where the line between v and p_1 intersects ray_3, if it exists
        end if
    end for
    p'_3 ← the closest intersection point to f
    find p''_3 by handling IH_3 analogously
    p_3 ← closer of p'_3 and p''_3 to f
    Triangles ← Triangles ∪ {(p_1, p_2, p_3)}
end for
Handle IH_2 and IH_3 analogously
```

The basic idea of the heuristic is to walk down the integer hull of one of the sectors and for every pair of adjacent vertices on the hull, to compute the line that passes between them. We then see where this line strikes the two rays that constitute the sector to obtain the points p_1 and p_2. Then we determine point p_3 by respecting the integer hulls of the two other sectors.

4.2.1 Numerical example

Let

$$f = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, r_1 = \begin{bmatrix} -5/4 \\ -1/4 \end{bmatrix}, r_2 = \begin{bmatrix} -1/4 \\ 3/4 \end{bmatrix}, r_3 = \begin{bmatrix} 3/4 \\ -11/12 \end{bmatrix}.$$

The sector formed by r_1 and r_3 can be written as

$$\begin{aligned} 11x + 9y &\leq 10 \\ -x + 5y &\leq 2 \end{aligned}$$

and we determined earlier using Harvey's algorithm that the vertices on its integer hull are $(-2, 0), (0, 0)$, and $(5, -5)$. The sector formed by r_1 and r_2 has $(-2, 0), (0, 1)$ and $(0, 2)$ as the vertices on its integer hull. The sector formed by r_2 and r_3 has $(0, 2), (1, 0)$ and $(5, -5)$ as the vertices on its integer hull. See Figure 22. In the figure, the length of the vectors

r_i have been increased for visual purposes. Observe how in this example that an integral point can belong to more than one integer hull.

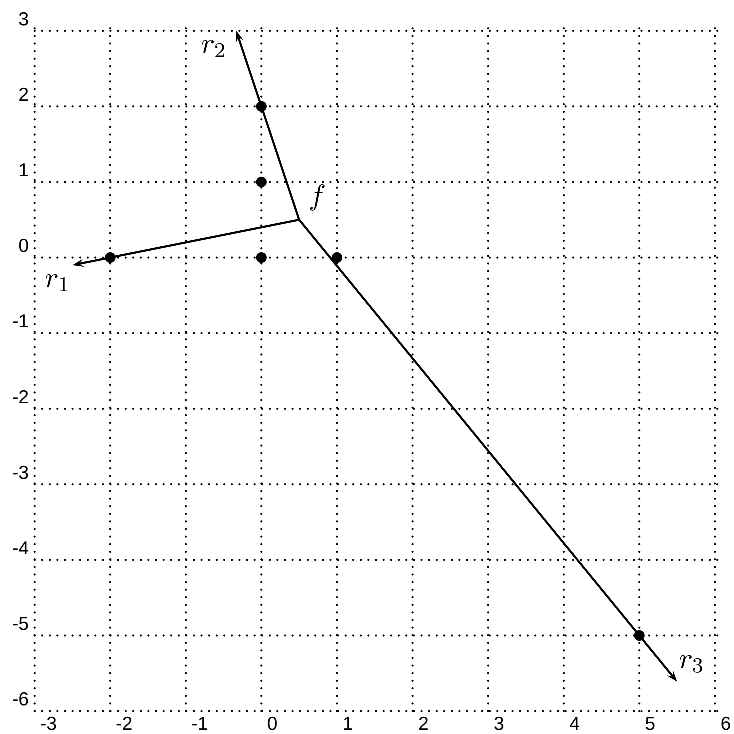

Figure 22: Numerical instance used to illustrate the heuristic algorithm.

Now we will give an example of a triangle found by the heuristic algorithm. Suppose that $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (0, 0)$ in Line 5 of the algorithm, while handling IH_3. The point p_1 is simply $(-2, 0)$ on ray_1 and the point p_2 is $(10/11, 0)$ on ray_3.

The point p'_3 is $(2/7, 8/7)$ on ray_2 that is intersected by the line between $(-2, 0)$ and $(0, 1)$. The point p''_3 is the point on ray_2 that is intersected by the line between $(10/11, 0)$ and $(0, 2)$, namely $(0, 2)$ itself. Since p'_3 is closer to f, the computed triangle has vertices $\{p_1, p_2, p'_3\} = \{(-2, 0), (10/11, 0), (2/7, 8/7)\}$ and is shown in Figure 23. Observe that the computed triangle is not quite a Type 2 triangle since the edge joining ray_2 and ray_3 does not contain an integer point in its relative interior.

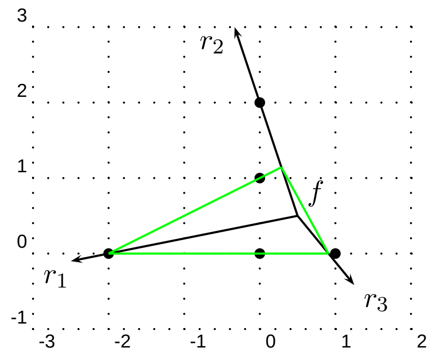

Figure 23: Triangle found by the heuristic algorithm.

CHAPTER V

EXACT TRIANGLES AND QUADRILATERALS

5.1 Gröbner bases

We first give an overview of the theory of Gröbner bases originally developed by B. Buchberger in his 1965 Ph.D. thesis at the University of Innsbruck in Austria. The theory is now widely used in symbolic computation and implemented in popular mathematical software packages such as Maple and Mathematica. The theory is attractive in that it can essentially be applied with knowledge of just polynomial arithmetic. Our notation and development closely follows that of [15].

Let $[x_1, \ldots, x_n]$ denote the set of monomials with coefficient 1 over the variables $x_1, \ldots, x_n$. If R is a field, let $R[x_1, \ldots, x_n]$ denote the ring of polynomials in $x_1, \ldots, x_n$ with coefficients from R. As usual, let $p \mid q$ denote that q is a multiple of p, p/q denote the quotient of p divided by q, and $\mathrm{LCM}(p, q)$ denote the least common multiple of p and q.

Define

$$C(p, t) = \text{coefficient of } t \text{ in } p \in R[x_1, \ldots, x_n].$$

and define

$$M(p, t) = C(p, t) \cdot t$$

to be the monomial of t in p. We also define

$$S(p) = \{t : C(p, t) \neq 0\}$$

to be the support of p.

In the sequel, the ordering of the monomials in a polynomial is important. Suppose that $\prec$ is a total ordering on $[x_1, \ldots, x_n]$. Then $\prec$ is defined to be *admissable* if

$$\text{for all } t \neq 1,\ t \succ 1$$

and for all t, u, v, we have that

$$u \prec v \text{ implies that } t \cdot u \prec t \cdot v.$$

As an example, the lexicographic ordering on $[x_1, x_2]$ with $x_1 \prec x_2$ orders the power products of $[x_1, x_2]$ as follows:

$$\begin{array}{ccccccccc}
1 & \prec & x_1 & \prec & x_1^2 & \prec & x_1^3 & \prec & \cdots \\
x_2 & \prec & x_1x_2 & \prec & x_1^2x_2 & \prec & x_1^3x_2 & \prec & \cdots \\
x_2^2 & \prec & x_1x_2^2 & \prec & x_1^2x_2^2 & \prec & x_1^3x_2^2 & \prec & \cdots
\end{array}$$

and is an example of an admissible ordering. Observe that if $p \mid q$, then $p \preceq q$.

Another admissible ordering is the "total degree lexicographic" ordering where the terms are first ordered by their total degree, and then terms with the same degree are ordered lexicographically. For example, the total degree lexicographic ordering on $[x_1, x_2]$ with $x_1 \prec x_2$ orders the power products of $[x_1, x_2]$ as follows:

$$\begin{array}{ccccccccc}
1 & \prec & x_1 & \prec & x_2 & \prec & & & \\
x_1^2 & \prec & x_1x_2 & \prec & x_2^2 & \prec & & & \\
x_1^3 & \prec & x_1^2x_2 & \prec & x_1x_2^2 & \prec & x_2^3 & \prec & \cdots
\end{array}$$

It is an easy fact that for any admissible ordering $\prec$, that for all u, v,

$$u \mid v \text{ implies } u \preceq v.$$

By a combinatorial result known as Dickson's lemma, it can be shown that for any admissible ordering $\prec$, there does not exist an infinite descending chain. A relation with such a property is called *Noetherian*.

Suppose we fix some admissable ordering $\prec$ on $[x_1, \ldots, x_n]$. Then given a polynomial p, it is desirable to have notation for various parts of p. The Leading Power Product of p is defined to be

$$LPP_{\prec}(p) = \max_{\prec} S(p)$$

and the Leading Coefficient of p is defined to be

$$LC_{\prec}(p) = C(p, LPP_{\prec}(p)).$$

These two are used to define the Leading Monomial of p which is simply

$$LM_{\prec}(p) = LC_{\prec}(p) \cdot LPP_{\prec}(p).$$

Now, the part of p Higher than t is defined to be

$$H_{\prec}(p,t) = \sum_{u \in S(p), u \succ t} C(p,u) \cdot u$$

With these definitions, we can now show how $\prec$ can be extended to a relation on $R[x_1, \ldots, x_n]$. For any u, v, we have that $u \prec v$ if there exists

$$t \in S(v) \setminus S(u)$$

such that

$$H_{\prec}(u,t) = H_{\prec}(v,t).$$

The extended relation $\prec$ can be shown to be a partial order on $R[x_1, \ldots, x_n]$ and to be Noetherian. In addition, $\prec$ has the property that for all non-zero polynomials p, we have $p \succ 0$.

Given a polynomial g and a set of polynomials F, it is of interest to reduce g modulo the polynomials in F to a smaller polynomial with respect to the ordering $\prec$. We will make this notion precise and set a notation.

For $f \in F$, the polynomial g reduces to h modulo f, written

$$g \rightarrow_f h,$$

if there exists $t \in S(g)$ such that

$$LPP(f) \mid t \text{ and } h = g - f \cdot M(g,t)/LM(f).$$

We say that g reduces to h modulo F, written

$$g \rightarrow_F h,$$

if there exists $f \in F$ such that $g \rightarrow_f h$. In general, a polynomial g can be repeatedly reduced until no terms are divisible by any of the leading power products of any $f \in F$. In this case, where there does not exist h such that

$$g \rightarrow_F h,$$

we say that g is reduced modulo F and write

$$\underline{g}_F.$$

Now it can be shown that $\rightarrow_F$ is a Noetherian relation and that if $g \rightarrow_F h$, then $g \succ h$. Now let $\rightarrow_F^*$ be the reflexive-transitive closure of $\rightarrow_F$. By the absence of any infinite chains of reductions modulo F,

$$g \rightarrow_F^* h$$

is equivalent to g reducing to h by finitely-many reduction steps modulo F. By repeatedly performing reductions steps until no further reductions are possible, it follows that there exists an algorithm RF such that

$$g \rightarrow_F^* \underline{RF(F,g)}_F.$$

$RF(F,g)$ is said to be a Reduced Form of g modulo F. In general, given a g and F, a reduced form is not unique. However, the case where for a given F, the reduced form is always unique is important and motivates the definition of a Gröbner basis.

To make this notion precise, let $\longleftrightarrow^*$ denote the reflexive-symmetric-transitive closure of a Noetherian relation $\rightarrow$. Then, $\rightarrow$ is said to have the *Church-Rosser property* if

$$x \longleftrightarrow^* y \text{ implies that there exists z such that } x \rightarrow^* z \leftarrow^* y.$$

Then F is defined to be a *Gröbner basis* if $\rightarrow_F$ has the Church-Rosser property. For a given F, we are interested in the algorithmic problem of finding a Gröbner basis G such that

$$\longleftrightarrow_F^* = \longleftrightarrow_G^*$$

It turns out that an algorithm for computing a Gröbner basis is easily found. An important concept in the algorithm is the notion of an S-polynomial. If f_1, f_2 are two monic polynomials, then their S-polynomial is defined to be

$$SP(f_1, f_2) = LCM \cdot f_1/LPP(f_1) - LCM \cdot f_2/LPP(f_2)$$

where

$$LCM = LCM(LPP(f_1), LPP(f_2)).$$

The S-polynomial basically multiplies f_i by a monomial such that the leading term of f_i is equal to the least common multiple of f_1 and f_2, and then takes the difference so that the least common multiple vanishes.

The central theorem of Gröbner bases is the following: F is a Gröbner basis if and only for all $f_1, f_2 \in F$,

$$RF(F, SP(f_1, f_2)) = 0.$$

Given some F, we can use this theorem to test if it is a Gröbner basis, but more importantly, we can also compute a Gröbner basis for F. A naive algorithm is:

Algorithm 3 Naive algorithm for computing a Gröbner basis

$G \leftarrow F$
for any $f_1, f_2 \in G$ **do**
    $h \leftarrow RF(G, SP(f_1, f_2))$
    if $h = 0$ **then**
        do nothing
    else if $h \neq 0$ **then**
        $G \leftarrow G \cup \{h\}$
    end if
end for

This algorithm can be shown to be correct, in that at the termination of the algorithm, G is a Gröbner basis and $\text{Ideal}(F) = \text{Ideal}(G)$. Given some finite F, the Gröbner basis G computed by the algorithm may not be unique. However, there is a canonical form that is guaranteed to be unique. G is said to be a *reduced Gröbner basis* if all the polynomials in G are monic and

$$\text{for all } g \in G, \underline{g}_{G \setminus g}.$$

An important property of a Gröbner basis is that the question of whether $f \in \text{Ideal}(F)$ is easily decided.

$$f \in \text{Ideal}(F) \text{ if and only if } RF(\text{Gröbner-Basis}(F), f) = 0.$$

For a general set of polynomials F, this is difficult to decide.

If F is a finite subset of $R[x_1, \ldots, x_n]$ and $i \leq n$ where $\prec$ is such that

$$x_1 \prec x_2 \prec \ldots \prec x_i,$$

then

$$\text{Gröbner-Basis}_{\prec}(F) \cap R[x_1, \ldots, x_i]$$

is a Gröbner basis for

$$\text{Ideal}(F) \cap R[x_1, \ldots, x_i].$$

This is known as the "Elimination Problem" as it allows us to determine the solutions of the system of equations F just by first finding the solutions of the polynomial in just the first variable, then substituting those values into the next polynomial and so forth. It is extremely powerful to be able to find all solutions by solving "variable by variable." This certainly does not work in general.

5.2 Exact formula for a triangle problem

In the sequel, we will need a solution to the following problem in order to be able to do computations. Consider a point

$$f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} \in \mathbb{R}^2,$$

and let

$$r_1 = \begin{bmatrix} r_{1x} \\ r_{1y} \end{bmatrix}, r_2 = \begin{bmatrix} r_{2x} \\ r_{2y} \end{bmatrix}, r_3 = \begin{bmatrix} r_{3x} \\ r_{3y} \end{bmatrix}$$

be three vectors in $\mathbb{R}^2$ whose non-negative cone is all of $\mathbb{R}^2$. We are interested in determining a triangle with vertices

$$\{v_1, v_2, v_3\}$$

such that vertex v_i lies on the open ray

$$\{x \in \mathbb{R}^2 : x = f + \lambda_i r_i \text{ for } \lambda_i > 0\}$$

and the line segments

$$[v_1, v_2], [v_2, v_3], \text{ and } [v_3, v_1]$$

contain the points

$$p_1 = \begin{pmatrix} p_{1x} \\ p_{1y} \end{pmatrix}, \; p_2 = \begin{pmatrix} p_{2x} \\ p_{2y} \end{pmatrix}, \text{ and } p_3 = \begin{pmatrix} p_{3x} \\ p_{3y} \end{pmatrix}$$

respectively. To this end, we let

$$v_i = \begin{bmatrix} f_x \\ f_y \end{bmatrix} + \lambda_i \begin{bmatrix} r_{ix} \\ r_{iy} \end{bmatrix}.$$

Now the vector from v_2 to v_1 is

$$(f + \lambda_1 r_1) - (f + \lambda_2 r_2) = \lambda_1 r_1 - \lambda_2 r_2$$

and the vector from p_1 to v_1 is

$$f + \lambda_1 r_1 - p_1.$$

To model that p_1 lies on the line segment $[v_1, v_2]$, we write

$$(f + \lambda_1 r_1 - p_1)\mu_1 = \lambda_1 r_1 - \lambda_2 r_2$$

for some μ_1. Similarly, for the point p_2 we obtain

$$(f + \lambda_2 r_2 - p_2)\mu_2 = \lambda_2 r_2 - \lambda_3 r_3$$

for some μ_2 and for the point p_3, we have

$$(f + \lambda_3 r_3 - p_3)\mu_3 = \lambda_3 r_3 - \lambda_1 r_1$$

for some μ_3. So we have obtained a system of six equations in the six unknowns

$$\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3$$

and the system is not linear in these variables. So we compute a Gröbner basis. In Mathematica, the command is

```
GroebnerBasis[{(fx + l1*r1x - p1x)*mu1 == l1*r1x - l2*r2x,
                (fy + l1*r1y - p1y)*mu1 == l1*r1y - l2*r2y,
                (fx + l2*r2x - p2x)*mu2 == l2*r2x - l3*r3x,
                (fy + l2*r2y - p2y)*mu2 == l2*r2y - l3*r3y,
                (fx + l3*r3x - p3x)*mu3 == l3*r3x - l1*r1x,
                (fy + l3*r3y - p3y)*mu3 == l3*r3y - l1*r1y},
                {mu1,mu2,mu3, l3,l2,l1}]
```

The last line of the command indicates that we want a lexicographic ordering with

$$\mu_1 \succ \mu_2 \succ \mu_3 \succ \lambda_3 \succ \lambda_2 \succ \lambda_1.$$

Running this command on Mathematica 6.0 on a Linux machine resulted in an output with 32 polynomials $\{g_1, \ldots, g_{32}\}$. The output of `GroebnerBasis[]` in Mathematica is in general not a reduced basis. Wolfram Research, the publisher of Mathematica, calls the output a "semi-reduced" Gröbner basis. The first polynomial g_1 that was returned is

$$\begin{array}{cccccc}
- & \lambda_1 p_{2y} r_{1y} r_{2y} r_{3x} {f_x}^2 & + & \lambda_1 p_{3y} r_{1y} r_{2y} r_{3x} {f_x}^2 & - & \lambda_1 p_{1y} r_{1y} r_{2x} r_{3y} {f_x}^2 \\
+ & \lambda_1 p_{2y} r_{1y} r_{2x} r_{3y} {f_x}^2 & + & \lambda_1 p_{1y} r_{1x} r_{2y} r_{3y} {f_x}^2 & - & \lambda_1 p_{3y} r_{1x} r_{2y} r_{3y} {f_x}^2 \\
+ & f_y \lambda_1 p_{1y} r_{1y} r_{2x} r_{3x} f_x & - & \lambda_1 p_{1y} p_{2y} r_{1y} r_{2x} r_{3x} f_x & - & f_y \lambda_1 p_{3y} r_{1y} r_{2x} r_{3x} f_x \\
+ & \lambda_1 p_{2y} p_{3y} r_{1y} r_{2x} r_{3x} f_x & + & {\lambda_1}^2 p_{2x} {r_{1y}}^2 r_{2y} r_{3x} f_x & - & {\lambda_1}^2 p_{3x} {r_{1y}}^2 r_{2y} r_{3x} f_x \\
- & f_y \lambda_1 p_{1y} r_{1x} r_{2y} r_{3x} f_x & + & f_y \lambda_1 p_{2y} r_{1x} r_{2y} r_{3x} f_x & + & \lambda_1 p_{1y} p_{3y} r_{1x} r_{2y} r_{3x} f_x \\
- & \lambda_1 p_{2y} p_{3y} r_{1x} r_{2y} r_{3x} f_x & + & f_y \lambda_1 p_{2x} r_{1y} r_{2y} r_{3x} f_x & + & \lambda_1 p_{1x} p_{2y} r_{1y} r_{2y} r_{3x} f_x \\
- & f_y \lambda_1 p_{3x} r_{1y} r_{2y} r_{3x} f_x & + & \lambda_1 p_{2y} p_{3x} r_{1y} r_{2y} r_{3x} f_x & - & \lambda_1 p_{1x} p_{3y} r_{1y} r_{2y} r_{3x} f_x \\
- & \lambda_1 p_{2x} p_{3y} r_{1y} r_{2y} r_{3x} f_x & - & {\lambda_1}^2 p_{2y} r_{1x} r_{1y} r_{2y} r_{3x} f_x & + & {\lambda_1}^2 p_{3y} r_{1x} r_{1y} r_{2y} r_{3x} f_x \\
+ & {\lambda_1}^2 p_{1x} {r_{1y}}^2 r_{2x} r_{3y} f_x & - & {\lambda_1}^2 p_{2x} {r_{1y}}^2 r_{2x} r_{3y} f_x & - & f_y \lambda_1 p_{2y} r_{1x} r_{2x} r_{3y} f_x \\
+ & \lambda_1 p_{1y} p_{2y} r_{1x} r_{2x} r_{3y} f_x & + & f_y \lambda_1 p_{3y} r_{1x} r_{2x} r_{3y} f_x & - & \lambda_1 p_{1y} p_{3y} r_{1x} r_{2x} r_{3y} f_x \\
+ & f_y \lambda_1 p_{1x} r_{1y} r_{2x} r_{3y} f_x & - & f_y \lambda_1 p_{2x} r_{1y} r_{2x} r_{3y} f_x & + & \lambda_1 p_{1y} p_{2x} r_{1y} r_{2x} r_{3y} f_x \\
- & \lambda_1 p_{1x} p_{2y} r_{1y} r_{2x} r_{3y} f_x & + & \lambda_1 p_{1y} p_{3x} r_{1y} r_{2x} r_{3y} f_x & - & \lambda_1 p_{2y} p_{3x} r_{1y} r_{2x} r_{3y} f_x \\
- & {\lambda_1}^2 p_{1y} r_{1x} r_{1y} r_{2x} r_{3y} f_x & + & {\lambda_1}^2 p_{2y} r_{1x} r_{1y} r_{2x} r_{3y} f_x & + & {\lambda_1}^2 p_{1y} {r_{1x}}^2 r_{2y} r_{3y} f_x \\
- & {\lambda_1}^2 p_{3y} {r_{1x}}^2 r_{2y} r_{3y} f_x & - & f_y \lambda_1 p_{1x} r_{1x} r_{2y} r_{3y} f_x & - & \lambda_1 p_{1y} p_{2x} r_{1x} r_{2y} r_{3y} f_x \\
+ & f_y \lambda_1 p_{3x} r_{1x} r_{2y} r_{3y} f_x & - & \lambda_1 p_{1y} p_{3x} r_{1x} r_{2y} r_{3y} f_x & + & \lambda_1 p_{1x} p_{3y} r_{1x} r_{2y} r_{3y} f_x \\
+ & \lambda_1 p_{2x} p_{3y} r_{1x} r_{2y} r_{3y} f_x & - & {\lambda_1}^2 p_{1x} r_{1x} r_{1y} r_{2y} r_{3y} f_x & + & {\lambda_1}^2 p_{3x} r_{1x} r_{1y} r_{2y} r_{3y} f_x \\
- & f_y {\lambda_1}^2 p_{1x} {r_{1y}}^2 r_{2x} r_{3x} & + & {\lambda_1}^2 p_{1x} p_{2y} {r_{1y}}^2 r_{2x} r_{3x} & + & f_y {\lambda_1}^2 p_{3x} {r_{1y}}^2 r_{2x} r_{3x} \\
- & {\lambda_1}^2 p_{2y} p_{3x} {r_{1y}}^2 r_{2x} r_{3x} & - & {f_y}^2 \lambda_1 p_{1x} r_{1y} r_{2x} r_{3x} & + & f_y \lambda_1 p_{1x} p_{2y} r_{1y} r_{2x} r_{3x} \\
+ & {f_y}^2 \lambda_1 p_{3x} r_{1y} r_{2x} r_{3x} & - & f_y \lambda_1 p_{1y} p_{3x} r_{1y} r_{2x} r_{3x} & - & f_y \lambda_1 p_{2y} p_{3x} r_{1y} r_{2x} r_{3x} \\
+ & \lambda_1 p_{1y} p_{2y} p_{3x} r_{1y} r_{2x} r_{3x} & + & f_y \lambda_1 p_{1x} p_{3y} r_{1y} r_{2x} r_{3x} & - & \lambda_1 p_{1x} p_{2y} p_{3y} r_{1y} r_{2x} r_{3x} \\
+ & f_y {\lambda_1}^2 p_{1y} r_{1x} r_{1y} r_{2x} r_{3x} & - & {\lambda_1}^2 p_{1y} p_{2y} r_{1x} r_{1y} r_{2x} r_{3x} & - & f_y {\lambda_1}^2 p_{3y} r_{1x} r_{1y} r_{2x} r_{3x}
\end{array}$$

$$
\begin{array}{llllll}
+ & {\lambda_1}^2 p_{2y}p_{3y}r_{1x}r_{1y}r_{2x}r_{3x} & - & f_y{\lambda_1}^2 p_{1y}{r_{1x}}^2 r_{2y}r_{3x} & + & f_y{\lambda_1}^2 p_{2y}{r_{1x}}^2 r_{2y}r_{3x} \\
+ & {\lambda_1}^2 p_{1y}p_{3y}{r_{1x}}^2 r_{2y}r_{3x} & - & {\lambda_1}^2 p_{2y}p_{3y}{r_{1x}}^2 r_{2y}r_{3x} & - & {\lambda_1}^2 p_{1x}p_{2x}{r_{1y}}^2 r_{2y}r_{3x} \\
+ & {\lambda_1}^2 p_{1x}p_{3x}{r_{1y}}^2 r_{2y}r_{3x} & + & {f_y}^2\lambda_1 p_{1x}r_{1x}r_{2y}r_{3x} & - & {f_y}^2\lambda_1 p_{2x}r_{1x}r_{2y}r_{3x} \\
+ & f_y\lambda_1 p_{1y}p_{2x}r_{1x}r_{2y}r_{3x} & - & f_y\lambda_1 p_{1x}p_{2y}r_{1x}r_{2y}r_{3x} & - & f_y\lambda_1 p_{1x}p_{3y}r_{1x}r_{2y}r_{3x} \\
+ & f_y\lambda_1 p_{2x}p_{3y}r_{1x}r_{2y}r_{3x} & - & \lambda_1 p_{1y}p_{2x}p_{3y}r_{1x}r_{2y}r_{3x} & + & \lambda_1 p_{1x}p_{2y}p_{3y}r_{1x}r_{2y}r_{3x} \\
- & f_y\lambda_1 p_{1x}p_{2x}r_{1y}r_{2y}r_{3x} & + & f_y\lambda_1 p_{1x}p_{3x}r_{1y}r_{2y}r_{3x} & - & \lambda_1 p_{1x}p_{2y}p_{3x}r_{1y}r_{2y}r_{3x} \\
+ & \lambda_1 p_{1x}p_{2x}p_{3y}r_{1y}r_{2y}r_{3x} & + & f_y{\lambda_1}^2 p_{1x}r_{1x}r_{1y}r_{2y}r_{3x} & - & f_y{\lambda_1}^2 p_{2x}r_{1x}r_{1y}r_{2y}r_{3x} \\
+ & {\lambda_1}^2 p_{1y}p_{2x}r_{1x}r_{1y}r_{2y}r_{3x} & - & {\lambda_1}^2 p_{1y}p_{3x}r_{1x}r_{1y}r_{2y}r_{3x} & + & {\lambda_1}^2 p_{2y}p_{3x}r_{1x}r_{1y}r_{2y}r_{3x} \\
- & {\lambda_1}^2 p_{1x}p_{3y}r_{1x}r_{1y}r_{2y}r_{3x} & - & f_y{\lambda_1}^2 p_{2y}{r_{1x}}^2 r_{2x}r_{3y} & + & {\lambda_1}^2 p_{1y}p_{2y}{r_{1x}}^2 r_{2x}r_{3y} \\
+ & f_y{\lambda_1}^2 p_{3y}{r_{1x}}^2 r_{2x}r_{3y} & - & {\lambda_1}^2 p_{1y}p_{3y}{r_{1x}}^2 r_{2x}r_{3y} & - & {\lambda_1}^2 p_{1x}p_{3x}{r_{1y}}^2 r_{2x}r_{3y} \\
+ & {\lambda_1}^2 p_{2x}p_{3x}{r_{1y}}^2 r_{2x}r_{3y} & + & {f_y}^2\lambda_1 p_{2x}r_{1x}r_{2x}r_{3y} & - & f_y\lambda_1 p_{1y}p_{2x}r_{1x}r_{2x}r_{3y} \\
- & {f_y}^2\lambda_1 p_{3x}r_{1x}r_{2x}r_{3y} & + & f_y\lambda_1 p_{1y}p_{3x}r_{1x}r_{2x}r_{3y} & + & f_y\lambda_1 p_{2y}p_{3x}r_{1x}r_{2x}r_{3y} \\
- & \lambda_1 p_{1y}p_{2y}p_{3x}r_{1x}r_{2x}r_{3y} & - & f_y\lambda_1 p_{2x}p_{3y}r_{1x}r_{2x}r_{3y} & + & \lambda_1 p_{1y}p_{2x}p_{3y}r_{1x}r_{2x}r_{3y} \\
- & f_y\lambda_1 p_{1x}p_{3x}r_{1y}r_{2x}r_{3y} & + & f_y\lambda_1 p_{2x}p_{3x}r_{1y}r_{2x}r_{3y} & - & \lambda_1 p_{1y}p_{2x}p_{3x}r_{1y}r_{2x}r_{3y} \\
+ & \lambda_1 p_{1x}p_{2y}p_{3x}r_{1y}r_{2x}r_{3y} & + & f_y{\lambda_1}^2 p_{2x}r_{1x}r_{1y}r_{2x}r_{3y} & - & {\lambda_1}^2 p_{1x}p_{2y}r_{1x}r_{1y}r_{2x}r_{3y} \\
- & f_y{\lambda_1}^2 p_{3x}r_{1x}r_{1y}r_{2x}r_{3y} & + & {\lambda_1}^2 p_{1y}p_{3x}r_{1x}r_{1y}r_{2x}r_{3y} & + & {\lambda_1}^2 p_{1x}p_{3y}r_{1x}r_{1y}r_{2x}r_{3y} \\
- & {\lambda_1}^2 p_{2x}p_{3y}r_{1x}r_{1y}r_{2x}r_{3y} & - & {\lambda_1}^2 p_{1y}p_{2x}{r_{1x}}^2 r_{2y}r_{3y} & + & {\lambda_1}^2 p_{2x}p_{3y}{r_{1x}}^2 r_{2y}r_{3y} \\
+ & f_y\lambda_1 p_{1x}p_{2x}r_{1x}r_{2y}r_{3y} & - & f_y\lambda_1 p_{2x}p_{3x}r_{1x}r_{2y}r_{3y} & + & \lambda_1 p_{1y}p_{2x}p_{3x}r_{1x}r_{2y}r_{3y} \\
- & \lambda_1 p_{1x}p_{2x}p_{3y}r_{1x}r_{2y}r_{3y} & + & {\lambda_1}^2 p_{1x}p_{2x}r_{1x}r_{1y}r_{2y}r_{3y} & - & {\lambda_1}^2 p_{2x}p_{3x}r_{1x}r_{1y}r_{2y}r_{3y}.
\end{array}
$$

As expected, observe that this polynomial is a univariate polynomial in the variable λ_1. Fortunately, it is just a quadratic equation. Before solving it, we list the remaining 31 polynomials that were computed.

The next polynomial g_2 is

$$
\begin{array}{llllll}
- & f_y\lambda_1 p_{1y}r_{2y}r_{3x}{r_{1x}}^2 & + & f_y\lambda_1 p_{2y}r_{2y}r_{3x}{r_{1x}}^2 & + & \lambda_1 p_{1y}p_{3y}r_{2y}r_{3x}{r_{1x}}^2 \\
- & \lambda_1 p_{2y}p_{3y}r_{2y}r_{3x}{r_{1x}}^2 & - & f_y\lambda_1 p_{2y}r_{2x}r_{3y}{r_{1x}}^2 & + & \lambda_1 p_{1y}p_{2y}r_{2x}r_{3y}{r_{1x}}^2 \\
+ & f_y\lambda_1 p_{3y}r_{2x}r_{3y}{r_{1x}}^2 & - & \lambda_1 p_{1y}p_{3y}r_{2x}r_{3y}{r_{1x}}^2 & + & f_x\lambda_1 p_{1y}r_{2y}r_{3y}{r_{1x}}^2 \\
- & \lambda_1 p_{1y}p_{2x}r_{2y}r_{3y}{r_{1x}}^2 & - & f_x\lambda_1 p_{3y}r_{2y}r_{3y}{r_{1x}}^2 & + & \lambda_1 p_{2x}p_{3y}r_{2y}r_{3y}{r_{1x}}^2 \\
- & f_y\lambda_2 p_{1x}{r_{2y}}^2 r_{3x}r_{1x} & + & f_y\lambda_2 p_{2x}{r_{2y}}^2 r_{3x}r_{1x} & + & \lambda_2 p_{1x}p_{3y}{r_{2y}}^2 r_{3x}r_{1x}
\end{array}
$$

$$
\begin{array}{llllll}
- & \lambda_2 p_{2x} p_{3y} {r_{2y}}^2 r_{3x} r_{1x} & + & f_y \lambda_1 p_{1y} r_{1y} r_{2x} r_{3x} r_{1x} & - & \lambda_1 p_{1y} p_{2y} r_{1y} r_{2x} r_{3x} r_{1x} \\
- & f_y \lambda_1 p_{3y} r_{1y} r_{2x} r_{3x} r_{1x} & + & \lambda_1 p_{2y} p_{3y} r_{1y} r_{2x} r_{3x} r_{1x} & + & f_y \lambda_1 p_{1x} r_{1y} r_{2y} r_{3x} r_{1x} \\
- & f_y \lambda_1 p_{2x} r_{1y} r_{2y} r_{3x} r_{1x} & + & \lambda_1 p_{1y} p_{2x} r_{1y} r_{2y} r_{3x} r_{1x} & - & f_x \lambda_1 p_{2y} r_{1y} r_{2y} r_{3x} r_{1x} \\
- & \lambda_1 p_{1y} p_{3x} r_{1y} r_{2y} r_{3x} r_{1x} & + & \lambda_1 p_{2y} p_{3x} r_{1y} r_{2y} r_{3x} r_{1x} & + & f_x \lambda_1 p_{3y} r_{1y} r_{2y} r_{3x} r_{1x} \\
- & \lambda_1 p_{1x} p_{3y} r_{1y} r_{2y} r_{3x} r_{1x} & + & f_y \lambda_2 p_{1y} r_{2x} r_{2y} r_{3x} r_{1x} & - & f_y \lambda_2 p_{2y} r_{2x} r_{2y} r_{3x} r_{1x} \\
- & \lambda_2 p_{1y} p_{3y} r_{2x} r_{2y} r_{3x} r_{1x} & + & \lambda_2 p_{2y} p_{3y} r_{2x} r_{2y} r_{3x} r_{1x} & + & f_y \lambda_2 p_{2y} {r_{2x}}^2 r_{3y} r_{1x} \\
- & \lambda_2 p_{1y} p_{2y} {r_{2x}}^2 r_{3y} r_{1x} & - & f_y \lambda_2 p_{3y} {r_{2x}}^2 r_{3y} r_{1x} & + & \lambda_2 p_{1y} p_{3y} {r_{2x}}^2 r_{3y} r_{1x} \\
+ & f_x \lambda_2 p_{1x} {r_{2y}}^2 r_{3y} r_{1x} & - & \lambda_2 p_{1x} p_{2x} {r_{2y}}^2 r_{3y} r_{1x} & - & f_x \lambda_2 p_{3x} {r_{2y}}^2 r_{3y} r_{1x} \\
+ & \lambda_2 p_{2x} p_{3x} {r_{2y}}^2 r_{3y} r_{1x} & - & f_x \lambda_1 p_{1y} r_{1y} r_{2x} r_{3y} r_{1x} & + & f_y \lambda_1 p_{2x} r_{1y} r_{2x} r_{3y} r_{1x} \\
+ & f_x \lambda_1 p_{2y} r_{1y} r_{2x} r_{3y} r_{1x} & - & \lambda_1 p_{1x} p_{2y} r_{1y} r_{2x} r_{3y} r_{1x} & - & f_y \lambda_1 p_{3x} r_{1y} r_{2x} r_{3y} r_{1x} \\
+ & \lambda_1 p_{1y} p_{3x} r_{1y} r_{2x} r_{3y} r_{1x} & + & \lambda_1 p_{1x} p_{3y} r_{1y} r_{2x} r_{3y} r_{1x} & - & \lambda_1 p_{2x} p_{3y} r_{1y} r_{2x} r_{3y} r_{1x} \\
- & f_x \lambda_1 p_{1x} r_{1y} r_{2y} r_{3y} r_{1x} & + & \lambda_1 p_{1x} p_{2x} r_{1y} r_{2y} r_{3y} r_{1x} & + & f_x \lambda_1 p_{3x} r_{1y} r_{2y} r_{3y} r_{1x} \\
- & \lambda_1 p_{2x} p_{3x} r_{1y} r_{2y} r_{3y} r_{1x} & - & f_x \lambda_2 p_{1y} r_{2x} r_{2y} r_{3y} r_{1x} & - & f_y \lambda_2 p_{2x} r_{2x} r_{2y} r_{3y} r_{1x} \\
+ & \lambda_2 p_{1y} p_{2x} r_{2x} r_{2y} r_{3y} r_{1x} & + & \lambda_2 p_{1x} p_{2y} r_{2x} r_{2y} r_{3y} r_{1x} & + & f_y \lambda_2 p_{3x} r_{2x} r_{2y} r_{3y} r_{1x} \\
- & \lambda_2 p_{2y} p_{3x} r_{2x} r_{2y} r_{3y} r_{1x} & + & f_x \lambda_2 p_{3y} r_{2x} r_{2y} r_{3y} r_{1x} & - & \lambda_2 p_{1x} p_{3y} r_{2x} r_{2y} r_{3y} r_{1x} \\
- & f_y \lambda_2 p_{1y} r_{1y} {r_{2x}}^2 r_{3x} & + & \lambda_2 p_{1y} p_{2y} r_{1y} {r_{2x}}^2 r_{3x} & + & f_y \lambda_2 p_{3y} r_{1y} {r_{2x}}^2 r_{3x} \\
- & \lambda_2 p_{2y} p_{3y} r_{1y} {r_{2x}}^2 r_{3x} & - & f_x \lambda_2 p_{2x} r_{1y} {r_{2y}}^2 r_{3x} & + & \lambda_2 p_{1x} p_{2x} r_{1y} {r_{2y}}^2 r_{3x} \\
+ & f_x \lambda_2 p_{3x} r_{1y} {r_{2y}}^2 r_{3x} & - & \lambda_2 p_{1x} p_{3x} r_{1y} {r_{2y}}^2 r_{3x} & - & f_y \lambda_1 p_{1x} {r_{1y}}^2 r_{2x} r_{3x} \\
+ & \lambda_1 p_{1x} p_{2y} {r_{1y}}^2 r_{2x} r_{3x} & + & f_y \lambda_1 p_{3x} {r_{1y}}^2 r_{2x} r_{3x} & - & \lambda_1 p_{2y} p_{3x} {r_{1y}}^2 r_{2x} r_{3x} \\
+ & f_x \lambda_1 p_{2x} {r_{1y}}^2 r_{2y} r_{3x} & - & \lambda_1 p_{1x} p_{2x} {r_{1y}}^2 r_{2y} r_{3x} & - & f_x \lambda_1 p_{3x} {r_{1y}}^2 r_{2y} r_{3x} \\
+ & \lambda_1 p_{1x} p_{3x} {r_{1y}}^2 r_{2y} r_{3x} & + & f_y \lambda_2 p_{1x} r_{1y} r_{2x} r_{2y} r_{3x} & - & \lambda_2 p_{1y} p_{2x} r_{1y} r_{2x} r_{2y} r_{3x} \\
+ & f_x \lambda_2 p_{2y} r_{1y} r_{2x} r_{2y} r_{3x} & - & \lambda_2 p_{1x} p_{2y} r_{1y} r_{2x} r_{2y} r_{3x} & - & f_y \lambda_2 p_{3x} r_{1y} r_{2x} r_{2y} r_{3x} \\
+ & \lambda_2 p_{1y} p_{3x} r_{1y} r_{2x} r_{2y} r_{3x} & - & f_x \lambda_2 p_{3y} r_{1y} r_{2x} r_{2y} r_{3x} & + & \lambda_2 p_{2x} p_{3y} r_{1y} r_{2x} r_{2y} r_{3x} \\
+ & f_x \lambda_2 p_{1y} r_{1y} {r_{2x}}^2 r_{3y} & - & f_x \lambda_2 p_{2y} r_{1y} {r_{2x}}^2 r_{3y} & - & \lambda_2 p_{1y} p_{3x} r_{1y} {r_{2x}}^2 r_{3y} \\
+ & \lambda_2 p_{2y} p_{3x} r_{1y} {r_{2x}}^2 r_{3y} & + & f_x \lambda_1 p_{1x} {r_{1y}}^2 r_{2x} r_{3y} & - & f_x \lambda_1 p_{2x} {r_{1y}}^2 r_{2x} r_{3y} \\
- & \lambda_1 p_{1x} p_{3x} {r_{1y}}^2 r_{2x} r_{3y} & + & \lambda_1 p_{2x} p_{3x} {r_{1y}}^2 r_{2x} r_{3y} & - & f_x \lambda_2 p_{1x} r_{1y} r_{2x} r_{2y} r_{3y} \\
+ & f_x \lambda_2 p_{2x} r_{1y} r_{2x} r_{2y} r_{3y} & + & \lambda_2 p_{1x} p_{3x} r_{1y} r_{2x} r_{2y} r_{3y} & - & \lambda_2 p_{2x} p_{3x} r_{1y} r_{2x} r_{2y} r_{3y}.
