THE KEYING PROPERTIES OF QUARTZ CRYSTAL OSCILLATORS

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SUMMARY

This report describes an extended investigation of the problems which arise when crystal controlled oscillators are keyed for the transmission of intelligence. The scope of the problem was deliberately limited to systems in which the crystal comes to rest during each key-up interval and is set into vibration during each key-down interval. The most difficult single problem is to establish these vibrations or oscillations sufficiently rapidly to permit keying rates suitable for fast communication. A closely related and comparable difficult problem is to stop or damp out the existing vibrations at the end of each character. If either of these conditions is not met the resulting signals are blurred and lack intelligibility.

The problems of securing adequate rates of rise and decay were studied extensively, both theoretically and experimentally. This work has resulted in design procedures, applicable over the specified frequency range of 1 to 10 mc, which lead to oscillators which may be keyed at teletype speeds, i.e., 43 dots per second or 130 words per minute. These design procedures are given in Chap. VII. A slightly modified Pierce circuit is favored, especially for wide-band operation, because of its simplicity and freedom from adjustments. The Miller circuit permits greater keying speeds but requires adjustments when used over a band of frequencies. The conventional forms are somewhat superior to the electron-coupled versions.

Incidental to the investigation of available keying speeds and the design of circuits for high speed keying the need arose for establishing a criterion to judge the suitability of a given wave form for the transmission...
mission of intelligence. It was found that extremely square waves produce un-
pleasant clicks and unnecessary interference with other facilities. On the
other hand, loss of intelligibility results from an excessive rounding of the
wave shapes. A recommended wave form is presented in Chap. I.

It was found that keying has no appreciable effect on crystal life. This
result is shown to be a natural consequence of the type of operation employed.
No significant frequency departures have been found to result directly from
keying.

Much work was devoted to the problem of damping out the vibrations of a
crystal plate. It is shown in Chaps. III and IV that very effective damping
can be produced by shunting a suitable electrical network across the crystal
terminals, and that this need not produce a significant change in frequency.
The Q of a typical crystal can be lowered from about 100,000 to about 15 in
this way. In addition, it is possible to damp crystals of different frequencies
with a single fixed network if extreme damping rates are not required.

To facilitate the exposition of the analysis it was found desirable to com-
pile a systematic review of a number of related topics concerning simple cir-
cuits and transients. This material is presented in Chap. II.

Three new oscillator circuits for operating the crystal at series resonance
are presented in Chap. IX. Time did not permit a thorough investigation of
their properties. However at least one of them appears promising for keyed ap-
lications. The others possess certain features which may make them useful for
continuous operation.

It was found that the performance of a keyed oscillator is nearly inde-
pendent of the point at which the key is inserted. Accordingly the keying
point may be selected on the basis of convenience.
The final chapter (X) presents suggestions for future investigation of a few topics which were not adequately covered. It is pointed out that some modification of present crystal impedance levels will be desirable for a number of applications.
LIST OF SYMBOLS

Note: Section Nos. refer to the first usage of the symbol or to the first formal definition of it appearing in the text.

A - Time Decrement, nepers/sec. Sec. 2-6.
A - Amplification or Voltage Gain. Sec. 9-3.
B - Susceptance. Sec. 2-19.
B - Damped Angular Frequency, radians/sec. Sec. 2-2.

B_h - Hyperbolic "Frequency", nepers/sec. Sec. 2-1.
C - Capacitance.
D - Decrement Ratio. Sec. 4-5.
E - E.m.f.
F - Fractional Frequency Difference. Sec. 4-5.
F - Frequency Ratio. Sec. 9-1.
G - Conductance.
H - Angular Decrement, nepers/radian. Sec. 4-8.
I - Current.
J - Generalized Network Co-ordinate. Sec. 2-17.

J - Resistance Factor. App. M.
K - An Arbitrary Constant.
K - Symbol for Kilohms. (On Figures only.)
K - Crystal Load Capacitance Symbol, alternatively for C_k. Sec. 7-1.
L - Inductance.
M - A Special Decrement Ratio. Sec. 4-2.
M - Symbol for Megohms. (On Figures only.)
N - A Special Frequency Ratio. Sec. 4-2.
LIST OF SYMBOLS

P - Resistance Factor. Sec. 7-1, App. A.
P - Power. App. A.
Q - Quality Factor or Selectivity. Sec. 2-9.
R - Resistance.
S - Resistance Factor. Sec. 3-3.
T - The Period of a Repeating Phenomenon.
U - A Specific Variable. Sec. 4-5.
W - Energy. Sec. 2-9.
X - Reactance.
Y - Admittance.
Z - Impedance.

a - Capacitance Ratio. Sec. 3-3.
a - Resistance Ratio. Sec. 9-3.
b - Capacitance Ratio. Sec. 7-1.
c - Frequency Multiplier. Sec. 3-3.
d - Capacitance Ratio. Sec. 9-3.
d - Differential Operator.
e = 2.7128..., Base of Natural Logarithms.
e - Instantaneous E.m.f.
f - Undamped or Driven Cyclic Frequency, cycles/sec.
f_d - Damped Cyclic Frequency, cycles/sec. Sec. 2-8.
g - Dimensionless Conductance Factor. Sec. 2-13.
h - Decrement Multiplier. Sec. 4-4.
h - Inductance Multiplier. Sec. 9-1.
i - Instantaneous Current.
i - Subscript for General Term in a Series.
j = $\sqrt{-1}$.
### LIST OF SYMBOLS

- **k** - An Arbitrary Constant.
- **k** - Coefficient of Coupling for Transformers. Sec. 9-3.
- **m** - A Crystal Constant. Sec. 2-20.
- **m** - Abbreviation for Milli, as in mh (millihenry).
- **n** - Crystal Capacitance Ratio. Sec. 2-20.
- **p** - Variable in Auxiliary Equations. Sec. 2-1.
- **p** - Time Derivative Operator, d/dt. Sec. 4-1.
- **p** - Inductance Multiplier. Sec. 9-1.
- **q** - Instantaneous Charge.
- **r** - Dimensionless Resistance Factor. Sec. 2-3.
- **r** - Resistance Multiplier. Sec. 9-1.
- **s** - Inductance Multiplier. Sec. 4-5.
- **t** - Time.
- **u = tan θ**. App. P.
- **w** - Undamped or Driven Angular Frequency, radians/sec. Sec. 2-2.
- **x** - A Small Increment of Reactance. Sec. 3-7.
- **x** - A Variable.
- **y** - A Variable.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>δ</td>
<td>Cyclic Decrement, db/cycle. Sec. 2-3.</td>
</tr>
<tr>
<td>Ε</td>
<td>Error Term. Sec. 4-7, App. J-2.</td>
</tr>
<tr>
<td>θ</td>
<td>Impedance Phase Angle. App. P.</td>
</tr>
<tr>
<td>μ</td>
<td>Tube Amplification Factor.</td>
</tr>
<tr>
<td>ρ</td>
<td>Abbreviation for Micro, as in μh (microhenry).</td>
</tr>
<tr>
<td>ν</td>
<td>Inductance Ratio. Sec. 4-13.</td>
</tr>
<tr>
<td>λ</td>
<td>A Crystal Constant. App. P.</td>
</tr>
<tr>
<td>$\phi = \tan^{-1} B/A.$</td>
<td>Sec. 2-1.</td>
</tr>
<tr>
<td>φ</td>
<td>Transformer Turns Ratio. Sec. 9-3.</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS

ψ - An Arbitrary Angle. Sec. 2-17.

Δ - Time Decrement, db/sec. Sec. 2-6.
Λ - Cyclic Decrement, nepers/cycle. Sec. 2-8.
Σ - Summation of . . .

* Is approximately equal to
≠ Is not equal to
≡ Is defined as

> Is greater than
< Is less than
≥ Is equal to or greater than
≫ Is much greater than

|x| The Absolute Magnitude of x

log x = log_{10} x
ln x = log_{e} x
CHAPTER I

INTRODUCTION

Need for a study of the keying properties of quartz crystal controlled oscillators was recognized several years ago by the personnel of the Frequency Control Branch of Squier Laboratory. However no opportunity to conduct this study occurred until the beginning of 1946.

In February of 1946 a conference was held to determine the scope of the present study. It was attended by Messrs. Johnson, Pritchard, Bower, and Ross of the Frequency Control Branch and Drs. Rosselot and Edson of Georgia Tech. A formal contract authorizing the work agreed upon was completed in the following months. Work on the project commenced during May of 1946.

1-1. The Problem.

Transmission of intelligence by means of interrupted continuous waves (code transmission) has proved very reliable and satisfactory for low keying rates. Crystal controlled oscillators have become universal in commercial transmitters because of the excellent frequency stability so achieved. Research and improved manufacturing techniques have resulted in crystals of improved uniformity, temperature coefficients, and high Q's. The use of these crystals has still further improved the degree of frequency stability. Unfortunately these crystals also seriously limit the maximum keying rate which may be obtained in conventional crystal oscillator circuits. In addition to the communication aspects of the problem, there are other possible applications of crystal controlled oscillators which require rapid
buildup or decay rates of the oscillations of the crystal.

Because of the limitations imposed on keying speeds by high $Q$ crystals, and for other considerations, it was decided that fundamental research should be undertaken to determine the factors governing the keying properties of crystal controlled oscillators. Specifically, the following information was to be secured for Pierce, Miller, and electron-coupled oscillators:

1. The point in the circuit that is to be keyed.
2. The value of the parameters of the equivalent circuit of crystal units.
3. The value of the parameters of the oscillator circuit other than the crystal units.
4. The maximum rate at which keying is possible.
5. The frequency stability obtained during the keying process.
6. The effect of keying on the life of the crystal.
7. The envelope of the output during the keying.

A tentative upper limit of 43 dots per second (teletype speed) was established as the greatest speed likely to be of interest. Information was also desired concerning the behavior of crystal controlled oscillators which operate the crystal at its series resonant frequency. Because the results of this study should be applicable to military use, it was evident that some consideration must be given to size, weight, simplicity of operation, and power requirements. These factors were kept in mind in the examination of the oscillator keying characteristics.

Essentially oscillator keying is a process by which the oscillator is turned on and off. However, because the oscillations require a finite time to build up and decay, there is a period after the oscillator is turned on during which the oscillations build up in amplitude before they reach a constant amplitude condition. A similar period exists after the oscillator is turned off during which
the oscillations decay to a negligible value. These buildup and decay periods represent the obvious limitations on keying speed. Academically, the keying problem is solved when the desired buildup and decay periods (rates) are obtained with a fixed frequency output during the buildup, steady-state, and decay portions of the keying cycle. A sufficient engineering solution is obtained when the desired buildup and decay rates are obtained without a detectable frequency change during the keying cycle; however, the circuit should be capable of operating over a band of frequencies and yet meet these conditions.

An examination of the keying characteristics of crystal controlled oscillators presents several problems as direct consequence of the buildup and decay rates of existing oscillators. The relative importance of these problems can be determined by an extensive theoretical examination of the keying operation; however, this process is involved and tedious. For this reason a series of experiments was performed on a simplified Pierce oscillator to evaluate these problems. These experiments are presented in detail in Appendix A.

The experimental results pertinent to the general keying problem reveal that the oscillation decay rate is much lower than the buildup rate and, therefore, is the prime factor limiting the keying speed. Furthermore, the decay rate is inversely proportional to the $Q$ of the crystal and is relatively independent of the remaining circuit parameters. Thus, the use of higher $Q$ crystals to improve frequency stability results in a decrease in the maximum keying speed.

1-2. The Gating Solution.

In an early experiment a Miller oscillator employing a 6J5 triode was keyed up to a rate of 500 dots per second. The key was applied in the plate supply lead, and nearly square wave envelopes were observed at the plate. However, this did not constitute true off-on keying because the crystal vi-
brated (rang) continuously throughout the keying cycle. Therefore, the process was really one of gating by which the tube was turned on and off, thus permitting the RF voltage to appear in its plate circuit. During the time the tube was on, excess energy was fed back to the crystal to replace that lost during the off period. This method has two inherent disadvantages. When the oscillator is first started, the crystal is not vibrating; therefore, for several keying cycles, the amplitude of oscillations increases. For communication applications this would result in the loss of the first several characters. Also, since the crystal vibrates continuously during operation, RF energy is present at all times. This will require certain shielding precautions to insure satisfactory break-in operation. It is obvious that high-speed keying can be obtained by employing a continuous wave oscillator and suitable gating at some subsequent stage in the transmitter. However, because of shielding difficulties above, the gating process is not considered a solution to the oscillator keying problem.


Since the keying problem resulted from difficulties experienced with code transmission, and the solution of the problem is directly applicable to this type of transmission, it is evident that information about the waveforms required for satisfactory code transmission is needed. This information is obtained from the qualitative experiments presented in Appendix B. From these experiments it was found that an adequate keying waveform has a decay rate considerably greater than the buildup rate. The modulation envelope for a barely readable signal is shown in Fig. 1.1. From this waveform and the established keying speed of 43 dots per second, a value of 6 db per millisecond is calculated for the minimum decay rate. The buildup rate is not readily expressible as a decrement because of the complicated buildup waveform which
exists in actual oscillators; however, the buildup time should be approximately four times the time required for the oscillations to decrease to \( \frac{1}{3} \) of the maximum value.

1-4. **Initial Method of Attack.**

Because existing oscillators have a buildup rate considerably larger than the decay rate, it appears that the first step is a study of means to increase the oscillation decay rate - that is, to damp the crystal vibrations. Thus, the initial work was directed towards the development of methods to damp the vibrations of a quartz crystal and towards the determination of the characteristics of the networks which produce damping.

Before presenting the development and analysis of damping circuits, it is considered desirable to present a discussion of certain basic material which is widely used, often with conflicting meanings. Chapter II presents this basic material with definitive discussions about the manner in which these terms will be used throughout the remainder of this report.
CHAPTER II

BASIC MATERIAL

The extensive detail with which various investigators have treated the properties of resonant circuits has resulted in conflicting terminology appearing in the literature. In addition, certain definitions have been used in a loose and quite often incorrect manner. This chapter provides a review of basic material and a definition of terms as they will be used in this report.

SERIES RLC CIRCUIT ANALYSIS


Consider the circuit shown in Fig. 2.1, and let the condenser be initially uncharged. The switch is closed at time \( t = 0 \). Then by Kirchhoff’s First Law

\[
\frac{d}{dt} R i + \frac{1}{C} \int i \, dt = E. \tag{2.1}
\]

Replacing \( i \) by \( \frac{dq}{dt} \), where \( q \) is the instantaneous charge on the condenser,

\[
L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E; \tag{2.2}
\]

and differentiating (2.1) with respect to \( t \),

\[
L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{i}{C} = 0. \tag{2.3}
\]
These are the differential equations for charge and current in the simple series RLC circuit. Their complete solutions consist of a "particular integral" and a "complementary function". The former is the steady-state solution; the latter is the transient solution.

Considering only (2.2) and (2.3), which involve no integrals, it is evident that the charge equation is the more fundamental one, because when $q$ is known as a function of time, $i$ may be obtained by differentiation, whereas, the converse is not true - that is, if $i$ were known as a function of time, the determination of $q$ would involve a constant of integration which could not be secured from (2.3) alone. Accordingly, we shall examine (2.2) first.

The steady-state condition is, by definition, such that there is no variation of $q$ with time. Then, setting both $dq/dt$ and $d^2q/dt^2$ equal to zero, we obtain

$$q = CE.$$  
(2.4)

This is the steady-state solution or particular integral of (2.2), and it states simply that when the circuit is in static equilibrium, all the voltage $E$ appears across the condenser terminals; consequently, charges of $+CE$ and $-CE$ rest on the opposing plates.

It takes a certain amount of time for the charge on the condenser to change from zero to the value $CE$; we know that $q$ must be definable in that interval by some function of time. Since equations of the form of (2.1), (2.2), and (2.3) are known to have solutions involving exponentials (and this is confirmed by the knowledge that almost any function of time met with in network theory may be expressed as a summation of $a_ie^{bit}$), we shall assume that

$$q = Ke^{-Pt}.$$  
(2.5)
Then, by successive differentiation,

$$\frac{dq}{dt} = -pK e^{-pt},$$

and

$$\frac{d^2q}{dt^2} = p^2K e^{-pt}. \tag{2.6}$$

Consider the equation obtained by setting the right-hand side of (2.2) equal to zero:

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0. \tag{2.7}$$

Now, if by properly choosing $K$ and $p$, the value of $q$ given by (2.5) could be made to satisfy this modified or "reduced equation" (2.7); then, obviously, the sum of the solutions, (2.4) and (2.5), would satisfy the original equation (2.2).

Substituting (2.5) and (2.6) into (2.7), we have as the required condition:

$$K e^{-pt} (L p^2 - R p + 1/C) = 0. \tag{2.8}$$

Either ($K = 0$), ($e^{-pt} = 0$), or ($L p^2 - R p + 1/C = 0$) will meet our needs. The first is a trivial solution, since it causes the reduced equation to vanish. The second, $e^{-pt} = 0$, is true only provided that $t = \infty$; this means that, in theory at least, the steady-state solution, $q = C E$, for (2.2) applies only after an infinite interval of time. The final possibility is termed the "auxiliary equation":

$$L p^2 - R p + 1/C = 0. \tag{2.9}$$

This is a second-degree algebraic equation in $p$ and yields the roots

$$p_1 = \frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}},$$

and

$$p_2 = \frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}. \tag{2.10}$$
Let
\[
A = \frac{R}{2L}, \quad w = \frac{1}{\sqrt{LC}}, \quad \text{and} \quad B_h = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \sqrt{A^2 - w^2}.
\] (2.11)

and

Note that \( w \) is the steady-state or undamped natural radian frequency of the circuit.

Then (2.10) becomes
\[
p_1 = A + B_h, \quad \text{and} \quad p_2 = A - B_h.
\] (2.12)

Hence, the complete solution of (2.7) may be written
\[
q = K_1 e^{-pt} + K_2 e^{-p_2t},
\] (2.13)

where \( K_1 \) and \( K_2 \) have yet to be defined. This solution, (2.13), is also the transient solution or complementary function of (2.2), whose complete solution is
\[
q = C e^{-pt} + K_1 e^{-p_1 t} + K_2 e^{-p_2 t}.
\] (2.14)

It will be shown in Sec. 2-16 that the complementary function is complete when the number of irreducible "arbitrary" constants contained in it is equal to the order of the differential equation. We have in (2.13) the two constants \( K_1 \) and \( K_2 \) which cannot be combined in any way to give a single constant, and the original equation (2.2) is of second order. When the particular integral is added to the complementary function, a complete solution results. Hence the assertion that (2.14) is the complete solution of (2.2).
The foregoing discussion tacitly assumes that \( p_1 \) and \( p_2 \) are different. In the special case where \( B_n = 0 \) and \( p_1 = p_2 = A \), it is seen that \( K_1 \) and \( K_2 \) may be combined to give

\[
q = C E + (K_1 + K_2) e^{-At} = C E + K_3 e^{-At}.
\]  

This cannot be the complete solution of (2.2) because only one arbitrary constant is present.

There are several ways of obtaining a second constant. Without going into a rigorous development, we may assume that \( K_3 \) is not necessarily a constant but is a function of time. Then it may be expanded into a series of the form

\[
K_3 = k_0 + k_1 t + k_2 t^2 + \ldots + k_n t^n + \ldots.
\]  

Inasmuch as we need only two non-combinable terms, we discard all but the first two. This gives us

\[
q = C E + (k_0 + k_1 t) e^{-At},
\]  

which is the complete solution for the case where \( p_1 = p_2 \).

The current equation (2.3) will have similar solutions corresponding to (2.14) and (2.17) but with different arbitrary constants and zero steady-state terms.

For \( p_1 \neq p_2 \),

\[
i = \frac{1}{p_1} e^{-p_1 t} + \frac{1}{p_2} e^{-p_2 t};
\]  

while for \( p_1 = p_2 \),

\[
i = \left(k_0 + k_1 t\right) e^{-At}.
\]  

Since \( i = dq/dt \), we may readily derive the relationships between the corresponding constants. Differentiating (2.14) and equating the result to (2.18), and differentiating (2.17) and equating the result to (2.19), we obtain
\[ K_1' = -p_1 K_1 , \]
\[ K_2' = -p_2 K_2 , \]
\[ k_0' = -A k_0 + k_1 , \]
\[ k_1' = -A k_1 . \]

To evaluate the constants we employ initial circuit conditions. Current cannot flow immediately upon closure of the switch because of the presence of inductance in the circuit, nor can charge accumulate immediately on the condenser. Hence we have:

\[ t = 0 , \]
\[ q = 0 , \]
\[ i = 0 . \]

From introducing these boundary values into (2.14) and (2.18), and using the relationships of (2.20), there results

\[ 0 = CE + K_1 + K_2 , \]
\[ 0 = K_1' + K_2' = -p_1 K_1 - p_2 K_2 , \]
whence

\[ K_1 = CE \frac{p_2}{p_1 - p_2} , \]
\[ K_2 = -CE \frac{p_1}{p_1 - p_2} . \]

Then from (2.14)

\[ q = CE + CE \left[ \frac{p_2}{p_1 - p_2} e^{-p_1 t} - \frac{p_1}{p_1 - p_2} e^{-p_2 t} \right] . \]

Substituting for \( p_1 \) and \( p_2 \) from (2.12),
\[
q = CE + \frac{CE}{2B_h} \left[ (A - B_h) e^{-(A + B_h)t} - (A + B_h) e^{-(A - B_h)t} \right],
\]

\[
= CE + \frac{CE e^{-At}}{2B_h} \left[ -A (e^{B_h t} - e^{-B_h t}) - B_h (e^{B_h t} + e^{-B_h t}) \right],
\]

\[
= CE \left[ 1 - \frac{e^{-At}}{B_h} (A \sinh B_h t + B_h \cosh B_h t) \right]. \tag{2.25}
\]

Through the use of a transformation (see Appendix G-4), (2.25) may be written

\[
q = CE \left[ 1 - \frac{e^{-At}}{B_h} \sqrt{A^2 - B_h^2} \sinh(B_h t + \phi_h) \right]. \tag{2.26}
\]

Now by introducing \(B_h^2 = A^2 - w^2\), from (2.11), under the radical in (2.26), we have

\[
q = CE \left[ 1 - \frac{w}{B_h} e^{-At} \sinh(B_h t + \phi_h) \right], \tag{2.27}
\]

where \(\phi_h = \tanh^{-1} \frac{B_h}{A}\). \(\tag{2.28}\)

In the foregoing work we have considered \(p_1\) and \(p_2\) to be pure real numbers - that is, \(B_h\) is a real number, \(A^2\) is greater than \(w^2\), and therefore from (2.11) \(\frac{R^2}{4L^2}\) is greater than \(\frac{1}{LC}\).

When \(\frac{R^2}{4L^2}\) is less than \(\frac{1}{LC}\) we write the definitions

\[
\begin{align*}
p_1 &= A + j\sqrt{w^2 - A^2} = A + jB, \\
p_2 &= A - j\sqrt{w^2 - A^2} = A - jB.
\end{align*} \tag{2.29}
\]

Comparing this with (2.12) we see that

\[
B_h = jB. \tag{2.30}
\]

Then, for the case where \(\frac{R^2}{4L^2} \ll \frac{1}{LC}\) (2.30) may be substituted into (2.25) to give
SOLUTION OF THE DIFFERENTIAL EQUATIONS

\[ q = CE \left[ 1 - \frac{e^{-At}}{jB} (A \sinh jBt + jB \cosh jBt) \right]. \quad (2.31) \]

From the relationships between hyperbolic and circular functions (see Appendix C-3) it is evident that (2.31) is equivalent to

\[ q = CE \left[ 1 - \frac{e^{-At}}{B} (A \sin Bt + B \cos Bt) \right]. \quad (2.32) \]

Again, through the use of a trigonometric transformation (see Appendix C-5), (2.32) may be written

\[ q = CE \left[ 1 - \frac{e^{-At}}{B} \sqrt{A^2 + B^2} \sin(Bt + \tan^{-1} \frac{B}{A}) \right]. \quad (2.33) \]

By substituting \( w^2 = A^2 + B^2 \), from (2.29), under the radical in (2.33) we obtain

\[ q = CE \left[ 1 - \frac{w}{B} e^{-At} \sin(Bt + \phi) \right], \quad (2.34) \]

where \( \phi = \tan^{-1} \frac{B}{A} \). \quad (2.35)

In the special case where \( \frac{w^2}{4L^2} = \frac{1}{LC} \) we see from (2.11) and (2.12) that \( B_1 = 0 \) and \( p_1 = p_2 = A \). To evaluate the constants \( k_0 \) and \( k_1 \) in (2.17) we employ again the boundary conditions given by (2.21). Substituting these in (2.17) and (2.19), and using the relationships of (2.20), we obtain

\[ 0 = CE + k_0, \quad (2.36) \]

and

\[ 0 = k_0' = -Ak_0 + k_1; \]

whence

\[ k_0 = -CE, \quad (2.37) \]

and

\[ k_1 = -CEA. \]

Then from (2.17)

\[ q = CE \left[ 1 - (1 + At) e^{-At} \right]. \quad (2.38) \]
Solutions of the current equation (2.3) in terms of the circuit parameters $A$, $w$, $B$, $B_h$, etc., may be procured by a similar treatment of (2.18) and (2.19), or more readily by differentiation of the $q$ solutions (2.27), (2.34), and (2.38). They are:

1. If $\frac{R^2}{4L^2} > \frac{1}{LC}$,
   \[ i = \frac{E}{LB_h} e^{-At} \sinh B_h t; \]  
2. If $\frac{R^2}{4L^2} = \frac{1}{LC}$,
   \[ i = \frac{E}{L} te^{-At}; \]  
3. If $\frac{R^2}{4L^2} < \frac{1}{LC}$,
   \[ i = \frac{E}{LB} e^{-At} \sin Bt. \]

For convenient reference the solutions of the charge equation (2.2) already derived are retabulated here:

1. If $\frac{R^2}{4L^2} > \frac{1}{LC}$,
   \[ q = CE \left[ 1 - \frac{w}{B_h} e^{-At} \sinh (B_h t + \tanh^{-1} \frac{B_h}{A}) \right]; \]  
2. If $\frac{R^2}{4L^2} = \frac{1}{LC}$,
   \[ q = CE \left[ 1 - (1 + At) e^{-At} \right]; \]  
3. If $\frac{R^2}{4L^2} < \frac{1}{LC}$,
   \[ q = CE \left[ 1 - \frac{w}{B} e^{-At} \sin (Bt + \tan^{-1} \frac{B}{A}) \right]. \]

2-2. Plot of the Current and Charge Solutions.

The curves of Fig. 2.2 show $q$ and $i$ plotted versus time, for the three cases. $L$ was arbitrarily chosen to be .0253 henry, and $C$ was taken as 1 microfarad. Then we have as conditions on $R$:

(a) when \( \frac{R^2}{4L^2} < \frac{1}{LC} \), \( R < 318 \text{ ohms} \);
(b) when \( \frac{R^2}{4L^2} > \frac{1}{LC} \), \( R > 318 \text{ ohms} \);
(c) when \( \frac{R^2}{4L^2} = \frac{1}{LC} \), \( R = 318 \text{ ohms} \).
Fig. 2.2 - Transient Variation of Current and Charge in the Series RLC Circuit of Fig. 2.1. L = 0.0253 henry, C = 1 microfarad.
Case(a) corresponds to (2.41) and (2.44). It is termed the "oscillatory" or "under-damped" case, because $q$ and $i$ undergo exponentially damped sinusoidal variations with time. The period $T$ of one cycle is seen to be $2\pi/B$, and $B$ is called the "damped radian frequency" of the circuit, as opposed to $\omega$, the undamped radian frequency. Referring to (2.29), it is noted that $B$ is always less than $\omega$.

Case(b) corresponds to (2.39) and (2.42). It is termed the "aperiodic" or "over-damped" case, because $q$ builds up to but never exceeds the maximum final charge on the condenser, and $i$ consists merely of a pulse which is sufficient to charge the condenser and which in falling off approaches but never goes beyond zero.

Case(c) is the boundary between (a) and (b). It corresponds to (2.40) and (2.43), and is termed the "critically-damped" case. The charge and current are aperiodic, but $q$ reaches its maximum and $i$ falls back to zero faster than for any other aperiodic case.

If the solutions for $q$ are divided by $C$ we have an expression for the voltage across the condenser at any time $t > 0$. Similarly, if the solutions for $i$ are multiplied by $R$, the voltage across the resistor as a function of time results. In general, then, measurement of these two voltages would provide the easiest experimental means of obtaining the curves shown in Fig. 2.2.

2-3. Plot of the p Roots as R Is Varied.

We have shown what values $p$ must take in order to allow the assumed solution of (2.5) to satisfy the charge and current equations for the series RLC circuit. See (2.10), (2.12), or (2.29).

We wish to examine the behavior of $p_1$ and $p_2$ as $R$ is varied over a wide range of values while $L$ and $C$ are held constant. To make the plotting easier we shall employ a modified resistance term as variable.
Sec. 2-3  PLOT OF THE P ROOTS AS R IS VARIED

Let

\[ R = r \sqrt{\frac{4L}{C}} \]

then

\[ r = R \sqrt{\frac{C}{4L}} = R \times \text{constant.} \tag{2.46} \]

Substituting for \( R \) in (2.10) and simplifying, we have

\[ p_1 = \frac{r}{\sqrt{LC}} + \sqrt{\frac{1}{LC}} (r^2 - 1) \]

and

\[ p_2 = \frac{r}{\sqrt{LC}} - \sqrt{\frac{1}{LC}} (r^2 - 1) \tag{2.47} \]

And, substituting for \( 1/\sqrt{LC} \) from (2.11), (2.47) becomes

\[ p_1 = w(r + \sqrt{r^2 - 1}) \]

and

\[ p_2 = w(r - \sqrt{r^2 - 1}) \tag{2.48} \]

From (2.48) \( p_1 \) and \( p_2 \) are evaluated in TABLE 2.1 below for values of \( r \) ranging from \(-\infty\) to \(+\infty\). A plot of \( p_1 \) and \( p_2 \) in the complex \( p \) plane is given in Fig. 2.3. It should be noted from (2.46) that \( r \) is actually a dimensionless resistance factor and measures the circuit resistance \( R \) in units of \( \sqrt{\frac{4L}{C}} \) ohms.

**TABLE 2.1**

SOLUTIONS OF EQUATIONS (2.48)

<table>
<thead>
<tr>
<th>Pt. Ref. No.</th>
<th>( r )</th>
<th>( r^2 )</th>
<th>( p_1 )</th>
<th>( p_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-\infty)</td>
<td>(+\infty)</td>
<td>0</td>
<td>(-\infty)</td>
</tr>
<tr>
<td>2</td>
<td>(-\sqrt{2})</td>
<td>2</td>
<td>(-0.41w)</td>
<td>(-2.41w)</td>
</tr>
<tr>
<td>3</td>
<td>(-1)</td>
<td>1</td>
<td>(-w)</td>
<td>(-w)</td>
</tr>
<tr>
<td>4</td>
<td>(-1/\sqrt{2})</td>
<td>1/2</td>
<td>(-w/\sqrt{2} [1 - j])</td>
<td>(-w/\sqrt{2} [1 + j])</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>(-jw)</td>
<td>(-jw)</td>
</tr>
<tr>
<td>6</td>
<td>1/\sqrt{2}</td>
<td>1/2</td>
<td>(w/\sqrt{2} [1 + j])</td>
<td>(w/\sqrt{2} [1 - j])</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>(w)</td>
<td>(w)</td>
</tr>
<tr>
<td>8</td>
<td>(\sqrt{2})</td>
<td>2</td>
<td>2.41w</td>
<td>0.41w</td>
</tr>
<tr>
<td>9</td>
<td>(+\infty)</td>
<td>(+\infty)</td>
<td>(+\infty)</td>
<td>0</td>
</tr>
</tbody>
</table>
\[ p_1 = w(r + \sqrt{r^2 - 1}) \]
\[ p_2 = w(r - \sqrt{r^2 - 1}) \]
\[ r = \frac{R}{\sqrt{4L/C}} \]

Notes:
1) Arrowheads on curves indicate \( r \) increasing.
2) Numbers at circled points refer to values of \( r \) in TABLE 2.1.

Fig. 2.3 - Plot of \( p \) Roots in Complex Plane vs. Resistance Factor \( r \), for Series RLC Circuit.
Equations (2.48) may be rewritten for the range of $r$ in which $p$ becomes complex—that is, when $|r|$ is less than unity:

$$p_1 = w(r + j\sqrt{1 - r^2}),$$

and

$$p_2 = w(r - j\sqrt{1 - r^2}).$$

Comparing this with (2.29), we see that

$$wr = \text{Re}(p) = A,$$

and

$$w\sqrt{1 - r^2} = \text{Im}(p) = B.$$  

This region of $r^2 < 1$ corresponds to $\frac{R^2}{4L^2} < \frac{1}{IL}$, or the Oscillatory Case.

We may derive the following circular relationships for it. Since $p_1$ and $p_2$ are conjugates, consider only one of the roots. Draw a radius vector on the graph of Fig. 2.3 from the origin to this root. Denote by $\phi$ the value of the angle which this vector makes with the positive $\text{Re}(p)$ axis. Then

$$A = w \cos \phi,$$

$$B = w \sin \phi,$$

$$w^2 = A^2 + B^2,$$

and

$$\frac{B}{A} = \tan \phi.$$  

The last result was obtained earlier in making the transformation to reach (2.34), although $\phi$ as defined by (2.35) was not graphically identified then.

2-4. **General Circuit Behavior.**

From (2.50) it is evident that if $r$ is negative, $A$ will be negative. Considering this condition in (2.39) to (2.41), we see that in all three cases the current will increase in amplitude indefinitely with time.

Similarly, $r$ positive and hence $A$ positive gives the exponentially-decaying currents shown on the graphs of Fig. 2.2.
When \( r \) is zero and, therefore, \( A \) also zero, the current should show no tendency either to decay or expand. However, the condition \( r = 0 \) represents physical instability, and in practice \( r \) must actually be slightly greater than or slightly less than zero. The phenomenon of oscillations having uniform amplitude may be explained as follows. The generator portion of the circuit (vacuum-tube oscillator) is representable by a non-linear negative resistance. The passive dissipative portion of the circuit (tank or load) is effectively a linear positive resistance. Initially the magnitude of the negative resistance is somewhat greater than that of the positive resistance; hence the net \( r \) in the circuit is negative, and if \( |r| < 1 \), the current is oscillatory and of course increasing in amplitude. At some value of amplitude the non-linearity of the negative resistance will cause it to decrease in value until it just equals the positive resistance. Then \( r \) remains effectively zero, and oscillations persist indefinitely at the amplitude of this equilibrium state dictated by the exact character of the generator non-linearity. There is a continuous tiny regulating action in which the net circuit \( r \) vacillates slightly to both sides of zero, but the effective \( r \) is zero.

It should be noted from (2.50) that when \(|r|\) is greater than unity, \( B \) becomes imaginary, and hence the quantity \( jB \) is real. By (2.30) this represents \( B_h \), and of course the circuit is non-oscillatory, as seen in (2.39) to (2.44).

The foregoing remarks lead to a diagram of general circuit behavior (character of current) as the net circuit \( r \) is varied from \(-\infty \) to \(+\infty \). This diagram is shown in Fig. 2.4; \( r \) is still defined by (2.46) and \( R \) in that equation is the actual net circuit resistance, positive or negative.

A sinusoidal oscillator operates about the center 0 axis. During buildup the net circuit \( r \) is in region II and during decay in region III.

A relaxation oscillator is characterized by net circuit \( r's \) in regions I and IV.
DEFINITIONS OF TERMS

Certain conventionalized terms have been employed thus far in the development of the differential equations pertaining to a series RLC circuit - for example, "particular integral," "reduced equation," and "auxiliary equation" - and, as these mathematical expressions are so generally accepted we proffer no further comment on them. However, several important circuit quantities which appear in nearly all mathematical treatments of electrical networks, and a few other common technical expressions not too well agreed upon, all of which are used frequently throughout this report, will be given definitive discussions here.

2-5. Damping Factor.

The solutions (2.39) through (2.44) of the differential equations developed in Sec. 2-1 are seen to contain a common factor e^{-At}. This
factor is directly responsible for the increase or decrease with time of the amplitude of the variable with which it is associated, and it will be called the Damping Factor.

2-6. Time Decrement.

The quantity $A$ (previously defined as $R/2L$ for the series RLC circuit) is known variously as the "damping term," "attenuation term," "decrement," and "time decrement." Inasmuch as the natural logarithm ($-At$) of the Damping Factor will have units of Nepers, we see that $A$ is expressible in units of Nepers/sec. On this basis we shall call the positive quantity $A$ the Time Decrement, or simply the Decrement when no distinction is to be drawn between it and other similar definitions of circuit loss.

It is often desirable to express $A$ in units of Decibels/sec. Now, by definition,

$$\left\{ \begin{array}{l}
(\text{No. of nepers}) = \ln \frac{i_1}{i_2} = 2.303 \log \frac{i_1}{i_2}, \\
(\text{No. of db}) = 20 \log \frac{i_1}{i_2},
\end{array} \right. \quad (2.52a)$$

and

where $i_1$ and $i_2$ are two current (or voltage or charge) values being compared.

Then

$$1 \text{ neper} = 8.686 \text{ decibels.} \quad (2.52b)$$

When the Time Decrement is given in terms of db/sec. it will be denoted by the symbol $\Delta$. 

2-7. Damping Ratio.

No restrictions were placed on the quantities $i_1$ and $i_2$ in the preceding section; they are any two values which occur in the system under consideration. Now, if the circuit is capable of oscillation it will do so at a radian frequency
Sec. 2-9 QUALITY FACTOR

\[ B \] during transient decay or expansion. Letting \( T \) be the period of one cycle (\( T = \frac{2\pi}{B} \)), we find that for a single damped current wave, the ratio of two instantaneous values separated in time by one period (or by one cycle) during transient simplifies to the expression

\[ \frac{I_1}{I_2} = \frac{e^{-A(t-T)}}{e^{-AT}} = e^{-AT}. \]  

(2.53)

See (2.41) which applies to an oscillatory condition. This quantity, \( e^{-AT} \), will be called the Damping Ratio.

2-8. Cyclic Decrements.

Upon taking the natural logarithm of the Damping Ratio we obtain

\[ \Lambda = \ln e^{AT} = AT. \]

And, if \( f_d \) is the damped frequency at which the circuit can oscillate (\( f_d = \frac{1}{\pi} \)), we have \( \Lambda = \frac{A}{f_d} \). From this latter equivalence it is apparent that the units of \( \Lambda \) are Néperes/cycle. The quantity \( \Lambda \) is often termed the "logarithmic decrement," but we shall call it the Cyclic Decrement.

From Sec. 2-6 we see that the Cyclic Decrement may also be expressed in Decibels/cycle, which is frequently more convenient. When given in terms of db/cycle it will be denoted by the symbol \( \delta \).

It is obvious from this and the two preceding sections that the Damping Ratio and the Cyclic Decrements \( \Lambda \) and \( \delta \) may be applied only to circuits in which the variable goes through a clearly defined and repeated cycle - that is, only to a periodic oscillatory system. On the other hand, \( A \) and \( \Delta \) are not so limited.


A circuit quantity which is used extensively in all electrical engineering work is the Quality Factor, or Selectivity, \( Q \). In the past the
term and the quantity itself have often been applied very loosely, an abuse which has probably been encouraged by the fact that physical condensers may be made substantially loss-free whereas physical coils have considerable losses. Although the particular looseness of definition for $Q$ which is encountered seldom has serious consequences when dealing with low-loss circuits, in the study of high-loss circuits it becomes necessary to define $Q$ in a unique manner and then to determine the range of application of this definition. It is also desirable to determine how $Q$ is associated with other circuit quantities, like the decrements discussed in Secs. 2-6 and 2-8 and also whether or not $Q$ may justifiably be used to describe a single circuit element.

One of the more general definitions for $Q$ is:

$$Q = 2 \pi \frac{\text{Energy Stored in the Circuit}}{\text{Energy Lost per Cycle}} \quad (2.54)$$

Because of the term "Cycle" in the denominator, it is evident that $Q$ so defined is applicable only to a system capable of oscillating in a periodic fashion.

Now the Energy lost per Cycle in a circuit is a function of the current through the dissipative elements (resistors); since it has been found that the Decrement express the decrease of current with time during transients, it appears reasonable to expect that $Q$ will be some inverse function of the Cyclic Decrement $\Lambda$ or $\delta$ and of the Time Decrement $\Delta$ or $\Delta$.

To determine the exact relationship we shall derive a formula for $Q$ for a simple RLC circuit similar to that already analyzed (see Fig. 2.5).

The general expression for the current $i$ is determined from the "auxiliary equation" for the circuit. The condenser was taken to have an initial charge,

---

but no emf. acts in the circuit. The assumed solution is

\[ i = I e^{-pt} + k. \]  \hspace{1cm} (2.55)

The auxiliary equation and its solution are

\[ L p^2 - R p + \frac{1}{C} = 0, \]

and

\[ p = \frac{R}{2L} + j \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \] \hspace{1cm} (2.56)

As before, let

\[ A = \frac{R}{2L}, \]

\[ w^2 = \frac{1}{LC}, \] \hspace{1cm} (2.57)

and

\[ B^2 = w^2 - A^2. \]

Then

\[ i = I e^{(-A + jB)t} + k, \] \hspace{1cm} (2.58)

where \( I \) is the initial current magnitude. It is evident that when \( t = \infty \), \( i = 0 \), and hence the constant \( k \) must be zero.

Because the energy stored in the circuit decreases in a non-uniform manner as time increases, it is necessary to specify in what way the stored energy will be determined. Two possible methods may be presented: either the energy is determined at some specific time during the cycle, or the energy is determined as the average stored energy for one cycle. This latter method will be used in the present examination because it is believed that the results will be more representative. The basic definition of \( Q \) \hspace{1cm} (2.54) is rewritten:
\[ Q = 2\pi \frac{\text{Average (Energy in L + Energy in C) for One Cycle}}{\text{Energy Lost in R for One Cycle}} \]  

(2.59)

It is obviously necessary that the derived equation be independent of the time during the cycle when the measurements are started. If the stored energy and lost energy are determined between \( t_1 \) and \( (t_1 + T) \) where \( t_1 \) is any arbitrary time and \( T \) is the period, the mathematical solution will satisfy its requirements provided \( t_1 \) does not appear in the final form.

The instantaneous energy stored in the inductance is

\[ W_{IL} = \frac{1}{2} LI^2 = \frac{L I^2}{2} e^{2(-A + jB)t} \]  

(2.60)

and the average energy stored for one cycle is

\[ W_L = \frac{\int_{t_1}^{t_1+T} W_{IL} \, dt}{T} \]

\[ = \frac{L I^2}{4T(-A + jB)} \left[ \frac{2(-A + jB)t_1}{e} \right] \left[ \frac{2(-A + jB)T - 1}{e} \right] . \]  

(2.61)

The instantaneous energy stored in the capacitance is

\[ W_{ic} = \frac{1}{2} CE^2 \]  

(2.62)

However

\[ E = \frac{1}{C} \int i \, dt = \frac{I}{C(-A + jB)} e^{(-A + jB)t} + K' . \]  

(2.63)

Since the maximum voltage across the condenser approaches zero as time approaches infinity, it is necessary that the constant of integration, \( K' \), be zero. Then (2.62) becomes

\[ W_{ic} = \frac{I^2}{2C(-A + jB)^2} e^{2(-A + jB)t} . \]  

(2.64)
and the average energy stored for one cycle is
\[ W_c = \int_{t_1}^{t_1+T} W_{ic} \, dt, \]
\[ = \frac{I^2}{4TC(-A + jB)^3} \left[ e^{2(-A + jB)t_1} \right] \left[ e^{2(-A + jB)T} - 1 \right]. \tag{2.65} \]

The energy lost per cycle is
\[ W_L = \int_{t_1}^{t_1+T} R i^2 \, dt, \]
\[ = R I^2 \int_{t_1}^{t_1+T} \frac{2(-A + jB)t_1}{e^{2(-A + jB)t_1} \left[ e^{2(-A + jB)T} - 1 \right]} \, dt, \tag{2.66} \]

The \( Q \) is given by
\[ Q = \frac{W_L + W_c}{W_L}, \]
\[ = \frac{L + \frac{1}{C(-A + jB)^2}}{2 \pi \frac{I^2}{2TR}}. \tag{2.67} \]

It is noted that \( t_1 \) disappears; therefore, the solution is independent of the point on the cycle which is considered as the starting point. \( T \) has been defined as the period of one cycle; but the actual angular frequency of oscillation is \( B \), and so \( T = \frac{2\pi}{B} \). The term \((-A + jB)^2\) will be expanded to \((A^2 + B^2)\). It will be acknowledged that such an expression must be correct.
if \( Q \) is to be single-valued. Upon making these substitutions,

\[
Q = \frac{HL + \frac{B}{Gw^2}}{2R}.
\]  

(2.68)

Substituting for \( w^2 \) from (2.57) and simplifying, we obtain by further use of (2.57),

\[
Q = \frac{HL}{R} = \frac{B}{2A} = \frac{\sqrt{w^2 - A^2}}{2A} = \frac{\sqrt{4L - R^2}}{2R}.
\]

(2.69)

When the system is loss-free, \( R = 0 \), and \( Q = \infty \). When \( R^2 = 4L/C \), \( Q = 0 \). Should \( R^2 \) be greater than \( 4L/C \), \( Q \) becomes imaginary. Since this condition corresponds to an over-damped or non-oscillatory system, it has here been reshown that \( Q \) is real and meaningful only when associated with a circuit capable of oscillating.

By considering a loss of energy as taking place through a positive resistance and a gain of energy through a negative resistance, it is seen that \( Q \) may assume positive or negative values as the system loses or gains energy. These remarks should be compared with those on the decrement \( A \) given in Sec. 2-4.

Since \( Q \) has now been positively defined, it is desirable to correlate the definition with that in general usage. If the system we have discussed has very low losses - that is, \( R^2 \ll 4L/C \) or \( A^2 \ll w^2 \) - then (2.69) reduces by (2.57) to

\[
Q = \frac{w}{2A} = \frac{wL}{R} = \frac{1}{wOR}.
\]

(2.70)

This is an approximate form of good engineering accuracy for \( Q \gg 10 \).

We may derive (2.70) in another way. Going back to the basic definition of \( Q \) given by (2.54), let us imagine our condenser as loss-free and consider the stored energy at an instant when it is all in the inductance.

\[
Q = 2\pi \frac{\frac{1}{2} L \frac{I^2}{\text{max}}}{\frac{1}{2} \frac{R^2}{\text{max}} f} = 2\pi \frac{fL}{R} = \frac{wL}{R}.
\]

(2.71)
Now the conditions under which this $Q$ would normally be measured are effectively such that the coil and its associated resistance are resonating under the influence of a generator having a series capacitance $C$ equal to $\frac{1}{4\pi^2 f^2 L}$ and a negative resistance equal to the coil resistance $R$, as shown in Fig. 2.6.

![Circuit Diagram](image)

**Fig. 2.6 - Circuit for Measurement of Coil $Q$.**

That is to say, the system is oscillating at uniform current amplitude and hence at the undamped radian frequency, $\omega = 2\pi f = \frac{1}{\sqrt{LC}}$. The overall $Q$ of this system is actually infinite because, considered as a whole, there is no net loss of energy in it. If we measure the power delivered by the generator we can compute a $Q$ at the measured radian frequency $\omega$, and since the coil is the only lossy element we can say that the coil has a Quality Factor, $Q$, given by (2.70).

