March 7, 1966

National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 1, Project A-318
"Application of Dimensional Analysis and Group Theory
to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the Period from February 1 to February 28, 1966

Gentlemen:

Activities were initiated on this project starting February 1, 1966.

During this reporting period a search and cataloging of the literature was begun. Approximately fifty references were found relating to this subject matter. Of these, one of the most useful was:


This contains a summary of many applications of group theory and a good bibliography.

Several members of the mathematics department at Georgia Tech were contacted and this project was discussed.

During the following month the literature survey will continue.

Sincerely,

L. J. Gallagher
Project Director
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 2, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20296
Covering the Period from March 1 to March 31, 1966

Gentlemen:

During the reporting period the searching and reading of the literature on group theory and differential equations was continued.

In addition a study was made of a class of partial differential equations that commonly arise in physical problems. These are the Euler-Lagrange type equations that are derivable from the variation of an action integral and occur in considerations of wave motion, heat flow and hydrodynamics.

This work will continue and a quarterly report will be prepared with next months report.

Respectfully submitted,

L. J. Gallagher
Project Director
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 3, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the Period from April 1 to April 30, 1966.

Gentlemen:

During the reporting period the study of the Euler-Lagrange type equations was continued. Application of Lie groups to the study of systems of ordinary differential equation of the Euler-Lagrange type can be found in the literature but it is not yet clear how these methods are extended to partial differential equations.

A quarterly report for the first quarter will follow under separate cover.

In the following month work will continue on Euler-Lagrange equations.

Respectfully submitted,

L.J. Gallagher
Project Director
National Aeronautics and Space Administration  
George C. Marshall Space Flight Center  
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 4, Project A-918  
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"  
Contract No. NASA-20286  
Covering the Period from May 1 to May 31, 1966.

Gentlemen:

During the reporting period the study of the Euler-Lagrange type equations was continued. In particular the properties of the Poisson bracket \{A,B\} for A and B general dynamic variables, was investigated. (See quarterly report for definitions and details on Poisson brackets and dynamic variables.) Whether the Poisson brackets of general dynamic variables forms a Lie algebra or not is still an open question.

Work will continue along these lines during the following month. Beginning June 13th, Miss M. J. Russell, a graduate student in mathematics at Emory University will be assisting on this project.

Respectfully submitted,

L. J. Gallahee  
Project Director

LJG:cm
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 5, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the Period from June 1 to June 30, 1966.

Gentlemen:

During the reporting period the study of Euler-Lagrange equations continued.

A proof was found to show that under a rather wide variety of circumstances the Poisson brackets of general dynamic variables does indeed form a Lie algebra. This question was left open in the quarterly report and can now be considered settled.

During the following month work will continue on Euler-Lagrange equations. In particular, an investigation into the circumstances under which partial differential equations can be reduced to ordinary differential equations will be investigated.

Respectfully submitted,

L. J. Gallaher
Project Director

LJC:cm
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 6, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the Period from July 1 to July 31, 1966

Gentlemen:

During the reporting period work was done on the application of group theory to the solution of ordinary and partial differential equations. In particular, the papers of Morgan* and Brand** were examined in some detail.

In regards to Brand's results, it has been possible to extend his work so as to include differential equations other than those invariant under the magnification group (the group associated with dimensional analysis). Thus, for any total differential equation invariant under a one parameter continuous group of transformations, an integrating factor can be given in terms of the functions characterizing the infinitesimal transformation. Also, if the equation is invariant under two distinct transformation groups, the ratio of the two associated integrating factors is an integral of the equation.

With respect to Morgan's theorems it was concluded that these can be applied to ordinary differential equations also, although he does not mention this.

It is not yet clear what the relation is between Morgan's theorems and those of Brand, nor is it yet clear how Brand's work can be extended to partial differential equations.
Investigation of Brand's and Morgan's theorems will continue during the following reporting period.

Respectfully submitted,

L. J. Gallaher
Project Director


National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 7, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the Period from August 1 to August 31, 1966

Gentlemen:

During the reporting period study continued on the relation between Morgan's theorem and the work of Brand. The application of Morgan's theorem to ordinary differential equations was studied. Writing of the final report was begun.

In the following reporting period the writing of the final report will continue.

Respectfully submitted,

L. J. Gallaher
Project Director

LJG:gd
October 8, 1966

National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 3, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20285
Covering the Period From September 1 to September 30, 1966.

Gentlemen:

During the reporting period, the study of Morgan's theorem was continued. Also the writing of the final report was continued. The following is a brief tentative outline of the technical subject matter in the final report.

I. Ordinary Differential Equations
   A. Group Theory and Ordinary Differential Equations
   B. Total Differential Equations
      1. Group Theory
      2. Dimension Analysis
      3. Examples
   C. Morgan's Theorem Applied to Ordinary Differential Equations

II. Partial Differential Equations
   A. Morgan's Theorem
   B. Examples

III. Euler-Lagrange Equations and the Relation to Lie Algebras and Groups

IV. Appendices
   A. Groups
   B. Lie Algebras and Groups
November 8, 1966

National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 9, Project A-918
"Application of Dimensional Analysis and
Group Theory to the Solution of Ordinary
and Partial Differential Equations"
Contract No. NASA-20286
Covering the Period from October 1 to
October 31, 1966.

Gentlemen:

During the period covered by this letter, progress
continued on the writing of the final report.

Work was also done on the application of the group
theoretic and dimensional analysis techniques to specific
problems.

Respectfully submitted,

L. J. Gallaher
Project Director

LJG:cm
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 10, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the Period From November 1 to November 30, 1966.

Gentlemen:

During the reporting period work continued on the preparation of the final report.

One item of interest came up concerning the extension of Brand's theorem on the application of dimensional analysis to total differential equations. It has been possible to show that systems of total differential equations can be treated either by dimensional analysis or the transformation group theory in a manner analogous to the treatment of single equations. The basic theorem is that if the system of $M$ total differential equations

$$
\sum_{\ell=1}^{n} p_{\ell}^{\beta} \frac{dx_{\ell}}{dx} = 0 \quad \beta = 1, 2, \ldots, M \\
(M < n)
$$

is invariant under the transformation

$$
U = \sum_{\ell=1}^{n} \alpha_{\ell}^{\gamma} \frac{\partial}{\partial x_{\ell}} \quad \gamma = 1, 2, \ldots, M
$$

then an integration factor is 

$$
\left((\alpha P)^{-1}\right)^{\gamma}_{\beta}
$$

where 

$$
\left((\alpha P)^{-1}\right)^{\gamma}_{\beta}
$$

is the $\gamma, \beta$ element of the inverse of the matrix product

$$
\sum_{\ell=1}^{n} \alpha_{\ell}^{\gamma} P_{\ell}^{\beta}, \text{ provided such an inverse exists.}$$
While this theorem may prove difficult to apply in practice it suggests the possibility of extending a similar theorem to partial differential equations by considering them as a continuously infinite system of ordinary differential equations.

During the following report period work will continue on the final report and a continuation of this contract will be submitted.

Respectfully submitted,

L. J. Gallagher
Project Director

LJG:cm
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 11, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NASA-20286
Covering the Period from January 17 to April 30, 1967

Gentlemen:

In previous work under this contract an investigation of Lie's theorems for the application of transformation groups to the solution of ordinary differential equations was carried out. Lie's results were shown to be applicable to total differential equations and systems of total differential. (For details see Final Report Contract NASA-20286, 17 January 1966 to 16 January 1967.)

Under the present contract an effort is being made to extend Lie's theorems to partial differential equations. This is being tried in two different ways. First, Lie's results can be used on the discrete approximation of a partial differential equation by treating the discrete approximations as a system of ordinary differential equations. If a solution can be found to the discrete system then the limit can be taken to obtain a solution to the original partial differential equation. This is a straightforward but rather cumbersome and difficult approach to a problem.

A second approach is to find a proof of a form of Lie's theorem directly for the partial differential equations and apply the group theoretic prescription directly without the discretizing and limiting process.

During the reporting period for this letter efforts have been made on both of the above approaches. An effort to apply Lie's theorem to a discretized version of the heat flow equation was tried and found to be both awkward and difficult. A proof of Lie's theorem for continuously infinite systems of equations has been given but its interpretation is still not understood nor is it yet clearly understood how this can be applied to solving partial differential equations.

In the following reporting period study will continue on both of these approaches.

Respectfully submitted,

Lawrence J. Gallagher
Project Director
June 6, 1967

National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 12, Project A-918
"Application of Dimensional Analysis and Group Theory to the
Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the period from 1 May to 1 June 1967

Dear Sir:

During this reporting period work continued on the extension of Lie's theorem to partial differential equations. As reported in last month's letter, a proof of Lie's theorem can be given for partial differential equations. The meaning and interpretation is now better understood and an application to the elementary form of the heat flow equation in one dimension has been made in such a way as to solve the initial value problem for this equation.

While this method appears to be quite powerful and general, it also has some drawbacks. Lie's theorem states that if the appropriate invariance group or groups can be found for a system of ordinary or partial differential equations, then an integration factor can be given and the original differential equation problem reduced to a problem in quadrature. The problems in using Lie's theorem are: first, finding the appropriate groups; second, constructing the integration factor; and third, carrying out the quadrature. In practical cases, each of these steps can be expected to prove quite difficult. Still it is encouraging that it can be done for the elementary heat flow equation and a search will continue for more interesting problems amenable to solution by the application of Lie's theorem.

Respectfully submitted,

L. J. Gallaher
Project Director

LJG: lg
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 13, Project A-918
"Application of Dimensional Analysis and Group Theory to the
Solution of Ordinary and Partial Differential Equations"
Contract No. NASA-20286
Covering the period from 1 June to 1 July 1967

Dear Sir:

During this reporting period work continued on the application of
Lie's theorem to partial differential equations. Many of the elementary
linear partial differential equations arising in theoretical physics;
such as, the wave equation, Laplace's equation, Schroedinger's equation,
the Klein-Gordon equation and others, can be solved in this manner. All
of these equations have been solved by other methods and the solutions
obtained by Lie's theorem are neither new nor different. But there does
appear to be emerging a wide class of equations for which this method is
effective. The class of problems solved so far are all linear partial
differential equations, but Lie's method is quite general and can be
applied to non-linear equations provided suitable transformation groups
can be found.

For the summer months, three graduate students in mathematics will be
working part time on this work. They are:

   Charles N. Driskell
   Robert H. Martin
   Mary J. Russell

During the following reporting period work will continue on the
application of Lie's theorems.

Respectfully submitted,

Lawrence J. Giffaber
Project Director

LJG/hh
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 14, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the period from 1 July to 1 August 1967

Gentlemen:

During this reporting period work was carried out on the mathematical foundations of Lie's theorem for partial differential equations. In particular it has been necessary to look into the foundations of the algebra of continuously infinite matrices. This in turn has required a study of distributions\(^1\) or generalized functions\(^2\). While the theory is somewhat involved, it does seem possible to give Lie's theorem for partial differential equations a firm mathematical foundation.