\end{array}
$$

The polynomial g_3 is

$$
\begin{array}{rcrcrcr}
 & & f_y\lambda_2 p_{2y} r_{3y} {r_{2x}}^2 & - & \lambda_2 p_{1y} p_{2y} r_{3y} {r_{2x}}^2 & - & f_y\lambda_2 p_{3y} r_{3y} {r_{2x}}^2 \\
+ & & \lambda_2 p_{1y} p_{3y} r_{3y} {r_{2x}}^2 & + & \lambda_1\lambda_2 p_{2y} r_{1y} r_{3y} {r_{2x}}^2 & - & \lambda_1\lambda_2 p_{3y} r_{1y} r_{3y} {r_{2x}}^2 \\
+ & & f_y\lambda_2 p_{1y} r_{2y} r_{3x} r_{2x} & - & f_y\lambda_2 p_{2y} r_{2y} r_{3x} r_{2x} & - & \lambda_2 p_{1y} p_{3y} r_{2y} r_{3x} r_{2x} \\
+ & & \lambda_2 p_{2y} p_{3y} r_{2y} r_{3x} r_{2x} & - & \lambda_1\lambda_2 p_{2y} r_{1y} r_{2y} r_{3x} r_{2x} & + & \lambda_1\lambda_2 p_{3y} r_{1y} r_{2y} r_{3x} r_{2x} \\
- & & f_y\lambda_1 p_{2y} r_{1x} r_{3y} r_{2x} & + & \lambda_1 p_{1y} p_{2y} r_{1x} r_{3y} r_{2x} & + & f_y\lambda_1 p_{3y} r_{1x} r_{3y} r_{2x} \\
- & & \lambda_1 p_{1y} p_{3y} r_{1x} r_{3y} r_{2x} & + & f_x\lambda_1 p_{2y} r_{1y} r_{3y} r_{2x} & - & \lambda_1 p_{1x} p_{2y} r_{1y} r_{3y} r_{2x} \\
- & & f_x\lambda_1 p_{3y} r_{1y} r_{3y} r_{2x} & + & \lambda_1 p_{1x} p_{3y} r_{1y} r_{3y} r_{2x} & - & f_x\lambda_2 p_{1y} r_{2y} r_{3y} r_{2x} \\
- & & f_y\lambda_2 p_{2x} r_{2y} r_{3y} r_{2x} & + & \lambda_2 p_{1y} p_{2x} r_{2y} r_{3y} r_{2x} & + & \lambda_2 p_{1x} p_{2y} r_{2y} r_{3y} r_{2x} \\
+ & & f_y\lambda_2 p_{3x} r_{2y} r_{3y} r_{2x} & - & \lambda_2 p_{2y} p_{3x} r_{2y} r_{3y} r_{2x} & + & f_x\lambda_2 p_{3y} r_{2y} r_{3y} r_{2x} \\
- & & \lambda_2 p_{1x} p_{3y} r_{2y} r_{3y} r_{2x} & - & \lambda_1\lambda_2 p_{2x} r_{1y} r_{2y} r_{3y} r_{2x} & + & \lambda_1\lambda_2 p_{3x} r_{1y} r_{2y} r_{3y} r_{2x} \\
- & & f_y\lambda_2 p_{1x} {r_{2y}}^2 r_{3x} & + & f_y\lambda_2 p_{2x} {r_{2y}}^2 r_{3x} & + & \lambda_2 p_{1x} p_{3y} {r_{2y}}^2 r_{3x} \\
- & & \lambda_2 p_{2x} p_{3y} {r_{2y}}^2 r_{3x} & + & \lambda_1\lambda_2 p_{2x} r_{1y} {r_{2y}}^2 r_{3x} & - & \lambda_1\lambda_2 p_{3x} r_{1y} {r_{2y}}^2 r_{3x} \\
- & & f_y\lambda_1 p_{1y} r_{1x} r_{2y} r_{3x} & + & f_y\lambda_1 p_{2y} r_{1x} r_{2y} r_{3x} & + & \lambda_1 p_{1y} p_{3y} r_{1x} r_{2y} r_{3x} \\
- & & \lambda_1 p_{2y} p_{3y} r_{1x} r_{2y} r_{3x} & + & f_y\lambda_1 p_{1x} r_{1y} r_{2y} r_{3x} & - & f_x\lambda_1 p_{2y} r_{1y} r_{2y} r_{3x} \\
- & & f_y\lambda_1 p_{3x} r_{1y} r_{2y} r_{3x} & + & \lambda_1 p_{2y} p_{3x} r_{1y} r_{2y} r_{3x} & + & f_x\lambda_1 p_{3y} r_{1y} r_{2y} r_{3x} \\
- & & \lambda_1 p_{1x} p_{3y} r_{1y} r_{2y} r_{3x} & + & f_x\lambda_2 p_{1x} {r_{2y}}^2 r_{3y} & - & \lambda_2 p_{1x} p_{2x} {r_{2y}}^2 r_{3y} \\
- & & f_x\lambda_2 p_{3x} {r_{2y}}^2 r_{3y} & + & \lambda_2 p_{2x} p_{3x} {r_{2y}}^2 r_{3y} & + & f_x\lambda_1 p_{1y} r_{1x} r_{2y} r_{3y} \\
- & & \lambda_1 p_{1y} p_{2x} r_{1x} r_{2y} r_{3y} & - & f_x\lambda_1 p_{3y} r_{1x} r_{2y} r_{3y} & + & \lambda_1 p_{2x} p_{3y} r_{1x} r_{2y} r_{3y} \\
- & & f_x\lambda_1 p_{1x} r_{1y} r_{2y} r_{3y} & + & \lambda_1 p_{1x} p_{2x} r_{1y} r_{2y} r_{3y} & + & f_x\lambda_1 p_{3x} r_{1y} r_{2y} r_{3y} \\
- & & \lambda_1 p_{2x} p_{3x} r_{1y} r_{2y} r_{3y}. & & & &
\end{array}
$$

The polynomial g_4 is

$$
\begin{array}{rcrcrcrcrcr}
 & f_y\lambda_1 r_{1x} & - & \lambda_1 p_{1y} r_{1x} & + & \lambda_1\lambda_2 r_{2y} r_{1x} & - & f_x\lambda_1 r_{1y} & + & \lambda_1 p_{1x} r_{1y} \\
- & f_y\lambda_2 r_{2x} & + & \lambda_2 p_{1y} r_{2x} & - & \lambda_1\lambda_2 r_{1y} r_{2x} & + & f_x\lambda_2 r_{2y} & - & \lambda_2 p_{1x} r_{2y}.
\end{array}
$$

The polynomial g_5 is

$$
\begin{array}{llllll}
- & f_y\lambda_1 p_{1y} r_{1x} r_{3x} & + & f_y\lambda_1 p_{2y} r_{1x} r_{3x} & + & \lambda_1 p_{1y} p_{3y} r_{1x} r_{3x} \\
- & \lambda_1 p_{2y} p_{3y} r_{1x} r_{3x} & + & f_y\lambda_1 p_{1x} r_{1y} r_{3x} & - & f_x\lambda_1 p_{2y} r_{1y} r_{3x} \\
- & f_y\lambda_1 p_{3x} r_{1y} r_{3x} & + & \lambda_1 p_{2y} p_{3x} r_{1y} r_{3x} & + & f_x\lambda_1 p_{3y} r_{1y} r_{3x} \\
- & \lambda_1 p_{1x} p_{3y} r_{1y} r_{3x} & + & f_y\lambda_2 p_{1y} r_{2x} r_{3x} & - & f_y\lambda_2 p_{2y} r_{2x} r_{3x} \\
- & \lambda_2 p_{1y} p_{3y} r_{2x} r_{3x} & + & \lambda_2 p_{2y} p_{3y} r_{2x} r_{3x} & - & \lambda_1\lambda_2 p_{2y} r_{1y} r_{2x} r_{3x} \\
+ & \lambda_1\lambda_2 p_{3y} r_{1y} r_{2x} r_{3x} & - & f_y\lambda_2 p_{1x} r_{2y} r_{3x} & + & f_y\lambda_2 p_{2x} r_{2y} r_{3x} \\
+ & \lambda_2 p_{1x} p_{3y} r_{2y} r_{3x} & - & \lambda_2 p_{2x} p_{3y} r_{2y} r_{3x} & + & \lambda_1\lambda_2 p_{2x} r_{1y} r_{2y} r_{3x} \\
- & \lambda_1\lambda_2 p_{3x} r_{1y} r_{2y} r_{3x} & + & f_x\lambda_1 p_{1y} r_{1x} r_{3y} & - & \lambda_1 p_{1y} p_{2x} r_{1x} r_{3y} \\
- & f_x\lambda_1 p_{3y} r_{1x} r_{3y} & + & \lambda_1 p_{2x} p_{3y} r_{1x} r_{3y} & - & f_x\lambda_1 p_{1x} r_{1y} r_{3y} \\
+ & \lambda_1 p_{1x} p_{2x} r_{1y} r_{3y} & + & f_x\lambda_1 p_{3x} r_{1y} r_{3y} & - & \lambda_1 p_{2x} p_{3x} r_{1y} r_{3y} \\
- & f_x\lambda_2 p_{1y} r_{2x} r_{3y} & - & f_y\lambda_2 p_{2x} r_{2x} r_{3y} & + & \lambda_2 p_{1y} p_{2x} r_{2x} r_{3y} \\
+ & f_x\lambda_2 p_{2y} r_{2x} r_{3y} & + & f_y\lambda_2 p_{3x} r_{2x} r_{3y} & - & \lambda_2 p_{2y} p_{3x} r_{2x} r_{3y} \\
+ & \lambda_1\lambda_2 p_{2y} r_{1x} r_{2x} r_{3y} & - & \lambda_1\lambda_2 p_{3y} r_{1x} r_{2x} r_{3y} & - & \lambda_1\lambda_2 p_{2x} r_{1y} r_{2x} r_{3y} \\
+ & \lambda_1\lambda_2 p_{3x} r_{1y} r_{2x} r_{3y} & + & f_x\lambda_2 p_{1x} r_{2y} r_{3y} & - & \lambda_2 p_{1x} p_{2x} r_{2y} r_{3y} \\
- & f_x\lambda_2 p_{3x} r_{2y} r_{3y} & + & \lambda_2 p_{2x} p_{3x} r_{2y} r_{3y}. & &
\end{array}
$$

The polynomial g_6 is

$$
\begin{array}{llllll}
- & f_y\lambda_1 p_{1y} r_{2y} r_{3x} {r_{1x}}^2 & + & f_y\lambda_1 p_{2y} r_{2y} r_{3x} {r_{1x}}^2 & + & \lambda_1 p_{1y} p_{3y} r_{2y} r_{3x} {r_{1x}}^2 \\
- & \lambda_1 p_{2y} p_{3y} r_{2y} r_{3x} {r_{1x}}^2 & - & f_y\lambda_1 p_{2y} r_{2x} r_{3y} {r_{1x}}^2 & + & \lambda_1 p_{1y} p_{2y} r_{2x} r_{3y} {r_{1x}}^2 \\
+ & f_y\lambda_1 p_{3y} r_{2x} r_{3y} {r_{1x}}^2 & - & \lambda_1 p_{1y} p_{3y} r_{2x} r_{3y} {r_{1x}}^2 & + & f_x\lambda_1 p_{1y} r_{2y} r_{3y} {r_{1x}}^2 \\
- & \lambda_1 p_{1y} p_{2x} r_{2y} r_{3y} {r_{1x}}^2 & - & f_x\lambda_1 p_{3y} r_{2y} r_{3y} {r_{1x}}^2 & + & \lambda_1 p_{2x} p_{3y} r_{2y} r_{3y} {r_{1x}}^2 \\
+ & f_y\lambda_3 p_{1y} r_{2y} {r_{3x}}^2 r_{1x} & - & f_y\lambda_3 p_{2y} r_{2y} {r_{3x}}^2 r_{1x} & - & \lambda_3 p_{1y} p_{3y} r_{2y} {r_{3x}}^2 r_{1x} \\
+ & \lambda_3 p_{2y} p_{3y} r_{2y} {r_{3x}}^2 r_{1x} & - & f_y\lambda_3 p_{2x} r_{2x} {r_{3y}}^2 r_{1x} & + & \lambda_3 p_{1y} p_{2x} r_{2x} {r_{3y}}^2 r_{1x} \\
+ & f_y\lambda_3 p_{3x} r_{2x} {r_{3y}}^2 r_{1x} & - & \lambda_3 p_{1y} p_{3x} r_{2x} {r_{3y}}^2 r_{1x} & + & f_x\lambda_3 p_{1x} r_{2y} {r_{3y}}^2 r_{1x} \\
- & \lambda_3 p_{1x} p_{2x} r_{2y} {r_{3y}}^2 r_{1x} & - & f_x\lambda_3 p_{3x} r_{2y} {r_{3y}}^2 r_{1x} & + & \lambda_3 p_{2x} p_{3x} r_{2y} {r_{3y}}^2 r_{1x} \\
+ & f_y\lambda_1 p_{1y} r_{1y} r_{2x} r_{3x} r_{1x} & - & \lambda_1 p_{1y} p_{2y} r_{1y} r_{2x} r_{3x} r_{1x} & - & f_y\lambda_1 p_{3y} r_{1y} r_{2x} r_{3x} r_{1x}
\end{array}
$$

$$
\begin{aligned}
&+ \lambda_1 p_{2y} p_{3y} r_{1y} r_{2x} r_{3x} r_{1x} + f_y \lambda_1 p_{1x} r_{1y} r_{2y} r_{3x} r_{1x} - f_y \lambda_1 p_{2x} r_{1y} r_{2y} r_{3x} r_{1x} \\
&+ \lambda_1 p_{1y} p_{2x} r_{1y} r_{2y} r_{3x} r_{1x} - f_x \lambda_1 p_{2y} r_{1y} r_{2y} r_{3x} r_{1x} - \lambda_1 p_{1y} p_{3x} r_{1y} r_{2y} r_{3x} r_{1x} \\
&+ \lambda_1 p_{2y} p_{3x} r_{1y} r_{2y} r_{3x} r_{1x} + f_x \lambda_1 p_{3y} r_{1y} r_{2y} r_{3x} r_{1x} - \lambda_1 p_{1x} p_{3y} r_{1y} r_{2y} r_{3x} r_{1x} \\
&- f_x \lambda_1 p_{1y} r_{1y} r_{2x} r_{3y} r_{1x} + f_y \lambda_1 p_{2x} r_{1y} r_{2x} r_{3y} r_{1x} + f_x \lambda_1 p_{2y} r_{1y} r_{2x} r_{3y} r_{1x} \\
&- \lambda_1 p_{1x} p_{2y} r_{1y} r_{2x} r_{3y} r_{1x} - f_y \lambda_1 p_{3x} r_{1y} r_{2x} r_{3y} r_{1x} + \lambda_1 p_{1y} p_{3x} r_{1y} r_{2x} r_{3y} r_{1x} \\
&+ \lambda_1 p_{1x} p_{3y} r_{1y} r_{2x} r_{3y} r_{1x} - \lambda_1 p_{2x} p_{3y} r_{1y} r_{2x} r_{3y} r_{1x} - f_x \lambda_1 p_{1x} r_{1y} r_{2y} r_{3y} r_{1x} \\
&+ \lambda_1 p_{1x} p_{2x} r_{1y} r_{2y} r_{3y} r_{1x} + f_x \lambda_1 p_{3x} r_{1y} r_{2y} r_{3y} r_{1x} - \lambda_1 p_{2x} p_{3x} r_{1y} r_{2y} r_{3y} r_{1x} \\
&+ f_y \lambda_3 p_{2y} r_{2x} r_{3x} r_{3y} r_{1x} - \lambda_3 p_{1y} p_{2y} r_{2x} r_{3x} r_{3y} r_{1x} - f_y \lambda_3 p_{3y} r_{2x} r_{3x} r_{3y} r_{1x} \\
&+ \lambda_3 p_{1y} p_{3y} r_{2x} r_{3x} r_{3y} r_{1x} - f_y \lambda_3 p_{1x} r_{2y} r_{3x} r_{3y} r_{1x} - f_x \lambda_3 p_{1y} r_{2y} r_{3x} r_{3y} r_{1x} \\
&+ f_y \lambda_3 p_{2x} r_{2y} r_{3x} r_{3y} r_{1x} + \lambda_3 p_{1x} p_{2y} r_{2y} r_{3x} r_{3y} r_{1x} + \lambda_3 p_{1y} p_{3x} r_{2y} r_{3x} r_{3y} r_{1x} \\
&- \lambda_3 p_{2y} p_{3x} r_{2y} r_{3x} r_{3y} r_{1x} + f_x \lambda_3 p_{3y} r_{2y} r_{3x} r_{3y} r_{1x} - \lambda_3 p_{2x} p_{3y} r_{2y} r_{3x} r_{3y} r_{1x} \\
&- f_y \lambda_3 p_{1y} r_{1y} r_{2x} {r_{3x}}^2 + \lambda_3 p_{1y} p_{2y} r_{1y} r_{2x} {r_{3x}}^2 + f_y \lambda_3 p_{3y} r_{1y} r_{2x} {r_{3x}}^2 \\
&- \lambda_3 p_{2y} p_{3y} r_{1y} r_{2x} {r_{3x}}^2 + f_x \lambda_3 p_{2y} r_{1y} r_{2y} {r_{3x}}^2 - \lambda_3 p_{1x} p_{2y} r_{1y} r_{2y} {r_{3x}}^2 \\
&- f_x \lambda_3 p_{3y} r_{1y} r_{2y} {r_{3x}}^2 + \lambda_3 p_{1x} p_{3y} r_{1y} r_{2y} {r_{3x}}^2 - f_x \lambda_3 p_{1x} r_{1y} r_{2x} {r_{3y}}^2 \\
&+ f_x \lambda_3 p_{2x} r_{1y} r_{2x} {r_{3y}}^2 + \lambda_3 p_{1x} p_{3x} r_{1y} r_{2x} {r_{3y}}^2 - \lambda_3 p_{2x} p_{3x} r_{1y} r_{2x} {r_{3y}}^2 \\
&- f_y \lambda_1 p_{1x} {r_{1y}}^2 r_{2x} r_{3x} + \lambda_1 p_{1x} p_{2y} {r_{1y}}^2 r_{2x} r_{3x} + f_y \lambda_1 p_{3x} {r_{1y}}^2 r_{2x} r_{3x} \\
&- \lambda_1 p_{2y} p_{3x} {r_{1y}}^2 r_{2x} r_{3x} + f_x \lambda_1 p_{2x} {r_{1y}}^2 r_{2y} r_{3x} - \lambda_1 p_{1x} p_{2x} {r_{1y}}^2 r_{2y} r_{3x} \\
&- f_x \lambda_1 p_{3x} {r_{1y}}^2 r_{2y} r_{3x} + \lambda_1 p_{1x} p_{3x} {r_{1y}}^2 r_{2y} r_{3x} + f_x \lambda_1 p_{1x} {r_{1y}}^2 r_{2x} r_{3y} \\
&- f_x \lambda_1 p_{2x} {r_{1y}}^2 r_{2x} r_{3y} - \lambda_1 p_{1x} p_{3x} {r_{1y}}^2 r_{2x} r_{3y} + \lambda_1 p_{2x} p_{3x} {r_{1y}}^2 r_{2x} r_{3y} \\
&+ f_y \lambda_3 p_{1x} r_{1y} r_{2x} r_{3x} r_{3y} + f_x \lambda_3 p_{1y} r_{1y} r_{2x} r_{3x} r_{3y} - \lambda_3 p_{1y} p_{2x} r_{1y} r_{2x} r_{3x} r_{3y} \\
&- f_x \lambda_3 p_{2y} r_{1y} r_{2x} r_{3x} r_{3y} - f_y \lambda_3 p_{3x} r_{1y} r_{2x} r_{3x} r_{3y} + \lambda_3 p_{2y} p_{3x} r_{1y} r_{2x} r_{3x} r_{3y} \\
&- \lambda_3 p_{1x} p_{3y} r_{1y} r_{2x} r_{3x} r_{3y} + \lambda_3 p_{2x} p_{3y} r_{1y} r_{2x} r_{3x} r_{3y} - f_x \lambda_3 p_{2x} r_{1y} r_{2y} r_{3x} r_{3y} \\
&+ \lambda_3 p_{1x} p_{2x} r_{1y} r_{2y} r_{3x} r_{3y} + f_x \lambda_3 p_{3x} r_{1y} r_{2y} r_{3x} r_{3y} - \lambda_3 p_{1x} p_{3x} r_{1y} r_{2y} r_{3x} r_{3y}.
\end{aligned}
$$

The polynomial g_7 is

$$
\begin{aligned}
& f_y\lambda_3 p_{1y} r_{2y} {r_{3x}}^2 && - f_y\lambda_3 p_{2y} r_{2y} {r_{3x}}^2 && - \lambda_3 p_{1y} p_{3y} r_{2y} {r_{3x}}^2 \\
+ & \lambda_3 p_{2y} p_{3y} r_{2y} {r_{3x}}^2 && + \lambda_1\lambda_3 p_{1y} r_{1y} r_{2y} {r_{3x}}^2 && - \lambda_1\lambda_3 p_{2y} r_{1y} r_{2y} {r_{3x}}^2 \\
- & f_y\lambda_1 p_{1y} r_{1x} r_{2y} r_{3x} && + f_y\lambda_1 p_{2y} r_{1x} r_{2y} r_{3x} && + \lambda_1 p_{1y} p_{3y} r_{1x} r_{2y} r_{3x} \\
- & \lambda_1 p_{2y} p_{3y} r_{1x} r_{2y} r_{3x} && + f_x\lambda_1 p_{1y} r_{1y} r_{2y} r_{3x} && - f_x\lambda_1 p_{2y} r_{1y} r_{2y} r_{3x} \\
- & \lambda_1 p_{1y} p_{3x} r_{1y} r_{2y} r_{3x} && + \lambda_1 p_{2y} p_{3x} r_{1y} r_{2y} r_{3x} && + f_y\lambda_3 p_{2y} r_{2x} r_{3y} r_{3x} \\
- & \lambda_3 p_{1y} p_{2y} r_{2x} r_{3y} r_{3x} && - f_y\lambda_3 p_{3y} r_{2x} r_{3y} r_{3x} && + \lambda_3 p_{1y} p_{3y} r_{2x} r_{3y} r_{3x} \\
- & \lambda_1\lambda_3 p_{1y} r_{1y} r_{2x} r_{3y} r_{3x} && + \lambda_1\lambda_3 p_{2y} r_{1y} r_{2x} r_{3y} r_{3x} && - f_y\lambda_3 p_{1x} r_{2y} r_{3y} r_{3x} \\
- & f_x\lambda_3 p_{1y} r_{2y} r_{3y} r_{3x} && + f_y\lambda_3 p_{2x} r_{2y} r_{3y} r_{3x} && + \lambda_3 p_{1x} p_{2y} r_{2y} r_{3y} r_{3x} \\
+ & \lambda_3 p_{1y} p_{3x} r_{2y} r_{3y} r_{3x} && - \lambda_3 p_{2y} p_{3x} r_{2y} r_{3y} r_{3x} && + f_x\lambda_3 p_{3y} r_{2y} r_{3y} r_{3x} \\
- & \lambda_3 p_{2x} p_{3y} r_{2y} r_{3y} r_{3x} && - \lambda_1\lambda_3 p_{1x} r_{1y} r_{2y} r_{3y} r_{3x} && + \lambda_1\lambda_3 p_{2x} r_{1y} r_{2y} r_{3y} r_{3x} \\
- & f_y\lambda_3 p_{2x} r_{2x} {r_{3y}}^2 && + \lambda_3 p_{1y} p_{2x} r_{2x} {r_{3y}}^2 && + f_y\lambda_3 p_{3x} r_{2x} {r_{3y}}^2 \\
- & \lambda_3 p_{1y} p_{3x} r_{2x} {r_{3y}}^2 && + \lambda_1\lambda_3 p_{1x} r_{1y} r_{2x} {r_{3y}}^2 && - \lambda_1\lambda_3 p_{2x} r_{1y} r_{2x} {r_{3y}}^2 \\
+ & f_x\lambda_3 p_{1x} r_{2y} {r_{3y}}^2 && - \lambda_3 p_{1x} p_{2x} r_{2y} {r_{3y}}^2 && - f_x\lambda_3 p_{3x} r_{2y} {r_{3y}}^2 \\
+ & \lambda_3 p_{2x} p_{3x} r_{2y} {r_{3y}}^2 && - f_y\lambda_1 p_{2y} r_{1x} r_{2x} r_{3y} && + \lambda_1 p_{1y} p_{2y} r_{1x} r_{2x} r_{3y} \\
+ & f_y\lambda_1 p_{3y} r_{1x} r_{2x} r_{3y} && - \lambda_1 p_{1y} p_{3y} r_{1x} r_{2x} r_{3y} && + f_y\lambda_1 p_{1x} r_{1y} r_{2x} r_{3y} \\
- & f_x\lambda_1 p_{1y} r_{1y} r_{2x} r_{3y} && + f_x\lambda_1 p_{2y} r_{1y} r_{2x} r_{3y} && - \lambda_1 p_{1x} p_{2y} r_{1y} r_{2x} r_{3y} \\
- & f_y\lambda_1 p_{3x} r_{1y} r_{2x} r_{3y} && + \lambda_1 p_{1y} p_{3x} r_{1y} r_{2x} r_{3y} && + f_x\lambda_1 p_{1y} r_{1x} r_{2y} r_{3y} \\
- & \lambda_1 p_{1y} p_{2x} r_{1x} r_{2y} r_{3y} && - f_x\lambda_1 p_{3y} r_{1x} r_{2y} r_{3y} && + \lambda_1 p_{2x} p_{3y} r_{1x} r_{2y} r_{3y} \\
- & f_x\lambda_1 p_{1x} r_{1y} r_{2y} r_{3y} && + \lambda_1 p_{1x} p_{2x} r_{1y} r_{2y} r_{3y} && + f_x\lambda_1 p_{3x} r_{1y} r_{2y} r_{3y} \\
- & \lambda_1 p_{2x} p_{3x} r_{1y} r_{2y} r_{3y}.
\end{aligned}
$$

The polynomial g_8 is

$$
\begin{aligned}
& f_y\lambda_1 r_{1x} - \lambda_1 p_{3y} r_{1x} + \lambda_1\lambda_3 r_{3y} r_{1x} - f_x\lambda_1 r_{1y} + \lambda_1 p_{3x} r_{1y} \\
- & f_y\lambda_3 r_{3x} + \lambda_3 p_{3y} r_{3x} - \lambda_1\lambda_3 r_{1y} r_{3x} + f_x\lambda_3 r_{3y} - \lambda_3 p_{3x} r_{3y}.
\end{aligned}
$$

The polynomial g_9 is

$$\begin{array}{cccccc}
- & f_y\lambda_1 p_{2y} r_{1x} r_{2x} & + & \lambda_1 p_{1y} p_{2y} r_{1x} r_{2x} & + & f_y \lambda_1 p_{3y} r_{1x} r_{2x} \\
- & \lambda_1 p_{1y} p_{3y} r_{1x} r_{2x} & + & f_y \lambda_1 p_{1x} r_{1y} r_{2x} & - & f_x \lambda_1 p_{1y} r_{1y} r_{2x} \\
+ & f_x \lambda_1 p_{2y} r_{1y} r_{2x} & - & \lambda_1 p_{1x} p_{2y} r_{1y} r_{2x} & - & f_y \lambda_1 p_{3x} r_{1y} r_{2x} \\
+ & \lambda_1 p_{1y} p_{3x} r_{1y} r_{2x} & + & f_y \lambda_3 p_{2y} r_{3x} r_{2x} & - & \lambda_3 p_{1y} p_{2y} r_{3x} r_{2x} \\
- & f_y \lambda_3 p_{3y} r_{3x} r_{2x} & + & \lambda_3 p_{1y} p_{3y} r_{3x} r_{2x} & - & \lambda_1 \lambda_3 p_{1y} r_{1y} r_{3x} r_{2x} \\
+ & \lambda_1 \lambda_3 p_{2y} r_{1y} r_{3x} r_{2x} & - & f_y \lambda_3 p_{2x} r_{3y} r_{2x} & + & \lambda_3 p_{1y} p_{2x} r_{3y} r_{2x} \\
+ & f_y \lambda_3 p_{3x} r_{3y} r_{2x} & - & \lambda_3 p_{1y} p_{3x} r_{3y} r_{2x} & + & \lambda_1 \lambda_3 p_{1x} r_{1y} r_{3y} r_{2x} \\
- & \lambda_1 \lambda_3 p_{2x} r_{1y} r_{3y} r_{2x} & + & f_x \lambda_1 p_{1y} r_{1x} r_{2y} & - & \lambda_1 p_{1y} p_{2x} r_{1x} r_{2y} \\
- & f_x \lambda_1 p_{3y} r_{1x} r_{2y} & + & \lambda_1 p_{2x} p_{3y} r_{1x} r_{2y} & - & f_x \lambda_1 p_{1x} r_{1y} r_{2y} \\
+ & \lambda_1 p_{1x} p_{2x} r_{1y} r_{2y} & + & f_x \lambda_1 p_{3x} r_{1y} r_{2y} & - & \lambda_1 p_{2x} p_{3x} r_{1y} r_{2y} \\
- & f_y \lambda_3 p_{1x} r_{2y} r_{3x} & + & f_y \lambda_3 p_{2x} r_{2y} r_{3x} & - & f_x \lambda_3 p_{2y} r_{2y} r_{3x} \\
+ & \lambda_3 p_{1x} p_{2y} r_{2y} r_{3x} & + & f_x \lambda_3 p_{3y} r_{2y} r_{3x} & - & \lambda_3 p_{2x} p_{3y} r_{2y} r_{3x} \\
+ & \lambda_1 \lambda_3 p_{1y} r_{1x} r_{2y} r_{3x} & - & \lambda_1 \lambda_3 p_{2y} r_{1x} r_{2y} r_{3x} & - & \lambda_1 \lambda_3 p_{1x} r_{1y} r_{2y} r_{3x} \\
+ & \lambda_1 \lambda_3 p_{2x} r_{1y} r_{2y} r_{3x} & + & f_x \lambda_3 p_{1x} r_{2y} r_{3y} & - & \lambda_3 p_{1x} p_{2x} r_{2y} r_{3y} \\
- & f_x \lambda_3 p_{3x} r_{2y} r_{3y} & + & \lambda_3 p_{2x} p_{3x} r_{2y} r_{3y}. & &
\end{array}$$

The polynomial g_{10} is

$$\begin{array}{cccccccccc}
 & f_y \lambda_2 r_{2x} & - & \lambda_2 p_{2y} r_{2x} & + & \lambda_2 \lambda_3 r_{3y} r_{2x} & - & f_x \lambda_2 r_{2y} & + & \lambda_2 p_{2x} r_{2y} \\
- & f_y \lambda_3 r_{3x} & + & \lambda_3 p_{2y} r_{3x} & - & \lambda_2 \lambda_3 r_{2y} r_{3x} & + & f_x \lambda_3 r_{3y} & - & \lambda_3 p_{2x} r_{3y}.
\end{array}$$

The polynomial g_{11} is

$$\begin{array}{cccccccc}
- & f_y \lambda_1 p_{1x} r_{1y} & + & f_x \lambda_1 p_{1y} r_{1y} & + & f_y \lambda_1 p_{3x} r_{1y} & - & \lambda_1 p_{1y} p_{3x} r_{1y} \\
- & f_x \lambda_1 p_{3y} r_{1y} & + & \lambda_1 p_{1x} p_{3y} r_{1y} & + & \lambda_1 \lambda_2 p_{2y} r_{2x} r_{1y} & - & \lambda_1 \lambda_2 p_{3y} r_{2x} r_{1y} \\
- & \lambda_1 \lambda_2 p_{2x} r_{2y} r_{1y} & + & \lambda_1 \lambda_2 p_{3x} r_{2y} r_{1y} & + & \lambda_1 \lambda_3 p_{1y} r_{3x} r_{1y} & - & \lambda_1 \lambda_3 p_{2y} r_{3x} r_{1y} \\
- & \lambda_1 \lambda_3 p_{1x} r_{3y} r_{1y} & + & \lambda_1 \lambda_3 p_{2x} r_{3y} r_{1y} & + & f_y \lambda_2 p_{2y} r_{2x} & - & \lambda_2 p_{1y} p_{2y} r_{2x} \\
- & f_y \lambda_2 p_{3y} r_{2x} & + & \lambda_2 p_{1y} p_{3y} r_{2x} & + & f_y \lambda_2 p_{1x} r_{2y} & - & f_x \lambda_2 p_{1y} r_{2y} \\
- & f_y \lambda_2 p_{2x} r_{2y} & + & \lambda_2 p_{1y} p_{2x} r_{2y} & + & f_x \lambda_2 p_{3y} r_{2y} & - & \lambda_2 p_{1x} p_{3y} r_{2y} \\
- & f_y \lambda_3 p_{2y} r_{3x} & + & \lambda_3 p_{1y} p_{2y} r_{3x} & + & f_y \lambda_3 p_{3y} r_{3x} & - & \lambda_3 p_{1y} p_{3y} r_{3x} \\
- & \lambda_2 \lambda_3 p_{1y} r_{2y} r_{3x} & + & \lambda_2 \lambda_3 p_{3y} r_{2y} r_{3x} & + & f_y \lambda_3 p_{2x} r_{3y} & - & \lambda_3 p_{1y} p_{2x} r_{3y} \\
- & f_y \lambda_3 p_{3x} r_{3y} & + & \lambda_3 p_{1y} p_{3x} r_{3y} & + & \lambda_2 \lambda_3 p_{1x} r_{2y} r_{3y} & - & \lambda_2 \lambda_3 p_{3x} r_{2y} r_{3y}.
\end{array}$$

The polynomial g_{12} is

$$\begin{array}{ccccccc}
 & f_x\lambda_1 p_{1y} r_{1x} & - & \lambda_1 p_{1y} p_{2x} r_{1x} & - & f_x\lambda_1 p_{3y} r_{1x} & + & \lambda_1 p_{2x} p_{3y} r_{1x} \\
+ & \lambda_1\lambda_2 p_{2y} r_{2x} r_{1x} & - & \lambda_1\lambda_2 p_{3y} r_{2x} r_{1x} & + & \lambda_1\lambda_3 p_{1y} r_{3x} r_{1x} & - & \lambda_1\lambda_3 p_{2y} r_{3x} r_{1x} \\
- & f_x\lambda_1 p_{1x} r_{1y} & + & \lambda_1 p_{1x} p_{2x} r_{1y} & + & f_x\lambda_1 p_{3x} r_{1y} & - & \lambda_1 p_{2x} p_{3x} r_{1y} \\
- & f_x\lambda_2 p_{1y} r_{2x} & - & f_y\lambda_2 p_{2x} r_{2x} & + & \lambda_2 p_{1y} p_{2x} r_{2x} & + & f_x\lambda_2 p_{2y} r_{2x} \\
+ & f_y\lambda_2 p_{3x} r_{2x} & - & \lambda_2 p_{2y} p_{3x} r_{2x} & - & \lambda_1\lambda_2 p_{2x} r_{1y} r_{2x} & + & \lambda_1\lambda_2 p_{3x} r_{1y} r_{2x} \\
+ & f_x\lambda_2 p_{1x} r_{2y} & - & \lambda_2 p_{1x} p_{2x} r_{2y} & - & f_x\lambda_2 p_{3x} r_{2y} & + & \lambda_2 p_{2x} p_{3x} r_{2y} \\
+ & f_y\lambda_3 p_{2x} r_{3x} & - & f_x\lambda_3 p_{2y} r_{3x} & - & f_y\lambda_3 p_{3x} r_{3x} & + & \lambda_3 p_{2y} p_{3x} r_{3x} \\
+ & f_x\lambda_3 p_{3y} r_{3x} & - & \lambda_3 p_{2x} p_{3y} r_{3x} & - & \lambda_1\lambda_3 p_{1x} r_{1y} r_{3x} & + & \lambda_1\lambda_3 p_{2x} r_{1y} r_{3x} \\
- & \lambda_2\lambda_3 p_{1y} r_{2x} r_{3x} & + & \lambda_2\lambda_3 p_{3y} r_{2x} r_{3x} & + & \lambda_2\lambda_3 p_{1x} r_{2y} r_{3x} & - & \lambda_2\lambda_3 p_{3x} r_{2y} r_{3x}.
\end{array}$$

The polynomial g_{13} is

$$\begin{array}{cccccc}
- & \lambda_1 p_{1y} r_{1x} {f_x}^2 & + & \lambda_1 p_{3y} r_{1x} {f_x}^2 & + & \lambda_2 p_{1y} r_{2x} {f_x}^2 \\
- & \lambda_2 p_{2y} r_{2x} {f_x}^2 & + & \lambda_3 p_{2y} r_{3x} {f_x}^2 & - & \lambda_3 p_{3y} r_{3x} {f_x}^2 \\
- & {\lambda_1}^2 p_{1y} {r_{1x}}^2 f_x & + & {\lambda_1}^2 p_{3y} {r_{1x}}^2 f_x & + & f_y\lambda_1 p_{1x} r_{1x} f_x \\
+ & \lambda_1 p_{1y} p_{2x} r_{1x} f_x & - & f_y\lambda_1 p_{3x} r_{1x} f_x & + & \lambda_1 p_{1y} p_{3x} r_{1x} f_x \\
- & \lambda_1 p_{1x} p_{3y} r_{1x} f_x & - & \lambda_1 p_{2x} p_{3y} r_{1x} f_x & + & {\lambda_1}^2 p_{1x} r_{1x} r_{1y} f_x \\
- & {\lambda_1}^2 p_{3x} r_{1x} r_{1y} f_x & - & f_y\lambda_2 p_{1x} r_{2x} f_x & + & f_y\lambda_2 p_{2x} r_{2x} f_x \\
- & \lambda_2 p_{1y} p_{2x} r_{2x} f_x & + & \lambda_2 p_{1x} p_{2y} r_{2x} f_x & - & \lambda_2 p_{1y} p_{3x} r_{2x} f_x \\
+ & \lambda_2 p_{2y} p_{3x} r_{2x} f_x & + & \lambda_1\lambda_2 p_{1y} r_{1x} r_{2x} f_x & - & 2\lambda_1\lambda_2 p_{2y} r_{1x} r_{2x} f_x \\
+ & \lambda_1\lambda_2 p_{3y} r_{1x} r_{2x} f_x & - & \lambda_1\lambda_2 p_{1x} r_{1y} r_{2x} f_x & + & \lambda_1\lambda_2 p_{2x} r_{1y} r_{2x} f_x \\
- & f_y\lambda_3 p_{2x} r_{3x} f_x & - & \lambda_3 p_{1x} p_{2y} r_{3x} f_x & + & f_y\lambda_3 p_{3x} r_{3x} f_x \\
- & \lambda_3 p_{2y} p_{3x} r_{3x} f_x & + & \lambda_3 p_{1x} p_{3y} r_{3x} f_x & + & \lambda_3 p_{2x} p_{3y} r_{3x} f_x \\
- & \lambda_1\lambda_3 p_{1y} r_{1x} r_{3x} f_x & + & 2\lambda_1\lambda_3 p_{2y} r_{1x} r_{3x} f_x & - & \lambda_1\lambda_3 p_{3y} r_{1x} r_{3x} f_x \\
- & \lambda_1\lambda_3 p_{2x} r_{1y} r_{3x} f_x & + & \lambda_1\lambda_3 p_{3x} r_{1y} r_{3x} f_x & + & \lambda_2\lambda_3 p_{1y} r_{2x} r_{3x} f_x \\
- & \lambda_2\lambda_3 p_{3y} r_{2x} r_{3x} f_x & + & {\lambda_1}^2 p_{1y} p_{2x} {r_{1x}}^2 & - & {\lambda_1}^2 p_{2x} p_{3y} {r_{1x}}^2 \\
- & f_y\lambda_1 p_{1x} p_{2x} r_{1x} & + & f_y\lambda_1 p_{2x} p_{3x} r_{1x} & - & \lambda_1 p_{1y} p_{2x} p_{3x} r_{1x}
\end{array}$$

$$
\begin{array}{llllll}
+ & \lambda_1 p_{1x} p_{2x} p_{3y} r_{1x} & - & {\lambda_1}^2 p_{1x} p_{2x} r_{1x} r_{1y} & + & {\lambda_1}^2 p_{2x} p_{3x} r_{1x} r_{1y} \\
- & {\lambda_1}^2 \lambda_2 p_{2y} {r_{1x}}^2 r_{2x} & + & {\lambda_1}^2 \lambda_2 p_{3y} {r_{1x}}^2 r_{2x} & + & f_y \lambda_2 p_{1x} p_{3x} r_{2x} \\
- & f_y \lambda_2 p_{2x} p_{3x} r_{2x} & + & \lambda_2 p_{1y} p_{2x} p_{3x} r_{2x} & - & \lambda_2 p_{1x} p_{2y} p_{3x} r_{2x} \\
+ & f_y \lambda_1 \lambda_2 p_{2x} r_{1x} r_{2x} & - & \lambda_1 \lambda_2 p_{1y} p_{2x} r_{1x} r_{2x} & + & \lambda_1 \lambda_2 p_{1x} p_{2y} r_{1x} r_{2x} \\
- & f_y \lambda_1 \lambda_2 p_{3x} r_{1x} r_{2x} & + & \lambda_1 \lambda_2 p_{2y} p_{3x} r_{1x} r_{2x} & - & \lambda_1 \lambda_2 p_{1x} p_{3y} r_{1x} r_{2x} \\
+ & \lambda_1 \lambda_2 p_{1x} p_{3x} r_{1y} r_{2x} & - & \lambda_1 \lambda_2 p_{2x} p_{3x} r_{1y} r_{2x} & + & {\lambda_1}^2 \lambda_2 p_{2x} r_{1x} r_{1y} r_{2x} \\
- & {\lambda_1}^2 \lambda_2 p_{3x} r_{1x} r_{1y} r_{2x} & - & {\lambda_1}^2 \lambda_3 p_{1y} {r_{1x}}^2 r_{3x} & + & {\lambda_1}^2 \lambda_3 p_{2y} {r_{1x}}^2 r_{3x} \\
+ & f_y \lambda_3 p_{1x} p_{2x} r_{3x} & - & f_y \lambda_3 p_{1x} p_{3x} r_{3x} & + & \lambda_3 p_{1x} p_{2y} p_{3x} r_{3x} \\
- & \lambda_3 p_{1x} p_{2x} p_{3y} r_{3x} & + & f_y \lambda_1 \lambda_3 p_{1x} r_{1x} r_{3x} & - & f_y \lambda_1 \lambda_3 p_{2x} r_{1x} r_{3x} \\
- & \lambda_1 \lambda_3 p_{1x} p_{2y} r_{1x} r_{3x} & + & \lambda_1 \lambda_3 p_{1y} p_{3x} r_{1x} r_{3x} & - & \lambda_1 \lambda_3 p_{2y} p_{3x} r_{1x} r_{3x} \\
+ & \lambda_1 \lambda_3 p_{2x} p_{3y} r_{1x} r_{3x} & + & \lambda_1 \lambda_3 p_{1x} p_{2x} r_{1y} r_{3x} & - & \lambda_1 \lambda_3 p_{1x} p_{3x} r_{1y} r_{3x} \\
+ & {\lambda_1}^2 \lambda_3 p_{1x} r_{1x} r_{1y} r_{3x} & - & {\lambda_1}^2 \lambda_3 p_{2x} r_{1x} r_{1y} r_{3x} & - & f_y \lambda_2 \lambda_3 p_{1x} r_{2x} r_{3x} \\
+ & f_y \lambda_2 \lambda_3 p_{3x} r_{2x} r_{3x} & - & \lambda_2 \lambda_3 p_{1y} p_{3x} r_{2x} r_{3x} & + & \lambda_2 \lambda_3 p_{1x} p_{3y} r_{2x} r_{3x} \\
+ & \lambda_1 \lambda_2 \lambda_3 p_{1y} r_{1x} r_{2x} r_{3x} & - & \lambda_1 \lambda_2 \lambda_3 p_{3y} r_{1x} r_{2x} r_{3x} & - & \lambda_1 \lambda_2 \lambda_3 p_{1x} r_{1y} r_{2x} r_{3x} \\
+ & \lambda_1 \lambda_2 \lambda_3 p_{3x} r_{1y} r_{2x} r_{3x}. & & & &
\end{array}
$$

The polynomial g_{14} is

$$
\begin{array}{llllll}
- & f_y \mu_3 r_{3x} & + & \mu_3 p_{3y} r_{3x} & - & \lambda_1 r_{1y} r_{3x} \\
+ & f_x \mu_3 r_{3y} & - & \mu_3 p_{3x} r_{3y} & + & \lambda_1 r_{1x} r_{3y}.