However, it must be remembered that this measurement of $Q$ was performed under conditions such that the overall $Q$ was infinite. If the circuit $Q$ is to have a finite value the actual radian frequency must be $\omega$ and not $\omega$.

Furthermore, in ascribing a $Q$ to the coil we have presupposed that it was being resonated with a loss-free condenser.
The desired association between the Selectivity $Q$ and the Time Decrement $A$ is given by (2.69). When $A^2 \ll w^2$, $Q$ by approximation becomes proportional to the inverse of $A$ as in (2.70).

We now tabulate the relationships between $Q$ and all four Decrements defined in the Secs. 2-6 and 2-8.

\begin{align*}
(i) \quad A &= \frac{B}{2Q} = \frac{\pi f_d}{Q} ; \\
(ii) \quad \Delta &= 8.686 \frac{A}{Q} = \frac{4.343B}{27.3}f_d ; \\
(iii) \quad \Lambda &= \frac{A}{f_d} = \frac{2\pi A}{B} = \frac{\pi}{Q} ; \\
(iv) \quad \delta &= 8.686 \Lambda = \frac{54.6A}{B} = \frac{27.3}{Q} .
\end{align*}

Here, as earlier, $f_d$ is the damped cyclic frequency of oscillation ($f_d = \frac{B}{2\pi}$). For ordinary engineering work with low-loss circuits $f_d$ is replaced by the steady-state circuit frequency $f$, corresponding to the substitution of $w$ for $B$.

The Selectivity $Q$ may be related also to the complex roots of the auxiliary equation as plotted in Fig. 2.3, Sec. 2-3.

There it was shown that if $\phi$ is the angle between the positive real axis and a radius vector drawn from the origin to one of the conjugate complex values of $\rho$, then $\tan \phi = B/A$. But by (2.69), $Q = B/2A$. Therefore

\begin{align*}
\tan \phi &= 2Q, \\
\phi &= \frac{1}{2}(\tan \phi).
\end{align*}

To summarize the above, these statements may be made.

The Quality Factor or Selectivity $Q$ can be applied strictly only to a circuit capable of oscillating by itself when excited. In "high-$Q$" systems in
which the losses are effectively confined to a single circuit element, that
circuit element may be described by a $Q$, but it is tacitly assumed that the
element will be used in an oscillatory circuit. $Q$ may be positive or nega-
tive depending on whether the circuit loses or gains energy. $Q$ varies from
zero for the critically-damped system to infinity for the loss-free system.
$Q$ is not real for over-damped systems.

2-10. The Term Frequency.

Frequency may be defined as "the number of times a given event re-
peats itself within a specified time interval." It is common in engineer-
ing practice to use the word "frequency" to refer to the number of cycles
of a current wave, or other variation, occurring in one second, and this
quantity is usually denoted by the symbol $f$. Strictly speaking, $f$ should
be called the "cyclic frequency," as opposed to $\omega$ or $\Omega$ which are "radian or
angular frequencies." In this report both $f$ (or $f_d$) and $\omega$ (or $\Omega$) will often
be spoken of simply as "frequencies," where the use of the appropriate sym-
bol serves to indicate whether reference is made to cyclic or to radian fre-
quency.

PARALLEL RLC CIRCUIT ANALYSIS


Consider the circuit shown in Fig. 2.7, and let the condenser have
an initial charge $q_0$. The switch is closed at time $t = 0$, whereupon current
flow takes place as illustrated. It will suffice to solve for the condenser
branch current $i$ as a function of time.
By Kirchhoff's Laws the following equations hold at any instant:

\[ i = i_1 + i_2 , \quad (2.74) \]

\[ \frac{1}{C} \int i \, dt + R \, i_1 = 0, \quad (2.75) \]

and

\[ \frac{1}{C} \int i \, dt + L \frac{d i_2}{dt} = 0. \quad (2.76) \]

Differentiating all three equations once with respect to \( t \), and (2.74) twice with respect to \( t \),

\[ \frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}, \quad (2.77) \]

\[ \frac{i}{C} + R \frac{di_1}{dt} = 0, \quad (2.78) \]

\[ \frac{i}{C} + L \frac{d^2i_2}{dt^2} = 0, \quad (2.79) \]

and

\[ \frac{d^2i}{dt^2} = \frac{d^2i_1}{dt^2} + \frac{d^2i_2}{dt^2}. \quad (2.80) \]

Substituting \( \frac{di_1}{dt} \) from (2.77) into (2.78),

\[ \frac{i}{C} + R \frac{di}{dt} - R \frac{d^2i_2}{dt} = 0. \quad (2.81) \]

Dividing (2.81) through by \( R \) and differentiating once more,

\[ \frac{1}{RC} \frac{di}{dt} + \frac{d^2i_1}{dt^2} - \frac{d^2i_2}{dt^2} = 0. \quad (2.82) \]

Substituting \( \frac{d^2i_2}{dt^2} \) from (2.82) into (2.79),

\[ \frac{i}{C} + L \frac{d^2i_1}{dt^2} + \frac{L}{RC} \frac{di}{dt} = 0. \quad (2.83) \]
Dividing this equation through by \( L \) and rearranging,

\[
\frac{d^2i}{dt^2} + \frac{1}{RC} \frac{di}{dt} + \frac{1}{LC} = 0.
\]  

(2.84)

We assume the solution

\[
i = I e^{-pt} + K.
\]

(2.85)

The "auxiliary equation" associated with (2.84) is

\[
p^2 - \frac{p}{RC} + \frac{1}{LC} = 0,
\]

(2.86)

whence the following conditions on \( p \) are established:

\[
p_1 = \frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} - \frac{1}{LC}},
\]

(2.87)

and

\[
p_2 = \frac{1}{2RC} - \sqrt{\frac{1}{4R^2C^2} - \frac{1}{LC}}.
\]

Letting

\[
A = \frac{1}{2RC},
\]

(2.88)

\[
w = \frac{1}{LC},
\]

and

\[
B_h = \sqrt{A^2 - w^2},
\]

we have

\[
p_1 = A + \sqrt{A^2 - w^2} = A + B_h,
\]

(2.89)

and

\[
p_2 = A - \sqrt{A^2 - w^2} = A - B_h.
\]
As we are most interested in the oscillatory case, where \( w^2 \) is greater than \( A^2 \), we shall change the \( p \) solutions at once into a more convenient form.

Let

\[
B = \sqrt{w^2 - A^2} = -j B_h.
\]  

(2.90)

Then

\[
P_1 = A + j \sqrt{w^2 - A^2} = A + j B,
\]

and

(2.91)

\[
P_2 = A - j \sqrt{w^2 - A^2} = A - j B.
\]

(2.91)

The current solution (2.85) becomes

\[
i = I_1 e^{(-A + jB)t} + K.
\]  

(2.92)

When \( t = 0 \), the current is zero; hence, \( K = 0 \). Now (2.92) may be written

\[
i = I_1 e^{(-A - jB)t} + I_2 e^{(-A + jB)t}.
\]  

(2.93)

Since we have two arbitrary constants \( I_1 \) and \( I_2 \) for the solution of a second-order differential equation, we can say that (2.93) represents a complete solution. The steady-state solution is in this case zero. To evaluate the constants \( I_1 \) and \( I_2 \) we integrate (2.93) thus:

\[
\int i \, dt = q = \frac{I_1}{(-A - jB)} e^{(-A - jB)t} + \frac{I_2}{(-A + jB)} e^{(-A + jB)t} + K'.
\]  

(2.94)

When \( t = \infty \), \( q \) vanishes and hence \( K' = 0 \). At \( t = 0 \), \( q = q_0 \), which gives

\[
q_0 = \frac{I_1}{(-A - jB)} + \frac{I_2}{(-A + jB)}
\]  

\[
= \frac{I_1(-A + jB) + I_2(-A - jB)}{A^2 + B^2}
\]  

\[
= \frac{-A(I_1 + I_2) + jB(I_1 - I_2)}{A^2 + B^2}.
\]  

(2.95)
Substituting $w^2 = A^2 + B^2$ from (2.91) and equating the real and imaginary parts of (2.95),

$$q_0 = \frac{-A}{w^2} (I_1 + I_2),$$

and

$$0 = \frac{B}{w^2} (I_1 - I_2).$$

It follows that

$$I_1 = I_2,$$

and

$$I_1 = \frac{-w^2}{2A} q_0.$$  

Then (2.93) becomes

$$i = \frac{-w^2}{2A} q_0 \left[ e^{-(A - jB)t} + e^{-(A + jB)t} \right],$$

$$= \frac{-w^2}{A} q_0 e^{-At} \left[ \frac{e^{jBt} + e^{-jBt}}{2} \right],$$

$$= \frac{-w^2}{A} q_0 e^{-At} \cos Bt.$$  

The negative sign for the current is explained as follows. The charge on the condenser was taken to be $+q_0$ at $t = 0$. Since $q$ decreases from this value immediately after the switch is closed in the passive network, the slope of the curve $q$ versus $t$ will be negative for $t$ slightly greater than zero. But the current $i$ is exactly this slope $dq/dt$; consequently, $i$ is negative for $t$ slightly greater than zero, as shown by (2.98).

The charge on the condenser as a function of time could be obtained by integrating the current solutions. For the oscillatory case, we have from (2.98)

Sec. 2-11  SOLUTION OF THE DIFFERENTIAL EQUATIONS
\[ q = \int i \, dt = -\frac{e^2}{A} q_0 \int e^{-At} \cos Bt \, dt, \]
\[
= \frac{e^2}{A} q_0 \frac{e^{-At}}{(A^2 + B^2)} (A \cos Bt - B \sin Bt) + k, \]
\[
= \frac{q_0}{A} e^{-At} (A \cos Bt - B \sin Bt) + k. \quad (2.99)\]

When \( t = 0, \) \( q = q_0. \) Substituting these values in (2.99), we find that
\[
q_0 = q_0 + k, \]
and therefore
\[
k = 0. \quad (2.100)\]

Using a transformation (see Appendix C-6) on (2.99), we obtain
\[
q = \frac{q_0}{A} e^{-At} \sqrt{A^2 + B^2} \cos (Bt + \tan^{-1} \frac{B}{A}), \quad (2.101)\]

where
\[
\varphi = \tan^{-1} \frac{B}{A}. \quad (2.102)\]

For convenience the current and charge equations are tabulated here for the three conditions of damping in a parallel RLC circuit. These correspond to (2.39) through (2.44) for the series RLC circuit.

The current equation group is:

for \( \frac{1}{4R^2C^2} > \frac{1}{LC}, \)
\[ i = \frac{-e^2}{A} q_0 e^{-At} \cosh \frac{Bt}{A}; \quad (2.103) \]

for \( \frac{1}{4R^2C^2} = \frac{1}{LC}, \)
\[ w = A, \text{ and } i = -wq_0 e^{-At}; \quad (2.104) \]

for \( \frac{1}{4R^2C^2} < \frac{1}{LC}, \)
\[ i = \frac{-e^2}{A} q_0 e^{-At} \cos Bt. \quad (2.105) \]
The charge equation group is:

\[
\begin{align*}
\text{for } & \frac{1}{4R^2C^2} > \frac{1}{LC}, \quad q = \frac{W}{A} q_0 e^{-At} \cosh(B_i t + \tanh^{-1} \frac{B_i}{A}); \\
\text{for } & \frac{1}{4R^2C^2} = \frac{1}{LC}, \quad q = q_0 e^{-At}; \\
\text{for } & \frac{1}{4R^2C^2} < \frac{1}{LC}, \quad q = \frac{W}{A} q_0 e^{-At} \cos(Bt + \tanh^{-1} \frac{B}{A}).
\end{align*}
\]

(2.106) (2.107) (2.108)

2-12. Plot of the Current and Charge Solutions.

The curves of Fig. 2.6 show q and i plotted versus time, for the three cases. As in the series circuit, \(L\) was chosen to be .0253 henry and \(C\) to be 1 microfarad. Then we have as conditions on \(R\):

(a) Underdamped or Oscillatory Case: \(\frac{1}{4R^2C^2} < \frac{1}{LC}, \quad R > 79.6 \text{ ohms}; \)

(b) Overdamped Case: \(\frac{1}{4R^2C^2} > \frac{1}{LC}, \quad R < 79.6 \text{ ohms}; \)

(c) Critically Damped Case: \(\frac{1}{4R^2C^2} = \frac{1}{LC}, \quad R = 79.6 \text{ ohms}. \)

(2.109)

The six curves should be compared with those of Fig. 2.2. Note that the effect of increasing \(R\) in the parallel RLC circuit is opposite to that in the series RLC circuit. Note also that the \(R\) of critical damping is not the value which reduces the charge to zero most quickly (\(R = 0\) does this) but merely that value which satisfies the equation \(\frac{1}{4R^2C^2} = \frac{1}{LC}\). See Sec. 2-14 for further discussion on this point.


A plot of the \(p\) roots for the parallel RLC circuit as \(R\) is varied will be very much like that shown in Fig. 2.3 of Sec. 2-3.
Fig. 2.8 - Transient Variation of Current and Charge in the Parallel 
RLC Circuit of Fig. 2.7. \( L = 0.0253 \) henry, \( C = 1 \) microfarad.
If we let
\[ \frac{1}{R} = g\sqrt{\frac{C}{L}}, \] (2.110)
the possible values of \( p \) from (2.87) and (2.88) are
\[ p_1 = w(g + \sqrt{g^2 - 1}), \]
\[ p_2 = w(g - \sqrt{g^2 - 1}), \] (2.111)

If \( r \) in Sec. 2-3 is replaced by \( g \), Fig. 2.3 becomes the plot of (2.111) as \( g \) is varied from \(-\infty\) to \(+\infty\). From (2.110) it is seen that \( g \) is an arbitrarily chosen dimensionless conductance factor.

The \( \Phi \) for the parallel RLC circuit may be derived by a process similar to that given in Sec. 2-9. It is found that
\[ \Phi = \frac{R}{E} (BC + B/Lw^2), \] (2.112)
or
\[ \Phi = BCR = B/2A = \frac{\sqrt{w^2 - A^2}}{2A}. \] (2.113)

This reduces to the common form for a low-loss parallel circuit, i.e. \( A^2 \ll w^2 \):
\[ \Phi \approx wCR = R/wL. \] (2.113)
Note that this \( \Phi \) is the inverse of the function obtained for the series RLC circuit. See (2.70).

All the relationships (2.72) between \( \Phi \) and the decrements \( A, \Delta, \Lambda, \) and \( \delta \), hold for the parallel circuit, as does also (2.73) giving \( \Phi \) as a function of the angle \( \delta \). See (2.102).

The fact pointed out at the close of the preceding section — namely, that the Q's for simple series and parallel RLC circuits bear an inverse relationship to each other in their approximate forms (2.70) and (2.113) — serves to introduce the principle of duality. This principle states, in effect, that for each possible network there is a corresponding inverse or dual network such that the expression for the admittance of one is identical with the expression for the impedance of the other. TABLE 2.2 below gives a few of the dual quantities involved in the familiar "series-parallel" transformation process of obtaining the structural inverse of a given circuit.

**TABLE 2.2**

<table>
<thead>
<tr>
<th>Series</th>
<th>Parallel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current</td>
<td>Voltage</td>
</tr>
<tr>
<td>Impedance</td>
<td>Admittance</td>
</tr>
<tr>
<td>Resistance</td>
<td>Conductance</td>
</tr>
<tr>
<td>Inductive Reactance</td>
<td>Capacitive Susceptance</td>
</tr>
<tr>
<td>Capacitive Reactance</td>
<td>Inductive Susceptance</td>
</tr>
<tr>
<td>Etc.</td>
<td>1/Etc.</td>
</tr>
</tbody>
</table>

In circuits which are inverses numerically as well as structurally, the product of each pair of corresponding elements is equal to the same constant. Fig. 2.9 shows a pair of inverse configurations. Suppose we have obtained a solution for the voltage across the condenser in Fig. 2.9(a) as a function of time:

\[ e_c = f(L_1, R_1, C_1, q_0, t) . \]  

(2.114)

By the principle of duality we may then write the solution for the current through the inductance of Fig. 2.9(b) as

\[ i_L = f(C_2, C_2, L_2, i_c, t) , \]  

(2.115)
where the two functions are identical in form, and \( L_1 \) in (2.114) is replaced by \( C_2 \), \( R_1 \) by \( G_2 \), etc.

![Diagram of circuits](image)

**Fig. 2.9 - Simple Inverse Networks Which May be Solved by a Single Analysis Through the Principle of Duality.**

Note that in Fig. 2.9(a) we have a series LR combination which could be regarded as connected in parallel with a series condenser-switch combination. The condenser has initial charge \( q_0 \) (that is, an initial voltage \( q_0/C_1 \) across its terminals) and the switch is initially open. In Fig. 2.9(b) we have a parallel CG combination connected in series with a parallel inductance-switch combination. Here the inductance must have initial current \( i_0 \) and the switch must be initially closed. (The current \( i_0 \) might have been created by the removal of a small permanent-magnet from the coil interior at some instant prior to \( t = 0 \).)

At time \( t = 0 \), switch 1 is closed and switch 2 is opened. The value of \( R_1 \) in (a) which brings the circuit to rest most rapidly is the critical damping resistance, and the value of \( G_2 \) in circuit (b) which brings it to rest most rapidly is the critical damping conductance.

The simple series and parallel RLC circuits of Figs. 2.1 and 2.7, analysed earlier in this chapter, do not present such a point-for-point correspondence. However, it should be noticed that the actual element arrangement of Fig. 2.1 is identical with that of Fig. 2.9(a), while the element arrangement of Fig. 2.7 is identical with that of Fig. 2.9(b). We might therefore expect that certain of the circuit properties, in particular those which do not depend on initial conditions, will prove to be duals...
for the series and parallel configurations of Figs. 2.1 and 2.7. If we write $R$ as $1/G$ in the parallel arrangement, the decrements become identical in form. Thus, for the circuit of Fig. 2.1

$$A = \frac{R}{2\pi},$$

and for Fig. 2.7

$$A = \frac{G}{2\pi}.$$  \hspace{1cm} (2.116)

Similarly, if we take the approximate $Q$ of the series circuit (Fig. 2.1) given by $\omega L/R$ (2.70), and replace inductive reactance $\omega L$ by capacitive susceptance $\omega C$ and resistance $R$ by conductance $G$, we obtain $\omega C/G$ for the approximate $Q$ of the parallel circuit (Fig. 2.7). Referring to (2.113) we see that this is correct, since $\frac{\omega C}{G} = \frac{R}{\omega L}$.

With regard to the "critical" damping resistances obtained in Secs. 2-2 and 2-12, it will be found that they are identical in value with those obtained for the circuits of Fig. 2.9 (a) and (b), respectively, although the critical damping resistance of Fig. 2.7 could not be defined in the same manner as that for Fig. 2.9(b) because of the different initial conditions.

Another example of the concept of duality is observed in the study of vacuum tube equivalent plate circuits. Fig. 2.10(a) shows an elementary triode amplifier, which through the well-known Equivalent Plate Circuit Theorem may be represented by a series "constant-voltage" generator, $E = \mu e_g$, having zero internal resistance, as in (b) of Fig. 2.10. However, through Norton's Theorem it may also be represented by a parallel "constant-current" generator, $I = g_m e_g$, having infinite internal resistance (i.e., zero admittance) and with the tube resistance $r_p$ connected in parallel rather than in series - as shown in (c) of Fig. 2.10.
Finally, the Kirchhoff equations $\sum V = 0$ and $\sum I = 0$ are seen to be a dual pair, leading to Mesh and Nodal forms of network analysis. It should be pointed out in this connection, though, that the Nodal Analysis set forth by Bode is not the dual equivalent of the conventional Mesh Analysis.


The number of degrees of freedom of a system (mechanical, electrical, chemical, etc.) is equal to the least number of variables it takes to specify uniquely the behavior of the system. For example, if we know the charge as a function of time in a simple electrical mesh, we can determine the current, the electromagnetic energy, and the electrostatic energy as functions of time. That is, we are able to answer any question about this elementary network by means of a single coordinate.

In general, the number of degrees of freedom in a given network can be taken to be the same as the number of independent mesh equations. This latter quantity is defined in Appendix D-1.

(2) See Appendix D-2.

(3) Charges, currents, and voltages may be looked upon as network co-ordinates.
Exceptions to the above rule occur in what may be regarded as degenerate networks. Thus, the simple parallel RLC circuit can evidently be described uniquely by a knowledge of only the voltage across the branches as a function of time. (Since the current through any branch containing a single element gives the voltage quite directly, any one such co-ordinate will suffice.)


Suppose that in the case of the series RLC circuit of Fig. 2.1 the condenser was taken to have an initial charge $q_0$. The behavior of the circuit after the switch is closed would obviously not be the same as it was when the condenser was initially uncharged. Yet the differential equations are identical for both cases. And the final steady-state — namely, a static charge of $CE$ residing on the condenser — will also be the same. In order for the solutions of the differential equation to describe accurately and uniquely the circuit behavior at any instant, regardless of what state the circuit was in at $t = 0$, the boundary conditions must be included. These enter through the "arbitrary" integration constants. Integration constants permit the general form of the differential equation to fit the particular physical situation. The number of integration constants must always exactly equal the number of initial conditions that have to be met; otherwise solutions of the differential equation will not be unique for each different physical situation.

It is apparent that a given mesh may not contain more than one equivalent inductance and one equivalent capacitance. For, if the mesh consisted of two branches, each containing both a coil and a condenser, the mesh current would see

(4) It is interesting to note that the Nodal Analysis used by Bode (see Appendix D-2) gives the correct number of independent equations or degrees of freedom in this case; and we may conclude that where the branches all contain single elements, such a Nodal Analysis functions as the dual equivalent of conventional Mesh Analysis to the inverse circuit.
only one combined lump of inductance and one of capacitance. On this basis we may define a maximum number of independent equivalent L's and C's in a network. The total number of independent equivalent L's and C's in a network cannot exceed twice the number of degrees of freedom, by Sec. 2-15.

In general, it is necessary to specify the initial charge on every capacitance in the circuit and the initial current through every inductance. The number of independent equivalent L's and C's in the network gives the minimum number of initial conditions which have to be met. This is the same as saying that we must specify an initial charge and current in every independent path. If \( P \) is the number of independent paths, then there will not be more than \( 2P \) initial conditions required.

It is found that each additional non-reducible L or C in a network increases the order of the associated differential equation by one. Then the order of a differential equation is the same as the number of initial conditions to be met, which is the same as the number of integration constants in the equation — a fact which was utilized without explanation in the analyses of Secs. 2-1 and 2-11.

It follows, also, that the order of a differential equation cannot be more than twice the number of degrees of freedom inherent in the associated network.

Consider the circuit of Fig. 2.11, which is a flat network having four independent mesh currents. Let these be chosen as shown. In mesh I, the equivalent independent capacitance is \( C_I = \frac{C_1C_2}{C_1 + C_2} \), while in mesh IV, it

(5) This is, of course, true for resistances, also, but they are of no interest in the ensuing thought; since they do not affect the order of the associated network differential equation.

(6) i.e., independent equivalent (L or C)
is \( C_{IV} = \frac{C_4C_5}{C_4 + C_5} \). Now if all of the actual condensers have other than zero initial charge, the problem of evaluating the integration constants may become complicated. Usually, however, there are only one or two elements with initial conditions other than zero, and these may be isolated by suitably selecting the meshes. For example, if \( C_5 \) has an initial charge and \( L_1 \) an initial current, we choose one mesh such that \( C_5 \) is the only capacitance in it (say the mesh of branches 5 and 3), and another mesh such that \( L_1 \) is the only inductance in it (say the mesh of branches 1 and 2). The remaining two meshes must be chosen so that at least one mesh current traverses every circuit element.

While the concept of equivalent independent mesh inductances and capacitances is useful in that it relates the order of the differential equation of a network with the number of inherent degrees of freedom, the counting of the independent elements may be reduced to a rather simple process. The number may be found directly from the circuit \( L \)'s and \( C \)'s, and it is not necessary actually to formulate the fictitious independent elements. The method is based on the fact that if an element in a branch common to two meshes has been considered in one of the meshes it may not be considered in the other, for that would be equivalent to assigning it two specifications of initial condition.
Thus, if we traverse mesh IV of Fig. 2.11, we may count elements $L_5$ and $C_5$ as the two independent elements in that branch. In mesh III we count $C_4$ and $L_3$. In mesh II we may count only $C_2$, because $L_3$ has been used in mesh III and is no longer available. In mesh I we may not count $C_2$ but may count $C_1$ and $L_1$. The number of independent elements is then: 2 in mesh IV; 2 in mesh III; 1 in mesh II; and 2 in mesh I; or 7 in all. The differential equation of this network will accordingly be of 7th order. The network has, of course, 4 degrees of freedom.

2-17. Modes of Vibration.

We have repeatedly spoken of differential equations associated with a network. While a "useful" differential equation is necessarily written in terms of only one network coordinate (say, the current in one of the branches), the equation actually represents each and all of the network coordinates. Thus, in the example of Fig. 2.12, the differential equation for any one of the branch or mesh currents or nodal or internal voltages or charges is found to be:

![Fig. 2.12 - An Illustrative Network.](image)

$$\frac{d^3 J}{dt^3} + \frac{d^2 J}{dt^2} \left( \frac{R_1}{L_1} + \frac{R_2}{L_2} + \frac{R_2}{L_3} \right) + \frac{dJ}{dt} \left( \frac{R_1 R_2}{L_1 L_3} + \frac{1}{L_1 C_2} + \frac{1}{L_2 C_2} \right) + \frac{JR_1}{L_1 L_2 C_2} = 0, \quad (2.117)$$

where $J$ is representative of current, voltage, or charge, as the case may be.

The differential equation does not distinguish between network coordinates, but the integrated solutions do - through initial conditions and integration constants.

If we assume a solution $J = X e^{pt}$ for (2.117), the auxiliary or conditional equation is of the form

$$p^3 + p_2 p^2 + p_1 p + p_0 = 0, \quad (2.118)$$
where \( P \) are constant coefficients and are all positive numbers. Since there are no energy sources in the circuit of Fig. 2.12 we will expect the general coordinate \( \mathbf{J} \) to decrease with time. That is, the real parts of all the roots of (2.118) must be negative, giving rise to damping factors of the form \( e^{-A_1 t} \).

Because an LC pair (series or parallel) is capable of oscillating harmonically if the circuit resistance is favorable, and as there are two \( L \)'s and one \( C \) to pair off in Fig. 2.12, we can assume the following roots for (2.118):

\[
\begin{align*}
\rho &= -A_1 + jB_1, \\
\rho &= -A_2, \\
\end{align*}
\]

and

\[
\rho = -A_1 ± jB_1
\]

where \( B_1 \) may be pure real or pure imaginary, depending on the values of \( R_1 \) and \( R_2 \). The solution of (2.117) is then written

\[
\mathbf{J} = K_1 e^{-A_1 t} \sin(B_1 t + \phi_1) + K_2 e^{-A_2 t}.
\]

The necessary three integration constants are \( K_1, K_2, \) and \( \phi_1 \).

Now suppose a circuit similar to Fig. 2.12 does not contain any LC pairs, but has associated with it a 3rd order differential equation. The auxiliary equation will still be of the form of (2.118), but we will expect all three roots to be negative real. Let the roots be

\[
\rho = \rho_1, \rho_2, \rho_3,
\]

and the solution

\[
\mathbf{J} = K_1 e^{\rho_1 t} + K_2 e^{\rho_2 t} + K_3 e^{\rho_3 t}.
\]

Obviously we may find real numbers \( a_1 \) and \( b_1 \) such that

\[
\rho_1 = -a_1 + b_1
\]

and

\[
\rho_2 = -a_1 - b_1.
\]

All the \( \rho \)'s must be negative; then if \( a_1 \) and \( b_1 \) are positive numbers, it follows that \( a_1 \) is greater than \( b_1 \). The solution (2.121) now becomes
The $K_i$ are arbitrary constants and must all be real since the exponentials are real and $J$ is a real quantity. We write (2.123) as

$$J = e^{-a_1 t} (K_1 e^{b_1 t} + K_2 e^{-b_1 t}) + K_3 e^{P_3 t}.$$  \hspace{1cm} (2.124)

Again we may find real numbers $k_1$ and $k_2$ such that

$$K_1 = k_1 + k_2,$$

and

$$K_2 = k_1 - k_2.$$ \hspace{1cm} (2.125)

Then (2.124) becomes

$$J = e^{-a_1 t} \left[ k_1 (e^{b_1 t} + e^{-b_1 t}) + k_2 (e^{b_1 t} - e^{-b_1 t}) \right] + K_3 e^{P_3 t},$$

$$= 2e^{-a_1 t} (k_1 \cosh b_1 t + k_2 \sinh b_1 t) + K_3 e^{P_3 t}. \hspace{1cm} (2.126)$$

If we let

$$k_1 = r \cosh \psi_1,$$

and

$$k_2 = r \sinh \psi_1,$$

so that

$$r = \sqrt{k_1^2 - k_2^2} = \sqrt{K_1 K_2},$$ \hspace{1cm} (2.127)

and

$$\psi_1 = \tanh^{-1} \frac{k_2}{k_1};$$

we obtain

$$J = 2e^{-a_1 t} (r \cosh \psi_1 \cosh b_1 t + r \sinh \psi_1 \sinh b_1 t) + K_3 e^{P_3 t},$$

$$= 2\sqrt{K_1 K_2} e^{-a_1 t} \cosh (b_1 t + \psi_1) + K_3 e^{P_3 t},$$

$$= I_1 e^{-a_1 t} \cosh (b_1 t + \psi_1) + I_2 e^{-a_2 t}, \text{ for symmetry}, \hspace{1cm} (2.128)$$
where \( I_1, I_2 \) and \( \psi_1 \) are the new arbitrary constants.

We have shown that a pair of component functions of time appearing in the solution of the sort of differential equations under consideration here (i.e., linear and homogeneous, with constant coefficients) may always be combined to give a single function of time, regardless of the nature of the roots of the associated auxiliary equation.

Because a hyperbolic function becomes trigonometric as its argument goes imaginary, we shall apply the generic name Modes of Vibration to trigonometric and hyperbolic functions of time alike. It then follows in our definition that the number of modes of vibration of a network is always exactly one-half the order of the associated differential equation, neglecting fractional remainders.

As stated in Sec. 2-16, the order of the differential equation may not be more than twice the number of degrees of freedom; therefore, the number of modes of vibration for a system will be equal to or less than the number of degrees of freedom as defined in Sec. 2-15.

Modes of Vibration is a more uniquely defined quantity than Degrees of Freedom, since it depends only on the order of the network differential equation. On the other hand, the number of Degrees of Freedom appears to depend upon how we go about analysing the network.

The two quantities are often assumed to be synonymous. It is obvious, however, that the generally accepted definition of Sec. 2-15 does not allow such a correlation. If extreme care is taken in handling degenerate circuits and in applying the principle of duality, a different but consistent way of defining

---

(7) RC oscillators produce sinusoidal oscillations even though no inductance is present in circuit. Such a network with the tube dead could have a transient characteristic of the form \( (2.128) \). When the circuit is activated, neglecting certain other changes, we find that the current expands sinusoidally and remains sinusoidal in the steady-state.

(8) See also Appendices D-1 and D-2.
the number of independent network equations may be developed which will give complete agreement. Through that, however, one of the terms would become superfluous.

REPRESENTATION OF OSCILLATORS AND CRYSTALS


In any oscillator circuit it is necessary to provide some source of energy to replace the energy lost in the dissipative elements of the circuit; furthermore, this energy source must be capable of causing oscillations to build up from zero but not to exceed permanently some given value. Generally the source of energy is represented by a negative resistance - that is, a fictitious circuit element which has the properties of resistance but adds energy to the circuit instead of dissipating energy from the circuit. Since this negative resistance functions in a way to limit oscillations, it must be nonlinear with respect to the magnitude of the voltage across or current through it.

Fortunately, the negative resistance devices such as vacuum tube circuits and arcs exhibit the desired properties; however, the nonlinearity is not produced in the same manner in the various devices. In some negative resistances the voltage is a single-valued function of the current, but not conversely; in others the current is a single-valued function of the voltage, but not conversely. All physical negative resistance devices fall into one of these two classes. The characteristics described are shown in Fig. 2.13 (a) and (b).

The resistance represented in Fig. 2.13(a) is called a current-controlled negative resistance. If an operating point, $o$, is selected as the zero for an alternating current, the equivalent dynamic resistance, $\Delta e/\Delta i$, remains nearly constant as the magnitude of the current swing
increases to points \( a \). However, as the current swing exceeds points \( a \), losses are introduced and the negative resistance decreases in magnitude. Thus, for values of alternating current exceeding a certain value, the absolute magnitude of the negative resistance decreases as the current through the circuit increases.

On the other hand, the resistance represented in Fig. 2.13(b) is called a voltage-controlled negative resistance. Its action is similar to the one described above, with voltage substituted for current. For values of alternating voltage exceeding the points \( b \), the absolute magnitude of the negative resistance increases as the voltage across it increases. Generally, vacuum tube circuits exhibit the characteristics of voltage-controlled negative resistances.

It is known that voltage-controlled negative resistances produce stable oscillations in certain types of oscillatory circuits, but not in others. The same statement applies to current-controlled negative resistances. And, the two types of resistance are complementary - that is, one functions where the other does not.


Oscillatory circuits may be divided into two classifications: series-resonant, and parallel-resonant. Any network will exhibit the properties of one of these if a sufficiently small frequency range is taken.
It can be shown that voltage-controlled negative resistances produce stable oscillations only with parallel-resonant circuits and that current-controlled negative resistances produce stable oscillations only with series-resonant circuits. A theorem has been accepted, without complete mathematical proof, which permits the determination of possible points of stable oscillation in an oscillator circuit.

The oscillator is represented by a linear passive admittance $Y_f$, connected to a nonlinear negative conductance $Y_n$, as shown in Fig. 2.14.

\[
Y_f = G_f + j B_f \\
Y_n = -G_n
\]

Fig. 2.14 - Symbolic Representation of Oscillator and Load.

$Y_f$ is plotted as a function of increasing $w$, and the negative of $Y_n$ is plotted as a function of increasing voltage (or current). These are shown together in Fig. 2.15.

Fig. 2.15 - Admittance Characteristics of Oscillator and Load.

The possible points of stable oscillation are given by the theorem:

A point of stable oscillation is indicated when the plot of $Y_f$ with increasing frequency approaches and intersects from the left the plot of $-Y_n$. 
with increasing voltage. Intersections from the right are unstable points. Thus in Fig. 2.15 (1) and (2) are stable points and (3) is an unstable point.

2-20. The Crystal and Its Associated Parameters.

It was shown by K. S. Van Dyke that a crystal operating at or near its principal resonance can be accurately represented by the configuration given in Fig. 2.16. The shunt capacitance \( C_h \) is that of the holder plates with the crystal material as dielectric. Because it has been found desirable in many circuits to incorporate an additional padding capacitance \( C_k \) across the crystal terminals, we show our adopted notation for that case also, in Fig. 2.17 (a) and (b), and proceed to define some quantities which will be used throughout this report.

Definitions:

\[
\begin{align*}
A_0 &= \frac{R_0}{2L_0} \\
\omega_0^2 &= \frac{1}{L_0C_0} \\
\eta_0 &= \frac{C_h}{C_0} \\
\eta &= \frac{n_0 + 1}{n_0} \\
\end{align*}
\]

\[(2.129)
(2.130a)\]
There are a number of facts implicit in the representation of Fig. 2.17(b) which are not too generally appreciated.

(a) First of these is that the shunting capacitance $C_x$, which cannot be less than $C_h$, constitutes the most serious restriction on the resonator. Were it not for the existence of this capacitance, the problem of damping a crystal would be extremely easy.

(b) The principal features of the performance of a crystal depend largely upon the capacitance ratio, $n$. The crystalline structure of quartz is such that $n_0$ (the minimum of $n$) can never be less than 125. The natural quantity $n_0$ rarely exceeds 3000, and, using normal values of padding capacitance, the quantity $n$ is likely to be limited in most practical cases to 4000.

(c) It has been shown in Appendix E that the series resonant frequency of a crystal may be approximated be the frequency $w_0$ with a maximum error of 20 parts per million; and the parallel resonant frequency is
practically the same as the frequency $w$, the error involved being about 1 part in $10^{10}$. Furthermore, on this basis, we may define the series resonant frequency to be approximately that at which a short-circuited crystal will tend to resonate, while the parallel or antiresonant frequency is essentially that at which an open-circuited crystal will tend to resonate.

Finally, it is demonstrated that, because of the 125 minimum limit on $n$, the antiresonant frequency cannot exceed the series resonant frequency by more than about 0.4%.

(d) A property which is also of great interest is the sharpness of resonance or selectivity, commonly stated by the equation

$$Q_o = \frac{w_0 L_o}{R_o}.$$  \hspace{1cm} (2.133)

The above equation gives the selectivity at series resonance, but it may be shown that the selectivity at antiresonance is not significantly different. The crystal selectivity will often be written as

$$Q_y = \frac{w_y L_o}{R_o},$$  \hspace{1cm} (2.134)

which may be regarded as a purely formal definition of $Q$ at the crystal antiresonance.

The selectivity of crystals is characteristic of very high, between 10,000 and 1,000,000 constituting a convenient range to assume for quartz.

(e) A feature which is not so commonly understood is the impedance level. It is clear that neither the resonant frequency nor the selectivity will be modified if every impedance in the circuit is modified by some fixed constant. But the usefulness as resonator may be considerably modified. At series resonance the equivalent circuit reduces to the single resistance $R_o$, which is ordinarily in the order of 100 ohms. Vacuum tubes which are available at present do
not produce amplification when used in conjunction with circuits having an impedance level below a few hundred ohms. For this reason it is necessary to employ impedance transforming networks, or more than one tube, in oscillators which employ ordinary crystals at their series resonance.

At the antiresonant frequency a crystal also presents a purely resistive impedance. At this frequency, however, the impedance becomes very high. It is shown in Appendix F that at parallel resonance the impedance is given by the equation

\[ R_p = \frac{Q_0^2 R_0}{n^2} \]  

(2.135)

If \( Q_0 = 100,000 \), \( R_0 = 100 \), and \( n = 200 \), the impedance reaches the value of \( 2.5 \times 10^7 \) ohms or 25 megohms, which is, of course, larger than the impedance normally used in vacuum tube circuits. This equation indicates why large values of \( n \) are advantageous in promoting frequency stability.

From the foregoing it appears that the design of crystal units might profitably be modified. Crystals for use at series resonance would be more convenient if the impedance level were increased so as to make \( R_0 \) of the order if 1000 ohms. Crystals for use at antiresonance would be better if the impedance were something less than a megohm. A superficial consideration of the problem indicates that the increase in impedance level could probably be achieved by reducing the total area of crystals such as the AT or BT.
CHAPTER III
CRYSTAL DAMPING - PART I

Because crystal damping is a requirement for high speed keying, it is necessary to develop circuits which provide a sufficient degree of damping along with a negligible effect on the frequency of oscillation. The succeeding analyses will assume that the damping circuit is connected directly to the crystal; however, in practical oscillator circuits, provision must be made to switch the damping circuit in and out of the oscillator circuit to provide damping during the decay period and to remove the damping circuit when the oscillator is operating. Furthermore, the oscillator circuit external to the crystal and damping circuit will be represented by a capacitance across the crystal terminals. This representation is essentially correct because this portion of the circuit has a capacitive reactance throughout the keying cycle and relatively small losses during the decay period.

The only obvious requirement of a damping circuit is that it contain a dissipative element, that is, a resistor. The crystal equivalent circuit suggests the three damping circuit configurations shown in Fig. 3.1.

Fig. 3.1 - Resistance Damping Circuits.
A detailed theoretical analysis should reveal the advantages and limitations of these circuits. Two basic methods of attack exist. Complex algebra leads with moderate facility to solutions which are approximately correct. Differential equations are more difficult but give complete and exact solutions. Since the circuit using pure resistance for damping is capable of oscillating in only one mode, an analysis with complex algebra will produce verifiable results. Damping circuits containing both resistance and inductance are capable of oscillating in two modes. The analysis of such circuits is much more involved and will be conducted with both complex algebra and differential equations.

**RESISTANCE DAMPING**

The simplest and most obvious method of damping a quartz crystal is by the connection of a resistance across its terminals. The actual and equivalent circuits for this method are shown in Fig.3.2. Because any oscillations which occur will be at a frequency which is very near to the resonant frequency of $L_0$ and $C_0$, it is legitimate to use the series-parallel transformation shown.

![Circuit Diagram](image)

*Fig. 3.2 - Equivalent Damping Circuits.*

3-1. **Determination of Time Decrement.**

The highest decay rate is obtained when $R_z$ is a maximum; therefore, the analysis must determine the relationship between circuit parameters required to produce a maximum $R_z$. Using the definitions
\[ x_X = \frac{1}{wC_X}, \quad x_Z = \frac{1}{wC_Z}, \quad x_0 = \frac{1}{wC_0}, \quad \text{and} \quad n = \frac{C_X}{C_0}. \] (3.1)

The actual impedance, \( Z_X \), is equated to its equivalent impedance \( Z_z \). (9)

Then
\[ R_z - jx_z = \frac{-jX_x R_x}{R_x - jX_x} = \frac{x_X^2 R_x - jX_x R_x^2}{R_x^2 + X_x^2}. \] (3.2)

And by equating the real and imaginary components of \( Z_x \) and \( Z_z \) the equations
\[ R_z = \frac{x_X^2 R_x}{R_x^2 + X_x^2} \quad \text{and} \quad x_z = \frac{R_x x_X^2}{R_x^2 + X_x^2} \] (3.3)

are obtained. Now \( R_z \) is a maximum with respect to \( R_x \) when
\[ \frac{dR_z}{dR_x} = \frac{x_X^2 (R_x^2 + X_x^2) - 2R_x (X_x^2 R_x)}{(R_x^2 + X_x^2)^2} = 0. \] (3.4)

From the above it is seen that
\[ R_x = x_x \] (3.5)

for maximum \( R_z \). The maximum \( R_z \) is given by
\[ R_z = \frac{x_X^3}{2X_x^2} = \frac{x_X}{2} = \frac{R_x}{2}. \] (3.6)

Since \( R_Z \gg R_0 \), the resonant \( Q \) of the circuit is
\[ Q = \frac{x_0}{R_z} = \frac{2X_0}{X_x} = \frac{2C_X}{C_0} = 2n. \] (3.7)

(9) This transformation is valid if the frequency is not changed appreciably in the process of making the transformation. It is known that a large variation in \( C_X \) does not change the frequency noticeably. Since the \( Q \) of the circuit is large even though \( R_z \) is added, \( R_z \) has little effect on frequency. Because of these facts it is concluded that the transformation cannot affect the frequency enough to cause serious error.
Thus, the minimum $Q$ obtainable is equal numerically to twice the capacitance ratio $C_x/C_0$.

The decrement, $\Delta$, will be used to determine the effectiveness of this method. From (2.72-ii),

$$\Delta = \frac{4.34 \omega}{Q} \text{ dB/sec} = \frac{27.3f}{2\pi n} \text{ dB/sec}.$$  \hspace{1cm} (3.8)

In this case the damping rate increases as the frequency increases and decreases as $n$ increases. Since operations will be conducted over a considerable range of frequencies, it is encouraging to note that higher frequencies promote rapid damping. The use of a small capacitance ratio is recommended by (3.8).

It has been shown that $\Delta$ must be greater than 6 dB/millisecond if satisfactory damping is to be attained to permit keying at a rate of 43 dot-cycles/second. In actual practice, $n$ may be expected to vary between 250 and 2000. Assuming the larger value, when $f = 1 \text{ mcts}$, $\Delta$ is calculated as follows:

$$\Delta = \frac{27.3 \times 1 \times 10^6}{2 \times 2000} \text{ db/sec} = 6.83 \text{ db/millisecond}.$$  \hspace{1cm} (3.9)

It is seen, therefore, that satisfactory operation with respect to decay rates can be obtained by this method of damping.

3-2. Frequency Shift during Damping.

The second important consideration is the effect of damping upon the frequency of oscillation. Because of the inherent complexity of any analysis of oscillators and the approximations involved therein, it is elected to determine the difference in frequency between a damped and undamped vibrating crystal and then to interpret these findings in terms of an actual
oscillator. It was stated in footnote(9) that the frequency of oscillation is not changed appreciable in making the transformation shown in Fig. 3.2. This is correct in that the accuracy of the transformation is adequate for purposes of determining damping. It does not, however, mean that the exact frequency of the crystal oscillations is unaffected by the addition of the damping resistor, \( R_x \).

Consider then the two circuits shown in Fig. 3.3, where \( R_x \) has the value which produces maximum \( R_z \). Fig. 3.3(b) is the equivalent of Fig. 3.3(a) when SW is closed.

From (3.3) and (3.5)

\[
x_z = \frac{R_x^2 X_x}{R_x^2 + X_x^2} = \frac{X_x}{2}, \tag{3.10}
\]

\[
c_z = 2 c_x, \tag{3.11}
\]

and

\[
R_z = \frac{R_x}{2} = \frac{X_x}{2}. \tag{3.12}
\]

With the switch(SW) open, the frequency of the circuit in Fig. 3.3(a) is given by

\[
w_1 = \sqrt{\frac{1}{L_0} \left[ \frac{1}{c_0} + \frac{1}{c_x} \right] - \frac{R_0^2}{4L_0^2}}. \tag{3.13}
\]

The frequency of the circuit in Fig. 3.3(b) is given by

\[
w_2 = \sqrt{\frac{1}{L_0} \left[ \frac{1}{c_0} + \frac{1}{c_z} \right] - \frac{(R_0 + R_z)^2}{4L_0^2}}. \tag{3.14}
\]
However, because both circuits have high values for $Q$, certain approximations and simplifications may be made.

Let

$$Q_y = \frac{w_yL_0}{R_0} \quad \text{and} \quad Q_d = \frac{w_dL_o}{(R_o + R_z)} ,$$

(3.15)

and let

$$w_y = \sqrt{\frac{1}{L_o} \left[ \frac{1}{C_0} + \frac{1}{C_x} \right]} \quad \text{and} \quad w_d = \sqrt{\frac{1}{L_o} \left[ \frac{1}{C_0} + \frac{1}{C_z} \right]} .$$

(3.16)

Then, (3.13) and (3.14) become respectively,

$$w_1 = \sqrt{w_y^2 - \frac{w_y^2}{4Q_y^2}} = w_y \sqrt{1 - \frac{1}{4Q_y^2}} ,$$

(3.17)

and

$$w_2 = \sqrt{w_d^2 - \frac{w_d^2}{4Q_d^2}} = w_d \sqrt{1 - \frac{1}{4Q_d^2}} .$$

(3.18)

Now $Q_y$ may be expected to be greater than 10,000. The term

$$\frac{1}{4Q_y^2} \leq \frac{1}{4 \times (10^4)^2} = 2.5 \times 10^{-9} ,$$

is very much smaller than 1 and may be discarded. To an accuracy better than one part in $10^8$, $w_1$ may be defined as

$$w_1 = w_y = \sqrt{\frac{1}{L_o} \left[ \frac{1}{C_0} + \frac{1}{C_x} \right]} .$$

(3.19)

It has been estimated that the minimum practical value of $n$ is 250. Referring to (3.7), it is found that the minimum $Q$ obtainable with damping is given by

$$Q_d \geq 2n = 500 .$$

(3.20)

Then

$$\frac{1}{4Q_d^2} \leq \frac{1}{4 \times (500)^2} = 10^{-6} .$$

(3.21)

Since this value is very much smaller than 1, (3.18) becomes, with an error
of one-half part in $10^6$, or less,

$$w_2 = w_d = \sqrt{\frac{1}{L_0} \left[ \frac{1}{C_0} + \frac{1}{C_x} \right]}.$$  \hfill (3.22)

Thus it is seen that the purely resistive effects of damping have negligible effects on the frequency of oscillation. There remains the determination of the frequency shift which results from the reactance change.