As for applications, it has been possible to apply Lie's theorem to the solution of a rather large group of linear partial differential equations. While there is no reason why the method should not be applicable to the nonlinear equations, so far we have not been able to solve any nonlinear partial differential equations by Lie's theorem.

During the following reporting period work will continue on the applications of Lie's theorem.

Respectfully submitted,

L. J. Gallagher
Project Director

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National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-SC

Subject: Monthly Progress Letter 15, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20296
Covering the Period from 1 August to 1 September 1967

Gentlemen:

During the reporting period, work continued on the investigation of the foundations of Lie's theorem as applied to partial differential equations.

A satisfactory theory of continuously infinite matrices has been worked out in such a way that most of the rules for manipulating finite matrices can be extended to the continuous matrices. It has also been possible to embed the discrete matrices within the set of the continuous matrices so that the discrete becomes a special case of the continuous. A satisfactory theory of infinite matrices is necessary for a rigorous derivation of Lie's theorem for partial differential equations.

As for applications of Lie's theorem, a large class of linear problems can be solved but so far no successful application to nonlinear problems has been achieved.

In a letter, Professor Thomas J. Higgins of the Department of Electrical Engineering at the University of Wisconsin pointed out to us that in 1957 he published a rather complete bibliography on the subject of dimensional analysis and similitude containing 596 references on the subject. The reference for this bibliography is:

During the following reporting period work will continue on the rigorous proof of Lie's theorem for partial differential equations and applications.

Sincerely yours,

L. J. Gallagher
Project Director

LJC/hh
National Aeronautics and Space Administration  
George C. Marshall Space Flight Center  
Huntsville, Alabama 35812  

Attention: PR-SC  

Subject: Monthly Progress Letter 16, Project A-918  
"Application of Dimensional Analysis and Group Theory to the  
Solution of Ordinary and Partial Differential Equations"  
Contract No. NAS8-20286  
Covering the period from 1 September to 1 October 1967  

Gentlemen:  

During this reporting period work continued on the mathematical foundations  
of Lie's theorem and applications.  

Included as a part of this monthly report is a draft of Chapter II  
of the coming final report on this project. This chapter covers the proof  
of Lie's theorem for the finite discrete system of equations and shows how  
the method can be applied to solve partial differential equations.  

In October work will continue on the preparation of the final report.  

Respectfully submitted,  

L. J. Gallaher  
Project Director
I. INTRODUCTION

Notation

The notation used here in connection with matrices is as follows. A doubly indexed quantity will be called a matrix. If \( A_{ij} \) are the elements of a matrix then the matrix is referred to as \( A \). Singly indexed quantities will be called vectors so that \( \mathbf{B}_i \) are the elements of the vector called \( \mathbf{B} \). The transpose of \( A \) and \( B \) will be denoted \( A^T \) and \( B^T \) respectively. The inverse of \( A \) is \( A^{-1} \) and its elements written as \( A^{-1}_{ij} \).

The summation convention will be used so that any repeated index is understood to be summed unless stated otherwise. For example, if \( A \) and \( C \) are matrices, the product \( AC \) will be written as \( A_{im}C_{mj} \). The index \( m \) is understood to be summed. The product \( ATC \) is written as \( A_{ih}C_{mj} \), etc. These sums run over the entire range for which the index is defined.

To shorten notation when partial derivatives are used, the comma notation will be used. That is \( \frac{\partial \phi}{\partial t} (x, t) \) and \( \frac{\partial \phi}{\partial x} (x, t) \) will be written as \( \phi_t \) and \( \phi_x \) respectively. If a quantity is a function of set of indexed variables, for example

\[ \psi(y_1, y_2, y_3, \ldots) = \psi(y), \]

its partial derivatives \( \frac{\partial \psi}{\partial y_k} \) will be written as \( \psi_k \). For example

\[ \frac{d}{dt} Q(y_1(t), y_2(t), \ldots, t) = Q_t + Q_1 \dot{y}_i \]

where \( \dot{y}_i = \frac{dy_i(t)}{dt} \).
II. LIE'S THEOREM FOR A FINITE SYSTEM
OF ORDINARY DIFFERENTIAL EQUATIONS

Lie developed the theory of one parameter continuous transformation groups for the purpose of studying ordinary differential equations (1). This technique has become a standard tool for the solution of first order ordinary differential equations and is derived and discussed in most textbooks on the subject (2,3). For some reason not too apparent, the extension of Lie's theorem to systems of first order differential equations seems to have been neglected. This extension is made in the first report on this contract (4) where Lie's theorem is proved for systems of equations and examples given. The proofs will be repeated here in this report in a slightly altered form, one that is more easily extended to partial differential equations, and more examples given. However, we will not repeat here many of the definitions and elementary concepts of group and transformation theory discussed in the first report but will refer the reader to this report (4) or to the standard text books on these subjects.

A. Lie's Theorem for Systems of Ordinary Differential Equations

Consider the system of M (total) differential equations in M + 1 variables

\[ P_{jk}(y_1, \ldots, y_M, t) \frac{dy_k}{dt} + Q_j(y_1, y_2, \ldots, y_M, t) dt = 0, \quad (II-1) \]

\[ j = 1, 2, \ldots, M. \] The summation convention for repeated indexes is used here.

Let \( \phi_j(y_1, y_2, \ldots, y_M, t) = c_i \) (constants) \( j = 1, 2, \ldots, M, \) be the family of solutions to II-1. That is

\[ \phi_{j,k} = \frac{\partial \phi_j}{\partial y_k} = \lambda_{ji} P_{1k} \quad \text{(II-2a)} \]
\[ \phi_{j,t} = \frac{\partial \phi_j}{\partial t} = \lambda_{j_1} Q_1 \quad (\text{II-2b}) \]

where the \( \lambda_{j_1} \) may be functions of the \( y \) and \( t \) but are independent of the index \( k \). \( \lambda \) is called the integration factor and is an \( M \times M \) matrix. Thus if an integration factor exists that satisfies II-2, each \( \phi_j \) must satisfy the partial differential equation

\[ \phi_{j,k}^{P^{-1}} k_{i} Q_1 - \phi_{j,t} = 0 \quad (\text{II-4}) \]

where \( P^{-1}_{ki} \) is the \( k,i \) element of the inverse of the matrix \( P \) provided \( P^{-1} \) exists.

Assume that the \( \phi_j = c_j \) are invariant as a family under the groups \( U_n \), \( n = 1, 2, \ldots, M \),

\[ U_n = \alpha_{nk} (y_1, y_2, \ldots, y_M, t) \frac{\partial}{\partial y_k} + p_n (y_1, y_2, \ldots, y_M, t) \frac{\partial}{\partial t} \]

that is

\[ U_n \phi_j = g_{nj} (\phi) \quad (\text{II-3}) \]

for \( n = 1, 2, \ldots, M \), and \( j = 1, 2, \ldots, M \), where the \( g_{nj} \) are some functions of the \( \phi \)'s. Introduce \( \xi_s \) defined as

\[ \xi_s = \int g_{is}^I (\phi) \, d\phi_i \quad (\text{II-5}) \]

so that \( \xi_s = C_s \) is identical with the family, \( \phi_j = c_j \). The notation here is that \( g_{is}^I \) is the \( i, s \) component of the inverse of the matrix \( g(\phi) \), assuming that this inverse exists. The right hand side of II-5 is meant to indicate a line integral in \( \phi \) space, i.e.

\[ \int g_{is}^I \, d\phi_i = \int_{R_1}^{R_2} \int_{R_3}^{R_4} \, (w, R_2, R_3, \ldots) \, dw \]
\[ + \int_{R_2} \phi_2 g_{2s} (\phi_1, w, R_3, R_4, \ldots) \, dw \]
\[ + \int_{R_3} \phi_3 g_{3s} (\phi_1, \phi_2, w, R_4, \ldots) \, dw \]
\[ + \text{etc,} \]

so that

\[ \frac{\partial \hat{s}_s}{\partial \psi_i} = g_{tis}. \]

Here the \( R_i \) are arbitrary constants.

Then

\[ u_m \psi_s = u_m \phi_1 \frac{\partial \hat{s}_s}{\partial \phi_i} \]
\[ = g_{mi} \hat{g}_{tis} \]
\[ = \delta_{ms} \] (II-6)

where \( \delta_{ms} = 1 \) if \( m = s \) or 0 if \( m \neq s \). It is also seen that the \( \psi_s \) obey the same partial differential equation II-4 as do the \( \phi_i \), that is

\[ \psi_s, p^k_{ki} Q_i - \psi_s, t = 0. \] (II-7)

Equation II-6 and II-7 can be combined to solve for \( \hat{s}_{s,k} \) and \( \hat{s}_{s,t} \) in terms \( P, Q, \alpha \) and \( \beta \) giving

\[ \hat{s}_{s,k} = \frac{\partial \hat{s}_s}{\partial y_k} - (P \alpha^T + Q \beta^T)_{si} P_{ik} \] (II-8a)
and

\[ \dot{s}_s,t = \frac{\partial \dot{s}_s}{\partial t} = (P \alpha^T + Q \beta^T)_{si}^T \dot{Q}_i \]  

(II-8b)

Here the notation \((P \alpha^T + Q \beta^T)_{si}^T\) refers to the \(s, i\) component of the inverse of the sum of the matrix products \(P\) with the transpose of \(\alpha\) and \(Q\) with the transpose of \(\beta\), provided this inverse exists. (Note that \(Q^T\) is a square matrix).

From equation II-7 it is seen that under the assumptions made, an integration factor or matrix exists of the form \((P \alpha^T + Q \beta^T)_{si}^T\), and matrix multiplication with equation II-1 gives a perfect differential in the sense that

\[ d\dot{s}_s = \frac{\partial \dot{s}_s}{\partial y_k} dy_k + \frac{\partial \dot{s}_s}{\partial t} dt = (P \alpha^T + Q \beta^T)_{ij} \cdot P_{jk} dy_k + (P \alpha^T + Q \beta^T)_{sj} Q_j dt. \]

The function \(\dot{s}\) can be found by a line integral in the \(y, t\) space along some convenient path, represented by

\[ \dot{s}_s = \int \dot{s}_s = \int (P \alpha^T + Q \beta^T)_{sj} P_{jk} dy_k + \int (P \alpha^T + Q \beta^T)_{sj} Q_j dt \]

\[ = K_s \quad s = 1, 2, \ldots, M \]  

(II-9)

where the \(K_s\) are constants. The equations \(\dot{s}_s(y_1, y_2, \ldots, t) = K_s\) represent then the general solution to the set of equations II-1.

A formal statement of the theorem used in this report for the solution of differential equations, which we will refer to as Lie's Theorem then is as follows:

If the differential equation \(P(y(t), t) \frac{dy}{dt} + Q(y(t), t) = 0\), where \(P\)

is a square matrix, \(y\) and \(Q\) vectors, and \(t\) a scalar, is invariant with respect to the set of transformations specified by

\[ U_n = \alpha_{ns}(y, t) \frac{\partial}{\partial y_s} + \beta_{ns}(y, t) \frac{\partial}{\partial t} \]

5
where $\alpha$ is a square matrix, $\beta$ a vector operator, and $\beta$ a vector, then provided that $P^T$ and $(P \alpha^T + \alpha Q^T)^T$ exist the general solution to the differential equation is

$$\int (P \alpha^T + \alpha Q)^T (P \beta + \alpha Q \beta) = K$$

where the integral is understood as a line integral in $y, t$ space along any convenient path, and $K$ is an arbitrary vector constant.