\end{array}
$$

The polynomial g_{15} is

$$
\begin{array}{llllll}
 & \mu_3 p_{2y} r_{2x} r_{3x} {f_y}^2 & - & \mu_3 p_{3y} r_{2x} r_{3x} {f_y}^2 & + & \mu_3 p_{2x} r_{2y} r_{3x} {f_y}^2 \\
- & \mu_3 p_{3x} r_{2y} r_{3x} {f_y}^2 & - & \mu_3 p_{2x} r_{2x} r_{3y} {f_y}^2 & + & \mu_3 p_{3x} r_{2x} r_{3y} {f_y}^2 \\
+ & \mu_3 {p_{3y}}^2 r_{2x} r_{3x} f_y & - & \mu_3 p_{1y} p_{2y} r_{2x} r_{3x} f_y & + & \mu_3 p_{1y} p_{3y} r_{2x} r_{3x} f_y \\
- & \mu_3 p_{2y} p_{3y} r_{2x} r_{3x} f_y & + & \lambda_1 p_{2y} r_{1y} r_{2x} r_{3x} f_y & - & \lambda_1 p_{3y} r_{1y} r_{2x} r_{3x} f_y \\
- & f_x \mu_3 p_{2y} r_{2y} r_{3x} f_y & + & \mu_3 p_{1x} p_{2y} r_{2y} r_{3x} f_y & + & f_x \mu_3 p_{3y} r_{2y} r_{3x} f_y \\
- & \mu_3 p_{1x} p_{3y} r_{2y} r_{3x} f_y & - & 2 \mu_3 p_{2x} p_{3y} r_{2y} r_{3x} f_y & + & 2 \mu_3 p_{3x} p_{3y} r_{2y} r_{3x} f_y \\
- & \lambda_1 \mu_3 p_{2y} r_{1x} r_{2y} r_{3x} f_y & + & \lambda_1 \mu_3 p_{3y} r_{1x} r_{2y} r_{3x} f_y & + & \lambda_1 p_{2x} r_{1y} r_{2y} r_{3x} f_y \\
+ & \lambda_1 \mu_3 p_{2x} r_{1y} r_{2y} r_{3x} f_y & - & \lambda_1 p_{3x} r_{1y} r_{2y} r_{3x} f_y & - & \lambda_1 \mu_3 p_{3x} r_{1y} r_{2y} r_{3x} f_y
\end{array}
$$

$$
\begin{array}{llllll}
+ & \mu_3 p_{1y} p_{2x} r_{2x} r_{3y} f_y & - & \mu_3 p_{1y} p_{3x} r_{2x} r_{3y} f_y & + & \mu_3 p_{2x} p_{3y} r_{2x} r_{3y} f_y \\
- & \mu_3 p_{3x} p_{3y} r_{2x} r_{3y} f_y & - & \lambda_1 p_{2y} r_{1x} r_{2x} r_{3y} f_y & + & \lambda_1 \mu_3 p_{2y} r_{1x} r_{2x} r_{3y} f_y \\
+ & \lambda_1 p_{3y} r_{1x} r_{2x} r_{3y} f_y & - & \lambda_1 \mu_3 p_{3y} r_{1x} r_{2x} r_{3y} f_y & + & \lambda_1 p_{1x} r_{1y} r_{2x} r_{3y} f_y \\
- & \lambda_1 p_{2x} r_{1y} r_{2x} r_{3y} f_y & - & \lambda_1 \mu_3 p_{2x} r_{1y} r_{2x} r_{3y} f_y & + & \lambda_1 \mu_3 p_{3x} r_{1y} r_{2x} r_{3y} f_y \\
- & \mu_3 {p_{3x}}^2 r_{2y} r_{3y} f_y & - & \mu_3 p_{1x} p_{2x} r_{2y} r_{3y} f_y & + & \mu_3 p_{1x} p_{3x} r_{2y} r_{3y} f_y \\
+ & \mu_3 p_{2x} p_{3x} r_{2y} r_{3y} f_y & - & \lambda_1 p_{1x} r_{1x} r_{2y} r_{3y} f_y & + & \lambda_1 p_{3x} r_{1x} r_{2y} r_{3y} f_y \\
- & \mu_3 p_{1y} {p_{3y}}^2 r_{2x} r_{3x} & + & \mu_3 p_{1y} p_{2y} p_{3y} r_{2x} r_{3x} & - & \lambda_1 p_{1y} p_{2y} r_{1y} r_{2x} r_{3x} \\
+ & \lambda_1 p_{1y} p_{3y} r_{1y} r_{2x} r_{3x} & - & f_x \mu_3 {p_{3y}}^2 r_{2y} r_{3x} & + & \mu_3 p_{1x} {p_{3y}}^2 r_{2y} r_{3x} \\
+ & \mu_3 p_{2x} {p_{3y}}^2 r_{2y} r_{3x} & - & \mu_3 p_{3x} {p_{3y}}^2 r_{2y} r_{3x} & + & {\lambda_1}^2 p_{2x} {r_{1y}}^2 r_{2y} r_{3x} \\
- & {\lambda_1}^2 p_{3x} {r_{1y}}^2 r_{2y} r_{3x} & + & f_x \mu_3 p_{2y} p_{3y} r_{2y} r_{3x} & - & \mu_3 p_{1x} p_{2y} p_{3y} r_{2y} r_{3x} \\
- & \lambda_1 \mu_3 {p_{3y}}^2 r_{1x} r_{2y} r_{3x} & + & \lambda_1 \mu_3 p_{2y} p_{3y} r_{1x} r_{2y} r_{3x} & - & f_x \lambda_1 p_{2y} r_{1y} r_{2y} r_{3x} \\
+ & \lambda_1 p_{1x} p_{2y} r_{1y} r_{2y} r_{3x} & + & f_x \lambda_1 p_{3y} r_{1y} r_{2y} r_{3x} & - & \lambda_1 p_{1x} p_{3y} r_{1y} r_{2y} r_{3x} \\
- & \lambda_1 p_{2x} p_{3y} r_{1y} r_{2y} r_{3x} & - & \lambda_1 \mu_3 p_{2x} p_{3y} r_{1y} r_{2y} r_{3x} & + & \lambda_1 p_{3x} p_{3y} r_{1y} r_{2y} r_{3x} \\
+ & \lambda_1 \mu_3 p_{3x} p_{3y} r_{1y} r_{2y} r_{3x} & - & {\lambda_1}^2 p_{2y} r_{1x} r_{1y} r_{2y} r_{3x} & + & {\lambda_1}^2 p_{3y} r_{1x} r_{1y} r_{2y} r_{3x} \\
+ & {\lambda_1}^2 p_{1x} {r_{1y}}^2 r_{2x} r_{3y} & - & {\lambda_1}^2 p_{2x} {r_{1y}}^2 r_{2x} r_{3y} & - & \mu_3 p_{1y} p_{2x} p_{3y} r_{2x} r_{3y} \\
+ & \mu_3 p_{1y} p_{3x} p_{3y} r_{2x} r_{3y} & + & \lambda_1 p_{1y} p_{2y} r_{1x} r_{2x} r_{3y} & - & \lambda_1 \mu_3 p_{1y} p_{2y} r_{1x} r_{2x} r_{3y} \\
- & \lambda_1 p_{1y} p_{3y} r_{1x} r_{2x} r_{3y} & + & \lambda_1 \mu_3 p_{1y} p_{3y} r_{1x} r_{2x} r_{3y} & - & f_x \lambda_1 p_{1y} r_{1y} r_{2x} r_{3y} \\
+ & \lambda_1 p_{1y} p_{2x} r_{1y} r_{2x} r_{3y} & + & f_x \lambda_1 p_{2y} r_{1y} r_{2x} r_{3y} & - & \lambda_1 p_{1x} p_{2y} r_{1y} r_{2x} r_{3y} \\
+ & \lambda_1 \mu_3 p_{1x} p_{2y} r_{1y} r_{2x} r_{3y} & - & \lambda_1 \mu_3 p_{2y} p_{3x} r_{1y} r_{2x} r_{3y} & - & \lambda_1 \mu_3 p_{1x} p_{3y} r_{1y} r_{2x} r_{3y} \\
+ & \lambda_1 \mu_3 p_{2x} p_{3y} r_{1y} r_{2x} r_{3y} & - & {\lambda_1}^2 p_{1y} r_{1x} r_{1y} r_{2x} r_{3y} & + & {\lambda_1}^2 p_{2y} r_{1x} r_{1y} r_{2x} r_{3y} \\
+ & {\lambda_1}^2 p_{1y} {r_{1x}}^2 r_{2y} r_{3y} & - & {\lambda_1}^2 p_{3y} {r_{1x}}^2 r_{2y} r_{3y} & + & \mu_3 {p_{3x}}^2 p_{3y} r_{2y} r_{3y} \\
+ & \mu_3 p_{1x} p_{2x} p_{3y} r_{2y} r_{3y} & - & \mu_3 p_{1x} p_{3x} p_{3y} r_{2y} r_{3y} & - & \mu_3 p_{2x} p_{3x} p_{3y} r_{2y} r_{3y} \\
+ & f_x \lambda_1 p_{1y} r_{1x} r_{2y} r_{3y} & - & \lambda_1 p_{1y} p_{2x} r_{1x} r_{2y} r_{3y} & + & \lambda_1 \mu_3 p_{1y} p_{2x} r_{1x} r_{2y} r_{3y} \\
- & \lambda_1 \mu_3 p_{1y} p_{3x} r_{1x} r_{2y} r_{3y} & - & f_x \lambda_1 p_{3y} r_{1x} r_{2y} r_{3y} & + & \lambda_1 p_{1x} p_{3y} r_{1x} r_{2y} r_{3y} \\
+ & \lambda_1 p_{2x} p_{3y} r_{1x} r_{2y} r_{3y} & - & \lambda_1 \mu_3 p_{2x} p_{3y} r_{1x} r_{2y} r_{3y} & - & \lambda_1 p_{3x} p_{3y} r_{1x} r_{2y} r_{3y} \\
+ & \lambda_1 \mu_3 p_{3x} p_{3y} r_{1x} r_{2y} r_{3y} & - & \lambda_1 \mu_3 {p_{3x}}^2 r_{1y} r_{2y} r_{3y} & - & \lambda_1 \mu_3 p_{1x} p_{2x} r_{1y} r_{2y} r_{3y} \\
+ & \lambda_1 \mu_3 p_{1x} p_{3x} r_{1y} r_{2y} r_{3y} & + & \lambda_1 \mu_3 p_{2x} p_{3x} r_{1y} r_{2y} r_{3y} & - & {\lambda_1}^2 p_{1x} r_{1x} r_{1y} r_{2y} r_{3y} \\
+ & {\lambda_1}^2 p_{3x} r_{1x} r_{1y} r_{2y} r_{3y}. & & & &
\end{array}
$$

The polynomial g_{16} is

$$
\begin{array}{cccccc}
 & \mu_3 p_{2y} r_{2y} {f_x}^2 & - & \mu_3 p_{3y} r_{2y} {f_x}^2 & - & f_y \mu_3 p_{2y} r_{2x} f_x \\
+ & \mu_3 p_{1y} p_{2y} r_{2x} f_x & + & f_y \mu_3 p_{3y} r_{2x} f_x & - & \mu_3 p_{1y} p_{3y} r_{2x} f_x \\
+ & \lambda_1 p_{1y} r_{1y} r_{2x} f_x & - & \lambda_1 p_{2y} r_{1y} r_{2x} f_x & - & f_y \mu_3 p_{2x} r_{2y} f_x \\
- & \mu_3 p_{1x} p_{2y} r_{2y} f_x & + & f_y \mu_3 p_{3x} r_{2y} f_x & - & \mu_3 p_{2y} p_{3x} r_{2y} f_x \\
+ & \mu_3 p_{1x} p_{3y} r_{2y} f_x & + & \mu_3 p_{2x} p_{3y} r_{2y} f_x & - & \lambda_1 p_{1y} r_{1x} r_{2y} f_x \\
+ & \lambda_1 p_{2y} r_{1x} r_{2y} f_x & + & \lambda_1 \mu_3 p_{2y} r_{1x} r_{2y} f_x & - & \lambda_1 \mu_3 p_{3y} r_{1x} r_{2y} f_x \\
- & \lambda_1 \mu_3 p_{2x} r_{1y} r_{2y} f_x & + & \lambda_1 \mu_3 p_{3x} r_{1y} r_{2y} f_x & - & {\lambda_1}^2 p_{1x} {r_{1y}}^2 r_{2x} \\
+ & {\lambda_1}^2 p_{2x} {r_{1y}}^2 r_{2x} & + & {f_y}^2 \mu_3 p_{2x} r_{2x} & - & f_y \mu_3 p_{1y} p_{2x} r_{2x} \\
- & {f_y}^2 \mu_3 p_{3x} r_{2x} & + & f_y \mu_3 p_{1y} p_{3x} r_{2x} & + & f_y \mu_3 p_{2y} p_{3x} r_{2x} \\
- & \mu_3 p_{1y} p_{2y} p_{3x} r_{2x} & - & f_y \mu_3 p_{2x} p_{3y} r_{2x} & + & \mu_3 p_{1y} p_{2x} p_{3y} r_{2x} \\
- & f_y \lambda_1 \mu_3 p_{2y} r_{1x} r_{2x} & + & \lambda_1 \mu_3 p_{1y} p_{2y} r_{1x} r_{2x} & + & f_y \lambda_1 \mu_3 p_{3y} r_{1x} r_{2x} \\
- & \lambda_1 \mu_3 p_{1y} p_{3y} r_{1x} r_{2x} & - & f_y \lambda_1 p_{1x} r_{1y} r_{2x} & + & f_y \lambda_1 p_{2x} r_{1y} r_{2x} \\
+ & f_y \lambda_1 \mu_3 p_{2x} r_{1y} r_{2x} & - & \lambda_1 p_{1y} p_{2x} r_{1y} r_{2x} & + & \lambda_1 p_{1x} p_{2y} r_{1y} r_{2x} \\
- & \lambda_1 \mu_3 p_{1x} p_{2y} r_{1y} r_{2x} & - & f_y \lambda_1 \mu_3 p_{3x} r_{1y} r_{2x} & + & \lambda_1 \mu_3 p_{2y} p_{3x} r_{1y} r_{2x} \\
+ & \lambda_1 \mu_3 p_{1x} p_{3y} r_{1y} r_{2x} & - & \lambda_1 \mu_3 p_{2x} p_{3y} r_{1y} r_{2x} & + & {\lambda_1}^2 p_{1y} r_{1x} r_{1y} r_{2x} \\
- & {\lambda_1}^2 p_{2y} r_{1x} r_{1y} r_{2x} & - & {\lambda_1}^2 p_{1y} {r_{1x}}^2 r_{2y} & + & {\lambda_1}^2 p_{2y} {r_{1x}}^2 r_{2y} \\
+ & f_y \mu_3 p_{1x} p_{2x} r_{2y} & - & f_y \mu_3 p_{1x} p_{3x} r_{2y} & + & \mu_3 p_{1x} p_{2y} p_{3x} r_{2y} \\
- & \mu_3 p_{1x} p_{2x} p_{3y} r_{2y} & + & f_y \lambda_1 p_{1x} r_{1x} r_{2y} & - & f_y \lambda_1 p_{2x} r_{1x} r_{2y} \\
+ & \lambda_1 p_{1y} p_{2x} r_{1x} r_{2y} & - & \lambda_1 \mu_3 p_{1y} p_{2x} r_{1x} r_{2y} & - & \lambda_1 p_{1x} p_{2y} r_{1x} r_{2y} \\
+ & \lambda_1 \mu_3 p_{1y} p_{3x} r_{1x} r_{2y} & - & \lambda_1 \mu_3 p_{2y} p_{3x} r_{1x} r_{2y} & + & \lambda_1 \mu_3 p_{2x} p_{3y} r_{1x} r_{2y} \\
+ & \lambda_1 \mu_3 p_{1x} p_{2x} r_{1y} r_{2y} & - & \lambda_1 \mu_3 p_{1x} p_{3x} r_{1y} r_{2y} & + & {\lambda_1}^2 p_{1x} r_{1x} r_{1y} r_{2y} \\
- & {\lambda_1}^2 p_{2x} r_{1x} r_{1y} r_{2y}. & & & &
\end{array}
$$

The polynomial g_{17} is

$$
\begin{array}{cccccccc}
- & f_y \mu_3 p_{2x} & + & \mu_3 p_{3y} p_{2x} & - & \lambda_1 r_{1y} p_{2x} & + & \lambda_2 r_{2y} p_{2x} \\
- & \lambda_2 \mu_3 r_{2y} p_{2x} & + & f_x \mu_3 p_{2y} & + & f_y \mu_3 p_{3x} & - & \mu_3 p_{2y} p_{3x} \\
- & f_x \mu_3 p_{3y} & - & \lambda_1 p_{1y} r_{1x} & + & \lambda_1 p_{2y} r_{1x} & + & \lambda_1 p_{1x} r_{1y} \\
+ & \lambda_2 p_{1y} r_{2x} & - & \lambda_2 p_{2y} r_{2x} & + & \lambda_2 \mu_3 p_{2y} r_{2x} & - & \lambda_2 \mu_3 p_{3y} r_{2x} \\
- & \lambda_2 p_{1x} r_{2y} & + & \lambda_2 \mu_3 p_{3x} r_{2y}. & & & &
\end{array}
$$

The polynomial g_{18} is

$$f_y\mu_3 - p_{3y}\mu_3 + \lambda_3 r_{3y}\mu_3 + \lambda_1 r_{1y} - \lambda_3 r_{3y}.$$

The polynomial g_{19} is

$$f_x\mu_3 - p_{3x}\mu_3 + \lambda_3 r_{3x}\mu_3 + \lambda_1 r_{1x} - \lambda_3 r_{3x}.$$

The polynomial g_{20} is

$$-f_y\mu_2 r_{2x} + \mu_2 p_{2y} r_{2x} - \lambda_3 r_{3y} r_{2x} + f_x\mu_2 r_{2y} - \mu_2 p_{2x} r_{2y} + \lambda_3 r_{2y} r_{3x}.$$

The polynomial g_{21} is

$$\begin{array}{cccccccc}
- & f_y\mu_2 p_{1x} & + & \mu_2 p_{2y} p_{1x} & + & \lambda_1 r_{1y} p_{1x} & - & \lambda_1\mu_2 r_{1y} p_{1x} \\
- & \lambda_3 r_{3y} p_{1x} & + & f_x\mu_2 p_{1y} & + & f_y\mu_2 p_{2x} & - & \mu_2 p_{1y} p_{2x} \\
- & f_x\mu_2 p_{2y} & - & \lambda_1 p_{1y} r_{1x} & + & \lambda_1\mu_2 p_{1y} r_{1x} & - & \lambda_1\mu_2 p_{2y} r_{1x} \\
+ & \lambda_1 p_{3y} r_{1x} & + & \lambda_1\mu_2 p_{2x} r_{1y} & - & \lambda_1 p_{3x} r_{1y} & + & \lambda_3 p_{1y} r_{3x} \\
- & \lambda_3 p_{3y} r_{3x} & + & \lambda_3 p_{3x} r_{3y}. & & & &
\end{array}$$

The polynomial g_{22} is

$$f_y\mu_2 - p_{2y}\mu_2 + \lambda_2 r_{2y}\mu_2 - \lambda_2 r_{2y} + \lambda_3 r_{3y}.$$

The polynomial g_{23} is

$$f_x\mu_2 - p_{2x}\mu_2 + \lambda_2 r_{2x}\mu_2 - \lambda_2 r_{2x} + \lambda_3 r_{3x}.$$

The polynomial g_{24} is

$$
\begin{array}{rrrrrr}
 & \mu_2 p_{1y} r_{1y} {f_x}^2 & - & \mu_2 p_{2y} r_{1y} {f_x}^2 & - & f_y \mu_2 p_{1y} r_{1x} f_x \\
+ & f_y \mu_2 p_{2y} r_{1x} f_x & + & \mu_2 p_{1y} p_{3y} r_{1x} f_x & - & \mu_2 p_{2y} p_{3y} r_{1x} f_x \\
- & f_y \mu_2 p_{1x} r_{1y} f_x & + & f_y \mu_2 p_{2x} r_{1y} f_x & - & \mu_2 p_{1y} p_{2x} r_{1y} f_x \\
+ & \mu_2 p_{1x} p_{2y} r_{1y} f_x & - & \mu_2 p_{1y} p_{3x} r_{1y} f_x & + & \mu_2 p_{2y} p_{3x} r_{1y} f_x \\
+ & \lambda_3 p_{1y} r_{1y} r_{3x} f_x & + & \lambda_3 \mu_2 p_{1y} r_{1y} r_{3x} f_x & - & \lambda_3 \mu_2 p_{2y} r_{1y} r_{3x} f_x \\
- & \lambda_3 p_{3y} r_{1y} r_{3x} f_x & - & \lambda_3 p_{1y} r_{1x} r_{3y} f_x & + & \lambda_3 p_{3y} r_{1x} r_{3y} f_x \\
- & \lambda_3 \mu_2 p_{1x} r_{1y} r_{3y} f_x & + & \lambda_3 \mu_2 p_{2x} r_{1y} r_{3y} f_x & + & {\lambda_3}^2 p_{1y} r_{1y} {r_{3x}}^2 \\
- & {\lambda_3}^2 p_{3y} r_{1y} {r_{3x}}^2 & + & {\lambda_3}^2 p_{1x} r_{1x} {r_{3y}}^2 & - & {\lambda_3}^2 p_{3x} r_{1x} {r_{3y}}^2 \\
+ & {f_y}^2 \mu_2 p_{1x} r_{1x} & - & {f_y}^2 \mu_2 p_{2x} r_{1x} & + & f_y \mu_2 p_{1y} p_{2x} r_{1x} \\
- & f_y \mu_2 p_{1x} p_{2y} r_{1x} & - & f_y \mu_2 p_{1x} p_{3y} r_{1x} & + & f_y \mu_2 p_{2x} p_{3y} r_{1x} \\
- & \mu_2 p_{1y} p_{2x} p_{3y} r_{1x} & + & \mu_2 p_{1x} p_{2y} p_{3y} r_{1x} & + & f_y \mu_2 p_{1x} p_{3x} r_{1y} \\
- & f_y \mu_2 p_{2x} p_{3x} r_{1y} & + & \mu_2 p_{1y} p_{2x} p_{3x} r_{1y} & - & \mu_2 p_{1x} p_{2y} p_{3x} r_{1y} \\
- & f_y \lambda_3 \mu_2 p_{1y} r_{1x} r_{3x} & + & f_y \lambda_3 \mu_2 p_{2y} r_{1x} r_{3x} & + & \lambda_3 \mu_2 p_{1y} p_{3y} r_{1x} r_{3x} \\
- & \lambda_3 \mu_2 p_{2y} p_{3y} r_{1x} r_{3x} & - & f_y \lambda_3 p_{1x} r_{1y} r_{3x} & - & \lambda_3 \mu_2 p_{1y} p_{2x} r_{1y} r_{3x} \\
+ & \lambda_3 \mu_2 p_{1x} p_{2y} r_{1y} r_{3x} & + & f_y \lambda_3 p_{3x} r_{1y} r_{3x} & - & \lambda_3 p_{1y} p_{3x} r_{1y} r_{3x} \\
+ & \lambda_3 p_{1x} p_{3y} r_{1y} r_{3x} & - & \lambda_3 \mu_2 p_{1x} p_{3y} r_{1y} r_{3x} & + & \lambda_3 \mu_2 p_{2x} p_{3y} r_{1y} r_{3x} \\
+ & f_y \lambda_3 p_{1x} r_{1x} r_{3y} & + & f_y \lambda_3 \mu_2 p_{1x} r_{1x} r_{3y} & - & f_y \lambda_3 \mu_2 p_{2x} r_{1x} r_{3y} \\
+ & \lambda_3 \mu_2 p_{1y} p_{2x} r_{1x} r_{3y} & - & \lambda_3 \mu_2 p_{1x} p_{2y} r_{1x} r_{3y} & - & f_y \lambda_3 p_{3x} r_{1x} r_{3y} \\
+ & \lambda_3 p_{1y} p_{3x} r_{1x} r_{3y} & - & \lambda_3 \mu_2 p_{1y} p_{3x} r_{1x} r_{3y} & + & \lambda_3 \mu_2 p_{2y} p_{3x} r_{1x} r_{3y} \\
- & \lambda_3 p_{1x} p_{3y} r_{1x} r_{3y} & + & \lambda_3 \mu_2 p_{1x} p_{3x} r_{1y} r_{3y} & - & \lambda_3 \mu_2 p_{2x} p_{3x} r_{1y} r_{3y} \\
- & {\lambda_3}^2 p_{1y} r_{1x} r_{3x} r_{3y} & + & {\lambda_3}^2 p_{3y} r_{1x} r_{3x} r_{3y} & - & {\lambda_3}^2 p_{1x} r_{1y} r_{3x} r_{3y} \\
+ & {\lambda_3}^2 p_{3x} r_{1y} r_{3x} r_{3y}. & & & &
\end{array}
$$

The polynomial g_{25} is

$$
\begin{array}{ccccccccc}
 & \mu_2 p_{1x} {f_y}^2 & - & \mu_2 p_{2x} {f_y}^2 & - & f_x \mu_2 p_{1y} f_y & + & \mu_2 p_{1y} p_{2x} f_y \\
+ & f_x \mu_2 p_{2y} f_y & - & \mu_2 p_{1x} p_{2y} f_y & - & \mu_2 p_{1x} p_{3y} f_y & + & \mu_2 p_{2x} p_{3y} f_y \\
- & \lambda_1 p_{1x} r_{1y} f_y & + & \lambda_1 \mu_2 p_{1x} r_{1y} f_y & - & \lambda_1 \mu_2 p_{2x} r_{1y} f_y & + & \lambda_1 p_{3x} r_{1y} f_y \\
- & \lambda_3 \mu_2 p_{1y} r_{3x} f_y & + & \lambda_3 \mu_2 p_{2y} r_{3x} f_y & + & \lambda_3 p_{1x} r_{3y} f_y & + & \lambda_3 \mu_2 p_{1x} r_{3y} f_y \\
- & \lambda_3 \mu_2 p_{2x} r_{3y} f_y & - & \lambda_3 p_{3x} r_{3y} f_y & + & {\lambda_3}^2 p_{1x} {r_{3y}}^2 & - & {\lambda_3}^2 p_{3x} {r_{3y}}^2 \\
+ & f_x \mu_2 p_{1y} p_{3y} & - & \mu_2 p_{1y} p_{2x} p_{3y} & - & f_x \mu_2 p_{2y} p_{3y} & + & \mu_2 p_{1x} p_{2y} p_{3y} \\
+ & f_x \lambda_1 p_{1y} r_{1y} & - & f_x \lambda_1 \mu_2 p_{1y} r_{1y} & + & f_x \lambda_1 \mu_2 p_{2y} r_{1y} & - & \lambda_1 p_{1y} p_{3x} r_{1y} \\
+ & \lambda_1 \mu_2 p_{1y} p_{3x} r_{1y} & - & \lambda_1 \mu_2 p_{2y} p_{3x} r_{1y} & - & f_x \lambda_1 p_{3y} r_{1y} & + & \lambda_1 p_{1x} p_{3y} r_{1y} \\
- & \lambda_1 \mu_2 p_{1x} p_{3y} r_{1y} & + & \lambda_1 \mu_2 p_{2x} p_{3y} r_{1y} & + & \lambda_3 \mu_2 p_{1y} p_{3y} r_{3x} & - & \lambda_3 \mu_2 p_{2y} p_{3y} r_{3x} \\
+ & \lambda_1 \lambda_3 p_{1y} r_{1y} r_{3x} & - & \lambda_1 \lambda_3 \mu_2 p_{1y} r_{1y} r_{3x} & + & \lambda_1 \lambda_3 \mu_2 p_{2y} r_{1y} r_{3x} & - & \lambda_1 \lambda_3 p_{3y} r_{1y} r_{3x} \\
- & f_x \lambda_3 p_{1y} r_{3y} & + & \lambda_3 \mu_2 p_{1y} p_{2x} r_{3y} & - & \lambda_3 \mu_2 p_{1x} p_{2y} r_{3y} & + & \lambda_3 p_{1y} p_{3x} r_{3y} \\
- & \lambda_3 \mu_2 p_{1y} p_{3x} r_{3y} & + & \lambda_3 \mu_2 p_{2y} p_{3x} r_{3y} & + & f_x \lambda_3 p_{3y} r_{3y} & - & \lambda_3 p_{1x} p_{3y} r_{3y} \\
- & \lambda_1 \lambda_3 p_{1x} r_{1y} r_{3y} & + & \lambda_1 \lambda_3 \mu_2 p_{1x} r_{1y} r_{3y} & - & \lambda_1 \lambda_3 \mu_2 p_{2x} r_{1y} r_{3y} & + & \lambda_1 \lambda_3 p_{3x} r_{1y} r_{3y} \\
- & {\lambda_3}^2 p_{1y} r_{3x} r_{3y} & + & {\lambda_3}^2 p_{3y} r_{3x} r_{3y}. & & & &
\end{array}
$$

The polynomial g_{26} is

$$
\begin{array}{ccccccccc}
 & \lambda_3 r_{2x} {r_{3y}}^2 & - & f_y \mu_3 r_{2x} r_{3y} & + & f_y \mu_2 \mu_3 r_{2x} r_{3y} & - & \mu_2 \mu_3 p_{2y} r_{2x} r_{3y} \\
+ & \mu_3 p_{3y} r_{2x} r_{3y} & - & \lambda_1 r_{1y} r_{2x} r_{3y} & + & \mu_2 \mu_3 p_{2x} r_{2y} r_{3y} & - & \mu_2 \mu_3 p_{3x} r_{2y} r_{3y} \\
+ & \lambda_1 \mu_2 r_{1x} r_{2y} r_{3y} & - & \lambda_3 r_{2y} r_{3x} r_{3y} & + & f_y \mu_3 r_{2y} r_{3x} & - & f_y \mu_2 \mu_3 r_{2y} r_{3x} \\
+ & \mu_2 \mu_3 p_{3y} r_{2y} r_{3x} & - & \mu_3 p_{3y} r_{2y} r_{3x} & + & \lambda_1 r_{1y} r_{2y} r_{3x} & - & \lambda_1 \mu_2 r_{1y} r_{2y} r_{3x}.
\end{array}
$$

The polynomial g_{27} is

$$f_y \mu_1 r_{1x} - \mu_1 p_{1y} r_{1x} + \lambda_2 r_{2y} r_{1x} - f_x \mu_1 r_{1y} + \mu_1 p_{1x} r_{1y} - \lambda_2 r_{1y} r_{2x}.$$

The polynomial g_{28} is

$$f_y \mu_1 - p_{1y} \mu_1 + \lambda_1 r_{1y} \mu_1 - \lambda_1 r_{1y} + \lambda_2 r_{2y}.$$

The polynomial g_{29} is

$$f_x \mu_1 - p_{1x} \mu_1 + \lambda_1 r_{1x} \mu_1 - \lambda_1 r_{1x} + \lambda_2 r_{2x}.$$

The polynomial g_{30} is

$$
\begin{array}{llllll}
- & \mu_1 p_{1y} r_{3y} {f_x}^2 & + & \mu_1 p_{3y} r_{3y} {f_x}^2 & + & f_y \mu_1 p_{1y} r_{3x} f_x \\
- & \mu_1 p_{1y} p_{2y} r_{3x} f_x & - & f_y \mu_1 p_{3y} r_{3x} f_x & + & \mu_1 p_{2y} p_{3y} r_{3x} f_x \\
+ & \lambda_2 p_{2y} r_{2y} r_{3x} f_x & - & \lambda_2 p_{3y} r_{2y} r_{3x} f_x & + & f_y \mu_1 p_{1x} r_{3y} f_x \\
+ & \mu_1 p_{1y} p_{2x} r_{3y} f_x & - & f_y \mu_1 p_{3x} r_{3y} f_x & + & \mu_1 p_{1y} p_{3x} r_{3y} f_x \\
- & \mu_1 p_{1x} p_{3y} r_{3y} f_x & - & \mu_1 p_{2x} p_{3y} r_{3y} f_x & - & \lambda_2 \mu_1 p_{1y} r_{2x} r_{3y} f_x \\
- & \lambda_2 p_{2y} r_{2x} r_{3y} f_x & + & \lambda_2 p_{3y} r_{2x} r_{3y} f_x & + & \lambda_2 \mu_1 p_{3y} r_{2x} r_{3y} f_x \\
+ & \lambda_2 \mu_1 p_{1x} r_{2y} r_{3y} f_x & - & \lambda_2 \mu_1 p_{3x} r_{2y} r_{3y} f_x & - & {\lambda_2}^2 p_{2x} {r_{2y}}^2 r_{3x} \\
+ & {\lambda_2}^2 p_{3x} {r_{2y}}^2 r_{3x} & - & {f_y}^2 \mu_1 p_{1x} r_{3x} & + & f_y \mu_1 p_{1x} p_{2y} r_{3x} \\
+ & {f_y}^2 \mu_1 p_{3x} r_{3x} & - & f_y \mu_1 p_{1y} p_{3x} r_{3x} & - & f_y \mu_1 p_{2y} p_{3x} r_{3x} \\
+ & \mu_1 p_{1y} p_{2y} p_{3x} r_{3x} & + & f_y \mu_1 p_{1x} p_{3y} r_{3x} & - & \mu_1 p_{1x} p_{2y} p_{3y} r_{3x} \\
+ & f_y \lambda_2 \mu_1 p_{1y} r_{2x} r_{3x} & - & \lambda_2 \mu_1 p_{1y} p_{2y} r_{2x} r_{3x} & - & f_y \lambda_2 \mu_1 p_{3y} r_{2x} r_{3x} \\
+ & \lambda_2 \mu_1 p_{2y} p_{3y} r_{2x} r_{3x} & - & f_y \lambda_2 \mu_1 p_{1x} r_{2y} r_{3x} & - & f_y \lambda_2 p_{2x} r_{2y} r_{3x} \\
+ & \lambda_2 \mu_1 p_{1y} p_{2x} r_{2y} r_{3x} & + & f_y \lambda_2 p_{3x} r_{2y} r_{3x} & + & f_y \lambda_2 \mu_1 p_{3x} r_{2y} r_{3x} \\
- & \lambda_2 \mu_1 p_{1y} p_{3x} r_{2y} r_{3x} & - & \lambda_2 p_{2y} p_{3x} r_{2y} r_{3x} & + & \lambda_2 \mu_1 p_{1x} p_{3y} r_{2y} r_{3x} \\
+ & \lambda_2 p_{2x} p_{3y} r_{2y} r_{3x} & - & \lambda_2 \mu_1 p_{2x} p_{3y} r_{2y} r_{3x} & + & {\lambda_2}^2 p_{2y} r_{2x} r_{2y} r_{3x} \\
- & {\lambda_2}^2 p_{3y} r_{2x} r_{2y} r_{3x} & - & {\lambda_2}^2 p_{2y} {r_{2x}}^2 r_{3y} & + & {\lambda_2}^2 p_{3y} {r_{2x}}^2 r_{3y} \\
- & f_y \mu_1 p_{1x} p_{2x} r_{3y} & + & f_y \mu_1 p_{2x} p_{3x} r_{3y} & - & \mu_1 p_{1y} p_{2x} p_{3x} r_{3y} \\
+ & \mu_1 p_{1x} p_{2x} p_{3y} r_{3y} & + & f_y \lambda_2 p_{2x} r_{2x} r_{3y} & + & \lambda_2 \mu_1 p_{1x} p_{2y} r_{2x} r_{3y} \\
- & f_y \lambda_2 p_{3x} r_{2x} r_{3y} & + & \lambda_2 \mu_1 p_{1y} p_{3x} r_{2x} r_{3y} & + & \lambda_2 p_{2y} p_{3x} r_{2x} r_{3y} \\
- & \lambda_2 \mu_1 p_{2y} p_{3x} r_{2x} r_{3y} & - & \lambda_2 \mu_1 p_{1x} p_{3y} r_{2x} r_{3y} & - & \lambda_2 p_{2x} p_{3y} r_{2x} r_{3y} \\
- & \lambda_2 \mu_1 p_{1x} p_{2x} r_{2y} r_{3y} & + & \lambda_2 \mu_1 p_{2x} p_{3x} r_{2y} r_{3y} & + & {\lambda_2}^2 p_{2x} r_{2x} r_{2y} r_{3y} \\
- & {\lambda_2}^2 p_{3x} r_{2x} r_{2y} r_{3y}. & & & &
\end{array}
$$

The polynomial g_{31} is

$$
\begin{array}{llllllll}
 & f_y \mu_1 p_{1x} & - & \mu_1 p_{3y} p_{1x} & + & \lambda_3 \mu_1 r_{3y} p_{1x} & - & f_x \mu_1 p_{1y} \\
- & f_y \mu_1 p_{3x} & + & \mu_1 p_{1y} p_{3x} & + & f_x \mu_1 p_{3y} & - & \lambda_2 p_{2y} r_{2x} \\
+ & \lambda_2 p_{3y} r_{2x} & + & \lambda_2 p_{2x} r_{2y} & - & \lambda_2 p_{3x} r_{2y} & - & \lambda_3 \mu_1 p_{1y} r_{3x} \\
+ & \lambda_3 p_{2y} r_{3x} & - & \lambda_3 p_{3y} r_{3x} & + & \lambda_3 \mu_1 p_{3y} r_{3x} & - & \lambda_3 p_{2x} r_{3y} \\
+ & \lambda_3 p_{3x} r_{3y} & - & \lambda_3 \mu_1 p_{3x} r_{3y}. & & & &
\end{array}
$$

The final polynomial g_{32} is

$$
\begin{array}{rrrrrr}
 & \mu_1\mu_2 p_{1y} r_{1y} r_{2x} r_{3x} & - & \mu_1\mu_2 p_{2y} r_{1y} r_{2x} r_{3x} & + & \mu_1\mu_2\mu_3 p_{2y} r_{1y} r_{2x} r_{3x} \\
- & \mu_2\mu_3 p_{2y} r_{1y} r_{2x} r_{3x} & - & \mu_1\mu_2\mu_3 p_{3y} r_{1y} r_{2x} r_{3x} & + & \mu_2\mu_3 p_{3y} r_{1y} r_{2x} r_{3x} \\
+ & \mu_1\mu_3 p_{1y} r_{1x} r_{2y} r_{3x} & - & \mu_1\mu_2\mu_3 p_{1y} r_{1x} r_{2y} r_{3x} & + & \mu_2\mu_3 p_{2y} r_{1x} r_{2y} r_{3x} \\
- & \mu_1\mu_3 p_{3y} r_{1x} r_{2y} r_{3x} & + & \mu_1\mu_2\mu_3 p_{3y} r_{1x} r_{2y} r_{3x} & - & \mu_2\mu_3 p_{3y} r_{1x} r_{2y} r_{3x} \\
- & \mu_1\mu_2 p_{1x} r_{1y} r_{2y} r_{3x} & - & \mu_1\mu_3 p_{1x} r_{1y} r_{2y} r_{3x} & + & \mu_1\mu_2\mu_3 p_{1x} r_{1y} r_{2y} r_{3x} \\
+ & \mu_1\mu_2 p_{2x} r_{1y} r_{2y} r_{3x} & - & \mu_1\mu_2\mu_3 p_{2x} r_{1y} r_{2y} r_{3x} & + & \mu_1\mu_3 p_{3x} r_{1y} r_{2y} r_{3x} \\
- & \mu_1\mu_2 p_{1y} r_{1x} r_{2x} r_{3y} & - & \mu_1\mu_3 p_{1y} r_{1x} r_{2x} r_{3y} & + & \mu_1\mu_2\mu_3 p_{1y} r_{1x} r_{2x} r_{3y} \\
+ & \mu_1\mu_2 p_{2y} r_{1x} r_{2x} r_{3y} & - & \mu_1\mu_2\mu_3 p_{2y} r_{1x} r_{2x} r_{3y} & + & \mu_1\mu_3 p_{3y} r_{1x} r_{2x} r_{3y} \\
+ & \mu_1\mu_3 p_{1x} r_{1y} r_{2x} r_{3y} & - & \mu_1\mu_2\mu_3 p_{1x} r_{1y} r_{2x} r_{3y} & + & \mu_2\mu_3 p_{2x} r_{1y} r_{2x} r_{3y} \\
- & \mu_1\mu_3 p_{3x} r_{1y} r_{2x} r_{3y} & + & \mu_1\mu_2\mu_3 p_{3x} r_{1y} r_{2x} r_{3y} & - & \mu_2\mu_3 p_{3x} r_{1y} r_{2x} r_{3y} \\
+ & \mu_1\mu_2 p_{1x} r_{1x} r_{2y} r_{3y} & - & \mu_1\mu_2 p_{2x} r_{1x} r_{2y} r_{3y} & + & \mu_1\mu_2\mu_3 p_{2x} r_{1x} r_{2y} r_{3y} \\
- & \mu_2\mu_3 p_{2x} r_{1x} r_{2y} r_{3y} & - & \mu_1\mu_2\mu_3 p_{3x} r_{1x} r_{2y} r_{3y} & + & \mu_2\mu_3 p_{3x} r_{1x} r_{2y} r_{3y}.