Substituting (3.11) in (3.22), we have

$$w_d = \sqrt{\frac{1}{L_0} \left[ \frac{1}{C_0} + \frac{1}{2C_x} \right]}. \hfill (3.23)$$

But $C_x = n C_0$, so from (3.16)

$$w_y = \sqrt{\frac{1}{L_0} \left[ 1 + \frac{1}{n} \right]}, \hfill (3.24)$$

and from (3.23)

$$w_d = \sqrt{\frac{1}{L_0 C_0} \left[ 1 + \frac{1}{2n} \right]}; \hfill (3.25)$$

therefore,

$$\frac{w_d}{w_y} = \sqrt{\frac{n + 0.5}{n + 1}} = \frac{(n + 0.5)^{1/2}}{(n + 1)^{1/2}}.$$  \hfill (3.26)

Expanding the numerator and denominator of (3.26) in a binomial series, there results

$$\frac{w_d}{w_y} = \frac{n^{1/2} + \frac{1}{4} n^{-1/2} - \frac{1}{32} n^{-3/2} + \cdots}{n^{1/2} + \frac{1}{2} n^{-1/2} - \frac{1}{8} n^{-3/2} + \cdots}. \hfill (3.27)$$

But, since $n \gg 1$, the terms beyond the second may be neglected; so, dividing by $n^{1/2}$, we have

$$\frac{w_d}{w_0} = \frac{1 + \frac{1}{4n}}{1 + \frac{1}{2n}}.$$  \hfill (3.28)

From a similar expansion, $1 - \varepsilon = \frac{1}{1 + \varepsilon}$, if $\varepsilon \ll 1$. Let $\varepsilon = \frac{1}{2n}$. Then
For engineering application of this damping method, certain circuit and operational requirements must be considered. It is desirable that the oscillator circuit parameters be fixed and that changing frequency be accomplished by changing the crystal only. This introduces two effects: (a) the

deviation of $R_z$ from the maximum $R_z$, and that of the damping frequency from oscillator frequency when $R_x$ is selected for some specific frequency and is not varied as the frequency is varied; and (b) the effect of a variation in the crystal holder capacitance, $C_h$, on $R_z$ and on the damping frequency.

From Fig. 3.2, it is seen that $C_x = C_h + C_k$. So, let

$$a = \frac{C_h}{C_k}$$

then $C_x = C_k(1 + a)$. \hspace{1cm} (3.32)

The $R_x$ required for maximum damping is

$$R_x = \frac{1}{\omega C_x} = \frac{1}{\omega C_k(1 + a)}.$$ \hspace{1cm} (3.33)

Then

$$R_z = \frac{1}{2\omega C_k(1 + a)}.$$ \hspace{1cm} (3.34)

Let us determine the effect on $R_z$ of a variation in frequency when $R_x$ is constant. The calculation is facilitated by substituting $\omega = \omega_a$, where $\omega_a$ is the angular frequency specified in calculating $R_x$, and $\omega$ is the actual frequency. Then, from (3.33),

$$R_x = \frac{1}{\omega_a C_k(1 + a)} \equiv S; \hspace{1cm} (3.35)$$

and substituting $R_x$ from (3.35) and $X_x$ from (3.33) into (3.3),

$$R_z = \frac{\left[ \frac{1}{\omega_a C_k(1 + a)} \right]^2 + \left[ \frac{1}{\omega C_k(1 + a)} \right]^2}{\left[ \frac{1}{\omega_a C_k(1 + a)} \right]^2 + \left[ \frac{1}{\omega C_k(1 + a)} \right]^2} = \frac{\left[ \frac{1}{\omega_a C_k(1 + a)} \right]^3 \frac{1}{c^2}}{\left[ \frac{1}{\omega_a C_k(1 + a)} \right]^2 \left[ 1 + \frac{1}{c^2} \right]}, \hspace{1cm} (3.36)$$

or

$$R_z = \frac{1}{\omega_a C_k(1 + a)} \frac{1}{1 + c^2} = S \frac{1}{1 + c^2}. \hspace{1cm} (3.37)$$
If \( R_x \) is readjusted for optimum damping at each frequency, the following modified equation for \( R_x \) is obtained:

\[
R_x = \frac{1}{w_a C_k (1 + a)} = \frac{S}{c} .
\]  

(3.38)

From (3.6) and (3.38), then, the maximum equivalent resistance is

\[
R_z = \frac{R_x}{2} = \frac{S}{2c} .
\]  

(3.39)

Plots of (3.37) and (3.39) are given in Fig. 3.4.

From Fig. 3.4 it is seen that \( R_z \) does not deviate from the maximum \( R_z \) obtainable by more than 10\% for a frequency range of 2.5, or more than 20\% for a range of 4, even though \( R_x \) is selected at one specific frequency. Because optimum damping is unlikely to be needed at all points in the band, it is probable that engineering application will permit a considerable extension of this ratio. Thus it is evident that satisfactory damping can be obtained over a reasonable frequency range with one specific damping resistance.

3-4. Variation of Frequency Shift.

An examination of the frequency shift over this frequency range is in order. As with the damping study, \( R_x \) is chosen for \( w_a \). Then

\[
R_x = \frac{1}{w_a C_x} \text{ and } X_x = \frac{1}{c w_a C_x} .
\]  

(3.40)

Substituting (3.40) in (3.3),

\[
X_z = \frac{1}{w_a C_x} \left( \frac{1}{c w_a C_x} \right) = \frac{1}{c w_a C_z} .
\]  

(3.41)
Fig. 3.4 - Variation of $R_z/S$ with Frequency
or
\[ c_z = c_x \frac{1 + c^2}{c^2} = c_x \left(1 + \frac{1}{c^2}\right). \]  

(3.42)

But, since \( c_x = nC_0 \),

we have

\[ c_z = nC_0 \left(1 + \frac{1}{c^2}\right). \]  

(3.43)

Then (3.22) becomes

\[ w_d = \sqrt{\frac{L_0C_0}{1 + \frac{1}{n(1 + \frac{1}{c^2})}}} \].  

(3.44)

From (3.24) and (3.44), we have

\[ \frac{w_d}{w_y} = \frac{\left[1 + \frac{1}{n(1 + \frac{1}{c^2})}\right]^{\frac{1}{2}}}{\left[1 + \frac{1}{n}\right]^{\frac{1}{2}}}. \]  

(3.45)

Upon expansion using the binomial theorem and neglecting high order terms, this equation becomes

\[ \frac{w_d}{w_y} = \frac{1 + \frac{1}{2n(1 + \frac{1}{c^2})}}{1 + \frac{1}{2n}}. \]  

(3.46)

For the same reason that (3.29) is valid, (3.46) becomes

\[ \frac{w_d}{w_y} = \left[1 + \frac{1}{2n(1 + \frac{1}{c^2})}\right] \left[1 - \frac{1}{2n}\right]. \]  

(3.47)
Multiplying and eliminating terms in \( \frac{1}{n^2} \), we have

\[
\frac{w_d}{w_y} = 1 - \frac{1}{2n(1 + c^2)}
\]  

(3.48)

From (3.48) it is evident that the frequency shift is less than that resulting from maximum damping when \( c \) is greater than 1, and vice versa. Refer to (3.30).

It is estimated that \( C_h \) may vary between 5 mmfd. and 25 mmfd. A load capacitance of \( C_k = 32 \) mmfd. has been tentatively specified as standard. From (3.32) it is seen that under these conditions \( a \) may vary between 0.16 and 0.78.

To determine the effect on \( R_z \) of holder capacitance variation, let \( a_o \) be the value of \( a \) for which \( R \) is calculated. From this and (3.33)

\[
R_x = \frac{1}{wC_k(1 + a_o)} = \frac{x_k}{1 + a_o}
\]  

(3.49)

Substituting \( R_x \) from (3.49) and \( X_x \) from (3.33) into (3.3),

\[
R_z = \frac{x_k}{(1 + a_o)^2 + (1 + a)^2}
\]  

(3.50)

Plots of (3.49) and (3.50) are presented in Fig. 3.5 for \( a_o = 0.25, 0.5, 0.75 \), and all values of \( a \) between 0 and 1. The curve \( a = a_o \) represents the maximum damping obtainable (\( R_x \) is readjusted as \( a \) changes). From this graph it is evident that the damping obtained for a fixed value of \( R_x \) does not deviate appreciably from the maximum obtainable; specifically, for \( a_o = .5 \), the actual damping will not vary more than 3% from maximum even though \( a \) is varied from 0.16 to 0.78. Therefore, by a judicious selection of \( a_o \), the damping may be made essentially independent of crystal holder capacities encountered with present crystals. It should be noted that as \( a \) increases, \( R_z \) decreases; so, larger holder capacities are unfavorable to rapid damping.
Capacitance Ratio, $a = C_h/C_k$

Fig. 3.5 - Plot of $\frac{R_z}{X_k} = \frac{1 + a_0}{(1 + a_0)^2 + (1 + a)^2}$.
The effect on the frequency shift, as the holder capacity is varied, is readily determined from (3.1), (3.3), and (3.32). The resulting equation for $C_z$ is

\[
\frac{1}{\omega C_z} = \frac{1}{w^2c_k^2(1 + a_0)^2} \frac{1}{2c_k(1 + a)} \cdot \frac{1}{w^2c_k^2(1 + a_0)^2} + \frac{1}{2c_k^2(1 + a)^2}
\]  \hspace{1cm} (3.51)

But $c_k = \frac{c_x}{1 + a}$; so,

\[
C_z = c_x \left[ 1 + \frac{(1 + a_0)^2}{(1 + a)^2} \right]
\]  \hspace{1cm} (3.52)

Then, from (3.22), (3.24), and (3.52),

\[
\frac{w_d}{w_y} = \sqrt{\frac{1}{L_0C_0} \left[ 1 + \frac{1}{n \left( 1 + \frac{1}{1 + a_0} \right) \left( 1 + \frac{(1 + a_0)^2}{(1 + a)^2} \right)} \right]}
\]  \hspace{1cm} (3.53)

or expanding in a binomial series and dropping the inconsequential higher order terms

\[
\frac{w_d}{w_y} = \frac{1 + \frac{1}{2n} \left[ 1 + \frac{(1 + a_0)^2}{(1 + a)^2} \right]}{1 + \frac{1}{2n}}
\]  \hspace{1cm} (3.54)
For the same reason that (3.29) is valid, (3.54) becomes

\[ \frac{w_d}{w_y} = \left[ 1 + \frac{1}{2n(1 + \frac{1}{(1 + a_o)^2})} \right] \left[ 1 - \frac{1}{2n} \right]. \quad (3.55) \]

By multiplying, eliminating terms in \( \frac{1}{n^2} \), and simplifying, we have

\[ \frac{w_d}{w_y} = 1 - \frac{(1 + a_o)^2}{2n \left[ (1 + a)^2 + (1 + a_o)^2 \right]} \quad . \quad (3.56) \]

The last term of (3.56) represents the frequency departure. We shall denote the multiplier of \( \frac{1}{2n} \) by \( f(a) \) - that is,

\[ f(a) = \frac{(1 + a_o)^2}{(1 + a)^2 + (1 + a_o)^2} \quad . \quad (3.57) \]

This function is plotted versus \( a \) in Fig. 3.6 for \( a_o = 0.25, 0.50, 0.75, \) and \( a \). The line, \( a_o = a \), corresponds to the frequency departure when \( R_x \) is adjusted to give maximum \( R_z \) for all values of \( a \). It is seen that the frequency shift is greater than that for optimum damping when \( a < a_o \), but smaller when \( a > a_o \).

From this theoretical examination the effectiveness and limitations of resistance damping have been determined. A damping resistance will provide satisfactory damping at moderate keying speeds over a considerable frequency range; however, the frequency shift during damping is dependent on the operating frequency and the frequency for which \( R_x \) is calculated. If the operating frequency is greater than the frequency for which \( R_x \) is calculated, the frequency shift is less than that resulting when \( R_x \) has a value to produce maximum \( R_z \); but when the operating frequency is less, the frequency shift is greater. Variation of the crystal holder capacitance
Fig. 3.6 - Plot of \( f(a) = \frac{(1 + a_o)^2}{(1 + a)^2 + (1 + a_o)^2} \)

Capacitance Ratio, \( a = \frac{C_h}{C_k} \)

\( a_o = a \)
\( a_o = 0.75 \)
\( a_o = 0.50 \)
\( a_o = 0.25 \)
has little effect on damping, but it does affect the frequency shift. Where frequency range requirements are not too great, \( R_x \) should be calculated for the minimum operating frequency and the minimum anticipated holder capacitance.

It is believed that this circuit will provide satisfactory performance at keying speeds for which the decay period represents a small portion of the entire keying cycle. Under these conditions the frequency shift during damping will be lost in the general decay transient.

Experimental verification of the decay rate phase of this study is presented in Appendix G. Considering the difficulties encountered in performing the experimental measurements, the experimental and theoretical conclusions were in complete agreement. Numerous attempts have been made to measure the frequency shift which theoretically occurs. Up to the present it has been experimentally impossible to obtain data which indicate that a frequency shift exists during damping.

FREQUENCY COMPENSATED DAMPING

During the study of resistance damping the frequency shift which occurs when the damping resistor is applied was found to be caused by a change in the equivalent reactance in series with \( L_o, R_o, \) and \( C_o \). This is a direct result of having the holder capacitance inseparable from \( L_o, R_o, \) and \( C_o \). To obtain zero frequency-shift damping requires an impedance which can be connected to the crystal terminals as shown in Fig. 3.7 and yet not affect the equivalent reactance \( X_z \) in series with \( L_o, R_o, \) and \( C_o \).

\[
Z = R - j X_z
\]

**Fig. 3.7 - Circuit Configurations.**

\[
Z = R - j X_z
\]

**Fig. 3.8 - Network.**
In the circuit of Fig. 3.8 it can be shown \(^{(11)}\) that if \( \frac{1}{wC} = wL \), the reactance is independent of \( R \) for all values of \( R \neq 0 \). The impedance of this circuit is given by the equation

\[
Z = \frac{1}{w^2C^2R} - j \frac{1}{wC} \quad (3.57)
\]

Thus, a circuit is available to perform the function required of \( Z_x \). This circuit and the conditions imposed upon it are shown in Fig. 3.9.

As \( R_x \) decreases, \( R_z \) increases and the damping increases. It is obvious that if \( R_x \) is too small the circuit will oscillate in two modes. Under this condition the results obtained by complex algebra are not useful. Because of this and the fact that \( L_x \) must be adjusted as the frequency changes, it has been decided that this circuit should be studied with the objective of obtaining extreme damping rates at a specific frequency, and that the examination is best conducted using differential equations. That study is presented in Chap. IV.

3-5. Wide Band Frequency Compensated Damping.

It will be shown that extreme damping rates can be obtained by the circuit of Fig. 3.9(a) with no frequency shift for one specific frequency; however, that method requires that the damping network be readjusted as the operating frequency

\(^{(11)}\) See Appendix H.
changes. For certain applications a damping circuit is required which produces negligible frequency shift and moderate damping over a band of frequencies.

In Fig. 3.8 a circuit was presented for which the reactance is independent of $R (R \neq 0)$, if $\omega L = \frac{1}{\omega C}$. Extension of this theorem to obtain a damping circuit for a frequency band necessitates the use of an inductive impedance which varies inversely with frequency, in place of $L$. The ideal impedance is a negative capacitance, which is non-physical.

Fig. 3.10 - Network and Impedance Plot.

Fig. 3.10 presents a circuit and associated impedance plot for which the inductive reactance varies more or less inversely with frequency. This circuit will be examined to determine the values for $L$, $R$, and $C$ which will most satisfactorily approximate a negative capacitance in series with a resistance. The equivalent impedance will be the $Z_x$ of Fig. 3.7.

3-6. Design Procedure.

In Fig. 3.10 we have three elements to which values must be assigned. Therefore it is necessary to supply three independent data or numbers. More constructively, it is possible to apply three independent requirements. One useful interpretation of these requirements is as follows:

The $LC$ product will evidently determine a sort of resonant frequency which may be associated with the lowest frequency at which the network is to be used. The $L/C$ ratio will produce a sort of sharpness or shape factor.
which will determine the range of frequencies over which the approximation is acceptable. And the L/CR ratio will determine the impedance level, which in turn will depend upon that of the crystals to be damped.

In order to proceed it is necessary to assume that all the crystals of one set, which are to be damped by a given network, have a value of holder capacitance, $C_h$, which varies in some simple way with the crystal frequency. For simplicity it will be assumed that all have equal values of $C_h$. A modified network design will be required for any other relationship.

To simplify the design it is convenient to assign the value of unity to the resistance. This is a form of normalization. Actual designs are obtained from the normal form by modifying all three impedances in the same ratio. If we will leave the operating frequency range as an undetermined coefficient we may also assign the value of unity to the condenser. This leaves only the inductance and the operating frequency to be determined.

A convenient form for this sort of calculation appears in TABLE 3.1. The impedance of the series combination of $R_d$ and $C_d$ is identified as $Z_1$. The admittance of the coil above is identified by $Y_2$. Fig. 3.11 shows the variation of the equivalent series $X$ plotted against the reciprocal of frequency for three values of $L_d$. It also shows the variation of equivalent series resistance for the intermediate value of inductance.

<table>
<thead>
<tr>
<th>$1/w$</th>
<th>$Z_1$</th>
<th>$Y_1 = 1/Z_1$</th>
<th>$Y_2$</th>
<th>$Y_{eq}$</th>
<th>$Z_{eq}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.05</td>
<td>1 - j.05</td>
<td>.999 + j.049</td>
<td>-j0.115</td>
<td>.999 - j.066</td>
<td>.999 + j.066</td>
</tr>
<tr>
<td>.10</td>
<td>1 - j.10</td>
<td>.990 + j.099</td>
<td>-j0.230</td>
<td>.990 - j.131</td>
<td>.990 + j.131</td>
</tr>
<tr>
<td>.15</td>
<td>1 - j.15</td>
<td>.978 + j.147</td>
<td>-j0.345</td>
<td>.978 - j.198</td>
<td>.982 + j.199</td>
</tr>
<tr>
<td>.20</td>
<td>1 - j.20</td>
<td>.962 + j.192</td>
<td>-j0.460</td>
<td>.962 - j.268</td>
<td>.964 + j.269</td>
</tr>
<tr>
<td>.25</td>
<td>1 - j.25</td>
<td>.942 + j.236</td>
<td>-j0.575</td>
<td>.942 - j.333</td>
<td>.940 + j.338</td>
</tr>
<tr>
<td>.30</td>
<td>1 - j.30</td>
<td>.918 + j.276</td>
<td>-j0.690</td>
<td>.918 - j.414</td>
<td>.905 + j.408</td>
</tr>
<tr>
<td>.35</td>
<td>1 - j.35</td>
<td>.892 + j.312</td>
<td>-j0.805</td>
<td>.892 - j.493</td>
<td>.858 + j.474</td>
</tr>
<tr>
<td>.40</td>
<td>1 - j.40</td>
<td>.862 + j.344</td>
<td>-j0.920</td>
<td>.862 - j.576</td>
<td>.801 + j.535</td>
</tr>
<tr>
<td>.45</td>
<td>1 - j.45</td>
<td>.829 + j.373</td>
<td>-j1.035</td>
<td>.829 - j.662</td>
<td>.738 + j.590</td>
</tr>
<tr>
<td>.50</td>
<td>1 - j.50</td>
<td>.800 + j.400</td>
<td>-j1.150</td>
<td>.800 - j.750</td>
<td>.665 + j.622</td>
</tr>
</tbody>
</table>

TABLE 3.1
CALCULATION OF A NETWORK IMPEDANCE
($R_d = 1$ ohm, $C_d = 1$ farad, $L_d = 1/2.3$ henries)
Fig. 3.11 - Wide-Band Frequency Compensated Damping.
It is seen that for \( L_d = 1/2.3 \) henries the value of \( X_{eq} \) is almost proportional to \( 1/w \) over a wide range of frequencies. Therefore we may conclude that this network will provide damping which is virtually free from frequency shift for all crystals having a given holder capacitance and a frequency above some minimum value.

3-7. Example.

To illustrate the technique suppose that the total shunting capacitance, \( C_x \), including holder and circuit capacitances, is 50 mmf., and that the lowest frequency is to be 500 kc.

We know that for \( R_d = 1, Q_d = 1, \) and \( I_d = 1/2.3 \) we have an equivalent series reactance of 0.54 at \( w = 1/40 = 2.5 \). We therefore wish to transform the frequency by a ratio of \( 2005 \times 10^5 /2.5 \) and the impedance level by the ratio of the reactance of 50 mmf. at 500 kc to the value 0.54. The frequency ratio when evaluated is \( 1.255 \times 10^6 \). The impedance ratio is

\[
\frac{10^{12}}{10^6 \pi \times 50 \times 0.54} = 11,800.
\]

Therefore the resistance in the final network will be 11,800 ohms. The capacitance in the final network must be reduced by the product of the above factors, to

\[
\frac{1}{11800 \times 1.255 \times 10^6}
\]

farads or 67 mmf. The inductance must be increased from 1/2.3 by the ratio 11,800 but diminished by the ratio \( 1.255 \times 10^6 \).

The final value is 4.08 mh.

3-8. Degree of Frequency Shift.

Inspection of the plot of Fig. 3.11 leads to the expectation that the frequency shift during damping will be quite small. The largest departures are to be expected for \( 1/w = \) about 0.325 or about 0.10. The two are best considered separately.
For $1/w = 0.325$ we have $X_{eq} = 0.44$ and $R_{eq} = 0.88$. The value of $X_{eq}$ departs by about 2% from the desired value, given by the dotted line in Fig. 3.11. The associated frequency shift is calculated by reference to (2) of Appendix H. There $R_a$ corresponds to $R_{eq}$, and $X_a$ and $X_b$ correspond to the reactance of $C_x$ and to $X_{eq}$, respectively; also, $R$ and $X$ are the same as $R_z$ and $X_z$ of Fig. 3.10(b). Let the reactance of $C_x$ be $X_x = 1/wC_x$.

Replacing $X_b$ by $x - X_a$ we obtain

$$X_z = X = \frac{R_a^2 X_a + x^2 X_a - xx_a^2}{R_a^2 + x^2} = X_a - \frac{xx_a^2}{R_a^2 + x^2}.$$  (3.58)

In the present case, the 2% reactance deviation means that $x = 0.02X_a$.

Furthermore, from the values of $X_{eq}$ and $R_{eq}$ cited above, $R_a = 2X_b$, and

$$R_a^2 = 4X_b^2 = 4(x - X_a)^2,$$

$$= 4X_a^2(.0004 - .04 + 1),$$

$$= 3.64X_a^2.$$  (3.59)

Neglecting the $x^2$ in the denominator, (3.58) becomes

$$X_z = X_a - \frac{.02}{3.84}X_a,$$

or

$$X_z = X_a(1 - 1/192).$$  (3.60)

Then, referring to Fig. 3.10,

$$C_z = \frac{C_x}{(1 - 1/192)},$$

$$= C_x(1 + 1/192).$$  (3.61)

Now, since $C_z \neq C_x = nC_0$, the deviation in $C_z$ represented by the final term in (3.61), is approximately $n$ times the deviation of the total
effective capacity \( C_t = \frac{C_o C_z}{C_o + C_z} \approx C_o \). Thus the deviation in \( C_t \) is \( 1/n \) that in \( C_z \), or \( 1/192n \). As the frequency varies inversely with the square root of \( C \), the corresponding frequency deviation is essentially

\[
\Delta f = \frac{1}{334n}.
\]

In a typical case where \( n = 260 \) the deviation is only 10 parts in a million.

For \( 1/w = 0.10 \) we have \( X_{eq} = 0.13 \) and \( R_{eq} = 0.99 \). The value of \( X_{eq} \) departs by about 1.5% from the desired value. Accordingly we may write \( R_a = 7.6X_b \),

\[
R_a^2 = 56X_a^2
\]

by (3.59), and from (3.58)

\[
X_z = X_a - \frac{0.15}{56} X_a,
\]

or

\[
X_z = X_x (1 - 1/3730).
\]

The frequency deviation becomes essentially

\[
\Delta f = \frac{1}{7460n}.
\]

Because the frequency is 3.25 times as high as that of the former case, the actual deviation in cycles is about 0.6 times that given by (3.62).

It is unlikely that the frequency error will be objectionable in either case.


To discuss the damping it is necessary to make some assumption as to the capacitance ratio. For simplicity it is assumed that all crystals which are to be damped by a given network have the same value of capacitance ratio. The degree of damping produced by this network is best determined by reference to (2) of Appendix B. It is seen that the real part of the impedance is given by the equation

\[
R = \frac{X_a^2}{R_a}, \text{ or } R_z = X_x \frac{X_x}{R_{eq}}.
\]

The values which apply at the lowest frequency of interest are \( 1/w = 0.40 \), \( X_{eq} = 0.54 \), and \( R_{eq} = 0.80 \). Substituting these values and recalling that
$X_{eq} = X_x$, we have

$$R_z = \frac{X_x 0.54}{0.80} = 0.68X_x.$$ (3.66)

It will be recalled that the maximum value achievable with pure resistance damping is $0.5X_x$. (See (3.6) of Sec. 3-1). Therefore at the lowest frequency the decrement available with the recommended network is $36\%$ greater than that available with optimum resistive damping. The decrement $A$ in a series circuit is $R/2L$. Since $R_o$ is small, we have for Fig. 3.9(b)

$$A = \frac{R_z}{2L_o}.$$

Two assumptions have been made which apply to this section: first, all the crystals under consideration must have the same value of shunting capacitance, $C_x$; and second, the capacitance ratio, $n$, must be the same in all. These two conditions automatically fix $C_o = C_x/n$. It is evident therefore, that the value of $L_o$ must vary between crystals inversely with the square of the frequency, inasmuch as crystal frequency is determined almost exclusively by the quantity $1/\sqrt{L_oC_o}$. Because $X_x$ is an inverse function of frequency, $R_z$ varies inversely with the square of frequency and with $R_{eq}$, as seen in (3.65). Consequently the decrement becomes

$$A = \frac{K}{R_{eq}}, \quad [C_x \text{ and } C_o \text{ constant}].$$ (3.67)

Since the useful range of $R_{eq}$ is from 0.8 to 1.0, the decrement at the highest frequencies is only $80\%$ of that at the lowest. It is concluded that at all crystal frequencies above the low frequency limit the decrement achieved with the network of Fig. 3.10 is in excess of that which can be achieved with pure resistance damping at the lowest frequency, provided $C_x$ and $C_o$ are substantially constant between crystals. If the capacitance ratio decreases ($C_x$ decreases or $C_o$ increases) as the rated crystal frequency is raised, the decrement will improve with the higher frequency crystals and vice versa.
CHAPTER IV
CRYSTAL DAMPING - PART II

The circuit shown in Fig. 4.1 is that of a crystal under damping by an external coil and resistor in series. This type of damping circuit is the only one which will be examined by the method of differential equations. The capacitance $C_x$ consists of the crystal holder capacitance $C_h$ plus any working or loading capacitance $C_k$ added to the circuit in shunt with the crystal.

DERIVATION OF EQUATIONS

4-1. The Basic Differential Equation.

The general differential equation for the circuit or for the current $i$ which represents the mechanical motion of the crystal, is most easily derived using operational methods or extensions of Heaviside's Calculus. (12) We first write the total A.C. admittance seen in the terminals. Let $\omega$ be the radian frequency at which this admittance $Y$ is taken. Then

$$Y = Y_0 + Y_x + Y_d; \quad (4.1)$$

(12) See, for example, Carter, G. W., The Simple Calculation of Electrical Transients. Cambridge University Press, 1944.
where

\[ Y_o = \frac{1}{R_o + jwL_o + \frac{1}{jwC_o}} \]

\[ Y_x = jwC_x \] \hspace{1cm} \{ (4.2) \}

and \[ Y_d = \frac{1}{R_d + jwL_d} \]

Replacing \( jw \) by \( p \), the Heaviside differential operator - a procedure which, in effect, alters the functions from steady-state to transient form - we obtain from (4.1) and (4.2)

\[ Y = \frac{p}{p^2L_o + pR_o + \frac{1}{C_o}} + \frac{pC_x}{pL_d + R_d} + \frac{1}{pL_d + R_d} \]

Replacing \( jw \) by \( p \), the Heaviside differential operator - a procedure which, in effect, alters the functions from steady-state to transient form - we obtain from (4.1) and (4.2)

\[ Y = \frac{p}{p^2L_o + pR_o + \frac{1}{C_o}} + \frac{pC_x}{pL_d + R_d} + \frac{1}{pL_d + R_d} \]

The numerator of the admittance function (4.3) when equated to zero turns out to be the "auxiliary equation" associated with the network differential equation. The denominator is not significant in this particular operation. Thus, the auxiliary equation is, when rearranged,

\[ p^4L_oL_dC_x + p^3(R_oL_x + R_oL_dC_x) + p^2\left(\frac{L_dC_x}{C_o} + L_x + L_o + R_oR_dC_x\right) \]

\[ + p\left(\frac{R_dC_x}{C_o} + R_d + R_o\right) + \frac{1}{C_o} = 0 \] \hspace{1cm} \{ (4.4) \}

Since the coefficient of \( p^4 \) is for our study always non-zero, we may divide the equation by it. Then if we write for \( p \) its equivalent,
d/dt, and let this operate on the desired current - that is, multiply each term of the equation by \( \frac{1}{i} \) - we obtain the actual differential equation:

\[
\frac{d^4i}{dt^4} + \left[ \frac{R}{L_d} + \frac{R_0}{L_o} \right] \frac{d^3i}{dt^3} + \left[ \left( \frac{1}{L_o C_o} + \frac{1}{L_o C_x} \right) + \frac{1}{L_d C_x} + \frac{R_d R_0}{L_d L_o} \right] \frac{d^2i}{dt^2} \\
+ \left[ \left( \frac{1}{L_o C_o} + \frac{1}{L_o C_x} \right) \frac{R_d}{L_d} + \frac{1}{L_d C_x} \frac{R_0}{L_0} \right] \frac{di}{dt} + \frac{1}{L_o C_o L_d C_x} = 0. \tag{4.5}
\]

The foregoing development does not, of course, pretend to be rigorous. However, (4.5) has been carefully derived by classical methods in Appendix I.

4-2. The Modified Differential Equation.

We now employ the following substitutions:

\[ A_0 = \frac{R_0}{2L_o} , \]

\[ A_d = \frac{R_d}{2L_d} , \]

\[ w_o^2 = \frac{1}{L_o C_o} , \]

\[ w_x^2 = \frac{1}{L_d C_x} , \]

\[ n = \frac{C_x}{C_o} , \]

\[ m = \frac{n + \frac{1}{n}}{n} , \]

\[ C_y = \frac{C_o C_x}{C_o + C_x} = \frac{C_o}{m} , \]

and

\[ w_y^2 = \frac{1}{L_o C_y} = m w_o^2 . \]
Also, for convenience only, let,

\[ M = \frac{A_d}{A_o} \]

and

\[ N = \frac{w_y^2}{w_y^2} \]

The use of these new parameters will reduce the complexity of the ensuing work. It should be noted that \( w_y \) is essentially the frequency at which the crystal becomes antiresonant, differing from the approximate series resonant frequency \( w_o \) by the factor \( \sqrt{m} \). (See Sec. 2-20). Furthermore, \( A_o \) and \( A_d \) may be regarded as the Decrements associated, respectively, with the isolated crystal and the damping coil alone.

Through the use of (4.6) and (4.7), (4.5) becomes

\[
\frac{d^4 i}{dt^4} + 2A_o (M + 1) \frac{d^3 i}{dt^3} + \left[ w_y^2 (N + 1) + 4A_o^2 M \right] \frac{d^2 i}{dt^2} + 2w_y^2 A_o (M + N) \frac{d i}{dt} + \frac{w_y^4}{m} N i = 0 .
\]  

4-3. The Conditional Equations.

If the solution of (4.8) is assumed to be

\[ i = I e^{pt}, \]  

the auxiliary equation becomes

\[
p^4 + 2A_o (M + 1)p^3 + \left[ w_y^2 (N + 1) + 4A_o^2 M \right] p^2 + 2w_y^2 A_o (M + N) p + \frac{w_y^4}{m} N = 0 .
\]  

which is, of course, the same as (4.4) using the modified parameters.
Because this fourth-degree algebraic equation is not readily factored, we proceed to make some assumptions about the roots. Since the configuration of Fig. 4.1 contains two L's and two C's, the circuit is capable of vibrating harmonically in two modes, and, consequently, (4.10) may have two complex pairs of roots. Again, since all the coefficients are positive, the real parts of at least one pair of the roots of (4.10) must be negative, from fundamental algebra. Moreover, because positive real parts correspond to expanding currents, which are non-physical in the purely passive circuit of Fig. 4.1, we assume that the real parts of all the roots are negative. Therefore, the general solution of (4.8) is written as

\[ i = I_1 e^{-A_1 t} \sin(B_1 t + \phi_1) + I_2 e^{-A_2 t} \sin(B_2 t + \phi_2) , \]

where \( I_1, I_2, \phi_1, \) and \( \phi_2 \) are arbitrary constants of integration, and \( A_1, B_1, \) and \( A_2, B_2 \) are positive circuit constants as yet unknown.

The auxiliary equation may be built up in terms of these last, non-arbitrary constants, from the form of (4.11). In factored form it is

\[ (p + A_1 + jB_1)(p + A_1 + jB_1)(p + A_2 - jB_2)(p + A_2 + jB_2) = 0 . \]

The conditions under which (4.11) is the solution of (4.8) are obtained by expanding (4.12) \(^{(13)}\) and equating coefficients with those of like powers of \( p \) in (4.10).

The result of this operation is the system of conditional equations (4.13) through (4.16):

\(^{(13)}\) See Appendix J-1.
These four equations are still not easy to solve as they stand because the A's and B's prove to be of vastly different magnitudes. However, if a judicious selection of fixed values for one pair of roots of (4.12) is made, we can eventually determine the remaining unknowns with less difficulty and with greater accuracy.

EXACT SOLUTION WITH ONE PAIR OF ROOTS FIXED

4-4. Choice of Fixed Roots.

The desired performance of the circuit of Fig. 4.1 is that during damping the crystal frequency shall remain the same as it was during operation but that the decrement of the passive portion of the oscillator network shall be made much greater.

There will actually be two frequencies present during damping, as indicated by (4.11). It is shown in Appendix K that for any significant improvement in damping rate to be achieved these two frequencies may not both be equal to \( w_y \), which is essentially the antiresonant operating frequency of the crystal. Our first step will be to fix one of the frequencies of (4.11) at this working frequency, \( w_y \). We then observe what deviation from this value occurs in the other part of (4.11) which may be looked upon as a new frequency introduced by the presence of \( L_d \) in the circuit.
Naturally, the decrements are of primary interest in a damping process. Because our objective is to cause the crystal to come to rest as rapidly as possible, it appears desirable to make the two current components of (4.11) decay at equal rates. However, this condition \( A_1 = A_2 \) may be imposed later on. In order to obtain a more general solution we shall let the decrement associated with the fixed frequency be adjustable by means of a factor \( h \), which refers it to the decrement \( A_0 \) of the isolated crystal; and we shall allow the other decrement to remain a dependent variable, along with its associated unknown frequency.

Thus, let

\[
A_2 = hA_0, \quad B_2 = w_y.
\]

The dependent variables are now \( A_1, B_1 \), and the two circuit parameters \( R_d \) and \( L_d \) which take on values, as prescribed by \( h \), through the quantities \( A_d \) and \( w_x \).

Substitution of (4.17) into (4.13) through (4.16) yields

\[
A_1 + hA_0 = A_0(M + 1); \quad (4.18)
\]

\[
A_1^2 + B_1^2 + h^2A_0^2 + 4hA_0A_1 = w_y^2N + 4A_0^2M; \quad (4.19)
\]

\[
A_1(h^2A_0^2 + w_y^2) + hA_0(A_1^2 + B_1^2) = w_y^2A_0(M + N); \quad (4.20)
\]

and

\[
(A_1^2 + B_1^2)(h^2A_0^2 + w_y^2) = \frac{w_y^4N}{m}. \quad (4.21)
\]

4-5. Exact Formulas in Terms of Engineering Quantities.

We are now prepared to combine and solve this system of equations. The following quantities will be involved:
\[ Q_y = \frac{w_y}{2A_0} = \frac{w_y L_0}{R_0}; \]
\[ Q_d = \frac{w_y}{2A_d} = \frac{w_y L_d}{R_d}; \]
\[ Q_1 = \frac{B_1}{2A_1}; \]
\[ Q_2 = \frac{B_2}{2A_2}; \]

\[
\begin{align*}
&Q_y = \frac{w_y}{2A_0} = \frac{w_y L_0}{R_0}; \\
&Q_d = \frac{w_y}{2A_d} = \frac{w_y L_d}{R_d}; \\
&Q_1 = \frac{B_1}{2A_1}; \\
&Q_2 = \frac{B_2}{2A_2}; \\
&\text{and} \\
&\text{Here } Q_y \text{ is the approximate } Q \text{ of the crystal at its working frequency } w_y. \text{ (See Sec. 2-20.) } Q_d \text{ is the nominal } Q \text{ of the damping coil at the crystal frequency } w_y. \text{ } Q_1 \text{ and } Q_2 \text{ are } "\text{circuit } Q\text{'s}" \text{ at the resultant damped frequencies of the circuit, } Q_2 \text{ of course being at the fixed desired frequency } B_2 = w_y. \\
\text{The quantities in which we will be eventually interested are:} \\
D = \frac{A_1}{A_2}, \text{ the ratio of the circuit decrements}; \quad (4.23) \\
F = \frac{w_y - B_1}{w_y}, \text{ the fractional frequency deviation of } B_1; \quad (4.24) \\
s = \frac{L_d}{L_0/\pi}, \text{ the ratio of damping inductance to an approximately requisite inductance } L_0/\pi; \quad (4.25)
\]

also \( Q_d, Q_1, \) and \( Q_2, \) as defined by \((4.22);\) and \( A_2 \) and \( B_2 \) as defined by \((4.17).\)

From \((4.18),\)
\[ A_1 = A_0 [M - (h - 1)], \quad (4.26) \]
and from \((4.19)\)
\[ A_1^2 + B_1^2 = N w_y^2 + A_0^2 (4M - h^2) - 4h A_0 A_1. \quad (4.27) \]
Substituting (4.27) in (4.20),

\[ A_1 \left( h^2 A_0^2 + w_y^2 \right) + h A_0 \left[ \frac{N w_y^2 + A_0^2 (4M - h^2)}{w_y^2} - 4h A_0 A_1 \right] - w_y^2 A_0 (N + M) = 0. \]  

(4.28)

Substituting for \( A_1 \) from (4.26) and dividing through by \( A_0 \),

\[ \left[ M - (h - 1) \right] \left( h^2 A_0^2 + w_y^2 \right) + h \left[ N w_y^2 + A_0^2 (4M - h^2) \right] - 4h A_0^2 \left[ M - (h - 1) \right] - w_y^2 (N + M) = 0. \]  

(4.29)

Expanding and simplifying,

\[ h A_0^2 \left[ h(2h - 3) - M(3h - 4) \right] + w_y^2 (h - 1)(N - 1) = 0. \]  

(4.30)

Substituting (4.27) in (4.21),

\[ \left[ N w_y^2 + A_0^2 (4M - h^2) - 4h A_0 A_1 \right] \left( h^2 A_0^2 + w_y^2 \right) = \frac{N w_y^4}{m}. \]  

(4.31)

Substituting for \( A_1 \) from (4.26),

\[ \left\{ N w_y^2 + A_0^2 (4M - h^2) - 4h A_0^2 \left[ M - (h - 1) \right] \right\} \left( h^2 A_0^2 + w_y^2 \right) = \frac{N w_y^4}{m}. \]  

(4.32)

Expanding and simplifying,

\[ N \left( \frac{m - 1}{m} \right) \frac{w_y^4}{A_0^4} + \left[ h^2 N + h(3h - 4) - 4M(h - 1) \right] \frac{w_y^2}{A_0^2} + h^2 \left[ h(3h - 4) - 4M(h - 1) \right] = 0. \]  

(4.33)

Substituting for \( \frac{w_y}{A_0} \) from (4.22) and for \( m \) from (4.6),

\[ 16q_y^4 \frac{N}{n + 1} + 4q_y^2 h^2 N + (4q_y^2 + h^2) \left[ h(3h - 4) - 4M(h - 1) \right] = 0, \]  

(4.34)
whence

\[
N = \frac{(n + 1)(4Qy^2 + h^2) \left[4M(h - 1) - h(3h - 4)\right]}{4Qy^2 \left[4Qy^2 + h^2(n + 1)\right]}. \tag{4.35}
\]

Substituting (4.35) into (4.30),

\[
4Qy^2 \left[4Qy^2 + h^2(n + 1)\right] \left[h(2h - 3) - M(3h - 4)\right] \\
+ w_y^2(h - 1) \left\{(n + 1)(4Qy^2 + h^2) \left[4M(h - 1) - h(3h - 4)\right] \\
- 4Qy^2 \left[4Qy^2 + h^2(n + 1)\right]\right\} = 0. \tag{4.36}
\]

Dividing through by \(w_y^2\) and using (4.22) again,

\[
h \left[4Qy^2 + h^2(n + 1)\right] \left[h(2h - 3) - M(3h - 4)\right] \\
+ (h - 1) \left\{(n + 1)(4Qy^2 + h^2) \left[4M(h - 1) - h(3h - 4)\right] \\
- 4Qy^2 \left[4Qy^2 + h^2(n + 1)\right]\right\} = 0. \tag{4.37}
\]

Rearranging terms,

\[
M \left\{h(3h - 4) \left[4Qy^2 + h(n + 1)\right] - 4(h - 1)^2(n + 1)(4Qy^2 + h^2)\right\} \\
= h^2(2h - 3) \left[4Qy^2 + h^2(n + 1)\right] - (h - 1) \left\{h(3h - 4)(n + 1)(4Qy^2 + h^2) \\
+ 4Qy^2 \left[4Qy^2 + h^2(n + 1)\right]\right\}. \tag{4.38}
\]

With further manipulation the right hand side of (4.38) becomes

\[
h \left\{h(3h - 4) \left[4Qy^2 + h(n + 1)\right] - 4(h - 1)^2(n + 1)(4Qy^2 + h^2)\right\} \\
- h^2(h - 1) \left[4Qy^2 + h^2(n + 1)\right] + h^2(h - 1)(n + 1)(4Qy^2 + h^2) \\
- 4Qy^2(h - 1) \left[4Qy^2 + h^2(n + 1)\right] ; \tag{4.39}
\]
whence we obtain

\[ M = h + U \]  

where

\[ U = \frac{4Q_y^2(h - 1)(4Q_y^2 + h^2)}{[4Q_y^2 + h^2(n + 1)](h - 2)^2 + 16Q_y^2n(h - 1)^2} \]  

Substituting (4.40) in (4.35),

\[ N = \frac{(n + 1)(4Q_y^2 + h^2) [4U(h - 1) + h^2]}{4Q_y^2 [4Q_y^2 + h^2(n + 1)]} \]  

Substituting (4.40) in (4.26),

\[ A_1 = A_0(U + 1) \]  

Combining this with (4.23) and (4.17),

\[ D = \frac{U + 1}{h} \]  

Substituting (4.43) in (4.27),

\[ B_1^2 = Nw_y^2 - A_0^2[(h + U)^2 + 2U(h - 1) + 1] \]  

Dividing through by \( w_y^2 \) and using (4.22),

\[ \frac{B_1^2}{w_y^2} = N - \frac{1}{4Q_y^2} [(h + U)^2 + 2U(h - 1) + 1] \]  

Substituting for \( N \) from (4.42),

\[ \frac{B_1^2}{w_y^2} = n \left[ \frac{4U(h - 1) + h^2}{4Q_y^2 + h^2(n + 1)} \right] - \frac{(U + 1)^2}{4Q_y^2} \]
From (4.6) and (4.7)

\[ L_d = \frac{L_0}{(n + 1)N} \]  \hspace{1cm} (4.48)

From (4.25), (4.48), and (4.6),

\[ s = \frac{1}{mN} \]  \hspace{1cm} (4.49)

From (4.6), (4.7), and (4.48),

\[ R_d = \frac{R_0M}{(n + 1)N} \]  \hspace{1cm} (4.50)

From (4.22), (4.48), and (4.50),

\[ Q_d = \frac{Q_y}{M} \]  \hspace{1cm} (4.51)

Finally, from (4.24) and (4.46) we may obtain

\[ F = \frac{w_y - B_1}{w_y} = 1 - \sqrt{\frac{B_1^2}{w_y^2}} = 1 - \sqrt{\frac{n \left[ 4U(h - 1) + h^2 \right]}{4Q_y^2 + h^2(n + 1)}} - \frac{(U + 1)^2}{4Q_y^2} \]  \hspace{1cm} (4.52)

All the equations thus far are exact.

If the capacitance ratio \( n \) and the crystal Quality Factor \( Q_y \) are specified, and a value of the decrement multiplier \( h \) for the damping term \( A_2 \) associated with the fixed desired frequency \( B_2 = w_y \) is selected; then (4.40) through (4.42) permit (4.44), (4.47) or (4.52), (4.49), and (4.51) to be solved for exact numerical values of \( A_1 \) and \( B_1 \) and for the values of damping parameters \( L_d \) and \( Q_d \) which are necessary to produce the performance. \( (w_y \) is, of course, known. \)
4-6. Ranges of the Variables.

In order to plot engineering curves more readily from the formulas just derived, we shall see if they may be simplified by neglecting small terms. To do this we have to examine the useful ranges of the several parameters involved.

The quantity \( n \) is an important crystal constant. We have defined it to include a standard loading capacitance, \( C_h \), (32 \( \mu \)pf has been suggested) placed in parallel with the crystal and the damping coil. However, there may be occasions when it is more advantageous to have the loading capacitance disconnected during periods of damping. In that case, \( n \) reduces to the simple ratio, \( n_0 = \frac{C_h}{C_0} \). As pointed out in Sec. 2-20, this natural crystal capacitance ratio may never be less than about 125, owing to the peculiar physical properties of quartz. An upper limit of 4000 for \( n = \frac{C_h}{C_0} \) should accommodate nearly all crystals found in practice and include a normal loading capacitance. In plotting the data, where necessary, we shall use six typical values for \( n \): 125, 250, 500, 1000, 2000, and 4000.

The quantity \( Q_y \), as defined by (4.22), will in general lie between the limits 10,000 and 1,000,000, although it is entirely possible to produce crystals having a \( Q \) outside of this range. Where it is necessary to specify \( Q_y \) we shall take the values 10,000, 100,000, and 1,000,000 as being representative.

The quantity \( h \) is actually the ratio of decrements associated with the crystal frequency, \( w_Y \), before and after damping is applied; it measures one of the circuit decrements during damping as a number of times greater than the decrement of the isolated crystal. From previous work we know that 200 to 2,000 is a good useful range of \( h \) for our present interests.