The paragraphs in this chapter leading up to a statement of this theorem can in fact be considered a proof of the theorem, but an alternate form of the proof will now be given.

The differential equation to be integrated,

$$P(y, t) \frac{dy}{dt} + Q(y, t) = 0 \quad (II-10)$$

can be written as

$$\frac{dy}{dt} + \bar{Q} = 0 \quad (II-10a)$$

where $\bar{Q} = P^T Q$, using the assumption that $P^T$ exists. If this equation (II-10a) is to be variant with respect to the transformation specified by

$$U_n = \alpha_{nk} \frac{\partial}{\partial y_k} + \beta_n \frac{\partial}{\partial t} \quad (II-11)$$

for all $n$ it must be invariant with respect to the infinitesimal transformations

$$y_k \rightarrow y_k + \epsilon \alpha_{nk} (y, t)$$

$$t \rightarrow t + \epsilon \beta_n (y, t) \quad (II-12)$$

$$\bar{Q}_k (y, t) \rightarrow \bar{Q}_k (y_j + \epsilon \alpha_{nj}, t + \epsilon \beta_n)$$

to first order in $\epsilon$ for all $k$ and $n$. Here $\epsilon$ is an infinitesimal parameter.
Making this transformation gives

\[ \frac{dy_k}{dt} + \bar{\alpha}_k + \varepsilon \left\{ (\alpha_{nk,m} + \bar{\alpha}_k \beta_{n,m}) \frac{dy_m}{dt} \right\} + \alpha_{nk} = 0 \]  

for all \( k \) and \( n \).

This equation will invariant up to first order in \( \varepsilon \) if and only if

\[ (\alpha_{nk,m} + \bar{\alpha}_k \beta_{n,m}) \bar{\alpha}_m = \alpha_{nk} \frac{dt}{t} + \alpha_{nm} \beta_{k,m} + \bar{\alpha}_k \beta_n + \beta_{n,t} \]

or

\[ (\alpha_{nk} + \bar{\alpha}_k \beta_n) \frac{dt}{t} + \beta_{k,m} (\alpha_{nm} + \bar{\alpha}_m \beta_n) - (\alpha_{nk} + \bar{\alpha}_k \beta_n) \frac{dt}{m} \bar{\alpha}_m = 0. \]  

Let the matrix \( \alpha^T + \bar{\alpha}^T \) be designated by \( A \) such that

\[ A_{kn} = \alpha_{nk} + \bar{\alpha}_k \beta_n, \]

and let \( A^{-1}_{s,j} \) be the \( s, j \) component of the inverse of \( A \). Then multiplying II-14 by \( A^T \) on the right gives

\[ A_{kn,t} A^T_{ns} + \bar{\alpha}_k A^T_{s,n} - A_{kn,m} \bar{\alpha}_m A^T_{ns} = 0 \]

or

\[ A_{kn} \begin{bmatrix} \frac{\partial}{\partial t} - \bar{\alpha}_m \frac{\partial}{\partial y_m} \end{bmatrix} A^T_{ns} = \bar{\alpha}_k A^T_{s,n}. \]  

Now \( A^T \) is a function of \( t \) explicitly and it is implicitly a function of \( t \) through \( y_m \). Since
- \frac{\ddot{Q}_m}{\dot{y}_m(t)} = \frac{d y_m(t)}{dt},

the operator in square brackets in II-15 is the total derivative with respect to t, i.e.

\left[ \frac{\partial}{\partial t} + \frac{d y_m(t)}{dt} \frac{\partial}{\partial y_m} \right] A^I(y(t), t) = \frac{d A^I}{dt}(y(t), t)

Equation II-15 then is

A_{kn} \frac{d}{dt} A_{ns} = \ddot{Q}_{k,s} \quad \text{(II-15a)}

The left side of this equation is the right Volterra derivative* of $A^I$.

It will now be shown that every solution to II-15 (or II-15a) is an integrating factor of II-10a.

Let $\lambda_{km}$ be an integrating factor of II-10a for each k; that is

\[
\frac{d \phi_k}{\dot{y}_m} = \lambda_{km} \frac{d y_m}{dt} + \lambda_{km} \ddot{Q}_m \frac{d t}{dt}
\]

where $\frac{d \phi_k}{\dot{y}_m}$ is a perfect differential for each k.

Then

\[
\lambda_{km} = \phi_{k,m}
\]

and

\[
\lambda_{km} \ddot{Q}_m = \phi_{k,t}.
\]

Thus $\phi_k$ is a solution of the partial differential equation

\[
\phi_{k,t} - \ddot{Q}_m \phi_{k,m} = 0. \quad \text{(II-16)}
\]

Differentiating (II-16) with respect to $y_s$ gives the partial differential equation for $\lambda$,

$$\lambda_{ks,t} - \delta^m_n \lambda_{ks,m} = \lambda_{km} \delta^m_n$$

or

$$\lambda^I_{kn} \left[ \frac{\partial}{\partial t} - \delta^m_n \frac{\partial}{\partial y_m} \right] \lambda_{ns} = \delta^s_k$$

Again the operator in square brackets is the total derivative with respect to $t$, so that $\lambda$ satisfies the equation

$$\lambda^I_{kn} \frac{d}{dt} \lambda_{ns} = \delta^s_k.$$  \hspace{1cm} (II-17)

Now if $C$ is an invertible constant matrix then

$$\lambda^I_{kn} C^T_{np} \frac{d}{dt} (C_{pq} \lambda_{qs}) = \delta^s_k,$$

and

$$C_{pq} \lambda_{qs} = (C_{pq} \phi_q)'_s,$$

so that $C\lambda$ is both an integrating factor and a solution to II-17 for every invertible constant $C$.

Suppose two invertible matrices $R(t)$ and $S(t)$ have equal Volterra derivatives, that is

$$R^I \frac{d}{dt} R = S^I \frac{d}{dt} S.$$
Then
\[ R^I \frac{d}{dt} R + \left( \frac{d}{dt} S^I \right) S = 0 \]

provided \( \frac{d}{dt} S^I \) exists (See Appendix ). Multiplying on the left by \( R \) and on the right by \( \frac{d}{dt} S^I \) gives
\[ \left( \frac{dR}{dt} \right) S^I + \frac{dR}{dt} S^I = 0 \]
or
\[ \frac{d}{dt} \left( R S^I \right) = 0. \]

Then
\[ R S^I = C \]
where \( C \) is some invertible constant matrix. That is, if two matrices have the same Volterra derivative, they are proportional to each other through some invertible constant matrix.

In this way it is shown that every solution of II-17 (or II-15a) is proportional to every other solution through some invertible constant matrix.

Thus if some solution is an integrating factor, every solution is an integrating factor. The matrix \( A^I = (\alpha^T + Q^T)^I \) is then an integrating factor of the matrix equation
\[ \frac{dy(t)}{dt} + \bar{Q}(y, t) = 0. \]

It is also clear that since \( \bar{Q} = P^I Q \)
\[ A^I = (P \alpha^T + Q^T)^I P \]
and that \( (P \alpha^T + Q^T)^I \) is an integrating factor of
\[ P_{km} \frac{dy}{dt} + Q_k = 0. \]
Then since

\[ d\phi_m = \frac{\partial \phi_m}{\partial y_k} dy_k + \frac{\partial \phi_m}{\partial t} dt \]

\[ = (P \alpha^T + Q \beta^T)^I_{mj} (P_{jk} dy_k + Q_j dt), \tag{II-18} \]

the line integral in \( y, t \) space

\[ \phi_m = \int \left( (P \alpha^T + Q \beta^T)^{-1} \right)_{mj} (P_{jk} dy_k + Q_j dt) \]

\[ = K_m \tag{II-19} \]

is a solution to the system of differential equations II-10 for each constant vector \( K \). That is, the \( K_m \) are the constants of integration.

This completes the proof of Lie's theorem for systems of differential equations, the basic theorem on which the methods and results of this report are based.

We note at this point that the Lie's theorem is proved here for a finite system of equations. The extension to countably infinite systems depends on an adequate theory of countably infinite matrices. The proof would be unchanged for a system of countably infinite matrices that form an algebra, that is, a system which is closed under addition and multiplication.* The extension to continuously infinite matrices which forms the basis of the application to partial differential equations will be discussed in a later chapter.

* C.C. MacDuffee (op. cit.,) page 106.
B. EXAMPLES

1. The One-dimensional Heat Flow Equation

As an example of the use of Lie's theorem to solve systems of ordinary differential equations the method will be applied to the discrete form of the one-dimensional heat flow equation.

Consider the partial differential equation

\[ \frac{\partial y(x,t)}{\partial t} - \frac{\partial^2 y(x,t)}{\partial x^2} = 0 \]  \hspace{1cm} (II-20)

with initial conditions

\[ y(x,0) = y^0(x) \]

and the periodic boundary conditions

\[ y(x+2L,t) = y(x,t). \]

Using the lowest order difference approximation for the derivative with respect to \( x \) gives the system of equations

\[ \frac{d}{dt} y_n(t) - \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = 0 \quad n = 0, \pm 1, \pm 2, \cdots N \]  \hspace{1cm} (II-4)

Here \( y_n(t) = y(nh,t) \) where \( h \) is the discretization interval \( (Nh = L) \). There are only \( 2N \) independent equations since \( y_n(t) = y_{n+2N}(t) \) by virtue of the periodic boundary conditions. The initial conditions are

\[ y_n(0) = y^0_n. \]
Considered as a system of coupled differential equations, II-21 is of the form

\[ P_{nk} \frac{\partial}{\partial t} y_k + Q_n(y, t) = 0 \]

where

\[ P_{nk} = \delta_{nk} \]

and

\[ Q_n(y, t) = -\frac{(y_{n+1} - 2y_n + y_{n-1})}{h^2}. \]

Equation II-21 is invariant with respect to the transformation

\[ y_n(t) \rightarrow y_n(t) + \epsilon y_{n-j}(t) \]

for \( n = 0, 1, \ldots, N \) and \( j = 0, 1, 2, \ldots, N \).

The operators characterizing the set of transformations are

\[ U_j = y_s - \frac{\partial}{\partial y_s} \frac{\partial}{\partial y_s} \quad j = 0, 1, \ldots, N. \]

This is of the form

\[ U_j = \alpha_j \frac{\partial}{\partial y_s} + \beta_j \frac{\partial}{\partial t} \]

with

\[ \alpha_j = y_{s-j} \text{ and } \beta_j = 0. \]

\( \alpha \) is a square \( 2N \) by \( 2N \) matrix.
Since \( P \) is the identity matrix, an integrating factor for (II-21) is the inverse of the transpose of \( \alpha \), i.e. \( \alpha^{TI} \).