\end{array}
$$

Now returning to the quadratic polynomial g_1, we observe that since there is no constant term, $\lambda_1 = 0$ is necessarily one of the solutions. This solution is not feasible to our problem unless

$$f = p_1 = p_2 = p_3.$$

The other solution is factored out to be n_1/d_1 where n_1 is

$$
\begin{array}{llllll}
- & p_{2y}r_{1y}r_{2y}r_{3x}f_x^2 & + & p_{3y}r_{1y}r_{2y}r_{3x}f_x^2 & - & p_{1y}r_{1y}r_{2x}r_{3y}f_x^2 \\
+ & p_{2y}r_{1y}r_{2x}r_{3y}f_x^2 & + & p_{1y}r_{1x}r_{2y}r_{3y}f_x^2 & - & p_{3y}r_{1x}r_{2y}r_{3y}f_x^2 \\
+ & f_y p_{1y}r_{1y}r_{2x}r_{3x}f_x & - & p_{1y}p_{2y}r_{1y}r_{2x}r_{3x}f_x & - & f_y p_{3y}r_{1y}r_{2x}r_{3x}f_x \\
+ & p_{2y}p_{3y}r_{1y}r_{2x}r_{3x}f_x & - & f_y p_{1y}r_{1x}r_{2y}r_{3x}f_x & + & f_y p_{2y}r_{1x}r_{2y}r_{3x}f_x \\
+ & p_{1y}p_{3y}r_{1x}r_{2y}r_{3x}f_x & - & p_{2y}p_{3y}r_{1x}r_{2y}r_{3x}f_x & + & f_y p_{2x}r_{1y}r_{2y}r_{3x}f_x \\
+ & p_{1x}p_{2y}r_{1y}r_{2y}r_{3x}f_x & - & f_y p_{3x}r_{1y}r_{2y}r_{3x}f_x & + & p_{2y}p_{3x}r_{1y}r_{2y}r_{3x}f_x \\
- & p_{1x}p_{3y}r_{1y}r_{2y}r_{3x}f_x & - & p_{2x}p_{3y}r_{1y}r_{2y}r_{3x}f_x & - & f_y p_{2y}r_{1x}r_{2x}r_{3y}f_x \\
+ & p_{1y}p_{2y}r_{1x}r_{2x}r_{3y}f_x & + & f_y p_{3y}r_{1x}r_{2x}r_{3y}f_x & - & p_{1y}p_{3y}r_{1x}r_{2x}r_{3y}f_x \\
+ & f_y p_{1x}r_{1y}r_{2x}r_{3y}f_x & - & f_y p_{2x}r_{1y}r_{2x}r_{3y}f_x & + & p_{1y}p_{2x}r_{1y}r_{2x}r_{3y}f_x \\
- & p_{1x}p_{2y}r_{1y}r_{2x}r_{3y}f_x & + & p_{1y}p_{3x}r_{1y}r_{2x}r_{3y}f_x & - & p_{2y}p_{3x}r_{1y}r_{2x}r_{3y}f_x \\
- & f_y p_{1x}r_{1x}r_{2y}r_{3y}f_x & - & p_{1y}p_{2x}r_{1x}r_{2y}r_{3y}f_x & + & f_y p_{3x}r_{1x}r_{2y}r_{3y}f_x \\
- & p_{1y}p_{3x}r_{1x}r_{2y}r_{3y}f_x & + & p_{1x}p_{3y}r_{1x}r_{2y}r_{3y}f_x & + & p_{2x}p_{3y}r_{1x}r_{2y}r_{3y}f_x \\
- & f_y^2 p_{1x}r_{1y}r_{2x}r_{3x} & + & f_y p_{1x}p_{2y}r_{1y}r_{2x}r_{3x} & + & f_y^2 p_{3x}r_{1y}r_{2x}r_{3x} \\
- & f_y p_{1y}p_{3x}r_{1y}r_{2x}r_{3x} & - & f_y p_{2y}p_{3x}r_{1y}r_{2x}r_{3x} & + & p_{1y}p_{2y}p_{3x}r_{1y}r_{2x}r_{3x} \\
+ & f_y p_{1x}p_{3y}r_{1y}r_{2x}r_{3x} & - & p_{1x}p_{2y}p_{3y}r_{1y}r_{2x}r_{3x} & + & f_y^2 p_{1x}r_{1x}r_{2y}r_{3x} \\
- & f_y^2 p_{2x}r_{1x}r_{2y}r_{3x} & + & f_y p_{1y}p_{2x}r_{1x}r_{2y}r_{3x} & - & f_y p_{1x}p_{2y}r_{1x}r_{2y}r_{3x} \\
- & f_y p_{1x}p_{3y}r_{1x}r_{2y}r_{3x} & + & f_y p_{2x}p_{3y}r_{1x}r_{2y}r_{3x} & - & p_{1y}p_{2x}p_{3y}r_{1x}r_{2y}r_{3x} \\
+ & p_{1x}p_{2y}p_{3y}r_{1x}r_{2y}r_{3x} & - & f_y p_{1x}p_{2x}r_{1y}r_{2y}r_{3x} & + & f_y p_{1x}p_{3x}r_{1y}r_{2y}r_{3x} \\
- & p_{1x}p_{2y}p_{3x}r_{1y}r_{2y}r_{3x} & + & p_{1x}p_{2x}p_{3y}r_{1y}r_{2y}r_{3x} & + & f_y^2 p_{2x}r_{1x}r_{2x}r_{3y} \\
- & f_y p_{1y}p_{2x}r_{1x}r_{2x}r_{3y} & - & f_y^2 p_{3x}r_{1x}r_{2x}r_{3y} & + & f_y p_{1y}p_{3x}r_{1x}r_{2x}r_{3y} \\
+ & f_y p_{2y}p_{3x}r_{1x}r_{2x}r_{3y} & - & p_{1y}p_{2y}p_{3x}r_{1x}r_{2x}r_{3y} & - & f_y p_{2x}p_{3y}r_{1x}r_{2x}r_{3y} \\
+ & p_{1y}p_{2x}p_{3y}r_{1x}r_{2x}r_{3y} & - & f_y p_{1x}p_{3x}r_{1y}r_{2x}r_{3y} & + & f_y p_{2x}p_{3x}r_{1y}r_{2x}r_{3y} \\
- & p_{1y}p_{2x}p_{3x}r_{1y}r_{2x}r_{3y} & + & p_{1x}p_{2y}p_{3x}r_{1y}r_{2x}r_{3y} & + & f_y p_{1x}p_{2x}r_{1x}r_{2y}r_{3y} \\
- & f_y p_{2x}p_{3x}r_{1x}r_{2y}r_{3y} & + & p_{1y}p_{2x}p_{3x}r_{1x}r_{2y}r_{3y} & - & p_{1x}p_{2x}p_{3y}r_{1x}r_{2y}r_{3y}
\end{array}
$$

and d_1 is

$$
\begin{array}{rcrcrcr}
 & & f_y p_{1y} r_{2y} r_{3x} r_{1x}^2 & - & f_y p_{2y} r_{2y} r_{3x} r_{1x}^2 & - & p_{1y} p_{3y} r_{2y} r_{3x} r_{1x}^2 \\
 & + & p_{2y} p_{3y} r_{2y} r_{3x} r_{1x}^2 & + & f_y p_{2y} r_{2x} r_{3y} r_{1x}^2 & - & p_{1y} p_{2y} r_{2x} r_{3y} r_{1x}^2 \\
 & - & f_y p_{3y} r_{2x} r_{3y} r_{1x}^2 & + & p_{1y} p_{3y} r_{2x} r_{3y} r_{1x}^2 & - & f_x p_{1y} r_{2y} r_{3y} r_{1x}^2 \\
 & + & p_{1y} p_{2x} r_{2y} r_{3y} r_{1x}^2 & + & f_x p_{3y} r_{2y} r_{3y} r_{1x}^2 & - & p_{2x} p_{3y} r_{2y} r_{3y} r_{1x}^2 \\
 & - & f_y p_{1y} r_{1y} r_{2x} r_{3x} r_{1x} & + & p_{1y} p_{2y} r_{1y} r_{2x} r_{3x} r_{1x} & + & f_y p_{3y} r_{1y} r_{2x} r_{3x} r_{1x} \\
 & - & p_{2y} p_{3y} r_{1y} r_{2x} r_{3x} r_{1x} & - & f_y p_{1x} r_{1y} r_{2y} r_{3x} r_{1x} & + & f_y p_{2x} r_{1y} r_{2y} r_{3x} r_{1x} \\
 & - & p_{1y} p_{2x} r_{1y} r_{2y} r_{3x} r_{1x} & + & f_x p_{2y} r_{1y} r_{2y} r_{3x} r_{1x} & + & p_{1y} p_{3x} r_{1y} r_{2y} r_{3x} r_{1x} \\
 & - & p_{2y} p_{3x} r_{1y} r_{2y} r_{3x} r_{1x} & - & f_x p_{3y} r_{1y} r_{2y} r_{3x} r_{1x} & + & p_{1x} p_{3y} r_{1y} r_{2y} r_{3x} r_{1x} \\
 & + & f_x p_{1y} r_{1y} r_{2x} r_{3y} r_{1x} & - & f_y p_{2x} r_{1y} r_{2x} r_{3y} r_{1x} & - & f_x p_{2y} r_{1y} r_{2x} r_{3y} r_{1x} \\
 & + & p_{1x} p_{2y} r_{1y} r_{2x} r_{3y} r_{1x} & + & f_y p_{3x} r_{1y} r_{2x} r_{3y} r_{1x} & - & p_{1y} p_{3x} r_{1y} r_{2x} r_{3y} r_{1x} \\
 & - & p_{1x} p_{3y} r_{1y} r_{2x} r_{3y} r_{1x} & + & p_{2x} p_{3y} r_{1y} r_{2x} r_{3y} r_{1x} & + & f_x p_{1x} r_{1y} r_{2y} r_{3y} r_{1x} \\
 & - & p_{1x} p_{2x} r_{1y} r_{2y} r_{3y} r_{1x} & - & f_x p_{3x} r_{1y} r_{2y} r_{3y} r_{1x} & + & p_{2x} p_{3x} r_{1y} r_{2y} r_{3y} r_{1x} \\
 & + & f_y p_{1x} r_{1y}^2 r_{2x} r_{3x} & - & p_{1x} p_{2y} r_{1y}^2 r_{2x} r_{3x} & - & f_y p_{3x} r_{1y}^2 r_{2x} r_{3x} \\
 & + & p_{2y} p_{3x} r_{1y}^2 r_{2x} r_{3x} & - & f_x p_{2x} r_{1y}^2 r_{2y} r_{3x} & + & p_{1x} p_{2x} r_{1y}^2 r_{2y} r_{3x} \\
 & + & f_x p_{3x} r_{1y}^2 r_{2y} r_{3x} & - & p_{1x} p_{3x} r_{1y}^2 r_{2y} r_{3x} & - & f_x p_{1x} r_{1y}^2 r_{2x} r_{3y} \\
 & + & f_x p_{2x} r_{1y}^2 r_{2x} r_{3y} & + & p_{1x} p_{3x} r_{1y}^2 r_{2x} r_{3y} & - & p_{2x} p_{3x} r_{1y}^2 r_{2x} r_{3y}
\end{array}
$$

At this point, the natural next step is to use the second polynomial g_2 from the Gröbner basis that we computed. Having already computed λ_1, observe that the only unknown in polynomial g_2 is λ_2. (We may also use polynomials g_3, g_4, or g_5 instead of g_2). In addition, because of the nature of our problem, we can also compute λ_2 by solving the following 2×2 system of equations

$$
\begin{aligned}
(f_x + \lambda_1 r_{1x} - p_{1x})\mu_1 &= \lambda_1 r_{1x} - \lambda_2 r_{2x} \\
(f_y + \lambda_1 r_{1y} - p_{1y})\mu_1 &= \lambda_1 r_{1y} - \lambda_2 r_{2y}
\end{aligned}
$$

for the unknowns λ_2 and μ_1. Geometrically, this would correspond to starting from the point v_1 and then in a straight line, passing through the point p_1 until we hit the ray

$$\{x \in \mathbb{R}^2 : x = f + \lambda_2 r_2 \text{ for } \lambda_2 > 0\}.$$

If we solve for λ_2 by either method, then we get $\lambda_2 = n_2/d_2$ where n_2 is

$$
\begin{array}{rcrcrc}
- & p_{2y}r_{1y}r_{2y}r_{3x}{f_x}^2 & + & p_{3y}r_{1y}r_{2y}r_{3x}{f_x}^2 & - & p_{1y}r_{1y}r_{2x}r_{3y}{f_x}^2 \\
+ & p_{2y}r_{1y}r_{2x}r_{3y}{f_x}^2 & + & p_{1y}r_{1x}r_{2y}r_{3y}{f_x}^2 & - & p_{3y}r_{1x}r_{2y}r_{3y}{f_x}^2 \\
+ & f_yp_{1y}r_{1y}r_{2x}r_{3x}f_x & - & p_{1y}p_{2y}r_{1y}r_{2x}r_{3x}f_x & - & f_yp_{3y}r_{1y}r_{2x}r_{3x}f_x \\
+ & p_{2y}p_{3y}r_{1y}r_{2x}r_{3x}f_x & - & f_yp_{1y}r_{1x}r_{2y}r_{3x}f_x & + & f_yp_{2y}r_{1x}r_{2y}r_{3x}f_x \\
+ & p_{1y}p_{3y}r_{1x}r_{2y}r_{3x}f_x & - & p_{2y}p_{3y}r_{1x}r_{2y}r_{3x}f_x & + & f_yp_{2x}r_{1y}r_{2y}r_{3x}f_x \\
+ & p_{1x}p_{2y}r_{1y}r_{2y}r_{3x}f_x & - & f_yp_{3x}r_{1y}r_{2y}r_{3x}f_x & + & p_{2y}p_{3x}r_{1y}r_{2y}r_{3x}f_x \\
- & p_{1x}p_{3y}r_{1y}r_{2y}r_{3x}f_x & - & p_{2x}p_{3y}r_{1y}r_{2y}r_{3x}f_x & - & f_yp_{2y}r_{1x}r_{2x}r_{3y}f_x \\
+ & p_{1y}p_{2y}r_{1x}r_{2x}r_{3y}f_x & + & f_yp_{3y}r_{1x}r_{2x}r_{3y}f_x & - & p_{1y}p_{3y}r_{1x}r_{2x}r_{3y}f_x \\
+ & f_yp_{1x}r_{1y}r_{2x}r_{3y}f_x & - & f_yp_{2x}r_{1y}r_{2x}r_{3y}f_x & + & p_{1y}p_{2x}r_{1y}r_{2x}r_{3y}f_x \\
- & p_{1x}p_{2y}r_{1y}r_{2x}r_{3y}f_x & + & p_{1y}p_{3x}r_{1y}r_{2x}r_{3y}f_x & - & p_{2y}p_{3x}r_{1y}r_{2x}r_{3y}f_x \\
- & f_yp_{1x}r_{1x}r_{2y}r_{3y}f_x & - & p_{1y}p_{2x}r_{1x}r_{2y}r_{3y}f_x & + & f_yp_{3x}r_{1x}r_{2y}r_{3y}f_x \\
- & p_{1y}p_{3x}r_{1x}r_{2y}r_{3y}f_x & + & p_{1x}p_{3y}r_{1x}r_{2y}r_{3y}f_x & + & p_{2x}p_{3y}r_{1x}r_{2y}r_{3y}f_x \\
- & {f_y}^2p_{1x}r_{1y}r_{2x}r_{3x} & + & f_yp_{1x}p_{2y}r_{1y}r_{2x}r_{3x} & + & {f_y}^2p_{3x}r_{1y}r_{2x}r_{3x} \\
- & f_yp_{1y}p_{3x}r_{1y}r_{2x}r_{3x} & - & f_yp_{2y}p_{3x}r_{1y}r_{2x}r_{3x} & + & p_{1y}p_{2y}p_{3x}r_{1y}r_{2x}r_{3x} \\
+ & f_yp_{1x}p_{3y}r_{1y}r_{2x}r_{3x} & - & p_{1x}p_{2y}p_{3y}r_{1y}r_{2x}r_{3x} & + & {f_y}^2p_{1x}r_{1x}r_{2y}r_{3x} \\
- & {f_y}^2p_{2x}r_{1x}r_{2y}r_{3x} & + & f_yp_{1y}p_{2x}r_{1x}r_{2y}r_{3x} & - & f_yp_{1x}p_{2y}r_{1x}r_{2y}r_{3x} \\
- & f_yp_{1x}p_{3y}r_{1x}r_{2y}r_{3x} & + & f_yp_{2x}p_{3y}r_{1x}r_{2y}r_{3x} & - & p_{1y}p_{2x}p_{3y}r_{1x}r_{2y}r_{3x} \\
+ & p_{1x}p_{2y}p_{3y}r_{1x}r_{2y}r_{3x} & - & f_yp_{1x}p_{2x}r_{1y}r_{2y}r_{3x} & + & f_yp_{1x}p_{3x}r_{1y}r_{2y}r_{3x} \\
- & p_{1x}p_{2y}p_{3x}r_{1y}r_{2y}r_{3x} & + & p_{1x}p_{2x}p_{3y}r_{1y}r_{2y}r_{3x} & + & {f_y}^2p_{2x}r_{1x}r_{2x}r_{3y} \\
- & f_yp_{1y}p_{2x}r_{1x}r_{2x}r_{3y} & - & {f_y}^2p_{3x}r_{1x}r_{2x}r_{3y} & + & f_yp_{1y}p_{3x}r_{1x}r_{2x}r_{3y} \\
+ & f_yp_{2y}p_{3x}r_{1x}r_{2x}r_{3y} & - & p_{1y}p_{2y}p_{3x}r_{1x}r_{2x}r_{3y} & - & f_yp_{2x}p_{3y}r_{1x}r_{2x}r_{3y} \\
+ & p_{1y}p_{2x}p_{3y}r_{1x}r_{2x}r_{3y} & - & f_yp_{1x}p_{3x}r_{1y}r_{2x}r_{3y} & + & f_yp_{2x}p_{3x}r_{1y}r_{2x}r_{3y} \\
- & p_{1y}p_{2x}p_{3x}r_{1y}r_{2x}r_{3y} & + & p_{1x}p_{2y}p_{3x}r_{1y}r_{2x}r_{3y} & + & f_yp_{1x}p_{2x}r_{1x}r_{2y}r_{3y} \\
- & f_yp_{2x}p_{3x}r_{1x}r_{2y}r_{3y} & + & p_{1y}p_{2x}p_{3x}r_{1x}r_{2y}r_{3y} & - & p_{1x}p_{2x}p_{3y}r_{1x}r_{2y}r_{3y}
\end{array}
$$

and d_2 is

$$
\begin{array}{cccccc}
- & f_y p_{1y} r_{1y} r_{3x} {r_{2x}}^2 & + & p_{1y} p_{2y} r_{1y} r_{3x} {r_{2x}}^2 & + & f_y p_{3y} r_{1y} r_{3x} {r_{2x}}^2 \\
- & p_{2y} p_{3y} r_{1y} r_{3x} {r_{2x}}^2 & + & f_y p_{2y} r_{1x} r_{3y} {r_{2x}}^2 & - & p_{1y} p_{2y} r_{1x} r_{3y} {r_{2x}}^2 \\
- & f_y p_{3y} r_{1x} r_{3y} {r_{2x}}^2 & + & p_{1y} p_{3y} r_{1x} r_{3y} {r_{2x}}^2 & + & f_x p_{1y} r_{1y} r_{3y} {r_{2x}}^2 \\
- & f_x p_{2y} r_{1y} r_{3y} {r_{2x}}^2 & - & p_{1y} p_{3x} r_{1y} r_{3y} {r_{2x}}^2 & + & p_{2y} p_{3x} r_{1y} r_{3y} {r_{2x}}^2 \\
+ & f_y p_{1y} r_{1x} r_{2y} r_{3x} r_{2x} & - & f_y p_{2y} r_{1x} r_{2y} r_{3x} r_{2x} & - & p_{1y} p_{3y} r_{1x} r_{2y} r_{3x} r_{2x} \\
+ & p_{2y} p_{3y} r_{1x} r_{2y} r_{3x} r_{2x} & + & f_y p_{1x} r_{1y} r_{2y} r_{3x} r_{2x} & - & p_{1y} p_{2x} r_{1y} r_{2y} r_{3x} r_{2x} \\
+ & f_x p_{2y} r_{1y} r_{2y} r_{3x} r_{2x} & - & p_{1x} p_{2y} r_{1y} r_{2y} r_{3x} r_{2x} & - & f_y p_{3x} r_{1y} r_{2y} r_{3x} r_{2x} \\
+ & p_{1y} p_{3x} r_{1y} r_{2y} r_{3x} r_{2x} & - & f_x p_{3y} r_{1y} r_{2y} r_{3x} r_{2x} & + & p_{2x} p_{3y} r_{1y} r_{2y} r_{3x} r_{2x} \\
- & f_x p_{1y} r_{1x} r_{2y} r_{3y} r_{2x} & - & f_y p_{2x} r_{1x} r_{2y} r_{3y} r_{2x} & + & p_{1y} p_{2x} r_{1x} r_{2y} r_{3y} r_{2x} \\
+ & p_{1x} p_{2y} r_{1x} r_{2y} r_{3y} r_{2x} & + & f_y p_{3x} r_{1x} r_{2y} r_{3y} r_{2x} & - & p_{2y} p_{3x} r_{1x} r_{2y} r_{3y} r_{2x} \\
+ & f_x p_{3y} r_{1x} r_{2y} r_{3y} r_{2x} & - & p_{1x} p_{3y} r_{1x} r_{2y} r_{3y} r_{2x} & - & f_x p_{1x} r_{1y} r_{2y} r_{3y} r_{2x} \\
+ & f_x p_{2x} r_{1y} r_{2y} r_{3y} r_{2x} & + & p_{1x} p_{3x} r_{1y} r_{2y} r_{3y} r_{2x} & - & p_{2x} p_{3x} r_{1y} r_{2y} r_{3y} r_{2x} \\
- & f_y p_{1x} r_{1x} {r_{2y}}^2 r_{3x} & + & f_y p_{2x} r_{1x} {r_{2y}}^2 r_{3x} & + & p_{1x} p_{3y} r_{1x} {r_{2y}}^2 r_{3x} \\
- & p_{2x} p_{3y} r_{1x} {r_{2y}}^2 r_{3x} & - & f_x p_{2x} r_{1y} {r_{2y}}^2 r_{3x} & + & p_{1x} p_{2x} r_{1y} {r_{2y}}^2 r_{3x} \\
+ & f_x p_{3x} r_{1y} {r_{2y}}^2 r_{3x} & - & p_{1x} p_{3x} r_{1y} {r_{2y}}^2 r_{3x} & + & f_x p_{1x} r_{1x} {r_{2y}}^2 r_{3y} \\
- & p_{1x} p_{2x} r_{1x} {r_{2y}}^2 r_{3y} & - & f_x p_{3x} r_{1x} {r_{2y}}^2 r_{3y} & + & p_{2x} p_{3x} r_{1x} {r_{2y}}^2 r_{3y}.
\end{array}
$$

Having determined λ_1 and λ_2, we may now solve for λ_3 using any of the polynomials $g_6, g_7, g_8, g_9, g_{10}, g_{11}$ or g_{12}. We may also solve for λ_3 by solving the following 2×2 system of equations

$$
\begin{aligned}
(f_x + \lambda_2 r_{2x} - p_{2x})\mu_2 &= \lambda_2 r_{2x} - \lambda_3 r_{3x} \\
(f_y + \lambda_2 r_{2y} - p_{2y})\mu_2 &= \lambda_2 r_{2y} - \lambda_3 r_{3y}
\end{aligned}
$$

for the unknowns λ_3 and μ_2. All of these methods will give us $\lambda_3 = n_3/d_3$ where n_3 is

$$
\begin{array}{cccccc}
- & p_{2y}r_{1y}r_{2y}r_{3x}{f_x}^2 & + & p_{3y}r_{1y}r_{2y}r_{3x}{f_x}^2 & - & p_{1y}r_{1y}r_{2x}r_{3y}{f_x}^2 \\
+ & p_{2y}r_{1y}r_{2x}r_{3y}{f_x}^2 & + & p_{1y}r_{1x}r_{2y}r_{3y}{f_x}^2 & - & p_{3y}r_{1x}r_{2y}r_{3y}{f_x}^2 \\
+ & f_y p_{1y}r_{1y}r_{2x}r_{3x}f_x & - & p_{1y}p_{2y}r_{1y}r_{2x}r_{3x}f_x & - & f_y p_{3y}r_{1y}r_{2x}r_{3x}f_x \\
+ & p_{2y}p_{3y}r_{1y}r_{2x}r_{3x}f_x & - & f_y p_{1y}r_{1x}r_{2y}r_{3x}f_x & + & f_y p_{2y}r_{1x}r_{2y}r_{3x}f_x \\
+ & p_{1y}p_{3y}r_{1x}r_{2y}r_{3x}f_x & - & p_{2y}p_{3y}r_{1x}r_{2y}r_{3x}f_x & + & f_y p_{2x}r_{1y}r_{2y}r_{3x}f_x \\
+ & p_{1x}p_{2y}r_{1y}r_{2y}r_{3x}f_x & - & f_y p_{3x}r_{1y}r_{2y}r_{3x}f_x & + & p_{2y}p_{3x}r_{1y}r_{2y}r_{3x}f_x \\
- & p_{1x}p_{3y}r_{1y}r_{2y}r_{3x}f_x & - & p_{2x}p_{3y}r_{1y}r_{2y}r_{3x}f_x & - & f_y p_{2y}r_{1x}r_{2x}r_{3y}f_x \\
+ & p_{1y}p_{2y}r_{1x}r_{2x}r_{3y}f_x & + & f_y p_{3y}r_{1x}r_{2x}r_{3y}f_x & - & p_{1y}p_{3y}r_{1x}r_{2x}r_{3y}f_x \\
+ & f_y p_{1x}r_{1y}r_{2x}r_{3y}f_x & - & f_y p_{2x}r_{1y}r_{2x}r_{3y}f_x & + & p_{1y}p_{2x}r_{1y}r_{2x}r_{3y}f_x \\
- & p_{1x}p_{2y}r_{1y}r_{2x}r_{3y}f_x & + & p_{1y}p_{3x}r_{1y}r_{2x}r_{3y}f_x & - & p_{2y}p_{3x}r_{1y}r_{2x}r_{3y}f_x \\
- & f_y p_{1x}r_{1x}r_{2y}r_{3y}f_x & - & p_{1y}p_{2x}r_{1x}r_{2y}r_{3y}f_x & + & f_y p_{3x}r_{1x}r_{2y}r_{3y}f_x \\
- & p_{1y}p_{3x}r_{1x}r_{2y}r_{3y}f_x & + & p_{1x}p_{3y}r_{1x}r_{2y}r_{3y}f_x & + & p_{2x}p_{3y}r_{1x}r_{2y}r_{3y}f_x \\
- & {f_y}^2 p_{1x}r_{1y}r_{2x}r_{3x} & + & f_y p_{1x}p_{2y}r_{1y}r_{2x}r_{3x} & + & {f_y}^2 p_{3x}r_{1y}r_{2x}r_{3x} \\
- & f_y p_{1y}p_{3x}r_{1y}r_{2x}r_{3x} & - & f_y p_{2y}p_{3x}r_{1y}r_{2x}r_{3x} & + & p_{1y}p_{2y}p_{3x}r_{1y}r_{2x}r_{3x} \\
+ & f_y p_{1x}p_{3y}r_{1y}r_{2x}r_{3x} & - & p_{1x}p_{2y}p_{3y}r_{1y}r_{2x}r_{3x} & + & {f_y}^2 p_{1x}r_{1x}r_{2y}r_{3x} \\
- & {f_y}^2 p_{2x}r_{1x}r_{2y}r_{3x} & + & f_y p_{1y}p_{2x}r_{1x}r_{2y}r_{3x} & - & f_y p_{1x}p_{2y}r_{1x}r_{2y}r_{3x} \\
- & f_y p_{1x}p_{3y}r_{1x}r_{2y}r_{3x} & + & f_y p_{2x}p_{3y}r_{1x}r_{2y}r_{3x} & - & p_{1y}p_{2x}p_{3y}r_{1x}r_{2y}r_{3x} \\
+ & p_{1x}p_{2y}p_{3y}r_{1x}r_{2y}r_{3x} & - & f_y p_{1x}p_{2x}r_{1y}r_{2y}r_{3x} & + & f_y p_{1x}p_{3x}r_{1y}r_{2y}r_{3x} \\
- & p_{1x}p_{2y}p_{3x}r_{1y}r_{2y}r_{3x} & + & p_{1x}p_{2x}p_{3y}r_{1y}r_{2y}r_{3x} & + & {f_y}^2 p_{2x}r_{1x}r_{2x}r_{3y} \\
- & f_y p_{1y}p_{2x}r_{1x}r_{2x}r_{3y} & - & {f_y}^2 p_{3x}r_{1x}r_{2x}r_{3y} & + & f_y p_{1y}p_{3x}r_{1x}r_{2x}r_{3y} \\
+ & f_y p_{2y}p_{3x}r_{1x}r_{2x}r_{3y} & - & p_{1y}p_{2y}p_{3x}r_{1x}r_{2x}r_{3y} & - & f_y p_{2x}p_{3y}r_{1x}r_{2x}r_{3y} \\
+ & p_{1y}p_{2x}p_{3y}r_{1x}r_{2x}r_{3y} & - & f_y p_{1x}p_{3x}r_{1y}r_{2x}r_{3y} & + & f_y p_{2x}p_{3x}r_{1y}r_{2x}r_{3y} \\
- & p_{1y}p_{2x}p_{3x}r_{1y}r_{2x}r_{3y} & + & p_{1x}p_{2y}p_{3x}r_{1y}r_{2x}r_{3y} & + & f_y p_{1x}p_{2x}r_{1x}r_{2y}r_{3y} \\
- & f_y p_{2x}p_{3x}r_{1x}r_{2y}r_{3y} & + & p_{1y}p_{2x}p_{3x}r_{1x}r_{2y}r_{3y} & - & p_{1x}p_{2x}p_{3y}r_{1x}r_{2y}r_{3y}
\end{array}
$$

and d_3 is

$$
\begin{array}{cccccc}
- & f_y p_{1y} r_{1y} r_{2x} {r_{3x}}^2 & + & p_{1y} p_{2y} r_{1y} r_{2x} {r_{3x}}^2 & + & f_y p_{3y} r_{1y} r_{2x} {r_{3x}}^2 \\
- & p_{2y} p_{3y} r_{1y} r_{2x} {r_{3x}}^2 & + & f_y p_{1y} r_{1x} r_{2y} {r_{3x}}^2 & - & f_y p_{2y} r_{1x} r_{2y} {r_{3x}}^2 \\
- & p_{1y} p_{3y} r_{1x} r_{2y} {r_{3x}}^2 & + & p_{2y} p_{3y} r_{1x} r_{2y} {r_{3x}}^2 & + & f_x p_{2y} r_{1y} r_{2y} {r_{3x}}^2 \\
- & p_{1x} p_{2y} r_{1y} r_{2y} {r_{3x}}^2 & - & f_x p_{3y} r_{1y} r_{2y} {r_{3x}}^2 & + & p_{1x} p_{3y} r_{1y} r_{2y} {r_{3x}}^2 \\
+ & f_y p_{2y} r_{1x} r_{2x} r_{3y} r_{3x} & - & p_{1y} p_{2y} r_{1x} r_{2x} r_{3y} r_{3x} & - & f_y p_{3y} r_{1x} r_{2x} r_{3y} r_{3x} \\
+ & p_{1y} p_{3y} r_{1x} r_{2x} r_{3y} r_{3x} & + & f_y p_{1x} r_{1y} r_{2x} r_{3y} r_{3x} & + & f_x p_{1y} r_{1y} r_{2x} r_{3y} r_{3x} \\
- & p_{1y} p_{2x} r_{1y} r_{2x} r_{3y} r_{3x} & - & f_x p_{2y} r_{1y} r_{2x} r_{3y} r_{3x} & - & f_y p_{3x} r_{1y} r_{2x} r_{3y} r_{3x} \\
+ & p_{2y} p_{3x} r_{1y} r_{2x} r_{3y} r_{3x} & - & p_{1x} p_{3y} r_{1y} r_{2x} r_{3y} r_{3x} & + & p_{2x} p_{3y} r_{1y} r_{2x} r_{3y} r_{3x} \\
- & f_y p_{1x} r_{1x} r_{2y} r_{3y} r_{3x} & - & f_x p_{1y} r_{1x} r_{2y} r_{3y} r_{3x} & + & f_y p_{2x} r_{1x} r_{2y} r_{3y} r_{3x} \\
+ & p_{1x} p_{2y} r_{1x} r_{2y} r_{3y} r_{3x} & + & p_{1y} p_{3x} r_{1x} r_{2y} r_{3y} r_{3x} & - & p_{2y} p_{3x} r_{1x} r_{2y} r_{3y} r_{3x} \\
+ & f_x p_{3y} r_{1x} r_{2y} r_{3y} r_{3x} & - & p_{2x} p_{3y} r_{1x} r_{2y} r_{3y} r_{3x} & - & f_x p_{2x} r_{1y} r_{2y} r_{3y} r_{3x} \\
+ & p_{1x} p_{2x} r_{1y} r_{2y} r_{3y} r_{3x} & + & f_x p_{3x} r_{1y} r_{2y} r_{3y} r_{3x} & - & p_{1x} p_{3x} r_{1y} r_{2y} r_{3y} r_{3x} \\
- & f_y p_{2x} r_{1x} r_{2x} {r_{3y}}^2 & + & p_{1y} p_{2x} r_{1x} r_{2x} {r_{3y}}^2 & + & f_y p_{3x} r_{1x} r_{2x} {r_{3y}}^2 \\
- & p_{1y} p_{3x} r_{1x} r_{2x} {r_{3y}}^2 & - & f_x p_{1x} r_{1y} r_{2x} {r_{3y}}^2 & + & f_x p_{2x} r_{1y} r_{2x} {r_{3y}}^2 \\
+ & p_{1x} p_{3x} r_{1y} r_{2x} {r_{3y}}^2 & - & p_{2x} p_{3x} r_{1y} r_{2x} {r_{3y}}^2 & + & f_x p_{1x} r_{1x} r_{2y} {r_{3y}}^2 \\
- & p_{1x} p_{2x} r_{1x} r_{2y} {r_{3y}}^2 & - & f_x p_{3x} r_{1x} r_{2y} {r_{3y}}^2 & + & p_{2x} p_{3x} r_{1x} r_{2y} {r_{3y}}^2
\end{array}
$$

As a sanity check, we also compute a Gröbner basis using the software package Maple from Maplesoft. We ran the following command

```
with(Groebner);
gbasis([r1x*m1*l1 +  fx*m1 -  p1x*m1 -  r1x*l1 +  r2x*l2,
        r1y*m1*l1 +  fy*m1 -  p1y*m1 -  r1y*l1 +  r2y*l2,
        r2x*m2*l2 +  fx*m2 -  p2x*m2 -  r2x*l2 +  r3x*l3,
        r2y*m2*l2 +  fy*m2 -  p2y*m2 -  r2y*l2 +  r3y*l3,
        r3x*m3*l3 +  fx*m3 -  p3x*m3 -  r3x*l3 +  r1x*l1,
        r3y*m3*l3 +  fy*m3 -  p3y*m3 -  r3y*l3 +  r1y*l1],
        plex(m1,m2,m3,l3,l2,l1));
```

using Maple 12 on a Linux machine. The first polynomial in the returned Gröbner basis is

$$
\begin{aligned}
& (r_{3y}p_{3y}r_{1x}p_{1x}r_{2y}p_{2x} - r_{3y}p_{1y}r_{1x}p_{3x}r_{2y}p_{2x} - p_{1x}r_{2y}{f_y}^2r_{3x}r_{1x} \\
&+ r_{2y}r_{3x}p_{1y}r_{1x}f_xf_y + r_{2y}r_{3x}r_{1y}f_xp_{3x}f_y - f_yr_{1x}p_{3y}r_{3x}r_{2y}p_{2x} \\
&- r_{2y}r_{3x}r_{1y}p_{1x}p_{3x}f_y + r_{2x}r_{3y}p_{3y}r_{1x}p_{2x}f_y - r_{2x}r_{3y}r_{1y}f_yp_{3x}p_{2x} \\
&- r_{1y}p_{3y}r_{3x}{f_x}^2r_{2y} + r_{2x}p_{1y}f_xr_{1y}r_{3x}p_{2y} - r_{2x}r_{3y}r_{1y}f_xp_{1y}p_{2x} \\
&- r_{2x}p_{1y}f_xr_{1y}r_{3x}f_y - r_{2y}f_xp_{2x}r_{1y}r_{3x}f_y - r_{2x}r_{1y}p_{3y}r_{3x}p_{2y}f_x \\
&+ r_{2x}r_{1y}p_{3y}r_{3x}f_yf_x - r_{2x}r_{3y}p_{1y}r_{1x}p_{3x}f_y - r_{2x}r_{3y}r_{1x}p_{2y}p_{3x}f_y \\
&+ r_{3y}p_{3y}r_{1x}{f_x}^2r_{2y} - r_{2x}r_{3y}p_{1y}r_{1x}p_{3y}p_{2x} - r_{2x}r_{3y}p_{3x}r_{1y}p_{1x}p_{2y} \\
&+ r_{1y}r_{3x}p_{2y}{f_x}^2r_{2y} - r_{2y}r_{3x}r_{1y}f_xp_{3x}p_{2y} + r_{1y}p_{1x}p_{3y}r_{3x}f_xr_{2y} \\
&+ r_{2x}r_{3y}p_{3x}r_{1y}p_{1y}p_{2x} + r_{2x}r_{3y}p_{1y}r_{1x}p_{3x}p_{2y} - r_{3y}r_{2y}{f_x}^2p_{1y}r_{1x} \\
&+ r_{1x}f_xr_{2y}p_{3y}r_{3x}p_{2y} - r_{3y}p_{3y}r_{1x}p_{1x}r_{2y}f_x - r_{3y}r_{1x}f_xr_{2y}p_{3x}f_y \\
&+ r_{3y}p_{1y}r_{1x}p_{3x}f_xr_{2y} + r_{3y}p_{1y}p_{2x}r_{1x}f_xr_{2y} + r_{3y}p_{1x}r_{2y}f_xr_{1x}f_y \\
&- r_{3y}p_{3y}r_{1x}f_xr_{2y}p_{2x} - r_{2x}r_{3y}p_{3x}r_{1y}p_{1y}f_x - r_{2x}r_{3y}r_{1y}p_{1x}f_xf_y \\
&- r_{2x}r_{3y}p_{2y}p_{1y}r_{1x}f_x - r_{2x}r_{3y}p_{3y}r_{1x}f_yf_x + r_{2x}r_{3y}r_{1y}p_{1x}f_xp_{2y} \\
&- p_{1y}r_{1x}p_{3y}r_{3x}f_xr_{2y} + r_{2x}r_{3y}p_{3x}r_{1y}p_{2y}f_x + r_{1y}p_{2x}p_{3y}r_{3x}f_xr_{2y} \\
&+ r_{2x}r_{3y}p_{3y}r_{1x}p_{1y}f_x + r_{2x}r_{3y}p_{2y}f_yr_{1x}f_x - r_{2x}p_{3x}r_{1y}p_{1y}p_{2y}r_{3x} \\
&+ r_{2x}r_{3y}r_{1y}p_{2x}f_xf_y + p_{1x}r_{2y}p_{2y}r_{3x}r_{1x}f_y - p_{2y}p_{1x}r_{2y}r_{1y}r_{3x}f_x \\
&- f_yp_{2y}r_{3x}r_{1x}f_xr_{2y} + r_{2x}r_{3y}r_{1y}{f_x}^2p_{1y} + p_{3y}r_{1x}p_{1x}r_{2y}f_yr_{3x} \\
&- p_{3y}r_{1x}p_{1x}r_{2y}p_{2y}r_{3x} - r_{2x}r_{3y}r_{1y}{f_x}^2p_{2y} + r_{2x}r_{1y}p_{1x}{f_y}^2r_{3x} \\
&+ r_{2x}p_{3x}r_{1y}p_{1y}f_yr_{3x} - r_{2x}r_{1y}r_{3x}{f_y}^2p_{3x} - r_{2x}r_{3y}{f_y}^2r_{1x}p_{2x} \\
&+ r_{1x}r_{2y}p_{2x}{f_y}^2r_{3x} - r_{2x}r_{1y}p_{1x}f_yr_{3x}p_{2y} + r_{2x}r_{1y}p_{1x}p_{3y}r_{3x}p_{2y} \\
&- r_{2x}r_{1y}p_{1x}p_{3y}r_{3x}f_y - r_{1y}p_{1x}p_{3y}r_{3x}r_{2y}p_{2x} + r_{2x}r_{1y}r_{3x}f_yp_{3x}p_{2y} \\
&- r_{2y}p_{2x}p_{1y}r_{1x}f_yr_{3x} + p_{1x}r_{2y}p_{2x}r_{1y}r_{3x}f_y + r_{2x}r_{3y}p_{1y}p_{2x}r_{1x}f_y \\
&+ r_{3y}f_yr_{1x}p_{3x}r_{2y}p_{2x} + p_{1y}r_{1x}p_{3y}r_{3x}r_{2y}p_{2x} + r_{2x}r_{3y}p_{3x}r_{1y}p_{1x}f_y \\
&- r_{3y}p_{1x}r_{2y}p_{2x}r_{1x}f_y + r_{2x}r_{3y}r_{1x}p_{3x}{f_y}^2 + r_{2y}r_{3x}r_{1y}p_{1x}p_{3x}p_{2y})\lambda_1 \\
&+ (r_{2y}f_xr_{1y}p_{1x}r_{3y}r_{1x} - r_{2y}f_xp_{1y}{r_{1x}}^2r_{3y} - r_{2y}p_{2x}{r_{1y}}^2f_xr_{3x} \\
&+ r_{2y}p_{2x}{r_{1y}}^2p_{1x}r_{3x} - r_{2y}p_{2x}r_{1y}p_{1x}r_{3y}r_{1x} + r_{2y}p_{2x}f_yr_{1x}r_{1y}r_{3x} \\
&- r_{2y}p_{2x}p_{1y}r_{1x}r_{1y}r_{3x} + r_{2y}p_{2x}p_{1y}{r_{1x}}^2r_{3y} - r_{2y}r_{3x}{r_{1y}}^2p_{1x}p_{3x} \\
&+ r_{1y}r_{3x}f_yr_{2x}p_{3y}r_{1x} - {r_{1y}}^2r_{3x}f_yr_{2x}p_{3x} - r_{2y}r_{3x}r_{1y}f_xp_{3y}r_{1x}
\end{aligned}
$$

$$
\begin{array}{llllll}
+ & r_{2y}r_{3x}{r_{1y}}^2 f_x p_{3x} & - & r_{3y}r_{2x}f_y{r_{1x}}^2 p_{3y} & + & r_{3y}r_{2x}f_y r_{1x}p_{3x}r_{1y} \\
+ & r_{3y}r_{2x}p_{1y}{r_{1x}}^2 p_{3y} & + & r_{3y}r_{2x}p_{1y}r_{1x}r_{1y}f_x & - & r_{3y}r_{2x}p_{1y}r_{1x}p_{3x}r_{1y} \\
- & r_{1y}r_{3x}p_{2y}r_{2x}p_{3y}r_{1x} & + & {r_{1y}}^2 r_{3x}p_{2y}r_{2x}p_{3x} & - & r_{1y}r_{2x}p_{2x}r_{3y}f_y r_{1x} \\
+ & r_{1y}r_{2x}p_{2x}r_{3y}p_{3y}r_{1x} & + & {r_{1y}}^2 r_{2x}p_{2x}r_{3y}f_x & - & {r_{1y}}^2 r_{2x}p_{2x}r_{3y}p_{3x} \\
- & r_{2y}{r_{1x}}^2 p_{2y}r_{3x}f_y & + & r_{2y}{r_{1x}}^2 p_{2y}r_{3x}p_{3y} & + & r_{2y}r_{1x}p_{2y}r_{3x}r_{1y}f_x \\
- & r_{2y}r_{1x}p_{2y}r_{3x}p_{3x}r_{1y} & + & r_{3y}{r_{1x}}^2 f_x r_{2y}p_{3y} & - & r_{3y}r_{1x}f_x r_{2y}p_{3x}r_{1y} \\
- & r_{3y}{r_{1x}}^2 r_{2y}p_{2x}p_{3y} & + & f_y r_{2x}{r_{1y}}^2 p_{1x}r_{3x} & - & f_y r_{2x}p_{1y}r_{1x}r_{1y}r_{3x} \\
- & p_{2y}r_{2x}r_{1y}f_x r_{3y}r_{1x} & - & p_{2y}r_{2x}{r_{1y}}^2 p_{1x}r_{3x} & + & p_{2y}r_{2x}r_{1y}p_{1x}r_{3y}r_{1x} \\
+ & p_{2y}r_{2x}f_y{r_{1x}}^2 r_{3y} & + & p_{2y}r_{2x}p_{1y}r_{1x}r_{1y}r_{3x} & - & p_{2y}r_{2x}p_{1y}{r_{1x}}^2 r_{3y} \\
+ & r_{2y}r_{3x}p_{1y}{r_{1x}}^2 f_y & - & r_{2y}r_{3x}p_{1y}{r_{1x}}^2 p_{3y} & + & r_{2y}r_{3x}p_{1y}r_{1x}p_{3x}r_{1y} \\
- & r_{3y}r_{2x}r_{1y}p_{1x}p_{3y}r_{1x} & - & r_{3y}r_{2x}{r_{1y}}^2 p_{1x}f_x & + & r_{3y}r_{2x}{r_{1y}}^2 p_{1x}p_{3x} \\
- & r_{2y}r_{3x}r_{1y}p_{1x}f_y r_{1x} & + & r_{2y}r_{3x}r_{1y}p_{1x}p_{3y}r_{1x} & + & r_{3y}r_{1x}r_{2y}p_{2x}p_{3x}r_{1y}){\lambda_1}^2
\end{array}
$$

which coincides with the polynomial g_1 computed by Mathematica. This gives us confidence in the correctness of these results. We omit the remaining polynomials in the Gröbner basis produced by Maple.