The various ranges are tabulated below. Using them we proceed to modify the formulas derived in the preceding section.
Sec. 4-7

APPROXIMATE FORMULAS

\[
\begin{align*}
125 < n < 4000 \\
10^4 < Q_y < 10^6 \\
200 < h < 2000
\end{align*}
\] (4.53)

4-7. Approximate Formulas.

Equation (4.41) is rewritten as follows:

\[
U = \frac{4Q_y^2(4Q_y^2 + h^2)}{4Q_y^2 + h^2(n + 1)} \left( \frac{h - 2}{h - 1} \right)^2 + 16Q_y^2n(h - 1)
\]

\[
16Q_y^4 \left[ 1 + \frac{h^2}{4Q_y^2} \right] 
\]

\[
16Q_y^2nh \left[ 1 - \frac{1}{h} + \frac{4Q_y^2 + h^2(n + 1)}{16Q_y^2nh} \frac{(h - 2)^2}{(h - 1)} \right]
\]

\[
Q_y^2 \left[ 1 + \frac{h^2}{4Q_y^2} \right] 
\]

\[
\frac{Q_y^2}{nh} \left[ 1 - \frac{1}{h} + \frac{1}{4n} + \frac{h^2}{16Q_y^2} \left( 1 + \frac{1}{n} \right) \right] \left[ 1 - \frac{3h - 4}{h(h - 1)} \right]
\] (4.54)

Because all the terms in the first bracket in the denominator are small, and because in its multiplier the quantity \(\frac{3h - 4}{h(h - 1)}\) is much less than 1, we may safely write 1 in place of the last bracket. Furthermore, in the first bracket we may neglect \(\frac{1}{h}\) compared to 1 as the multiplier of a small term. Thus

\[
U = \frac{Q_y^2 \left[ 1 + \frac{h^2}{4Q_y^2} \right]}{nh \left( 1 - \frac{1}{h} + \frac{1}{4n} + \frac{h^2}{16Q_y^2} \left( 1 + \frac{1}{n} \right) \right)}
\]

\[
= \frac{Q_y^2 \left[ 1 + \frac{h^2}{4Q_y^2} \right]}{nh \left[ 1 - \frac{1}{h} + \frac{1}{4n} + \frac{h^2}{16Q_y^2} \right]}
\] (4.55)
The errors involved in these simplifications may be neglected in view of much grosser approximations to be made.

Since the fractional terms in the denominator of (4.55) are, combined, much less than unity, we use the device given in Appendix J-3 to write

\[
U = \frac{Q_y^2}{nh} \left[ 1 + \frac{h^2}{4Q_y^2} \right] \left[ 1 + \frac{1}{h} - \frac{1}{4h} - \frac{h^2}{16Q_y^2} \right],
\]

\[
= \frac{Q_y^2}{nh} \left[ 1 + \frac{1}{h} + \frac{3h^2}{16Q_y^2} - \frac{1}{4h} \right].
\]

(4.56)

Now \(\frac{1}{h} + \frac{3h^2}{16Q_y^2}\) has a maximum value of 8/1000 and a minimum of about 1/2000, while 1/4h has a maximum value of 2/1000 and a minimum of 1/16,000. Consequently,

\[
U = \frac{Q_y^2}{nh},
\]

(4.57a)

with a greatest error of -0.8% approximately.

In accordance with Appendix J-2 we write this:

\[
U = \frac{Q_y^2}{nh} \begin{bmatrix} \mathcal{E} \max = -0.8\% \end{bmatrix} \quad n = 4000, \quad Q_y = 10^4, \quad h = 2000
\]

(4.57b)

where the final numbers indicate the values of the parameters which lead to greatest error. This notation will be employed throughout the remaining sections of Chapter IV.

Substituting in (4.42) the exact expression (4.41) for \(U\), and simplifying, we have

\[
N = \frac{(n + 1)(4Q_y^2 + h^2) \left[ h^2(h - 2)^2 + 16Q_y^2(h - 1)^2 \right]}{4Q_y^2 \left[ \frac{4Q_y^2 + h^2(n + 1)}{h^2(n + 1)} (h - 2)^2 + 16Q_y^2n(h - 1)^2 \right]}.
\]

(4.58)
This is rewritten as follows:

\[
N = \frac{(n + 1) \left[ 1 + \frac{h^2}{4Q_y^2} \right] \left[ h^2(h - 2)^2 + 16Q_y^2(h - 1)^2 \right]}{16Q_y^2n(h - 1)^2 + \left[ 4Q_y^2 + h^2(n + 1) \right](h - 2)^2},
\]

\[
= \frac{(n + 1) \left[ 1 + \frac{h^2}{4Q_y^2} \right] \left[ \frac{h^2(h - 2)^2}{h - 1} + 16Q_y^2(h - 1) \right]}{16Q_y^2n(h - 1) + \left[ 4Q_y^2 + h^2(n + 1) \right] \left[ \frac{(h - 2)^2}{h - 1} \right]},
\]

\[
= \frac{(n + 1) \left[ 1 + \frac{h^2}{4Q_y^2} \right] \left[ \frac{h - 1}{h} + \frac{h^2(h - 2)^2}{16Q_y^2h(h - 1)} \right]}{n \left\{ 1 - \frac{1}{h} + \frac{\left[ 4Q_y^2 + h^2(n + 1) \right](h - 2)^2}{16Q_y^2nh(h - 1)} \right\}}.
\]

\[
(4.59)
\]

The denominator is similar to that of (4.54) and hence will reduce to the approximate form given in (4.55). And, by the same arguments which were applied to \( U \), the exact equation (4.59) becomes

\[
N = \frac{(n + 1) \left[ 1 + \frac{h^2}{4Q_y^2} \right] \left\{ 1 - \frac{1}{h} + \frac{h^2}{16Q_y^2} \left[ 1 - \frac{3h - 4}{h(h - 1)} \right] \right\}}{n \left[ 1 - \frac{1}{h} + \frac{1}{4n} + \frac{h^2}{16Q_y^2} \right]},
\]

\[
= \frac{\left[ 1 + \frac{1}{n} \right] \left[ 1 + \frac{h^2}{4Q_y^2} \right] \left[ 1 - \frac{1}{h} + \frac{h^2}{16Q_y^2} \right]}{1 - \frac{1}{h} + \frac{1}{4n} + \frac{h^2}{16Q_y^2}},
\]

\[
= \frac{\left[ 1 + \frac{1}{n} \right] \left[ 1 + \frac{h^2}{4Q_y^2} \right] \left[ 1 - \frac{1}{h} + \frac{h^2}{16Q_y^2} \right] \left[ 1 + \frac{1}{h} - \frac{1}{4n} - \frac{h^2}{16Q_y^2} \right]}{1 + \frac{1}{n} + \frac{h^2}{4Q_y^2} - \frac{1}{h} + \frac{h^2}{16Q_y^2} + \frac{1}{h} - \frac{1}{4n} - \frac{h^2}{16Q_y^2}};
\]

\[
(4.60a)
\]
or
\[ N = 1 + \frac{3}{4n} + \frac{h^2}{4Q_y^2}, \quad [0.0\%] \quad (14) \]

Since \(3/4n\) has a maximum value of 0.6/1000, and \(h^2 / 4Q_y^2\) a maximum of 10/1000,
we have
\[ N = 1, \quad [\xi_{\text{max}} = -1.6\%] \quad q_y = 10^4 \quad h = 2000 \quad (4.61) \]

Equation (4.42) can be rewritten as
\[
\frac{4U(h - 1) + h^2}{4Q_y^2 + h^2(n + 1)} = \frac{4Q_y^2N}{(n + 1)(4Q_y^2 + h^2)} \quad (4.62)
\]

Substituting (4.62) into (4.47) gives
\[
\frac{B_1^2}{w_y^2} = \frac{4Q_y^2nN}{(n + 1)(h^2 + 4Q_y^2)} - \frac{(U + 1)^2}{4Q_y^2} \]
\[
= \frac{N}{1 + \frac{1}{n}} \left[ 1 + \frac{h^2}{4Q_y^2} \right] - \frac{(U + 1)^2}{4Q_y^2} \quad (4.63)
\]

Substituting for \(N\) from (4.60b) and making the usual approximations of
negligible error,
\[
\frac{B_1^2}{w_y^2} \doteq \left[ 1 + \frac{3}{4n} + \frac{h^2}{4Q_y^2} - \frac{1}{n} \frac{h^2}{4Q_y^2} \right] - \frac{(U + 1)^2}{4Q_y^2} \]

or
\[
\frac{B_1^2}{w_y^2} \doteq 1 - \frac{1}{4n} - \frac{(U + 1)^2}{4Q_y^2}, \quad [0.0\%] \]
\[ (4.64) \]

(14) This symbol will be used after approximate equations in which the error
may be considered negligible.
The final term varies over a considerable range of magnitudes. And because the whole right-hand side may become zero, no simplifications subject to known limited errors can be made in (4.64).

Substituting for $N$ from (4.60b) into (4.48), we obtain

$$L_d = \frac{L_0}{n} \left[ 1 + \frac{1}{n} \left[ 1 + \frac{3}{4n} \frac{h^2}{4} \right] \right]$$

Now, $7/4n$ has a maximum value of $14/1000$, and $h^2/4Q_y^2$ a maximum of $10/1000$. Then

$$L_d = \frac{L_0}{n} \left[ 1 - \frac{7}{4n} \frac{h^2}{4Q_y^2} \right] . \quad (4.65)$$

And, of course, from the result of (4.66), (4.25) becomes

$$S = 1, \left[ \varepsilon_{\text{max}} = +2.4\% \right] \quad \frac{n = 125}{\bar{Q}_y} = 10^4 \quad \frac{h = 2000}{\bar{h}} . \quad (4.66)$$

Using $U$ as given by (4.57) we may write for (4.44),

$$M = \frac{Q_y^2 + nh}{nh^2} , \left[ \varepsilon_{\text{max}} < -0.8\% \right] \quad (15) \quad . \quad (4.68)$$

Again, substituting for $U$ from (4.57) into (4.40), we obtain

$$M = h + \frac{Q_y^2}{nh} , \left[ \varepsilon_{\text{max}} < -0.8\% \right] . \quad (4.69)$$

(15) Note that $U$ has its greatest error, given by (4.57b), when $U = 12.5$. The greatest error in $(U + 1)$ is, therefore, something less than $-0.8\%$ and may occur at a different set of parameter values. However, the error in $U$ provides a convenient upper limit for the error in $(U + 1)$. These remarks apply also to (4.69) and (4.70).
Similarly, from (4.51) and (4.69)

\[
Q_d = \frac{Q_y n h}{Q_y^2 + n h^2}, \quad \left[ \epsilon_{\text{max}} = +0.3\% \right]\quad (4.70)
\]

4-8. Further Modification.

We shall now introduce a modified parameter \( H \) to use in place of \( h \) as an independent variable. Thus, let

\[
h = 2 Q_y H. \quad (4.71)
\]

From (4.17) and (4.22) we see that

\[
A_2 = H w_y, \quad (4.72)
\]

or \( H \) measures the circuit decrement \( A_2 \) during damping as a number of times the frequency \( B_2 \) (= \( w_y \)).

The chief advantage in using \( H \) is that some of the engineering formulas just derived will become independent of \( Q_y \). However, where this simplification is not effected \( h \) will be retained, as its nature is more obvious than that of \( H \). Note that \( h \) (4.17) is dimensionless, being the ratio of two decrements; while \( H \) (4.72) has the units of nepers/radian, and is, in fact, an "angular decrement" or a sort of inverse \( Q \).

Substituting (4.71) into (4.57),

\[
U = \frac{Q_y}{2 n H}, \quad \left[ \epsilon_{\text{max}} = -0.8\% \right] \quad \frac{Q_y}{n} = 4000 \quad \frac{Q_y}{h} = 2000
\]

Then, substituting (4.71) into (4.64),

\[
\frac{B_1}{w_y} = 1 - \frac{1}{4n} - \frac{1}{4} \left( \frac{1}{2 n H} + \frac{1}{Q_y} \right)^2, \quad \left[ \epsilon_{\text{max}} = \text{not determined} \right] \quad (4.74)
\]

(16) In accordance with Appendix J-2, we write the error as positive, implying that (4.70) in its approximate form is too large. This follows because the quantity \( M \) appears in the denominator of (4.51), and by (4.69) the approximate value of \( M \) used is too small by 0.8% or thereabouts.
Substituting (4.71) into (4.68), (4.69), and (4.70), we have

\[ D = \frac{1}{2H} \left( \frac{1}{2nH} + \frac{1}{Q'_{y}} \right), \quad \left[ E_{\text{max}} < -0.8\% \right]; \quad (4.75) \]

\[ M = Q'_{y} \left( 2H + \frac{1}{2nH} \right), \quad \left[ E_{\text{max}} < -0.8\% \right]; \quad (4.76) \]

\[ Q_{d} = \frac{2nH}{4nH^2 + 1}, \quad \left[ E_{\text{max}} < +0.8\% \right]; \quad (4.77) \]

From (4.71) and (4.53) we see that the range for \( H \) is:

when \( Q'_{y} = 10^4 \), \( 0.01 < H < 0.1 \);

when \( Q'_{y} = 10^5 \), \( 0.001 < H < 0.01 \);

when \( Q'_{y} = 10^6 \), \( 0.0001 < H < 0.001 \);

\[ \text{OPTIMUM DAMPING CONDITION, } A_1 = A_2 \]

As mentioned in the paragraph immediately preceding (4.17), the optimum condition for damping would appear to be that the two components of the current (4.11) decay at equal rates.

4-9. \textbf{Engineering Formulas.}

From (4.44), when \( D = A_1/A_2 = 1 \), we have

\[ h = U + 1. \quad (4.79) \]

Using (4.56) for \( U \), and writing the total correction factor (17) as \( (1 + c_1) \), where \( c_{1\text{max}} = +.008 \) by (4.57),

\[ h - 1 = \frac{Q'_{y}^2}{nh} (1 + c_1), \quad \left[ \text{[0.0\%]} \right] \quad (18) \]

(17) See Appendix J-2.

(18) See Footnote (14).
Substituting for \( h \) from (4.71),

\[
2Q_y H - 1 = \frac{Q_y}{2nH} (1 + c_1),
\]

or

\[
4nQ_y H^2 - 2nH - Q_y (1 + c_1) = 0. \tag{4.81}
\]

The solution of this quadratic in \( H \) is

\[
H = \frac{1}{4Q_y} \pm \sqrt{\frac{1}{16Q_y^2} + \frac{(1 + c_1)^2}{4n}}, \quad [0.0\%]. \tag{4.82}
\]

Because \( H \) must be positive, by definition, we can discard the negative root and rewrite (4.82) as

\[
H = \frac{1}{4Q_y} + \frac{(1 + c_1)^{1/2}}{2\sqrt{n}} \left[ 1 + \frac{n}{4Q_y^2(1 + c_1)} \right] \quad [0.0\%]. \tag{4.83}
\]

Since \( c_1 \) and \( \frac{n}{4Q_y^2(1 + c_1)} \) are small, we use the devices given in Appendices J-4 and J-3 to write

\[
H = \frac{1}{4Q_y} + \frac{(1 + c_1/2)}{2\sqrt{n}} \left[ 1 + \frac{n}{8Q_y^2(1 + c_1)} \right],
\]

\[
= \frac{1}{4Q_y} + \frac{1}{2\sqrt{n}} \left[ 1 + \frac{c_1}{2} + \frac{n}{8Q_y^2(1 + c_1)} \right],
\]

\[
= \frac{1}{2\sqrt{n}} \left[ 1 + \frac{c_1}{2} + \frac{n}{8Q_y^2(1 + c_1)} + \frac{\sqrt{n}}{2Q_y} \right], \quad [0.0\%], \tag{4.84}
\]

in which, from (4.56),

\[
c_1 = \frac{1}{h} + \frac{3h^2}{16Q_y^2} - \frac{1}{4n},
\]

\[
= \frac{1}{2Q_y H} + \frac{3h^2}{4} - \frac{1}{4n}, \quad \text{by (4.71).} \tag{4.85}
\]

Then

\[
H = \frac{1}{2\sqrt{n}} (1 + c_2), \quad [0.0\%], \tag{4.86}
\]

where

\[
c_2 = \frac{1}{4Q_y H} + \frac{3h^2}{8} - \frac{1}{8n} + \frac{n}{8Q_y^2(1 + c_1)} + \frac{\sqrt{n}}{2Q_y}. \]
As we have been considering only first-order effects, we neglect $c_1$ in the fourth term of $c_2$ (4.86) and substitute $H = 1/2\sqrt{n}$ in the first and second terms. Accordingly,

$$c_2 = \frac{\sqrt{n}}{Q_y} + \frac{3}{8n} - \frac{1}{8Q_y^2} + \frac{\sqrt{n}}{Q_y},$$

$$= \frac{\sqrt{n}}{Q_y} + \frac{n}{8Q_y^2} - \frac{1}{32n};$$

(4.87)

and (4.86) becomes

$$H = \frac{1}{2\sqrt{n}}, \quad \left[\varepsilon_{\text{max}} = -0.6\%\right], \quad \begin{cases} n = 4000 \quad \text{for} \quad Q_y = 10^4 \end{cases}$$

(4.88)

Substituting $H = 1/2\sqrt{n}$ in the correction term, $c_1$ (4.85),

$$c_1 = \frac{\sqrt{n}}{Q_y} + \frac{3}{16n} - \frac{1}{4n},$$

$$= \frac{\sqrt{n}}{Q_y} - \frac{1}{16n};$$

(4.89)

Now, $c_1$ is the correction term for $U$, from (4.56). We may then write for (4.73):

$$U = \frac{Q_y}{2H} (1 + c_1), \quad \left[0.0\%\right].$$

(4.90)

Substituting for $H$ from (4.86),

$$U = \frac{Q_y}{\sqrt{n}} \left(1 + \frac{c_1}{1 + c_2}\right),$$

$$= \frac{Q_y}{\sqrt{n}} (1 + c_1 - c_2),$$

$$= \frac{Q_y}{\sqrt{n}} \left[1 - \frac{n}{8Q_y^2} - \frac{1}{32n}\right], \quad \text{by (4.89) and (4.87)};$$

and

$$U = \frac{Q_y}{\sqrt{n}}, \quad \left[\varepsilon_{\text{max}} = +0.03\%\right], \quad \begin{cases} n = 125 \quad \text{for} \quad Q_y = 10^4 \end{cases}$$

(4.91)
This error is decreased only slightly by an increase in $Q_y$, but falls off inversely with $n$. The quantity $U$ itself is inversely proportional to the square root of $n$ and directly proportional to $Q_y$. Consequently, the maximum error in the quantity $(U + 1)$ will occur when $n$ and $Q_y$ are the values given in (4.91), in which case $U = 893$. The error in $(U + 1)$ at optimum damping can therefore be considered the same as that in $U$ alone.

From the foregoing we see that the worst error due to $U$ in (4.64) at optimum damping occurs when the final term has the value 0.002. The error in this term is then $0.06\%$ of 0.002, which is clearly negligible compared to $1 - 1/4n$. So, substituting for $U$ from (4.91) in (4.64),

$$\frac{B_1^2}{w_y^2} = 1 - \frac{1}{4n} - \frac{1}{4n} \left[ \frac{1}{\sqrt{n}} + \frac{1}{Q_y} \right]^2$$

$$= 1 - \frac{1}{4n} - \frac{1}{4n} \left[ 1 + \frac{\sqrt{n}}{Q_y} \right]^2$$

$$\approx 1 - \frac{1}{4n} - \frac{1}{4n} \left[ 1 + \frac{\sqrt{n}}{Q_y} \right] \text{ by Appendix J-4},$$

or

$$\frac{B_1^2}{w_y^2} = 1 - \frac{1}{2n} \left[ 1 + \frac{\sqrt{n}}{Q_y} \right], \quad \left[ 0.0\% \right] \quad (4.92)$$

Taking the square root of (4.92) by Appendix J-4, since the whole final expression is small,

$$\frac{B_1}{w_y} \approx 1 - \frac{1}{4n} \left[ 1 + \frac{\sqrt{n}}{Q_y} \right], \quad \left[ 0.0\% \right]$$

$$\approx 1 - \frac{1}{4n}, \quad \left[ \varepsilon_{max} \approx +0.0002\% \right], \quad n = 125 \quad Q_y = 10^4 \quad (4.93)$$

Finally, substituting this in (4.24), we obtain

$$F \approx \frac{1}{4n} \left[ 1 + \frac{\sqrt{n}}{Q_y} \right],$$

$$\approx \frac{1}{4n}, \quad \left[ \varepsilon_{max} \approx -0.6\% \right], \quad n = 4000 \quad Q_y = 10^4 \quad (4.94)$$
Using the same methods, we may develop (4.40) as follows:

\[ M = 2Q_y h + U, \text{ by (4.71)}, \]
\[ = \frac{Q_y}{\sqrt{n}} (1 + c_2) + \frac{Q_y}{\sqrt{n}} (1 + c_1 - c_2), \text{ by (4.86) and (4.91)}, \]
\[ = \frac{Q_y}{\sqrt{n}} (2 + c_1), \]
\[ = \frac{2Q_y}{\sqrt{n}} \left[ 1 + \frac{\sqrt{n}}{32Q_y} - \frac{1}{32n} \right], \text{ by (4.89)}, \]

or

\[ M = \frac{2Q_y}{\sqrt{n}}, \left[ \varepsilon_{\max} = -0.3\% \right] n = 4000 \cdot \]
\[ \text{by (4.95)} \]

Then (4.51) becomes, by introducing \( M \) from (4.95),

\[ Q_d = \frac{\sqrt{n}}{2}, \left[ \varepsilon_{\max} = +0.3\% \right] n = 4000 \cdot \]
\[ \text{by (4.96)} \]

Substituting (4.88) into (4.72), we have for optimum damping

\[ A_1 = A_2 = \frac{w_y}{2\sqrt{n}}, \left[ \varepsilon_{\max} = -0.6\% \right] n = 4000 \cdot \]
\[ \text{by (4.97)} \]

The circuit \( Q_e^a \) (4.22) which result from the use of the optimum damping coil are:

from (4.17) and (4.97),

\[ Q_e = \sqrt{n}, \left[ \varepsilon_{\max} = +0.6\% \right] n = 4000 \cdot \]
\[ \text{by (4.98a)} \]

and, from (4.93) and (4.97),

\[ Q_1 = \sqrt{n} \left[ 1 - \frac{1}{4n} \right], \left[ \varepsilon_{\max} = +0.6\% \right] n = 4000 \cdot \]
\[ \text{by (4.98b)} \]
It is evident from (4.94) that larger values of \( n \) give less frequency deviation during damping. However, lower circuit Q's are obtained with smaller values of \( n \), as shown by (4.95a) and (4.95b). Now, while the ratio \( n_0 (= C_h/C_0) \) is fixed for a given crystal, the ratio \( n (= C_x/C_0) \) may be greater or the same depending on whether or not \( C_x \) includes standard or other loading capacitance, \( C_k \). For minimum frequency deviation a large \( C_k \) should be included in the circuit during damping. For maximum decay rates, on the other hand, \( C_k \) should be omitted and \( C_x \) consist of only the holder capacitance \( C_h \).

It should be noted in this connection that the decay rates vary as the square root of \( n \) while the frequency deviation is inversely proportional to the first power of \( n \).

4-10. Special Considerations on \( L_d \).

The design of damping coils to produce the performances indicated by the formulas just given requires a knowledge of the inductance \( L_d \), to be used in conjunction with the nominal \( Q \) of the coil, \( Q_d \).

From (4.48) we have

\[
N = \frac{L_0}{(n + 1)L_d}
\]  

(4.99)

Thus \( N \) is inversely proportional to \( L_d \). The value of \( N \) itself is approximately unity, as seen in (4.61). Therefore, it is evident (from (4.46), for example) that the frequency deviation of \( B_1 \) will depend closely on \( L_d \). The quantity \( N \) enters, in a less obvious way, in many of the subsequent equations also. The approximate formulas of Sec. 4-9 were developed on the basis of \( N \) carrying a negligible error. Accordingly, \( L_d \) in (4.99) must be determined to the same accuracy as \( N \) in (4.60). Equation (4.65), giving \( L_d \) as a constant independent of \( n, Q_y, \) and \( h \), cannot be used, since it involves a possible error of 2.4%. Instead we employ the more exact formula (4.65).
Introducing $H$ from (4.71) into (4.65), we obtain

$$L_d = \frac{L_0}{n} \left( 1 - \frac{7}{4n} - H^2 \right), \quad \left[ 0.0\% \right]. \quad (4.100)$$

Then (4.25) becomes

$$s = \frac{L_d}{L_0/n} = \left( 1 - \frac{7}{4n} - H^2 \right), \quad \left[ 0.0\% \right]. \quad (4.101)$$

At optimum damping, $H$ is given by (4.88), so that (4.101) can be written

$$s = 1 - \frac{2}{n}, \quad \left[ 0.0\% \right]. \quad (4.102)$$

4-11. ENGINEERING CURVES FOR OPTIMUM DAMPING CONDITION.

The damping coil constants are defined by the following expressions:

$$L_d = sL_0/n,$$

and

$$R_d = w_y L_d/Q_d. \quad (4.103)$$

Fig. 4.2 shows the $Q$ of the damping coil, (4.96), and the inductance multiplier $s$, (4.102), both plotted against $n$. Thereby, the damping coil required for optimum damping in a given case is completely specified if the crystal antiresonant frequency, $w_y$, the crystal inductance, $L_0$, and the capacitance ratio, $n$, are known.

The circuit $Q$'s (4.98a) and (4.98b) \(^{(19)}\), also are plotted in Fig. 4.2; and the decrement multiplier $H$, (4.88), versus $n$ is given in Fig. 4.3. From these we determine the effectiveness of the damping circuit, in

---

\(^{(19)}\) These equations indicate that $Q_2$ is always greater than $Q_1$ at optimum damping. The larger of the two, $Q_2$, is plotted. But, because $Q_2$ and $Q_1$ differ from each other by a negligible amount, the curve of Fig. 4.2 may be said to represent both for all practical purposes.
Fig. 4.2 - Inductance Multiplier, $s$, Damning Coil $Q$ and Resultant Circuit $Q'$s for Optimum Damping Conditions: $A_1 = A_2$, $B_2 = w_y$. 

Circuit $Q'$s:

$Q_1 = Q_2 = B_2 / 2A_2 = \sqrt{n}$

$[\tau_{max} = +0.5\%]$

$Q_1 = w_y L_d / R_d = \sqrt{n} / 2$

$[\tau_{max} = +0.3\%]$
$n = \frac{C_x}{C_0}$

Fig. 4.3 - Frequency Deviation of $B_1$, and Decrement Multiplier, $H$, at Optimum Damping: $A_1 = A_2$, $B_2 = w_y$. 

$H = \frac{A_1}{w_y} = \frac{1}{2\sqrt{n}}$,

$\varepsilon_{\text{max}} = -0.6\%$
accordance with the definitions
\[
A_1 = A_2 = H w_y ;
\]
and
\[
Q_1 \cdot Q_2 = \frac{B_2}{2A_2} .
\]

Finally, Fig. 4.3 shows the frequency deviation, (4.94), of the unwanted component of current during damping, as a function of \( n \).

The values of \( Q_y \) shown on both sets of curves for various portions of the range of \( n \) are obtained from the cross-relationships (4.78) and (4.88). They denote that the curves have their specified accuracy only for the values of \( Q_y \) indicated. Actually, the intrinsic errors are small enough to permit considerable extension of the \( Q_y \) restrictions. Consequently, the assumed lower limit of \( Q_y = 10^4 \) may be taken to apply over the whole range of \( n \) for most purposes. There is no upper limit for \( Q_y \), other than the natural physical limit.

**Illustrative Example.**

To illustrate the use of the curves, suppose we have a crystal whose measured parameters are:

\[
\begin{align*}
R_0 &= 30 \text{ ohms,} \\
L_0 &= 1/16 \text{ henry,} \\
C_0 &= 0.04 \mu\text{f,} \\
C_h &= 28 \mu\text{f,}
\end{align*}
\]

and 
\[
w_a = 20,014,280 \text{ rad/sec (true antiresonant frequency).}
\]

This is to be damped by means of a coil, and a padding capacitance of \( 32 \mu\text{f} \) remains across the crystal terminals during damping.

We calculate
\[
\begin{align*}
C_x &= C_h + C_k = 28 + 32 = 60 \mu\text{f;}
\end{align*}
\]
\[
\begin{align*}
n &= C_x/C_0 = 60/0.04 = 1500;
\end{align*}
\]
\[
\begin{align*}
m &= n + 1/n = 1 + 1/1500;
\end{align*}
\]
\[
\begin{align*}
w_0 &= 1/\sqrt{L_0 C_0} = 20 \times 10^5 \text{ rad/sec .}
\end{align*}
\]
and \[ w_y = w_0 \sqrt{m} = w_0 (1 + 1/3000), \] by Appendix J-4,
\[ = 20,000,670 \text{ rad/sec}. \]

From Fig. 4.2, we find that when \( n = 1500 \), \( s = 0.9987 \) and \( Q_d = 19.4 \).

Then
\[ L_d = sL_0/n = \frac{0.9987 \times 1/16}{1500} = 41.57 \mu \text{H}, \]
and
\[ R_d = w_Ld/Q_d = \frac{20.00 \times 10^6 \times 41.57 \times 10^{-6}}{19.4} = 21.42 \text{ ohms}. \]

From Fig. 4.3, when \( n = 1500 \), \( H = 0.013 \). Then the circuit decrements are:
\[ A_1 = A_2 = Hw_y = 0.013 \times 20.00 \times 10^6 = 0.26 \times 10^6 \text{ nepers/sec}; \quad \text{and, as defined in Sec. 2-6,} \]
\[ \Delta_1 = \Delta_2 = 8.686 \times A_1 = 8.686 \times 0.32 \times 10^6 = 2.26 \times 10^6 \text{ db/sec}. \]

These correspond to circuit \( Q \)'s, from Fig. 4.2, of
\[ Q_1 = Q_2 = 39. \]

The frequency deviation of the component \( B_1 \) is, from Fig. 4.3,
\[ F = 167 \text{ parts per million}. \]

The total deviation may be expressed in cycles as
\[ \frac{Fw_y}{2\pi} = \frac{167 \times 10^{-6} \times 20.00 \times 10^6}{2\pi} = 532 \text{ cycles}, \]
out of a working frequency of
\[ \frac{w_y}{2\pi} = \frac{20,000,670}{2\pi} = 3.182 \times 10^6 \text{ cycles/sec}. \]

4-12. **Beat Phenomena.**

From the frequency deviation curve of Fig. 4.3 it is seen that \( B_1 \) differs from \( w_y \) or \( B_2 \) by an amount which lies between 60 and 2000 parts per million. This frequency difference results in interference between the
two components of the crystal current (4.11). It would seem impractical, if not impossible, to determine the initial magnitudes, \( I_1 \) and \( I_2 \), of the components. However, if we make the simplifying assumptions that \( I_1 = I_2 \) and that the phase angles are approximately \( \phi_1 = \phi_2 = 90^\circ \), we have from (4.11)

\[
i = I_1 e^{-A_1 t} (\cos B_1 t + \cos B_2 t),
\]

by a trigonometric identity. But,

\[
B_1 - B_2 = B_1 - w_y = -F w_y; \quad \text{and} \quad B_1 + B_2 = -F w_y + 2 w_y.
\]

Then

\[
i = 2I_1 e^{-A_1 t} \cos \left[ 1 - \frac{F}{2} \right] w_y t \cos \left( \frac{F}{2} w_y t \right),
\]

since \( F/2 \) is ordinarily much less than unity; or

\[
i = I \cos w_y t,
\]

where the envelope is

\[
I = 2I_1 e^{-A_1 t} \cos \left( \frac{F}{2} w_y t \right).
\]

(4.107)

If we let \( N_c \) be the number of cycles of damping, we may write for (4.107)

\[
I = 2I_1 e^{-A_1 t} \cos \left( \frac{F}{2} \cdot 2\pi N_c \right),
\]

where

\[
N_c = \frac{w_y t}{2\pi}.
\]

(4.108)

Fig. 4.4 shows the sort of beat pattern which is obtained. A decrement of \( \delta = 1 \text{ db/cycle} \) was used (corresponding to \( n = 750 \) from Fig. 4.3, and equivalent to \( A_1 = 0.0183 \text{ w nepers/sec} \)). The effect is exaggerated by choosing \( F = 1 \text{ part in 20} \) (note that from Fig. 4.3, at \( n = 750, F = 333 \text{ parts per million.} \) The actual
equation of the envelope plotted in Fig. 4.4 is

\[ I = 2I_1 \cdot 10^{-\left(\frac{N_c}{20}\right)} \cos \frac{2\pi N_c}{40}. \]  

\[ (4.109) \]

\[ \delta = 1 \text{ db/cycle} \]

\[ H = 0.0183 \text{ nepers/rad} \]

\[ F = 0.05 \]

\[ N_c - \text{Cycles of Damping of Carrier Frequency, } w_y \]

**Fig. 4.4 - Type of Decaying Current Pattern Obtained with R-L Crystal Damping.**

It is obvious that whenever there exists a difference in the frequencies of the components of the damped crystal current, the resultant
wave tends to be smaller than that which would be obtained if there were no frequency difference. Note that if the correct value of $F$ ($333 \times 10^{-6}$) had been used in Fig. 4.4 and (4.108), the first "node" would have appeared at $N_c = 1500$ cycles of the carrier wave (instead of after 10 cycles, as shown in the graph). But, owing to the high decrement, the total current becomes unrecognizable after only a few cycles, anyway. The effect of the beat phenomenon, then, is to reduce the current magnitude below the level of recognition sooner than can be accomplished by the simple damping factor alone (see dotted line envelope, representing the case of zero beat frequency). Evidently, beat phenomena aid rather than oppose rapid crystal damping. The deviation of $B_1$ should, therefore, not be considered objectionable unless it is large enough to cause a perceptible change in the "carrier" frequency, given by (4.106) as $(1 - F/2) \omega_c$.

4-13. General Design Considerations.

From a design point of view it is undesirable to incorporate a variable inductance in crystal oscillator equipment for field or other service. Secs. 4-9 and 4-10 show that under optimum damping conditions all important quantities involved in the theory may be reduced to simple functions of $n$ alone - with the exception of $L_d$ which depends on $L_0$ also, in direct proportion. This means that multiple crystal substitution, even at a single frequency, would require adjustments in $L_d$; for no two crystals are likely to have identical values of $L_0$. Furthermore, any change in $L_d$ would necessitate a compensating resistance change to bring $Q_d$ to its specified value.

Because $n$ may be altered by means of a variable shunting capacitance, some flexibility in the damping circuit is possible, permitting the substitution of crystals having reasonable uniform values of $L_0$. In addition, operation over a band of frequencies might be admissible. A fixed inductance could be used in conjunction with a variable series resistance and a variable shunting capacitance.
The equations developed in the preceding sections may be modified as follows for such a method of adjusting the damping elements.

Let

\[ \checkmark = \frac{L_\circ}{L_d} \]  \hspace{1cm} (4.110)

Note that \( \checkmark \) is, in general, a large quantity. Then from (4.25) and (4.102),

\[ n = l - \frac{2}{n} \]

\[ \frac{1}{1 + 2\checkmark} \]  \hspace{1cm} (4.111)

or

\[ n = \checkmark - 2 \]  \hspace{1cm} (4.112)

Using this relationship we can rewrite the important equations:

\[ Q_d = \frac{\sqrt{\checkmark - 2}}{2} = \frac{\sqrt{\checkmark}}{2} \left[ 1 - \frac{1}{\sqrt{\checkmark}} \right] = \frac{\sqrt{\checkmark}}{2} \]  \hspace{1cm} (4.113)

\[ A_1 = A_2 = \frac{w_v}{2(\checkmark - 2)} = \frac{w_v}{2\checkmark} \]  \hspace{1cm} (4.114)

\[ Q_1 = Q_2 = \sqrt{\checkmark - 2} = \sqrt{\checkmark} \]  \hspace{1cm} (4.115)

\[ F = \frac{1}{4n} = \frac{1}{4\checkmark(1 - 2/\checkmark)} \]

\[ = \frac{1 + 2\checkmark}{4\checkmark} \]  \hspace{1cm} (4.116)

\[ \frac{R_d}{Q_d} = \frac{w_LL_d}{Q_d} = \frac{\checkmark w_0L_0}{Q_d\checkmark} = \frac{\checkmark}{Q_d} \left( \frac{L_0}{c_0} \right) \]  \hspace{1cm} (4.117)

Defining

\[ Z_0 = \sqrt{\frac{L_0}{c_0}} \]  \hspace{1cm} (4.118)
we have

\[ R_d = \frac{2Z_0 \sqrt{1 + \frac{1}{\sqrt{2} - 2}}}{\sqrt{\sqrt{2} - 2}}, \text{ by (4.6), (4.112), and (4.113)}, \]

\[ = \frac{2Z_0 (1 - \frac{1}{\sqrt{2} - 2})}{\sqrt{\frac{3}{2}} \left(1 - \frac{2}{\sqrt{2}}\right)}, \text{ by Appendix J-4}, \]

\[ = \frac{2Z_0}{\sqrt{\frac{3}{2}}} \left(1 + \frac{3}{2} \sqrt{2}\right) = \frac{2Z_0}{\sqrt{3/2}}. \]

It is evident that the curves of Figs. 4.2 and 4.3 remain applicable if the independent variable is changed from \( n \) to \( (\sqrt{2} - 2) \), or even simply \( \sqrt{2} \) since the quantities involved are not critical as to accuracy. The need for accuracy has been transferred from the coil to the tuning condenser and tuning resistor. It is now necessary to know or calculate the quantities \( \sqrt{2} \) and \( Z_0 \) for a given crystal (with fixed \( L_d \) in the oscillator equipment) and then to adjust the variables in accordance with the following specifications.

Let the coil resistance be \( R_L \), the tuning resistance \( R_k \), and the tuning condenser \( C_k \) as before. Then

\[ C_k = C_x - C_h = C_h (n - 1), \]

\[ = C_h (\sqrt{2} - 3), \text{ by (4.112)}; \]

and

\[ R_k = R_d - R_L \]

\[ = \frac{2Z_0}{\sqrt{\frac{3}{2}}} - R_L, \text{ by (4.119)}. \]

The fixed damping inductance \( L_d \) would be designed on the basis of a minimum \( L_o \) likely to be encountered among all crystals for the contemplated range of frequencies. Suppose this were .0127 henries. Then, since \( n_{\text{min}} = 125, L_d \) should be about 100 \( \mu \)h, by (4.110) and (4.112). The resistance \( R_L \) should be fairly low at the frequency of maximum \( R_L \). The \( Q \) of the damping circuit is thus high for
zero added tuning resistance allowing a good range of $Q_d$.

While it is possible to employ such a tuning arrangement as has been outlined, the difficulties in adjusting the elements to even approximate the requirements for optimum damping given by (4.120) and (4.121) would very likely be prohibitive. Moreover, it would almost certainly be necessary to include some switching apparatus, such as decade condensers and resistors, in order to provide adequate ranges of the variables for all crystals of a wide band of frequency operation.

The problems mentioned are, of course, inherent in the production of high damping rates by the use of inductance.

**UNEQUAL DECAY RATES, $A_1 \neq A_2$**

In the preceding sections we have been primarily concerned with the so-called optimum damping case, in which the two components of crystal current, (4.11), decay at equal rates though at slightly different frequencies. To conclude this study of R-L damping we shall briefly examine the possibilities of causing the unwanted lower frequency component to decay more rapidly than the desired fixed frequency component.

As pointed out in Sec. 4-12, the need for unequal decay rates might arise when the deviation, $F$, is large enough to change the "carrier" frequency, $w_0(1 - F/2)$, appreciably. No experimental evidence of a relevant criterion on the magnitude of $F$ has been produced, so that this final section is presented, in a sense, only for the sake of completeness.

4-14. Formulas and Curves.

From (4.23), the condition $A_1 \neq A_2$ is equivalent to $D \neq 1$. As $A_1$ is the decrement associated with the unwanted lower frequency, $B_1$, we are interested only in $D > 1$ and, say, $D < 10$. 
We have seen that the approximate equations involve very small errors, and that only \( L_d \) (and possibly \( R_d \)) need be determined accurately in the case of variable \( R-L \) damping. Accordingly, we shall omit statements of the errors contained in the various approximate equations which follow.

We begin with (4.68), which is rewritten as

\[
D = \frac{Q_y + 2nH}{4nQ_yH^2} \quad (4.122)
\]

By (4.78) we see that \( 2nH \) may be neglected compared with \( Q_y \), so that

\[
D = \frac{1}{4nH^2} \quad (4.123)
\]

from which

\[
H = \frac{1}{2\sqrt{nD}} \quad (4.124)
\]

Substituting this in (4.77)

\[
Q_d = \frac{\sqrt{nD}}{D + 1} \quad (4.125)
\]

Substituting (4.124) in (4.74),

\[
\frac{B_l}{w_y} = 1 - \frac{1}{4n} - \frac{1}{4} \left[ \frac{D}{\sqrt{n}} + \frac{1}{Q_y} \right]^2,
\]

\[
= 1 - \frac{1}{4n} - \frac{1}{4} \left[ \frac{D}{n} + \frac{2\sqrt{nD}}{\sqrt{n} Q_y} \right], \text{ by Appendix J-4;}
\]

and

\[
\frac{B_h}{w_y} = 1 - \frac{1}{8n} - \frac{1}{8n} \left[ D + \frac{2\sqrt{nD}}{Q_y} \right], \text{ by Appendix J-4,}
\]

\[
= 1 - \frac{D + 1}{8n} \quad (4.126)
\]

Then (4.24) becomes

\[
F = \frac{D + 1}{8n} \quad (4.128)
\]
The circuit Q's, (4.22), are as follows:

from (4.17), (4.72), and (4.124),

\[ Q_2 = \frac{B_2}{2A_2} = \sqrt{\frac{n}{D}} \]  \hspace{1cm} (4.129)

from (4.127), (4.23), (4.72), and (4.124),

\[ Q_1 = \frac{B_1}{2A_1} = \sqrt{\frac{n}{D}} \left[ 1 - \frac{D + 1}{8n} \right] \]  \hspace{1cm} (4.130)

As before, \( L_d \) needs special consideration, and we use the exact formula (4.100). Substituting in this for \( H \) from (4.124),

\[ s = 1 - \frac{7}{4n} - \frac{1}{4nD} \],

or

\[ s = 1 - \frac{7D + 1}{4nD} \]  \hspace{1cm} (4.131)

Curves corresponding to those of Figs. 4.2 and 4.3 are given in Figs. 4.5, 4.6, and 4.7. These are used as in the illustrative example of Sec. 4-11, with the additional fact that \( A_1 = DA_2 \).
Fig. 4.5 - Design Data for Damping Coil to Give Unequal Decay Rates.
$$Q_1 = \frac{B_1}{2A_1}$$

Circuit $Q_1$ at Unwanted Lower Frequency.

$$Q_2 = \frac{B_2}{2A_2}$$

Circuit $Q_2$ at Desired Frequency.

Fig. 4.6 - Resultant Circuit $Q$'s for Condition of Unequal Damping Rates.
Fig. 4.7 - Circuit Characteristics for Condition of Unequal Decay Rates.
CHAPTER V
A REVIEW OF DAMPING AND ITS PRACTICAL DIFFICULTIES

The two foregoing chapters are devoted to detailed analyses of various damping circuits. Because the various results which were derived are scattered it appears desirable to present a compilation and summary at this point. The difficulties encountered when these circuits are incorporated in practical oscillators are discussed.

5-1. Resistive Damping.

Resistance damping provides satisfactory damping for operation up to 43 dot-cycles per second for oscillator frequencies above about 1 mops. To obtain maximum damping the damping resistance must equal the reactance of the total capacitance, $C_x$, shunting the crystal series arm. When this optimum resistance is used for damping, the resulting $Q$ is equal to $2n$, and the damped frequency differs from the steady-state frequency by 1 part in $4n$.

If the damping resistance is selected for some nominal frequency the damping rate decreases from the maximum obtainable as the operating frequency departs from the nominal frequency. This decrease is gradual; so, a four to one frequency range may be covered without exceeding a twenty percent decrease from maximum damping. The frequency shift is greater than that associated with maximum damping if the operating frequency is less than nominal but less when the operating frequency is greater than nominal.

If the damping resistance is selected for a nominal holder capacitance the damping rate is practically independent of any anticipated variations.
in the actual holder capacity; however, the frequency shift is greater if the actual holder capacity is less than the nominal and smaller if the holder capacity is greater than nominal.

When the required frequency coverage is not large (approximately $2:1$), it is recommended that the damping resistance used be the optimum resistance for the lowest operating frequency and lowest holder capacitance to be encountered. Under this condition, as the operating frequency and holder capacitance increase, the damping rate remains reasonably close to the maximum obtainable, and the frequency departure never exceeds 1 part in $4\pi$ of the operating frequency. The primary advantages of resistance damping are the circuit simplicity and the wide frequency band covered.

5-2. **Wide Band Frequency Compensated Damping.**

In order to produce considerable damping rates without an associated frequency shift it is necessary to use a resistance in conjunction with a suitable reactance. A single reactance element is suitable for only a single operating frequency, but by combining two reactances with a resistance it is possible to secure a negligible frequency shift during damping with crystals over a considerable band of frequencies.

For a set of crystals having uniform values of capacitance ratio, $R$, and of shunting capacitance, $C_x$, the damping and frequency shift can be made satisfactory for all frequencies above about 1 mcps. The damping rate which can be achieved for all frequencies is in excess of that which can be secured by pure resistance damping at the lowest frequency in the band. The greatest frequency departure is about ten parts per million, again referred to the lowest frequency in the band.

The elements of the damping circuit must evidently be reasonably accurate in order to obtain the desired results. Allowable errors have not been worked
out in detail, but experience with the calculation of required values indicates that an accuracy of about 1% is sufficient to produce the desired results.

5-3. Extreme Damping.

Extreme damping rates may be produced by the use of either a series or a parallel combination of resistance and inductance. The series connection only has been given extensive treatment - in Chapter IV; however, the results obtained with the other connection are substantially equivalent.

Maximum damping is achieved when the $Q$ of the added coil is $\sqrt{n}/2$ at the operating frequency. Under this condition the oscillation takes the form of two frequencies equally damped, corresponding to circuit $Q$'s of $\sqrt{n}$ in value. One of these frequencies is essentially the same as the operating (antiresonant) frequency of the crystal, while the other differs from it by one part in $4n$. With an $n$ of 760 and a frequency of 1 mc/sec, the decrement is about $5 \times 10^5$ db/sec ($Q = 23$) and the frequency shift is 325 cycles.

These high damping rates can be secured only if the added elements are closely controlled. In particular, the coil inductance must be quite exactly $L_0/n (1 - 2/n)$ for the damping circuit to behave as described. Use of a variable condenser in parallel with the crystal to adjust $n$, and a variable resistor in series with the coil to alter its $Q$ will permit a single well-designed inductance to be employed with crystals over a frequency range of the order of 2:1.

However, it is unlikely that any simple tuning arrangement would be adequate for operation over a wide band of frequencies. Moreover no simple criterion for judging correct tuning has been found. It there-
foreappears that this form of damping is not suitable for use in field equipment.

5-4. Feedback Damping.