The matrix \( \alpha \), whose elements are \( y_{s,j'} \), is called a circulant matrix. Much is known about circulants.* The circulants of a given order form an algebra, and the inverse of a circulant is a circulant, if it exists. It is straightforward to show that if \( q_{k-m} \) are the elements of the inverse of the matrix whose elements are \( y_{j-s} \) (i.e. \( \alpha_{js}^T = \alpha_{sj} \)),

\[
q_{m-k} y_{j-m} = \delta_{kj},
\]

then

\[
q_k = \frac{1}{2N} \frac{e^{-\pi i ks/n}}{e^{\pi i js/n}} y_j
\]

(Note here that the summation convention is used on all repeated indices, the sums running from \(-N+1\) to \(N\), and \( q_k \) is periodic with period \(2N\).) This inverse can be obtained in a variety of ways. It can be obtained from the theory of finite Fourier expansions. Also, by showing that powers of the \( N \)th roots of unity form a unitary matrix that diagonalizes every \( N \) by \( N \) circulant, this inverse can be obtained from the reciprocal of the eigenvalues of \( y_{m-j} \).

With this inverse then the integration factor is

---

\[ \alpha_{km} = q_{k-m} = \frac{1}{2N} \frac{e^{-\pi i (k-m)s/N}}{e^{\pi i sj/N} y_j}. \]

There exist then a perfect differential \( \delta \)

\[ \frac{d\delta}{\delta m} = \delta_m, p \frac{dy}{p} + \delta_m, t \frac{dt}{t} \]

such that

\[ \frac{\delta \delta_m}{\delta y_k} = \alpha_{km} \]

\[ \frac{\delta \delta_m}{\delta t} = \alpha_{km} q_k \]

\[ = - \alpha_{km} (y_{k+1} - 2y_k + y_{k-1})/h^2 \]

\[ = - (\delta_{m+1} - 2\delta_m + \delta_{m-1})/h^2 . \]

The functions \( \delta_m(y_{-N+1}, y_{-N+2}, \ldots, y_0, \ldots, y_N, t) \)

can then be calculated by integrating \( \delta_m \) along some convenient path in the \( 2N + 1 \) dimensional space of the \( y \)'s and \( t \). A convenient path of integration is as follows:

1) \( y_k = \delta_{k0} \) along \( t \) from \( t = 0 \) to \( t \)

2) \( t = t, y_k = 0 \) (\( k \neq 0 \)) along \( y_o \) from \( y_o = 1 \) to \( y_o(t) \)

3) \( t = t, y_k = 0(k \neq 0, l) y_o = y_o(t), \) along \( y_l \) from \( y_l = 0 \) to \( y_l(t) \)
4) \( t = t, \ y_k = 0(k \neq 0, 1, -1) \ y_0 = y_0(t), \ y_1 = y_1(t), \) along \( y_1 \) from \( y_1 = 0 \) to \( y_1(t) \)

etc.

Written out with the summation signs, this is

\[
- \int_0^t dt \ (s_{ml} - 2s_{m0} + s_{m-1})/h^2
\]

\[+ \frac{1}{2N} \sum_{N < j < N} \int_1^{y_0} dw \ e^{-\pi j (0-m)s/N} \sum_{-1 < j < 0} e^{\pi j s/N} y_j + e^{\pi 10s/N} w\]

\[+ \int_1^{y_1} dw \ e^{-\pi j (1-m)s/N} \sum_{-1 < j < 0} e^{\pi j s/N} y_j + e^{\pi 1s/N} w\]

\[+ \int_0^{y_{-1}} dw \ e^{-\pi j (-1-m)s/N} \sum_{-2 < j < 1} e^{\pi j s/N} y_j + e^{\pi (-1)s/N} w\]

\[+ \cdots , \]

\[+ \int_0^{y_N} dw \ e^{-\pi j (N-m)s/N} \sum_{-N < j < N} e^{\pi j s/N} y_j + e^{\pi Ns/N} w\]

This particular path of integration gives

\[\phi_m(y, t) = \frac{1}{2N} \ln(e^{\pi j s/N} y_j) - \frac{t}{h^2} (s_{ml} - 2s_{m0} + s_{m-1}).\]
That this is in fact $\tilde{\phi}_m(y,t)$ can be readily verified by calculating that

$$\tilde{\phi}_{m,k} = \Omega_0^{k_l}$$

and

$$\tilde{\phi}_{m,t} = -\left(\delta_{m,1} - 2\delta_{m,0} + \delta_{m,-1}\right)/\hbar^2.$$

A completely general solution to equation II-21 is given by $\tilde{\phi}_m = K_m$ where the $K_m$ are arbitrary constants.

While this solution does not look particularly useful, it can be solved explicitly for the $y_j$ by introducing new integration constants $M_s$ where

$$\ln M_s = e^{\pi i m s/N} K_m.$$

Then taking (discrete) Fourier transforms of both sides of

$$e^{-\pi i m s/N} \left( \ln e^{\pi i j /N} y_j \right) - \frac{t}{\hbar^2} \left(\delta_{m,1} - 2\delta_{m,0} + \delta_{m,-1}\right) = K_m$$

gives

$$\ln(e^{\pi i j /N} y_j) + \hbar t \sin^2(\pi s / 2N)/\hbar^2 = \ln M_s$$

or

$$y_j(t) = e^{\pi i j s/N - \hbar t \sin^2(\pi s / 2N)/\hbar^2} M_s.$$

The constants $M_s$ can then be related to initial conditions by noting that at $t = 0$

$$y_j(0) = \frac{1}{2N} e^{-\pi i j s/N} M_s.$$
or

\[ M_s = e^{\frac{n\pi m s}{N}} y_m^o. \]

In terms of the initial conditions then the solution to \( II - 21 \) is

\[ y_n(t) = \frac{e^{-n\pi (n-m)s/N}}{2N} e^{-4t\sin^2(\pi s/2N)/h^2} y_m^o. \]

(Note the sum over both the repeated indices \( m \) and \( s \).)

While this solution to the one-dimensional heat flow equation may not look familiar, by passing to the limits \( h \to 0, N \to \infty \) with \( Nh = L \) it can be seen that this is the usual solution for the initial value problem.

Introducing the notation

\[ x_n = nh, \]
\[ x_m' = mh, \]
\[ \Delta x' = h, \]

and writing out the summation signs explicitly

\[ y(x,t) = \lim_{N \to \infty} \sum_{N < m < N} \sum_{-N < s < N} \frac{e^{-2\pi i (x_m - x_m') s/2L}}{2Nh} e^{-4t\sin^2(\pi s/2N)/h^2} y_m^o. \]

\[ = \int_{-L}^{+L} \frac{dx'}{2L} \sum_{-\infty < s < \infty} e^{-2\pi i (x-x') s/2L} e^{-4t(\pi s/2L)^2} y(x',0) \]

If the limit \( L \to \infty \) is now taken one obtains the solution to the heat flow equation valid over the entire real axis for the initial value problem.

Introducing the notation
\[ p_s \equiv s/2L \quad , \quad \Delta p = 1/2L, \]
gives

\[
\lim_{L \to \infty} \int_{-L}^{+L} dx \sum_{\Delta p} e^{-2\pi i (x-x')} p_s - \frac{4t(\pi p_s)^2}{2} \cdot y(x',0)
\]
\[
\Delta p \to 0
\]

\[
= \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dp \cdot e^{-2\pi i (x-x')p} - (2\pi p)^2 \cdot t \cdot y(x',0)
\]

This is the usual Fourier transform solution for the infinite interval.

The integration over \( p \) can be carried out and gives

\[
y(x,t) = \int_{-\infty}^{+\infty} dx' \cdot e^{-\frac{(x-x')^2}{4t}} \cdot \frac{y(x',0)}{2\sqrt{\pi t}}
\]

This is the standard solution to the initial value problem for the infinite interval.
1.1 Discussion

The above technique for using Lie's theorem to solve the heat flow equation is quite complicated and gives well known solutions that are much more easily obtained in other ways. It is used here only to illustrate this method. The equation II-20 is linear but Lie's theorem can be applied to the non-linear problems. The technique of discretization of the partial differential equation, followed by the application of Lie's theorem to a finite system of ordinary differential equations, followed in turn by taking the limit back to the continuous system, can be applied to other partial differential equations but is an extremely awkward way of proceeding. A more desirable method would be to obtain a form of Lie's theorem applicable directly to partial differential equation without introducing the discrete approximation. This subject will be taken up in later chapters of this report.

2. The Wave Equation

The second example given here will be the application to a second order partial differential equation, the wave equation

\[
\frac{\partial^2}{\partial t^2} y(x,t) - \frac{\partial^2}{\partial x^2} y(x,t) = 0 \quad (\text{II-25})
\]

with initial values \(y(x,0) = y^0(x)\) and \(\frac{\partial}{\partial t} y(x,t) \big|_{t=0} = \dot{y}^0(x)\).

Since Lie's theorem is applicable to first order differential equations it will be necessary to reduce II-25 to a pair of first order differential equations. This can be done by introducing two new variables \(\dot{y}\) and \(y'\) defined as \(\dot{y} = \frac{\partial y}{\partial t}(x,t)\), \(y' = \frac{\partial y}{\partial x}(x,t)\).
Then the single 2nd order partial differential equation II-25 can be written as a pair of first order coupled differential equations

\[
\frac{\partial}{\partial t} \dot{y} - \frac{\partial}{\partial x} y' = 0 \quad (\text{II-26a})
\]

\[
\frac{\partial}{\partial t} y' - \frac{\partial}{\partial x} \dot{y} = 0. \quad (\text{II-26b})
\]

This pair of coupled equations can be reduced to a pair of uncoupled equations defining

\[
u = \dot{y} + y'
\]

\[
v = \dot{y} - y'
\]

which satisfy the equations

\[
\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 \quad (\text{II-27a})
\]

\[
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0 \quad (\text{II-27b})
\]

The solution of II-25 is related to the solution of II-27 by

\[
y(x,t) = y(x,0) + \frac{1}{2} \int_0^t [u(x,\xi) + v(x,\xi)] \, d\xi.
\]

Equations II-27 represent a pair of uncoupled first order partial differential equations equivalent to II-25. These can be solved separately and the equation for \( u \) only will be solved here since that for \( v \) can be obtained by reversing the sign of \( t \) in the solution for \( u \).

The discrete version of II-27a is

\[
\frac{d}{dt} u_n - \frac{u_{n+1} - u_{n-1}}{2h} = 0 \quad (\text{II-28})
\]
If the periodic boundary conditions \( u_n = u_{n+2N} \) are used, this equation is invariant with respect to the set of infinitesimal transformations

\[
    u_n \rightarrow u_n + cu_{n+k}
\]

\[
    k = 0, \pm 1, \pm 2, \ldots, \pm N
\]

characterized by the operator

\[
    U_k = u_{k+n} \frac{\partial}{\partial y_n}
\]

Equation II-28 is in the form

\[
    P_n \frac{du_k}{dt} + Q_n = 0
\]

where

\[
    P_{nk} = \delta_{nk}
\]

\[
    Q_n = - \frac{(u_{n+1} - u_{n-1})}{2h}
\]

and \( U \) is in the form

\[
    U_k = \alpha_{kn} \frac{\partial}{\partial u_n} + \beta_k \frac{\partial}{\partial t}
\]

where \( \alpha_{kn} = u_{k+n} \), \( \beta_k = 0 \).