For our purposes, we are not concerned with the values of the μ_i, but if needed, they can be calculated. μ_3 can be determined from any of $g_{14}, g_{15}, \ldots, g_{19}$, μ_2 can be determined from any of $g_{20}, g_{21}, \ldots, g_{26}$, and μ_1 can be determined from any of $g_{27}, g_{28}, \ldots, g_{32}$. Observe the sequence of the determination of the indeterminates and our chosen lexicographic ordering. As expected, they coincide.

We have proven the following theorem.

Theorem 5.2.1 *For $f, r_1, r_2, r_3, p_1, p_2, p_3 \in \mathbb{R}^2$, the unique triangle if it exists with vertices $v_i = f + \lambda_i r_i$ where edge $[v_1, v_2]$ contains point p_1, edge $[v_2, v_3]$ contains point p_2, edge $[v_3, v_1]$ contains point p_3, is given by $\lambda_i = n_i/d_i$.*

5.2.1 Numerical example

Let

$$
f = \begin{bmatrix} 18/100 \\ 29/100 \end{bmatrix}, r_1 = \begin{bmatrix} 0 \\ 3/4 \end{bmatrix}, r_2 = \begin{bmatrix} 57/80 \\ -57/80 \end{bmatrix}, r_3 = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}
$$

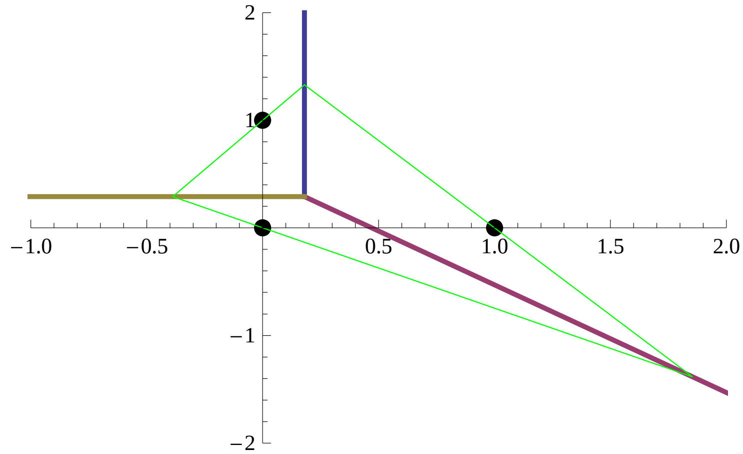

Figure 24: Triangle example

and let

$$p_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ p_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \text{and } p_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Substituting these values into the formulas that we have derived for λ_i, we get

$$\lambda_1 = \frac{3013}{2175}, \lambda_2 = \frac{6026}{2565},\ \text{and } \lambda_3 = \frac{3013}{2650}.$$

The input and the solution are shown in Figure 24. In the figure, the length of the vectors r_i have been increased for visual purposes.

5.3 Solution of a quadrilateral problem

The quadrilateral problem we wish to solve is similar to our triangle problem. We simply have one more vector r_4. Again, we have the point

$$f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} \in \mathbb{R}^2,$$

and let

$$r_1 = \begin{bmatrix} r_{1x} \\ r_{1y} \end{bmatrix}, r_2 = \begin{bmatrix} r_{2x} \\ r_{2y} \end{bmatrix}, r_3 = \begin{bmatrix} r_{3x} \\ r_{3y} \end{bmatrix} r_4 = \begin{bmatrix} r_{4x} \\ r_{4y} \end{bmatrix}$$

be four vectors in $\mathbb{R}^2$ whose non-negative cone is all of $\mathbb{R}^2$. We are interested in determining a quadrilateral with vertices

$$\{v_1, v_2, v_3, v_4\}$$

such that vertex v_i lies on the open ray

$$\{x \in \mathbb{R}^2 : x = f + \lambda_i r_i \text{ for } \lambda_i > 0\}$$

and the line segments

$$[v_1, v_2], [v_2, v_3], [v_3, v_4] \text{ and } [v_4, v_1]$$

contain the points

$$p_1 = \begin{pmatrix} p_{1x} \\ p_{1y} \end{pmatrix}, \; p_2 = \begin{pmatrix} p_{2x} \\ p_{2y} \end{pmatrix}, \; p_3 = \begin{pmatrix} p_{3x} \\ p_{3y} \end{pmatrix}, \text{ and } p_4 = \begin{pmatrix} p_{4x} \\ p_{4y} \end{pmatrix}$$

respectively. To this end, we again let

$$v_i = \begin{bmatrix} f_x \\ f_y \end{bmatrix} + \lambda_i \begin{bmatrix} r_{ix} \\ r_{iy} \end{bmatrix}.$$

and similar to the triangle case, we obtain the following non-linear system of eight equations in eight unknowns.

$$\begin{aligned} (f + \lambda_1 r_1 - p_1)\mu_1 &= \lambda_1 r_1 - \lambda_2 r_2 \\ (f + \lambda_2 r_2 - p_2)\mu_2 &= \lambda_2 r_2 - \lambda_3 r_3 \\ (f + \lambda_3 r_3 - p_3)\mu_3 &= \lambda_3 r_3 - \lambda_4 r_4 \\ (f + \lambda_4 r_4 - p_4)\mu_4 &= \lambda_4 r_4 - \lambda_1 r_1 \end{aligned}$$

We compute a Gröbner basis in Mathematica using the command

```
G = GroebnerBasis[{
(fx + l1*r1x - p1x)*mu1 == l1*r1x - l2*r2x,
(fy + l1*r1y - p1y)*mu1 == l1*r1y - l2*r2y,
(fx + l2*r2x - p2x)*mu2 == l2*r2x - l3*r3x,
(fy + l2*r2y - p2y)*mu2 == l2*r2y - l3*r3y,
(fx + l3*r3x - p3x)*mu3 == l3*r3x - l4*r4x,
(fy + l3*r3y - p3y)*mu3 == l3*r3y - l4*r4y,
(fx + l4*r4x - p4x)*mu4 == l4*r4x - l1*r1x,
(fy + l4*r4y - p4y)*mu4 == l4*r4y - l1*r1y},
{mu1,mu2,mu3,mu4,l4,l3,l2,l1}]
```

resulting in a set $\{g'_1, g'_2, \ldots, g'_{93}\}$ of polynomials. The first polynomial in the Gröbner basis is:

$$
\begin{array}{clcl}
- & \lambda_1 p_{3y} r_{1y} r_{2y} r_{3y} r_{4x} {f_x}^3 & - & \lambda_1 p_{2y} r_{1y} r_{2y} r_{3x} r_{4y} {f_x}^3 \\
+ & \lambda_1 p_{3y} r_{1y} r_{2y} r_{3x} r_{4y} {f_x}^3 & + & \lambda_1 p_{2y} r_{1y} r_{2x} r_{3y} r_{4y} {f_x}^3 \\
+ & \lambda_1 p_{1y} r_{1x} r_{2y} r_{3y} r_{4y} {f_x}^3 & + & f_y \lambda_1 p_{2y} r_{1y} r_{2y} r_{3x} r_{4x} {f_x}^2 \\
- & \lambda_1 p_{2y} p_{3y} r_{1y} r_{2y} r_{3x} r_{4x} {f_x}^2 & + & \lambda_1 p_{3y} p_{4y} r_{1y} r_{2y} r_{3x} r_{4x} {f_x}^2 \\
+ & f_y \lambda_1 p_{1y} r_{1y} r_{2x} r_{3y} r_{4x} {f_x}^2 & + & f_y \lambda_1 p_{3y} r_{1y} r_{2x} r_{3y} r_{4x} {f_x}^2 \\
- & \lambda_1 p_{1y} p_{3y} r_{1y} r_{2x} r_{3y} r_{4x} {f_x}^2 & + & \lambda_1 p_{2y} p_{4y} r_{1y} r_{2x} r_{3y} r_{4x} {f_x}^2 \\
+ & {\lambda_1}^2 p_{3x} {r_{1y}}^2 r_{2y} r_{3y} r_{4x} {f_x}^2 & - & f_y \lambda_1 p_{1y} r_{1x} r_{2y} r_{3y} r_{4x} {f_x}^2 \\
+ & f_y \lambda_1 p_{3y} r_{1x} r_{2y} r_{3y} r_{4x} {f_x}^2 & - & \lambda_1 p_{3y} p_{4y} r_{1x} r_{2y} r_{3y} r_{4x} {f_x}^2 \\
+ & f_y \lambda_1 p_{3x} r_{1y} r_{2y} r_{3y} r_{4x} {f_x}^2 & + & \lambda_1 p_{2x} p_{3y} r_{1y} r_{2y} r_{3y} r_{4x} {f_x}^2 \\
- & f_y \lambda_1 p_{4x} r_{1y} r_{2y} r_{3y} r_{4x} {f_x}^2 & - & \lambda_1 p_{1x} p_{4y} r_{1y} r_{2y} r_{3y} r_{4x} {f_x}^2 \\
- & \lambda_1 p_{2x} p_{4y} r_{1y} r_{2y} r_{3y} r_{4x} {f_x}^2 & - & {\lambda_1}^2 p_{3y} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} {f_x}^2 \\
+ & {\lambda_1}^2 p_{4y} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} {f_x}^2 & - & \lambda_1 p_{1y} p_{2y} r_{1y} r_{2x} r_{3x} r_{4y} {f_x}^2 \\
- & f_y \lambda_1 p_{3y} r_{1y} r_{2x} r_{3x} r_{4y} {f_x}^2 & + & {\lambda_1}^2 p_{2x} {r_{1y}}^2 r_{2y} r_{3x} r_{4y} {f_x}^2 \\
- & {\lambda_1}^2 p_{3x} {r_{1y}}^2 r_{2y} r_{3x} r_{4y} {f_x}^2 & + & f_y \lambda_1 p_{2y} r_{1x} r_{2y} r_{3x} r_{4y} {f_x}^2 \\
- & f_y \lambda_1 p_{3y} r_{1x} r_{2y} r_{3x} r_{4y} {f_x}^2 & + & f_y \lambda_1 p_{4y} r_{1x} r_{2y} r_{3x} r_{4y} {f_x}^2 \\
- & \lambda_1 p_{2y} p_{4y} r_{1x} r_{2y} r_{3x} r_{4y} {f_x}^2 & + & \lambda_1 p_{1x} p_{2y} r_{1y} r_{2y} r_{3x} r_{4y} {f_x}^2 \\
- & f_y \lambda_1 p_{3x} r_{1y} r_{2y} r_{3x} r_{4y} {f_x}^2 & - & \lambda_1 p_{1x} p_{3y} r_{1y} r_{2y} r_{3x} r_{4y} {f_x}^2 \\
- & \lambda_1 p_{2x} p_{3y} r_{1y} r_{2y} r_{3x} r_{4y} {f_x}^2 & - & \lambda_1 p_{3y} p_{4x} r_{1y} r_{2y} r_{3x} r_{4y} {f_x}^2 \\
- & {\lambda_1}^2 p_{2y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} {f_x}^2 & + & {\lambda_1}^2 p_{1x} {r_{1y}}^2 r_{2x} r_{3y} r_{4y} {f_x}^2 \\
- & {\lambda_1}^2 p_{2x} {r_{1y}}^2 r_{2x} r_{3y} r_{4y} {f_x}^2 & + & \lambda_1 p_{1y} p_{2y} r_{1x} r_{2x} r_{3y} r_{4y} {f_x}^2 \\
+ & f_y \lambda_1 p_{4y} r_{1x} r_{2x} r_{3y} r_{4y} {f_x}^2 & + & f_y \lambda_1 p_{1x} r_{1y} r_{2x} r_{3y} r_{4y} {f_x}^2 \\
- & f_y \lambda_1 p_{2x} r_{1y} r_{2x} r_{3y} r_{4y} {f_x}^2 & - & \lambda_1 p_{1x} p_{2y} r_{1y} r_{2x} r_{3y} r_{4y} {f_x}^2 \\
+ & \lambda_1 p_{1y} p_{3x} r_{1y} r_{2x} r_{3y} r_{4y} {f_x}^2 & + & \lambda_1 p_{1y} p_{4x} r_{1y} r_{2x} r_{3y} r_{4y} {f_x}^2 \\
- & \lambda_1 p_{2y} p_{4x} r_{1y} r_{2x} r_{3y} r_{4y} {f_x}^2 & + & {\lambda_1}^2 p_{2y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} {f_x}^2 \\
+ & {\lambda_1}^2 p_{1y} {r_{1x}}^2 r_{2y} r_{3y} r_{4y} {f_x}^2 & - & f_y \lambda_1 p_{1x} r_{1x} r_{2y} r_{3y} r_{4y} {f_x}^2
\end{array}
$$

$$
\begin{aligned}
&- \lambda_1 p_{1y} p_{2x} r_{1x} r_{2y} r_{3y} r_{4y} {f_x}^2 && + f_y \lambda_1 p_{4x} r_{1x} r_{2y} r_{3y} r_{4y} {f_x}^2 \\
&- \lambda_1 p_{1y} p_{4x} r_{1x} r_{2y} r_{3y} r_{4y} {f_x}^2 && + \lambda_1 p_{2x} p_{4y} r_{1x} r_{2y} r_{3y} r_{4y} {f_x}^2 \\
&+ \lambda_1 p_{3x} p_{4y} r_{1x} r_{2y} r_{3y} r_{4y} {f_x}^2 && + {\lambda_1}^2 p_{4x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4y} {f_x}^2 \\
&- {f_y}^2 \lambda_1 p_{1y} r_{1y} r_{2x} r_{3x} r_{4x} f_x && + f_y \lambda_1 p_{1y} p_{3y} r_{1y} r_{2x} r_{3x} r_{4x} f_x \\
&- \lambda_1 p_{1y} p_{2y} p_{3y} r_{1y} r_{2x} r_{3x} r_{4x} f_x && - f_y \lambda_1 p_{2y} p_{4y} r_{1y} r_{2x} r_{3x} r_{4x} f_x \\
&- f_y \lambda_1 p_{3y} p_{4y} r_{1y} r_{2x} r_{3x} r_{4x} f_x && - f_y {\lambda_1}^2 p_{2x} {r_{1y}}^2 r_{2y} r_{3x} r_{4x} f_x \\
&+ {\lambda_1}^2 p_{2x} p_{3y} {r_{1y}}^2 r_{2y} r_{3x} r_{4x} f_x && - {\lambda_1}^2 p_{3y} p_{4x} {r_{1y}}^2 r_{2y} r_{3x} r_{4x} f_x \\
&+ {f_y}^2 \lambda_1 p_{1y} r_{1x} r_{2y} r_{3x} r_{4x} f_x && - f_y \lambda_1 p_{1y} p_{3y} r_{1x} r_{2y} r_{3x} r_{4x} f_x \\
&+ f_y \lambda_1 p_{2y} p_{3y} r_{1x} r_{2y} r_{3x} r_{4x} f_x && + f_y \lambda_1 p_{2y} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} f_x \\
&+ \lambda_1 p_{1y} p_{3y} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} f_x && - {f_y}^2 \lambda_1 p_{2x} r_{1y} r_{2y} r_{3x} r_{4x} f_x \\
&- f_y \lambda_1 p_{1x} p_{2y} r_{1y} r_{2y} r_{3x} r_{4x} f_x && + \lambda_1 p_{1x} p_{2y} p_{3y} r_{1y} r_{2y} r_{3x} r_{4x} f_x \\
&+ {f_y}^2 \lambda_1 p_{4x} r_{1y} r_{2y} r_{3x} r_{4x} f_x && - f_y \lambda_1 p_{3y} p_{4x} r_{1y} r_{2y} r_{3x} r_{4x} f_x \\
&+ \lambda_1 p_{2y} p_{3y} p_{4x} r_{1y} r_{2y} r_{3x} r_{4x} f_x && + f_y \lambda_1 p_{2x} p_{4y} r_{1y} r_{2y} r_{3x} r_{4x} f_x \\
&- \lambda_1 p_{1x} p_{3y} p_{4y} r_{1y} r_{2y} r_{3x} r_{4x} f_x && + f_y {\lambda_1}^2 p_{2y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} f_x \\
&- {\lambda_1}^2 p_{2y} p_{3y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} f_x && + {\lambda_1}^2 p_{3y} p_{4y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} f_x \\
&- f_y {\lambda_1}^2 p_{1x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} f_x && - f_y {\lambda_1}^2 p_{3x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} f_x \\
&+ {\lambda_1}^2 p_{2y} p_{3x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} f_x && - {\lambda_1}^2 p_{2x} p_{3y} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} f_x \\
&+ f_y {\lambda_1}^2 p_{4x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} f_x && + {f_y}^2 \lambda_1 p_{2y} r_{1x} r_{2x} r_{3y} r_{4x} f_x \\
&- f_y \lambda_1 p_{1y} p_{2y} r_{1x} r_{2x} r_{3y} r_{4x} f_x && + f_y \lambda_1 p_{1y} p_{3y} r_{1x} r_{2x} r_{3y} r_{4x} f_x \\
&- f_y \lambda_1 p_{2y} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} f_x && + f_y \lambda_1 p_{3y} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} f_x \\
&- \lambda_1 p_{1y} p_{3y} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} f_x && + {f_y}^2 \lambda_1 p_{2x} r_{1y} r_{2x} r_{3y} r_{4x} f_x \\
&- f_y \lambda_1 p_{1y} p_{2x} r_{1y} r_{2x} r_{3y} r_{4x} f_x && - {f_y}^2 \lambda_1 p_{3x} r_{1y} r_{2x} r_{3y} r_{4x} f_x \\
&+ f_y \lambda_1 p_{2y} p_{3x} r_{1y} r_{2x} r_{3y} r_{4x} f_x && + \lambda_1 p_{1y} p_{2x} p_{3y} r_{1y} r_{2x} r_{3y} r_{4x} f_x \\
&+ {f_y}^2 \lambda_1 p_{4x} r_{1y} r_{2x} r_{3y} r_{4x} f_x && - f_y \lambda_1 p_{3y} p_{4x} r_{1y} r_{2x} r_{3y} r_{4x} f_x \\
&+ \lambda_1 p_{1y} p_{3y} p_{4x} r_{1y} r_{2x} r_{3y} r_{4x} f_x && - \lambda_1 p_{1x} p_{2y} p_{4y} r_{1y} r_{2x} r_{3y} r_{4x} f_x \\
&+ f_y \lambda_1 p_{3x} p_{4y} r_{1y} r_{2x} r_{3y} r_{4x} f_x && + f_y {\lambda_1}^2 p_{1y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} f_x \\
&- f_y {\lambda_1}^2 p_{2y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} f_x && - {\lambda_1}^2 p_{1y} p_{3y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} f_x \\
&- f_y {\lambda_1}^2 p_{4y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} f_x && - f_y {\lambda_1}^2 p_{1y} {r_{1x}}^2 r_{2y} r_{3y} r_{4x} f_x
\end{aligned}
$$

$$
\begin{array}{llll}
+ & f_y{\lambda_1}^2p_{3y}{r_{1x}}^2r_{2y}r_{3y}r_{4x}f_x & - & {\lambda_1}^2p_{3y}p_{4y}{r_{1x}}^2r_{2y}r_{3y}r_{4x}f_x \\
- & {\lambda_1}^2p_{1x}p_{3x}{r_{1y}}^2r_{2y}r_{3y}r_{4x}f_x & + & {\lambda_1}^2p_{1x}p_{4x}{r_{1y}}^2r_{2y}r_{3y}r_{4x}f_x \\
+ & {\lambda_1}^2p_{2x}p_{4x}{r_{1y}}^2r_{2y}r_{3y}r_{4x}f_x & + & f_y\lambda_1p_{1y}p_{2x}r_{1x}r_{2y}r_{3y}r_{4x}f_x \\
- & {f_y}^2\lambda_1p_{3x}r_{1x}r_{2y}r_{3y}r_{4x}f_x & - & f_y\lambda_1p_{1x}p_{3y}r_{1x}r_{2y}r_{3y}r_{4x}f_x \\
- & f_y\lambda_1p_{2x}p_{3y}r_{1x}r_{2y}r_{3y}r_{4x}f_x & - & \lambda_1p_{1y}p_{2x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4x}f_x \\
+ & f_y\lambda_1p_{3x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4x}f_x & + & \lambda_1p_{1x}p_{3y}p_{4y}r_{1x}r_{2y}r_{3y}r_{4x}f_x \\
+ & \lambda_1p_{2x}p_{3y}p_{4y}r_{1x}r_{2y}r_{3y}r_{4x}f_x & - & f_y\lambda_1p_{2x}p_{3x}r_{1y}r_{2y}r_{3y}r_{4x}f_x \\
- & \lambda_1p_{1x}p_{2x}p_{3y}r_{1y}r_{2y}r_{3y}r_{4x}f_x & + & f_y\lambda_1p_{2x}p_{4x}r_{1y}r_{2y}r_{3y}r_{4x}f_x \\
- & \lambda_1p_{1x}p_{3y}p_{4x}r_{1y}r_{2y}r_{3y}r_{4x}f_x & + & \lambda_1p_{1x}p_{2x}p_{4y}r_{1y}r_{2y}r_{3y}r_{4x}f_x \\
+ & \lambda_1p_{1x}p_{3x}p_{4y}r_{1y}r_{2y}r_{3y}r_{4x}f_x & + & f_y{\lambda_1}^2p_{1x}r_{1x}r_{1y}r_{2y}r_{3y}r_{4x}f_x \\
- & f_y{\lambda_1}^2p_{3x}r_{1x}r_{1y}r_{2y}r_{3y}r_{4x}f_x & + & {\lambda_1}^2p_{2x}p_{3y}r_{1x}r_{1y}r_{2y}r_{3y}r_{4x}f_x \\
- & {\lambda_1}^2p_{1y}p_{4x}r_{1x}r_{1y}r_{2y}r_{3y}r_{4x}f_x & - & {\lambda_1}^2p_{1x}p_{4y}r_{1x}r_{1y}r_{2y}r_{3y}r_{4x}f_x \\
- & {\lambda_1}^2p_{2x}p_{4y}r_{1x}r_{1y}r_{2y}r_{3y}r_{4x}f_x & + & {\lambda_1}^2p_{1x}p_{2y}{r_{1y}}^2r_{2x}r_{3x}r_{4y}f_x \\
+ & f_y{\lambda_1}^2p_{3x}{r_{1y}}^2r_{2x}r_{3x}r_{4y}f_x & + & {f_y}^2\lambda_1p_{3y}r_{1x}r_{2x}r_{3x}r_{4y}f_x \\
- & f_y\lambda_1p_{1y}p_{3y}r_{1x}r_{2x}r_{3x}r_{4y}f_x & + & \lambda_1p_{1y}p_{2y}p_{3y}r_{1x}r_{2x}r_{3x}r_{4y}f_x \\
- & {f_y}^2\lambda_1p_{4y}r_{1x}r_{2x}r_{3x}r_{4y}f_x & + & f_y\lambda_1p_{2y}p_{4y}r_{1x}r_{2x}r_{3x}r_{4y}f_x \\
- & \lambda_1p_{1y}p_{2y}p_{4y}r_{1x}r_{2x}r_{3x}r_{4y}f_x & + & f_y\lambda_1p_{1x}p_{2y}r_{1y}r_{2x}r_{3x}r_{4y}f_x \\
+ & {f_y}^2\lambda_1p_{3x}r_{1y}r_{2x}r_{3x}r_{4y}f_x & - & f_y\lambda_1p_{2y}p_{3x}r_{1y}r_{2x}r_{3x}r_{4y}f_x \\
+ & \lambda_1p_{1y}p_{2y}p_{3x}r_{1y}r_{2x}r_{3x}r_{4y}f_x & - & \lambda_1p_{1x}p_{2y}p_{3y}r_{1y}r_{2x}r_{3x}r_{4y}f_x \\
- & f_y\lambda_1p_{1y}p_{4x}r_{1y}r_{2x}r_{3x}r_{4y}f_x & + & f_y\lambda_1p_{3y}p_{4x}r_{1y}r_{2x}r_{3x}r_{4y}f_x \\
- & \lambda_1p_{2y}p_{3y}p_{4x}r_{1y}r_{2x}r_{3x}r_{4y}f_x & - & {\lambda_1}^2p_{1y}p_{2y}r_{1x}r_{1y}r_{2x}r_{3x}r_{4y}f_x \\
- & f_y{\lambda_1}^2p_{3y}r_{1x}r_{1y}r_{2x}r_{3x}r_{4y}f_x & - & f_y{\lambda_1}^2p_{1y}{r_{1x}}^2r_{2y}r_{3x}r_{4y}f_x \\
+ & f_y{\lambda_1}^2p_{2y}{r_{1x}}^2r_{2y}r_{3x}r_{4y}f_x & + & {\lambda_1}^2p_{1y}p_{3y}{r_{1x}}^2r_{2y}r_{3x}r_{4y}f_x \\
+ & f_y{\lambda_1}^2p_{4y}{r_{1x}}^2r_{2y}r_{3x}r_{4y}f_x & - & {\lambda_1}^2p_{1x}p_{2x}{r_{1y}}^2r_{2y}r_{3x}r_{4y}f_x \\
+ & {\lambda_1}^2p_{1x}p_{3x}{r_{1y}}^2r_{2y}r_{3x}r_{4y}f_x & + & {\lambda_1}^2p_{3x}p_{4x}{r_{1y}}^2r_{2y}r_{3x}r_{4y}f_x \\
+ & {f_y}^2\lambda_1p_{1x}r_{1x}r_{2y}r_{3x}r_{4y}f_x & + & f_y\lambda_1p_{1y}p_{2x}r_{1x}r_{2y}r_{3x}r_{4y}f_x \\
- & f_y\lambda_1p_{1x}p_{2y}r_{1x}r_{2y}r_{3x}r_{4y}f_x & - & f_y\lambda_1p_{2y}p_{3x}r_{1x}r_{2y}r_{3x}r_{4y}f_x \\
+ & f_y\lambda_1p_{2x}p_{3y}r_{1x}r_{2y}r_{3x}r_{4y}f_x & - & {f_y}^2\lambda_1p_{4x}r_{1x}r_{2y}r_{3x}r_{4y}f_x
\end{array}
$$

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\begin{array}{llll}
+ & f_y\lambda_1 p_{1y}p_{4x}r_{1x}r_{2y}r_{3x}r_{4y}f_x & - & \lambda_1 p_{1y}p_{3y}p_{4x}r_{1x}r_{2y}r_{3x}r_{4y}f_x \\
- & f_y\lambda_1 p_{1x}p_{4y}r_{1x}r_{2y}r_{3x}r_{4y}f_x & - & f_y\lambda_1 p_{3x}p_{4y}r_{1x}r_{2y}r_{3x}r_{4y}f_x \\
+ & \lambda_1 p_{2y}p_{3x}p_{4y}r_{1x}r_{2y}r_{3x}r_{4y}f_x & + & f_y\lambda_1 p_{1x}p_{3x}r_{1y}r_{2y}r_{3x}r_{4y}f_x \\
- & \lambda_1 p_{1x}p_{2y}p_{3x}r_{1y}r_{2y}r_{3x}r_{4y}f_x & - & f_y\lambda_1 p_{2x}p_{4x}r_{1y}r_{2y}r_{3x}r_{4y}f_x \\
- & \lambda_1 p_{1x}p_{2y}p_{4x}r_{1y}r_{2y}r_{3x}r_{4y}f_x & - & \lambda_1 p_{2y}p_{3x}p_{4x}r_{1y}r_{2y}r_{3x}r_{4y}f_x \\
+ & \lambda_1 p_{1x}p_{3y}p_{4x}r_{1y}r_{2y}r_{3x}r_{4y}f_x & + & f_y{\lambda_1}^2 p_{1x}r_{1x}r_{1y}r_{2y}r_{3x}r_{4y}f_x \\
- & f_y{\lambda_1}^2 p_{2x}r_{1x}r_{1y}r_{2y}r_{3x}r_{4y}f_x & + & f_y{\lambda_1}^2 p_{3x}r_{1x}r_{1y}r_{2y}r_{3x}r_{4y}f_x \\
- & {\lambda_1}^2 p_{1y}p_{3x}r_{1x}r_{1y}r_{2y}r_{3x}r_{4y}f_x & - & {\lambda_1}^2 p_{1x}p_{3y}r_{1x}r_{1y}r_{2y}r_{3x}r_{4y}f_x \\
- & {\lambda_1}^2 p_{2x}p_{3y}r_{1x}r_{1y}r_{2y}r_{3x}r_{4y}f_x & + & {\lambda_1}^2 p_{2y}p_{4x}r_{1x}r_{1y}r_{2y}r_{3x}r_{4y}f_x \\
+ & {\lambda_1}^2 p_{2x}p_{4y}r_{1x}r_{1y}r_{2y}r_{3x}r_{4y}f_x & - & f_y{\lambda_1}^2 p_{2y}{r_{1x}}^2 r_{2x}r_{3y}r_{4y}f_x \\
+ & {\lambda_1}^2 p_{1y}p_{2y}{r_{1x}}^2 r_{2x}r_{3y}r_{4y}f_x & - & {\lambda_1}^2 p_{1y}p_{4y}{r_{1x}}^2 r_{2x}r_{3y}r_{4y}f_x \\
- & {\lambda_1}^2 p_{1x}p_{3x}{r_{1y}}^2 r_{2x}r_{3y}r_{4y}f_x & - & {\lambda_1}^2 p_{1x}p_{4x}{r_{1y}}^2 r_{2x}r_{3y}r_{4y}f_x \\
+ & {\lambda_1}^2 p_{2x}p_{4x}{r_{1y}}^2 r_{2x}r_{3y}r_{4y}f_x & - & f_y\lambda_1 p_{1y}p_{2x}r_{1x}r_{2x}r_{3y}r_{4y}f_x \\
+ & f_y\lambda_1 p_{2y}p_{3x}r_{1x}r_{2x}r_{3y}r_{4y}f_x & - & {f_y}^2\lambda_1 p_{4x}r_{1x}r_{2x}r_{3y}r_{4y}f_x \\
+ & f_y\lambda_1 p_{1y}p_{4x}r_{1x}r_{2x}r_{3y}r_{4y}f_x & - & \lambda_1 p_{1y}p_{2y}p_{4x}r_{1x}r_{2x}r_{3y}r_{4y}f_x \\
- & f_y\lambda_1 p_{2x}p_{4y}r_{1x}r_{2x}r_{3y}r_{4y}f_x & - & f_y\lambda_1 p_{3x}p_{4y}r_{1x}r_{2x}r_{3y}r_{4y}f_x \\
+ & \lambda_1 p_{1y}p_{3x}p_{4y}r_{1x}r_{2x}r_{3y}r_{4y}f_x & + & f_y\lambda_1 p_{2x}p_{3x}r_{1y}r_{2x}r_{3y}r_{4y}f_x \\
- & \lambda_1 p_{1y}p_{2x}p_{3x}r_{1y}r_{2x}r_{3y}r_{4y}f_x & - & f_y\lambda_1 p_{1x}p_{4x}r_{1y}r_{2x}r_{3y}r_{4y}f_x \\
+ & f_y\lambda_1 p_{2x}p_{4x}r_{1y}r_{2x}r_{3y}r_{4y}f_x & + & \lambda_1 p_{1x}p_{2y}p_{4x}r_{1y}r_{2x}r_{3y}r_{4y}f_x \\
- & \lambda_1 p_{1y}p_{3x}p_{4x}r_{1y}r_{2x}r_{3y}r_{4y}f_x & + & f_y{\lambda_1}^2 p_{2x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4y}f_x \\
- & {\lambda_1}^2 p_{1x}p_{2y}r_{1x}r_{1y}r_{2x}r_{3y}r_{4y}f_x & - & {\lambda_1}^2 p_{2y}p_{3x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4y}f_x \\
- & f_y{\lambda_1}^2 p_{4x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4y}f_x & + & {\lambda_1}^2 p_{1x}p_{4y}r_{1x}r_{1y}r_{2x}r_{3y}r_{4y}f_x \\
- & {\lambda_1}^2 p_{2x}p_{4y}r_{1x}r_{1y}r_{2x}r_{3y}r_{4y}f_x & - & {\lambda_1}^2 p_{1y}p_{3x}{r_{1x}}^2 r_{2y}r_{3y}r_{4y}f_x \\
+ & {\lambda_1}^2 p_{2x}p_{4y}{r_{1x}}^2 r_{2y}r_{3y}r_{4y}f_x & + & f_y\lambda_1 p_{1x}p_{2x}r_{1x}r_{2y}r_{3y}r_{4y}f_x \\
+ & f_y\lambda_1 p_{1x}p_{3x}r_{1x}r_{2y}r_{3y}r_{4y}f_x & - & f_y\lambda_1 p_{2x}p_{4x}r_{1x}r_{2y}r_{3y}r_{4y}f_x \\
+ & \lambda_1 p_{1y}p_{2x}p_{4x}r_{1x}r_{2y}r_{3y}r_{4y}f_x & + & \lambda_1 p_{1y}p_{3x}p_{4x}r_{1x}r_{2y}r_{3y}r_{4y}f_x \\
- & \lambda_1 p_{1x}p_{2x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4y}f_x & - & \lambda_1 p_{2x}p_{3x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4y}f_x \\
+ & {\lambda_1}^2 p_{1x}p_{2x}r_{1x}r_{1y}r_{2y}r_{3y}r_{4y}f_x & - & {\lambda_1}^2 p_{2x}p_{4x}r_{1x}r_{1y}r_{2y}r_{3y}r_{4y}f_x
\end{array}
$$

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\begin{array}{llll}
- & \lambda_1{}^2 p_{3x} p_{4x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4y} f_x & - & f_y \lambda_1{}^2 p_{1x} p_{2y} r_{1y}{}^2 r_{2x} r_{3x} r_{4x} \\
- & f_y \lambda_1{}^2 p_{1x} p_{3y} r_{1y}{}^2 r_{2x} r_{3x} r_{4x} & - & f_y{}^2 \lambda_1{}^2 p_{4x} r_{1y}{}^2 r_{2x} r_{3x} r_{4x} \\
+ & f_y \lambda_1{}^2 p_{2y} p_{4x} r_{1y}{}^2 r_{2x} r_{3x} r_{4x} & - & \lambda_1{}^2 p_{2y} p_{3y} p_{4x} r_{1y}{}^2 r_{2x} r_{3x} r_{4x} \\
+ & f_y{}^3 \lambda_1 p_{1x} r_{1y} r_{2x} r_{3x} r_{4x} & - & f_y{}^2 \lambda_1 p_{1x} p_{3y} r_{1y} r_{2x} r_{3x} r_{4x} \\
+ & f_y \lambda_1 p_{1x} p_{2y} p_{3y} r_{1y} r_{2x} r_{3x} r_{4x} & + & f_y{}^2 \lambda_1 p_{1y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4x} \\
+ & f_y{}^2 \lambda_1 p_{2y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4x} & + & f_y{}^2 \lambda_1 p_{3y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4x} \\
- & f_y \lambda_1 p_{1y} p_{3y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4x} & + & \lambda_1 p_{1y} p_{2y} p_{3y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4x} \\
- & f_y{}^2 \lambda_1 p_{1x} p_{4y} r_{1y} r_{2x} r_{3x} r_{4x} & + & f_y \lambda_1 p_{1x} p_{3y} p_{4y} r_{1y} r_{2x} r_{3x} r_{4x} \\
- & \lambda_1 p_{1x} p_{2y} p_{3y} p_{4y} r_{1y} r_{2x} r_{3x} r_{4x} & + & f_y \lambda_1{}^2 p_{1y} p_{2y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4x} \\
+ & f_y \lambda_1{}^2 p_{1y} p_{3y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4x} & + & f_y{}^2 \lambda_1{}^2 p_{4y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4x} \\
- & f_y \lambda_1{}^2 p_{2y} p_{4y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4x} & + & \lambda_1{}^2 p_{2y} p_{3y} p_{4y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4x} \\
+ & f_y{}^2 \lambda_1{}^2 p_{1y} r_{1x}{}^2 r_{2y} r_{3x} r_{4x} & - & f_y \lambda_1{}^2 p_{1y} p_{3y} r_{1x}{}^2 r_{2y} r_{3x} r_{4x} \\
+ & f_y \lambda_1{}^2 p_{2y} p_{3y} r_{1x}{}^2 r_{2y} r_{3x} r_{4x} & + & f_y \lambda_1{}^2 p_{2y} p_{4y} r_{1x}{}^2 r_{2y} r_{3x} r_{4x} \\
+ & \lambda_1{}^2 p_{1y} p_{3y} p_{4y} r_{1x}{}^2 r_{2y} r_{3x} r_{4x} & + & f_y \lambda_1{}^2 p_{1x} p_{2x} r_{1y}{}^2 r_{2y} r_{3x} r_{4x} \\
- & \lambda_1{}^2 p_{1x} p_{2x} p_{3y} r_{1y}{}^2 r_{2y} r_{3x} r_{4x} & + & \lambda_1{}^2 p_{1x} p_{3y} p_{4x} r_{1y}{}^2 r_{2y} r_{3x} r_{4x} \\
- & f_y{}^3 \lambda_1 p_{1x} r_{1x} r_{2y} r_{3x} r_{4x} & - & f_y{}^2 \lambda_1 p_{1y} p_{2x} r_{1x} r_{2y} r_{3x} r_{4x} \\
+ & f_y{}^2 \lambda_1 p_{1x} p_{2y} r_{1x} r_{2y} r_{3x} r_{4x} & - & f_y{}^2 \lambda_1 p_{2x} p_{3y} r_{1x} r_{2y} r_{3x} r_{4x} \\
+ & f_y \lambda_1 p_{1y} p_{2x} p_{3y} r_{1x} r_{2y} r_{3x} r_{4x} & + & f_y{}^2 \lambda_1 p_{1x} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} \\
- & f_y{}^2 \lambda_1 p_{2x} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} & - & f_y \lambda_1 p_{1x} p_{2y} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} \\
- & f_y \lambda_1 p_{1x} p_{3y} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} & - & \lambda_1 p_{1y} p_{2x} p_{3y} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} \\
+ & \lambda_1 p_{1x} p_{2y} p_{3y} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} & - & f_y \lambda_1 p_{1x} p_{2x} p_{3y} r_{1y} r_{2y} r_{3x} r_{4x} \\
- & f_y{}^2 \lambda_1 p_{1x} p_{4x} r_{1y} r_{2y} r_{3x} r_{4x} & + & f_y \lambda_1 p_{1x} p_{3y} p_{4x} r_{1y} r_{2y} r_{3x} r_{4x} \\
- & \lambda_1 p_{1x} p_{2y} p_{3y} p_{4x} r_{1y} r_{2y} r_{3x} r_{4x} & + & \lambda_1 p_{1x} p_{2x} p_{3y} p_{4y} r_{1y} r_{2y} r_{3x} r_{4x} \\
- & f_y{}^2 \lambda_1{}^2 p_{1x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} & - & f_y \lambda_1{}^2 p_{1y} p_{2x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} \\
+ & f_y \lambda_1{}^2 p_{1x} p_{3y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} & + & \lambda_1{}^2 p_{1y} p_{2x} p_{3y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} \\
+ & f_y \lambda_1{}^2 p_{1y} p_{4x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} & - & \lambda_1{}^2 p_{1y} p_{3y} p_{4x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} \\
+ & \lambda_1{}^2 p_{2y} p_{3y} p_{4x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} & - & \lambda_1{}^2 p_{1x} p_{3y} p_{4y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} \\
+ & f_y{}^2 \lambda_1{}^2 p_{2y} r_{1x}{}^2 r_{2x} r_{3y} r_{4x} & - & f_y{}^2 \lambda_1{}^2 p_{3y} r_{1x}{}^2 r_{2x} r_{3y} r_{4x}
\end{array}
$$

$$
\begin{array}{llll}
+ & f_y{\lambda_1}^2 p_{1y} p_{3y} {r_{1x}}^2 r_{2x} r_{3y} r_{4x} & + & {\lambda_1}^2 p_{1y} p_{2y} p_{4y} {r_{1x}}^2 r_{2x} r_{3y} r_{4x} \\
+ & f_y{\lambda_1}^2 p_{3y} p_{4y} {r_{1x}}^2 r_{2x} r_{3y} r_{4x} & + & f_y{\lambda_1}^2 p_{1x} p_{3x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} \\
- & {\lambda_1}^2 p_{1x} p_{2y} p_{3x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} & + & {\lambda_1}^2 p_{1x} p_{2y} p_{4x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} \\
- & {\lambda_1}^2 p_{1x} p_{3y} p_{4x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} & - & {f_y}^3 \lambda_1 p_{2x} r_{1x} r_{2x} r_{3y} r_{4x} \\
+ & {f_y}^2 \lambda_1 p_{1y} p_{2x} r_{1x} r_{2x} r_{3y} r_{4x} & - & {f_y}^2 \lambda_1 p_{1y} p_{3x} r_{1x} r_{2x} r_{3y} r_{4x} \\
- & {f_y}^2 \lambda_1 p_{2y} p_{3x} r_{1x} r_{2x} r_{3y} r_{4x} & + & {f_y}^2 \lambda_1 p_{2x} p_{3y} r_{1x} r_{2x} r_{3y} r_{4x} \\
- & f_y \lambda_1 p_{1y} p_{2x} p_{3y} r_{1x} r_{2x} r_{3y} r_{4x} & - & f_y \lambda_1 p_{1y} p_{2x} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} \\
- & {f_y}^2 \lambda_1 p_{3x} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} & + & f_y \lambda_1 p_{2y} p_{3x} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} \\
- & \lambda_1 p_{1y} p_{2y} p_{3x} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} & + & \lambda_1 p_{1y} p_{2x} p_{3y} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} \\
+ & {f_y}^2 \lambda_1 p_{1x} p_{3x} r_{1y} r_{2x} r_{3y} r_{4x} & - & {f_y}^2 \lambda_1 p_{2x} p_{4x} r_{1y} r_{2x} r_{3y} r_{4x} \\
+ & f_y \lambda_1 p_{1y} p_{2x} p_{4x} r_{1y} r_{2x} r_{3y} r_{4x} & - & \lambda_1 p_{1y} p_{2x} p_{3y} p_{4x} r_{1y} r_{2x} r_{3y} r_{4x} \\
- & f_y \lambda_1 p_{1x} p_{3x} p_{4y} r_{1y} r_{2x} r_{3y} r_{4x} & - & {f_y}^2 {\lambda_1}^2 p_{2x} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} \\
+ & f_y {\lambda_1}^2 p_{1x} p_{2y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} & - & f_y {\lambda_1}^2 p_{1y} p_{3x} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} \\
- & f_y {\lambda_1}^2 p_{2y} p_{3x} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} & - & f_y {\lambda_1}^2 p_{1x} p_{3y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} \\
+ & f_y {\lambda_1}^2 p_{2x} p_{3y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} & - & {\lambda_1}^2 p_{1y} p_{2y} p_{4x} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} \\
- & f_y {\lambda_1}^2 p_{3y} p_{4x} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} & + & f_y {\lambda_1}^2 p_{2x} p_{4y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} \\
- & {\lambda_1}^2 p_{1x} p_{2y} p_{4y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} & - & {\lambda_1}^2 p_{2x} p_{3y} p_{4y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} \\
+ & f_y {\lambda_1}^2 p_{1y} p_{2x} {r_{1x}}^2 r_{2y} r_{3y} r_{4x} & - & {\lambda_1}^2 p_{1y} p_{2x} p_{4y} {r_{1x}}^2 r_{2y} r_{3y} r_{4x} \\
+ & {\lambda_1}^2 p_{2x} p_{3y} p_{4y} {r_{1x}}^2 r_{2y} r_{3y} r_{4x} & - & {\lambda_1}^2 p_{1x} p_{2x} p_{4x} {r_{1y}}^2 r_{2y} r_{3y} r_{4x} \\
- & {f_y}^2 \lambda_1 p_{1x} p_{2x} r_{1x} r_{2y} r_{3y} r_{4x} & - & f_y \lambda_1 p_{1y} p_{2x} p_{3x} r_{1x} r_{2y} r_{3y} r_{4x} \\
+ & f_y \lambda_1 p_{1x} p_{2x} p_{3y} r_{1x} r_{2y} r_{3y} r_{4x} & - & f_y \lambda_1 p_{2x} p_{3x} p_{4y} r_{1x} r_{2y} r_{3y} r_{4x} \\
+ & \lambda_1 p_{1y} p_{2x} p_{3x} p_{4y} r_{1x} r_{2y} r_{3y} r_{4x} & + & f_y \lambda_1 p_{1x} p_{2x} p_{3x} r_{1y} r_{2y} r_{3y} r_{4x} \\
- & f_y \lambda_1 p_{1x} p_{2x} p_{4x} r_{1y} r_{2y} r_{3y} r_{4x} & - & \lambda_1 p_{1x} p_{2x} p_{3x} p_{4y} r_{1y} r_{2y} r_{3y} r_{4x} \\
- & f_y {\lambda_1}^2 p_{1x} p_{2x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} & - & {\lambda_1}^2 p_{1y} p_{2x} p_{3x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} \\
+ & {\lambda_1}^2 p_{1y} p_{2x} p_{4x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} & + & {\lambda_1}^2 p_{1x} p_{2x} p_{4y} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} \\
+ & {f_y}^2 {\lambda_1}^2 p_{3y} {r_{1x}}^2 r_{2x} r_{3x} r_{4y} & - & f_y {\lambda_1}^2 p_{2y} p_{3y} {r_{1x}}^2 r_{2x} r_{3x} r_{4y} \\
+ & {\lambda_1}^2 p_{1y} p_{2y} p_{3y} {r_{1x}}^2 r_{2x} r_{3x} r_{4y} & + & f_y {\lambda_1}^2 p_{1y} p_{4y} {r_{1x}}^2 r_{2x} r_{3x} r_{4y} \\
+ & f_y {\lambda_1}^2 p_{2y} p_{4y} {r_{1x}}^2 r_{2x} r_{3x} r_{4y} & + & f_y {\lambda_1}^2 p_{1x} p_{4x} {r_{1y}}^2 r_{2x} r_{3x} r_{4y}
\end{array}
$$

$$- {\lambda_1}^2 p_{1x} p_{2y} p_{4x} {r_{1y}}^2 r_{2x} r_{3x} r_{4y} + {\lambda_1}^2 p_{2y} p_{3x} p_{4x} {r_{1y}}^2 r_{2x} r_{3x} r_{4y}$$

$$- {f_y}^3 \lambda_1 p_{3x} r_{1x} r_{2x} r_{3x} r_{4y} + {f_y}^2 \lambda_1 p_{2y} p_{3x} r_{1x} r_{2x} r_{3x} r_{4y}$$

$$- f_y \lambda_1 p_{1y} p_{2y} p_{3x} r_{1x} r_{2x} r_{3x} r_{4y} - {f_y}^2 \lambda_1 p_{1y} p_{4x} r_{1x} r_{2x} r_{3x} r_{4y}$$

$$- {f_y}^2 \lambda_1 p_{2y} p_{4x} r_{1x} r_{2x} r_{3x} r_{4y} - {f_y}^2 \lambda_1 p_{3y} p_{4x} r_{1x} r_{2x} r_{3x} r_{4y}$$

$$+ f_y \lambda_1 p_{1y} p_{3y} p_{4x} r_{1x} r_{2x} r_{3x} r_{4y} - \lambda_1 p_{1y} p_{2y} p_{3y} p_{4x} r_{1x} r_{2x} r_{3x} r_{4y}$$

$$+ {f_y}^2 \lambda_1 p_{3x} p_{4y} r_{1x} r_{2x} r_{3x} r_{4y} - f_y \lambda_1 p_{2y} p_{3x} p_{4y} r_{1x} r_{2x} r_{3x} r_{4y}$$

$$+ \lambda_1 p_{1y} p_{2y} p_{3x} p_{4y} r_{1x} r_{2x} r_{3x} r_{4y} - f_y \lambda_1 p_{1x} p_{2y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4y}$$