The systems so far discussed have been restricted to passive networks. Damping networks have been developed which use active parameters (tubes) in an amplifier feedback network. The operation is accomplished by amplifying the crystal output and feeding back this amplified output to oppose the crystal oscillations. Very low circuit Q's (as low as 8) have been obtained with this method. Further investigation of feedback damping was not undertaken because the circuit is regarded as too complex for use in military equipment. In addition, resistance damping is sufficient for operation with the buildup rates obtainable with present oscillators.

It is noted that this method may be useful in producing high decay rates if the buildup rate problem is solved and if complexity of the oscillator circuit may be tolerated.

5-5. Switches.

Three types of damping circuits have been developed to fulfill different primary requirements. The use of these circuits with an oscillator presents one common problem, namely some method for switching the damping network into and out of the oscillator circuit. For keying purposes the damping circuit must be connected during the time the oscillator is cut off, and disconnected when the oscillator is operative. Ideally these switching operations are coincident with the application to and removal from the vacuum tube of the D.C. polarizing potentials.

Relays and electronic switches (vacuum tubes) are two basic mechanisms with which this switching operation may be performed. Relays give satisfactory

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(20) A circuit of this type is described by P. F. Brown, "Pulsed Quartz Crystal Oscillator". MIT Radiation Laboratory Report 803, August 21, 1945.
performance at low keying speeds; however, as the keying speed increases most relays exhibit undesirable phenomena such as mechanical slowness and bouncing. Electronic switches will operate at extremely high keying speeds but impose other limitations because tubes require D.C. biases.

The addition of a simple electronic switch to an oscillator circuit considerably increases the circuit complexity and the circuit losses. Moreover the electronic switches which were tested either produced unreasonable circuit losses or produced unstable oscillators in which the crystal exerted inadequate frequency control. After several unsuccessful attempts were made to devise effective electronic switches it was decided to use relays. At this time only one type of simple relay which will operate at 100 dot-cycles per second is available. Two of these relays are required. One relay is connected in series with the damping network and is closed when the oscillator is inoperative. A second relay of this type is used to apply and remove the keying voltage from the tube.

5-6. Difficulties Encountered in Practical Circuits.

The use of a relay to switch the damping circuit was found to be impractical because of the extremely long rise times which resulted. Further investigation showed that the small high speed relays have appreciable intercontact capacities. As a result the damping network produces large losses in the oscillator even though the relay is open, thus causing long rise times. In addition it was found that relay operation is critical with respect to the relay coil operating voltage. Moderate deviations from the optimum operating voltage results in slowness or bouncing. For these reasons the damping approach is not considered to produce a satisfactory solution to the keying problem.

CHAPTER VI

THE KEYED OSCILLATOR

It has been stated in Chapter V that suitable decay rates can be obtained with damping circuits; however, the practical use of these damping circuits greatly decreases the oscillation buildup rates. In actual oscillator circuits the presence of a damping circuit lowers the buildup rates to a point where keying speeds are seriously limited. Accordingly, an experiment was performed to determine the tube characteristics which govern the buildup rate. Oscillation rise times were measured for many tubes of several different types in two different oscillator circuits. The experimental results are somewhat contradictory; however, there is indication that beam power tetrodes produce the fastest buildup rates. The difficulties encountered in the practical application of damping circuits and the inconclusiveness of this experiment recommend a re-examination of the damping philosophy as a solution to the keying problem.

It is evident that the keying rate of an oscillator is limited by the buildup and decay rates of the oscillations; so, the factors which limit both rates are the same factors which limit the keying rate. It is necessary to evaluate these limiting factors and determine whether any relationship exists between the buildup and decay rates. In addition, the experience with damping circuits indicates that the oscillator circuit should be a complete unit in which keying is accomplished simply by turning it on and off — that is, by switching on and off one of the D.C. potentials. These requirements can be fulfilled if the oscillator is studied as a complete unit. As a result, the properties of a keyed oscillator

(22) The experiment is described in Appendix L.
will be examined with a basic requirement that the oscillator be considered as an indivisible unit whose characteristics are determined for the various portions of the keying cycle.

6-1. The Use of Q to Study a Keyed Oscillator.

The rate at which oscillations build up or decay is adequately defined by the decrement $\Delta$ which, for oscillatory circuits, is related by definition to the circuit $Q$ by the equation

$$\Delta = \frac{27.3 f}{Q} \text{ db/sec}.$$  \hspace{1cm} (6.1)

For the examination of oscillatory circuits, $Q$ is easier to manipulate than $\Delta$, so $Q$ will be used in this study. From (6.1) it is seen that $\Delta$ increases as $Q$ decreases; therefore, for high buildup and decay rates it is necessary for the absolute magnitude of $Q$ to be small. The limiting factors for buildup and decay may be defined as the minimum $Q$'s obtained during buildup and decay.

An indication of the requirements placed on a keyed oscillator is obtained from a consideration of variations in $Q$ over a keying cycle. A study of performance admits the existence of positive, negative, and infinite $Q$'s. The variation of the oscillator circuit $Q$ during a keying cycle is as follows. (23)

For oscillations to build up, $Q$ must be negative and finite. During the period of steady-state oscillation the energy level remains constant, so $Q$ is infinite. During periods of oscillation decay $Q$ is positive and finite. For high buildup and decay rates, it is necessary for $Q$ to be small; therefore, during buildup $Q$ should be small and negative, and during

(23) It is emphasized that the $Q$ in question refers to the entire circuit and not to the resonator or any other separable component.
decay $Q$ should be small and positive.

In simple oscillators employing quartz crystals as frequency control devices, essentially all the circuit energy is stored in the crystal. In addition, the crystal has a very low loss characteristic; thus it is a high $Q$ device. The oscillator circuit, exclusive of the crystal, stores very little of the total circuit energy, produces the majority of the energy loss, and supplies all of the energy gain. Therefore, with sufficient accuracy, the crystal may be assumed to store all the circuit energy and the remainder of the oscillator to produce the energy loss and gain.

6-2. The Simplest Keyed Oscillator.

It is necessary to set up circuits which will furnish the desired positive and negative circuit $Q$'s for the various portions of the keying cycle. Positive $Q$'s are obtained by the proper use of positive resistances. Negative $Q$'s require the use of negative resistance devices. Several methods of producing negative resistances and their inherent limitations have been presented by Herold. Of these methods, the only ones which appear applicable to keyed oscillators are the circuits employing vacuum tubes. An inherent limitation on the use of vacuum tubes to provide negative resistance is the fact that in every case the negative resistance is unavoidably accompanied by a shunting capacitance. We may interpret this to mean that the combination of resistance and capacitance, with crystal, constitutes the most elementary possible representation of a crystal oscillator.

Even in the event that the nonlinear negative resistance could be obtained as a simple circuit element, it has been shown by Llewellyn that for steady-


state conditions any nonlinear negative resistance can be represented by a linear negative resistance shunted by a reactance. It is also noted that this resistance is voltage-controlled, a fact which automatically restricts the types of circuits in which the resistance will cause stable oscillations.

The criterion for determining the points of stable oscillation with voltage-controlled and current-controlled negative resistances has been presented in Sec. 2-19. This criterion for stability permits the determination of the frequency or frequencies at which a circuit will oscillate.

The desired keying method recommends the permanent incorporation of the loss-producing elements in the circuit and the use of an appropriate negative resistance to cancel the effects of these elements and to furnish the gain necessary to give the desired buildup rate. A representative circuit configuration illustrating these features is shown in Fig. 6.1, where $R_x$ is the loss-producing resistance, $R_n$ is a negative resistance which may vary from some finite value to infinity, and $C_n$ is the shunt capacitance inherently associated with $R_n$. It is evident that this circuit is entirely equivalent to the resistance-damping circuit for which a solution was developed in Sec. 3-1. As a direct consequence, there is a value of negative resistance which will produce the maximum possible buildup rate. If $C_n$ is assigned the value for $C_k$, the crystal load capacitance, the circuit can be redrawn as shown in Fig. 6.2.

Now, from Sec. 3-1 it is known that $R_x = 1/\omega C_x$ produces the maximum decay rate; similarly, the maximum buildup rate will be obtained when the
net negative resistance shunting $C_x$ equals $1/wC_x$. This condition is given by

$$\frac{R_nR_x}{R_n + R_x} = -\frac{1}{wC_x} = -R_x, \quad (6.2)$$

which yields the solution

$$-R_n = \frac{R_x}{2} = \frac{1}{2wC_x} \quad (6.3)$$

for the value of $R_n$ which results in maximum buildup rate. The negative resistance $R_n$ must be voltage-controlled so that its magnitude approaches and exceeds that of $R_x$ as the voltage across $R_n$ increases; (then the circuit becomes infinite and a constant amplitude condition of oscillation is reached).

![Fig. 6.2 - Equivalent Circuits for Fig. 6.1.](image)

It is apparent that the parameters can readily be calculated for this circuit. However, an undesirable frequency shift occurs during buildup and decay, the magnitude of which may be directly deduced from the information obtained in the study of resistive damping.

6-3. Consideration of a Constant Frequency Keyed Oscillator.

To produce an oscillator which has a constant frequency output, it is necessary that the impedance in series with $L_o$, $R_o$, and $C_o$ have a reactive component which is invariant to the positive and negative values assumed by the effective resistance $R_e$, as shown in Fig. 6.3. Furthermore, in the interests of stability, the circuit must not oscillate when $L_o$, $R_o$, and $C_o$ are removed.
This condition exists only if the circuit is capable of oscillating in just one mode.

Previous analyses directed towards synthesizing constant frequency damping networks have called for the incorporation of inductance into the network. A reasonable extension of the theory leads to the belief that any network used to effect a constant frequency throughout the keying cycle must contain an inductance. It is concluded, therefore, that the undetermined four terminal network of Fig. 6.3 will contain at least an inductance. The presence of this inductance in the network immediately provides sufficient conditions for the existence of two modes of oscillation. Furthermore, the presence of the capacitance $C$ shunting $R_e$ causes both modes of oscillation to be voltage-controlled, so that it is not possible to exclude one of the modes by the choice of a voltage-controlled negative resistance. Thus the stability requirement is violated.

If the circuit parameters are such that a given positive $R_e$ produces two modes, the location of the complex conjugate roots on the p-plane will, in general, be as shown by the $z$'s in Fig. 6.4. These represent two different associated decay rates. Now, if $R_e$ is made negative, the roots
will cross over to the positive half of the p-plane as shown by the o's. It is immediately apparent that the frequency which decays the fastest also builds up the fastest. In order for the crystal to exercise frequency control during the constant amplitude period, it is necessary that the mode determined by the crystal have a much slower decay rate than the other mode. But in that event the undesired frequency will predominate during buildup.

If, on the other hand, the circuit parameters are such that only one mode of vibration exists, an additional real root is obtained, with the attendant possibility of relaxation oscillations. The study of extreme damping indicates that the circuit parameters will have values to permit two modes of oscillation if high buildup and decay rates are achieved.

The above arguments lead to the conclusion that the objective cannot be realized when Re merely assumes alternate positive and negative values so as to order decay and buildup.

Conceivably, the loss element may be included in the network (isolated from Re) in such a manner that it will produce oscillation decay as desired, with Re required only to have negative values and give the desired buildup. So far a network of this type has not been devised which results in constant frequency operation.
CHAPTER VII
KEYED OSCILLATOR CIRCUITS

The use of a resistance to damp the crystal produces sufficient decay rates for most crystals when keyed at 43 dots per second. So far no practical method has been given for incorporating the damping resistor in the oscillator circuit or for determining the circuit parameters required for a good keyed oscillator. This chapter is devoted to the theoretical analyses and design procedures for keyed oscillators of the Pierce, Miller, and electron-coupled types.

7-1. The Keyed Pierce Oscillator.

When the Pierce Oscillator is to be keyed it should be modified to provide plate circuit damping. The chief advantages of this circuit are its extreme simplicity and the broad band operation which can be obtained without readjusting any circuit parameters. The basic circuit, which differs only slightly from the conventional one, is presented in Fig. 7.1.

![Fig. 7.1 - Pierce Oscillator for Keying Service.](image)

The plate load resistor, $R_2$, has two functions: one, to provide damping of the crystal during the "off" period; and, the other, to provide an A.C. load for the "on" period. The fact that a relatively low
resistance is used for a plate load, instead of an inductance or high resistance, is the only difference between this circuit and the conventional Pierce Oscillator.

Because the oscillation decay rate is of more concern than the buildup rate, the first step is the determination of the factors controlling the damping. For this purpose the circuit is redrawn in Fig. 7.2 omitting the tube. $R_1$ also is omitted since it is large and has negligible effect.

\[ Z_2 = \frac{1}{C_1 + j \omega C_2} \]

Fig. 7.2 - Equivalent Damping Circuit of Fig. 7.1.

Since the crystal is designed to operate into a load capacitance, $K$, the three capacitances, $C_1$, $C_2$, and $C_h$, can be related to this load capacitance by the definitions:

\[ C_1 = (b + 1) K, \]

\[ C_2 = \frac{C_1}{b} = \frac{b + 1}{b} K, \]

and

\[ C_h = a K; \]

where

\[ K = \frac{C_1 C_2}{C_1 + C_2}. \]

The value of $R_2$ which will produce maximum damping of the crystal can be calculated from a knowledge of $a$, $b$, $K$, and the operating frequency, $\omega$. The mathematical determination of $R_2$ to produce maximum damping is presented in Appendix M. The results of these calculations are exhibited in Table 7.1 and Fig. 7.3, from which $R_2$ is given as the constant $P$ times $1/\omega K$. The effect on damping when the frequency varies, with $R_2$ selected for some given frequency $\omega_r$, is shown in Fig. 7.4, where the curve gives the ratio of actual damping to the

(26) Note that $K$ is here written in place of the usual load capacitance symbol $C_L$. 
Fig. 7.3 - Resistance Factor $P$ as a Function of $a$ and $b$, $(R_2 = P \frac{1}{\omega K})$. 

Capacitance Ratio, $b = C_2/C_1$
Actual Damping Obtained with Fixed $R_2$ in Percent of the Maximum Obtainable with Variable $R_2$.
damping which could be obtained if $R_2$ varied with frequency. From this curve it is seen that a frequency range of 4 to 1 can be used without producing a serious degradation of the damping rate. With the above in-

| TABLE 7.1 |
| VALUES OF RESISTANCE FACTOR, $P$, FOR VARIOUS VALUES OF $a$ AND $b$ |

<table>
<thead>
<tr>
<th>b</th>
<th>$a = .25$</th>
<th>$a = .5$</th>
<th>$a = .75$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.4494</td>
<td>.4171</td>
<td>.3924</td>
</tr>
<tr>
<td>2</td>
<td>.5779</td>
<td>.5187</td>
<td>.4764</td>
</tr>
<tr>
<td>3</td>
<td>.6372</td>
<td>.5621</td>
<td>.5089</td>
</tr>
<tr>
<td>5</td>
<td>.6943</td>
<td>.6017</td>
<td>.5357</td>
</tr>
<tr>
<td>8</td>
<td>.7309</td>
<td>.6253</td>
<td>.5505</td>
</tr>
<tr>
<td>12</td>
<td>.7530</td>
<td>.6387</td>
<td>.5577</td>
</tr>
<tr>
<td>$\infty$</td>
<td>.8000</td>
<td>.6667</td>
<td>.5710</td>
</tr>
</tbody>
</table>

formation a value of $R_2$ can be calculated and the performance of the oscillator during the damping period may be predicted.

There now remains the problem of determining the requirement which the vacuum tube must meet in order to produce a satisfactory buildup rate. The equivalent circuit of the oscillator is drawn in Fig. 7.5.

Fig. 7.5 - Equivalent Circuit of the Pierce Oscillator of Fig. 7.1.

Because no satisfactory method is known for the analysis of the oscillation buildup when the tube is operated under nonlinear conditions, the analysis will be restricted to small signal amplitudes where the tube is linear. Experience indicates that this solution will give a satisfactory indication of performance as the tube is driven into nonlinear conditions.
In Fig. 7.5, it has been assumed that the tube resistance, $r_t$, is very large compared with $R_2$ and that the grid to plate capacitance may be lumped with the crystal holder capacitance. These assumptions are certainly valid if the tube is a pentode or high-mu triode.

An expression is now derived for the impedance $Z$ seen by the crystal. From the equivalent circuit the following facts are evident:

\[ I_3 = I_1 - I_2 - e_g \varepsilon_m, \quad (7.5) \]
\[ E = -j \frac{1}{wC_2} I_2 - j \frac{1}{wC_1} I_1, \quad (7.6) \]
\[ Z = \frac{E}{I_1} = -j \frac{1}{wC_1} - j \frac{1}{wC_2} \frac{I_2}{I_1}, \quad (7.7) \]
\[-j \frac{1}{wC_2} I_2 = R_2 I_3 = R_2 (I_1 - I_2 - e_g \varepsilon_m), \quad (7.8)\]

and
\[ e_g = -I_1 \left[ -\frac{1}{wC_1} \right]. \quad (7.9) \]

The unknown currents may be eliminated to give

\[ Z = -j \frac{1}{wC_1} - j \frac{1}{wC_2} \left[ \frac{R_2 \left( 1 - \frac{1}{wC_1} \varepsilon_m \right)}{R_2 - j \frac{1}{wC_2}} \right]. \quad (7.10) \]

After rationalizing and substituting for $C_1$ from (7.2), there results

\[ Z = R - j X = \frac{R_2 - \frac{R_2 \varepsilon_m}{b} - j \frac{1}{wC_2} \left[ w^2C_2^2R_2^2 (1 + \frac{1}{b}) + \frac{1}{b} + \frac{R_2 \varepsilon_m}{b} \right]}{w^2C_2^2R_2^2 + 1}, \quad (7.11) \]

which constitutes an exact expression for the impedance seen by the crystal.

We next draw the equivalent circuit of the crystal and connect $Z$ across it as shown in Fig. 7.6. The resistance $R_x$, seen by the crystal series arm($L_0R_0C_0$),
Sec. 7-1  THE KEYED PIERCE OSCILLATOR

is given by

\[ R_z = \frac{Xh^2 R}{R^2 + (X + Xh)^2}, \]  \hspace{1cm} (7.12)

where \( X = 1/wC \) and \( Xh = 1/wCh \).

It is found by differentiation of (7.12) that \( R_z \) is a maximum with respect to \( R \) when

\[ R = X + Xh. \]  \hspace{1cm} (7.13)

Now, \( R_2 \) has already been fixed so as to produce a maximum \( R_z \) in the absence of \( g_m \); then, substituting in (7.13) for \( R \) and \( X \) from (7.11) with \( g_m = 0 \), we have

\[ \frac{R_2}{w^2C_2R_2^2 + 1} = \frac{1}{wCh} + \frac{1}{wC_2} \cdot \frac{w^2C_2R_2^2(1 + \frac{1}{b}) + \frac{1}{b}}{w^2C_2R_2^2 + 1}. \]  \hspace{1cm} (7.14)

The value of \( R \) given by (7.13) is for maximum decay rate. If the buildup rate is to be a maximum, it is necessary that

\[ R = -(X + Xh). \]  \hspace{1cm} (7.15)

Hence, from (7.11) with \( g_m \neq 0 \),

\[ \frac{R_2 - \frac{R_2 g_m}{b}}{w^2C_2R_2^2 + 1} = - \left[ \frac{1}{wCh} + \frac{1}{wC_2} \cdot \frac{w^2C_2R_2^2(1 + \frac{1}{b}) + \frac{1}{b} + \frac{R_2 g_m}{b}}{w^2C_2R_2^2 + 1} \right]. \]  \hspace{1cm} (7.16)
The combination of (7.14) and (7.16) gives
\[ g_m \left[ \frac{R_2^2}{b} - \frac{R_2}{bwC_2} \right] = 2 R_2; \]  
(7.17)
or, eliminating \( C_2 \) by (7.2),
\[ g_m = \frac{2b}{R_2} \left[ \frac{1}{1 - \frac{b}{P(b + 1)}} \right], \]  
(7.18)
in which \( P \) is the function of \( a \) and \( b \) defined by the equation
\[ R_2 = \frac{P}{\pi K}, \]  
(7.19)
and is evaluated for various values of \( a \) and \( b \) in TABLE 7.1. By substituting in (7.18) various values for \( P \) from the Table, it is seen that the \( g_m \) required for maximum buildup rate is negative if \( b \geq 1 \) and \( 0.25 < a < 1 \). Since negative \( g_m \)'s are not possible with the Pierce Oscillator and since \( a \) will ordinarily lie within the limits stated, it follows that the maximum buildup rate cannot be achieved for \( b \geq 1 \).

If the \( g_m \) is calculated for a buildup rate 0.4 times the maximum, the following expression is obtained:
\[ g_m = \frac{1.5b}{R_2} \left[ \frac{1}{1 - \frac{b}{2P(b + 1)}} \right]. \]  
(7.20)
When \( b = 1 \) and \( a = .5 \) this yields
\[ g_m = \frac{3.75}{R_2}. \]  
(7.21)
The value of \( g_m \) required rapidly approaches infinity as \( b \) is increased.

The foregoing mathematical study shows that tubes having large values of transconductance are necessary if satisfactory buildup rates are to be achieved along with good decay rates. Because of the many factors involved it appears that
the actual circuit parameters are best determined by experiment.

Values of $a$ and $b$ to give a suitable value of $P$ from TABLE 7.1 are assumed. Knowing the crystal holder capacitance, we can determine $K$ by (7.3) and then $R_2$ from (7.19). Values of $a = 0.5$ and $3 < b < 5$ are suitable for a first choice. Next the tube is selected on the basis of highest $g_m$ consistent with its size and power consumption. Among presently available tubes the 6AC7 and 6AK5 are likely choices. The oscillator is set up with the selected values of elements and tested. Finally, $R_2$ is adjusted to give the best overall performance over the specified frequency range. Consideration should be given to the fact that the decay rate ought to be somewhat higher than the buildup rate.

7-2. Calculation and Experimental Design of a Keyed Pierce Oscillator.

Use of the foregoing method of design is now illustrated. The values of $b = 5$ and $a = 0.5$ are assumed as representative of the crystals to be used. The load capacitance is assumed to be $25 \mu F$, corresponding to a $C_h$ of $12.5 \mu F$. Let the nominal frequency be $4 \text{ mc}$. Then

$$\frac{1}{wK} = \frac{1}{2 \pi (4 \times 10^6) \times 25 \times 10^{-12}} = 1600 \text{ ohms}.$$ 

From TABLE 7.1, $P = .602$, and by (7.19), $R_2 = .602(1600) = 1000 \text{ ohms}$. 

From (7.1), $C_1 = (b + 1)K = 6(25) = 150 \mu F$, and from (7.2), $C_2 = C_1/b = 150/5 = 30 \mu F$.

Allowing for the tube capacities the circuit shown in Fig. 7.7 is obtained. This circuit was tested at a keying rate of 45 dot-cycles per second. It was found that the decay rate was excellent but the buildup rate was much too low. By increasing $R_2$ to $3300 \text{ ohms}$, the buildup rate
was improved sufficiently to produce satisfactory operation. The decay rate was decreased only slightly. Fig. 7.8 presents the control grid voltage waveforms obtained when the oscillator described above was used with crystals from 2,030kc to 8,400kc.

![Diagram of Experimental Keyed Pierce Oscillator](image)

Because the oscilloscope with which these observations were made has an upper frequency limit at 2 mc., the r.f. is attenuated. For this reason the oscillograms obtained are not a true representation of the grid voltage. The crystals for 4830 kc, 5570 kc, and 6091 kc have extremely high $Q$ values - in the neighborhood of 200,000. The remainder of the crystals have $Q$'s between 50,000 and 100,000. The relatively long rise and fall times obtained for the 4830, 5570, and 6091 kc crystals are explained on this basis.

7-3. The Keyed Electron Coupled Pierce Oscillator.

For many applications it is desirable to isolate the plate circuit of the tube from the remainder of the oscillator circuit. This objective is accomplished by using electron coupling. In one common arrangement the suppressor grid of a pentode is utilized as an electrostatic shield between the plate and

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*The capacity between the key and cathode is shown as infinite. The condenser must actually be of such magnitude that it provides a virtual short-circuit to the crystal oscillation frequency but a high impedance to the "frequency" of keying. This notation is to have the same significance in all the keyed circuit diagrams which follow.*
Fig. 7.8 - Oscillograms of Control Grid Waveforms for Various Crystals Keyed at 45 Dot-Cycles per Second in the Keyed Pierce Oscillator Circuit of Fig. 7.7.
the remaining tube elements. It is necessary that the suppressor be effective as an electrostatic shield and that it also be at ground potential. A modified Pierce circuit which satisfies these requirements is given in Fig. 7.9. Note that a separate suppressor lead is needed because the cathode is not connected to ground.

Fig. 7.9 - Keyed Electron Coupled Pierce Oscillator.

It can be shown that this circuit is reducible to that of the keyed Pierce Oscillator of Fig. 7.1. Therefore, \( R_1, R_2, C_1, \) and \( C_2 \) are the same in both cases, and it follows that the development presented in Sec. 7-1 is directly applicable to the electron coupled version of the Pierce Oscillator. The circuit provides a transconductance for buildup which is the sum of the plate and screen grid \( g_m \)'s. The presence of \( Z_L \) in the plate circuit has only one effect: the power dissipated in \( Z_L \) is subtracted from the power dissipated by the plate of the tube.

Fig. 7.10 - Experimental Keyed Electron Coupled Pierce Oscillator.
The experimental oscillator of Fig. 7.10 was set up to test the behavior of this circuit. Consistent with the preceding discussion, the element values correspond to those of Fig. 7.7. The cathode voltage waveforms obtained for various crystals are presented in Fig. 7.11. Here again the upper frequency limit of the oscilloscope causes the RF to be attenuated and results in waveforms which are not representative of the true condition. The voltage step which exists before the oscillations begin is due to the flow of current through the cathode resistor. The high $Q$ of the 4830, 5570, and 6091 kc crystals accounts for their long rise and fall times. For reasons that are not understood, the 2030 kc crystal would not oscillate in this particular circuit.

The oscillograms indicate that the electron coupled Pierce does not have as short a rise time as the conventional Pierce, but performance is satisfactory.

7-4. The Keyed Miller Oscillator.

Because the Miller Oscillator requires an inductive plate circuit load, it is not possible to replace the plate load impedance by a pure resistance as was done in the Pierce. Accordingly, the damping resistance is connected across the crystal terminals. The basic circuit for the keyed Miller Oscillator is shown in Fig. 7.12. In this circuit $R_2$ and $C_4$ act...
Fig. 7.11 - Oscillograms of Cathode Waveforms for Various Crystals Keyed at 45 Dot-Cycles per Second in the Keyed Electron Coupled Pierce Oscillator Circuit of Fig. 7.10.
serve to furnish and control the negative bias developed at the grid during the "on" period. The value of $R_1$ which produces maximum damping is equal to the absolute magnitude of the reactance shunting the crystal terminals, as indicated by the analysis of Chap. III.

Because $R_2$ and $C_4$ are large the circuit of Fig. 7.12 may be reduced to the equivalent circuit shown in Fig. 7.13, in which $C_g = C_h + C_l$.

![Equivalent Circuit of Keyed Miller Oscillator.](image)

The input admittance, $Y$, of the tube and the associated elements have been derived by many investigators. If the $r_p$ of the tube is large and there are negligible losses in the plate circuit, the input admittance is given by the expression

$$Y = G + jB = \left[ C_1 - \frac{wC_2C_m}{1/wL - w(C_2 + C_3)} \right] + j \left[ w(C_g + c_2) + \frac{w^2C_2}{1/wL - w(C_2 + C_3)} \right], \quad (7.22)$$

where $C_1$ has been written in place of $1/R_1$. The first condition is satisfied through the use of a pentode. The equivalent conductance shunting $L$ and $C_3$ will be very small compared to the other admittances if a good coil is used for $L$; in this manner the second condition may also be satisfied. From (7.22) it is seen that the capacitance shunting the crystal

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terminals is independent of the tube $g_m$; therefore, this shunt capacitance is constant throughout the keying cycle.

Then, to produce maximum damping, as shown in Sec. 3-1,

$$1/R_1 = B,$$

or

$$G_1 = w(C_g + C_2) + \frac{w^2 C_2}{\frac{1}{wL} - w(C_2 + C_3)}.$$

And, to produce maximum buildup,

$$G = -1/R_1,$$

or

$$G_1 - \frac{wC_2 g_m}{\frac{1}{wL} - w(C_2 + C_3)} = -G_1,$$

whence

$$g_m = \frac{2G_1 \left[ \frac{1}{wL} - w(C_2 + C_3) \right]}{wC_2}.$$

Substituting for $G_1$ from (7.23), we obtain

$$g_m = 2 \left\{ \frac{w(C_g + C_2)}{wC_2} \left[ \frac{1}{wL} - w(C_2 + C_3) \right] \right\}.$$

If the definition

$$\frac{1}{wL} - w(C_2 + C_3) = wC_2$$

is employed, (7.23) and (7.25) reduce to

$$G_1 = w(C_g + 2C_2)$$

and

$$g_m = 2G_1 = w(C_g + 2C_2);$$

and the imaginary part of (7.22) becomes

$$B = w(C_g + 2C_2).$$
Now, the crystal is designed to operate into a load capacitance, \( K \).

In terms of \( C_h \) and \( K \), the susceptance seen by the crystal series arm \((L_o R_o C_o)\) is

\[ B = \omega (C_h + K) \]  

(7.31)

But, in Fig. 7.13, \( C_g = C_h + C_1 \). Combining this with (7.30) and (7.31), we have

\[ K = C_1 + 2C_2 \]  

(7.32)

For the oscillator to produce maximum buildup and decay rates, (7.27), (7.28), (7.29), and (7.32) must be satisfied. Although the selection of a tube specifies a nominal \( g_m \), this \( g_m \) may be varied over a considerable range by the proper choice of bias potentials. On the other hand, the tube capacities are fixed. Therefore, by selecting a tube (and considering stray wiring capacities) minimum values are obtained for \( C_1 \) and \( C_3 \). The crystal itself will prescribe \( \omega \), \( C_h \), and \( K \). Values for \( C_2 \), \( L \), and \( g_m \) remain to be determined. Before proceeding, it should be remembered that (as indicated by the analysis of Chap. III) the buildup and decay rates decrease as the capacitance shunting the crystal increases. Therefore, it is desirable to have \( C_h \) and \( K \) as small as possible.

7-5. Design of a Keyed Miller Oscillator.

The calculation of circuit parameters is a simple process, and an illustration of the procedure is given here. A 6SJ7 tube and a 3680 kc crystal (with \( C_h = 14 \mu\text{f} \) and \( K = 25 \mu\text{f} \)) are arbitrarily selected. Making allowance for stray wiring capacities, the following values are determined:

\[ C_1 = 6 \mu\text{f} \]  
\[ K = 25 \mu\text{f} \]
\[ C_3 = 9 \mu\text{f} \]  
\[ \omega = 2 \pi \text{f} \]
\[ C_h = 14 \mu\text{f} \]  
\[ = 23.15 \times 10^6 \text{c} \]
Then, from (7.32),
\[ C_2 = 8.5 \mu \text{pf}; \]
from (7.27),
\[ L = 72 \mu \text{h}; \]
from (7.28),
\[ R_1 = 1110 \text{ ohms}; \]
and from (7.29)
\[ g_m = 1800 \text{ micromhos}. \]

The tube manuals indicate that a 6SJ7 has a \( g_m \) of 1800 with 50 volts applied to the screen grid. From the above data the oscillator shown in Fig. 7.14 is obtained. The value of \( C_4 \) is chosen arbitrarily, the requirement being that this capacitance be large compared to the input capacitance of the tube. \( R_2 \) is chosen to have an impedance large compared to \( C_4 \) and to \( R_1 \).

This oscillator was tested and found to give satisfactory operation at 45 dot-cycles per second. Crystals for 4080 kc and 3230 kc would operate but produced longer rise times. Crystals for frequencies further removed from 3680 kc would not operate.

Changing the \( g_m \) by varying the screen supply voltage from 25 volts to 150 volts did not appreciably affect the rise time. This was expected because variation of the damping resistor (\( R_1 \) or \( G_1 \)) over a limited range does not appreciably affect the decay \(^{(29)}\), and because \( g_m \) is related to \( G_1 \) as shown in (7.29).

\(^{(29)}\) See Appendix G.
Sec. 7-6  KEYED ELECTRON-COUPLED MILLER OSCILLATOR

Good operation was obtained successively on 4830 kc and 5570 kc by decreasing \(L\). The results of this experiment indicate that the Miller Oscillator can be operated over a range of frequencies only by changing \(L\). As this is not usually feasible, it appears that the Miller Oscillator is restricted to substantially fixed frequency operation. Oscillograms of the plate voltage waveforms obtained are presented in Fig. 7.15.

As with the Pierce Oscillator, it is recommended that final values of the parameters be obtained by varying the damping resistance (\(R_1\) of Figs. 7.12 and 7.13) in the vicinity of the calculated value.

7-6. The Keyed Electron-Coupled Miller Oscillator.

The same relationship exists between the electron-coupled Miller Oscillator and the conventional Miller as exists between the electron-coupled Pierce and the conventional Pierce. Theoretically, the two versions are equivalent in operation, provided the assumptions made in analyzing the conventional Miller are true for the electron-coupled Miller. Accordingly, in the oscillator presented in Fig. 7.16, the parameters \(R_1, C_1, C_2, C_3, \) and \(L\) correspond to the same elements in Fig. 7.12.

It is recognized that equivalence of the two circuits requires the dynamic plate resistance to be large compared with the other impedances in the circuit; we may reasonably assume that the use of a pentode ensures the fulfillment of this condition. The second prerequisite, that the conductance shunting \(L\) be small, may or may not be satisfied. This must remain, for the present, an unknown factor.
Crystal Frequency: 3680 kilocycles
Plate Load Inductance: 72 microhenries

Crystal Frequency: 4830 kilocycles
Plate Load Inductance: 41.6 microhenries

Crystal Frequency: 5570 kilocycles
Plate Load Inductance: 31.4 microhenries

Fig. 7.15 - Oscillograms of the Plate Waveforms for Various Crystals Keyed at 45 Dot-Cycles per Second in the Keyed Miller Circuit of Fig. 7.14.
The circuit in Fig. 7.17, which is a transform of Fig. 7.14 was set up and tested. It was found that the circuit would not operate well because of an extremely long rise time. Further investigation revealed that good keying waveforms were obtained if the 1.2K damping resistor was removed. This suggested that considerable losses exist in the circuit. The heater to cathode conductance was measured and found to be approximately 20,000 ohms, and the assumption of negligible shunt conductance across L was therefore incorrect.

Like the conventional Miller, this circuit operates properly only for a limited frequency range. Because of the indeterminate factors (such as
heater to cathode conductance) the required parameter values are probably best obtained experimentally. The keying speeds obtained are somewhat inferior to those of the normal Miller Oscillator but are adequate for most applications.

7-7. **General Considerations for Keyed Oscillators.**

During the study of keyed oscillators the basic circuit components alone were considered. The design procedures given in the first six sections of this chapter lead to oscillators which incorporate only the minimum essential circuit elements. Obviously, circuits used in practical operating equipment should have certain refinements, particularly those which will protect the tube in case of circuit failure.

Protection of the tube is best accomplished by the use of cathode bias - that is, by the insertion of an adequately bypassed resistor in the cathode lead. This resistor, \( R_k \), must be between the cathode and the grid return resistor, \( R_g \), as shown in Fig. 7.18. The value of \( R_k \) is calculated from the tube characteristics and is selected to limit the tube dissipations to allowable values with zero volts on the control grid. The bypass condenser should be large enough to provide an effective A.C. short-circuit path over the frequency range of the oscillator.

It should be mentioned that a tube may also be protected from excessive dissipation by inserting a suitable resistor in the B supply lead. In a triode this resistor carries the total plate current and must therefore be capable of dissipating considerable power. Consideration of the maximum power transfer theorem shows that if the voltage drop across the resistor under normal conditions is half of the supply voltage, then any change of the tube conditions will decrease
the power dissipated in the tube. The power in the resistor, however, may be considerably increased. In the event that the tube is accidentally shorted the power into the resistor is quadrupled from its normal value. The same advantage can be secured in a pentode by putting the resistor in the screen circuit only. In general, the latter arrangement reduces the power dissipation required of the resistor. When used with triodes this method of tube protection requires a considerably greater power loss than cathode biasing. However, in circuits using pentodes with screen dropping resistors, it is advantageous to make the circuit protect the tube by the proper selection of the screen dropping resistor alone.

The circuits given previously are drawn with batteries providing the D.C. power. These batteries are considered ideal—that is, they are assumed to possess no internal impedance. In practical circuits it is necessary to provide adequate bypass condensers in shunt with the various sources of D.C. potential.

The method of obtaining voltage or power output will vary with the particular oscillator used. Generally speaking, the conventional Pierce and Miller Oscillators will employ capacitive coupling, whereas the electron coupled oscillators are readily adaptable to inductive coupling. The use of capacitive coupling entails consideration of the coupling circuit and its load in selecting values for the oscillator parameters. For example, if a conventional Pierce Oscillator is coupled to a buffer stage by a capacitance, \( C_c \), as in Fig. 7.19, the additional capacitance of \( C_c \) and \( C_1 \) in series must be subtracted from \( C_2 \). The grid resistance, \( R_3 \), of the buffer should be large.

The use of inductive coupling with an electron coupled oscillator is not so critical. The plate tank circuit and its load should ordinarily
be designed to produce the maximum power output. However, the nature of electron coupling prevents the plate circuit load from affecting the rest of the oscillator. Inductive coupling is illustrated in Fig. 7.20.

Because of its simplicity and wide band performance without adjustment, the keyed Pierce Oscillator is favored over the keyed Miller Oscillator. For fixed frequency operation the faster buildup rate of the Miller would be advantageous. It is recommended that the Pierce Oscillator be used exclusively for band operation and that the Miller Oscillator be considered only if fixed frequency operation is contemplated.
CHAPTER VIII
GENERAL TOPICS

This chapter is devoted to the examination of various subjects, some of which were originally specified for study, the others being pertinent to the overall investigation.

8-1. Shock Excitation of Crystals.

The application of a sudden voltage change across the terminals of a crystal will produce a transient mechanical vibration of the crystal plate. It is conceivable that this effect can be used to start an oscillator rapidly, if not instantaneously. The familiar circuit of Fig. 8.1 shows the electrical parameters which are equivalent to the actual mechanical parameters. This representation is possible because of the piezoelectric phenomena exhibited by quartz.

It has been shown by Van Dyke and others that this circuit is the electrical equivalent of a crystal and that the electrical parameters $C_o$, $L_o$, and $R_o$ may be identified with the crystal compliance, mass, and friction, respectively. Therefore, a displacement of charge across $C_o$ actually represents a mechanical displacement in the crystal. The only electrical measurement which can conveniently be made is the voltage across the crystal terminals; but, if the crystal is assumed to be vibrating with constant amplitude at its resonant frequency, the maximum charge displacement across $C_o$ can be calculated in terms of the terminal voltage. To
secure this relationship let \( n_0 = \frac{C_h}{C_o} \) and let the terminal voltage be

\[ E_t = \hat{E} \sin \omega_n t, \]

where \( \omega_n \) is the crystal resonant frequency (actually the approximate natural antiresonant frequency, defined in Sec. 2-20). The circulating current, which is large compared to the line current and is therefore essentially the same in all the elements, may be written

\[ I = E_t \omega_n C_h. \tag{8.1} \]

The voltage developed across \( C_o \) is

\[ E_o = -\frac{I}{\omega_n C_o}, \tag{8.2} \]

or, substituting for \( I \) from (8.1),

\[ E_o = -\frac{C_h}{C_o} E_t = -n_0 E_t, \]

\[ = -n_0 \hat{E} \sin \omega_n t. \tag{8.3} \]

For a crystal driven at its resonant frequency, the peak mechanical displacement is represented by \( n_0 \hat{E} \), where \( \hat{E} \) is the peak terminal voltage. To produce the same mechanical displacement with a D.C. voltage, then, a potential of \( n_0 \hat{E} \) must be applied across the terminals. With practical oscillators and crystals \( \hat{E} \) is of the order of 100 volts and \( n_0 \) of the order of 1000. Thus, to produce a static displacement equal to the resonant displacement requires something like 100 kilovolts across the crystal terminals. This is clearly an unreasonable figure for practical apparatus.

It is impossible to apply a discontinuous voltage step to the crystal terminals because of the holder capacity; except for the academic case of an infinite power source, the result will always be a voltage of the form \( E(1 - e^{-\sigma t}) \). Certainly the crystal can be shocked into vibration by the application of such a voltage to its terminals; however, in order to obtain a good transfer of energy
to the crystal, an extremely critical adjustment of $\omega$ and large values of $E$ would be required. A consideration of realizable numerical values shows that relatively little energy (compared to the resonant energy storage) can be transferred to the crystal with practical keying voltages and present circuit parameters.

8-2. The Point in the Circuit to Key.

Two effects are produced when the D.C. keying voltage is applied to the vacuum tube. The vacuum tube becomes operative and provides an effective negative resistance to produce oscillations. In addition, a transient is set up throughout the entire oscillator circuit. The effect of this transient on the buildup of oscillators is best determined by considering the crystal and the remainder of the oscillator separately.

The duration of transients which affect the tube operation are readily calculated from the time constants of the circuit. If the keyed Pierce Oscillator shown in Fig. 8.2 is used as an example, it is found that the time constants are negligible compared to the period of one dot-cycle.

Assume: $C_1 = 150 \mu\text{pf}$ $R_1 = 100,000 \text{ ohms}$
$C_2 = 30 \mu\text{pf}$ $R_2 = 3,300 \text{ ohms}$
$C_h = 20 \mu\text{pf}$

Fig. 8.2 - Keyed Pierce Oscillator.

The time constant of the grid circuit is

$$T \leq R_1(C_1 + C_h). \quad (3.4)$$
For the assumed values,

\[ T = 10^5(150 + 20)10^{-12} = 17\mu\text{sec}. \]

The time constant of the plate circuit is

\[ T = R_2 (C_2 + C_h), \]  \hspace{1cm} (8.5)

which for the given case becomes

\[ T = 3300(30 + 20)10^{-12} = .2\mu\text{sec}. \]

But at a keying rate of 50 dot-cycles per second the "on" period has a duration of 10,000 \( \mu\) secs. Therefore, the total "on" time is very much greater than the transient duration, and it can be stated that these transients have practically no effect on the operation of tube and its associated circuits.

As was noted in Sec. 8-1, application of the keying wave to the oscillator will shock the crystal into damped vibrations. Because the resulting vibrations will be symmetrical about the rest position it is evident that the direction of the initial displacement is unimportant and, therefore, the polarity of the keying transient is of little consequence. However, the magnitude of the shock displacement may be important under certain conditions. If the original shock produces crystal output voltages somewhat greater than the noise level, the oscillation will start from this higher level rather than from the noise voltage level. The tube can be regarded as linear under the conditions which exist at the beginning of buildup; initial buildup will thus be of the form \( E(e^{\alpha t} - 1)\sin \omega t \).

If the initial displacement produces a voltage greater than the noise voltage it is clear that the time required for buildup to a given amplitude will decrease as the initial crystal displacement increases. Furthermore, in this case the starting phase relationships will be the same for each successive keying cycle. But, if the buildup starts from the noise voltage the rise time will be independent of the keying pulse magnitude.
It has been pointed out that the polarity of the keying pulse is not important but that the magnitude may be. Accordingly, an experiment was performed to test the effect of applying different pulse voltage amplitudes to a crystal. Various keying methods were used to produce the several pulse voltage amplitudes. The experiment is described in Appendix N. The results indicate that the crystal shock does not result in crystal voltages in excess of noise voltages and, as a consequence, the buildup time is substantially independent of the keying voltage amplitude, provided the keying voltage is of the order of 100 volts.

The preceding discussion leads to the recommendation that the circuit be keyed at any convenient point which provides protection for the tube.

8-3. The Effect of Keying Upon Crystal Life.

A practical keyed oscillator must operate the crystal in a manner which will not appreciably shorten the life of the crystal. In any keyed oscillator there are two possible conditions which might damage crystals. The oscillations might momentarily build up to an excessive amplitude before settling down to steady-state conditions. Or, alternatively, the keying transient might stress the crystal enough to break it.

In the oscillator circuits studied, the vacuum tube limits the maximum amplitude of oscillation in such a manner that the amplitude never exceeds that of the steady-state. Although it is known that some special crystal oscillators (e.g., the Meacham) build up to relatively large amplitudes before settling down to the steady-state, this effect has not been found in any of the oscillators studied. Difficulty from this direction is not anticipated in any oscillators applicable to the present problem.

If it is assumed that the keying transient produces stresses which are uniform throughout the crystal, it is apparent from the arguments
presented in Sec. 8-1 and a knowledge of the amplitude of the keying transients that the shock displacement experienced by the crystal is very much smaller than the resonant displacement, and, therefore, can not destroy the crystal. The assumption of uniform crystal stress is reasonable for plates which are "shear excited" over their entire surface. This type of plate is used almost exclusively in the frequency range of interest (i.e., 1 mc. to 10 mc.).

The experimental determination of the effect of keying upon crystal life is presented in Appendix 0. The data, taken over a period of five months, indicates that keying has no detrimental effect on crystal life.

8-4. Frequency Stability During Keying.

Considerable effort has been expended to develop circuits which, in theory, produce a constant frequency during the complete keying cycle. Because the calculations of resistive damping predict a relatively large frequency shift during damping, experiments were performed in an attempt to measure this frequency shift.

Several variations of the beat frequency technique which is normally extremely sensitive were used in the attempt to observe a frequency variation. Nevertheless, the observations gave no indication of a frequency deviation during the damping or decay period. Similar experiments performed to determine the amount of frequency shift during oscillation buildup also yielded negative results. No completely satisfactory explanation of the discrepancy between the theoretical and experimental results has yet been offered. The most plausible explanation is that the expected frequency shift is masked by buildup and decay transients.

The bandwidth required by these transients in a keyed oscillator is easily estimated from the output waveform and the keying rate. The output waveform may be represented by a Fourier Series summation of the fundamental and harmonics of a sinusoidal wave having a frequency equal to the keying rate. For the oscillators described in Chap. VII, whose typical keying waveforms are presented in Figs.
7.8, 7.11, and 7.15, a sufficiently close approximation to the keying waveform is obtained by carrying the Fourier Series to the tenth harmonic. For a keying rate of 45 dot-cycles per second, the tenth harmonic produces modulation sidebands displaced from the carrier by 450 cycles, and the output of the oscillator occupies a total of about 1 kc of the frequency spectrum. It is seen that this is comparable to the frequency shift predicted for resistive damping of crystals having typical values of frequency and capacitance ratio.

8-5. Long Time Stability.

With unimportant exceptions, the factors which govern the frequency of CW crystal oscillators control the mean frequency of keyed crystal oscillators. These factors include temperature, applied voltages, crystal aging, and variation of circuit reactances. It is therefore clear that the long time stability of a keyed oscillator is comparable to a CW oscillator with the same circuit components.
CHAPTER IX
SERIES RESONANT CRYSTAL CONTROLLED OSCILLATORS

The operation of a crystal at its series resonant frequency appears to have several advantages in providing frequency control for the oscillator. Of those, the most important is a marked reduction of the effect upon frequency of stray admittances. This is true because the feedback path through $L_0$, $R_0$, and $C_0$ has a resonant impedance which is much smaller than any of the shunting impedances. In the ideal case, the frequency would therefore be independent of the so-called load capacitance, which is so important to the operation of oscillators of the Pierce and Miller type.