The integrating factor \((Pa^T + Qb^T)^I\)
then is just \(\alpha^{TI}\). Since \(\alpha = \alpha^T\) and \(\alpha^T = \alpha^{TI}\), the \(T\) superscript will be dropped from \(\alpha\).

The matrix \(\alpha\) whose elements are \(u_{k+n}\) is called an anticirculant matrix.

The inverse of an anticirculant is an anticirculant if it exists. It is
straightforward to show that if \( q_{j+k} \) are the elements of the inverse of the matrix whose elements are \( u_{k+n} \) so that

\[
q_{j+k} u_{k+n} = \delta_{jn},
\]

then \( q_k = \frac{1}{2N} e^{\frac{\pi i k s}{N}} \) \( u_j \), and \( q_{km} = \frac{1}{2N} e^{\frac{\pi i (k+m)s}{N}} u_j \).

(Note the summation over both repeated indexes \( s \) and \( j \), and that \( q \) is periodic with period \( 2N \).)

There exists then a function

\[
\hat{\varphi}_m (u_{-N+1}, \ldots, u_{-1}, u_0, u_1, \ldots, u_N, t)
\]

such that

\[
\frac{\partial \hat{\varphi}_m}{\partial u_k} = q_{k+m}
\]

and

\[
\frac{\partial \hat{\varphi}_m}{\partial t} = -q_{m+n} \frac{(u_{n+1} - u_{n-1})}{2h}
\]

\( = -(\delta_{ml} - \delta_{m-1})/2h. \)

\( \hat{\varphi}_m \) can be obtained from a line integral in the \( 2N + 1 \) dimensional space of \( u \) and \( t \). A convenient path of integration is the same used above for the heat flow problem, (page ). This can be written
\[
\sum_{0 \leq k < N} \int_0^{y_k} \delta_{ko} dw \left\{ \frac{1}{2N} \sum_{-N \leq s \leq N} \frac{e^{-\pi (k-s)/N}}{\Sigma e^{\pi js/N} y_j + e^{\pi ks/N} w} \right\} \\
+ \sum_{-N < k \leq -1} \int_{y_k}^0 \delta_{ko} dw \left\{ \frac{1}{2N} \sum_{-N \leq s \leq N} \frac{e^{-\pi (k-s)/N}}{\Sigma e^{\pi js/N} y_j + e^{\pi ks/N} w} \right\} \\
+ \int_{0}^{t} dt \frac{\delta_{ml} - \delta_{n-1}}{2h}.
\]

Integrating along this path gives

\[
\hat{\phi}_m (u,t) = \frac{e^{\pi ms/N}}{2N} \ln(e^{\pi js/N} u_j) - t(\delta_{ml} - \delta_{n-1})/2h.
\]

It is readily verified that this \( \hat{\phi}_m \) gives the correct \( \partial \hat{\phi}_m / \partial u_k \) and \( \partial \hat{\phi}_m / \partial t \). The relations

\[
\hat{\phi}_m (u,t) = K_m
\]

give the general solutions to equations II-26 where the \( K_m \) are the 2N arbitrary constants. Introducing new constants \( M_s \) such that

\[
\ln M_s = e^{-\pi ms/N} K_m
\]

one can solve for \( u_j \) as

\[
u_j(t) = e^{-\pi js/N + t \sin(\pi s/N)/h} M_s
\]

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The $M_s$ are related to the initial values of $u(x,t) = \frac{\partial}{\partial t} y(x,t) + \frac{\partial}{\partial x} y(x,t)$ by $M_s = e^{\frac{\pi is}{N}} u_0^0$ where $u_0^0 = u_j(0)$ is the value of $u$ at $t = 0$. The $M_s$ are the discrete Fourier transforms of the initial $u_j$. In terms of the $u_j^0$ then

$$u_j(t) = e^{\frac{-\pi (j-k)s}{N}} e^{-t \sin(\pi s/N)/h} u_k^0.$$  

Taking the limit as $h \to 0$, $N \to \infty$, with $hN = L$ gives the continuous solution to II-28 for periodic boundary conditions with period $2L$. Introducing the notation

$$x_j = jh$$
$$x_k' = kh$$
$$\Delta x' = h$$

and writing in the summations explicitly we have

$$u(x,t) = \lim_{h \to 0} \sum_{N=-\infty}^{N=\infty} \sum_{-N<s<N} e^{-\pi i (x_k-x_k')s/hN} e^{-t \sin(\pi hs/hN)/h} u_k^0$$

Taking the limit as $L \to \infty$ with

$$p = s/2L, \ \Delta p = 1/2L,$$

gives

$$u(x,t) = \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dp \ e^{2\pi i (x' - x + t)p} u_0^0(x')$$

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\[\begin{align*}
&= \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dp \quad e^{2\pi i x' p} u_0(x'+x-t) \\
&= u_0^*(x-t).
\end{align*}\]

The solution for \(v(s,t)\) is similar to the solution for \(u\) except with the sign of \(t\) reversed and can be worked out to give

\[v(x,t) = v_0^*(x+t).\]

Since

\[u(x,t) = \dot{y}(x,t) + y'(x,t)\]

\[v(x,t) = \dot{y}(x,t) - y'(x,t)\]

and

\[y(x,t) = y(x,0) + \int_0^t \frac{1}{2} \left[ u(x,T) + v(x,T) \right] \, dT,\]

we have

\[y(x,t) = y(x,0) + \int_0^t \frac{1}{2} \left[ \dot{y}(x+T,0) + \dot{y}(x-T,0) \right] \, dT + y'(x+T,0) - y'(x-T,0) \, dT\]

\[= \frac{1}{2} \left\{ y(x+t,0) + y(x-t,0) + \int_{x-t}^{x+t} \dot{y}(T,0) \, dT \right\}.\]

This is the usual form of d' Alambert's solution to the wave equation.

Again this is a long and involved way of finding a well known solution that is much more easily obtained by other methods. The purpose here is to illustrate the method and principles involved in applying Lie's theorem.
In treating second order differential equations by Lie's method it is necessary to first reduce the problem to a pair of first order equations. If these two equations can then be uncoupled as was done above the mechanics of the solution become much simpler, but the method still applies even if the equations remain coupled.
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National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-SC

Subject: Monthly Progress Letter 17, Project A-918
"Application of Dimensional Analysis and Group Theory to the
Solution of Ordinary and Partial Differential Equations"
Contact No. NAS8-20286
Covering the period from 1 October to 1 November 1967

Gentlemen:

During this reporting period work continued on the preparation of
the final report.

Last month's report contained a preliminary version of Chapter II
of the final report. Included as a part of this month's report is a
revised version of this chapter in which a large number of errors
have been corrected. Also the proof of Lie's Theorem is somewhat
simplified in this version.

In the following month work will continue on the final report.
Draft versions of those chapters completed will be included as a part
of future monthly letters.

Respectfully submitted,

[Signature]
L. J. Gallagher
Project Director

LJG/hh
I. INTRODUCTION

Notation

The notation used here in connection with matrices is as follows. A doubly indexed quantity will be called a matrix. If \( A_{ij} \) are the elements of a matrix then the matrix is referred to as \( A \). Singly indexed quantities will be called vectors so that \( B_i \) are the elements of the vector called \( B \).

The transpose of \( A \) and \( B \) will be denoted \( A^T \) and \( B^T \) respectively. The inverse of \( A \) is \( A^{-1} \) and its elements written as \( A_{ij}^{-1} \).

The summation convention will be used so that any repeated index is understood to be summed unless stated otherwise. For example, if \( A \) and \( C \) are matrices, the product \( AC \) will be written as \( A_{im}C_{mj} \). The index \( m \) is understood to be summed. The product \( A^T C \) is written as \( A_{mi}C_{mj} \), etc. These sums run over the entire range for which the index is defined.

To shorten notation when partial derivatives are used, the comma notation will be used. That is, \( \frac{\partial \phi}{\partial t} (x, t) \) and \( \frac{\partial \phi}{\partial x} (x, t) \) will be written as \( \phi_t \) and \( \phi_x \) respectively. If a quantity is a function of a set of indexed variables, for example

\[
y(\nu_1, \nu_2, \nu_3, \ldots) = \nu(y), \text{ then}
\]

its partial derivatives \( \frac{\partial \nu}{\partial \nu_k} \) will be written as \( \nu_{,k} \). For example

\[
\frac{d}{dt} Q(\nu_1(t), \nu_2(t), \ldots, t) = Q_t + Q_{,1} \dot{y}_1
\]

where \( \dot{y}_1 = \frac{dy_1(t)}{dt} \).
II. LIE'S THEOREM FOR A FINITE SYSTEM
OF ORDINARY DIFFERENTIAL EQUATIONS

Lie developed the theory of one parameter continuous transformation
groups for the purpose of studying ordinary differential equations (1).
This technique has become a standard tool for the solution of first order
ordinary differential equations and is derived and discussed in most text
books on the subject (2,3). For some reason not too apparent, the extension
of Lie's theorem to systems of first order differential equations seems to
have been neglected. This extension is made in the first report on this
contract (4) where Lie's theorem is proved for systems of equations and
examples given. The proofs will be repeated here in this report in a slightly
altered form, one that is more easily extended to partial differential equa-
tions, and more examples given. However, we will not repeat here many of
the definitions and elementary concepts of group and transformation theory
discussed in the first report but will refer the reader to this report (4)
or to the standard text books on these subjects.

A. Lie's Theorem for Systems of Ordinary Differential Equations

Consider the system of M (total) differential equations in M + 1 variables

\[ P_{jk}(y_1, \ldots, y_M, t) \frac{dy_k}{dt} + Q_{j}(y_1, y_2, \ldots, y_M, t)dt = 0, \quad (II-1) \]

\( j = 1, 2, \ldots, M. \) The summation convention for repeated indexes is used here.
Let \( \phi_j(y_1, y_2, \ldots, y_M, t) = c_i \) (constants) \( j = 1, 2, \ldots, M, \) be the family
of solutions to II-1. That is

\[ \phi_{j,k} = \frac{\partial \phi_j}{\partial y_k} = \lambda_j P_{jk} \quad (II-2a) \]
\[ \phi_j, t = \frac{\partial \phi_j}{\partial t} \cdot \lambda_{j1} Q_1 \]  

(II-2b)

where the \( \lambda_{j1} \) may be functions of the \( y \) and \( t \) but are independent of the index \( k \). \( \lambda \) is called the integration factor and is an \( M \) by \( M \) matrix. Thus if an integration factor exists that satisfies II-2, each \( \phi_j \) must satisfy the partial differential equation

\[ \phi_j, k P^I_{k1} Q_1 - \phi_j', t = 0 \]  

(II-4)

where \( P^I_{k1} \) is the \( k,i \) element of the inverse of the matrix \( P \) provided \( P^I \) exists.