$$- {f_y}^2 \lambda_1 p_{3x} p_{4x} r_{1y} r_{2x} r_{3x} r_{4y} + f_y \lambda_1 p_{2y} p_{3x} p_{4x} r_{1y} r_{2x} r_{3x} r_{4y}$$

$$- \lambda_1 p_{1y} p_{2y} p_{3x} p_{4x} r_{1y} r_{2x} r_{3x} r_{4y} + \lambda_1 p_{1x} p_{2y} p_{3y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4y}$$

$$- {f_y}^2 {\lambda_1}^2 p_{3x} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y} + f_y {\lambda_1}^2 p_{1x} p_{3y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y}$$

$$- {\lambda_1}^2 p_{1x} p_{2y} p_{3y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y} - f_y {\lambda_1}^2 p_{1y} p_{4x} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y}$$

$$- f_y {\lambda_1}^2 p_{2y} p_{4x} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y} - f_y {\lambda_1}^2 p_{1x} p_{4y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y}$$

$$+ {\lambda_1}^2 p_{1x} p_{2y} p_{4y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y} - {\lambda_1}^2 p_{2y} p_{3x} p_{4y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y}$$

$$+ f_y {\lambda_1}^2 p_{1y} p_{3x} {r_{1x}}^2 r_{2y} r_{3x} r_{4y} + f_y {\lambda_1}^2 p_{2x} p_{3y} {r_{1x}}^2 r_{2y} r_{3x} r_{4y}$$

$$- {\lambda_1}^2 p_{1y} p_{2x} p_{3y} {r_{1x}}^2 r_{2y} r_{3x} r_{4y} + {\lambda_1}^2 p_{1y} p_{2x} p_{4y} {r_{1x}}^2 r_{2y} r_{3x} r_{4y}$$

$$- {\lambda_1}^2 p_{1y} p_{3x} p_{4y} {r_{1x}}^2 r_{2y} r_{3x} r_{4y} + {\lambda_1}^2 p_{1x} p_{2x} p_{4x} {r_{1y}}^2 r_{2y} r_{3x} r_{4y}$$

$$- {\lambda_1}^2 p_{1x} p_{3x} p_{4x} {r_{1y}}^2 r_{2y} r_{3x} r_{4y} + f_y \lambda_1 p_{1x} p_{2y} p_{3x} r_{1x} r_{2y} r_{3x} r_{4y}$$

$$+ {f_y}^2 \lambda_1 p_{2x} p_{4x} r_{1x} r_{2y} r_{3x} r_{4y} - f_y \lambda_1 p_{2x} p_{3y} p_{4x} r_{1x} r_{2y} r_{3x} r_{4y}$$

$$+ \lambda_1 p_{1y} p_{2x} p_{3y} p_{4x} r_{1x} r_{2y} r_{3x} r_{4y} - \lambda_1 p_{1x} p_{2y} p_{3x} p_{4y} r_{1x} r_{2y} r_{3x} r_{4y}$$

$$+ f_y \lambda_1 p_{1x} p_{2x} p_{4x} r_{1y} r_{2y} r_{3x} r_{4y} + \lambda_1 p_{1x} p_{2y} p_{3x} p_{4x} r_{1y} r_{2y} r_{3x} r_{4y}$$

$$- \lambda_1 p_{1x} p_{2x} p_{3y} p_{4x} r_{1y} r_{2y} r_{3x} r_{4y} + {\lambda_1}^2 p_{1x} p_{2x} p_{3y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y}$$

$$+ f_y {\lambda_1}^2 p_{2x} p_{4x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} + {\lambda_1}^2 p_{1y} p_{3x} p_{4x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y}$$

$$- {\lambda_1}^2 p_{2y} p_{3x} p_{4x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} + {\lambda_1}^2 p_{1x} p_{3x} p_{4y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y}$$

$$+ f_y {\lambda_1}^2 p_{2y} p_{3x} {r_{1x}}^2 r_{2x} r_{3y} r_{4y} - f_y {\lambda_1}^2 p_{3x} p_{4y} {r_{1x}}^2 r_{2x} r_{3y} r_{4y}$$

$$+ {\lambda_1}^2 p_{1y} p_{3x} p_{4y} {r_{1x}}^2 r_{2x} r_{3y} r_{4y} - {\lambda_1}^2 p_{2x} p_{3x} p_{4x} {r_{1y}}^2 r_{2x} r_{3y} r_{4y}$$

$$- {f_y}^2 \lambda_1 p_{2x} p_{3x} r_{1x} r_{2x} r_{3y} r_{4y} + {f_y}^2 \lambda_1 p_{3x} p_{4x} r_{1x} r_{2x} r_{3y} r_{4y}$$

$$- f_y \lambda_1 p_{1y} p_{3x} p_{4x} r_{1x} r_{2x} r_{3y} r_{4y} + \lambda_1 p_{1y} p_{2y} p_{3x} p_{4x} r_{1x} r_{2x} r_{3y} r_{4y}$$

$$+ f_y \lambda_1 p_{2x} p_{3x} p_{4y} r_{1x} r_{2x} r_{3y} r_{4y} + f_y \lambda_1 p_{1x} p_{3x} p_{4x} r_{1y} r_{2x} r_{3y} r_{4y}$$

$$
\begin{aligned}
& - f_y\lambda_1 p_{2x}p_{3x}p_{4x}r_{1y}r_{2x}r_{3y}r_{4y} - \lambda_1 p_{1x}p_{2y}p_{3x}p_{4x}r_{1y}r_{2x}r_{3y}r_{4y} \\
& - f_y{\lambda_1}^2 p_{2x}p_{3x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4y} + f_y{\lambda_1}^2 p_{3x}p_{4x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4y} \\
& - {\lambda_1}^2 p_{1y}p_{3x}p_{4x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4y} + {\lambda_1}^2 p_{2x}p_{3x}p_{4y}r_{1x}r_{1y}r_{2x}r_{3y}r_{4y} \\
& + {\lambda_1}^2 p_{1y}p_{2x}p_{3x}{r_{1x}}^2 r_{2y}r_{3y}r_{4y} - f_y\lambda_1 p_{1x}p_{2x}p_{3x}r_{1x}r_{2y}r_{3y}r_{4y} \\
& + f_y\lambda_1 p_{2x}p_{3x}p_{4x}r_{1x}r_{2y}r_{3y}r_{4y} + \lambda_1 p_{1x}p_{2x}p_{3x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4y} \\
& - {\lambda_1}^2 p_{1x}p_{2x}p_{3x}r_{1x}r_{1y}r_{2y}r_{3y}r_{4y}.
\end{aligned}
$$

This is a quadratic polynomial in λ_1 without a constant term. Ignoring the solution $\lambda_1 = 0$, we find that $\lambda_1 = n_1'/d_1'$ where n_1' is

$$
\begin{aligned}
& p_{3y}r_{1y}r_{2y}r_{3y}r_{4x}{f_x}^3 + p_{2y}r_{1y}r_{2y}r_{3x}r_{4y}{f_x}^3 \\
& - p_{3y}r_{1y}r_{2y}r_{3x}r_{4y}{f_x}^3 - p_{2y}r_{1y}r_{2x}r_{3y}r_{4y}{f_x}^3 \\
& - p_{1y}r_{1x}r_{2y}r_{3y}r_{4y}{f_x}^3 - f_y p_{2y}r_{1y}r_{2y}r_{3x}r_{4x}{f_x}^2 \\
& + p_{2y}p_{3y}r_{1y}r_{2y}r_{3x}r_{4x}{f_x}^2 - p_{3y}p_{4y}r_{1y}r_{2y}r_{3x}r_{4x}{f_x}^2 \\
& - f_y p_{1y}r_{1y}r_{2x}r_{3y}r_{4x}{f_x}^2 - f_y p_{3y}r_{1y}r_{2x}r_{3y}r_{4x}{f_x}^2 \\
& + p_{1y}p_{3y}r_{1y}r_{2x}r_{3y}r_{4x}{f_x}^2 - p_{2y}p_{4y}r_{1y}r_{2x}r_{3y}r_{4x}{f_x}^2 \\
& + f_y p_{1y}r_{1x}r_{2y}r_{3y}r_{4x}{f_x}^2 - p_{1y}p_{4y}r_{1x}r_{2y}r_{3y}r_{4x}{f_x}^2 \\
& + p_{3y}p_{4y}r_{1x}r_{2y}r_{3y}r_{4x}{f_x}^2 - p_{1x}p_{3y}r_{1y}r_{2y}r_{3y}r_{4x}{f_x}^2 \\
& - p_{2x}p_{3y}r_{1y}r_{2y}r_{3y}r_{4x}{f_x}^2 - p_{3y}p_{4x}r_{1y}r_{2y}r_{3y}r_{4x}{f_x}^2 \\
& + p_{1x}p_{4y}r_{1y}r_{2y}r_{3y}r_{4x}{f_x}^2 + p_{3x}p_{4y}r_{1y}r_{2y}r_{3y}r_{4x}{f_x}^2 \\
& - f_y p_{1y}r_{1y}r_{2x}r_{3x}r_{4y}{f_x}^2 + f_y p_{3y}r_{1y}r_{2x}r_{3x}r_{4y}{f_x}^2 \\
& - p_{2y}p_{3y}r_{1y}r_{2x}r_{3x}r_{4y}{f_x}^2 - f_y p_{2y}r_{1x}r_{2y}r_{3x}r_{4y}{f_x}^2 \\
& + f_y p_{3y}r_{1x}r_{2y}r_{3x}r_{4y}{f_x}^2 - f_y p_{4y}r_{1x}r_{2y}r_{3x}r_{4y}{f_x}^2 \\
& + p_{2y}p_{4y}r_{1x}r_{2y}r_{3x}r_{4y}{f_x}^2 - p_{1x}p_{2y}r_{1y}r_{2y}r_{3x}r_{4y}{f_x}^2 \\
& + f_y p_{3x}r_{1y}r_{2y}r_{3x}r_{4y}{f_x}^2 + p_{1x}p_{3y}r_{1y}r_{2y}r_{3x}r_{4y}{f_x}^2 \\
& + p_{2x}p_{3y}r_{1y}r_{2y}r_{3x}r_{4y}{f_x}^2 + p_{3y}p_{4x}r_{1y}r_{2y}r_{3x}r_{4y}{f_x}^2 \\
& + f_y p_{2y}r_{1x}r_{2x}r_{3y}r_{4y}{f_x}^2 - f_y p_{4y}r_{1x}r_{2x}r_{3y}r_{4y}{f_x}^2 \\
& + p_{1y}p_{4y}r_{1x}r_{2x}r_{3y}r_{4y}{f_x}^2 + f_y p_{2x}r_{1y}r_{2x}r_{3y}r_{4y}{f_x}^2 \\
& - p_{1y}p_{2x}r_{1y}r_{2x}r_{3y}r_{4y}{f_x}^2 - p_{1y}p_{3x}r_{1y}r_{2x}r_{3y}r_{4y}{f_x}^2
\end{aligned}
$$

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\begin{array}{llll}
+ & p_{2y}p_{3x}r_{1y}r_{2x}r_{3y}r_{4y}{f_x}^2 & + & p_{2y}p_{4x}r_{1y}r_{2x}r_{3y}r_{4y}{f_x}^2 \\
+ & f_yp_{1x}r_{1x}r_{2y}r_{3y}r_{4y}{f_x}^2 & + & p_{1y}p_{3x}r_{1x}r_{2y}r_{3y}r_{4y}{f_x}^2 \\
- & f_yp_{4x}r_{1x}r_{2y}r_{3y}r_{4y}{f_x}^2 & - & p_{1x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4y}{f_x}^2 \\
- & p_{2x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4y}{f_x}^2 & + & {f_y}^2p_{1y}r_{1y}r_{2x}r_{3x}r_{4x}f_x \\
- & f_yp_{1y}p_{2y}r_{1y}r_{2x}r_{3x}r_{4x}f_x & + & p_{1y}p_{2y}p_{3y}r_{1y}r_{2x}r_{3x}r_{4x}f_x \\
- & {f_y}^2p_{4y}r_{1y}r_{2x}r_{3x}r_{4x}f_x & + & f_yp_{3y}p_{4y}r_{1y}r_{2x}r_{3x}r_{4x}f_x \\
- & p_{2y}p_{3y}p_{4y}r_{1y}r_{2x}r_{3x}r_{4x}f_x & + & {f_y}^2p_{2y}r_{1x}r_{2y}r_{3x}r_{4x}f_x \\
+ & f_yp_{1y}p_{3y}r_{1x}r_{2y}r_{3x}r_{4x}f_x & + & f_yp_{1y}p_{4y}r_{1x}r_{2y}r_{3x}r_{4x}f_x \\
- & f_yp_{2y}p_{4y}r_{1x}r_{2y}r_{3x}r_{4x}f_x & + & p_{2y}p_{3y}p_{4y}r_{1x}r_{2y}r_{3x}r_{4x}f_x \\
+ & {f_y}^2p_{2x}r_{1y}r_{2y}r_{3x}r_{4x}f_x & - & f_yp_{2x}p_{3y}r_{1y}r_{2y}r_{3x}r_{4x}f_x \\
- & p_{1x}p_{2y}p_{3y}r_{1y}r_{2y}r_{3x}r_{4x}f_x & + & f_yp_{2y}p_{4x}r_{1y}r_{2y}r_{3x}r_{4x}f_x \\
+ & f_yp_{3y}p_{4x}r_{1y}r_{2y}r_{3x}r_{4x}f_x & - & f_yp_{1x}p_{4y}r_{1y}r_{2y}r_{3x}r_{4x}f_x \\
- & f_yp_{2x}p_{4y}r_{1y}r_{2y}r_{3x}r_{4x}f_x & + & p_{2x}p_{3y}p_{4y}r_{1y}r_{2y}r_{3x}r_{4x}f_x \\
- & {f_y}^2p_{2y}r_{1x}r_{2x}r_{3y}r_{4x}f_x & + & {f_y}^2p_{3y}r_{1x}r_{2x}r_{3y}r_{4x}f_x \\
- & f_yp_{1y}p_{3y}r_{1x}r_{2x}r_{3y}r_{4x}f_x & - & p_{1y}p_{2y}p_{4y}r_{1x}r_{2x}r_{3y}r_{4x}f_x \\
- & f_yp_{3y}p_{4y}r_{1x}r_{2x}r_{3y}r_{4x}f_x & + & {f_y}^2p_{1x}r_{1y}r_{2x}r_{3y}r_{4x}f_x \\
- & {f_y}^2p_{2x}r_{1y}r_{2x}r_{3y}r_{4x}f_x & - & f_yp_{1x}p_{2y}r_{1y}r_{2x}r_{3y}r_{4x}f_x \\
+ & {f_y}^2p_{3x}r_{1y}r_{2x}r_{3y}r_{4x}f_x & + & f_yp_{2x}p_{3y}r_{1y}r_{2x}r_{3y}r_{4x}f_x \\
- & p_{1y}p_{2x}p_{3y}r_{1y}r_{2x}r_{3y}r_{4x}f_x & + & f_yp_{1y}p_{4x}r_{1y}r_{2x}r_{3y}r_{4x}f_x \\
+ & f_yp_{3y}p_{4x}r_{1y}r_{2x}r_{3y}r_{4x}f_x & - & f_yp_{1x}p_{4y}r_{1y}r_{2x}r_{3y}r_{4x}f_x \\
+ & p_{1x}p_{2y}p_{4y}r_{1y}r_{2x}r_{3y}r_{4x}f_x & + & p_{2y}p_{3x}p_{4y}r_{1y}r_{2x}r_{3y}r_{4x}f_x \\
- & {f_y}^2p_{1x}r_{1x}r_{2y}r_{3y}r_{4x}f_x & + & {f_y}^2p_{3x}r_{1x}r_{2y}r_{3y}r_{4x}f_x \\
- & f_yp_{1y}p_{3x}r_{1x}r_{2y}r_{3y}r_{4x}f_x & + & f_yp_{2x}p_{3y}r_{1x}r_{2y}r_{3y}r_{4x}f_x \\
+ & f_yp_{1x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4x}f_x & - & f_yp_{3x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4x}f_x \\
+ & p_{1y}p_{3x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4x}f_x & - & p_{2x}p_{3y}p_{4y}r_{1x}r_{2y}r_{3y}r_{4x}f_x \\
+ & f_yp_{1x}p_{3x}r_{1y}r_{2y}r_{3y}r_{4x}f_x & + & p_{1x}p_{2x}p_{3y}r_{1y}r_{2y}r_{3y}r_{4x}f_x \\
- & f_yp_{1x}p_{4x}r_{1y}r_{2y}r_{3y}r_{4x}f_x & + & p_{1x}p_{3y}p_{4x}r_{1y}r_{2y}r_{3y}r_{4x}f_x \\
+ & p_{2x}p_{3y}p_{4x}r_{1y}r_{2y}r_{3y}r_{4x}f_x & - & p_{1x}p_{3x}p_{4y}r_{1y}r_{2y}r_{3y}r_{4x}f_x
\end{array}
$$

$$
\begin{array}{llll}
- & p_{2x}p_{3x}p_{4y}r_{1y}r_{2y}r_{3y}r_{4x}f_x & + & f_y p_{1y}p_{3y}r_{1x}r_{2x}r_{3x}r_{4y}f_x \\
+ & f_y p_{2y}p_{3y}r_{1x}r_{2x}r_{3x}r_{4y}f_x & + & {f_y}^2 p_{4y}r_{1x}r_{2x}r_{3x}r_{4y}f_x \\
- & f_y p_{1y}p_{4y}r_{1x}r_{2x}r_{3x}r_{4y}f_x & + & p_{1y}p_{2y}p_{4y}r_{1x}r_{2x}r_{3x}r_{4y}f_x \\
+ & {f_y}^2 p_{1x}r_{1y}r_{2x}r_{3x}r_{4y}f_x & - & {f_y}^2 p_{3x}r_{1y}r_{2x}r_{3x}r_{4y}f_x \\
+ & f_y p_{1y}p_{3x}r_{1y}r_{2x}r_{3x}r_{4y}f_x & - & p_{1y}p_{2y}p_{3x}r_{1y}r_{2x}r_{3x}r_{4y}f_x \\
- & f_y p_{1x}p_{3y}r_{1y}r_{2x}r_{3x}r_{4y}f_x & + & f_y p_{1y}p_{4x}r_{1y}r_{2x}r_{3x}r_{4y}f_x \\
- & p_{1y}p_{2y}p_{4x}r_{1y}r_{2x}r_{3x}r_{4y}f_x & + & p_{2y}p_{3y}p_{4x}r_{1y}r_{2x}r_{3x}r_{4y}f_x \\
- & {f_y}^2 p_{1x}r_{1x}r_{2y}r_{3x}r_{4y}f_x & - & f_y p_{1y}p_{2x}r_{1x}r_{2y}r_{3x}r_{4y}f_x \\
+ & f_y p_{1x}p_{2y}r_{1x}r_{2y}r_{3x}r_{4y}f_x & + & f_y p_{2y}p_{3x}r_{1x}r_{2y}r_{3x}r_{4y}f_x \\
- & f_y p_{2x}p_{3y}r_{1x}r_{2y}r_{3x}r_{4y}f_x & + & {f_y}^2 p_{4x}r_{1x}r_{2y}r_{3x}r_{4y}f_x \\
- & f_y p_{1y}p_{4x}r_{1x}r_{2y}r_{3x}r_{4y}f_x & + & p_{1y}p_{3y}p_{4x}r_{1x}r_{2y}r_{3x}r_{4y}f_x \\
+ & f_y p_{1x}p_{4y}r_{1x}r_{2y}r_{3x}r_{4y}f_x & + & f_y p_{3x}p_{4y}r_{1x}r_{2y}r_{3x}r_{4y}f_x \\
- & p_{2y}p_{3x}p_{4y}r_{1x}r_{2y}r_{3x}r_{4y}f_x & - & f_y p_{1x}p_{3x}r_{1y}r_{2y}r_{3x}r_{4y}f_x \\
+ & p_{1x}p_{2y}p_{3x}r_{1y}r_{2y}r_{3x}r_{4y}f_x & + & f_y p_{2x}p_{4x}r_{1y}r_{2y}r_{3x}r_{4y}f_x \\
+ & p_{1x}p_{2y}p_{4x}r_{1y}r_{2y}r_{3x}r_{4y}f_x & + & p_{2y}p_{3x}p_{4x}r_{1y}r_{2y}r_{3x}r_{4y}f_x \\
- & p_{1x}p_{3y}p_{4x}r_{1y}r_{2y}r_{3x}r_{4y}f_x & - & {f_y}^2 p_{2x}r_{1x}r_{2x}r_{3y}r_{4y}f_x \\
+ & f_y p_{1y}p_{2x}r_{1x}r_{2x}r_{3y}r_{4y}f_x & + & p_{1y}p_{2y}p_{3x}r_{1x}r_{2x}r_{3y}r_{4y}f_x \\
+ & {f_y}^2 p_{4x}r_{1x}r_{2x}r_{3y}r_{4y}f_x & - & f_y p_{2y}p_{4x}r_{1x}r_{2x}r_{3y}r_{4y}f_x \\
+ & p_{1y}p_{2y}p_{4x}r_{1x}r_{2x}r_{3y}r_{4y}f_x & - & p_{1y}p_{2x}p_{4y}r_{1x}r_{2x}r_{3y}r_{4y}f_x \\
+ & f_y p_{3x}p_{4y}r_{1x}r_{2x}r_{3y}r_{4y}f_x & + & f_y p_{1x}p_{3x}r_{1y}r_{2x}r_{3y}r_{4y}f_x \\
- & f_y p_{2x}p_{3x}r_{1y}r_{2x}r_{3y}r_{4y}f_x & - & p_{1x}p_{2y}p_{3x}r_{1y}r_{2x}r_{3y}r_{4y}f_x \\
+ & f_y p_{1x}p_{4x}r_{1y}r_{2x}r_{3y}r_{4y}f_x & + & p_{1y}p_{2x}p_{4x}r_{1y}r_{2x}r_{3y}r_{4y}f_x \\
- & p_{1x}p_{2y}p_{4x}r_{1y}r_{2x}r_{3y}r_{4y}f_x & - & p_{2y}p_{3x}p_{4x}r_{1y}r_{2x}r_{3y}r_{4y}f_x \\
- & f_y p_{1x}p_{2x}r_{1x}r_{2y}r_{3y}r_{4y}f_x & - & p_{1y}p_{2x}p_{3x}r_{1x}r_{2y}r_{3y}r_{4y}f_x \\
+ & f_y p_{2x}p_{4x}r_{1x}r_{2y}r_{3y}r_{4y}f_x & + & f_y p_{3x}p_{4x}r_{1x}r_{2y}r_{3y}r_{4y}f_x \\
- & p_{1y}p_{3x}p_{4x}r_{1x}r_{2y}r_{3y}r_{4y}f_x & + & p_{1x}p_{3x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4y}f_x \\
+ & p_{2x}p_{3x}p_{4y}r_{1x}r_{2y}r_{3y}r_{4y}f_x & + & {f_y}^2 p_{1x}p_{2y}r_{1y}r_{2x}r_{3x}r_{4x} \\
+ & {f_y}^2 p_{1x}p_{3y}r_{1y}r_{2x}r_{3x}r_{4x} & + & {f_y}^3 p_{4x}r_{1y}r_{2x}r_{3x}r_{4x}
\end{array}
$$

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\begin{array}{llll}
- & {f_y}^2 p_{1y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4x} & + & f_y p_{1y} p_{2y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4x} \\
- & {f_y}^2 p_{3y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4x} & + & f_y p_{2y} p_{3y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4x} \\
- & p_{1y} p_{2y} p_{3y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4x} & - & f_y p_{1x} p_{2y} p_{4y} r_{1y} r_{2x} r_{3x} r_{4x} \\
- & f_y p_{1x} p_{3y} p_{4y} r_{1y} r_{2x} r_{3x} r_{4x} & + & {f_y}^3 p_{1x} r_{1x} r_{2y} r_{3x} r_{4x} \\
- & {f_y}^3 p_{2x} r_{1x} r_{2y} r_{3x} r_{4x} & - & {f_y}^2 p_{1x} p_{2y} r_{1x} r_{2y} r_{3x} r_{4x} \\
- & {f_y}^2 p_{1x} p_{3y} r_{1x} r_{2y} r_{3x} r_{4x} & - & f_y p_{1y} p_{2x} p_{3y} r_{1x} r_{2y} r_{3x} r_{4x} \\
+ & f_y p_{1x} p_{2y} p_{3y} r_{1x} r_{2y} r_{3x} r_{4x} & + & {f_y}^2 p_{2x} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} \\
- & f_y p_{1y} p_{2x} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} & + & f_y p_{1x} p_{3y} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} \\
- & f_y p_{2x} p_{3y} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} & - & p_{1x} p_{2y} p_{3y} p_{4y} r_{1x} r_{2y} r_{3x} r_{4x} \\
- & {f_y}^2 p_{1x} p_{2x} r_{1y} r_{2y} r_{3x} r_{4x} & + & {f_y}^2 p_{1x} p_{4x} r_{1y} r_{2y} r_{3x} r_{4x} \\
- & f_y p_{1x} p_{2y} p_{4x} r_{1y} r_{2y} r_{3x} r_{4x} & + & p_{1x} p_{2y} p_{3y} p_{4x} r_{1y} r_{2y} r_{3x} r_{4x} \\
+ & f_y p_{1x} p_{2x} p_{4y} r_{1y} r_{2y} r_{3x} r_{4x} & + & {f_y}^3 p_{2x} r_{1x} r_{2x} r_{3y} r_{4x} \\
- & {f_y}^2 p_{1y} p_{2x} r_{1x} r_{2x} r_{3y} r_{4x} & + & {f_y}^2 p_{1y} p_{3x} r_{1x} r_{2x} r_{3y} r_{4x} \\
+ & {f_y}^2 p_{2y} p_{3x} r_{1x} r_{2x} r_{3y} r_{4x} & - & {f_y}^2 p_{2x} p_{3y} r_{1x} r_{2x} r_{3y} r_{4x} \\
+ & f_y p_{1y} p_{2x} p_{3y} r_{1x} r_{2x} r_{3y} r_{4x} & + & f_y p_{1y} p_{2x} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} \\
+ & {f_y}^2 p_{3x} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} & - & f_y p_{2y} p_{3x} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} \\
+ & p_{1y} p_{2y} p_{3x} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} & - & p_{1y} p_{2x} p_{3y} p_{4y} r_{1x} r_{2x} r_{3y} r_{4x} \\
- & {f_y}^2 p_{1x} p_{3x} r_{1y} r_{2x} r_{3y} r_{4x} & + & {f_y}^2 p_{2x} p_{4x} r_{1y} r_{2x} r_{3y} r_{4x} \\
- & f_y p_{1y} p_{2x} p_{4x} r_{1y} r_{2x} r_{3y} r_{4x} & + & p_{1y} p_{2x} p_{3y} p_{4x} r_{1y} r_{2x} r_{3y} r_{4x} \\
+ & f_y p_{1x} p_{3x} p_{4y} r_{1y} r_{2x} r_{3y} r_{4x} & + & {f_y}^2 p_{1x} p_{2x} r_{1x} r_{2y} r_{3y} r_{4x} \\
- & {f_y}^2 p_{2x} p_{3x} r_{1x} r_{2y} r_{3y} r_{4x} & - & f_y p_{1x} p_{2x} p_{3y} r_{1x} r_{2y} r_{3y} r_{4x} \\
- & f_y p_{1x} p_{2x} p_{4y} r_{1x} r_{2y} r_{3y} r_{4x} & - & p_{1y} p_{2x} p_{3x} p_{4y} r_{1x} r_{2y} r_{3y} r_{4x} \\
+ & p_{1x} p_{2x} p_{3y} p_{4y} r_{1x} r_{2y} r_{3y} r_{4x} & + & f_y p_{1x} p_{2x} p_{4x} r_{1y} r_{2y} r_{3y} r_{4x} \\
- & p_{1x} p_{2x} p_{3y} p_{4x} r_{1y} r_{2y} r_{3y} r_{4x} & + & {f_y}^3 p_{3x} r_{1x} r_{2x} r_{3x} r_{4y} \\
- & {f_y}^2 p_{1y} p_{3x} r_{1x} r_{2x} r_{3x} r_{4y} & + & f_y p_{1y} p_{2y} p_{3x} r_{1x} r_{2x} r_{3x} r_{4y} \\
- & {f_y}^3 p_{4x} r_{1x} r_{2x} r_{3x} r_{4y} & + & {f_y}^2 p_{2y} p_{4x} r_{1x} r_{2x} r_{3x} r_{4y} \\
- & f_y p_{1y} p_{2y} p_{4x} r_{1x} r_{2x} r_{3x} r_{4y} & - & f_y p_{1y} p_{3y} p_{4x} r_{1x} r_{2x} r_{3x} r_{4y} \\
- & f_y p_{2y} p_{3y} p_{4x} r_{1x} r_{2x} r_{3x} r_{4y} & - & {f_y}^2 p_{3x} p_{4y} r_{1x} r_{2x} r_{3x} r_{4y}
\end{array}
$$

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\begin{array}{clcl}
+ & f_y p_{1y} p_{3x} p_{4y} r_{1x} r_{2x} r_{3x} r_{4y} & - & p_{1y} p_{2y} p_{3x} p_{4y} r_{1x} r_{2x} r_{3x} r_{4y} \\
- & {f_y}^2 p_{1x} p_{4x} r_{1y} r_{2x} r_{3x} r_{4y} & + & {f_y}^2 p_{3x} p_{4x} r_{1y} r_{2x} r_{3x} r_{4y} \\
- & f_y p_{1y} p_{3x} p_{4x} r_{1y} r_{2x} r_{3x} r_{4y} & + & p_{1y} p_{2y} p_{3x} p_{4x} r_{1y} r_{2x} r_{3x} r_{4y} \\
+ & f_y p_{1x} p_{3y} p_{4x} r_{1y} r_{2x} r_{3x} r_{4y} & + & {f_y}^2 p_{1x} p_{3x} r_{1x} r_{2y} r_{3x} r_{4y} \\
- & f_y p_{1x} p_{2y} p_{3x} r_{1x} r_{2y} r_{3x} r_{4y} & + & f_y p_{1y} p_{2x} p_{4x} r_{1x} r_{2y} r_{3x} r_{4y} \\
+ & f_y p_{2x} p_{3y} p_{4x} r_{1x} r_{2y} r_{3x} r_{4y} & - & f_y p_{1x} p_{3x} p_{4y} r_{1x} r_{2y} r_{3x} r_{4y} \\
+ & p_{1x} p_{2y} p_{3x} p_{4y} r_{1x} r_{2y} r_{3x} r_{4y} & + & f_y p_{1x} p_{3x} p_{4x} r_{1y} r_{2y} r_{3x} r_{4y} \\
- & p_{1x} p_{2y} p_{3x} p_{4x} r_{1y} r_{2y} r_{3x} r_{4y} & + & {f_y}^2 p_{2x} p_{3x} r_{1x} r_{2x} r_{3y} r_{4y} \\
- & f_y p_{1y} p_{2x} p_{3x} r_{1x} r_{2x} r_{3y} r_{4y} & + & f_y p_{1y} p_{3x} p_{4x} r_{1x} r_{2x} r_{3y} r_{4y} \\
+ & f_y p_{2y} p_{3x} p_{4x} r_{1x} r_{2x} r_{3y} r_{4y} & - & f_y p_{2x} p_{3x} p_{4y} r_{1x} r_{2x} r_{3y} r_{4y} \\
+ & p_{1y} p_{2x} p_{3x} p_{4y} r_{1x} r_{2x} r_{3y} r_{4y} & + & f_y p_{2x} p_{3x} p_{4x} r_{1y} r_{2x} r_{3y} r_{4y} \\
- & p_{1y} p_{2x} p_{3x} p_{4x} r_{1y} r_{2x} r_{3y} r_{4y} & + & f_y p_{1x} p_{2x} p_{3x} r_{1x} r_{2y} r_{3y} r_{4y} \\
- & f_y p_{2x} p_{3x} p_{4x} r_{1x} r_{2y} r_{3y} r_{4y} & - & p_{1x} p_{2x} p_{3x} p_{4y} r_{1x} r_{2y} r_{3y} r_{4y}
\end{array}
$$

and d_1' is

$$
\begin{array}{clcl}
 & p_{3x} {r_{1y}}^2 r_{2y} r_{3y} r_{4x} {f_x}^2 & - & p_{3y} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} {f_x}^2 \\
+ & p_{4y} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} {f_x}^2 & - & p_{3x} {r_{1y}}^2 r_{2y} r_{3x} r_{4y} {f_x}^2 \\
- & p_{2y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} {f_x}^2 & + & p_{1x} {r_{1y}}^2 r_{2x} r_{3y} r_{4y} {f_x}^2 \\
- & p_{2x} {r_{1y}}^2 r_{2x} r_{3y} r_{4y} {f_x}^2 & + & p_{2y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} {f_x}^2 \\
+ & p_{1y} {r_{1x}}^2 r_{2y} r_{3y} r_{4y} {f_x}^2 & - & p_{1x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4y} {f_x}^2 \\
+ & p_{4x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4y} {f_x}^2 & + & p_{2x} p_{3y} {r_{1y}}^2 r_{2y} r_{3x} r_{4x} f_x \\
+ & f_y p_{4x} {r_{1y}}^2 r_{2y} r_{3x} r_{4x} f_x & + & f_y p_{2y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} f_x \\
- & p_{2y} p_{3y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} f_x & + & p_{3y} p_{4y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4x} f_x \\
- & f_y p_{1x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} f_x & - & f_y p_{3x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} f_x \\
+ & p_{2y} p_{3x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} f_x & - & p_{2x} p_{3y} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} f_x \\
+ & f_y p_{4x} {r_{1y}}^2 r_{2x} r_{3y} r_{4x} f_x & + & f_y p_{1y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} f_x \\
- & f_y p_{2y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} f_x & - & p_{1y} p_{3y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} f_x \\
- & f_y p_{4y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4x} f_x & - & f_y p_{1y} {r_{1x}}^2 r_{2y} r_{3y} r_{4x} f_x
\end{array}
$$

$$
\begin{array}{llll}
+ & f_y p_{3y} r_{1x}^2 r_{2y} r_{3y} r_{4x} f_x & - & p_{3y} p_{4y} r_{1x}^2 r_{2y} r_{3y} r_{4x} f_x \\
- & p_{1x} p_{3x} r_{1y}^2 r_{2y} r_{3y} r_{4x} f_x & + & p_{1x} p_{4x} r_{1y}^2 r_{2y} r_{3y} r_{4x} f_x \\
+ & p_{2x} p_{4x} r_{1y}^2 r_{2y} r_{3y} r_{4x} f_x & - & f_y p_{3x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} f_x \\
+ & p_{1y} p_{3x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} f_x & - & p_{1y} p_{4x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} f_x \\
+ & p_{3y} p_{4x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} f_x & - & p_{2x} p_{4y} r_{1x} r_{1y} r_{2y} r_{3y} r_{4x} f_x \\
- & f_y p_{1x} r_{1y}^2 r_{2x} r_{3x} r_{4y} f_x & + & f_y p_{3x} r_{1y}^2 r_{2x} r_{3x} r_{4y} f_x \\
- & p_{2y} p_{3x} r_{1y}^2 r_{2x} r_{3x} r_{4y} f_x & - & p_{1y} p_{2y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y} f_x \\
- & f_y p_{3y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y} f_x & - & f_y p_{1y} r_{1x}^2 r_{2y} r_{3x} r_{4y} f_x \\
+ & f_y p_{2y} r_{1x}^2 r_{2y} r_{3x} r_{4y} f_x & + & p_{1y} p_{3y} r_{1x}^2 r_{2y} r_{3x} r_{4y} f_x \\
+ & f_y p_{4y} r_{1x}^2 r_{2y} r_{3x} r_{4y} f_x & - & p_{1x} p_{2x} r_{1y}^2 r_{2y} r_{3x} r_{4y} f_x \\
+ & p_{1x} p_{3x} r_{1y}^2 r_{2y} r_{3x} r_{4y} f_x & + & p_{3x} p_{4x} r_{1y}^2 r_{2y} r_{3x} r_{4y} f_x \\
+ & f_y p_{1x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} f_x & + & p_{1y} p_{2x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} f_x \\
+ & f_y p_{3x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} f_x & + & p_{2y} p_{3x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} f_x \\
- & p_{1x} p_{3y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} f_x & - & f_y p_{4x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} f_x \\
+ & p_{2y} p_{4x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} f_x & - & p_{3x} p_{4y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} f_x \\
- & f_y p_{2y} r_{1x}^2 r_{2x} r_{3y} r_{4y} f_x & + & f_y p_{4y} r_{1x}^2 r_{2x} r_{3y} r_{4y} f_x \\
- & p_{1y} p_{4y} r_{1x}^2 r_{2x} r_{3y} r_{4y} f_x & + & p_{2x} p_{3x} r_{1y}^2 r_{2x} r_{3y} r_{4y} f_x \\
- & p_{1x} p_{4x} r_{1y}^2 r_{2x} r_{3y} r_{4y} f_x & + & f_y p_{2x} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} f_x \\
- & p_{1x} p_{2y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} f_x & - & p_{2y} p_{3x} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} f_x \\
- & f_y p_{4x} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} f_x & + & p_{1x} p_{4y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} f_x \\
- & p_{2x} p_{4y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} f_x & - & p_{1y} p_{3x} r_{1x}^2 r_{2y} r_{3y} r_{4y} f_x \\
+ & p_{2x} p_{4y} r_{1x}^2 r_{2y} r_{3y} r_{4y} f_x & + & p_{1x} p_{2x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4y} f_x \\
+ & p_{1x} p_{3x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4y} f_x & - & p_{3x} p_{4x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4y} f_x \\
+ & f_y^2 p_{1x} r_{1y}^2 r_{2x} r_{3x} r_{4x} & - & f_y p_{1x} p_{3y} r_{1y}^2 r_{2x} r_{3x} r_{4x} \\
+ & p_{1x} p_{2y} p_{3y} r_{1y}^2 r_{2x} r_{3x} r_{4x} & + & f_y p_{2y} p_{4x} r_{1y}^2 r_{2x} r_{3x} r_{4x} \\
+ & f_y p_{3y} p_{4x} r_{1y}^2 r_{2x} r_{3x} r_{4x} & - & f_y^2 p_{1y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4x} \\
+ & f_y p_{1y} p_{2y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4x} & - & p_{1y} p_{2y} p_{3y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4x} \\
+ & f_y^2 p_{4y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4x} & - & f_y p_{3y} p_{4y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4x}
\end{array}
$$

$$
\begin{array}{llll}
+ & p_{2y}p_{3y}p_{4y}r_{1x}r_{1y}r_{2x}r_{3x}r_{4x} & - & {f_y}^2p_{2y}{r_{1x}}^2r_{2y}r_{3x}r_{4x} \\
- & f_yp_{1y}p_{3y}{r_{1x}}^2r_{2y}r_{3x}r_{4x} & - & f_yp_{1y}p_{4y}{r_{1x}}^2r_{2y}r_{3x}r_{4x} \\
+ & f_yp_{2y}p_{4y}{r_{1x}}^2r_{2y}r_{3x}r_{4x} & - & p_{2y}p_{3y}p_{4y}{r_{1x}}^2r_{2y}r_{3x}r_{4x} \\
+ & f_yp_{1x}p_{2x}{r_{1y}}^2r_{2y}r_{3x}r_{4x} & - & f_yp_{1x}p_{4x}{r_{1y}}^2r_{2y}r_{3x}r_{4x} \\
+ & p_{1x}p_{3y}p_{4x}{r_{1y}}^2r_{2y}r_{3x}r_{4x} & + & {f_y}^2p_{2x}r_{1x}r_{1y}r_{2y}r_{3x}r_{4x} \\
- & f_yp_{1y}p_{2x}r_{1x}r_{1y}r_{2y}r_{3x}r_{4x} & - & f_yp_{2x}p_{3y}r_{1x}r_{1y}r_{2y}r_{3x}r_{4x} \\
+ & p_{1y}p_{2x}p_{3y}r_{1x}r_{1y}r_{2y}r_{3x}r_{4x} & - & f_yp_{2y}p_{4x}r_{1x}r_{1y}r_{2y}r_{3x}r_{4x} \\
- & p_{1y}p_{3y}p_{4x}r_{1x}r_{1y}r_{2y}r_{3x}r_{4x} & + & f_yp_{1x}p_{4y}r_{1x}r_{1y}r_{2y}r_{3x}r_{4x} \\
- & p_{1x}p_{3y}p_{4y}r_{1x}r_{1y}r_{2y}r_{3x}r_{4x} & - & f_yp_{1y}p_{2y}{r_{1x}}^2r_{2x}r_{3y}r_{4x} \\
- & {f_y}^2p_{3y}{r_{1x}}^2r_{2x}r_{3y}r_{4x} & - & f_yp_{2y}p_{4y}{r_{1x}}^2r_{2x}r_{3y}r_{4x} \\
+ & p_{1y}p_{2y}p_{4y}{r_{1x}}^2r_{2x}r_{3y}r_{4x} & - & p_{1y}p_{3y}p_{4y}{r_{1x}}^2r_{2x}r_{3y}r_{4x} \\
+ & f_yp_{1x}p_{3x}{r_{1y}}^2r_{2x}r_{3y}r_{4x} & - & f_yp_{2x}p_{4x}{r_{1y}}^2r_{2x}r_{3y}r_{4x} \\
+ & p_{1x}p_{2y}p_{4x}{r_{1y}}^2r_{2x}r_{3y}r_{4x} & + & p_{2x}p_{3y}p_{4x}{r_{1y}}^2r_{2x}r_{3y}r_{4x} \\
- & {f_y}^2p_{2x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4x} & + & {f_y}^2p_{3x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4x} \\
- & f_yp_{1y}p_{3x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4x} & + & p_{1y}p_{2y}p_{3x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4x} \\
- & f_yp_{1x}p_{3y}r_{1x}r_{1y}r_{2x}r_{3y}r_{4x} & + & f_yp_{2y}p_{4x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4x} \\
- & p_{1y}p_{2y}p_{4x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4x} & + & p_{1y}p_{3y}p_{4x}r_{1x}r_{1y}r_{2x}r_{3y}r_{4x} \\
+ & f_yp_{2x}p_{4y}r_{1x}r_{1y}r_{2x}r_{3y}r_{4x} & + & p_{1x}p_{3y}p_{4y}r_{1x}r_{1y}r_{2x}r_{3y}r_{4x} \\
- & p_{2x}p_{3y}p_{4y}r_{1x}r_{1y}r_{2x}r_{3y}r_{4x} & - & f_yp_{2x}p_{3y}{r_{1x}}^2r_{2y}r_{3y}r_{4x} \\
- & p_{1y}p_{2x}p_{4y}{r_{1x}}^2r_{2y}r_{3y}r_{4x} & + & p_{1x}p_{2x}p_{3x}{r_{1y}}^2r_{2y}r_{3y}r_{4x} \\
- & p_{1x}p_{2x}p_{4x}{r_{1y}}^2r_{2y}r_{3y}r_{4x} & + & f_yp_{2x}p_{3x}r_{1x}r_{1y}r_{2y}r_{3y}r_{4x} \\
- & p_{1y}p_{2x}p_{3x}r_{1x}r_{1y}r_{2y}r_{3y}r_{4x} & - & p_{2x}p_{3y}p_{4x}r_{1x}r_{1y}r_{2y}r_{3y}r_{4x} \\
+ & p_{1x}p_{2x}p_{4y}r_{1x}r_{1y}r_{2y}r_{3y}r_{4x} & - & f_yp_{1y}p_{3y}{r_{1x}}^2r_{2x}r_{3x}r_{4y} \\
- & f_yp_{2y}p_{3y}{r_{1x}}^2r_{2x}r_{3x}r_{4y} & - & {f_y}^2p_{4y}{r_{1x}}^2r_{2x}r_{3x}r_{4y} \\
+ & f_yp_{1y}p_{4y}{r_{1x}}^2r_{2x}r_{3x}r_{4y} & - & p_{1y}p_{2y}p_{4y}{r_{1x}}^2r_{2x}r_{3x}r_{4y} \\
+ & f_yp_{1x}p_{4x}{r_{1y}}^2r_{2x}r_{3x}r_{4y} & - & f_yp_{3x}p_{4x}{r_{1y}}^2r_{2x}r_{3x}r_{4y} \\
+ & p_{2y}p_{3x}p_{4x}{r_{1y}}^2r_{2x}r_{3x}r_{4y} & + & f_yp_{2y}p_{3x}r_{1x}r_{1y}r_{2x}r_{3x}r_{4y} \\
+ & f_yp_{1x}p_{3y}r_{1x}r_{1y}r_{2x}r_{3x}r_{4y} & + & {f_y}^2p_{4x}r_{1x}r_{1y}r_{2x}r_{3x}r_{4y}
\end{array}
$$

$$
\begin{array}{llll}
- & f_y p_{1y} p_{4x} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y} & + & p_{1y} p_{2y} p_{4x} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y} \\
- & f_y p_{1x} p_{4y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y} & + & f_y p_{3x} p_{4y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y} \\
- & p_{2y} p_{3x} p_{4y} r_{1x} r_{1y} r_{2x} r_{3x} r_{4y} & - & f_y p_{2y} p_{3x} {r_{1x}}^2 r_{2y} r_{3x} r_{4y} \\
+ & f_y p_{2x} p_{3y} {r_{1x}}^2 r_{2y} r_{3x} r_{4y} & - & f_y p_{2x} p_{4y} {r_{1x}}^2 r_{2y} r_{3x} r_{4y} \\
+ & p_{1y} p_{2x} p_{4y} {r_{1x}}^2 r_{2y} r_{3x} r_{4y} & + & p_{2y} p_{3x} p_{4y} {r_{1x}}^2 r_{2y} r_{3x} r_{4y} \\
+ & p_{1x} p_{2x} p_{4x} {r_{1y}}^2 r_{2y} r_{3x} r_{4y} & - & f_y p_{1x} p_{3x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} \\
+ & p_{1x} p_{2x} p_{3y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} & - & p_{1y} p_{2x} p_{4x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} \\
+ & p_{1y} p_{3x} p_{4x} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} & - & p_{1x} p_{2x} p_{4y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} \\
+ & p_{1x} p_{3x} p_{4y} r_{1x} r_{1y} r_{2y} r_{3x} r_{4y} & - & p_{1y} p_{2y} p_{3x} {r_{1x}}^2 r_{2x} r_{3y} r_{4y} \\
- & f_y p_{3x} p_{4y} {r_{1x}}^2 r_{2x} r_{3y} r_{4y} & + & p_{1x} p_{3x} p_{4x} {r_{1y}}^2 r_{2x} r_{3y} r_{4y} \\
- & p_{2x} p_{3x} p_{4x} {r_{1y}}^2 r_{2x} r_{3y} r_{4y} & + & p_{1x} p_{2y} p_{3x} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} \\
+ & f_y p_{3x} p_{4x} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} & - & p_{1x} p_{3x} p_{4y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} \\
+ & p_{2x} p_{3x} p_{4y} r_{1x} r_{1y} r_{2x} r_{3y} r_{4y} & - & p_{2x} p_{3x} p_{4y} {r_{1x}}^2 r_{2y} r_{3y} r_{4y} \\
- & p_{1x} p_{2x} p_{3x} r_{1x} r_{1y} r_{2y} r_{3y} r_{4y} & &
\end{array}
$$

Having computed λ_1, we can now use the polynomial g_2' to determine λ_2. If we do this, then we get an expression for λ_2 with more than 96,000 terms. If we use the polynomial g_3', then we get an expression with more than 70,000 terms. If we solve the following system of equations

$$
\begin{aligned}
(f_x + \lambda_1 r_{1x} - p_{1x})\mu_1 &= \lambda_1 r_{1x} - \lambda_2 r_{2x} \\
(f_y + \lambda_1 r_{1y} - p_{1y})\mu_1 &= \lambda_1 r_{1y} - \lambda_2 r_{2y}
\end{aligned}
$$

then the expression we get for λ_2 has more than 92,000 terms. Due to space constraints, it has become infeasible for us to give a closed-form expression for a solution of this problem.