Three types of oscillators which approach the objective of series resonant operation are described in this chapter. One of these employs transformer coupling to operate the components at favorable impedance levels. The other two employ a two-terminal negative resistance obtained with a pentode connected as a transitron.

9-1. Transformer Coupled Oscillator.

During the study of keyed oscillators it became apparent that neither the crystal nor the vacuum tube is operated at suitable impedance levels in the Pierce and Miller oscillators. For an average crystal the series resonant impedance is of the order of 100 ohms whereas the parallel resonant impedance is of the order of 5 megohms. On the other hand, a suitable plate load impedance for most vacuum tubes is in the region of 10,000 ohms. It is evident that the proper use of transformers will permit operation of the crystal at its series resonant frequency and the vacuum tube at an appropriate impedance level.
Fig. 9.1 is an example of such a transformer coupled oscillator. If the resistances, $R$, are low (of the order of the series resonant crystal impedance) the feedback path through the crystal will permit oscillations to be sustained only at frequencies very near the crystal series resonance. Furthermore, the circuit is incapable of oscillating in the absence of the quartz plate. Limitations on this type of circuit are imposed by the available tubes, parasitic capacities, and the transformers.

The circuit of Fig. 9.1, composed of transformers, capacitances, and resistances, may be represented by a pair of general 4-terminal networks and an ideal tube, as shown in Fig. 9.2. The analysis is simplified by initially assuming that the two networks are identical. True series resonant operation requires that the crystal be a pure resistance: that is, there must be no phase shift through the crystal. To approximate this ideal
condition the networks must produce negligible phase shift over the frequency band for which the oscillator is designed. The requirements on the gain characteristic are not so severe. Sufficient gain must be available over the band to cause oscillations to build up, but excessive gain at certain portions of the band is not detrimental. However, the gain should decrease to values which are insufficient to sustain oscillations at frequencies outside the operating band. 

It has been shown by Bode that any low pass network can be transformed to a band pass network, and that knowledge of the characteristics of the low pass network permits calculation of the band pass network characteristics. The chief advantage of this method is that the behavior of low pass networks is more familiar and is more easily studied than that of corresponding band pass structures. In terms of a low pass network, the requirement is that there shall be negligible phase shift and relatively constant gain from zero frequency up to some limiting frequency. Above this frequency there is no specific requirement on either characteristic; however, in practice the gain will ultimately fall off and the phase will approach some small integral multiple of $90^\circ$. One convenient idealized phase characteristic is shown in Fig. 9.3.

![Fig. 9.3 - Idealized Phase Characteristic for Low Pass Network.](image)

Bode's investigations show that the gain and phase characteristics are interdependent for a minimum phase shift network; furthermore, all passive ladder

networks are of the minimum phase type. Therefore, for any specified phase characteristic of a ladder network there is only one possible gain characteristic. Fig. 9.4 shows the gain characteristic which corresponds to the phase characteristic of Fig. 9.3. These gain and phase relationships are for the transmission characteristics of a specific 4-terminal low pass network. They represent only one of many possible engineering solutions to the problem. It should be noted that the gain and phase curves are proportional to each other, so that one will be doubled if the other is doubled.

![Gain Characteristic](image)

**Fig. 9.4 - Associated Gain Characteristic for Low Pass Network.**

Development of a suitable low pass network does not constitute a solution to the problem, because there is the additional requirement that the network be capable of transformation to a band pass impedance-transforming network by the use of physically realizable elements. In particular, the required transformers must have inductance values and coefficients of coupling which can be realized over the required frequency band.

In the equivalent circuit of Fig. 9.2 the right-hand 4-terminal network is driven by a pentode, which is assumed to have an infinite dynamic plate resistance and is, therefore, a constant-current generator. The amplifying property of the tube is unavoidably accompanied by capacitance, which is

---

(31) The definition of "minimum phase shift" networks and some material especially pertinent to this discussion are given on pp. 121 and 242 ff. of Bode.
indicated in Fig. 9.5 as being in shunt with terminals 1 and 2 of the network.
The tube may then be regarded as an ideal constant-current generator. It can
also be shown that corresponding to a practical transformer with a realizable
coefficient of coupling in the band pass network, the low pass network must have
an unshunted series inductance between terminals 1 and 3. Thus, the transfer
phase and gain characteristics and two elements of the network are known. Fig.
9.5 shows the partially determined 4-terminal network.

![Fig. 9.5 - Minimum Partial Low Pass Network, from Practical Considerations.](image)

Deduction of the remaining network elements involves design practices which
are not unique. It is therefore possible that the network derived in the follow-
ing paragraphs is not optimum in either performance or simplicity.

The design offered is based upon the gain characteristic of Fig. 9.4, from
which it is seen that the gain increases with frequency up to the point \( f_0 \). One
network completion of Fig. 9.5 which approximates that characteristic is shown
in Fig. 9.6. The phase characteristics of the transfer impedance of this network

![Fig. 9.6 - A Possible Solution for Desired 4-Terminal Low Pass Network.](image)

are presented in a normalized form in Fig. 9.7. In this normalized presenta-
tion the parameters \( L_1, L_2, \) and \( R_2 \) are related to \( R_1 \) and \( C_3 \) by the following
factors:
Fig. 9.7 - Normalized Transfer Phase Characteristics for Network of Fig. 9.6.

\[ F = \frac{f}{f_0} \]

\[ L_2 = \frac{p^2 C_2 R_1}{f_0} \]

\[ f_0 = \frac{1}{2\pi C_3 R_1} \]

\[ R_2 = r R_1 \]

\[ L_1 = h^2 C_3 R_1 \]

\[ R_1 = 1 \]
The value of $R_1$ is approximately fixed by the tube and the gain requirements, and the minimum $C_2$ is fixed by the tube. The effects of varying $L_1$, $L_2$, and $R_2$ are determined by varying the factors $h$, $\xi$, and $r$. From the curves of Fig. 9.7 it is seen that changing $h$ affects the bandwidth achieved for a given maximum phase shift. In addition it will be found that $h$ also controls the coefficient of coupling required in a practical transformer. Extensive calculations have indicated that with a fixed $h$ the maximum bandwidth which can be obtained for a given maximum allowable phase shift is determined quite accurately by the equation

$$P' = r x \text{ (a constant).} \quad (9.5)$$

Since it is desired to operate the crystal very close to series resonance, a maximum phase shift of \( \pm 5^\circ \) is arbitrarily specified. For a maximum phase shift of \( \pm 5^\circ \) the constant in (9.5) is approximately $\sqrt{2}$, so that

$$P = r \sqrt{2}. \quad (9.6)$$

Because the phase characteristic of the network is known, certain deductions and restrictions may be made concerning the gain characteristic. In the process of transforming a low pass network into a band pass network, the gain of the low pass network at zero frequency becomes the gain of the band pass network at its mean frequency. It can be shown that when the phase limitations are imposed the gain of the network of Fig. 9.6 is a minimum at zero frequency. Let us represent an equivalent crystal impedance (which will be essentially resistive when operated properly) by a resistance, $R_x$. The low pass equivalent of the oscillator is illustrated in Fig. 9.8.

\[
\begin{align*}
L_1 &= h^2 C_3 R_1, \quad (9.1) \\
L_2 &= p^2 C_3 R_1, \quad (9.2) \\
R_2 &= r R_1, \quad (9.3)
\end{align*}
\]

and

$$r_0 = \frac{1}{2\pi C_3 R_1}. \quad (9.4)$$
Calculations show that for a prescribed phase characteristic and overall gain at zero frequency, the bandwidth of this low pass equivalent is relatively independent of $\frac{R_2}{R_1}$. Good oscillator performance is anticipated if $R_x = R_1$. By making the restriction that $R_1 = R_2 = R_x$ the equivalent circuit of Fig. 9.9 is obtained for frequencies near zero.

The overall gain at low frequency is given by

$$\text{Gain} = \frac{e_o}{e_i} = \frac{R_1}{r_m}. \quad (9.7)$$

To provide a margin for operation, let us specify that the gain shall be 2. The resistance is then fixed by the relationship

$$R_1 = \frac{16}{r_m}. \quad (9.8)$$

The transformation of the network in Fig. 9.6 to a band pass network is accomplished by adding an inductance in shunt with each capacitance and a capacitance in series with each inductance in the original network. The added parameters are selected to resonate or antiresonate with their complementary
parameters at a frequency which is the geometric mean of the band to be covered. This transformation is shown in Fig. 9.10. The values for the various dependent parameters are given by the equations

\[ L_3 = \frac{1}{4\pi^2 f_m^2 C_3} \]  
(9.9)

\[ C_1 = \frac{1}{4\pi^2 f_m^2 L_1} \]  
(9.10)

and

\[ C_2 = \frac{1}{4\pi^2 f_m^2 L_2} \]  
(9.11)

where \( f_m \) is the geometric mean frequency of the band.

\[ \text{Fig. 9.10 - Band Pass Network Corresponding to Low Pass Network of Fig. 9.6.} \]

Certain arbitrary restrictions have been imposed on the various parameters. Since \( R_1 = R_2 \), then \( r = 1 \) by (9.3). Also, using (9.1) and (9.8), (9.10) becomes

\[ C_1 = \frac{\varepsilon_m}{64 \pi^2 f_m^2 g^2 C_3} \]  
(9.12)

Finally, when \( r = 1 \), (9.6) yields \( p = \sqrt{2} \). Taking this value in conjunction with (9.2) and (9.8), (9.11) becomes

\[ C_2 = \frac{\varepsilon_m}{128 \pi^2 f_m^2 C_3} \]  
(9.13)

The band pass network is converted into an impedance transforming network by inserting an ideal transformer at the dotted line in Fig. 9.10. This transformer has an impedance ratio of \( \phi^2 \) and is step-down from left to right.
By defining $\phi^2 < 1$ the network of Fig. 9.11 is obtained. By well known processes (32) the elements between a-b in Fig. 9.11 can be obtained by employing a practical transformer as shown in Fig. 9.12. The values for $L_p$, $L_s$, and $k$, the coefficient of coupling, are given by

\[ L_p = L_3, \quad (9.14) \]
\[ L_s = \phi^2(L_1 + L_3), \quad (9.15) \]
\[ k = \sqrt{\frac{L_3}{L_1 + L_3}}, \quad (9.16) \]

Because the restriction $R_1 = R_x$ was imposed, the value for $\phi^2$ can be determined. The impedance of the crystal at series resonance closely approximates the $R_0$ of the crystal. Therefore it is desired that $\phi^2$ have a value such that

\[ \phi^2 R_1 = R_0, \quad (9.17) \]

or from (9.8)

\[ \phi^2 = \frac{R_0 g_m}{16}. \]

We may now specify the values for the various oscillator components, shown in Fig. 9.13, in terms of $C_3$, $g_m$, $f_m$, and $h$.

![Actual Oscillator Diagram](image)

**Fig. 9.13 - Actual Oscillator.**

\[
L_p = \frac{1}{4\pi^2 f_m^2 C_3}, \quad (9.16)
\]

\[
L_s = \frac{R_0 g_m}{16} \left[ \frac{1}{4\pi^2 f_m^2 C_3} + \frac{h^2 g_m}{g_m} \right], \quad (9.19)
\]

\[
C_s = \frac{1}{4\pi^2 f_m^2 g_m^2 C_3 R_0}, \quad (9.20)
\]

\[
R_s = R_0, \quad (9.21)
\]

\[
R_a = R_0, \quad (9.22)
\]

\[
C_a = \frac{1}{8\pi^2 f_m^2 g_m^2 C_3 R_0}, \quad (9.23)
\]

\[
L_a = 2 C_3 R_0, \quad (9.24)
\]

and

\[
k^2 = \frac{1}{1 + \frac{64\pi^2 f_m^2 g_m^2 h^2}{g_m}}. \quad (9.25)
\]

It has been found that $k$ is the factor which limits wide band operation in the frequency range from 1 to 8 mc. Actually, for reasonable
values of \( h, C_3, \) and \( g_m, \) \( k \) must be greater than 0.8 if wide band operation is to be obtained in this frequency range. Specific calculations for an assumed \( k = 0.5 \) gives a band about 25 kc wide for which the phase shift does not exceed \( +5^\circ \) in each transformer. This calculation is illustrated below.

Assume: \( k = .5, \ h = 4, \ g_m = 10000 \) micromhos, and \( f_m = 2 \) mc. Then from (9.25)

\[
C_3^2 = \frac{(1 - k^2)g_m}{64\pi^2 f_m^2 h^2 k^2} \quad (9.26)
\]

The numerical solution is

\[
C_3^2 = \frac{(1 - .25) .01}{64\pi^2 (2 \times 10^6)^2 (4)^2 (.5)^2} = .74 \times 10^{-13},
\]

or

\[
C_3 = 860 \text{ pp } f.
\]

This value gives, by (9.4) and (9.8), a "critical" frequency of

\[
f_o = \frac{f_m}{3\pi C_3}
\]

\[
= \frac{.01}{32\pi \times 860 \times 10^{-12}} = 115 \text{ kc}.
\]

Referring to Fig. 9.7 it is seen that for \( h = 4 \) the frequency at which the phase deviates \(-5^\circ\) is

\[
f' = .2 f_o = 23 \text{ kc}.
\]

But the bandwidth of the low pass and band pass structures are the same, and so this is the band over which the oscillator will operate when referred to \( f_m \). That is,

\[
\text{Operating Band} = f_m + \frac{f'}{2},
\]

\[
= (2 + .0115) \text{ mc}.
\]
When the coefficient of coupling, $k$, exceeds about 0.9, the tube capacities limit the obtainable bandwidth. Whether such close coupling can be obtained with existing transformer construction techniques is not known. If it can, comparatively wide band operation may be secured in the frequency range from 1 to 8 mc.

The circuit of Fig. 9.13 does not indicate the manner in which D.C. tube biases are obtained. The operating grid bias is developed through an added $RC$ combination in the grid circuit. The values selected for this $RC$ circuit should be such that its impedance effects on the remainder of the circuit are negligible. Other tube biases are obtained in a conventional manner.

The circuit was tested with relatively crude hand-wound transformers. Very stable continuous oscillations were observed at a frequency substantially equal to the series-resonant frequency of the crystal. Because of the poor coefficient of coupling, however, it was necessary to readjust the transformer in order to operate with crystals having frequencies which differed by more than a few kilocycles from the original. No attempt to key this unit was made, because it was felt that the transformers were not good enough to warrant the test. In addition, it appears that this circuit is unlikely to be capable of high speed keying on account of the rather small gain margins which can be secured without going to excessively high impedances in the grid and plate circuits. However, due to the resistors attached to the crystal, rapid decay rates ought to be realized.

9-2. The Impedance Inverting Transitron Oscillator.

Present day circuits which operate crystals at series resonance solve the impedance level problem by employing transformers or more than one tube in the oscillator. Neither solution is entirely satisfactory.

In an attempt to evade the drawbacks associated with both of the above methods, the circuit shown in Fig. 9.14 was devised. By means of a quarter-wave
impedance inverting section it solves simultaneously the problems of impedance level and series resonant operation. Although the desired results could be achieved through the use of a quarter-wavelength transmission line, it is more satisfactory for the frequencies of present crystals to employ a system of lumped constants \((L, C_1\) and \(C_2\) in the figure). The oscillator circuit consists of three main parts: crystal, artificial line, and tube. It will be noticed that the line provides a shunting capacitance adjacent to both the crystal and the tube. Hence it is possible to absorb the holder capacitance of the crystal and the parasitic capacitance of the tube into the line elements.

![Diagram of impedance inverting transitron oscillator](image)

**Fig. 9.14 - Impedance Inverting Transitron Oscillator.**

The circuit of Fig. 9.14 retains a desirable feature of the Pierce and most other practical oscillators in that it will not oscillate in the absence of the crystal. This results from making the line characteristic impedance, \(Z_0\), and the load resistor, \(R_1\), somewhat lower than the minimum value at which the tube will operate as a transitron. For presently
available tubes this minimum impedance varies from about 600 to 6000 ohms.

It is important that the impedance of the terminated line, as seen by the tube terminals, is relatively independent of the capacitance C₂. If this were not so the removal of the crystal together with its holder capacitance would upset the relationships and introduce the possibility of unwanted oscillations. The line portion of C₂ should, of course, be much larger than the included crystal holder capacitance.

When a crystal is inserted in the circuit, oscillations occur at or very near the series resonant frequency of the crystal. At this frequency the crystal is essentially a pure resistance, R₀, and therefore the line is terminated by a pure resistance, the parallel combination of R₀ and R₁. Since R₀ is small compared to R₁, the combination has a value approximately equal to R₀. Because of the impedance inverting property of a quarter-wave line the impedance seen by the tube is a high pure resistance approximately equal to Z²/R₀. This can readily be made considerably larger than the limiting value given above, so that oscillations will start.

The fact that the resulting oscillations are stable arises from another property of quarter-wave lines. Not only is the low impedance of the crystal inverted into a high impedance by the line, but also the series resonance is inverted into an effective parallel resonance. That this is true can be seen in a general way by noting that at frequencies only slightly removed from the series resonant frequency of the crystal the line is still essentially a quarter-wave long, but the crystal acts virtually as an open circuit. Accordingly, the impedance seen by the tube falls to Z at frequencies only slightly removed from the series resonant frequency of the crystal. The property of passing through a maximum of impedance is, of course, characteristic of parallel resonance.

The behavior may be studied by reference to Fig. 9.15. The first plot shows the variation with respect to frequency of the admittance of the series elements
of the crystal and also of the admittance of these elements in conjunction with the load resistor, \( R_1 \). The second plot shows the impedance which results when this combined admittance is inverted about the impedance, \( Z \), of the quarter-wave line. It is seen that the total impedance reaches a maximum when it is a pure resistance equal to \( R_1 \left(1 + \frac{R_1}{R_0}\right) \). The behavior in this region is identical with that of any high-Q parallel resonant system. At frequencies well removed from \( w_o \) the line ceases to be effectively a quarter-wave long and the diagram is accordingly distorted. The figures are drawn for the reasonable ratio \( R_1 = 10 \, R_0 \).

The circuit of Fig. 9.14 was tested using a 6SJ7 tube and a 3230 kΩ crystal. The load resistor, \( R_1 \), was 1000 ohms, approximately equal to the \( Z \) of the line, which was composed of \( L = 50 \mu \)h and \( C_1 = C_2 = 50 \mu \mu F \) including holder and tube capacities. The plate potential was reduced to approximately 20 volts to promote the control action of the suppressor grid. The potential of the screen grid was varied from approximately 70 to 100 volts. The crystal had a series resistance, \( R_0 \), of approximately 30 ohms.

Fig. 9.15 - Plots of Admittance and Impedance Versus Frequency for The Circuit of Fig. 9.14.
Operation as a CW oscillator was stable and most satisfactory. Frequency departures not exceeding ±3 parts per million were observed for ±5 volts variation of the plate potential and ±15 volts variation of the screen grid potential. An increase of 50 ppf in \( C_2 \) resulted in a decrease of 60 parts per million in frequency. A similar variation was noted with respect to \( C_1 \). No significant change in amplitude or general behavior was observed when the original crystal was replaced by five other crystals having the same nominal frequency. Moreover, the circuit proved to be operative on a plug-in basis over at least a small frequency band without readjusting the line parameters. This was demonstrated by replacing the 3230 kc crystal by a 2905 kc crystal and observing that the oscillations were stable. It is probable that at least a ±10% frequency band could be covered by refining the design only very slightly.

The circuit was tested for keying behavior by controlling the bias applied through the grid leak resistor, \( R_3 \), to the suppressor grid. Oscillations were recognizable approximately 4 milliseconds after the voltage was applied, and they reached full amplitude only 2 milliseconds later. The decay times were comparable. Satisfactory keying with square waves up to 30 cycles per second was obtained.

From these experimental observations and theoretical deductions it appears that this transitron circuit may be useful for both CW and keyed applications over relatively narrow frequency bands.


Like the two preceding oscillators the circuit of Fig. 9.16 operates the crystal very near series resonance. In addition, the configuration is extremely simple. From the basic circuit shown in Fig. 9.16 it is apparent that the crystal serves as a feedback path between the screen grid and the suppressor grid of a
transitron-connected pentode.

A preliminary experiment was performed to explore the possibilities of this type of circuit. The tube used was a 6AS6. The experiment indicated that the crystal-coupled transitron oscillator is critical with respect to element values and is likely to operate as a relaxation oscillator, the crystal exercising no frequency control.

Proper operation was obtained with CT and DT cut crystals, in the frequency range from 200 to 500 kc. These crystals are characterized by unusually low values of holder capacitance (1 to 5\(\mu\)f); moreover, the holder capacities decrease with crystals of higher frequency rating. When it was attempted to use AT and BT cut crystals, for frequencies above 1 mc, the circuit either would not operate or else produced relaxation oscillations. These higher frequency crystals possess larger holder capacitances (10 to 20\(\mu\)f), and the holder capacities remain nearly constant as the rated crystal frequency increases.

To determine the nature of the difficulties obviously present and to evaluate the factors which cause or prevent proper operation, the following analysis was developed. Particular attention is devoted to a criterion as to whether or not relaxation oscillations will occur. Conditions for the production of crystal-controlled sinusoidal oscillations are also studied.

Since a transitron-connected pentode exhibits a negative \(\mu\) (that is,
no phase reversal exists between the suppressor and screen grids) but is otherwise similar to a conventionally connected triode, the effects of the crystal and of the elements $R_1$, $R_2$, $C_1$, and $C_2$ can be evaluated by considering these parameters to form an interstage network. This representation is shown in Fig. 9.17. The tube $r_p$ is assumed large compared to $R_1$ and has consequently been omitted.

![Fig. 9.17 - Interstage Network Equivalent for the Oscillator of Fig. 9.16.](image)

The voltage gain through the tube and network is

$$A = \frac{e_0}{e_1} = g_m \frac{Z_1 Z_2}{Z_1 + Z_2 + Z_3},$$

or

$$AZ_3 = g_m Z_1 Z_2 - A(Z_1 + Z_2).$$

The impedances $Z_1$ and $Z_2$ represent the element pairs $R_1 C_1$ and $R_2 C_2$, respectively, and are given by the expressions

$$Z_1 = \frac{R_1}{1 + j \omega R_1 C_1},$$

and

$$Z_2 = \frac{R_2}{1 + j \omega C_2 R_2}.$$

Two separate conditions on the crystal impedance, $Z_3$, exist. One results from the practical requirement that the circuit must not be capable of oscillation if the crystal plate fractures. With a broken plate the crystal unit reduces to a capacitance, $C_x$, which is the sum of the holder capacitance and parasitic capacities in shunt with the crystal terminals. Thus, a value for $Z_3$ is defined for which the circuit must not produce oscillations:
The other condition on $Z_3$ is that occurring with a good crystal. Vector diagrams show that the network can have zero phase shift only if the reactance of $Z_3$ is capacitive; therefore, the crystal must operate on the capacitive side of series resonance. That is to say, the operating frequency will always be slightly lower than that of series resonance. In Appendix P it is shown that near series resonance the resistive component of the crystal impedance is essentially $R_0$. Accordingly, the crystal may be represented by $R_0$ in series with an effective capacitance, $C_3$, (which is a function of frequency): the circuit must be capable of oscillating when

$$Z_3 = R_0 - \frac{1}{wC_3}, \quad (9.31)$$

where

$$\frac{1}{wC_3} = \frac{1}{wC_0} - wL_0.$$  

From the crystal equivalent circuit it is deduced that $C_3$ is very large when $w$ is infinitesimally below the crystal series resonant frequency and that as $w$ decreases $C_3$ decreases. In the limit, as $w$ approaches zero $C_3$ approaches $C_x$. Thus, any solution using (9.31) must submit to the restriction $C_3 > C_x$.

To simplify the equations let

$$R_2 = aR_1,$$

$$C_2 = bC_1,$$

$$C_3 = C_1/a,$$

and $$C_x = C_1/a'.$$

From substituting (9.28), (9.29), (9.30) and (9.32) into (9.27b), an expression is obtained which fixes the limiting condition for no oscillation with a fractured crystal:
\[
A \left[ \frac{-jd'}{wC_1} \right] = -A \left[ \frac{\text{Real} \left( R_1 (1 + a) + jwC_1R_1^2(1 + b) \right)}{1 - w^2C_1^2R_1^2ab + jwC_1R_1(1 + ab)} \right]
\]

(9.33)

For oscillations to persist the overall gain must be equal to or greater than unity. By setting \( A = 1 \) and equating the real components of (9.33), we find that

\[
R_1 = \frac{1}{\varepsilon_m} \left[ \frac{1 + \frac{1}{a} + d'(1 + \frac{1}{a} + b)}{a} \right]
\]

(9.34)

determines the threshold of sinusoidal oscillations when the crystal is broken. To exclude the possibility of relaxation oscillations, \( R_1 \) should be less than the value given by (9.34) for all values of \( \varepsilon_m \) which will be met. Finally, there exists a value of \( R_1 \) such that no oscillations whatever can occur for smaller values of \( R_1 \).

This limiting \( R_1 \) may be determined as follows. Assume \( C_1, C_2, \) and \( Z_3 \) in Fig. 9.16 are all zero. Then, from (9.27a),

\[
A = \varepsilon_m \frac{R_1R_2}{R_1 + R_2}
\]

(9.35)

Setting \( A = 1 \) and using (9.32), we have

\[
R_1 = \frac{1}{\varepsilon_m} \left[ \frac{1 + a}{a} \right]
\]

(9.36)

specifying the minimum \( R_1 \) which will permit oscillations of any type for given values of \( \varepsilon_m \) and \( a \).

The resistance ratio \( a \) has an important effect on the operation of the circuit. To explore this factor it is convenient and legitimate to neglect the effects of \( C_1 \) and \( C_2 \). It is seen that \( R_1 \) and \( R_2 \) in series constitute the resistance which damps the crystal and which limits its frequency control under steady oscillation. Both these considerations point to holding the sum \( S = R_1 + R_2 \) to a minimum.

The mathematics is facilitated by considering \( S \) constant and determining
the value of the resistance ratio \( a \) which leads to maximum overall gain for a given \( g_m \). This same value of \( a \) evidently leads to a minimum value of \( S \) for a given \( A \). Rewriting (9.27a) with the above qualification,

\[
A = g_m \frac{R_1(S - R_1)}{S + Z_3}.
\]

(9.37)

Setting the derivative equal to zero for a maximum gives

\[
\frac{dA}{dR_1} = g_m \frac{(S - R_1) - R_1}{(S + Z_3)} = 0;
\]

(9.37)

whence

\[
S = 2R_1.
\]

(9.38)

This yields

\[
R_1 = R_2,
\]

(9.39)

or

\[
a = 1,
\]

as the condition for maximum amplification, \( A \), for a given sum of \( R_1 + R_2 \). Detailed calculation shows that this problem is closely related to that of maximum power transfer and that \( A \) is decreased by only 10% when \( a = \frac{1}{2} \) or 2. It is concluded, therefore, that values of \( a \) in the neighborhood of unity are optimum.

It has been established that for stable oscillation \( C_3 \) must be greater than \( C_x \), or \( d < d' \) by (9.32). With this fact in mind, a critical study shows that (for \( R_0 > 0 \)) the phase requirements can be satisfied only if the nominal crystal frequency is lower than the frequency, \( \omega_a \), which would prevail if the circuit oscillated with \( C_x \). Setting \( A = 1 \) in (9.33) and equating the imaginary components, with the additional condition that \( R_1 \) have the value given by (9.34), we obtain an expression for \( \omega_a \):

\[
\omega_a^2 = \frac{d'}{C_1^2 R_1^2 a (1 + b + bd')}.
\]

(9.40)

An upper limit is placed on the crystal frequency by (9.40), but the crystal frequency should actually be a good deal lower than this value.
Consequently, it is desirable that $w_a$ be as large as possible.

The conditions which lead to a maximum value of $w_a$ in (9.40) are best explored by returning to the element values themselves. Making the indicated substitutions through (9.32), we have

$$
w_a^2 = \frac{1}{R_1R_2\left(C_1C_x + C_2C_x + C_1C_2\right)} \quad (9.41)
$$

From this equation it is seen that all values of capacitance should be held to a minimum and that, consistent with (9.39), $R_1$ and $R_2$ should be made equal.

In order to preserve some margin for variation of tube performance, however, it is necessary to be sure that the criteria of (9.34) and (9.36) differ by a considerable factor. Using 2 as a suitable safety factor for the transconductance of (9.34) over that of (9.36), we may write

$$
\frac{1}{R_1} \left[1 + \frac{1}{a} + d'(\frac{1}{a} + b)\right] = 2 \frac{(1 + a)}{aR_1} \quad (9.42)
$$

This requires that

$$d'(1 + ab) = 1 + a \quad (9.43)$$

and is identically satisfied if

$$d' = b = 1 \quad (9.44)$$

or

$$C_1 = C_2 = C_x \quad (9.45)$$

On the assumption that $a = 1$, we have from (9.42)

$$1 + b = \frac{2}{d'} \quad (9.46)$$

or

$$C_1 + C_2 = 2C_x \quad (9.47)$$

Taking as an alternative assumption $a = 2$, we have

$$1 + 2b = \frac{3}{d'} \quad (9.48)$$

or

$$C_1 + 2C_2 = 3C_x \quad (9.49)$$
From (9.45) and (9.46) it is evident that, if $C_x$ is larger than half the sum of the parasitic capacitances, the necessary margin between desired and undesired oscillations can be achieved only by padding $C_1$ or $C_2$. By (9.41) this is seen to effect a serious reduction of the band of frequencies over which oscillations may be secured. The present situation in which high frequency crystals have considerably larger holder capacitances than low frequency crystals is particularly unfortunate for this circuit.

The following design employing a 6AS6 tube will serve to illustrate the general features of the crystal coupled oscillator circuit. With this tube, values of $C_1$ and $C_2$ as low as 5 pF may be realized. The direct suppressor-to-screen capacitance contributes about 2 pF to $C_x$. The nominal transconductance is 1600 micromhos. It is therefore reasonable to specify that desired oscillations shall occur for all values of $g_m$ above 1000 but that undesired oscillations shall not occur for values of $g_m$ below 2000.

Let us further make the somewhat optimistic assumption that $C_h$ may be reduced to 3 pF. Thus, in accordance with (9.44),

$$C_x = C_1 = C_2 = 5 \text{ pF},$$

or $d' = b = 1$.

Choosing $a = 1$, we satisfy (9.45) also; and using $g_m = 1000$ in (9.36) gives, with $a = 1$ in (9.32),

$$R_1 = R_2 = 2000 \text{ ohms}.$$

From (9.40), assuming $g_m = 2000$, we have

$$\omega_a^2 = \frac{1}{(25 \times 10^{-24})(4 \times 10^6)(3)} = 33 \times 10^{14},$$

or $\omega_a = 5.7 \times 10^7$.

and

$$\omega_a \neq 9 \text{ mc}.$$
Crystals in the frequency range 0 - 5 mc which have the required low value of holder capacitance of not over 3 \( \mu \)pf should operate in this circuit at a frequency very near their series resonance.

The frequency range over which this circuit will operate may be extended in two ways. If operation at very low frequencies must be retained it is best to place small fixed inductances in series with the resistors \( R_1 \) and \( R_2 \). The resulting elements may be calculated in terms of two-terminal video interstage theory. It appears that values of inductance determined by the limits

\[
1 > \frac{L_1}{C_1R_1} > 1/2 ,
\]

and

\[
1 > \frac{L_2}{C_2R_2^2} > 1/2 ,
\]

should be suitable in most cases, and should extend the useful frequency band by a ratio in the order of two.

If operation at low frequencies is not required the circuit may be transformed by adding fixed coils in shunt with the resistors \( R_1 \) and \( R_2 \). These elements are determined by the equation

\[
\frac{1}{w_m^2} = L_1 C_1 = L_2 C_2 ,
\]

where \( w_m \) is the mean radian frequency of the band to be covered. It is clear that this circuit is capable of operating the crystal either above or below its resonant frequency. Moreover it appears that relaxation oscillations should not be troublesome in such a circuit because of the short time constants involved. Some tentative experiments indicate that this is indeed the case.

Finally, it is possible to effect the bandpass transformation of the extended low-pass structure. The configuration and pertinent equations are given in Fig. 9.18. This circuit has not been tested or calculated in detail.
It should, however, be capable of oscillating over a considerable band of frequencies without any retuning. And the element values should not be critical.

\[ L_1 C_1 = L_2 C_2 = L_3 C_3 = L_4 C_4 = \frac{1}{w_m^2} \]

\[ 1 > \frac{L_3}{C_1 R_1^2} > \frac{1}{2} \]

\[ 1 > \frac{L_4}{C_2 R_2^2} > \frac{1}{2} \]

Fig. 9.18 - Network for Broad-Band Crystal Coupled Transitron Oscillator.
CHAPTER X

TOPICS REMAINING TO BE INVESTIGATED

The keying speed of present oscillators is limited by the time required for oscillations to build up. It is therefore necessary to find some way of accelerating the buildup of oscillation in order to obtain keying speeds much greater than 50 dot-cycles per second. There are at least two methods which deserve consideration. First, methods for cancelling the detrimental effect of the crystal holder capacitance, consistent with the maintenance of good frequency control, should be sought. Second, methods for shocking the crystal into relatively vigorous oscillation should be investigated. It is felt that a circuit which operates the crystal at series resonance and uses some method of impedance inversion is one approach to cancelling the holder capacitance.

Very short rise times have been observed in a particular Pierce oscillator using a 6AK5 tube. Keying speeds with good waveforms up to 200 dot-cycles per second were obtained. However, the operation of the oscillator was unpredictable and various crystals gave quite different results. An investigation revealed that UHF parasitic oscillations were responsible for the short buildup times. Because this phenomenon was observed during the last stage of the program of study it was not examined in great detail. But it is interesting to note that such a condition can exist and that, contrary to what might be expected, it improves performance.

The determination of the proper impedance level at which a crystal should operate is another topic of interest. On several occasions it has been evident that the crystals available do not have suitable impedance levels for a particular application. Sometimes it was desirable to scale all the equivalent para-
meters in one direction. At other times it was desirable to change only one of the parameters.

It appears that a great deal of work remains to be done in the control and specification of quartz plates in terms of the elements of the equivalent electrical circuit. Not only is there need to study the behavior of the circuit associated with the principal mode of resonance, but also the behavior of the several parasitic resonances needs attention. It should be noted that the criterion for parasitic responses depends upon whether the crystal is to be used at series or parallel resonance. If the crystal is to be used at series resonance it is necessary that the undesired series resonances shall have impedances considerably higher than that of the desired one. If, on the other hand, the crystal is to be operated in antiresonance with some given padding capacitance it is necessary that the undesired antiresonances shall have impedances considerable lower than that of the desired one.

The three circuits presented in Chapter IX for operating a crystal at series resonance have not been completely explored. The transformer coupled oscillator depends for its success upon excellent transformers. In most applications the transformers need to work over a considerable band of frequencies. It is believed that this circuit is capable of performance which far exceeds that which has been obtained to date.

The impedance-inverting transitron oscillator also needs further study. In this case the work might well be directed towards producing a lumped line or phase-shifting circuit which maintains a phase shift near $90^\circ$ over a considerable band of frequencies.

The crystal coupled transitron circuit is attractive primarily because of its simplicity. Here the limitations of performance stem from the limited values of transconductance-to-capacitance ratio available in
existing tubes and from the generally low impedance level of available crystals. Crystals having small values of $n$, large values of $Q$, and a series resistance of the order of 1000 ohms are optimum for use with available tubes. Excellent results are anticipated from this circuit when suitable element values are employed.

The use of relays to connect and disconnect a damping circuit was not fully exploited because of the limitations of available relays. If relays having less distributed capacitance and more rapid operation become available it may be practical to employ them for high speed keying. The switching of the damping circuit is of course desirable because it permits the use of optimum elements for damping without any degradation of the rise time.
The first experimental work which was performed was of an exploratory character and utilized a Pierce Oscillator. The principal objectives were to secure a familiarity with the behaviour of a typical quartz oscillator and to determine the general scope and nature of the experimental problem.

The basic Pierce Oscillator circuit is shown in Fig. 1. For experimental purposes, it was desirable to use the simplest possible form of the circuit. Certain of the element values were selected to present a proper load capacitance to the crystal, the others because they gave a good stable oscillator. The circuit in Fig. 1 was designed to use a crystal requiring a load capacitance (given here by $C_1 C_2 / C_1 + C_2$) between 25 and 30 pF.

---

**Fig. 1 - Simple Pierce Oscillator.**

The tube used was either a 6AC7 or a 6L7 depending upon the particular phase of study. With a 6AC7 the oscillator was found to be stable over the
available range of plate supply voltages (210 to 350 volts) and screen grid supply voltages (105 to 350 volts, the screen grid supply being maintained at a D.C.
voltage smaller than the plate supply). The amplitude of oscillation was found to be independent of the plate supply voltage but was affected by the screen supply voltage.

The discussion of keying characteristics is facilitated by the use of two definitions:

(a) The rise time (abbreviated R.T.) is defined as the time required for the plate oscillations to reach 63.2% of the maximum value.

(b) The fall time (abbreviated F.T.) is defined as the time required for the plate oscillations to drop to 36.8% of the maximum value.

To improve high speed keying it is necessary to determine the limitations imposed on R.T. and F.T. by various circuit parameters and then to develop a method of overcoming these limitations.

1. Variation of Crystal Q.

Because quartz crystals have a very high Q, the crystal will undoubtedly be one of the prime factors in producing a long R.T. and F.T. The initial experiment was performed to determine the effect of variations in crystal Q. Five crystals (whose equivalent parameters were known) were used in the oscillator, and the R.T. and F.T. were measured for each crystal. The data obtained are tabulated in TABLE 1. From the data shown in TABLE 1, it can be seen that both R.T. and F.T. increase as Q increases; moreover, the fall time is almost proportional to Q. (33)

(33) Subsequent experiments indicate that the Q of crystal No. 12 is probably about 50,000 rather than the calculated value of 66,200.
TABLE 1

CORRELATION OF RISE TIME AND FALL TIME WITH Q

<table>
<thead>
<tr>
<th>Crystal No.</th>
<th>Crystal Q.</th>
<th>Rise Time</th>
<th>Fall Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>104,000</td>
<td>3.0 ms.</td>
<td>10.0 ms.</td>
</tr>
<tr>
<td>12</td>
<td>66,200</td>
<td>2.6 ms.</td>
<td>7.1 ms.</td>
</tr>
<tr>
<td>14</td>
<td>56,300</td>
<td>2.5 ms.</td>
<td>7.6 ms.</td>
</tr>
<tr>
<td>11</td>
<td>50,400</td>
<td>1.9 ms.</td>
<td>5.4 ms.</td>
</tr>
<tr>
<td>13</td>
<td>48,600</td>
<td>1.9 ms.</td>
<td>5.0 ms.</td>
</tr>
</tbody>
</table>

2. Variation of Capacitance Ratio.

Referring to Fig. 1, it is seen that the ratio of $C_1$ to $C_2$ may be varied and not appreciably affect the capacitive load presented to the crystal. This statement will hold if $\frac{C_1}{C_2} = K$, where $K$ is the load capacitance for that particular crystal. It can be shown for the Pierce Oscillator that the $g_m$ required for sustained oscillations is a minimum when $C_1 = C_2$. It was therefore suspected that the rise time would be a minimum for $C_1 = C_2$. It was desired to find the limits within which the ratio $C_1/C_2$ may be varied without seriously affecting the rise time.

The oscillator was set up using a 6L7 with a $g_m$ of approximately 1000. The data obtained are tabulated in TABLE 2 and plotted in Fig. 2.

The rise time is arbitrarily measured in distance on an oscilloscope trace which has a uniform velocity. From this graph it can be seen that $C_1/C_2$ may be varied from .5 to 2 without appreciably affecting the rise time.

(34) Note that $K$ is used here in place of the usual crystal load capacitance symbol $C_k$. 

(54)
Fig. 2 - Variation of Rise Time with Capacitance Ratio.

6L7 Tube
$g_m=1000\mu\text{mhos}$
3. Variation of Grid Return Resistor.

The effect of varying the grid return resistor, \( R \), was next determined. \( R \) was varied from a value below which no oscillations could be obtained up beyond the value at which self-blocking occurred, and measurements were made of the rise time and the amplitude of oscillation. The data obtained are presented in Figs. 3 and 4 which show respectively the rise time and amplitude of plate oscillations as a function of \( R \). It was observed that for very low values of \( R \) (\( R = 10,000 \) ohms) oscillations were very unstable and the rise time was very long. Depending upon the crystal, stable oscillations were obtained for \( R \) in the range, 20,000 to 50,000 ohms. Under this condition the amplitude of oscillation was large but the rise time was still long. As \( R \) was increased, the rise time decreased, but the amplitude also decreased. Finally, depending on the crystal, self-blocking occurred for \( R \) greater than about 20 megohms.

A qualitative explanation for this behaviour is offered. For low values of \( R \) the losses in the circuit are so great that oscillations cannot exist. As \( R \) increases these losses decrease, but there is still not
Fig. 3 - Variation of Rise Time with Grid Return Resistance.
Fig. 4 - Variation of Plate Voltage Amplitude with Grid Return Resistance.

Grid Resistance, $R$, in Megohms

Plate Voltage Amplitude in Arbitrary Units

6L7 Tube
$g_m=1000 \mu \text{mhos}$

Crystal #15

Crystal #14
sufficient gain left over (after the losses are cancelled) to build up oscillations at a rapid rate. This will account for the long rise time. To determine the amplitude effects, consider the self-biasing operation in an oscillator. The grid draws a small amount of current at the peak of each swing which charges the capacitance in the grid circuit to produce a negative D.C. bias. During the time the grid is not drawing current, some of the charge leaks off the capacitance through $R$. During the next positive half cycle the grid draws enough current to replace this lost charge. If $R$ is low a rather large portion of the charge leaks off when the grid is not conducting. It is then necessary for the grid to draw a relatively large current to replace this lost charge; to do so the grid must go considerably positive with respect to the cathode. This causes a large plate current to flow which in turn makes the plate voltage drop to a low value. It can be seen that increasing $R$ decreases the grid circuit loss and thus requires a smaller grid swing which produces a smaller plate swing.

It may be shown that the grid return resistor produces losses which are somewhat larger than might be anticipated, because of a rectification effect. This effect may be evaluated by use of the principle of conservation of energy. Referring to Fig. 5, the A.C. power delivered directly to $R$ through the large condenser $C$ is

$$P_{ac} = \frac{E^2}{2R},$$  \hspace{1cm} (1)

where $\hat{E}$ is the peak amplitude of the A.C. voltage.

![Fig. 5 - Loading Effect due to Grid Rectification.](image-url)
Sec. 4  

VARIATION OF PLATE LOAD IMPEDANCE  

The condenser will assume a D.C. voltage due to rectification by the grid. If $RC$ is large (and it usually is) compared to the period of $\omega$, this D.C. voltage will equal $\frac{E}{R}$. Then there is a D.C. component of power,

$$P_{dc} = \frac{E^2}{R}.$$  

The total power dissipated is

$$P = P_{ac} + P_{dc} = \frac{3}{2} \frac{E^2}{R}.$$  

A single resistor which will dissipate the same amount of power as the above rectifying circuit must be equal to $R/3$.

4. Variation of Plate Load Impedance.

The next step was to determine the effects of varying the plate load impedance, $L$. The data obtained are tabulated in TABLES 3 and 4.

### TABLE 3

<table>
<thead>
<tr>
<th>$L$</th>
<th>Crystal #11</th>
<th>Crystal #12</th>
<th>Crystal #13</th>
<th>Crystal #14</th>
<th>Crystal #15</th>
</tr>
</thead>
<tbody>
<tr>
<td>50.5 p.h.</td>
<td>1.1 ms.</td>
<td>1.2 ms.</td>
<td>1.2 ms.</td>
<td>1.3 ms.</td>
<td>1.3 ms.</td>
</tr>
<tr>
<td>221 p.h.</td>
<td>1.3 ms.</td>
<td>1.35 ms.</td>
<td>1.3 ms.</td>
<td>1.4 ms.</td>
<td>1.4 ms.</td>
</tr>
<tr>
<td>498 p.h.</td>
<td>1.3 ms.</td>
<td>1.4 ms.</td>
<td>1.35 ms.</td>
<td>1.55 ms.</td>
<td>1.45 ms.</td>
</tr>
<tr>
<td>750 p.h.</td>
<td>1.15 ms.</td>
<td>1.3 ms.</td>
<td>1.2 ms.</td>
<td>1.35 ms.</td>
<td>1.4 ms.</td>
</tr>
<tr>
<td>1.50 mh.</td>
<td>1.3 ms.</td>
<td>1.35 ms.</td>
<td>1.2 ms.</td>
<td>1.4 ms.</td>
<td>1.3 ms.</td>
</tr>
<tr>
<td>4.5 mh.</td>
<td>1.4 ms.</td>
<td>1.5 ms.</td>
<td>1.4 ms.</td>
<td>1.7 ms.</td>
<td>1.55 ms.</td>
</tr>
</tbody>
</table>

### TABLE 4

<table>
<thead>
<tr>
<th>Crystal #</th>
<th>11</th>
<th>11</th>
<th>11</th>
<th>11</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>50.5 p.h.</td>
<td>221 p.h.</td>
<td>498 p.h.</td>
<td>750 p.h.</td>
<td>1.5 mh.</td>
</tr>
<tr>
<td>Amplitude</td>
<td>10 div.</td>
<td>7.6 div.</td>
<td>7.4 div.</td>
<td>6.2 div.</td>
<td>7.0 div.</td>
</tr>
</tbody>
</table>
It is noted that the value of $L$ appears to have little effect on either R.T. or amplitude of plate oscillation as long as the resonant frequency of $L$ with $C_2$ and distributed capacity is reasonably lower than the crystal frequency. As the resonant frequency of $LC_2$ approaches the crystal frequency, the R.T. decreases and the amplitude of oscillations increases. However, if wide band operation is contemplated, in order to operate with the resonant frequency of $LC_2$ near the crystal frequency, some method for changing $L$ as the crystal frequency is changed would be required. One of the chief advantages of the Pierce oscillator is the fact that it can be made to operate over a large range of frequencies without requiring any tuning controls. It is believed that any advantage resulting from a decrease in rise time and increase in amplitude, obtained through the use of a tuning control, would not equal the disadvantage incurred by the resultant reduction in flexibility. For this reason it seems that $L$ should be of such a value that the resonant frequency of $LC_2$ would be considerably lower than the lowest crystal frequency contemplated.

5. Variation of Tube Parameters.

The final circuit parameter to be varied was the vacuum tube. The two tube characteristics of interest are the $r_p$ and $g_m$. Because the $r_p$ may be regarded as equivalent to the grid return resistor, $R$, insofar as its loss effects are concerned, the effect of varying $r_p$ is known.
In the special circuit shown in Fig. 6, a 6L7 tube was used to study the effect which $g_m$ has on rise time. This particular tube was chosen because it is possible to vary the transconductance between the first grid and the plate by changing the bias applied to the third grid. No nonlinear distortion nor significant variation of $r_p$ accompanies this control. The characteristic curves are shown in Fig. 7. The R.T. was measured for various values of bias on $G_3$; the data are tabulated in TABLE 5.

It is seen that the R.T. increases as the $g_m$ decreases. This recommends the use of high $g_m$ tubes to promote a short rise time.