Assume that the \( \phi_j = c_j \) are invariant as a family under the groups \( U_n \), \( n = 1, 2, \ldots, M, \)

\[ U_n = \alpha_{nk} (y_1, y_2, \ldots, y_M, t) \frac{\partial}{\partial y_k} + \varepsilon_n (y_1, y_2, \ldots, y_M, t) \frac{\partial}{\partial t} \]

that is

\[ U_n \phi_j = g_{nj} (\phi) \]  

(II-3)

for \( n = 1, 2, \ldots, M \), and \( j = 1, 2, \ldots, M \), where the \( g_{nj} \) are some functions of the \( \phi \)'s. Introduce \( \delta_s \) defined as

\[ \delta_s = \int g^I_{1s} (\phi) \ d\phi_1 \]  

(II-5)

so that \( \delta_s = C_s \) is identical with the family, \( \phi_j = c_j \). The notation here is that \( g^I_{1s} \) is the \( i, s \) component of the inverse of the matrix \( g(\phi) \), assuming that this inverse exists. The right hand side of II-5 is meant to indicate a line integral in \( \phi \) space, i.e.

\[ \int g^I_{1s} \ d\phi_1 = \]

\[ \int_{R_1}^{R_1} g^I_{1s} (w, R_2, R_3, \ldots) \ dw \]
\[ + \int_{R_2} \phi_2 \left. g_{2s} (\phi_1, w, R_3, R_4, \ldots) \right|_w dw + \int_{R_3} \phi_3 \left. g_{3s} (\phi_1, \phi_2, w, R_4, \ldots) \right|_w dw + \text{etc.,} \]

so that

\[ \frac{\partial \delta_s}{\partial \phi_i} = g_{is}. \]

Here the \( R_i \) are arbitrary constants.

Then

\[ U_{ms} \delta_s = U_{m, i} \phi_i \left. \frac{\partial \delta_s}{\partial \phi_i} \right|_w = g_{mi} g_{is} \]

\[ = \delta_{ms} \quad (\text{II-6}) \]

where \( \delta_{ms} = 1 \) if \( m = s \) or 0 if \( m \neq s \). It is also seen that the \( \delta_s \) obey the same partial differential equation II-4 as do the \( \phi_1 \), that is

\[ \delta_{s, k} P^I_{si} Q_i - \delta_{s, t} = 0. \quad (\text{II-7}) \]

Equation II-6 and II-7 can be combined to solve for \( \delta_{s, k} \) and \( \delta_{s, t} \) in terms \( P, Q, \alpha \) and \( \beta \) giving

\[ \delta_{s, k} = \frac{\partial \delta_s}{\partial y_k} = (P\alpha^T + Q\beta^T) s_i P_{ik} \quad (\text{II-8a}) \]
\[ \dot{s}_{s, t} = \frac{\partial \dot{s}_s}{\partial t} = (P \alpha^T + Q \beta^T)^T_{si} Q_i \]  \hspace{1cm} \text{(II-8b)}

Here the notation \((P \alpha^T + Q \beta^T)^T_{si}\) refers to the \(s, i\) component of the inverse of the sum of the matrix products \(P\) with the transpose of \(\alpha\) and \(Q\) with the transpose of \(\beta\), provided this inverse exists. (Note that \(Q \beta^T\) is a square matrix).

From equation II-7 it is seen that under the assumptions made, an integration factor or matrix exists of the form \((P \alpha^T + Q \beta^T)^T_{i}\), and matrix multiplication with equation II-1 gives a perfect differential in the sense that

\[ d\dot{s}_s = \frac{\partial \dot{s}_s}{\partial y_k} dy_k + \frac{\partial \dot{s}_s}{\partial t} dt = (P \alpha^T + Q \beta^T)^T_{ij} P_{jk} dy_k + (P \alpha^T + Q \beta^T)^T_{sj} Q_j dt. \]

The function \(s\) can be found by a line integral in the \(y, t\) space along some convenient path, represented by

\[ \dot{s}_s = \int d\dot{s}_s = \int (P \alpha^T + Q \beta^T)^T_{sj} P_{jk} dy_k + \int (P \alpha^T + Q \beta^T)^T_{sj} Q_j dt. \]

\[ = K_s \quad s = 1, 2, \ldots , M \]  \hspace{1cm} \text{(II-9)}

where the \(K_s\) are constants. The equations \(s_s(y_1, y_2, \ldots , t) = K_s\) represent then the general solution to the set of equations II-1.

There are two points to be noted in connection with this result. The first is that instead of the matrix equation \(P \frac{dy}{dt} + Q = 0\), it would be just as general to have used the equation \(\frac{dy}{dt} + \bar{Q} = 0\), where \(\bar{Q} = P^{-1}Q\) since a necessary and sufficient condition for the existence of a solution to the first is that \(P^{-1}\) exist.
The second point to note is that there is no need to consider transformations
of the variable \( t \). That is the transformation

\[
t = t + \varepsilon \beta_n
\]

is exactly the same transformation (as far as the equation \( \frac{dy}{dt} - Q(y,t) = 0 \))
is concerned) as the transformation

\[
y_k \rightarrow y_k(t - \varepsilon \beta_n)
\]

since

\[
y_k(t - \varepsilon \beta_n) \approx y_k(t) - \frac{dy}{dt} \varepsilon \beta_n = y_k + \varepsilon Q_k \beta_n.
\]

Thus the transformations

\[
U_n = \alpha_{nk} \frac{\partial}{\partial y_k} + \beta_{n}\frac{\partial}{\partial t}
\]

are identical to the transformations

\[
U_n = (\alpha_{nk} + Q_k \beta_n) \frac{\partial}{\partial y_k}
\]

with respect to the equation \( \frac{dy}{dt} + Q = 0 \).

The remainder of this chapter will be concerned with the equations
\( \frac{dy}{dt} + Q = 0 \) and transformations \( y \rightarrow y + \varepsilon \alpha \) only.

A formal statement of the theorem used in this report for the solution
of differential equations, which we will refer to as Lie's Theorem then is as
follows:

"If the differential equation \( \frac{dy(t)}{dt} + Q(y(t), t) = 0 \), where \( y \) and \( Q \) are vectors,
and \( t \) a scalar, is invariant with respect to the set of transformations specified by

\[
U_n = \alpha_{ns}(y, t) \frac{\partial}{\partial y_s}
\]

with
where $\alpha$ is a square matrix and $\frac{\partial}{\partial y}$ a vector operator, then provided $\alpha^{TI}$ exist, the general solution to the differential equation is

$$\int \alpha^{TI} (dy + Qdt) = K$$

where the integral is understood as a line integral in $y, t$ space along any convenient path, and $K$ is an arbitrary vector constant.

The paragraphs in this chapter leading up to a statement of this theorem can in fact be considered a proof of the theorem, but an alternate form of the proof will now be given.

The differential equation to be integrated is

$$\frac{dy}{dt} + Q(y,t) = 0.$$  \hspace{1cm} \text{(II-10)}

If this equation is to be invariant with respect to the transformation specified by

$$U_n = \alpha_{nk} \frac{\partial}{\partial y_k}$$  \hspace{1cm} \text{(II-11)}

for all $n$ it must be invariant with respect to the infinitesimal transformations

$$y_k \rightarrow y_k + \varepsilon \alpha_{nk}(y,t)$$  \hspace{1cm} \text{(II-12)}

$$Q_k(y,t) \rightarrow Q_k(y_j + \varepsilon \alpha_{nj}, t)$$

to first order in $\varepsilon$ for all $k$ and $n$. Here $\varepsilon$ is an infinitesimal parameter.
Making this transformation gives

$$\frac{dy_k}{dt} + \alpha_k + \varepsilon \left\{ \frac{d}{dt} \alpha_{nk} + \alpha_{nm} q_{k,m} \right\} + \varepsilon^2 (\ldots) = 0 \quad (II-13)$$

This equation is invariant up to first order in $\varepsilon$ if and only if

$$\frac{d}{dt} \alpha_{nk} (y(t), t) = - \alpha_{nm} q_{k,m} \quad (II-14)$$

Letting $\alpha_{kn}^{T} = \alpha_{nk}$ and $\alpha_{nm}^{TI}$ be the inverse of $\alpha^{T}$, $II-14$ gives

$$\alpha_{kn}^{T} \frac{d}{dt} \alpha_{ns}^{TI} = q_{k,s} \quad (II-15)$$

The left side of this equation is the right Volterra derivative* of $\alpha_{ns}^{TI}$.

It will now be shown that every solution to $II-15$ is an integrating factor of $II-10$.

Let $\lambda_{km}$ be an integrating factor of $II-10$ for each $k$; that is

$$d\phi_k = \lambda_{km} dy_m + \lambda_{km} q_m dt$$

where $d\phi_k$ is a perfect differential for each $k$.

Then

$$\lambda_{km} = \phi_{k,m}$$

and

$$\lambda_{km} q_m = \phi_{k,t}.$$ 

Thus $\phi_k$ is a solution of the partial differential equation

$$\phi_{k,t} - q_m \phi_{k,m} = 0. \quad (II-16)$$

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Then

\[ R^T \frac{d}{dt} R + \left( \frac{d}{dt} S^T \right) S = 0 \]

provided \( \frac{d}{dt} S^T \) exists (See Appendix ). Multiplying on the left by \( R \) and on the right by \( \frac{d}{dt} S^T \) gives

\[ \left( \frac{dR}{dt} \right) S^T + R \frac{dS^T}{dt} = 0 \]

or

\[ \frac{d}{dt} \left( RS^T \right) = 0. \]

Then

\[ RS^T = C \]

where \( C \) is some invertible constant matrix. That is, if two matrices have the same Volterra derivative, they are proportional to each other through some invertible constant matrix.

In this way it is shown that every solution of II-17 (or II-15a) is proportional to every other solution through some invertible constant matrix.

Thus if some solution is an integrating factor, every solution is an integrating factor. The matrix \( \sigma^T(y(t), t) \) is then an integrating factor of the matrix equation

\[ \frac{dy(t)}{dt} + Q(y, t) = 0. \]
Then since
\[ d\phi_m = \frac{\partial \phi_m}{\partial y_k} dy_k + \frac{\partial \phi_m}{\partial t} dt \]
\[ = \alpha_m^{TI} (dy_j + Q_j dt), \]  
(II-18)

the line integral in y, t space
\[ \phi_m = \int \alpha_m^{TI} (dy_j + Q_j dt) \]
\[ = K_m \]  
(II-19)
is a solution to the system of differential equations II-10 for each constant vector K. That is, the \( K_m \) are the constants of integration.

This completes the proof of Lie's theorem for systems of differential equations, the basic theorem on which the methods and results of this report are based.

We note at this point that the Lie's theorem is proved here for a finite system of equations. The extension to countably infinite systems depends on an adequate theory of countably infinite matrices. The proof would be unchanged for a system of countably infinite matrices that form an algebra, that is, a system which is closed under addition and multiplication.* The extension to continuously infinite matrices which forms the basis of the application to partial differential equations will be discussed in a later chapter.

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*C.C. MacDuffee (op. cit.) page 106.
B. EXAMPLES

1. The One-dimensional Heat Flow Equation

As an example of the use of Lie's theorem to solve systems of ordinary
differential equations the method will be applied to the discrete form of the
one-dimensional heat flow equation.