However, given an instance of this problem, we can solve it in practice by computing λ_2 by using the above system of equations with the specific numerical values of the instance. Then we can compute λ_3 by solving

$$
\begin{aligned}
(f_x + \lambda_2 r_{2x} - p_{2x})\mu_2 &= \lambda_2 r_{2x} - \lambda_3 r_{3x} \\
(f_y + \lambda_2 r_{2y} - p_{2y})\mu_2 &= \lambda_2 r_{2y} - \lambda_3 r_{3y}
\end{aligned}
$$

and then determine λ_4 by solving

$$\begin{aligned}(f_x + \lambda_3 r_{3x} - p_{3x})\mu_3 &= \lambda_3 r_{3x} - \lambda_4 r_{4x} \\ (f_y + \lambda_3 r_{3y} - p_{3y})\mu_3 &= \lambda_3 r_{3y} - \lambda_4 r_{4y}.\end{aligned}$$

We have proven the following theorem.

Theorem 5.3.1 *For $f, r_1, r_2, r_3, r_4, p_1, p_2, p_3, p_4 \in \mathbb{R}^2$, there is an algorithm that will compute the unique quadrilateral, if it exists, with vertices $v_i = f + \lambda_i r_i$ where the edges $[v_1, v_2]$, $[v_2, v_3]$, $[v_3, v_4]$, and $[v_4, v_1]$ contains the points p_1, p_2, p_3, and p_4 respectively.*

CHAPTER VI

COMPUTATIONAL RESULTS ON 2-ROW CUTS

6.1 Closures

Suppose A is an $m \times n$ integral matrix and $b \in \mathbb{Z}^m$. If

$$P = \{x \geq 0 : Ax \leq b\},$$

then it is well-known that

$$P_I = \mathrm{conv}(P \cap \mathbb{Z}^n)$$

is polyhedral. Recall that the Chvátal-Gomory procedure takes a vector $u \in \mathbb{R}^n$ with $u \geq 0$ and produces the inequality

$$\lfloor u^T A \rfloor x \quad \leq \quad \lfloor u^T b \rfloor \tag{8}$$

which is valid for P_I. Inequality (8) is said to have a Chvátal-Gomory rank of 1. The higher rank inequalities are derived recursively in that an inequality with rank $k \geq 2$ is derived using the Chvátal-Gomory procedure on a system containing all inequalities with rank less than k. If we add to P all possible Chvátal-Gomory inequalities that are obtained directly from the formulation (i.e. all the rank 1 inequalities), then the resulting set

$$P_1 = \{x \geq 0 : Ax \leq b, \lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor \text{ for all } u \geq 0\}$$

is called the *first Chvátal closure* and was shown to be a polyhedral set by Chvátal [18]. The closure is also sometimes termed *elementary*. Observe that

$$P_I \subseteq P_1 \subseteq P.$$

In the case of the matching polytope, it is known by a famous result of Edmonds that $P_I = P_1$.

In general, we have that $P_1 \subsetneq P$ and so Fischetti and Lodi [31] considered minimizing cx over P_1 in order to get a tighter bound on the optimal objective value of the integer program

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \\ & x \text{ integer.} \end{aligned}$$

Essentially, they were interested in how practical it is to approximate P_I by P_1.

Recall the equivalence between separation and optimization for polyhedra using the ellipsoid method by Grötschel, Lovász, and Schrijver [40]. The separation problem here is difficult. Given an arbitrary x (which we may assume is in P), Eisenbrand [29] showed in 1999 that it is NP-hard to either find a $u \geq 0$ with $u \in \mathbb{R}^m$ such that

$$\lfloor u^T A \rfloor x > \lfloor u^T b \rfloor$$

or determine that no such u exists. In 2003, Caprara and Letchford [16] strengthened Eisenbrand's result and in addition, showed the strong NP-completeness of separating split cuts [19], MIR-inequalities [46] and other inequalities.

Fischetti and Lodi deal with this difficulty by formulating the rank 1 Chvátal-Gomory separation problem as a mixed-integer program and solving this MIP at each iteration using a solver. Given an x^* that needs to be separated, they solve

$$\begin{aligned} \max \quad & \alpha^T x^* - \alpha_0 \\ \text{s.t.} \quad & \alpha_j \leq u^T A_j \text{ for } j = 1, \ldots, n \\ & \alpha_0 \geq u^T b - 1 + \epsilon \\ & u \geq 0 \\ & \alpha_j \text{ integer for } j = 0, \ldots, n. \end{aligned}$$

where A_j is the j-th column of A and $\epsilon > 0$ is a small fudge factor that prevents $\alpha = u^T b - 1$ when $u^T b$ is integral. The objective function is chosen so that the resulting cut is maximally violated by x^*.

In their computational experiment, Fischetti and Lodi found that points can be separated from the Chvátal closure in practice. For many of the pure-integer instances in MIPLIB 3.0 and MIPLIB 2003 that they considered, a decent percentage of the integrality gap was closed. In addition, Fischetti and Lodi were able to solve the difficult `nsrand-ipx` instance by cut preprocessing and may have found a new class of facets of the Asymmetric Traveling Salesman Problem by applying their separation procedure to a particular TSPLIB instance and analysis and clique lifting of one of the resulting Chvátal-Gomory cuts.

Fischetti and Lodi originally presented their approach at the 2005 IPCO conference and following this, a number of other researchers considered other closures for MIP problems. Bonami, Cornuéjols, Dash, Fischetti and Lodi[12] were subsequently interested in whether a similar result could be found for the mixed integer case.

The intersection of all Gomory mixed integer cuts with the non-negative orthant is known as the *Gomory mixed integer closure*. It was shown by Nemhauser and Wolsey [46] in 1990 that this closure is identical to the split closure. However, Fischetti and Lodi's approach cannot be directly applied. The separation problem is NP-hard and does not have a known MIP formulation. Its solution involves solving a non-linear MIP or a parametric mixed integer linear program. So instead, Bonami et al. take the LP relaxation of the problem they want to solve, project it onto the integer variables and then determine Chvátal-Gomory cuts for the resulting system.

Suppose that the MIP that is desired to be solved is

$$\begin{aligned} \min \quad & cx + fy \\ \text{s.t.} \quad & Ax + Cy \leq b \\ & x \geq 0 \\ & x \text{ integer} \\ & y \geq 0 \end{aligned}$$

where $A \in \mathbb{Q}^{m \times n}$, $C \in \mathbb{Q}^{m \times r}$, $c \in \mathbb{Q}^n$, $f \in \mathbb{Q}^r$, and $b \in \mathbb{Q}^m$. Then the LP relaxation is

$$P(x, y) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^r : Ax + Cy \leq b, x, y \geq 0\}$$

and the integer hull is

$$P_I(x, y) = \text{conv}\{(x, y) \in P(x, y) : x \in \mathbb{Z}^n, x \geq 0\}.$$

If the extreme rays of the cone

$$\{u \in \mathbb{R}^m : uC \geq 0, u \geq 0\}$$

are $u_1, \ldots, u_K$, then the projection of $P(x, y)$ onto the integer variables is

$$\begin{aligned} P(x) &= \{x \in \mathbb{R}^n_+ : Ax + Cy \leq b \text{ for some } y \in \mathbb{R}^r, y \geq 0\} \\ &= \{x \in \mathbb{R}^n_+ : u^k Ax \leq u^k b \text{ for } k = 1, \ldots, K\} \\ &= \{x \in \mathbb{R}^n_+ : \overline{A}x \leq \bar{b}\}. \end{aligned}$$

Now a *projected Chvátal-Gomory cut* is simply just a Chvátal-Gomory cut obtained from

$$\overline{A}x \leq \bar{b}, x \geq 0.$$

Equivalently, a projected Chvátal-Gomory cut can be found by taking

$$\lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor$$

for $u \geq 0$ satisfying $u^T C \geq 0$. In this case, then the separation problem can be handled by solving the MIP that Fischetti and Lodi solved with the additional constraint

$$u^T C_j \geq 0$$

for all $j = 1, \ldots, r$ where C_j is the j-th column of C.

For their computational experiment, Bonami et al. considered the mixed-integer instances from MIPLIB 3.0 and instances of the asymmetric traveling salesman problem with time windows (TW-ATSP). They argue that projected Chvátal-Gomory cuts would perform well on mixed-integer problems where the continuous variables have zero coefficient in the objective function, which accounts for their interest in TW-ATSP. For 41 mixed instances in MIPLIB 3.0 where `dsbmip` and `noswot` are excluded, the average gap closed was around 29%. On some instances, no projected Chvátal-Gomory cut could be found and on others, a large percentage of the gap was closed. On the TW-ATSP problems, a substantial percentage of the integrality gap was closed.

In 2008, Balas and Saxena [7] considered optimizing over the elementary split closure. Suppose the MIP in question is

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax \geq b \\ & x_j \text{ integer for } j \in N_1 \end{aligned}$$

where $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ and $N_1 \subseteq N = \{1, \ldots, n\}$. The constraints $Ax \geq b$ are assumed to contain any non-negativity constraints and upper-bound constraints. Then, if

$$P = \{x \in \mathbb{R}^n : Ax \geq b\},$$

the LP relaxation is

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & x \in P \end{aligned}$$

and the integer hull is

$$P_I = \text{conv}(\{x : x_j \in \mathbb{Z}, j \in N_1\} \cap P).$$

Now if $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$ where $\pi_j = 0$ for $j \in N \backslash N_1$, then the disjunction

$$\pi x \leq \pi_0 \ \vee \ \pi x \geq \pi_0 + 1$$

is satisfied by any feasible x. This is known as the *split disjunction*. If

$$\begin{aligned} \Pi_1 &= P \cap \{x : \pi x \leq \pi_0\} \\ \Pi_2 &= P \cap \{x : \pi x \geq \pi_0 + 1\} \end{aligned}$$

then

$$P_I \subseteq \Pi_1 \cup \Pi_2.$$

An inequality valid for $\Pi_1 \cup \Pi_2$ for some $(\pi, \pi_0) \in \mathbb{Z}^n \times Z$ is called a *split cut*. Here, a split cut will arise from the disjunction

$$\left\{ \begin{array}{rcl} Ax & \geq & b \\ -\pi x & \geq & -\pi_0 \end{array} \right\} \bigvee \left\{ \begin{array}{rcl} Ax & \geq & b \\ \pi x & \geq & \pi_0 + 1 \end{array} \right\}.$$

Split cuts that are directly obtainable from the above disjunction are *rank 1* or *elementary* split cuts. The intersection of all split cuts is a polyhedron called the *elementary split closure*, or simply the *split closure*. When P is a rational polyhedron, Cook, Kannan and Schrijver [19] showed that its split closure is rational. Recently, Dash, Günlük and Lodi [24] and Vielma [48] have given alternative proofs of this fact. Recall that Caprara and Letchford [16] have shown the strong NP-completeness of separation for split cuts.

Many well-known inequalities can be viewed as split cuts. The lift-and-project inequalities [5] by Balas, Ceria and Cornuéjols are split cuts since the "lift-and-project" procedure applied to a $0-1$ variable x_j can also be derived from the disjunction

$$x_j \leq 0 \ \vee \ x_j \geq 1.$$

GMI inequalities, K-cuts, and MIR inequalities and others can also be viewed as split cuts.

If $\alpha x \geq \beta$ is a split cut, then there exist $u, u_0, v, v_0 \geq 0$ such that

$$\begin{aligned} \alpha &= uA - u_0\pi \\ &= vA + v_0\pi \\ \beta &= ub - u_0\pi_0 \\ &= vb + v_0(\pi_0 + 1) \end{aligned}$$

If at some point in the cutting plane algorithm we have a fractional point $\hat{x}$, then the separation problem to be solved is

$$\begin{aligned} \min \quad & \alpha\hat{x} - \beta \\ \text{s.t.} \quad & uA - u_0\pi = \alpha \\ & vA + v_0\pi = \alpha \\ & ub - u_0\pi_0 = \beta \\ & vb + v_0(\pi_0 + 1) = \beta \\ & u_0 + v_0 = 1 \\ & \pi_j = 0 \text{ for } j \in N \backslash N_1 \\ & (\pi, \pi_0) \in \mathbb{Z}^n \times Z \\ & u, u_0, v, v_0 \geq 0. \end{aligned}$$

The constraint

$$u_0 + v_0 = 1$$

serves as a normalization constraint. If the optimal solution has a non-negative objective, then we have proven than $\hat{x}$ is in the elementary split closure. Otherwise, we have found a split cut that is violated by $\hat{x}$.

Observe however that this problem contains products of continuous and integer variables. By setting the two expressions for α equal to each other, and doing the same for β, rewriting the objective function, and using the normalization constraint, the problem can be rewritten

$$\begin{aligned} \min \quad & (uA - u_0\pi)\hat{x} - (ub - u_0\pi_0) \\ \text{s.t.} \quad & uA - vA - \pi = 0 \\ & -ub + vb + \pi_0 = u_0 - 1 \\ & \pi_j = 0 \text{ for } j \in N \setminus N_1 \\ & (\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z} \\ & u, v \geq 0 \\ & 0 \leq u_0 \leq 1 \end{aligned}$$

The problem is now a mixed integer linear program with the single parameter u_0 which occurs in the right hand side and in the objective function. Caprara and Letchford [16] also formulated an optimization problem for finding a violated split cut, but the form of their split cut, the disjunction used and the normalization constraint was slightly different. However, Dash, Günlük and Lodi [24] have shown that the set of optimal solutions are identical.

By defining $\hat{s}$ to be the surplus in the constraint $Ax \geq b$ from $\hat{x}$, and using an algebraic trick reminiscent of Gaussian elimination, the objective function can be rewritten

$$(v_0u + u_0v)\hat{s} - u_0v_0$$

without π and π_0. If u_0 is renamed θ, then $v_0 = 1 - \theta$ and the separation problem can be rewritten

$$\begin{aligned} \min \quad & z(\theta) = (1-\theta)u\hat{s} + \theta v\hat{s} - \theta(1-\theta) \\ \text{s.t.} \quad & uA - vA - \pi = 0 \\ & -ub + vb + \pi_0 = u_0 - 1 \\ & \pi_j = 0 \text{ for } j \in N \setminus N_1 \\ & (\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z} \\ & u, v \geq 0 \\ & 0 \leq \theta \leq 1. \end{aligned}$$

By non-negativity, it follows that $z(\theta) \geq -\theta(1-\theta)$ for all $0 \leq \theta \leq 1$. In addition, if (u, v, π, π_0) is a solution for the parameter θ, then $(v, u, -\pi, -\pi_0 - 1)$ is a solution for the parameter $1 - \theta$, and so it follows that

$$\min z(\theta) = \min z(1-\theta)$$

for all $0 \leq \theta \leq 1$. This symmetry in the problem allows Balas and Saxena to only consider θ in the interval $(0, 1/2]$.

In their computational experiment, Balas and Saxena initially consider

$$\theta \in \{0, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5\}$$

and as needed, introduce new values by taking the midpoint of two adjacent values. The parametric mixed integer program constituting the separating problem is not solved to full optimality. However, any feasible point of the separation problem with negative objective value yields a violated rank 1 split cut. Balas and Saxena don't directly take the cut $\alpha x \geq \beta$ from the solution of the separation problem, and instead use the disjunction and derive a lift-and-project cut with a different normalization constraint. In addition, they also tighten the separation problem by adding an integer "rounding" constraint and also impose another condition derived from a set-covering problem based on the disjunctions found so far.

On the instances in MIPLIB 3.0, strong bounds were obtained in their computational results. Balas and Saxena closed on average more than 72% of the duality gap on the 41 mixed integer instances in MIPLIB 3.0 with 15 instances having more than 98% of the gap closed. They also closed about 72% of the gap on average on the 24 pure integer instances. They also obtained results on a number of network, location and lot-sizing problems.

In 2010, Dash, Günlük and Lodi [24] considered the mixed-integer rounding closure of polyhedral sets. The MIR inequality was first introduced by Nemhauser and Wolsey [45, 46] in 1988 via the mixed-integer rounding procedure. Wolsey [49] subsequently defined the MIR inequality differently in his 1998 textbook on integer programming (See also Marchand and Wolsey [43]). It has been observed by Dash, Günlük and Lodi [24] and also by Bonami and Cornuéjols [11] that the closures from the two different definitions are not identical.

Dash et al. used the earlier definition in their work. In the earlier definition, the closure is identical to the Gomory mixed-integer closure and the split closure, and hence the difficulty of the separation problem follows from Caprara and Letchford [16]. In the later definition, the closure is in general larger. Dash et al. were motived by the results of Fischetti and Lodi [31] and the containment of the MIR closure in the first Chvátal closure.

If the mixed integer set is

$$P = \{(v, x) \in \mathbb{R}^{|J|} \times \mathbb{Z}^{|I|} : Cv + Ax \geq b, v, x \geq 0\},$$

then from the typical technique of combining variables to get the form of the basic mixed-integer set, applying the basic mixed-integer inequality and taking the strongest inequality, and then aggregating constraints using $\lambda \in \mathbb{R}^m$ for $\lambda \geq 0$, one obtains the MIR inequality

$$(\lambda C)^+ v + (-\lambda)^+ (Cv + Ax - b) + \min\{\lambda A - \lfloor \lambda A \rfloor, r1\}x + r\lfloor \lambda A \rfloor x \geq r\lceil \lambda b \rceil$$

where $(\cdot)^+ = \max\{0, \cdot\}$ and $r = \lambda b - \lfloor \lambda b \rfloor$. If P is in equality form, then for

$$\tilde{C} = (C, -I) \text{ and } \tilde{v} = (v, Cv + Ax - b),$$

the MIR inequality becomes

$$(\lambda \tilde{C})^+ \tilde{v} + \min\{\lambda A - \lfloor \lambda A \rfloor, r1\}x + r\lfloor \lambda A \rfloor x \geq r\lceil \lambda b \rceil.$$

Dash et al. define the notion of a *relaxed MIR inequality*. It is a somewhat technical definition that we have not seen elsewhere. If $\lambda \in \mathbb{R}^m, c^+ \in \mathbb{R}^l, \hat{\alpha} \in \mathbb{R}^n, \bar{\alpha} \in \mathbb{Z}^n, \hat{\beta} \in \mathbb{R}$, and $\bar{\beta} \in \mathbb{Z}$ satisfy

$$c^+ \geq \lambda C \tag{9}$$

$$\hat{\alpha} + \bar{\alpha} \geq \lambda A \tag{10}$$

$$\hat{\beta} + \bar{\beta} \leq \lambda b \tag{11}$$

$$c^+ \geq 0 \tag{12}$$

$$\hat{\alpha}, \hat{\beta} \in [0, 1] \tag{13}$$

then

$$c^+ v + (\hat{\alpha} + \bar{\alpha})x \geq \hat{\beta} + \bar{\beta} \tag{14}$$

is valid for the LP relaxation of P as it is a relaxation of the aggregation

$$\lambda C v + \lambda A x = \lambda b.$$

Now from (14), the inequality

$$c^{+} v + \hat{\alpha} x + \hat{\beta} \bar{\alpha} x \geq \hat{\beta}(\bar{\beta} + 1)$$

can be derived and is known as the relaxed MIR inequality using the base inequality (14).

Dash et al. show that a point in the LP relaxation satisfies all MIR inequalities if and only if all relaxed MIR inequalities are satisfied. If Π denotes the set of all $(\lambda, c^{+}, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta})$ which satisfy the constraints (9)-(13), then let $\bar{\Pi}$ be the projection of Π onto the $c^{+}, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}$ variables. The MIR closure of P can then be described as exactly those points (v, x) in the LP relaxation of P which satisfy the inequality (14) for all

$$(c^{+}, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \bar{\Pi}.$$

Now it is possible to test whether a point (v^*, x^*) is in the MIR closure by solving

$$\begin{aligned} \max \quad & -(c^{+} v^* + \hat{\alpha} x^* + \hat{\beta} \bar{\alpha} x^*) + \hat{\beta}(\bar{\beta} + 1) \\ \text{s.t.} \quad & (\lambda, c^{+}, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi \end{aligned}$$

which is a non-linear mixed integer program. When the optimal value is positive, the solution yields a maximally-violated MIR inequality. When the optimal value is non-positive, the point (v^*, x^*) is contained in the MIR closure.

In their computational experiment, Dash et al. linearize the objective of the separation problem using binary variables and solve it approximately at each iteration to approximately optimize over the MIR closure. They employ some heuristics to help find MIR cuts separation problem, such as using the RINS heuristic [22] after every 100 nodes. Their results were more or less comparable to that of Balas and Saxena [7] and used less computing time.

For 0-1 mixed integer programs, Bonami and Minoux [13] have studied optimizing over the rank 1 lift-and-project closure. On the 0-1 problems in MIPLIB 3.0 and some multi-dimensional 0-1 knapsack problems, they found that rank 1 lift-and-project closure was "computationally promising" compared with mixed-integer Gomory cuts and MIR cuts.

6.2 Balas-Jeroslow Lifting

One of the problems with the current 2-row theory is the presence of non-basic integer variables. Hence, it is of interest to strengthen their coefficients as much as possible. In 1980, Balas and Jeroslow [6] gave a method for strengthening the coefficients of integer variables in pure and mixed-integer programs. The coefficients of the continuous variables are not changed by the method.

Suppose that $y \in \mathbb{R}^q$ satisfies

$$y \in T \text{ and } y = a_0 + \sum_{j \in J} a_j t_j \tag{15}$$

where $T \subseteq \mathbb{R}^q, a_0 \in \mathbb{R}^q, a_j \in \mathbb{R}^q$ and $t_j \geq 0$ for $j \in J = \{1, \ldots, n\}$. The conditions (15) are said to imply

$$\sum_{j \in J} \pi_j(a_j) t_j \geq \pi_0$$

if this inequality holds for all $t \in \mathbb{R}^n$ satisfying (15).

Recall from geometry that the Minkowski sum $A + B$ of two non-empty sets A and B in Euclidean space is defined to be

$$A + B = \{a + b : a \in A, b \in B\}.$$

From algebra, a set with an associative, closed operation and an identity element with respect to the operation is called a *monoid*. If M is a set of vectors and forms a monoid, then $0 \in M$ and for any $u, v \in M$, we have $u + v \in M$. In addition, we have that $M + M = M$.

Suppose J is the disjoint union of J_1 and $J_2 = J \backslash J_1$ where t_j is integer constrained for $j \in J_1 \subseteq J$. In addition, suppose that for some monoid M, $T + M$ can replace T in (15). Under these conditions, the inequality can be strengthened. Suppose $m_j \in M$ for $j \in J_1$. Since t_j is a non-negative integer for $j \in J_1$, then

$$\sum_{j \in J_1} m_j t_j \in M$$

and so

$$y + \sum_{j \in J_1} m_j t_j \in (T + M) + M = T + M.$$

Now

$$y + \sum_{j \in J_1} m_j t_j = a_0 + \sum_{j \in J_1} (a_j + m_j) t_j + \sum_{j \in J_2} a_j t_j$$

and so

$$\sum_{j \in J_1} \pi_j(a_j + m_j) t_j + \sum_{j \in J_2} \pi_j(a_j) t_j \geq \pi_0.$$

Since the $m_j \in M$ for $j \in J_1$ were arbitrary, this is the straightforward proof of the following theorem.

Theorem 6.2.1 *(Balas and Jeroslow, 1979) If $y \in T$ and $y = a_0 + \sum_{j \in J} a_j t_j$ for $T \subseteq \mathbb{R}^q, a_0 \in \mathbb{R}^q, a_j \in \mathbb{R}^q$ and $t_j \geq 0$ for $j \in J = \{1, \ldots, n\}$ imply $\sum_{j \in J} \pi_j(a_j) t_j \geq \pi_0$, then adding the condition that t_j is integral for $j \in J_1 \subseteq J$ implies*

$$\sum_{j \in J_1} \left\{ \inf_{m \in M} \pi_j(a_j + m) \right\} t_j + \sum_{j \in J_2} \pi_j(a_j) t_j \geq \pi_0.$$

By letting $q = 1$ and T and M be the set of integers in the theorem, Gomory's mixed-integer cut can be derived although we do not get into the derivation here. Now consider the following disjunctive program

$$\bigvee_{i \in Q} \left\{ \begin{array}{rcl} A^i t & \geq & a_0^i \\ t & \geq & 0 \end{array} \right\}$$

where A^i has dimension $r_i \times n$ and $a_0^i \in \mathbb{R}^{r_i}$. The jth column of A^i is denoted by a_j^i.

Without loss of generality, suppose that for each $i \in Q$, there exists some $t \geq 0$ such that $A^i t \geq a_0^i$. In general, the convex hull of the union of polyhedra need not be closed, and hence is not necessarily a polyhedral set. The valid inequalities for disjunctions of polyhedra are easily characterized. The following theorem was shown by Balas in 1975 [4].

Theorem 6.2.2 *If $\{x \in \mathbb{R}^n : A^i t \geq a_0^i, t \geq 0\} \neq \emptyset$ for all $i \in Q$, then*

$$\sum_{j \in J} \alpha_j t_j \geq \alpha_0$$

is valid for the convex hull of

$$\bigcup_{i \in Q} \{x \in \mathbb{R}^n : A^i t \geq a_0^i, t \geq 0\}$$

if and only if there exists $\theta^i \geq 0$ *for* $i \in Q$ *such that*

$$\alpha_j \geq \sup_{i \in Q} \ \theta^i a^i_j \text{ for } j \in J \text{ and } \alpha_0 \leq \inf_{i \in Q} \ \theta^i a^i_0.$$

6.3 Prior experiments

In 2008, Espinoza performed a computational study of multi-row cuts that appeared in the IPCO conference [30]. Because a characterization of maximal lattice-free convex bodies is known only in two dimensions, Espinoza considered three families in his computational experiments. The first family $T1_n$ is simply a non-unit simplex

$$T1_n = \left\{ x \in \mathbb{R}^n : x \geq 0, \sum x_i \leq n \right\}.$$

In $\mathbb{R}^2$, $T1_2$ is the familiar Type 1 triangle with vertices $(0,0), (2,0), (0,2)$ and with the points $(0,1), (1,0), (1,1)$ being in the relative interior of the edges. This is a facet as long as f is in the interior. In $\mathbb{R}^3$, $T1_3$ is a sort of "Type 1 tetrahedron" with vertices $(0,0,0), (3,0,0), (0,3,0)$, and $(0,0,3)$.

The second family G_n is the translated hypercube

$$G_n = (1/2, 1/2, \ldots, 1/2) + \left\{ x : \delta^T x \leq n/2 \text{ for } \delta \in \{-1,1\}^n \right\}.$$

In $\mathbb{R}^2$, G_2 is the square with vertices

$$x_1 = (1/2, 3/2), x_2 = (3/2, 1/2), x_3 = (1/2, -1/2), x_4 = (-1/2, 1/2)$$

and the points

$$y_1 = (1,1), y_2 = (1,0), y_3 = (0,0), y_4 = (0,1)$$

in the relative interior of the edges. Now observe that we have

$$\frac{||y_i - x_i||}{||y_i - x_{i+1}||} = \begin{cases} t & \text{for } i = 1,3 \\ 1/t & \text{for } i = 2,4 \end{cases}$$

for $t = 1$, and hence the ratio condition fails to be satisfied. Despite this quadrilateral failing to be a facet, a slight perturbation of it, say by tilting one edge around its integral point, results in a facet. In $\mathbb{R}^3$, G_3 is a regular octahedron.

The third and final family considered is

$$T2'_n = \left\{ x : \sum_{i=1}^{n} x_i \leq 2^n - 1, \sum_{i=1}^{j-1} x_i \leq x_j \text{ for } j = 1, \ldots, n \right\}$$

which has as its vertices $\{v_{k,n}\}_{k=1}^{n+1}$ where

$$(v_{k,n})_i = \begin{cases} 0 & \text{if } i < k \\ 2^k(1 - 2^{-n}) & \text{if } i = k \\ 2^{i-1}(1 - 2^{-n}) & \text{if } i > k \end{cases}$$

On the instances from MIPLIB 3.0, MIPLIB 2003 and the literature that Espinoza considered, the improvements in the LP bound at the root node were not dramatic when found. Another computational result using multi-row cuts is the 2010 IPCO paper by Dey, Lodi, Tramontani and Wolsey [25]. They gave a heuristic for generating lattice-free Type 2 triangles and ran it on random multi-dimensional knapsack instances generated using software from A. Atamturk.

Given three vectors $r^{j_1}, r^{j_2}, r^{j_3}$ whose positive cone is $\mathbb{R}^2$, the idea of their heuristic is to first construct a facet

$$\alpha_{j_1} y_{j_1} + \alpha_{j_2} y_{j_2} \geq 1$$

of the convex hull of the set

$$\{(z, y) \in \mathbb{Z}^2 \times \mathbb{R}^2 : z = r^{j_1} y_{j_1} + r^{j_2} y_{j_2}\}$$

where $r^{j_1}, r^{j_2} \in \mathbb{Q}^2$. The line segment between the points $f + r^{j_1}/\alpha_{j_1}$ and $f + r^{j_2}/\alpha_{j_2}$ is checked to see that it contains at least two integer points by solving some subproblems and using an iterative process. Then the third continuous variable is lifted to obtain the inequality

$$\alpha_{j_1} y_{j_1} + \alpha_{j_2} y_{j_2} + \alpha_{j_3} y_{j_3} \geq 1$$

where either the line segment between $f + r^{j_1}/\alpha_{j_1}$ and $f + r^{j_3}/\alpha_{j_3}$ or the other line segment between $f + r^{j_2}/\alpha_{j_2}$ and $f + r^{j_3}/\alpha_{j_3}$ contains an integral point. Observe that the resulting inequality does not need to be a facet since there is no way to ensure that both line segments contain an integral point. (The triangle heuristic that we derived earlier in Chapter 4 using Harvey's algorithm also suffers from this exact same difficulty.)

Although Dey, Lodi, Tramontani and Wolsey only considered this particular class of two-row cut, they performed a very detailed and in-depth study of its performance. In lieu of the standard MIPLIB benchmarks, they constructed sets of random multi-dimensional knapsack instances designed to elicit answers to questions about the effectiveness of this class of cut. In one set called the A set, all of the basic variables are free and there is a small number of non-basic variables which are non-negative and continuous. In another set called the B set, the setup is the same as in A except that there are additional non-basic variables which are non-negative and integer constrained. The final set called the C set is the same as the B set except that the objective coefficients of the continuous variables are divided by 100 in order to increase the significance of the integer variables.

Because the two-row model assumes that the two integer basic variables are free and the non-basic variables are non-negative, Dey, Lodi, Tramontani and Wolsey also added bounds to some of the instances in the sets B and C to try to discover limitations of the model. They compared the integrality gap closed by one round of GMICs and one round of their heuristic triangles and considered a number of interesting questions about their cuts. They observed that important non-basic integer variables result in poorer performance, but otherwise they essentially found that there is still a lot of work to do.