![Oscillator Circuit Using Variable $g_m$ Tube.](image)
Fig. 7 - Variation of Transconductance with Respect to Grid Biases.
TABLE 5

CORRELATION OF TUBE $G_m$ WITH TIME DELAY

<table>
<thead>
<tr>
<th>$E_{G_3}$</th>
<th>Rise Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>35</td>
</tr>
<tr>
<td>-2</td>
<td>33</td>
</tr>
<tr>
<td>-4</td>
<td>53</td>
</tr>
<tr>
<td>-6</td>
<td>69</td>
</tr>
<tr>
<td>-8</td>
<td>92</td>
</tr>
</tbody>
</table>

Note:
The time delay is measured in arbitrary units. It was found that oscillations ceased when $E_{G_3}$ was more negative than $-8.5$ volts. Plate current became zero when $E_{G_3}$ was more negative than $-18$ volts.

Summary of Preliminary Observations.

It was found that:

(a) High $Q$ crystals produce long R.T.'s and F.T.'s.

(b) When the circuit parameters meet other restrictions, the $Q$ of the crystal determines the F.T.

(c) The ratio $C_1/C_2$ may vary between 1/2 and 2 without appreciably affecting the R. T. The value, $C_1/C_2 = 1$, produces minimum R. T.

(d) $R$ should be large for short rise time.

(e) $L$ should have a value such that $wL \gg 1/wC_2$ at the crystal frequency.

(f) Vacuum tubes having large values of $r_p$ and $g_m$ should be used.
Referring to TABLE 1, it is apparent that the F.T. is considerably longer than the R. T. This suggests the possibility that the fall time is the limiting factor in high speed keying.
APPENDIX B

OPTIMUM WAVEFORMS FOR KEYING

Since the initial experiments indicated that the rise time is much shorter than the fall time, the question arose concerning the type of keying waveform (or envelope) which is most desirable. In a conference held at Long Branch in December, 1946, this question was discussed, and it developed that there was no generally accepted criterion for the optimum waveform of a keyed oscillator. Accordingly, work was undertaken to determine those properties of a keyed wave which affect its usefulness. The broad objective which governed this work was to obtain maximum intelligibility of signals in conjunction with a minimum use of the frequency spectrum.

Because a keyed signal will be judged by the human ear (excluding machine recorders) and because the behavior of the ear is not readily expressed in mathematical terms, the study was conducted on a purely experimental basis. Signals of various forms were applied to the terminals of a loudspeaker and the resulting signal was judged by several observers. The actual waveform was recorded in each case by means of an oscilloscope. The methods used and the results obtained are described in the following paragraphs.

1. General Procedure.

Experimentally, a successful examination of the problem requires that some method be provided to obtain an audio frequency modulated with various keying waveforms. This objective is illustrated in Fig. 1.
The experimental setup is shown in Fig. 2. Circuit diagrams for the keyed oscillator and converter are shown in Figs. 3 and 4 respectively. The operation of the setup is relatively simple. The output of the keyed oscillator is combined in the mixer stage with the output of a crystal controlled CW oscillator which has an output frequency differing by a few hundred cycles from that of the keyed oscillator.

 Provision is made for varying the frequency of the CW oscillator to obtain a pleasing tone. Since the mixer is a nonlinear device, there exists in the plate circuit an audio frequency which is the difference of the two radio frequencies. This audio frequency is passed through filters to remove any RF present and is then amplified to drive a loudspeaker. Oscillograms are made of the voltage waveforms at the keyed oscillator plate and across the speaker terminals.

2. Observations and Data.

Oscilloscope photographs were taken at points (a) and (b), Fig. 2, for various keying waveforms. These photographs, with comments, are presented in Fig. 5. The comments are the result of averaging the opinions of several persons (acquainted with code transmission) who listened to the signals.
Fig. 2 - Experimental Circuit for Determining Optimum Keying Waveforms.

A) is connected direct to Y-axis input on oscilloscope and has loose capacitive coupling to the mixer.

Fig. 3 - Keyed Oscillator.

B) is loosely coupled to Keyed oscillator.
C) is connected to input of audio amplifier.

Fig. 4 - Frequency Converter.
Fig. 5 - Oscillograms of Keying Waveforms.
The following qualitative results were obtained:

(a) Essentially square wave keying gives a perfectly readable signal, but key clicks are produced; however, the majority of the observers expressed no objection to slight clicks, and a few expressed a preference for slight key clicks.

(b) A relatively long rise time does not degrade the readability of a signal; moreover, most of the noticeable clicks disappear.

(c) A relatively long fall time definitely degrades the readability.

(d) A long rise time and a long fall time produce unreadable signals.

(e) The most noticeable clicks are produced by a short rise time. The clicks produced by a short fall time are relatively unimportant.

The following conclusions are drawn:

(a) The fall time must be small compared to the period of one dot-cycle.

(b) The rise time should not be too short or objectionable clicks will be produced. A relatively long rise time will still produce readable signals.

3. Tentative Recommendations.

For the study of keyed crystal controlled oscillators it was decided arbitrarily to assign maximum values, in per cent of the period of a dot-cycle, for the rise and fall times. In view of the above conclusions it is believed that for adequate performance:

(a) The time for the keying waveform to reach 80% of maximum value should not exceed 25% of the period of one dot-cycle.

(b) The time for the keying waveform to decrease to 20% of maximum value should not exceed 10% of the period of one dot-cycle.
(c) The rise and fall curves should approximate \((1 - e^{-kt})\) and \((e^{-k't})\), respectively.  

This limiting condition is shown in Fig. 6. It is emphasized that these values are arbitrary, but it is believed that they will produce satisfactory operation.

![Fig. 6 - Modulation Envelope for a Barely Readable Signal.](image)

To apply these results to crystal damping, it is desirable to interpret them in terms of the time decrement, \(\Delta\). A specific tentative goal of 43 dot-cycles per second was given for keying oscillators.\(^{(35)}\) Then

\[
\Delta = 20 \log_{10} \left( \frac{E_2}{E_1} \cdot \frac{1}{\text{period} \times 10^6} \right) \text{ db/sec., or} \tag{1}
\]

\[
\Delta = 20 \log_{10} \left( \frac{2E_1}{E_1} \cdot \frac{1}{1/43 \times 1} \right) \text{ db/sec.} = 6011 \text{ db/sec.} \tag{2}
\]

A suitable and convenient figure will be

\[
\Delta = 6000 \text{ db/sec.} = 6 \text{ db/millisecond.} \tag{3}
\]

\(^{(35)}\) Conference at Long Branch Frequency Control Laboratory, December 10 and 11, 1946.
Thus it is seen that a $\Delta > \Delta_{0}$ is required for an oscillator to key properly at 43 dot-cycles per second. This establishes a value for the fall time. The rise time may not be expressed so simply because it usually involves two or more exponential terms having opposite curvature.

4. Effect of Keying on a Conventional Receiver.

Another method used in an attempt to obtain data on keying waveforms is shown in Fig. 7.

![Block Diagram for Circuit Using Receiver](image)

It was found that the effect of varying the waveform could be determined, but that the receiver seriously distorted the keying waveform. Because there was no way to evaluate the effects of the receiver distortion, this method was discarded. It should be noted that a radio receiver does introduce keying waveform distortion, particularly if a very sensitive and selective receiver is used. However, the effect of this distortion on readability did not seem to be appreciable.

An interesting implication of this work is that the receiver used, and probably many receivers of present design, produces appreciable distortion of the keyed waveform. For slow keying, this is probably unimportant, but for fast keying it may be necessary to employ specially designed or adjusted receivers.
APPENDIX C

HYPERBOLIC AND TRIGONOMETRIC RELATIONSHIPS
AND TRANSFORMATIONS

1. Hyperbolic Functions.

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \]

\[ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \]

\[ \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \]

\[ e^x = \cosh x + \sinh x \]

\[ \cosh x = \frac{e^x + e^{-x}}{2} \]

\[ \sinh x = \frac{e^x - e^{-x}}{2} \]

\[ \tanh x = \frac{\sinh x}{\cosh x} \]

\[ \cosh^2 x - \sinh^2 x = 1 \]

\[ \cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y \]

\[ \sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y \]

2. Trigonometric Functions.

\[ e^{jx} = 1 + jx - \frac{x^2}{2!} - j \frac{x^3}{3!} + \frac{x^4}{4!} + j \frac{x^5}{5!} + \ldots \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \]
4. Reduction of Equation (2.25).

Given: \( A \sinh B_t h + B_h \cosh B_t h \).

Let \( A = r \cosh \phi_h \),
\[ B_h = r \sinh \phi_h, \]
\[ r = \sqrt{A^2 - B_h^2}, \]
and \( \phi_h = \tanh \frac{-B_h}{A}. \)


\[ \cosh jx = \cos x \]
\[ \sinh jx = \sin x \]
\[ \cos jx = \cosh x \]
\[ \sin jx = \sinh x \]
Then

\[ A \sinh B \cosh B t + B \cosh B \sinh B t = r(\cosh \varphi \sinh B \cosh B t + \sinh \varphi \cosh B t), \]
\[ = r \sinh (B t + \varphi) \cosh t, \]
\[ = \sqrt{A^2 - B^2} \sinh (B t + \tanh^{-1} \frac{B}{A}). \]

5. Reduction of Equation (2.32).

Given: \( A \sin Bt + B \cos Bt. \)

Let \( A = r \cos \varphi \), \( B = r \sin \varphi \), \( r = \sqrt{A^2 + B^2} \), and \( \varphi = \tan^{-1} \frac{B}{A} \).

Then

\[ A \sin Bt + B \cos Bt = r(\cos \varphi \sin Bt + \sin \varphi \cos Bt), \]
\[ = r \sin(Bt + \varphi), \]
\[ = \sqrt{A^2 + B^2} \sin(Bt + \tan^{-1} \frac{B}{A}). \]


Given: \( A \cos Bt - B \sin Bt. \)

Using the above substitutions,

\[ A \cos Bt - B \sin Bt = r(\cos \varphi \cos Bt - \sin \varphi \sin Bt), \]
\[ = r \cos(Bt + \varphi), \]
\[ = \sqrt{A^2 + B^2} \cos(Bt + \tan^{-1} \frac{B}{A}). \]
APPENDIX D

INDEPENDENT NETWORK EQUATIONS

1 - Mesh Equations.

A mesh equation of the form \( \sum E = 0 \) may be written for each closed path which may be traced in a network. We quote a theorem, proved by Bode, regarding the number of independent meshes obtainable.

**Theorem:** In any conductively united network the number of independent mesh equations \( I_m \) which may be written is given by the equation

\[
I_m = B - N + 1,
\]

where \( B \) is the number of branches and \( N \) the number of nodes in the network.

A **node** is defined by Bode as the common termination of three or more circuit elements.

A **branch** is defined as a series group of circuit elements terminated at both ends by nodes.

Thus, in Fig. 1:

- \( a, c, g, \) and \( h \) are not nodes;
- \( b, d, e, \) and \( f \) are nodes;
- \( 1 + 2, 3, 4, 5, 6, \) and \( 7 + 8 \) are branches.

Then

\[
B = 6, \quad N = 4, \quad \text{and} \quad I_m = 3.
\]

Fig. 1 - Example of a Flat Network.

The type of configuration shown in Fig. 1 is termed a flat network, characterized by the fact that all closed paths may be traced without crossing any line. That is, when the network is spread out flat no branch crosses another branch. It can be proved that in a flat network the number of independent mesh equations is equal to the number of free areas bounded by the branches. Thus, Fig. 1 has 3 free areas and, hence, $I_M = 3$, as before.

2. Nodal Equations.

A nodal equation of the form $\Sigma I = 0$ may be written for each node in a network. It is also shown by Bode that the number of independent nodal equations, $I_N$, which may be written is given by the equation

$$I_N = N - 1.$$  \hspace{1cm} (2)

Then for the circuit of Fig. 1,

$$N = 4 \text{ and } I_N = 3.$$

It should be noted, however, that $I_M$ and $I_N$ are not necessarily equal. For example, if a branch is added between nodes $a$ and $b$, $I_M$ is raised to four, while $I_N$ remains three.

Apparently, then, Nodal Analysis as set forth by Bode often involves fewer equations; consequently, network determinantal solutions are frequently simplified by its use.
APPENDIX E

THE RESONANT FREQUENCIES OF A CRYSTAL

Fig. 1 - Equivalent Circuit of a Crystal.

Fig. 1 is the same as Fig. 2.17(b). From fundamental considerations, the impedance seen looking into the crystal terminals is

\[ Z(w) = R(w) + jX(w) , \]

provided that \( X_i \) is not zero and neither \( X_i \) nor \( X_0 \) become infinite.

\[ Z(w) = R_0^2 + \left[ R_0^2 + X_0 \left( X_0 + X_i \right) \right] \]

\[ R_0^2 + \left( X_0 + X_i \right)^2 \]

(1)

The resonances of the crystal are defined by the equation

\[ X(w) = 0. \]

(2)

From (1),

\[ 0 = \frac{X_i \left[ R_0^2 + X_0 \left( X_0 + X_i \right) \right]}{R_0^2 + \left( X_0 + X_i \right)^2} , \]

\[ = R_0^2 + X_0 \left( X_0 + X_i \right) \]

(3)

provided that \( X_i \) is not zero and neither \( X_i \) nor \( X_0 \) become infinite.

(37) Formula (1) for the impedance is given in Reference Data for Radio Engineers, published by the Federal Tel. & Radio Corp., New York, 1943, p. 52.
Substituting for the reactances in terms of the circuit parameters,

\[
0 = R_0^2 + \left( wL - \frac{1}{wC_0} \right) \left( wL - \frac{1}{wC_x} \right),
\]

\[
= R_0^2 + \left( wL - \frac{1}{wC_0} \right) \left( wL - \frac{1}{wC_x} \right),
\]

\[
= R_0^2 + \left( wL - \frac{1}{wC_0} \right) \left( wL - \frac{m}{wC_0} \right), \text{by (2.132)},
\]

\[
= R_0^2 + w^2L^2 - \frac{I_0}{C_0} - m \frac{I_0}{C_0} + \frac{m}{w^2C_0^2} \cdot (4)
\]

Multiplying by \(w_2^2/\omega_0^2\),

\[
0 = w^4 - w^2 \left[ \frac{(m + 1)}{\omega_0^2C_0} - \frac{R_0^2}{\omega_0^2C_0^2} \right] + \frac{m}{\omega_0^2C_0^2}
\]

\[
= w^4 - w^2\omega_0^2 \left[ (m + 1) - \frac{R_0^2}{\omega_0^2C_0^2} \right] + m\omega_0^4, \text{by (2.129)},
\]

\[
= w^4 - w^2\omega_0^2 (m + 1 - \frac{1}{\omega_0^2C_0^2}) + m\omega_0^4, \text{by (2.133)}. (5)
\]

Then

\[
2w^2 = \omega_0^2 (m + 1 - \frac{1}{\omega_0^2C_0^2}) \pm \sqrt{w_0^4(m + 1 - \frac{1}{\omega_0^2C_0^2})^2 - 4m\omega_0^4},
\]

\[
= \omega_0^2 \left[ (m + 1 - \frac{1}{\omega_0^2C_0^2}) \pm \sqrt{(m + 1 - \frac{1}{\omega_0^2C_0^2})^2 - 4m} \right]. (6)
\]

The radical may be expanded to give

\[
\sqrt{(m + 1 - \frac{1}{\omega_0^2C_0^2})^2 - 4m} = \left[ (m + 1)^2 - 2(m + 1) \frac{1}{\omega_0^2C_0^2} + \frac{1}{\omega_0^4C_0^4} - 4m \right]^{1/2},
\]

\[
= \left[ (m - 1)^2 - 2(m + 1) \frac{1}{\omega_0^2C_0^2} + \frac{1}{\omega_0^4C_0^4} \right]^{1/2},
\]

\[
= \left[ \frac{1}{n} - 2(2n + 1) \frac{1}{\omega_0^2C_0^2} + \frac{n^2}{\omega_0^4C_0^4} \right]^{1/2}, \text{by (2.132)},
\]

\[
= \frac{1}{n} \left[ 1 - 2(2n + 1) \frac{1}{\omega_0^2C_0^2} + \frac{n^2}{\omega_0^4C_0^4} \right]^{1/2}.
\]
Now, \( Q_0 \) is normally greater than \( 10^4 \) while \( n \) will rarely exceed 4000. Hence, we may neglect the final term with insignificant error (less than 1 part in \( 10^8 \) of the correction term \( \frac{4n^2}{Q_0^2} \)).

Then (7) becomes

\[
\sqrt{(m + 1 - \frac{1}{Q_0^2})^2 - 4m} \neq \frac{1}{n} \left[ 1 - \frac{4n^2}{Q_0^2} \left( 1 + \frac{1}{2n} \right) \right]^{\frac{1}{2}}; \tag{8}
\]

and (6) becomes

\[
2w^2 = w_o^2 \left[ \frac{(2m + 1) - \frac{1}{2Q_0^2}}{n} \right] \pm \frac{1}{n} \sqrt{1 - \frac{4n^2}{Q_0^2} \left( 1 + \frac{1}{2n} \right)}.
\]

\[
= 2w_o^2 \left[ (1 + \frac{1}{2n} - \frac{1}{2Q_0^2}) \pm \frac{1}{2n} \sqrt{1 - \frac{4n^2}{Q_0^2} \left( 1 + \frac{1}{2n} \right)} \right]. \tag{9}
\]

Since the radical appears as the multiplier of a small quantity, we may take the root to two terms of the binomial expansion, by Appendix J-4. Thus,

\[
w^2 = w_o^2 \left\{ (1 + \frac{1}{2n} - \frac{1}{2Q_0^2}) \pm \frac{1}{2n} \left[ 1 - \frac{2n^2}{Q_0^2} \left( 1 + \frac{1}{2n} \right) \right] \right\},
\]

\[
= w_o^2 \left\{ (1 + \frac{1}{2n} - \frac{1}{2Q_0^2}) \pm \frac{1}{2n} \left[ \frac{1}{2n} - \frac{n}{Q_0^2} \left( 1 + \frac{1}{2n} \right) \right] \right\},
\]

\[
= w_o^2 \left\{ (1 + \frac{1}{2n} - \frac{1}{2Q_0^2}) \pm \left( \frac{1}{2n} - \frac{n}{Q_0^2} - \frac{1}{2Q_0^2} \right) \right\}. \tag{10}
\]

Denoting by \( w_1 \) and \( w_2 \) the two roots of (5),

\[
w_1^2 = w_o^2 \left[ 1 + \frac{1}{2n} - \frac{1}{2Q_0^2} + \frac{1}{2n} - \frac{n}{Q_0^2} - \frac{1}{2Q_0^2} \right],
\]
\[ w_1 = w_0 \left[ 1 + \frac{1}{n} - \frac{(n+1)}{Q_0^2} \right], \]
\[ = w_0 \left[ 1 + \frac{1}{n} \left( 1 - \frac{m}{Q_0^2} \right) \right], \text{ by (2.132),} \]
\[ = w_0 \left[ 1 + \frac{1}{n} \right], \quad (11) \]

with an error of about 1 part in \(10^{10}\), which we neglect.

\[ w_2 = w_0 \left[ 1 + \frac{1}{2n} - \frac{1}{2Q_0^2} - \frac{1}{2n} + \frac{n}{2Q_0^2} + \frac{1}{2Q_0^2} \right], \]
\[ = w_0 \left[ 1 + \frac{n}{Q_0^2} \right]. \quad (12) \]

Taking the square root of (11) to three terms, by Appendix J-4,

\[ w_1 = w_0 \left( 1 + \frac{1}{2n} - \frac{1}{8n^2} \right), \quad (13) \]
\[ \text{or } w_1 = w_0 \left( 1 + \frac{1}{2n} \right), \quad (14) \]

with a maximum error of about \(+(8 \times 10^{-6})\).

Taking the square root of (12) to two terms, by Appendix J-4,

\[ w_2 = w_0 \left( 1 + \frac{n}{2Q_0^2} \right), \quad (15) \]
\[ = w_0, \quad (16) \]

with a maximum error of about \(-{(20 \times 10^{-6})}\).

From (13) and (15),

\[ \frac{w_1 - w_2}{w_0} = \left[ \frac{1}{2n} - \frac{1}{8n^2} - \frac{n}{2Q_0^2} \right], \]
\[ = \frac{1}{2n} \left[ 1 - \frac{1}{4n} - \frac{n^2}{Q_0^2} \right], \quad (17) \]
or

\[ \frac{w_1 - w_2}{w_0} = \frac{1}{2n}, \]  

(18)

with a maximum error of about +16% occurring when \( n = 4000 \) and \( Q_0 = 10,000 \). Actually, crystals with low \( Q \)'s would not usually have high values of \( n \) (although excessive padding capacitance could produce such a case.) Consequently, (18) holds normally with a small error.

At the minimum \( n \) of 125, the frequency difference is 0.4%, from (18). The error in this value as computed from (17) is +0.2%. Since the maximum frequency difference occurs with minimum \( n \), we may state that the difference between the two resonant frequencies of a crystal cannot exceed 0.4%.

From Fig. 2 we see that the larger root, \( w_1 \), is the antiresonant frequency of the crystal, and \( w_2 \) is the series resonant frequency. Thus, from (14),

\[ w_a = w_1 = w_0 (1 + 1/2n), \]  

(19)

and from (16),

\[ w_s = w_2 = w_0. \]  

(20)
Now, an isolated short-circuited crystal will tend to resonate at a frequency determined by the series arm parameters alone. This obviously is

\[ \omega_{sc} = \frac{1}{\sqrt{L_0 C_0}} = \omega_0 \text{ by definition.} \]  

(21)

An isolated open-circuited crystal, on the other hand, will tend to resonate at a frequency involving both the series and shunt arms. This frequency will be determined by the condition

\[ X_0 + X_x = 0. \]  

(22)

Introducing the circuit parameters and solving, we have

\[ \omega_{oc} = \sqrt{\frac{1}{L_0 \frac{C_o C_x}{C_o + C_x}}} = \sqrt{\frac{1}{L_0 C_y}} = \omega_y, \]

\[ = \omega_0 \left(1 + \frac{1}{n}\right)^{\frac{1}{2}}, \]  

(23)

from the definitions of (2.132).

But (23) is identical with (11) and hence, \( \omega_y \) may be regarded as the actual antiresonant frequency of a crystal with an error of about 1 part in \( 10^{10} \).

Then (19) and (20) become

\[ \omega_a \approx \omega_y = \omega_{oc}, \]  

(24)

\[ \omega_s \approx \omega_0 = \omega_{sc}. \]  

(25)
The quantities $w_y$ and $w_0$ provide accurate mathematical definitions of the crystal resonances, while $w_{oc}$ and $w_{sc}$ express operational definitions which are of some interest.
APPENDIX F

THE IMPEDANCE LEVEL OF A CRYSTAL AT ANTIRESONANCE

Equation (22) of Appendix E defines the frequency, \( w_y \), which is essentially the true antiresonant frequency, \( w_a \). Applying (22) to the impedance equation, (1), of Appendix E, we have

\[
Z(w_y) = \frac{X_x^2 R_0 + jX_x R_0^2}{R_0^2}
\]

\[
= \frac{X_x}{R_0} (X_x + jR_0).
\]  

Because at the true antiresonance \( Z(w_a) \) is a pure resistance, we write

\[
Z(w_y) \approx R(w_y) = \frac{X_x^2}{R_0}.
\]  

This is the same as saying that in (1) \( R_0 \) is small compared with \( X_x \). We note that since \( X_x \) and \( R_0 \) are in quadrature, \( X_x \) need be only about ten times greater than \( R_0 \) in order to neglect the reactance term of \( Z(w_y) \). \( R_0 \) is generally small. By (22) \( X_x \) must be the negative of \( X_0 \), which is itself zero at frequency \( w_0 \) but quite large at frequencies removed even as slightly as \( w_y \) \( (= w_0 \sqrt{1 + \frac{1}{n^2}}) \) on account of the enormous \( Q \)'s of crystals. Hence (2) is undoubtedly a good approximation for (1).

Then the impedance level at antiresonance is

\[
R(w_y) = \frac{X_x^2}{R_0},
\]

\[
= \frac{1}{w_y^2 C_x R_0},
\]  

(3)

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THE IMPEDANCE LEVEL OF A CRYSTAL AT ANTIRESONANCE

\[
\begin{align*}
&= \frac{1}{m w_0^2 C_0^2 n^2 R_0}, \text{ by (2.132)}, \\
&= \frac{1}{w_0^2 C_0^2 n^2 R_0}, \text{ since } m \text{ is essentially unity,} \\
&= \frac{w_0^2 L_0^2}{n^2 R_0}, \text{ by (2.129)}, \\
&= \frac{w_0^2 L_0^2}{R_0^2} \cdot \frac{R_0}{n^2}; \\
\end{align*}
\]

or

\[
R_y = \frac{Q_0^2 R_0}{n^2}, \text{ by (2.133),} \tag{4}
\]

where \( R_y \) is the symbol used in place of \( R(w_y) \) for the approximate antiresonant impedance of a crystal. This impedance (a pure resistance at the exact antiresonant frequency, \( w_a \)) is often called the Performance Index (PI) of the crystal unit. As such it is usually written in the form given by (3); or, more exactly,

\[
\text{PI} = \frac{1}{w_a^2 C_x^2 R_0}. \tag{5}
\]

It is immediately evident from (4) that the PI contains perhaps the three most important crystal unit constants: Quality Factor, Capacitance Ratio, and Crystal Resistance.
APPENDIX G

EXPERIMENT ON RESISTIVE DAMPING OF CRYSTALS

1. Damping an Isolated Crystal.

It has been shown theoretically that a vibrating quartz crystal can be damped by the connection of a resistance across the crystal terminals. For maximum damping this resistance should have a value equal to the reactance of the capacitance shunting the crystal plate. In an attempt to verify or refute these theoretical results, the following experiment was performed to evaluate the resistance producing maximum damping.

An oscillator was set up as shown in Fig. 1. The connections are such that with the relays actuated, the crystal is connected to the control grid of the tube, and voltage is applied to the screen of the tube.

Fig. 1 - Experimental Circuit.
When this occurs, the unit functions as an oscillator, and vibrations build up in the crystal. When the relays are in the opposite position, the damping resistance is connected across the crystal, and a damped decay transient is produced. The time required for the oscillations to decay to one-third of their initial value was measured by a precision timer. This timer is so constructed that large readings correspond to small time intervals, and vice versa.

Four crystals of different frequencies were used, and the optimum resistance value was determined for each. Difficulty was encountered in measuring time delays for resistance values near the optimum, because the difference in damping rate was very small; however, by careful observation, a small time difference could be noticed. The holder capacity, $C_h$, of each crystal was measured, and the capacity to ground, $C_k$, of the circuit connected to the crystal was measured at the resonant frequency of each crystal with the relay in the damping position. Because of the low $Q$ of this circuit, it was necessary to disconnect the damping resistance from ground before making the latter measurements.

The data obtained are presented in the following tables:

**TABLE 1**
(Crystal No. 15, 3680 kc., $C_h = 13$, $C_k = 50$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>500</td>
<td>84.9</td>
<td>85.2</td>
</tr>
<tr>
<td>600</td>
<td>85.5</td>
<td>85.5</td>
</tr>
<tr>
<td>700</td>
<td>85.6</td>
<td>85.7</td>
</tr>
<tr>
<td>800</td>
<td>85.5</td>
<td>85.6</td>
</tr>
<tr>
<td>900</td>
<td>85.3</td>
<td>85.4</td>
</tr>
<tr>
<td>1000</td>
<td>85.2</td>
<td>85.2</td>
</tr>
<tr>
<td>1100</td>
<td>85.0</td>
<td>85.0</td>
</tr>
<tr>
<td>1200</td>
<td>84.8</td>
<td>84.8</td>
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<tr>
<td>1300</td>
<td>84.5</td>
<td>84.5</td>
</tr>
<tr>
<td>1400</td>
<td>84.1</td>
<td>84.1</td>
</tr>
</tbody>
</table>

(38) See footnote on next page.
TABLE 2
(Crystal No. 10, 3590 kc., $C_h = 12, C_k = 61$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>100</td>
<td>79.3</td>
<td>79.5</td>
</tr>
<tr>
<td>200</td>
<td>81.4</td>
<td>81.5</td>
</tr>
<tr>
<td>300</td>
<td>83.7</td>
<td>83.1</td>
</tr>
<tr>
<td>400</td>
<td>85.0</td>
<td>84.6</td>
</tr>
<tr>
<td>500</td>
<td>85.3</td>
<td>85.2</td>
</tr>
<tr>
<td>600</td>
<td>85.1</td>
<td>85.2</td>
</tr>
<tr>
<td>700</td>
<td>85.1</td>
<td>85.1</td>
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<tr>
<td>800</td>
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<td>84.9</td>
</tr>
<tr>
<td>900</td>
<td>84.7</td>
<td>84.5</td>
</tr>
</tbody>
</table>

TABLE 3
(Crystal No. 94, 2410 kc., $C_h = 13, C_k = 50$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>500</td>
<td>79.8</td>
<td>80.1</td>
</tr>
<tr>
<td>600</td>
<td>81.6</td>
<td>81.7</td>
</tr>
<tr>
<td>700</td>
<td>82.5</td>
<td>82.4</td>
</tr>
<tr>
<td>800</td>
<td>83.0</td>
<td>83.1</td>
</tr>
<tr>
<td>900</td>
<td>83.2</td>
<td>83.2</td>
</tr>
<tr>
<td>1000</td>
<td>83.4</td>
<td>83.2</td>
</tr>
<tr>
<td>1100</td>
<td>83.5</td>
<td>83.5</td>
</tr>
<tr>
<td>1200</td>
<td>83.4</td>
<td>83.4</td>
</tr>
<tr>
<td>1300</td>
<td>83.4</td>
<td>83.5</td>
</tr>
<tr>
<td>1400</td>
<td>83.2</td>
<td>83.4</td>
</tr>
</tbody>
</table>

(39) Note that, because of the type of timer used, the maximum damping rate corresponds to the maximum Time Delay reading to TABLES 1 to 4 and Fig. 2.
TABLE 4

(Crystal No. 104, 2905 kc., $C_h = 12$, $C_k = 48$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>500</td>
<td>85.8</td>
<td>85.9</td>
</tr>
<tr>
<td>600</td>
<td>86.0</td>
<td>86.3</td>
</tr>
<tr>
<td>700</td>
<td>86.5</td>
<td>86.6</td>
</tr>
<tr>
<td>800</td>
<td>86.6</td>
<td>86.6</td>
</tr>
<tr>
<td>900</td>
<td>86.6</td>
<td>86.7</td>
</tr>
<tr>
<td>1000</td>
<td>86.6</td>
<td>86.6</td>
</tr>
<tr>
<td>1100</td>
<td>86.5</td>
<td>86.5</td>
</tr>
<tr>
<td>1200</td>
<td>86.5</td>
<td>86.5</td>
</tr>
<tr>
<td>1300</td>
<td>86.4</td>
<td>86.4</td>
</tr>
<tr>
<td>1400</td>
<td>86.3</td>
<td>86.4</td>
</tr>
</tbody>
</table>

The optimum damping resistance was obtained by plotting the average time delay against the damping resistance and by reading the value which produced maximum damping. An example is shown in Fig. 2.

![Graph showing the plot of damping data](image)

Fig. 2 - Plot of Damping Data of TABLE 1.

The calculated optimum resistance is given by the formula,

$$R_x = \frac{1}{wC_x},$$

where

$$C_x = C_h + C_k.$$  \hspace{1cm} (6)

Then,

$$R_x = \frac{1}{2\pi \times 3680 \times 10^5 \times 63 \times 10^{-12}} = 687 \text{ ohms.}$$

(7)
The percentage difference is calculated from the following equation:

\[
% \text{ difference} = \frac{R_{\text{exp.}} - R_{\text{calc.}}}{R_{\text{calc.}}} \times 100.
\]  \hfill (9)

For the example used

\[
% \text{ difference} = \frac{718 - 687}{687} = + 4.5\%.
\]  \hfill (10)

The experimental and calculated values of \( R_x \) are presented in Table 5.

<table>
<thead>
<tr>
<th>Crystal</th>
<th>( R_x ) (experimental)</th>
<th>( R_x ) (calculated)</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 15, 3680 kc.</td>
<td>718</td>
<td>687</td>
<td>+ 4.5%</td>
</tr>
<tr>
<td>No. 10, 3590 kc.</td>
<td>555</td>
<td>609</td>
<td>- 9.0%</td>
</tr>
<tr>
<td>No. 94, 2410 kc.</td>
<td>1145</td>
<td>1080</td>
<td>+ 6.0%</td>
</tr>
<tr>
<td>No. 104, 2905 kc.</td>
<td>880</td>
<td>910</td>
<td>- 3.0%</td>
</tr>
</tbody>
</table>

In view of the difficulty in determining the experimental resistance and the many minor, but indeterminate factors, it is believed that these data represent a close agreement of experimental and theoretical values. Therefore, the experimental results substantiate the theoretical study.

2. Damping a Crystal in an Oscillator.

The previous section showed that the calculated damping resistance is correct when used to damp a crystal alone. Now it is desired to determine the effect of the complete oscillator circuit on the damping resistance.
An oscillator was set up as shown in Fig. 3.

![Experimental Circuit](image)

Fig. 3 - Experimental Circuit.

The procedure followed was similar to that of the preceding experiment. The time required for the oscillations to decay to one-third of their initial value was measured for various values of $R_x$. Difficulty was again encountered in determining the optimum resistance because the fall time varied only slightly when $R_x$ was in the vicinity of optimum $R_x$. The capacity to ground, $C_k$, of the circuit across the crystal and the holder capacity, $C_h$, of the crystal were also measured. Because of the low $Q$ of the damping circuit it was necessary to disconnect the damping resistance when measuring $C_k$.

The experimental $R_x$ was determined by plotting the average of the fall times against $R_x$ and by reading the optimum value.

The data obtained are presented in the following tables:
### TABLE 6
(Crystal No. 9, 3590 kc., $C_h = 14$, $C_k = 73$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>100</td>
<td>67.9</td>
<td>64.2</td>
</tr>
<tr>
<td>200</td>
<td>81.7</td>
<td>80.5</td>
</tr>
<tr>
<td>300</td>
<td>84.2</td>
<td>84.2</td>
</tr>
<tr>
<td>400</td>
<td>85.8</td>
<td>86.1</td>
</tr>
<tr>
<td>500</td>
<td>86.2</td>
<td>85.7</td>
</tr>
<tr>
<td>600</td>
<td>85.6</td>
<td>86.2</td>
</tr>
<tr>
<td>700</td>
<td>85.9</td>
<td>86.3</td>
</tr>
<tr>
<td>800</td>
<td>85.9</td>
<td>86.1</td>
</tr>
<tr>
<td>900</td>
<td>85.6</td>
<td>85.6</td>
</tr>
</tbody>
</table>

Note that again, because of the type of timer used, the desired minimum fall times correspond to the maximum Times Delay readings of Tables 6 to 15.

### TABLE 7
(Crystal No. 10, 3590 kc., $C_h = 12$, $C_k = 75$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>100</td>
<td>71.6</td>
<td>67.5</td>
</tr>
<tr>
<td>200</td>
<td>81.4</td>
<td>81.1</td>
</tr>
<tr>
<td>300</td>
<td>84.5</td>
<td>84.9</td>
</tr>
<tr>
<td>400</td>
<td>85.8</td>
<td>85.8</td>
</tr>
<tr>
<td>500</td>
<td>86.3</td>
<td>86.2</td>
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<tr>
<td>600</td>
<td>85.7</td>
<td>85.5</td>
</tr>
<tr>
<td>700</td>
<td>85.2</td>
<td>86.4</td>
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<tr>
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<td>85.8</td>
</tr>
<tr>
<td>900</td>
<td>85.7</td>
<td>85.6</td>
</tr>
</tbody>
</table>

(40)
### Table 8
(Crystal No. 12, 3680 kc., $C_h = 15$, $C_k = 75$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>100</td>
<td>76.8</td>
<td>77.0</td>
</tr>
<tr>
<td>200</td>
<td>83.4</td>
<td>85.9</td>
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<tr>
<td>300</td>
<td>86.8</td>
<td>87.2</td>
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<tr>
<td>400</td>
<td>87.7</td>
<td>88.1</td>
</tr>
<tr>
<td>500</td>
<td>87.9</td>
<td>87.9</td>
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<tr>
<td>600</td>
<td>87.9</td>
<td>88.5</td>
</tr>
<tr>
<td>700</td>
<td>88.4</td>
<td>88.2</td>
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<tr>
<td>800</td>
<td>88.2</td>
<td>88.4</td>
</tr>
<tr>
<td>900</td>
<td>88.3</td>
<td>88.2</td>
</tr>
</tbody>
</table>

### Table 9
(Crystal No. 13, 3680 kc., $C_h = 12$, $C_k = 75$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>100</td>
<td>72.4</td>
<td>76.4</td>
</tr>
<tr>
<td>200</td>
<td>86.2</td>
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<td>88.8</td>
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<td>89.6</td>
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<td>90.2</td>
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<td>89.5</td>
<td>89.5</td>
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<tr>
<td>700</td>
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<td>89.4</td>
</tr>
<tr>
<td>900</td>
<td>89.1</td>
<td>88.9</td>
</tr>
</tbody>
</table>
### TABLE 10
(Crystal No. 14, 3680 kc., $C_h = 13$, $C_k = 75$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
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<tr>
<td>100</td>
<td>75.0</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>85.6</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>87.0</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>89.2</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>89.4</td>
<td>89.9</td>
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<tr>
<td>600</td>
<td>89.1</td>
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<tr>
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<td>89.9</td>
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<td>89.8</td>
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<td>88.4</td>
</tr>
<tr>
<td>1400</td>
<td>88.1</td>
<td>88.2</td>
</tr>
</tbody>
</table>

### TABLE 11
(Crystal No. 15, 3680 kc., $C_h = 12$, $C_k = 75$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>100</td>
<td>77.7</td>
<td>70.1</td>
</tr>
<tr>
<td>200</td>
<td>82.7</td>
<td>72.4</td>
</tr>
<tr>
<td>300</td>
<td>85.6</td>
<td>85.4</td>
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<tr>
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<td>87.1</td>
<td>87.6</td>
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<tr>
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<td>87.3</td>
</tr>
<tr>
<td>900</td>
<td>87.2</td>
<td>87.0</td>
</tr>
</tbody>
</table>
TABLE 12
(Crystal No. 87, 2410 kc., C_h = 14, C_k = 75)

<table>
<thead>
<tr>
<th>Resistance, R_x</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
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<td>500</td>
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<td>80.1</td>
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<tr>
<td>600</td>
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<td>82.1</td>
</tr>
<tr>
<td>700</td>
<td>82.6</td>
<td>82.4</td>
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<tr>
<td>800</td>
<td>82.7</td>
<td>82.7</td>
</tr>
<tr>
<td>900</td>
<td>82.9</td>
<td>83.1</td>
</tr>
<tr>
<td>1000</td>
<td>82.7</td>
<td>83.4</td>
</tr>
<tr>
<td>1100</td>
<td>82.8</td>
<td>83.2</td>
</tr>
<tr>
<td>1200</td>
<td>82.7</td>
<td>83.0</td>
</tr>
<tr>
<td>1300</td>
<td>82.3</td>
<td>82.9</td>
</tr>
<tr>
<td>1400</td>
<td>82.3</td>
<td>82.8</td>
</tr>
</tbody>
</table>

TABLE 13
(Crystal No. 92, 2410 kc., C_h = 15, C_k = 75)

<table>
<thead>
<tr>
<th>Resistance, R_x</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>500</td>
<td>80.1</td>
<td>81.0</td>
</tr>
<tr>
<td>600</td>
<td>82.5</td>
<td>82.5</td>
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<tr>
<td>700</td>
<td>84.2</td>
<td>83.6</td>
</tr>
<tr>
<td>800</td>
<td>84.6</td>
<td>83.8</td>
</tr>
<tr>
<td>900</td>
<td>84.4</td>
<td>83.5</td>
</tr>
<tr>
<td>1000</td>
<td>84.9</td>
<td>84.5</td>
</tr>
<tr>
<td>1100</td>
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<td>83.9</td>
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<tr>
<td>1200</td>
<td>83.4</td>
<td>83.5</td>
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<td>1300</td>
<td>83.6</td>
<td>83.2</td>
</tr>
<tr>
<td>1400</td>
<td>83.7</td>
<td>82.9</td>
</tr>
</tbody>
</table>
TABLE 14
(Crystal No. 109, 3230 kc., $C_h = 18, C_k = 72$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>100</td>
<td>77.6</td>
<td>77.3</td>
</tr>
<tr>
<td>200</td>
<td>82.5</td>
<td>84.1</td>
</tr>
<tr>
<td>300</td>
<td>86.2</td>
<td>85.9</td>
</tr>
<tr>
<td>400</td>
<td>85.7</td>
<td>86.4</td>
</tr>
<tr>
<td>500</td>
<td>86.6</td>
<td>86.5</td>
</tr>
<tr>
<td>600</td>
<td>86.7</td>
<td>87.3</td>
</tr>
<tr>
<td>700</td>
<td>87.6</td>
<td>87.0</td>
</tr>
<tr>
<td>800</td>
<td>86.9</td>
<td>86.4</td>
</tr>
<tr>
<td>900</td>
<td>86.5</td>
<td>86.3</td>
</tr>
</tbody>
</table>

TABLE 15
(Crystal No. 112, 3230 kc., $C_h = 18, C_k = 72$)

<table>
<thead>
<tr>
<th>Resistance, $R_x$</th>
<th>Time Delay (Arbitrary Units)</th>
<th>Average Time Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trial I</td>
<td>Trial II</td>
</tr>
<tr>
<td>100</td>
<td>77.4</td>
<td>78.5</td>
</tr>
<tr>
<td>200</td>
<td>84.6</td>
<td>85.1</td>
</tr>
<tr>
<td>300</td>
<td>85.3</td>
<td>85.8</td>
</tr>
<tr>
<td>400</td>
<td>85.8</td>
<td>86.1</td>
</tr>
<tr>
<td>500</td>
<td>86.3</td>
<td>86.9</td>
</tr>
<tr>
<td>600</td>
<td>86.5</td>
<td>86.6</td>
</tr>
<tr>
<td>700</td>
<td>86.4</td>
<td>86.4</td>
</tr>
<tr>
<td>800</td>
<td>86.2</td>
<td>86.2</td>
</tr>
<tr>
<td>900</td>
<td>86.1</td>
<td>86.1</td>
</tr>
</tbody>
</table>
The experimental results are compared with the calculated values in TABLE 16. It is apparent from these data that the experimental $R_x$ is greater than the calculated $R_x$. The average deviation is +22%. Two explanations for this are as follows: (1) the value obtained for $C_k$ is not the same as that which exists when the oscillator is operating; and (2) losses due to the tube and other circuit elements add a considerable amount of damping. It is believed that the discrepancy must be attributed to a combination of the above effects.

It has been pointed out that the variation of fall time from minimum is small even though $R_x$ is twice the optimum $R_x$. Because of this small variation and the experimental results, it seems reasonable to state that the $R_x$ used in a circuit should be about 1.3 times the calculated $R_x$.

<table>
<thead>
<tr>
<th>Crystal</th>
<th>$R_x$(Experimental)</th>
<th>$R_x$(Calculated)</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 9, 3590 kc</td>
<td>650</td>
<td>510</td>
<td>+27.5</td>
</tr>
<tr>
<td>No. 10, 3590 kc</td>
<td>590</td>
<td>510</td>
<td>+15.7</td>
</tr>
<tr>
<td>No. 12, 3680 kc</td>
<td>510</td>
<td>481</td>
<td>+6.0</td>
</tr>
<tr>
<td>No. 13, 3680 kc</td>
<td>630</td>
<td>499</td>
<td>+26.5</td>
</tr>
<tr>
<td>No. 14, 3680 kc</td>
<td>640</td>
<td>492</td>
<td>+30.2</td>
</tr>
<tr>
<td>No. 15, 3680 kc</td>
<td>584</td>
<td>500</td>
<td>+16.8</td>
</tr>
<tr>
<td>No. 87, 2410 kc</td>
<td>980</td>
<td>741</td>
<td>+32.3</td>
</tr>
<tr>
<td>No. 92, 2410 kc</td>
<td>1000</td>
<td>734</td>
<td>+56.0</td>
</tr>
<tr>
<td>No. 109, 3230 kc</td>
<td>675</td>
<td>548</td>
<td>+23.1</td>
</tr>
<tr>
<td>No. 112, 3230 kc</td>
<td>570</td>
<td>548</td>
<td>+4.0</td>
</tr>
</tbody>
</table>
APPENDIX H

A CONSTANT REACTANCE NETWORK

The circuit configuration and specifications for the constant reactance network shown in Fig. 1 are suggested in a paper by Mallett. Under certain conditions this circuit has a reactance independent of the resistance, $R_a$.

![Basic Network Diagram]

The impedance seen in the terminals of the network is

$$ Z = R + jX $$

or

$$ Z = \frac{jX_a(R_a + jX_b)}{R_a + j(X_a + X_b)} \quad (1) $$

Now, $X$ will be independent of $R_a$ when

$$ X_a = -X_b \quad (3) $$

and $R_a \neq 0$. (If $R_a = 0$, $Z$ becomes infinite.) Substituting (3) in (2), we have

$$ Z = \frac{X_a R_a + j[R_a^2 X_a + X_a X_b (X_a + X_b)]}{R_a^2 + (X_a + X_b)^2} = R + jX \quad (2) $$

\[
Z = \frac{X_a^2}{R_a} + j X_a = R + j X. \tag{4}
\]

It follows that if \(X_a\) is capacitive, \(X_b\) must be inductive.

Note that the reactance of \(Z\) has the same sign as that of the branch not containing resistance. Therefore, the network shown in Fig. 2 has a constant capacitive reactance for all values of \(R_a\) (\(R_a \neq 0\)).

![Diagram of network](image)

**Fig. 2.** Desired Network.

The impedance is, from (4),

\[
Z = \frac{1}{w^2 C^2 R} - \frac{1}{wC}. \tag{5}
\]
APPENDIX I

CLASSICAL DERIVATION OF EQUATION (4.5)

Fig. 1 is the same as Fig. 4.1, and shows the additional currents $i_1$ and $i_2$ necessary to arrive at a network solution by the method of differential equations.

![Equivalent Damping Circuit](image)

Fig. 1 - Equivalent Damping Circuit.