Consider the partial differential equation

$$\frac{\partial y(x,t)}{\partial t} - \frac{\partial^2 y(x,t)}{\partial x^2} = 0 \quad (II-20)$$

with initial conditions

$$y(x,0) = y^0(x)$$

and the periodic boundary conditions

$$y(x+2L,t) = y(x,t).$$

Using the lowest order difference approximation for the derivative with
respect to $x$ gives the system of equations

$$\frac{d}{dt} y_n(t) - \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = 0 \quad n = 0, 1, 2, \ldots, N \quad (II-4)$$

Here $y_n(t) = y(nh,t)$ where $h$ is the discretization interval ($Nh = L$). There
are only $2N$ independent equations since $y_n(t) = y_{n+2N}(t)$ by virtue of the
periodic boundary conditions. The initial conditions are

$$y_n(0) = y^0_n.$$
Considered as a system of coupled differential equations, II-21 is of the form

$$\frac{d}{dt} y_n + Q_n(y, t) = 0$$

where

$$Q_n(y, t) = -(y_{n+1} - 2y_n + y_{n-1})/h^2.$$ 

Equation II-21 is invariant with respect to the transformation

$$y_n(t) ightarrow y_n(t) + \epsilon \psi_{j+n}^*(t)$$

for $n = 0, 1, \ldots, N$ and $j = 0, 1, 2, \ldots, N$. The operators characterizing the set of transformations are

$$U_j = y_{j+s} \frac{\partial}{\partial y_s}$$

for $j = 0, 1, \ldots, N$. This is of the form

$$U_j = \alpha \frac{\partial}{\partial y_s}$$

with

$$\alpha_{js} = y_{j+s} = \alpha^*_s e_j.$$ 

$\alpha$ is a square $2N \times 2N$ matrix.
The matrix \( \alpha \), whose elements are \( y_{j+s} \), is called an anticirculant matrix. Much is known about anticirculants. In particular the inverse of an anticirculant is an anticirculant, if it exists. It is straightforward to show that if \( q_{m+k} \) are the elements of the inverse of the matrix whose elements are \( y_{m+j} \) (\( = \alpha_T^{jm} = \alpha_{mj} \)),

so that

\[
\alpha_T^{mk} \alpha_T^{jm} = q_{m+k} y_{m+j} = \delta_{kj},
\]

then

\[
q_k = \frac{1}{2N} e^{\frac{\pi i ks}{N}} \frac{e^{-\pi i js/N}}{y_j}
\]

(Note here that the summation convention is used on all repeated indicies, the sums running from \(-N+1\) to \(N\), and \(q_k\) is periodic with period \(2N\).) This inverse can be obtained in a variety of ways. It can be obtained from the theory of finite Fourier expansions. Also, by showing that powers of the \(N\)th roots of unity form a unitary matrix that diagonalizes every \(N\) by \(N\) anticirculant this inverse can be obtained from the reciprocal of the eigenvalues of \( y_{m+j} \).

With this inverse then the integration factor is

\[14\]
\[ \alpha_{mk} = q_{m+k} = 1 \left( \frac{n_i(k+m)s}{N} \right) \left( \frac{n_ijs}{N} \right) \frac{y_j}{y_j} . \]

There exist then perfect differentials \( d\hat{\delta}_m \)

\[ d\hat{\delta}_m = \hat{\delta}_{m,k} \frac{dy_k}{t} + \hat{\delta}_m \frac{dt}{t} \]

such that

\[ \frac{\partial \hat{\delta}_m}{\partial y_k} = \alpha_{mk} \hat{\delta}_m \]

\[ \frac{\partial \hat{\delta}_m}{\partial t} = \alpha_{mk} \hat{\delta}_m \]

\[ = -\alpha_{mk} \left( y_{k+1} - 2y_k + y_{k-1} \right)/\hbar^2 \]

\[ = -\left( \delta_{m1} - 2\delta_{m0} + \delta_{m-1} \right)/\hbar^2 . \]

The functions \( \hat{\delta}_m(y_{-N+1}, y_{-N}, \ldots, y_0, \ldots, y_N, t) \)

can then be calculated by integrating \( d\hat{\delta}_m \) along some convenient path in the \( 2N + 1 \) dimensional space of the \( y \)'s and \( t \). A convenient path of integration is as follows:

1) \( y_k = \delta_{k0} \) along \( t \) from \( t = 0 \) to \( t \)

2) \( t = t, \ y_k = 0 \ (k \neq 0) \) along \( y_0 \) from \( y_0 = 1 \) to \( y_0(t) \)

3) \( t = t, \ y_k = 0(k \neq 0, l) \ y_0 = y_0(t), \) along \( y_1 \) from \( y_1 = 0 \) to \( y_1(t) \)
\[ t = t, \quad y_k = 0(k \neq 0, 1, -1) \quad y_0 = y_0(t), \quad y_1 = y_1(t), \quad \text{along} \quad y_{-1} \quad \text{from} \quad y_{-1} = 0 \quad \text{to} \quad y_{-1}(t) \quad \]

etc.

Written out with the summation signs, this is

\[ -\int_0^t dt \left( \delta_{ml} - 2\delta_{mo} + \delta_{m1} \right) e^{\frac{2\pi i (0+m)s}{N}} \]

\[ + \frac{1}{2N} \sum_{-N \leq j < N} \sum_{0 \leq j < 1} \left( y_j e^{\frac{2\pi i j s}{N}} - e^{\frac{2\pi i j s}{N}} y_j + e^{\frac{2\pi i (0+s)j}{N}} w \right) \]

\[ + \int_0^1 dw \sum_{-1 \leq j < 1} \frac{e^{\frac{2\pi i (-1+m)s}{N}}}{\sum_{-N \leq j < N} e^{\frac{2\pi i j s}{N}} y_j + e^{\frac{\pi i (1-s)N}{N}} w} \]

\[ + \cdots , \quad \]

\[ + \int_0^1 dw \frac{e^{\frac{2\pi i (N+m)s}{N}}}{\sum_{-N \leq j < N} e^{\frac{2\pi i j s}{N}} y_j + e^{\frac{\pi i Ns}{N}} w} \}

This particular path of integration gives

\[ \hat{\xi}_m(y, t) = e^{\frac{2\pi i m s}{N}} \ln(e^{\frac{2\pi i j s}{N}} y_j) \]

\[ - \frac{t}{h^2} \left( \delta_{ml} - 2\delta_{mo} + \delta_{m1} \right) \]
That this is in fact $\hat{y}_m(y, t)$ can be readily verified by calculating that

$$\hat{y}_{m, k} = \frac{e^{\pi i m}}{k_m},$$

and

$$\hat{y}_{m, t} = -\left(\frac{\delta_{m_1} - 2\delta_{m_0} + \delta_{m-1}}{h^2}\right).$$

A completely general solution to equation II-21 is given by $\hat{y}_m = K_m$ where the $K_m$ are arbitrary constants.

While this solution does not look particularly useful, it can be solved explicitly for the $y_j$ by introducing new integration constants $M_s$ where

$$\ln M_s = e^{\pi i m s N} K_m.$$ 

Then taking (discrete) Fourier transforms of both sides of

$$e^{\pi i m s N} \left(\ln e^{\pi i j s N} y_j\right) - \frac{t}{h^2} (\delta_{m_1} - 2\delta_{m_0} + \delta_{m-1}) = K_m$$

gives

$$\ln(e^{\pi i j s N} y_j) + 4t \sin^2(\pi s/2N)/h^2 = \ln M_s$$

or

$$y_j(t) = \frac{e^{-\pi i j s N - 4t \sin^2(\pi s/2N)/h^2}}{M_s}.$$ 

The constants $M_s$ can then be related to initial conditions by noting that at $t = 0$

$$y_j(0) = \frac{1}{2N} e^{-\pi i j s N} M_s$$

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or
\[ M_s = e^{\frac{\pi im}{N}} y_m^s. \]

In terms of the initial conditions then the solution to II - 21 is
\[ y_n(t) = \frac{e^{-in(n-m)s/N}}{2N} e^{-4t\sin^2(\pi m/2N)/h^2} y_m^s. \]

(Note the sum over both the repeated indices m and n.)

While this solution to the one-dimensional heat flow equation may not look familiar, by passing to the limits \( h \to 0, N \to \infty \) with \( Nh = L \) it can be seen that this is the usual solution for the initial value problem.

Introducing the notation
\[ x_n = nh, \]
\[ x_m' = mh \]
\[ \Delta x' = h, \]

and writing out the summation signs explicitly
\[ y(x, t) = \lim_{N \to \infty} \sum_{-N < m < N} \sum_{-N < s < N} \frac{e^{-2\pi i(x-m)x'm)s/2L}}{2Nh} e^{-4ts\sin^2(\pi s/2N)/h^2} y_m^s. \]

If the limit \( L \to \infty \) is now taken one obtains the solution to the heat flow equation valid over the entire real axis for the initial value problem.

Introducing the notation
\[ p_s = s/2L, \quad \Delta p = 1/2L, \]
gives

\[
\lim_{L \to \infty} \int_{-L}^{+L} \, dx \sum_{-\infty<s<\infty} \Delta p \, e^{-2\pi i(x-x')p_s - 4t(\pi p)^2} y(x',0)
\]

\[
\approx \int_{-\infty}^{+\infty} \, dx \int_{-\infty}^{+\infty} \, dp \, e^{-2\pi i(x-x')p - (2\pi p)^2} y(x',0)
\]

This is the usual Fourier transform solution for the infinite interval.

The integration over \( p \) can be carried out and gives

\[
y(x,t) = \int_{-\infty}^{+\infty} \, dx' \frac{e^{-2\pi i(x-x')^2/4\sqrt{t}}}{2\sqrt{\pi t}} y(x',0)
\]

This is the standard solution to the initial value problem for the infinite interval.
1.1 Discussion

The above technique for using Lie's theorem to solve the heat flow equation is quite complicated and gives well known solutions that are much more easily obtained in other ways. It is used here only to illustrate this method. The equation II-20 is linear but Lie's theorem can be applied to the non-linear problems. The technique of discretization of the partial differential equation, followed by the application of Lie's theorem to a finite system of ordinary differential equations, followed in turn by taking the limit back to the continuous system, can be applied to other partial differential equations but is an extremely awkward way of proceeding. A more desirable method would be to obtain a form of Lie's theorem applicable directly to partial differential equation without introducing the discrete approximation. This subject will be taken up in later chapters of this report.

2. The Wave Equation

The second example given here will be the application to a second order partial differential equation, the wave equation

$$\frac{\partial^2}{\partial t^2} y(x,t) - \frac{\partial^2}{\partial x^2} y(x,t) = 0$$

(II-25)

with initial values \(y(x,0) = y^0(x)\) and \(\frac{\partial}{\partial t} y(x,t)|_{t=0} = y^0(x)\).