The final computational result using multi-row cuts that we are aware of is by Basu, Bonami, Cornuéjols and Margot [8] which is to appear in INFORMS Journal on Computing. In their experiment, they considered two rows from the tableau where one of the basic integer variables is integral and the other is fractional and then derived Type 1 and Type 2 triangle cuts. Using the non-negativity of the basic variable that is integer-valued and the integrality of some of the non-basic variables, they also derived expressions to strengthen their cuts. Their computational results showed that their two-row cuts closed less of the gap than GMI cuts.

6.4 Facets of the polyhedron $R_f(r_1, \ldots, r_k)$

Suppose that B_ψ is a maximal lattice-free triangle or a maximal lattice-free quadrilateral and that its vertices are $x_1, \ldots, x_h$. We have $h = 3$ in the triangle case and $h = 4$ in the quadrilateral case. Suppose further that the vertices are ordered, say clockwise on the boundary of B_ψ. The *corner rays* of B_ψ are defined to be the rays $r_j = x_j - f$ for $j = 1, \ldots, h$. Observe that $f + r_j$ is on ∂B_ψ and so $\psi(r_j) = 1$ for $j = 1, \ldots, h$.

Suppose that y_i is an integral point that is on the interior of the edge between x_i and x_{i+1}, where x_{h+1} is considered to be x_1. So we have $\alpha x_1 + (1-\alpha)x_2 = y_1$ for some $0 < \alpha < 1$. Similarly, for some β, γ with $0 < \beta, \gamma < 1$, we can obtain y_2 and y_3 as a convex combination of the appropriate x_i, and in the quadrilateral case, we can obtain y_4 for some $0 < \delta < 1$.

Now let X be the $2 \times h$ matrix where the i-th column is the vector x_i, and let Y be the $2 \times h$ matrix where the i-th column is the vector y_i. If S is the $h \times h$ matrix where the i-th column gives the convex combination yielding y_i from x_i and x_{i+1}, then we have $Y = XS$ where

$$S = \begin{pmatrix} \alpha & 0 & 1-\gamma \\ 1-\alpha & \beta & 0 \\ 0 & 1-\beta & \gamma \end{pmatrix}$$

in the triangle case and

$$S = \begin{pmatrix} \alpha & 0 & 0 & 1-\delta \\ 1-\alpha & \beta & 0 & 0 \\ 0 & 1-\beta & \gamma & 0 \\ 0 & 0 & 1-\gamma & \delta \end{pmatrix}$$

in the quadrilateral case. Now let $\overline{X}$ and $\overline{Y}$ denote the matrices X and Y respectively with a row of 1s added to the bottom. These matrices satisfy $\overline{Y} = \overline{X} \cdot S$. If $N(A)$ denotes the nullspace of a matrix A and $C(A)$ denotes the column space of A, then if A is an $m \times n$ matrix and B is an $n \times p$ matrix, recall from linear algebra that the rank of their product can be determined and is

$$\text{rank}(AB) = \text{rank}(B) - \dim(N(A) \cap C(B)).$$

In addition, if the first matrix has full column rank, then its nullspace has zero dimension and $\text{rank}(AB) = \text{rank}(B)$. For a triangle B_ψ, the matrices $\overline{X}, \overline{Y}$ have rank 3 and since $\overline{Y} = \overline{X}S$, S has rank 3 and the columns of S must be affinely independent. The columns of S satisfy $\sum \psi(r_i)s_i \geq 1$ with equality and so we have a facet of $R_f(r_1, r_2, r_3)$.

The quadrilateral case is slightly more complicated than the triangle case. In this case, both the matrices $\overline{X}$ and $\overline{Y}$ have rank 3. Since $\overline{X}$ has 4 columns, $N(\overline{X})$ has dimension 1 and is a line through the origin. We have

$$\text{rank}(\overline{Y}) = \text{rank}(\overline{X}S) = \text{rank}(S) - \dim(N(\overline{X}) \cap C(S))$$

and we want $\dim(N(\overline{X}) \cap C(S)) = 1$ and this happens if and only if $N(\overline{X}) \subseteq C(S))$. By a theorem of Lovász, y_1, y_2, y_3 and y_4 are the vertices of a parallelogram (having area 1) and so there exists $c, d_1, d_2 \in \mathbb{R}^2$ such that

$$(y_1, y_2, y_3, y_4) = (c + d_1, c + d_2, c - d_1, c - d_2).$$

Hence, if $u = (1, -1, 1, -1)$, then $\overline{Y}u = 0$ and we have that $\overline{X}Su = 0$. If we solve this system, we get that $\alpha = \gamma, \beta = 1 - \gamma$, and $\delta = 1 - \gamma$. In Mathematica, this can be done using

```
S = {{alpha, 0, 0, 1 - delta}, {1 - alpha, beta, 0, 0},
     {0, 1 - beta, gamma, 0}, {0, 0, 1 - gamma, delta}};
Solve[Thread[S.{1, -1, 1, -1} == {0, 0, 0, 0}]]
```

The solution of the linear system is precisely the quadrilateral ratio condition.

In the other case where $Su \neq 0$, we have that $N(\overline{X})$ is the line through the origin with direction Su. Now Su is necessarily in $C(S)$ and so $N(\overline{X}) \subseteq C(S)$ and we have $\text{rank}(S) = 4$. The columns of S are affinely independent points that satisfy $\sum \psi(r_i)s_i = 1$ and hence $\sum \psi(r_i)s_i \geq 1$ is a facet of $R_f(r_1, r_2, r_3, r_4)$.

Now that we have covered the "base case" to sort of speak, we can discuss a certain dimension reduction technique. If $f, r_1, r_2, \ldots, r_k \in \mathbb{Q}^2$ and B_ψ is either a split, a maximal lattice-free triangle or maximal lattice-free quadrilateral, Cornuéjols and Margot [21] have a characterization of the facets of $R_f(r_1, \ldots, r_k)$ that can algorithmically determine if B_ψ is a facet or not. Their algorithm is called the *Reduction Algorithm*. The first step in their algorithm is to start from the point f and shoot rays in the direction of $r_1, r_2, \ldots, r_k$ and find where they strike the boundary of B_ψ.

Let p_j be the intersection of the ray $\{f + \lambda r_j : \lambda \geq 0\}$ with ∂B_ψ. p_j is called the *boundary point* for r_j and the set $P = \{p_1, \ldots, p_k\}$ are the boundary points. A boundary point $p_j \in P$ is said to be *active* if there exists $\lambda_i \geq 0$ for $i = 1, \ldots, k$ where $\sum \lambda_i = 1$ and $\lambda_j > 0$ such that $\sum \lambda_i p_i$ is an integral point on ∂B_ψ. An active point p is said to be *uniquely active* if there is only one convex combination of points in P with p having positive coefficient yielding an integral point.

Algorithm 4 The Reduction Algorithm

Find the boundary points $P = \{p_1, \ldots, p_k\}$ for $r_1, r_2, \ldots, r_k$.
while P contains an active point p that can be expressed as a convex combination of the points in $P \backslash \{p\}$ **do**
    Delete p from P.
end while
Remove the uniquely active points from P.
if there are exactly two active points in P **then**
    Remove both of them from P.
end if

The reduction algorithm is shown in Algorithm 4. The algorithm starts with a ψ and a set of vectors $\{r_1, \ldots, r_k\}$ and steadily removes points r_j to reduce the dimension of the problem. When applied to a triangle or quadrilateral, no edge will have more than two active points on it at the end of the algorithm. In addition, every active point at the end of the algorithm will need to be present in two or more convex combinations since uniquely active points are removed. The last step ensures that if there are active points left at the end of the algorithm, there are more than 2 of them.

When the reduction algorithm is applied to some triangle or quadrilateral and P is the empty set at the end of the algorithm, it is said that *the ray condition holds* for the triangle or quadrilateral. In the case of splits, the ray condition is said to hold if P is the empty set at the end of the algorithm or consists of points $\{p_1, q_1, p_2, q_2\}$ such that p_1 and q_1 lie on one of the boundary lines of the split and p_2, q_2 lie on the other boundary line such that there are at least two integer points on the boundary lines between both pair of points.

The steps of the reduction algorithm may not at first glance seem to make much sense, but in fact, they actually do make sense. We now briefly describe the reasoning behind the steps of the algorithm. Suppose that $\{s_1, s_2, \ldots, s_t\}$ is a set of affinely independent points satisfying both $\sum \psi(r_j)s_j \geq 1$ with equality and $x = f + \sum r_j s_j \in \mathbb{Z}^2$. In addition, assume further that the choice is such that t is as large as possible.

Let S be the matrix $[s_1\ s_2\ \cdots\ s_t]$ with dimension $k \times t$, let R be the matrix $[r_1\ r_2\ \cdots\ r_k]$ with dimension $2 \times k$, and let D be the diagonal matrix with $\psi(r_1), \ldots, \psi(r_k)$ on the diagonal. Suppose that $\overline{S}$ is defined to be $D \cdot S$ and $\overline{R}$ is defined to be $R \cdot D^{-1}$. The affine dimension of the column space of S is the same as that of the column space of $\overline{S}$. Now, if $\overline{s}$ is a column vector of $\overline{S}$, we have $\overline{s} \geq 0$ and

$$\sum_{j=1}^{k} \overline{s}_j = \sum_{j=1}^{k} \psi(r_j)s_j = 1.$$

If $\overline{r}$ is a column vector of $\overline{R}$, then observe by the scaling that $\psi(\overline{r}) = 1$ and so $f + \overline{r}$ is on ∂B_ψ. This means that the boundary point for r is exactly $f + \overline{r}$. Summing over the columns, we have

$$\sum_{j=1}^{k} p_j \overline{s}_j = \sum_{j=1}^{k} (f + \overline{r}_j)\overline{s}_j = f\left(\sum_{j=1}^{k} \overline{s}_j\right) + \sum_{j=1}^{k} r_j s_j = f + \sum_{j=1}^{k} r_j s_j$$

which is integral and necessarily an integral point on the boundary of B_ψ. Hence, $\overline{s}$ yields a convex combination of the boundary points resulting in an integral point on the boundary of B_ψ.

Suppose that the reduction algorithm removes from P an active point which can be labeled p_k without any loss of generality. Then, it is immediate that ψ is a facet of $R_f(r_1, \ldots, r_k)$ if $k = 1$ and Cornuéjols and Margot show when $k > 1$, ψ is a face of $R_f(r_1, \ldots, r_k)$ with dimension w if and only if ψ is a face of $R_f(r_1, \ldots, r_{k-1})$ with dimension $w - 1$. In the $k > 1$ case, the argument can actually be viewed in terms of elementary column operations on the matrix $\overline{S}$. The boundary point being active corresponds to being able to find a non-zero entry in the corresponding row of $\overline{S}$.

Cornuéjols and Margot then similarly justify the removal of uniquely active points in the algorithm. That is, if p_k is a uniquely active point removed by the algorithm, then for $k = 1$, ψ is a facet of $R_f(r_1, \ldots, r_k)$ and when $k > 1$, ψ is a face of $R_f(r_1, \ldots, r_k)$ with dimension w if and only if ψ is a face of $R_f(r_1, \ldots, r_{k-1})$ with dimension $w - 1$. In addition, the final step of the Reduction Algorithm is justified since if p_1 and p_2 are active points removed at the end, ψ is a facet of $R_f(r_1, r_2)$. By a straightforward contradiction argument, Cornuéjols and Margot have shown that when the reduction algorithm is applied to a polytope B_ψ, the active points in P at the end of the algorithm must be either the vertices of B_ψ or the empty set.

Given some ψ, the idea of the reduction algorithm is to just recursively keep reducing the dimension knowing that the facetness of ψ cannot be lost if points r_j are removed appropriately. When points are removed by the algorithm, the dimension of the problem and the affine dimension of the column space of the modified matrix $\overline{S}$ are reduced equally. If at the end of the algorithm there is an inactive boundary point, then we know that ψ is not a facet of $R_f(r_1, \ldots, r_k)$. This corresponds to an all-zero row of the matrix that is obtained from $\overline{S}$. Suppose that $P' = \{p_{i_1}, \ldots, p_{i_{k'}}\}$ is the set of boundary points remaining at the end of the reduction algorithm. In the case of triangles or quadrilaterals, we have that ψ is a facet of $R_f(r_1, \ldots, r_k)$ if and only if $P' = \emptyset$ or all the points in P' are active and ψ is a facet of $R_f(r_{i_1}, \ldots, r_{i_{k'}})$. In the latter case where all the points in P' are active, they must be the vertices of B_ψ and we are then reduced to the earlier "base case."

We can now state the following theorem of Cornuéjols and Margot [21] which describes the facets of $R_f(r_1, \ldots, r_k)$. This description is more precise than what was described by Andersen, Louveaux, Weismantel and Wolsey [1].

Theorem 6.4.1 *The facets of $R_f(r_1, \ldots, r_k)$ consist of*

- *split inequalities parallel to the line $L = \{f + \lambda r_j : \lambda \in \mathbb{R}\}$ for some $j = 1, \ldots, k$ where $L \cap \mathbb{Z}^2 = \emptyset$; or where B_ψ satisfies the ray condition for split inequalities*
- *triangle inequalities such that the vertices of the triangle B_ψ lie on the rays $\{f + \lambda r_{j_i} : \lambda > 0\}$ for some j_1, j_2, j_3; or where B_ψ satisfies the ray condition*
- *quadrilateral inequalities such that the vertices of the quadrilateral B_ψ lie on the rays $\{f + \lambda r_{j_i} : \lambda > 0\}$ for some j_1, j_2, j_3, j_4 and satisfies the quadrilateral ratio condition*

When the point f lies on the boundary of $\mathrm{cl}B_\psi$, B_ψ is considered to be *degenerate*. The point f may be on the interior of one of the edges of $\mathrm{cl}B_\psi$ or f may be a vertex of $\mathrm{cl}B_\psi$.

For splits, degeneracy cannot occur at a vertex, but for triangles and quadrilaterals, we can have both vertex and edge degeneracy. In an implementation, degenerate cases are not desirable since the corresponding ψ is then not finite everywhere. When it comes to facets of $R_f(r_1, \ldots, r_k)$ however, the degenerate cases are not needed as proved by Cornuéjols and Margot [21]. In the case of two dimensions, the proof is relatively straightforward and given a degenerate minimal function (which necessarily falls into one of five possible cases), they find a nondegenerate minimal valid function that is equivalent in the directions $r_1, \ldots, r_k$. Zambelli [50] has shown that it is also true that for the general case that degenerate inequalities are not needed to define the facets of $R_f(r_1, \ldots, r_k)$.

For R_f, which is the Gomory and Johnson relaxation of $R_f(r_1, \ldots, r_k)$ yielding a problem with two integer variables and two constraints having infinite dimension, Cornuéjols and Margot [21] have shown that some degeneracy is required. All degenerate split inequalities and some degenerate triangle inequalities are facets. Since we concentrate on $R_f(r_1, \ldots, r_k)$, we won't discuss R_f any further.

6.5 Our experiments

For a cutting plane algorithm using a class of inequalities, an important question is the ease of generating the inequalities. As Caprara and Letchford showed for many classes of inequalities, given a fractional solution of the LP relaxation, it can be very difficult to compute a violated inequality.

Despite the theoretical difficulties, Fischetti and Lodi showed that the separation problem for the first Chvátal closure could be solved in practice. Their results then led to a number of subsequent results by a number of researchers. Motivated by the wealth of results on various closures, our desire was to optimize over the triangle and quadrilateral closures. However, for triangle and quadrilateral cuts, the complexity of separation is not known and there are no published results. Unfortunately, we were unable to formulate an optimization problem to model either triangle or quadrilateral separation. This is in sharp contrast to the relative ease of formulating a model for separating split cuts.

In fact, given a polyhedron, it is not even known whether the triangle closure or quadrilateral closure is even polyhedral. Recall that Basu et al. [9] avoided this issue in their study of how well the split, triangle and quadrilateral cuts approximate the integer hull by generalizing Goemans' theorem so that the relaxation of the integer hull that is being considered need not be polyhedral.

Hence, our approach is somewhat ad-hoc and we are not able to perform a true separation. It would not even be appropriate to call our method a separation heuristic. Using the formulas that we derived in Chapter 5, we performed a computational experiment to study the effectiveness of two row cuts derived from lattice-free triangles and quadrilaterals whose vertices lie on non-basic rays emanating from the fractional point. We also considered triangles computed by the heuristic that we derived in Chapter 4. Recall that the heuristic finds triangles that are "close" to being Type 2 and we explained in that chapter why the theory suggests that Type 2 triangles are of interest.

The code was implemented in C and C++ and we used IBM ILOG CPLEX 12.0 as our solver. We ran our code on Linux 2.6.18 machines with 2.4 GHz Intel Core 2 Quad Q6600 CPUs with 8 GB of RAM. Given the complexity of the formulas derived in Chapter 5, the computations were performed in exact arithmetic using the GNU Multiple Precision arithmetic library whenever possible.

For an instance, let z^*_{UB} denote the value of the optimal solution. If the optimal solution is unknown, we let it denote the value of the best known solution. Let z^*_{LP} denote the value of the LP relaxation and let $z^*_{LP+cuts}$ denote the value of the LP relaxation with the cuts added. Then the integrality gap closed is defined to be

$$100 \cdot \frac{z^*_{LP+cuts} - z^*_{LP}}{z^*_{UB} - z^*_{LP}}$$

and is the primary measure of performance for our experiments.

In the exact triangle and exact quadrilateral cuts that we derive for the computational experiment, we consider integer points whose L_1 distance from f is at most $\delta = 10$ and $\delta = 100$. Without some constraint on the considered integer points, the number of generated triangles on some instances can be truly out of control.

Algorithm 5 Computational Experiment for Exact Cuts

Solve the LP relaxation of the input mixed-integer program.
while the 4-hour time limit hasn't been reached **do**
  Generate a round of MIR inequalities.
  Generate a round of two-row inequalities.
  Add cuts and re-optimize.
end while

The setup for our experiment with exact triangle and exact quadrilateral cuts is shown in Algorithm 5. We setup the experiment in this way because it would seem reasonable for somebody solving a problem in practice to consider the marginal benefit of two-row cuts for their problem. We performed our experiment with the heuristic triangles differently so that our experimental setup was more similar to that of Dey, Lodi, Tramontani and Wolsey [25]. For these cuts, we considered the gap closed by one round of heuristic triangles with the gap closed by one round of MIRs. We performed the computations with MIRs first and with the number of MIRs generated limited to approximately 500. We then considered the heuristic triangles with the number of triangles approximately limited by the number of MIRs generated for the instance.

The non-basic integer variables are lifted using approximate Balas-Jeroslow lifting with a boxsize of 5. In the experiments, we don't report the separation time simply because it is significantly more expensive to compute these two-row cuts than Gomory cuts. The heuristic triangles, the exact triangles and exact quadrilaterals are all relatively expensive to compute. However, our goal in the first place was not to consider running time but rather the strength of the cuts.

The performance of the exact triangle cuts is shown in Tables 2 and 5. The performance of the exact quadrilateral cuts is shown in Tables 3 and 6. The performance of the heuristic triangle cuts is shown in Tables 4 and 7. The computational results were less than what we had hoped for given the expense and effort expended to compute the cuts. The performance of the quadrilateral cuts was especially poor. We separated out the pure-integer instances of the MIPLIB 3.0 library from the mixed-integer instances so that the poorer performance on the pure-integer instances is more apparent.

For exact triangles and quadrilaterals, the situation is complicated due to our model.

Table 2: Performance of exact triangle cuts on mixed integer instances.

					$\delta = 10$			$\delta = 100$		
Instance	Rows	Cols	Int	0/1	Cuts	Rnds	Gap	Cuts	Rnds	Gap
10teams	230	2025	1800	ALL	0	0	0	0	0	0
arki001	1048	1388	538	415	22	0	0	31	0	0
bell3a	123	133	71	39	183	2	48.7	182	1	45.59
bell5	91	104	58	30	300	5	22.11	321	3	23.86
blend2	274	353	264	231	1335	3	11.76	1417	3	16.18
dano3mip	3202	13873	552	ALL	0	0	0	0	0	0
danoint	664	521	56	ALL	0	0	0	0	0	0
dcmulti	290	548	75	ALL	505	1	34.64	502	1	34.76
dsbmip	1182	1886	192	160	1484	3	–	504	1	–
egout	98	141	55	ALL	945	7	72.01	936	5	72.51
fiber	363	1298	1254	ALL	596	1	53.7	503	1	52.29
fixnet6	478	878	378	ALL	486	1	7.88	494	1	7.88
flugpl	18	18	11	0	12	1	9.61	16	1	10.88
gen	780	870	150	144	598	1	42.39	658	1	42.34
gesa2	1392	1224	408	240	238	1	13.77	439	2	17.17
gesa2_o	1248	1224	720	384	868	1	27.77	1018	2	32.05
gesa3	1368	1152	384	216	1113	2	14.57	1008	2	15.25
gesa3_o	1224	1152	672	336	601	1	20.26	724	1	17.81
khb05250	101	1350	24	ALL	172	1	77.86	172	1	77.86
markshare1	6	62	50	ALL	0	0	0	0	0	0
markshare2	7	74	60	ALL	0	0	0	0	0	0
mas74	13	151	150	ALL	0	0	0	0	0	0
mas76	12	151	150	ALL	0	0	0	0	0	0
misc03	96	160	159	ALL	718	2	10.69	703	1	4.31
misc06	820	1808	112	ALL	683	4	63.14	1110	5	64.01
misc07	212	260	259	ALL	778	1	0.72	774	1	0.72
mkc	3411	5325	5323	ALL	915	1	0.07	939	1	0.07
mod011	4480	10958	96	ALL	140	1	17.41	143	1	17.43
modglob	291	422	98	ALL	324	1	17.65	297	1	17.89
noswot	182	128	100	75	1107	128	–	809	33	–
pk1	45	86	55	ALL	0	0	0	0	0	0
pp08a	136	240	64	ALL	343	1	53.07	366	1	53.07
pp08aCUTS	246	240	64	ALL	124	0	0	121	0	0
qiu	1192	840	48	ALL	5	0	0	7	0	0
qnet1	503	1541	1417	1288	516	1	8.31	501	1	14.09
qnet1_o	456	1541	1417	1288	536	1	18.96	508	1	18.7
rentacar	6803	9557	55	ALL	117	1	2.96	130	1	2.96
rgn	24	180	100	ALL	140	1	0	140	1	0
rout	291	556	315	300	71	0	0	76	0	0
set1ch	492	712	240	ALL	1143	2	61.95	1054	2	60.85
swath	884	6805	6724	ALL	176	0	0	176	0	0
vpm1	234	378	168	ALL	359	6	9.61	489	7	10.04
vpm2	234	378	168	ALL	275	2	16.06	317	2	16.06

Table 3: Performance of exact quadrilateral cuts on mixed integer instances.

					$\delta = 10$			$\delta = 100$		
Instance	Rows	Cols	Int	0/1	Cuts	Rnds	Gap	Cuts	Rnds	Gap
10teams	230	2025	1800	ALL	218	0	0.0	124	0	0.0
arki001	1048	1388	538	415	0	0	0.0	0	0	0.0
bell3a	123	133	71	39	7	0	0.0	16	0	0.0
bell5	91	104	58	30	7	0	0.0	13	0	0.0
blend2	274	353	264	231	0	0	0.0	0	0	0.0
dano3mip	3202	13873	552	ALL	0	0	0.0	0	0	0.0
danoint	664	521	56	ALL	139	0	0.0	78	0	0.0
dcmulti	290	548	75	ALL	25	0	0.0	21	0	0.0
dsbmip	1182	1886	192	160	0	0	–	1	0	–
egout	98	141	55	ALL	4	1	55.9	7	1	55.9
fiber	363	1298	1254	ALL	0	0	0.0	0	0	0.0
fixnet6	478	878	378	ALL	0	0	0.0	0	0	0.0
flugpl	18	18	11	0	285	7	19.4	52	1	12.6
gen	780	870	150	144	21	0	0.0	18	0	0.0
gesa2	1392	1224	408	240	31	0	0.0	27	0	0.0
gesa2_o	1248	1224	720	384	30	0	0.0	13	0	0.0
gesa3	1368	1152	384	216	3	0	0.0	3	0	0.0
gesa3_o	1224	1152	672	336	2	0	0.0	2	0	0.0
khb05250	101	1350	24	ALL	0	0	0.0	0	0	0.0
markshare1	6	62	50	ALL	1738	87	0.0	573	20	0.0
markshare2	7	74	60	ALL	1608	56	0.0	515	13	0.0
mas74	13	151	150	ALL	68	1	7.4	33	0	0.0
mas76	12	151	150	ALL	178	1	7.2	25	0	0.0
misc03	96	160	159	ALL	294	0	0.0	294	0	0.0
misc06	820	1808	112	ALL	6	0	0.0	6	0	0.0
misc07	212	260	259	ALL	0	0	0.0	0	0	0.0
mkc	3411	5325	5323	ALL	0	0	0.0	0	0	0.0
mod011	4480	10958	96	ALL	0	0	0.0	0	0	0.0
modglob	291	422	98	ALL	20	0	0.0	26	0	0.0
noswot	182	128	100	75	142	0	–	140	0	–
pk1	45	86	55	ALL	753	6	0.0	122	1	0.0
pp08a	136	240	64	ALL	80	1	53.9	89	1	53.9
pp08aCUTS	246	240	64	ALL	2	0	0.0	3	0	0.0
qiu	1192	840	48	ALL	970	0	0.0	300	0	0.0
qnet1	503	1541	1417	1288	14	0	0.0	80	0	0.0
qnet1_o	456	1541	1417	1288	0	0	0.0	0	0	0.0
rentacar	6803	9557	55	ALL	0	0	0.0	0	0	0.0
rgn	24	180	100	ALL	132	1	5.6	139	1	5.6
rout	291	556	315	300	5	0	0.0	6	0	0.0
set1ch	492	712	240	ALL	20	0	0.0	18	0	0.0
swath	884	6805	6724	ALL	0	0	0.0	0	0	0.0
vpm1	234	378	168	ALL	36	4	11.6	34	4	11.6
vpm2	234	378	168	ALL	12	0	0.0	13	0	0.0

Table 4: Performance of heuristic triangle cuts on mixed integer instances.

Instance	Rows	Cols	Int	0/1	MIRs	Gap	Δs	Gap
10teams	230	2025	1800	ALL	500	57.1	500	0.0
arki001	1048	1388	538	415	507	41.4	251	0.0
bell3a	123	133	71	39	178	60.4	4	1.1
bell5	91	104	58	30	131	14.5	29	2.5
blend2	274	353	264	231	49	16.4	49	5.3
dano3mip	3202	13873	552	ALL	500	0.1	0	0.0
danoint	664	521	56	ALL	506	1.7	102	0.0
dcmulti	290	548	75	ALL	219	43.8	221	21.1
dsbmip	1182	1886	192	160	219	–	22	–
egout	98	141	55	ALL	120	55.9	0	0.0
fiber	363	1298	1254	ALL	410	67.2	410	32.9
fixnet6	478	878	378	ALL	180	10.9	0	0.0
flugpl	18	18	11	0	34	11.7	36	7.6
gen	780	870	150	144	389	62.6	390	26.4
gesa2	1392	1224	408	240	508	30.5	111	9.8
gesa2_o	1248	1224	720	336	505	22.6	252	14.8
gesa3	1368	1152	384	216	505	37.9	505	6.0
gesa3_o	1224	1152	672	336	502	33.4	363	0.0
khb05250	101	1350	24	ALL	57	74.9	0	0.0
misc03	96	160	159	ALL	95	7.2	98	0.0
misc06	820	1808	112	ALL	67	29.4	67	13.3
misc07	212	260	259	ALL	217	0.7	218	0.7
mod011	4480	10958	96	ALL	48	17.1	48	12.1
modglob	291	422	98	ALL	152	15.9	105	13.3
noswot	182	128	100	75	249	–	128	–
pk1	45	86	55	ALL	150	0.0	153	0.0
pp08a	136	240	64	ALL	263	52.9	105	13.7
pp08aCUTS	246	240	64	ALL	152	30.4	265	8.7
qiu	1192	840	48	ALL	108	2.0	110	0.8
qnet1	503	1541	1417	1288	482	15.8	49	0.0
qnet1_o	456	1541	1417	1288	96	30.8	96	15.4
rentacar	6803	9557	55	ALL	51	26.9	3	0.0
rgn	24	180	100	ALL	72	4.5	72	0.0
rout	291	556	315	300	252	0.3	252	0.2
set1ch	492	712	240	ALL	482	38.1	435	43.1
vpm1	234	378	168	ALL	38	9.5	29	0.4
vpm2	234	378	168	ALL	89	12.6	89	6.7

Table 5: Performance of exact triangle cuts on pure integer instances.

					$\delta = 10$			$\delta = 100$		
Instance	Rows	Cols	Int	0/1	Cuts	Rnds	Gap	Cuts	Rnds	Gap
air03	124	10757	10757	ALL	0	0	0	0	0	0
air04	823	8904	8904	ALL	430	0	0	430	0	0
air05	426	7195	7195	ALL	212	0	0	221	0	0
cap6000	2176	6000	6000	ALL	44	0	0	59	0	0
enigma	21	100	100	ALL	635	8	–	411	5	–
fast0507	507	63009	63009	ALL	32	0	0	28	0	0
gt2	29	188	188	24	92	1	32.23	103	1	35.46
harp2	112	2993	2993	ALL	71	0	0	107	0	0
l152lav	97	1989	1989	ALL	37	0	0	37	0	0
lseu	28	89	89	ALL	50	2	44.66	60	3	43.45
mitre	2054	10724	10724	ALL	106	0	0	128	0	0
mod008	6	319	319	ALL	0	0	0	1	0	0
mod010	146	2655	2655	ALL	501	1	100	501	1	100
nw04	36	87482	87482	ALL	12	0	0	12	0	0
p0033	16	33	33	ALL	35	3	11.07	36	3	11.07
p0201	133	201	201	ALL	227	1	21.02	227	1	21.02
p0282	241	282	282	ALL	273	1	3.96	364	1	4.2
p0548	176	548	548	ALL	127	1	0.82	168	1	0.97
p2756	755	2756	2756	ALL	27	3	0.16	39	4	0.16
seymour	4944	1372	1372	ALL	220	0	0	221	0	0
stein27	118	27	27	ALL	27953	128	0	16131	73	0
stein45	331	45	45	ALL	45112	90	0	23831	48	0

Table 6: Performance of exact quadrilateral cuts on pure integer instances.

					$\delta = 10$			$\delta = 100$		
Instance	Rows	Cols	Int	0/1	Cuts	Rnds	Gap	Cuts	Rnds	Gap
air03	124	10757	10757	ALL	0	0	0.0	0	0	0.0
air04	823	8904	8904	ALL	1	0	0.0	2	0	0.0
air05	426	7195	7195	ALL	15	0	0.0	12	0	0.0
cap6000	2176	6000	6000	ALL	0	0	0.0	0	0	0.0
enigma	21	100	100	ALL	198	2	–	164	1	–
fast0507	507	63009	63009	ALL	3	0	0.0	3	0	0.0
gt2	29	188	188	24	25	1	68.2	13	1	68.2
harp2	112	2993	2993	ALL	0	0	0.0	1	0	0.0
l152lav	97	1989	1989	ALL	161	0	0.0	176	0	0.0
lseu	28	89	89	ALL	67	1	50.4	67	1	50.4
mitre	2054	10724	10724	ALL	0	0	0.0	1	0	0.0
mod008	6	319	319	ALL	149	5	33.8	31	1	20.9
mod010	146	2655	2655	ALL	38	0	0.0	38	0	0.0
nw04	36	87482	87482	ALL	1	0	0.0	1	0	0.0
p0033	16	33	33	ALL	346	7	74.1	99	3	68.2
p0201	133	201	201	ALL	17	0	0.0	17	0	0.0
p0282	241	282	282	ALL	54	0	0.0	68	0	0.0
p0548	176	548	548	ALL	0	0	0.0	1	0	0.0
p2756	755	2756	2756	ALL	1	0	0.0	1	0	0.0
seymour	4944	1372	1372	ALL	1	0	0.0	1	0	0.0
stein27	118	27	27	ALL	3058	13	0.0	2131	9	0.0
stein45	331	45	45	ALL	7242	11	0.0	2116	3	0.0

Table 7: Performance of heuristic triangle cuts on pure integer instances.

Instance	Rows	Cols	Int	0/1	MIRs	Gap	Δs	Gap
air03	124	10757	10757	ALL	275	100.0	0	0.0
air04	823	8904	8904	ALL	500	6.5	59	0.0
air05	426	7195	7195	ALL	500	4.9	3	0.0
cap6000	2176	6000	6000	ALL	20	41.6	22	41.6
enigma	21	100	100	ALL	60	–	60	–
fast0507	507	63009	63009	ALL	500	1.3	6	0.0
gt2	29	188	188	24	103	68.2	5	4.3
harp2	112	2993	2993	ALL	300	23.7	21	0.0
l152lav	97	1989	1989	ALL	500	4.6	0	0.0
lseu	28	89	89	ALL	120	50.4	67	17.1
mitre	2054	10724	10724	ALL	501	0.0	92	0.0
mod008	6	319	319	ALL	50	20.9	29	8.7
mod010	146	2655	2655	ALL	225	100.0	225	100.0
nw04	36	87482	87482	ALL	60	62.3	20	0.0
p0033	16	33	33	ALL	48	55.3	5	6.0
p0201	133	201	201	ALL	96	18.2	179	15.4
p0282	241	282	282	ALL	232	3.7	166	3.2
p0548	176	548	548	ALL	470	39.5	9	0.0
p2756	755	2756	2756	ALL	370	0.5	8	0.0
seymour	4944	1372	1372	ALL	500	0.5	500	0.5
stein27	118	27	27	ALL	42	0.0	39	0.0
stein45	331	45	45	ALL	70	0.0	70	0.0

Given the non-basic columns corresponding to a fractional solution, it may not be possible to find a violated triangle or quadrilateral inequality of the form that we desire. The non-linear system of equations that we considered in Chapter 5 need not have a solution. It is easy to construct an instance where this happens, say by putting one of the integer points very far from f and the rest close to f.

However, it can also be the case that a facet may exist, but simply cannot be found due to our model which requires vertices to lie on non-basic rays emanating from the fractional point f. Consider the following instance

$$f = (1, 1/2), r_1 = (1, -2), r_2 = (-3, -1), r_3 = (-3, 5), r_4 = (1, 1)$$

which is essentially from Cornuéjols and Margot [21]. The triangle that is shown on the left in Figure 25 cannot be found in our model since only two of its vertices lie on rays emanating from f, but the triangle is a facet of $R_f(r_1, \ldots, r_k)$ since the ray condition is satisfied. Two of the boundary points are uniquely active and the other two boundary points are both active and get removed at the last step of the Reduction Algorithm. In

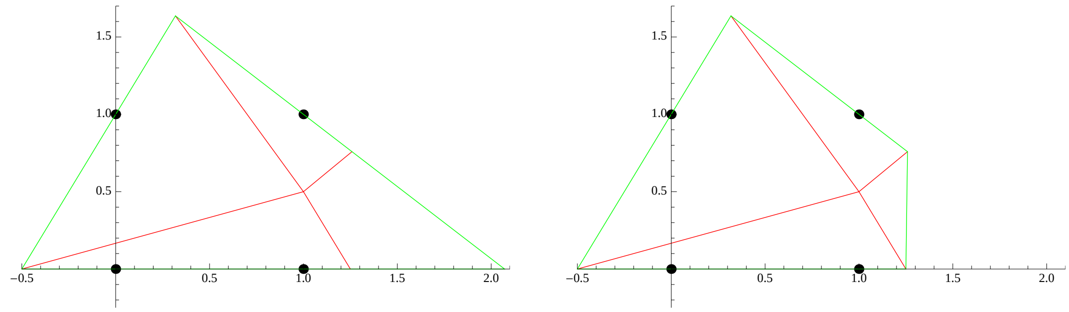

Figure 25: A non-exact facet-defining triangle.

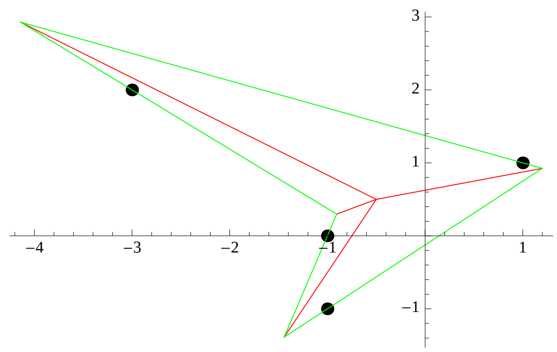

Figure 26: A non-convex quadrilateral.

Andersen, Louveaux, Weismantel and Wolsey [1]'s work, they are able to obtain this facet by considering a lattice-free quadrilateral that is not maximal. This quadrilateral is shown on the right in Figure 25 and defines the same facet as the triangle. However, we are not able to obtain this quadrilateral using our quadrilateral formula and the reason for this is that the right-most edge does not have an integral point in its relative interior.

Even when the derived formulas have a solution, we cannot immediately generate a cut as the systems do not model the convexity of the resulting polygon. We have to check that each computed polygon is in fact convex. For example, the instance

$$f = (-1/2, 1/2), r_1 = (-3, 2), r_2 = (-2, -1), r_3 = (-1, -2), r_4 = (4, 1)$$

gives a quadrilateral that is not convex as shown in Figure 26. (In addition, the quadrilateral is not lattice-free.) There are still a number of issues that are unresolved.

In the case of triangles, our computational experience with them has shown that they are easier to find and in a sense, more plentiful. This is essentially reflective of the fact that it is easier to force a polygon through three points than through four points. Since it is easier to find triangles, we can afford to be picky with them although it is not readily apparent which triangles are the most desirable. Dey, Lodi, Tramontani and Wolsey [25] considered the minimum angle of the triangles in their experiment which seems very reasonable. We believe that more research has to be done to determine which triangles are useful in computations. We rejected the triangles where f is on the boundary or too close to the boundary as we had no interest in dealing with infinite values from degenerate triangles or high coefficients from near-degenerate triangles.

In the case of quadrilaterals, the situation is different and in a sense, they appear to be less plentiful. The quadrilaterals are more difficult to find and so we cannot insist for example that the integer points in the relative interiors of the edges form a quadrilateral of unit area. Quadrilaterals are hard enough to find that we did not even use the ratio condition to disqualify the quadrilaterals that we did find. In our experiments, we still rejected quadrilaterals where f was too close to the boundary however.

Generally speaking, given an (f, r_1, r_2, r_3) instance, we can very often find a large number of exact triangles and approximate Type 2 triangles using our heuristic. However, given an (f, r_1, r_2, r_3, r_4) instance, we have found that the discovery of more than one exact maximal lattice-free quadrilateral is not common. An example of an instance where this does occur is

$$\begin{aligned} f &= (-699/422, -811/753) \\ r_1 &= (-414/557, 786/359) \\ r_2 &= (-122/715, -103/331) \\ r_3 &= (47/46, -653/157) \\ r_4 &= (467/206, -113/166). \end{aligned}$$

The instance along with the portion of the integer hull close to f is shown in Figure 27. The two distinct quadrilaterals are shown separately in Figure 28 and are shown together in Figure 29.

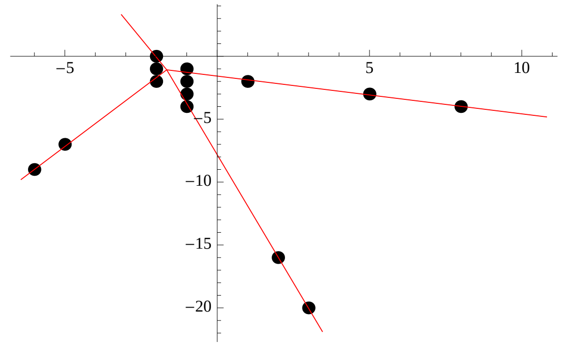

Figure 27: Quadrilateral instance with subset of integer hulls shown.

6.6 Conclusion

In the two-row model, the two integer basic variables are assumed to be free and the non-basic variables are non-negative. It is clear that problems encountered in the real world do not conform to this model. The presence of non-basic integer variables presents problems and until a complete description of the master polyhedron is discovered, such variables can be dealt with by lifting.

We showed how to exactly calculate lattice-free triangles and quadrilaterals whose vertices lie on non-basic rays emanating from the fractional point. We also gave a heuristic for calculating triangle inequalities which are approximately of Type 2.

We performed an experiment with two-row cuts from these derivations on instances from the MIPLIB 3.0 library. The performance of the triangle cuts on the mixed-integer instances was not impressive and there was a noticeable degradation in performance on the pure-integer instances. The performance of the quadrilateral cuts was fairly bad across the board however.

Although the performance of the cuts was not impressive, we would not immediately dismiss these cuts as not being useful. We had to limit the number of points considered on the integer hulls to keep things manageable and we wish that we knew how to compute non-exact triangles so that we could see how they perform. There are still many unsolved problems and a lot of work left to do.

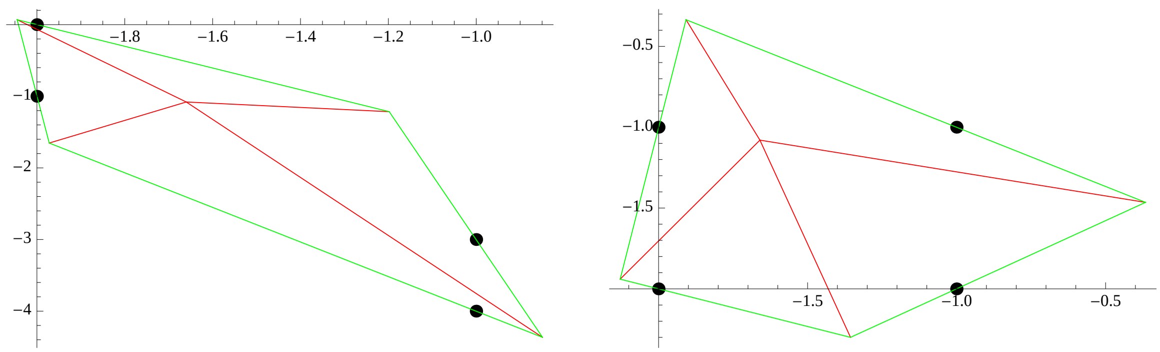

Figure 28: Two maximal lattice-free quadrilaterals for the same instance.

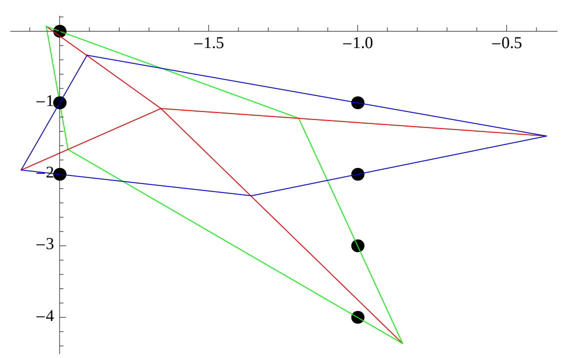

Figure 29: The two quadrilaterals shown together.

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