1. **Kirchhoff Equations.**

   \[
   i_2 = i - i_1 \quad (1)
   \]

   \[
   L_0 \frac{di}{dt} + R_0 i + \int \frac{i}{C_0} dt + \int \frac{i_1}{C_x} dt = C \quad (2)
   \]

   and \[
   L_d \frac{di_2}{dt} + R_d i_2 - \int \frac{i_1}{C_x} dt = 0 \quad (3)
   \]

2. **Elimination of $i_1$ and $i_2$.**

   From (2), by differentiation,

   \[
   -\frac{i_1}{C_x} = L_0 \frac{d}{dt^2} + R_0 \frac{di}{dt} + \frac{i}{C_0} \quad (4)
   \]
From (1) and (3),
\[ L_d \frac{di}{dt} + R_d \frac{di}{dt} = \int \frac{1}{C_x} \frac{dt}{dt} + L_d \frac{di}{dt} + \frac{1}{C_o} \frac{dt}{dt} . \] \tag{5}

Differentiating (5),
\[ L_d \frac{d^2i}{dt^2} + R_d \frac{di}{dt} = \frac{1}{C_x} + L_d \frac{d^2i}{dt^2} + \frac{1}{C_o} \frac{di}{dt} . \] \tag{6}

From (4) and (6),
\[ (L_o + L_d) \frac{d^2i}{dt^2} + (R_o + R_d) \frac{di}{dt} + \frac{1}{C_o} = L_d \frac{d^2i}{dt^2} + \frac{1}{C_o} \frac{di}{dt} . \] \tag{7}

From (4),
\[ \frac{L}{C_x} = \frac{1}{C_x} (L_o \frac{d^2i}{dt^2} + R_o \frac{di}{dt} + \frac{1}{C_o}) . \] \tag{8}

From (7) and (8),
\[ (L_o + L_d) \frac{d^2i}{dt^2} + (R_o + R_d) \frac{di}{dt} + \frac{1}{C_o} + L_d C_x \left[ \frac{L_o \frac{d^4i}{dt^4}}{C_x} + \frac{R_o \frac{d^3i}{dt^3}}{C_o} + \frac{1}{C_o} \frac{d^2i}{dt^2} \right] \\
+ R_d C_x \left[ \frac{L_o \frac{d^3i}{dt^3}}{C_x} + \frac{R_o \frac{d^2i}{dt^2}}{C_o} + \frac{1}{C_o} \frac{di}{dt} \right] = 0 . \] \tag{9}

Grouping terms of (9),
\[ L_o L_d C_x \frac{d^4i}{dt^4} + (R_o L_d C_x + R_d L_o C_x) \frac{d^3i}{dt^3} + \left[ \frac{L C_x}{C_o} + R_o R_d C_x + L_o + L_d \right] \frac{d^2i}{dt^2} \\
+ \left[ \frac{R d C_x}{C_o} + R_o + R_d \right] \frac{di}{dt} + \frac{1}{C_o} = 0 . \] \tag{10}

If $L_d$ is not zero (and our analysis is on this basis), we may divide...
out by the coefficient of $\frac{d^4i}{dt^4}$, obtaining the more useful form

$$\frac{d^4i}{dt^2} + \left[ \frac{R_o}{L_o} + \frac{R_d}{L_d} \right] \frac{d^3i}{dt^3} + \left[ \frac{1}{L_o C_d} + \frac{R_o R_d}{L_o L_d} + \frac{1}{L_d C_x} + \frac{1}{L_o C_x} \right] \frac{d^2i}{dt^2}$$

$$+ \left[ \frac{1}{L_o C_0} \frac{R_d}{L_d C_x} \right] \frac{di}{dt} + \left[ \frac{1}{L_o C_0 L_d C_x} \right] i = 0. \tag{11}$$

Equation (11) is seen to be identical with (4.5).
1. Expansion of Factors.

Given: \((p - r_1)(p - r_2)(p - r_3)(p - r_4) = 0\). \hspace{1cm} (1)

Then it may be shown \(^{(42)}\) that:

Coefficient of \(p^4 = 1\); \hspace{1cm} (2)

Coefficient of \(p^4 = -\sum r_1\),

\[= -(r_1 + r_2 + r_3 + r_4); \hspace{1cm} (3)\]

Coefficient of \(p^2 = \sum r_1 r_j\),

\[= r_1^2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4; \hspace{1cm} (4)\]

Coefficient of \(p^1 = -\sum r_1 r_j r_k\),

\[= -(r_1^2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4); \hspace{1cm} (5)\]

and Coefficient of \(p^0 = r_1 r_2 r_3 r_4\). \hspace{1cm} (6)

If the roots are complex, so that

\[
\begin{align*}
    r_1 &= -A_1 + jB_1, \\
    r_2 &= -A_1 - jB_1, \\
    r_3 &= -A_2 + jB_2, \\
    r_4 &= -A_2 - jB_2;
\end{align*}
\]

we find:

\[
\text{Coeff. } p^3 = -\left[-A_1 + jB_1 - A_1 - jB_1 - A_2 + jB_2 - A_2 - jB_2\right],
\]

\[= 2(A_1 + A_2); \hspace{1cm} (8)\]

Coeff. $p^2 = (-A_1 + jB_1)(-A_1 - jB_1) + (-A_1 + jB_1)(-A_2 + jB_2)$
$+ (-A_1 + jB_1)(-A_2 - jB_2) + (-A_2 + jB_2)(-A_2 + jB_2)$
$+ (-A_1 - jB_1)(-A_2 - jB_2) + (-A_2 + jB_2)(-A_2 - jB_2)$.

$$= A_1^2 + B_1^2 + A_2^2 + B_2^2 + 4A_1A_2; \quad (9)$$

Coeff. $p^1 = -\left[(-A_1 + jB_1)(-A_1 - jB_1)(-A_2 + jB_2) + (-A_1 + jB_1)(-A_1 - jB_1)(-A_2 - jB_2)
+ (-A_1 + jB_1)(-A_2 + jB_2)(-A_2 - jB_2) + (-A_1 + jB_1)(-A_2 + jB_2)(-A_2 - jB_2)\right],$
$$= 2 \left[A_2(A_1^2 + B_1^2) + A_1(A_2^2 + B_2^2)\right]; \quad (10)$$

Coeff. $p^0 = (-A_1 + jB_1)(-A_1 - jB_1)(-A_2 + jB_2)(-A_2 - jB_2)$
$$= (A_1^2 + B_1^2)(A_2^2 + B_2^2). \quad (11)$$

2. Errors and Correction Terms.

If $x = k(1 - \varepsilon)$, and we write $x \approx k$ when $\varepsilon$ is small, then $\varepsilon$ will be termed
the per unit error in the equation $x \approx k$. Where $\varepsilon$ is a variable quantity
having a known maximum, $\varepsilon_{\text{max}}$ will be called the greatest error of $x \approx k$. $\varepsilon$ may,
of course, be expressed in percent, in parts per million, etc.

The error will frequently be written after the approximate equation to
which it refers. Thus, consider the formula

$$x = k(1 + \frac{\varepsilon}{y^2}). \quad (12)$$

We may first write

$$x \approx k; \quad \left[\varepsilon = \frac{-700}{y^2}\right]. \quad (13)$$

Suppose $x$ cannot have a value less than 20. Then

$$x \approx k, \quad \left[\varepsilon_{\text{max}} = -1.75\%\right]. \quad (14)$$

An error is positive if the approximate form of the given equation is too
large, and negative if the approximate form is too small.

The negative of the error will be called the correction term, $c$. If the
correction term associated with a given approximate equation is stated, we mul-
tiply the approximate equation by $(1 + c)$ to obtain an accurate result. Simi-
larly, if the error is stated we multiply the approximate equation by $(1 - \varepsilon)$.
The quantity \((1 + c)\) or its equivalent \((1 - \epsilon)\) will be called the correction factor.

3. Approximations - Division and Multiplication.

If in the expression
\[ x = k \frac{(1 + y_1)}{(1 + y_2)} \]
the quantities \(y_1\) and \(y_2\) are small compared to 1, we may write with negligible error
\[ x \approx k (1 + y_1)(1 - y_2) \]
\[ \approx k (1 + y_1 - y_2) \]
and \(x \approx k\) with an approximate error of \(-y_1 - y_2\) or \(100(y_2 - y_1)\%\).

Using the notation of Appendix J-2:
\[ x \approx k, \quad \left[ \epsilon = (y_2 - y_1) \right] \]


By the Binomial Theorem,
\[ (1 + y)^{\frac{1}{2}} = 1 + \frac{y}{2} - \frac{y^2}{8} + \frac{y^3}{16} - \frac{5y^4}{128} + \ldots \]

If \(y\) is small compared to 1, we may neglect the second and higher order terms in \(y\). Thus,
\[ (1 + y)^{\frac{1}{2}} \approx 1 + \frac{y}{2}, \quad \left[ \epsilon = \frac{y^2}{8} \right] \]

Similarly, the quantity
\[ (1 + y)^2 = 1 + 2y + y^2 \]
\[ (1 + y)^2 \approx 1 + 2y, \quad \left[ \epsilon = -y^2 \right] \]
when \(y \ll 1\).

The errors involved in both these operations may often be regarded as negligible.
APPENDIX K

A THEOREM ON DAMPING

THEOREM: When a crystal is damped by means of a coil (R-L damping) no significant improvement in damping rate can be realized unless a frequency deviation from the working frequency of the crystal exists during transient.

The circuit under consideration is that of Fig. 4.1. We may prove the theorem by substituting the condition $B_1 = B_2 = W_y$ into the system of conditional equations (4.13) to (4.16). To facilitate the development we write

\[
\begin{align*}
A_2 &= h A_0, \\
A_1 &= k A_2 = k h A_0, \\
\frac{W_y}{A_0} &= 2Q_y.
\end{align*}
\]

The results of these substitutions into the four equations are:

\[
\begin{align*}
h(k + 1) &= M + 1 \\
h^2(k^2 + 1) &= 4Q_y^2(N - 1) + 4M, \\
kh(h^2 + 4Q_y^2) + h(k^2h^2 + 4Q_y^2) &= 4Q_y^2(M + N), \\
\text{and} \quad (k^2h^2 + 4Q_y^2)(h^2 + 4Q_y^2) &= 16 Q_y^4 \cdot \frac{N}{M}.
\end{align*}
\]

We are considering a transient current having two components of the same frequency but not necessarily equal decrements. We may express this current, from (4.11), as

\[
i = (I_1 e^{-A_1t} + I_2 e^{-A_2t}) \sin(W_y t + \phi).
\]
Assuming $I_1$ and $I_2$ to be of the same order of magnitude, it is obvious that if either $A_1$ or $A_2$ is not appreciably greater than $A_0$ (the decrement of the isolated crystal) no significant improvement in damping rate has been achieved. Let us require that $A_1$ and $A_2$ each be at least ten times as large as $A_0$.

Then we have the conditions, from (1), that

$$h \geq 10$$
$$kh \geq 10$$

(7)

We can rearrange (4) to obtain

$$4\zeta_y^2 \left[ M + N - h (k + 1) \right] = kh^3 (k + 1).$$

(8)

Substituting (2) in (8),

$$4\zeta_y^2 (N - 1) = kh^3 (k + 1).$$

(9)

Substituting (9) in (3),

$$h^2 (k^2 + 1) = kh^3 (k + 1) + 4M.$$\hspace{1cm} (10)

Substituting for $M$ from (2),

$$k^2 h^2 (h - 1) + kh (h^2 + 4) - (h^2 - 4h + 4) = 0$$

(11)

Solving this quadratic for $k$, we take only the positive root of the determinant, since $k$ is necessarily a positive quantity.

$$k = \frac{-h(h^2 + 4) + h \sqrt{h^4 + 4h^3 - 12h^2 + 32h}}{2h^2 (h - 1)}$$

(12)

Because $h$ must be equal to or greater than ten, we can neglect the last term in the radical and take an approximate root. Thus
This proves that $kh$ is less than unity for $h > 10$; and therefore, from (1), $A_1 < A_0$ — which constitutes a degradation rather than an improvement of the damping rate for the current (6).

While it has here been demonstrated that if no change in frequency is permitted during R-L damping no satisfactory improvement in damping rate is achieved; the correlative, that high damping rates involve a frequency deviation in at least one component of crystal current, is left to be shown in the body of Chapter IV.
APPENDIX L

AN EXPERIMENT TO DETERMINE VACUUM TUBE PREFERENCE
FOR USE IN KEYED OSCILLATORS

This experiment was performed in an attempt to determine the tube
types which produce fastest buildup rates, and from this information to
evaluate the tube characteristics which govern the buildup rates. The
procedure employed was the measurement of rise times for tubes of sever-
al types when used in the two oscillator circuits shown in Figs. 1 and 2.
Previous experiments on rise time indicated that there was very little
variation in the keying characteristics for several tubes of one type;
so, only one tube of each type was used for this experiment.

Fig. 1 - Pentode Oscillator.
These oscillators were set up, and the rise times were measured for both triode and pentode connections of the following tubes: 6L6, 6V6, 6Y6, 6AG7, 6SH7, 6SJ7, 6AK5, and 6AC7. The use of a 1/4 mh. RFC is required for the triode oscillator, so that it may have the same equivalent impedance as the pentode. The two RFC's in the pentode circuit are effectively in parallel. The 100 ohm resistance in the screen lead is added to discourage any tendency towards UHF parasitic oscillations. All tubes had the same D.C. biases for the pentode connection, except the screen grid bias which was made the maximum allowable for the particular type tube; the 6AG7 was an exception to this with 150V on the screen.

![Triode Oscillator Diagram](image)

**Fig. 2 - Triode Oscillator.**

When used as a triode, the screen was connected directly to the plate with all tubes using a plate supply voltage of 300V. It should be noted that the screen voltage is lower for the pentode than for the triode connection. This will affect the comparison between the pentode and triode connection of a given tube. In the tables the pentode connection is indicated by a "P" and the triode connection by a "T".
The rise time was measured from the time the key was closed until the oscillations were 1/3 or 1/2 of maximum amplitude. The rise times to 1/3 amplitude are tabulated in Table 1. The rise times to 1/2 amplitude are tabulated in Table 2.

**TABLE 1**

RISE TIMES TO 1/3 MAXIMUM AMPLITUDE

<table>
<thead>
<tr>
<th>Tube</th>
<th>6AG7</th>
<th>6AC7</th>
<th>6SJ7</th>
<th>6SH7</th>
<th>6AK5</th>
<th>6L6</th>
<th>6V6</th>
<th>6Y6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crystal</td>
<td>P</td>
<td>T</td>
<td>P</td>
<td>T</td>
<td>P</td>
<td>T</td>
<td>P</td>
<td>T</td>
</tr>
<tr>
<td>No. 12</td>
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<td>2.9</td>
<td>1.6</td>
<td>3.4</td>
<td>3.1</td>
<td>2.7</td>
<td>2.3</td>
<td>2.3</td>
</tr>
<tr>
<td>No. 13</td>
<td>1.5</td>
<td>3.0</td>
<td>1.7</td>
<td>3.4</td>
<td>3.3</td>
<td>2.9</td>
<td>2.4</td>
<td>2.3</td>
</tr>
<tr>
<td>No. 14</td>
<td>1.6</td>
<td>3.0</td>
<td>2.2</td>
<td>3.4</td>
<td>3.4</td>
<td>3.0</td>
<td>2.5</td>
<td>2.3</td>
</tr>
<tr>
<td>Average</td>
<td>1.5</td>
<td>3.0</td>
<td>1.8</td>
<td>3.4</td>
<td>3.3</td>
<td>2.9</td>
<td>2.4</td>
<td>2.3</td>
</tr>
<tr>
<td>O.P. *</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>7</td>
<td>8</td>
<td>4</td>
<td>7</td>
<td>2</td>
</tr>
</tbody>
</table>

*C.O.P.: Order of Preference

**TABLE 2**

RISE TIMES TO 1/2 MAXIMUM AMPLITUDE

<table>
<thead>
<tr>
<th>Tube</th>
<th>6AG7</th>
<th>6AC7</th>
<th>6SJ7</th>
<th>6SH7</th>
<th>6AK5</th>
<th>6L6</th>
<th>6V6</th>
<th>6Y6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crystal</td>
<td>P</td>
<td>T</td>
<td>P</td>
<td>T</td>
<td>P</td>
<td>T</td>
<td>P</td>
<td>T</td>
</tr>
<tr>
<td>No. 12</td>
<td>2.3</td>
<td>3.0</td>
<td>2.9</td>
<td>3.5</td>
<td>4.0</td>
<td>3.5</td>
<td>2.9</td>
<td>3.2</td>
</tr>
<tr>
<td>No. 13</td>
<td>2.4</td>
<td>3.1</td>
<td>3.1</td>
<td>4.5</td>
<td>4.1</td>
<td>3.7</td>
<td>3.2</td>
<td>3.3</td>
</tr>
<tr>
<td>No. 14</td>
<td>2.4</td>
<td>3.2</td>
<td>3.2</td>
<td>4.4</td>
<td>4.2</td>
<td>4.1</td>
<td>3.2</td>
<td>3.5</td>
</tr>
<tr>
<td>Average</td>
<td>2.4</td>
<td>3.1</td>
<td>3.1</td>
<td>4.1</td>
<td>4.1</td>
<td>3.8</td>
<td>3.1</td>
<td>3.3</td>
</tr>
<tr>
<td>O.P. *</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

*C.O.P.: Order of Preference

Table 3 presents a consolidation of the tube characteristics. Some of the values given in this table are derived from the other characteristics, but it is believed that all values are as accurate as the original data in the tube manuals.
TABLE 3
TUBE CHARACTERISTICS
(for eg₁ = 0 volts)

<table>
<thead>
<tr>
<th>Pentode (ep = 300v; eg₂ = design maximum for tube)</th>
<th>Triode (eg₂ = ep = 300V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tube</td>
<td>Gₘ</td>
</tr>
<tr>
<td>6L6</td>
<td>8800</td>
</tr>
<tr>
<td>6V6</td>
<td>6500</td>
</tr>
<tr>
<td>6Y6</td>
<td>10500</td>
</tr>
<tr>
<td>6AG7</td>
<td>15000</td>
</tr>
<tr>
<td>6AC7</td>
<td>16000</td>
</tr>
<tr>
<td>6SH7</td>
<td>5300</td>
</tr>
<tr>
<td>6AK5</td>
<td>9000</td>
</tr>
<tr>
<td>6SJ7</td>
<td>2650</td>
</tr>
</tbody>
</table>

Note: c-s, control grid to screen grid
s-p, screen grid to plate

The correlation of order of preference for rise times to 1/3 maximum amplitude and tube characteristics is presented in Table 4. The correlation of order of preference for rise times to 1/2 maximum amplitude and tube characteristics is presented in Table 5.

TABLE 4
O.P. AND TUBE CHARACTERISTICS FOR RISE TIME TO 1/3 MAX. AMPLITUDE

<table>
<thead>
<tr>
<th>Pentode</th>
<th>Triode</th>
</tr>
</thead>
<tbody>
<tr>
<td>O.P.</td>
<td>Gₘ</td>
</tr>
<tr>
<td>6Y6</td>
<td>10500</td>
</tr>
<tr>
<td>6V6</td>
<td>6500</td>
</tr>
<tr>
<td>6AG7</td>
<td>15000</td>
</tr>
<tr>
<td>6L6</td>
<td>8800</td>
</tr>
<tr>
<td>6AG7</td>
<td>16000</td>
</tr>
<tr>
<td>6AK5</td>
<td>9000</td>
</tr>
<tr>
<td>6SH7</td>
<td>5300</td>
</tr>
<tr>
<td>6SJ7</td>
<td>2650</td>
</tr>
</tbody>
</table>
From the data obtained it is seen that there is some variation of the relative positions of different tubes when the rise time is measured to 1/3 or 1/2 of maximum amplitude. Apparently this change in relative position is due to a difference in behaviour of various tube types as the tubes are driven far into the class C region. Since the rise time to 1/2 maximum amplitude is a better qualitative indication of a tube's value than the rise time to 1/3 maximum amplitude, the former is used as the determining factor of the value of a tube in a keyed oscillator.

For the pentode connection the data in Table 5 reveals that the power tubes, having low plate resistances and high transconductances, produce the shortest rise times. Quantitatively, the plate resistance appears to have a greater effect than the transconductance.

For the triode connection the data in Table 5 reveals no apparent correlation between tube characteristics and the order of preference. In fact, the magnitudes seem to have random distribution.

It is noted that the triode connection compares favorably with the pentode connection. However, in the oscillator circuits used, the
screen voltage for the pentode was generally much lower than the "screen voltage" for the triode. In the pentode connection the screen voltage has a much greater effect on the $g_m$ than the plate voltage, and the $g_m$ increases as the screen voltage increases. This may account for the relatively good showing the triode connection made when compared with the pentode connection.

In conclusion it may be stated that power pentodes appear to give the shortest rise times. However, the inconclusive nature of the remaining data recommends a theoretical examination of the problem to determine the individual effects of the tube parameters.
APPENDIX M

ANALYSIS OF PLATE CIRCUIT DAMPING

The circuit of Fig. 7.2 is here analyzed to determine the value of $R_0$ which produces maximum crystal damping. The equivalent circuit for the impedance, $Z_z$, seen by $R_0$, $L_0$, and $C_0$ is shown in Fig. 1.

Fig. 1 - Equivalent Circuits for Plate Circuit Damping.

Fig. 1(a) is redrawn in Fig. 2, in which the reactances are defined as follows, using (7.1) through (7.4):

$$\begin{align*}
X & = \frac{1}{wC_1} = \frac{1}{wK(b+1)}, \\
X_h & = \frac{1}{wC_h} = \frac{1}{wK}, \\
\text{and} \quad bX & = \frac{1}{wC_2}.
\end{align*} \tag{1}$$

Fig. 2 - Equivalent Network.
The impedance, $Z_z$, of this network is found to be

$$Z_z = R_z - jX_z = \frac{-[(b + 1)X X_h R_z^2 + b^2 X^3 X_h] - j b^2 X^2 X_h R_z}{b^2 X R_z - j [(b + 1)R_z^2 + X_h R_z^2 + b^2 X^2 + b^2 X X_h]}. \quad (2)$$

Rationalizing and solving for the resistive component,

$$R_z = \frac{b^2 X X_h^2 [R_z^3 + b^2 X X_h^2]}{b^4 X^4 R_z^2 + \left[R_z^2 \left[(b + 1)X + X_h\right] + b^2 X^2 \left(X + X_h\right)\right]^2}. \quad (3)$$

Substituting for all reactances in terms of $K$ from (1),

$$R_z = \frac{\frac{b^2}{w^2 K^2 (b + 1)^2 (a + 1)^2} \left[R_z^3 + \frac{b^2}{w^2 K^2 (b + 1)^2} R_z^2 \right]}{R_z^4 \left[\frac{2(a + 1)(a + b + 1)}{b + 1} \right] + \frac{a^2 b^2}{(b + 1)^2}} \left[R_z^2 + \frac{b^4}{w^4 K^4 (b + 1)^4 (a + 1)^2} \left[a + b + 1\right] \right]^2. \quad (4)$$

To obtain maximum damping $R_z$ must be a maximum. The classical method of obtaining a maximum $R_z$ is to set $\frac{dR_z}{dR_z} = 0$. However, when (4) is differentiated, the resulting equation is a cubic with literal coefficients. The solution of a complicated literal cubic equation is a laborious and usually unsatisfactory process.

Since the conditions $C_2 = \frac{C_1}{b}$, $C_1 = (b + 1)K$, and $C_h = b K$ were imposed, there are, besides $R_z$, three variables: $a$, $b$, and $w$.

Make these definitions:

$$B = \frac{b^2}{(b + 1)^2} ;$$

and $C = \frac{1}{(a + 1)^2}$ ;
\[
D = \left[ \frac{2(a + 1)(a + b + 1)}{b + 1} + \frac{a^2b^2}{(b + 1)^2} \right],
\]

and 
\[
E = \frac{a + b + 1}{b + 1}.
\]

Substituting (5) in (4),

\[
R_z = \frac{1}{w^2k^2} \frac{BO}{R_2} \left[ \frac{R_2^3 + \frac{E}{w^2k^2} R_2}{R_2^4 + \frac{BCD}{w^2k^2} R_2 + \frac{B^2CE^2}{w^4k^2}} \right].
\]

Then

\[
\frac{dR_z}{dR_2} = \frac{1}{w^2k^2} \frac{BO}{R_2} \left[ \frac{3R_2^2 + \frac{B}{w^2k^2} R_2^4 + \frac{BCD}{w^2k^2} R_2^2 + \frac{B^2CE^2}{w^4k^2}}{R_2^4 + \frac{BCD}{w^2k^2} R_2^2 + \frac{B^2CE^2}{w^4k^2}} \right]^2.
\]

The value of \( R_2 \) for maximum \( R_z \) is given by the equation

\[
R_2^6 + \frac{R_2^4}{w^2k^2} (3B - BCD) + \frac{R_2^2}{w^4k^2} (B^2CD - 3B^2CE^2) - \frac{1}{w^6k^4} B^3CE^2 = 0.
\]

This equation when factored will assume the form

\[
\left[ R_2^2 - p^2 \frac{1}{w^2k^2} \right] \left[ R_2^4 + MR_2^2 \frac{1}{w^2k^2} + N \frac{1}{w^4k^2} \right] = 0;
\]

whence

\[
R_2 = P \cdot \frac{1}{wK}
\]

where \( P \) is a yet undetermined coefficient.

The other roots are complex conjugate and are spurious roots intro-
duced in the rationalization processes.

An expression has been obtained for \( R_2 \); we next find a corresponding expression for \( R_z \). Substituting (10) in (6),

\[
R_z = \frac{BC}{w^2K^2} \left[ \frac{P^3 + BP}{w^3K^3} \right] \]

or \( R_z = \frac{1}{wK} \left[ \frac{BC (P^3 + BP)}{P^4 + BCDP^2 + B^2C^2} \right] \); (11)

\[
R_z = J = \frac{1}{wK}, \quad (12)
\]

where \( J \) is another function of \( a \) and \( b \). Using brute force methods, \( P \) and \( J \) were calculated for various values of \( a \) and \( b \), the results being tabulated in TABLES 1 and 2 and plotted in Figs. 3 and 4, respectively.

### TABLE 1

VALUES OF RESISTANCE FACTOR \( P \) FOR VARIOUS VALUES OF \( a \) AND \( b \)

<table>
<thead>
<tr>
<th>( b )</th>
<th>( a = .25 )</th>
<th>( a = .5 )</th>
<th>( a = .75 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.4494</td>
<td>.4171</td>
<td>.3924</td>
</tr>
<tr>
<td>2</td>
<td>.5779</td>
<td>.5187</td>
<td>.4764</td>
</tr>
<tr>
<td>3</td>
<td>.6372</td>
<td>.5621</td>
<td>.5089</td>
</tr>
<tr>
<td>5</td>
<td>.6943</td>
<td>.6017</td>
<td>.5357</td>
</tr>
<tr>
<td>8</td>
<td>.7309</td>
<td>.6253</td>
<td>.5505</td>
</tr>
<tr>
<td>12</td>
<td>.7530</td>
<td>.6387</td>
<td>.5577</td>
</tr>
<tr>
<td>( \infty )</td>
<td>.8000</td>
<td>.6667</td>
<td>.5710</td>
</tr>
</tbody>
</table>
Fig. 3 - Values of Resistance Factor $P$ resulting for various values of $a$ and $b$. ($R_2 = \frac{P}{WK}$).
Fig. 4 - Values of Resistance Factor \( J \) resulting for various values of \( a \) and \( b \). \( R_z = \frac{1}{\omega K} \).
To determine the effect on $R_z$ of variations in frequency without readjustment of $R_2$, values of $R_z$ were calculated for various $w$'s with $R_2$ fixed at the value corresponding to a maximum $R_z$ for some reference frequency $w_r$. The resulting curve is presented in Fig. 5. The frequency is given as a multiple, $c$, of the reference frequency. $R_z$ is shown in a normalized form. That is, inasmuch as only relative values are of interest, a quantity $\overline{R_z}$, which is proportional to $R_z$, has been plotted.

Also included in Fig. 5 is the curve of $\overline{R_z}$ for $R_2$ adjusted with frequency, corresponding to the maximum damping condition. It is seen that the two curves differ by less than 20% for a 4 to 1 frequency range.

![Graph](Fig. 5 - Normalized Equivalent Resistance, $\overline{R_z}$, Versus Frequency.)
The qualitative analytical consideration of the best circuit position to key an oscillator indicates that the transient effects upon the circuit other than the crystal are of no consequence. However, there is reason to believe that the keying transient may affect the rise time due to shock effects upon the crystal. This experiment was performed in an attempt to evaluate the crystal shock effects for keying at various points in the circuit.

A pentode was selected for the tube. Two restrictions are immediately placed upon the keying process if the tube is to be protected from damage - namely, the screen grid must not have positive voltage applied if the plate has no positive voltage, and the screen grid must not have positive voltage applied if the suppressor grid is sufficiently negative to cut off the plate current. Observing these precautions and discarding redundant keying methods, there are four basic methods by which keying may be accomplished.

1. Screen keying - where the screen lead is interrupted.
2. Plate and screen keying - where the plate and screen leads are interrupted together.
3. Grid keying - where the control grid is biased below cutoff.
4. Cathode keying - where the cathode lead is interrupted.

In addition to these there are various complex methods which may be used if recommended by experimental results.
For this experiment the basic E. C. Pierce circuit shown in Fig. 1 was used.

The suppressor is returned to a small positive voltage in an attempt to operate the tube at a higher $g_m$ and to isolate the plate from the remainder of the circuit. To insure effective cessation of crystal vibrations during the "OFF" period a damping network was incorporated in the experimental circuit. The circuit actually used is shown in Fig. 2.
In Fig. 2, a relay or relays were inserted at the appropriate symbol to obtain the various keying methods as given below.

- $X$ - screen keying
- $O$ - plate and screen keying
- $*$ - cathode keying (at cathode)
- $\bigcirc$ - cathode keying (at ground)

Control grid keying was obtained as shown in Fig. 3.

The damping resistor was selected for each crystal and was approximately 2000 ohms. Suitable care was taken to phase the keying and damping relays properly.

The rise time data (rise time measured to 2/3 maximum amplitude) obtained is presented in TABLE 1. It is seen that the maximum difference between the different methods is 12%. Evidently the shock effects on the crystal vary only slightly for the several keying methods. This recommends that the point at which the oscillator is keyed be determined as a matter of convenience.
TABLE 1

RISE TIMES

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Crystal</th>
<th>Method of Keying</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>S</td>
</tr>
<tr>
<td>2410</td>
<td>93</td>
<td>8.1 ms</td>
</tr>
<tr>
<td>2410</td>
<td>94</td>
<td>9.5 ms</td>
</tr>
<tr>
<td>2905</td>
<td>101</td>
<td>5.8 ms</td>
</tr>
<tr>
<td>2905</td>
<td>102</td>
<td>5.8 ms</td>
</tr>
<tr>
<td>2905</td>
<td>103</td>
<td>5.9 ms</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td>7.0 ms</td>
</tr>
</tbody>
</table>
One possible difficulty which might be experienced with keyed oscillators is a detrimental effect of the keying process on crystal life. Accordingly, an experiment was performed to determine what effect keying and damping have on crystal life. Since the circuits for this experiment were set up before the development of the final keyed oscillator circuits, the circuits used are not the same as those recommended for keyed oscillators; however, it is believed that the information obtained is representative.

The experimental setup consisted of five groups of three oscillators operating at five frequencies—namely, 4830 kc, 3680 kc, 3230 kc, 2905 kc, and 2030 kc. Thus, for each frequency there were two keyed oscillators and one continuous wave (CW) oscillator. The CW oscillator was provided as a control. In addition to the oscillators, suitable power supplies and a keyer were provided. The keyer produced a square wave of 20 cycles per second; so, the keying rate was 20 dots per second.

The basic oscillator circuit is the Electron Coupled Pierce. The circuit for the keyed oscillator is shown in Fig. 1. The circuit for the CW oscillator is shown in Fig. 2. It is evident that the keyed and CW oscillators are identical except for the damping resistor and keying relay.

The 330 ohm resistor in the cathode return protects the tube in the event of crystal failure. A 2200 ohm resistor in the plate lead simulates an external load. The suppressor is not connected to the cathode because it is desired to isolate the plate circuit and its load from the remainder of the circuit.
NOTE:
All resistors 1/2 watt unless otherwise noted. $R_d$ determined for crystal frequency from design formulas.

**Fig. 1 - Keyed Oscillator.**
NOTE:
All resistors 1/2 watt unless otherwise noted.

Fig. 2 - CW Oscillator.
For this reason the suppressor is returned to A.C. ground. However, to prevent a decrease in the control grid-to-plate transconductance caused by having the suppressor at a lower D.C. potential than the cathode, the suppressor is operated at a D.C. potential approximately equal to the most positive swing of the cathode.

In addition, the keyed oscillator has a keying relay in the screen grid lead and a damping relay and resistor between control grid and ground. These relays are driven so that one is open when the other is closed. Suitable square waves are applied to the relays by means of switching tubes. It was found necessary to control the amplitude of the square waves applied to the switching tubes in order to obtain proper operation of the relays.

Suitable RF filters have been provided to keep RF out of the D.C. lines. In Fig. 1, the following voltages are applied to the input terminals: 1)-ground; 2)-$E_1^+$; 3)-$E_2^+$; 5) and 6)-square waves of opposite polarity; 7) and 8)-heater voltage. In Fig. 2, the following voltages are applied to the input terminals: 1)-ground; 2)-$E_1^+$; 5) and 6)-heater voltage.

Fig. 3 shows the circuit diagram for the keyer and one power supply. The circuit diagram for the other power supply is entirely conventional and is not shown. The keyer includes a frequency-determining multivibrator operating at 20 cycles per second. Outputs are taken from the plates of both multivibrator tubes to obtain square waves of opposite polarity. These square wave outputs are run through separate cathode followers and thence to the oscillator switching tubes. Fig. 4 shows the inter-unit wiring diagram between the two power supply chassis.
NOTE:
All resistors 1/2 watt unless otherwise noted.
A) is connected to htrs. of keyer only. B) - Keyer output jacks.

Fig. 3 - Keyer and Power Supply.
Fig. 4 - Inter-Unit Wiring.
### TABLE 1

**CRYSTAL LIFE DATA**

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<thead>
<tr>
<th></th>
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<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>34</td>
<td>4830 kc</td>
<td>ok</td>
<td>ok</td>
<td>ok</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1B</td>
<td>32</td>
<td>4830 kc</td>
<td>ok</td>
<td>ok</td>
<td>ok</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1C</td>
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<td>4830 kc</td>
<td>ok</td>
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<td>2A</td>
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<tr>
<td>5C</td>
<td>2</td>
<td>2030 kc</td>
<td>ok</td>
<td>ok</td>
<td>ok</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**NOTES**

- **6/13** - All units put into operation.
- **6/16** - 2B, bad damping relay. 5A, intermittent circuit trouble.
- **7/21** - 4C, bad crystal, replaced 97 with 102.
- **7/26** - 4C, bad crystal, replaced 102 with 100.
- **8/9** - 4C, bad crystal, replaced 100 with 96, change tube.
- **8/14** - 4C, bad crystal, replaced 96 with 98.
- **8/21** - 5B, bad crystal, replaced 5 with 4.
- **11/11** - 3B, bad damping relay, cease taking data.
The complete unit was placed in operation June 13, 1947. The performance of the oscillators was checked daily from that date until August 29, 1947. Thereafter the performance was checked weekly until November 11, 1947. Because of the considerable amount of data obtained most of which is negative, this data is consolidated in TABLE 1 with only the dates of abnormal operation listed. The A and B oscillators are keyed, and the C oscillator is the CW control.

From the data presented in TABLE 1 it appears that keying has little if any effect on crystal life. It is noted that only one crystal failed in keyed service. The bad performance obtained with 4C is believed to have been caused by an abnormal tube. After the tube was replaced only one crystal failed.

From this experiment it is concluded that an oscillator circuit configuration which produces satisfactory CW operation with respect to crystal life will produce satisfactory keyed operation.
APPENDIX F

THE IMPEDANCE OF A CRYSTAL NEAR SERIES RESONANCE

It was stated without proof in Sec. 2-20 that at series resonance the equivalent crystal circuit, Fig. 2.17(b), reduces to the single resistance $R_0$. That this should be approximately true is an obvious fact; however, we shall now demonstrate its degree of validity. We shall also show that the resistive portion of the crystal impedance does not depart significantly from the value $R_0$ at frequencies slightly to either side of series resonance.

From Appendix E equation (1), the components of crystal impedance are

$$ R(w) = \frac{X_x^2 R_0}{R_0^2 + (X_0 + X_x)^2}, \quad (1) $$

and

$$ X(w) = \frac{X_x \left[ R_0^2 + X_0 (X_0 + X_x) \right]}{R_0^2 + (X_0 + X_x)^2}. \quad (2) $$

We may write the total impedance as

$$ Z(w) = R(w) + jX(w) = |Z(w)| \angle \theta, \quad (3) $$

with $\theta$ defined by

$$ \tan \theta = \frac{X(w)}{R(w)} = \frac{R_0^2 + X_0 (X_0 + X_x)}{X_x R_0}. \quad (4) $$

If we let

$$ w = cw_0 \quad (5) $$

and substitute the appropriate circuit parameters from Fig. 1 of Appendix E, we have for the reactances:

$$ X_x = -\frac{1}{w C_x}, $$

$$ = -\frac{1}{cw_0 C_0}, \text{ by (2.132)}, $$

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and, similarly,

\[ X_0 = \omega L_0 - \frac{1}{\omega C_0}, \]

\[ = \frac{\omega L_0}{c} \left( c^2 - 1 \right) . \]  \hspace{1cm} (7)

Combining (6) and (7),

\[ (X_0 + X_x) = \frac{\omega L_0}{c} \left[ c^2 - (1 + \frac{1}{n}) \right], \]

\[ = \frac{\omega L_0}{c} \left( c^2 - m \right), \] by (2.132).  \hspace{1cm} (8)

Then (1) becomes

\[ R(w) = \frac{R_0}{n^2 \left[ \frac{R_0^2 c^2}{\omega L_0^2} + (c^2 - m)^2 \right]}, \]

\[ = \frac{R_0}{n^2} \cdot \frac{1}{Q_o^2 + (m - c^2)^2}, \] by (2.133).  \hspace{1cm} (9)

Also, (4) may be written

\[ \tan \theta = \frac{R_0^2 + \omega L_0^2}{c^2} \left( c^2 - 1 \right) (c^2 - m), \]

\[ = \frac{n Q_o}{c} \left[ \left( c^2 - 1 \right) \left( c^2 - m \right) + \frac{c^2}{Q_o^2} \right], \]

\[ = \frac{n Q_o}{c} \left[ \left( c^2 - 1 \right) (m - c^2) - \frac{c^2}{Q_o^2} \right]. \]  \hspace{1cm} (10)

At series resonance, the reactive component of \( Z(w) \) is zero.

In Appendix E equations (20) and (12) we have the series resonant frequency already determined extremely accurately:

\[ w_s^2 = \omega_0^2 \left[ 1 + \frac{n}{Q_o^2} \right]; \]  \hspace{1cm} (11a)

or in terms of \( c_s \), by (5),

\[ c_s^2 = 1 + \frac{n}{Q_o^2} = \frac{Q_o^2 + n}{Q_o^2}. \]  \hspace{1cm} (11b)
Substituting this value into (9),
\[
R(w_s) = \frac{R_0}{n^2} \times \frac{1}{\frac{q_0^2 + n}{q_0^4} + \left[1 + \frac{n}{q_0^2} - (1 + \frac{1}{n})\right]^2} 
\]
\[
= \frac{R_0}{n^2} \times \frac{n^2 q_0^4}{n^2 q_0^2 + n^3 + n^4 - 2n^2 q_0^2 + q_0^4} 
\]
\[
= \frac{R_0}{1 - n^2 q_0^2 + n^4 q_0^4 / q_0^4} 
\]

Now let
\[
\lambda = \frac{n}{q_0^2} 
\]

This ratio is generally quite small, and since \(\lambda\) is essentially unity in value, we can rewrite (12) thus:
\[
\frac{R_s}{R_0} = \frac{R_0}{1 - \lambda^2 (1 - \lambda^2)} 
\]

where \(R_s\) is an abbreviated notation for the series resonant impedance, \(R(w_s)\).

The ratio \(R_s/R_0\) is shown plotted versus \(\lambda\) in Fig. 1. As discussed in Sec. 2-20, \(n\) will rarely exceed 4000 and \(Q_0\) will ordinarily be greater than 10,000. Thus, for our study \(\lambda\) is always less than 0.4 and will usually lie below the value 0.2. From Fig. 1 it is seen that the error in assuming \(R_0\) to be the series resonant impedance of a crystal is about 5% at \(\lambda = 0.2\) and only 1% at \(\lambda = 0.1\).
Now let us determine the equivalent series resistance of the crystal when it is operated at some frequency, \( w_u \), such that \( \tan \theta = u \).

Using (4) and (10) we write

\[
u = \frac{X(w_u)}{R(w_u)} = \frac{nQ_0}{c} \left[ (c^2 - 1)(m - c^2) - \frac{c^2}{Q_0^2} \right]; \quad (15)
\]

whence

\[
\frac{c^2}{Q_0^2} = (c^2 - 1)(m - c^2) - \frac{uc}{nQ_0}. \quad (16)
\]

Substituting this value into the denominator of (9) yields

\[
R(w_u) = \frac{R_0}{n^2} \cdot \frac{1}{(m - c^2)(m - 1) - \frac{uc}{nQ_0}},
\]

\[
= \frac{R_0}{n} \cdot \frac{1}{(m - c^2) - \frac{uc}{Q_0}}, \quad (17)
\]

since \((m - 1) = 1/n\).

The value \( c_u \) corresponding to \( w_u \) is obtained by expanding (15):

\[
c^4 - (m + 1 - \frac{1}{Q_0^2})c^2 + \frac{uc}{nQ_0} + m = 0. \quad (18)
\]

The solution of this quadratic is facilitated by the transformation:

\[
c = 1 + \alpha, \quad (19a)
\]

\[
c^2 = 1 + 2\alpha + \alpha^2, \quad (19b)
\]

\[
c^4 = 1 + 4\alpha + 6\alpha^2 + 4\alpha^3 + \alpha^4. \quad (19c)
\]

From equation (18) of Appendix E we know that the antiresonant frequency of a crystal differs from the series resonant frequency by about \( 1/2n \).

We are interested only in frequency departures less than this amount. Consequently, \( \alpha \) in (19) is established as a very small number, and we may make the approximation for (19c):

\[
c^4 = 1 + 4\alpha + 6\alpha^2. \quad (20)
\]
Substitution of (19) and (20) into (18) gives

\[ \alpha^2(4 - \frac{1}{n} + \frac{1}{Q_0^2}) - \frac{2\alpha}{n} \left[ 1 - \frac{1}{Q_0} \left( \frac{u}{n} + \frac{n}{Q_0} \right) \right] + \frac{1}{Q_0} \left( \frac{u}{n} + \frac{1}{Q_0} \right) = 0. \]  

(21)

As we are now, in effect, solving for a correction quantity, a sufficiently accurate solution will be obtained if (21) is simplified to

\[ 4\alpha^2 - \frac{2\alpha}{n} \left[ \frac{u}{n} + \frac{1}{Q_0} \right] = 0; \]

(22a)
or, multiplying by \( n/2 \) and introducing \( \lambda \) from (13),

\[ 2n\alpha^2 - \alpha + \frac{\lambda}{2n}(u + \lambda) = 0. \]

(22b)
The solution of this quadratic is

\[ \alpha = \frac{1}{4n} \left[ 1 + \sqrt{1 - 4\lambda(\lambda + u)} \right]. \]

(23)
The negative root evidently specifies the frequency nearer series resonance, which is the one of present interest.

There are three different conditions leading to approximate expressions for (23).

Case (a): \( u \ll \lambda \)

\[ \alpha = \frac{1}{4n} \left[ 1 - \sqrt{1 - 4\lambda^2} \right], \]

\[ \approx \frac{1}{4n} \left[ 1 - (1 - 2\lambda^2 + 2\lambda^4) \right], \text{ by App. J-4}, \]

\[ = \frac{\lambda^2}{2n} (1 - \lambda^2), \text{ by (13)}. \]

(24)

Case (b): \( u \approx \lambda \)

\[ \alpha = \frac{1}{4n} \left[ 1 - \sqrt{1 - 4\lambda(u + \lambda)} \right], \]

\[ \approx \frac{1}{4n} \left[ 1 - 1 + 2\lambda(u + \lambda) \right], \]

\[ = \frac{1}{2Q_0} \cdot (u + \lambda). \]

(25)

Case (c): \( u \gg \lambda \)

\[ \alpha = \frac{1}{4n} \left[ 1 - \sqrt{1 - 4\lambda u} \right], \]

\[ \approx \frac{1}{4n} \left[ 1 - (1 - 2\lambda u + 2\lambda^2 u^2) \right], \]

\[ = \frac{\lambda u}{2n} (1 - \lambda u) = \frac{u}{2Q_0} (1 - \lambda u). \]

(26)
In (b) \(4\lambda (u + \lambda)\) must be very small compared to 1, since the root of (23) was taken to only two terms of the binomial expansion. In (a) and (c) the quantities \(4\lambda^2\) and \(4\lambda u\), respectively, must be somewhat less than 1.

Equation (24) in conjunction with (19a) gives the frequency factor for small values of \(u = \tan \theta\) — that is, when the crystal operates close to its series resonance and is essentially a pure resistance. Equation (26) determines the frequency when \(\tan \theta\) is large compared with the ratio \(n/Q_0\) — that is, when the crystal impedance is definitely complex.

From (26) and (19) we have for case (c):

\[
\begin{align*}
  c &= 1 + \frac{u}{2Q_0} (1 - \lambda u) \\
  c^2 &= 1 + \frac{u}{Q_0} (1 - \lambda u), \text{ by Appendix J-4}
\end{align*}
\]

Substitution of these values into (17) yields

\[
R(wu) = \frac{R_0}{n} \cdot \frac{1}{\left[1 + \frac{1}{n} - 1 - \frac{u}{Q_0} (1 - \lambda u)\right] - \frac{u}{Q_0} \left(1 + \frac{u}{2Q_0}\right)}
\]

\[
= \frac{R_0}{n} \cdot \frac{1}{\frac{1}{n} - \frac{u}{Q_0} (2 - \lambda u)}, \quad \text{since } \lambda u \text{ is small;}
\]

finally,

\[
R(wu) = \frac{R_0}{1 - 2\lambda u} \cdot \left[\frac{u^{\gg \lambda}}{4\lambda u < 1}\right].
\] (28)

For case (b),

\[
R(wu) = \frac{R_0}{1 - \lambda (2u + \lambda)}, \quad \left[\frac{u \approx \lambda}{4\lambda (u + \lambda) \ll 1}\right].
\] (29)
For case (a),

\[ R(w_u) = \frac{R_0}{1 - \lambda^2(1 - \lambda^2)}, \quad \left[ u \ll \lambda \right] \]

This is seen to be identical with (14), which is consistent since at series resonance the condition \( u = 0 \) renders valid the approximations involved in (24).

It is thus apparent that in the vicinity of series resonance the equivalent series resistance of the crystal is essentially that shown in Fig. 1. Larger phase angles, corresponding to frequencies more removed from resonance, have the associated equivalent series resistances shown in Fig. 2 as functions of \( \lambda \).

In a typical case, \( n = 1000 \) and \( \omega_0 = 50,000 \), so that \( \lambda = 0.05 \). From Fig. 2 it is seen that the equivalent series resistance may be assumed equal to \( R_0 \) with an error of only 10% when the crystal is operating inductively or capacitively at a phase angle of as much as 45°, with \( n/\omega_0 = 0.05 \).

The equivalent series reactance is readily found from (15) to be

\[ X(w_u) = u \cdot R(w_u); \]

and the equivalent series impedance (3) becomes

\[
\begin{align*}
|Z(w_u)| &= R(w_u) \sqrt{1 + u^2} , \\
\end{align*}
\]

with \( \theta = \tan^{-1} u \).
Fig. 2 - Ratio of Equivalent Series Resistance to Actual Crystal Resistance, Versus $\frac{n}{Q_0}$ for Various Values of Equivalent Series Impedance Phase Angle.
BIBLIOGRAPHY


2. Articles.