Since Lie's theorem is applicable to first order differential equations it will be necessary to reduce II-25 to a pair of first order differential equations. This can be done by introducing two new variables \(\dot{y}\) and \(y'\) defined as \(\dot{y} = \frac{\partial y}{\partial t} (x,t)\), \(y' = \frac{\partial y}{\partial x} (x,t)\).
Then the single 2nd order partial differential equation II-25 can be written as a pair of first order coupled differential equations

\[
\frac{\partial}{\partial t} y - \frac{\partial}{\partial x} y' = 0 \quad \text{(II-26a)}
\]

\[
\frac{\partial}{\partial t} \dot{y} - \frac{\partial}{\partial x} \dot{y} = 0. \quad \text{(II-26b)}
\]

This pair of coupled equations can be reduced to a pair of uncoupled equations defining

\[
u = \dot{y} + y'
\]
\[
v = \dot{y} - y'
\]

which satisfy the equations

\[
\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 \quad \text{(II-27a)}
\]

\[
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0 \quad \text{(II-27b)}
\]

The solution of II-25 is related to the solution of II-27 by

\[
y(x,t) = y(x,0) + \frac{1}{2} \int_0^t \{u(x,\tilde{t}) + v(x,\tilde{t})\} \, d\tilde{t}.
\]

Equations II-27 represent a pair of uncoupled first order partial differential equations equivalent to II-25. These can be solved separately and the equation for \( u \) only will be solved here since that for \( v \) can be obtained by reversing the sign of \( t \) in the solution for \( u \).

The discrete version of II-27a is

\[
\frac{d}{dt} u_n - \left( \frac{u_{n+1} - u_{n-1}}{2h} \right) = 0 \quad \text{(II-28)}
\]
If the periodic boundary conditions \( u_n = y_{n+2N} \) are used, this equation is invariant with respect to the set of infinitesimal transformations

\[
  u_n \rightarrow u_n + cu_{n+k}, \quad k = 0, \pm 1, \pm 2, \ldots, \pm N
\]

characterized by the operator

\[
  U_k = u_{k+n} \frac{\partial}{\partial x_n}.
\]

Equation II-28 is in the form

\[
  \frac{d u_n}{dt} + Q_n = 0
\]

where

\[
  Q_n = -(u_{n+1} - u_{n-1})/2h
\]

and \( U \) is in the form

\[
  U_k = \alpha_{kn} \frac{\partial}{\partial u_n}
\]

where \( \alpha_{kn} = u_{k+n} \).

Since \( \alpha = \alpha^T \) and \( \alpha^T = \alpha \), the \( T \) superscript will be dropped from \( \alpha \).

The matrix \( \alpha \) whose elements are \( u_{k+n} \) is called an anticirculant matrix. The inverse of an anticirculant is an anticirculant if it exists. It is
straightforward to show that if $a_{j+k}$ are the elements of the inverse of the matrix whose elements are $u_{k+n}$ so that

$$a_{j+k} u_{k+n} = \delta_{jn}$$

then $a_k = \frac{1}{2N} e^{\frac{2\pi ik}{N} j} u_j$, and $a_{km} = \frac{1}{2N} e^{\frac{2\pi i(k+m)s}{N}} u_j$.

(Note the summation over both repeated indexes $s$ and $j$, and that $q$ is periodic with period $2N$.)

There exists then a function

$$\hat{\phi}_m(u_{-N+1}, \ldots, u_{-1}, u_0, u_1, \ldots, u_N, t)$$

such that

$$\frac{\partial \hat{\phi}_m}{\partial u_k} = a_{k+m}$$

and

$$\frac{\partial \hat{\phi}_m}{\partial t} = - a_{m+n} (u_{n+1} - u_{n-1}) / 2\hbar$$

$$= - (\delta_{ml} - \delta_{m-1}) / 2\hbar.$$ 

$\hat{\phi}_m$ can be obtained from a line integral in the $2N + 1$ dimensional space of $u$ and $t$. A convenient path of integration is the same used above for the heat flow problem, (page 15). This can be written
\[
\sum_{0 < k < N} \int_0^{u_k} dw \left\{ \frac{1}{2N} \sum_{-N < s < N} \frac{e^{\pi i (k+m)s/N}}{\Sigma e^{\pi i j s/N} u_j + e^{\pi i k s/N} w} \right\} \\
+ \sum_{-N < k < -1} \int_0^{u_k} dw \left\{ \frac{1}{2N} \sum_{-N < s < N} \frac{e^{\pi i (k+m)s/N}}{\Sigma e^{\pi i j s/N} u_j + e^{\pi i k s/N} w} \right\} \\
+ \int_0^t dt \left( \delta_{m-1}^l - \delta_{n-1}^l \right)/2h.
\]

Integrating along this path gives

\[
\hat{\phi}_m(u,t) = e^{\pi i m s/N} \ln(e^{\pi i j s/N} u_j) - t(\delta_{m-1}^l - \delta_{m-1}^l)/2h.
\]

It is readily verified that this \( \hat{\phi}_m \) gives the correct \( \partial \hat{\phi}_m / \partial u_k \) and \( \partial \hat{\phi}_m / \partial t \). The relations

\[
\hat{\phi}_m(u,t) = K_m
\]

give the general solutions to equations II-28 where the \( K_m \) are the \( 2N \) arbitrary constants. Introducing new constants \( M_s \) such that

\[
\ln M_s = e^{-\pi i m s/N} K_m
\]

one can solve for \( u_j \) as

\[
u_j(t) = e^{-(\pi i j s/N + t \sin(\pi s/N)/h)} M_s
\]

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The $M_s$ are related to the initial values of $u(u(x,t) = \frac{\partial}{\partial t} y(x,t) + \frac{\partial}{\partial x} y(x,t))$ by $M_s = e^{\frac{\pi i rs}{N}} u^I_k$ where $u^I_j = u_j(0)$ is the value of $u$ at $t = 0$. The $M_s$ are the discrete Fourier transforms of the initial $u_j$. In terms of the $u^I_j$ then

$$u_j(t) = e^{-\frac{\pi i (j-k)s}{N}} e^{-t \sin(\pi s/N)}/h_k \cdot u^I_k.$$  

Taking the limit as $h \to 0$, $N \to \infty$, with $hN = L$ gives the continuous solution to II-28 for periodic boundary conditions with period $2L$. Introducing the notation

$$x_j = jh$$

$$x'_{k} = kh$$

$$\Delta x' = h$$

and writing in the summations explicitly we have

$$u(x,t) = \lim_{h \to 0, N \to \infty} \sum_{-N < k < N} \frac{1}{2hN} \sum_{-N < \xi < N} e^{\frac{\pi i (x \cdot x' - x \cdot t)}{2hN}} u^I_k$$

$$= \sum_{-\infty < x' < \infty} \frac{1}{2L} \int_{-L}^{+L} dx' e^{\frac{\pi i (x' \cdot x - x \cdot t)}{L}} u^I(x')$$

Taking the limit as $L \to \infty$ with

$$p = s/2L, \, \Delta p = 1/2L,$$

gives

$$u(x,t) = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dx' e^{2\pi i (x' \cdot x - x \cdot t)} p u^I(x')$$
\[
\int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dx' e^{2\pi ix'p} u_0^0(x'+x-t)
= u_0^0(x-t).
\]

The solution for \( v(s,t) \) is similar to the solution for \( u \) except with the sign of \( t \) reversed and can be worked out to give

\[ v(x,t) = v_0^0(x+t).\]

Since

\[ u(x,t) = \dot{y}(x,t) + y'(x,t) \]
\[ v(x,t) = \dot{y}(x,t) - y'(x,t) \]

and

\[ y(x,t) = y(x,0) + \int_0^t \frac{1}{2} \{ u(x,T) + v(x,T) \} \, dT, \]

we have

\[ y(x,t) = y(x,0) + \int_0^t \frac{1}{2} \{ \dot{y}(x+T,0) + \dot{y}(x-T,0) \}
+ y'(x+T,0) - y'(x-T,0) \} \, dT \]

\[ = \frac{1}{2} \{ y(x+t,0) + y(x,t,0) + \int_{x-t}^{x+t} \dot{y}(T,0) \, dT \}. \]

This is the usual form of d' Alambert's solution to the wave equation.

Again this is a long and involved way of finding a well known solution that is much more easily obtained by other methods. The purpose here is to illustrate the method and principles involved in applying Lie's theorem.
In treating second order differential equations by Lie's method it is necessary to first reduce the problem to a pair of first order equations. If these two equations can then be uncoupled as was done above the mechanics of the solution become much simpler, but the method still applies even if the equations remain coupled.
REFERENCES


1b. Lie, S., Forhand Vid. -Selsk. Christiania, (1875) 1; Math. Ann. 9 (1876) 245; 11 (1877) 464; 24 (1884) 537; 25 (1885) 71.


National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-SC

Subject: Monthly Progress Letter 18, Project A-918
"Application of Dimensional Analysis and Group Theory to the
Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the period from 1 November to 1 December 1967

Gentlemen:

During this reporting period work continued on the preparation
of the final report. Enclosed is a draft of Chapter III containing the
discussion of the application of Lie's theorem to partial differential
equations.

Respectfully submitted,

/ L. J. Gallagher
Project Director

LJG/hh

Enclosure: Draft, Chapter III
III. LIE'S THEOREM FOR PARTIAL DIFFERENTIAL EQUATIONS

Notation

This chapter deals with the application of continuously infinite matrices to the solution of partial differential equations through transformation theory. The notation here can become quite complex and under some circumstances ambiguous. Some of the conventions and notations used in this chapter will be described as follows.

The partial derivative will have its usual meaning. That is if \( \varphi = \varphi(x, t, z) \) for example then

\[
\frac{\partial \varphi}{\partial x} = \frac{d}{dx} \quad \text{and} \quad \frac{\partial \varphi}{\partial t} = \frac{d}{dt}
\]

\[
\begin{align*}
& t = \text{constant} \\
& z = \text{constant} \\
& x = \text{constant} \\
& z = \text{constant}
\end{align*}
\]

If \( z \) happens to be a function of \( x \) and \( t \), that is \( z = z(x, t) \), then the total (partial) derivatives are defined as

\[
\frac{d \varphi}{dx} = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial z} \frac{\partial z(x, t)}{\partial x},
\]

\[
\frac{d \varphi}{dt} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial z} \frac{\partial z(x, t)}{\partial t}.
\]

The comma notation will also be used for the partial derivatives, i.e.

\[
\varphi_{,t} = \frac{\partial \varphi}{\partial t}, \quad \varphi_{,x} = \frac{\partial \varphi}{\partial x}, \quad \text{and}
\]

\[
f_{,j} = \frac{\partial}{\partial y_j} f(y_1, y_2, y_3 \cdots).
\]
Where this will cause no confusion, the prime and dot notation will also
be used for partial or total derivatives with respect to $x$ and $t$
respectively. That is

\[
\dot{y} = \frac{\partial y}{\partial t}(x,t), \quad \ddot{y} = \frac{\partial^2 y}{\partial t^2}(x,t), \quad y' = \frac{\partial y}{\partial x}(x,t) \quad \text{etc.}
\]

A functional notation will also be used. Round parenns will be used
to indicate parameters of functions or distributions and square brackets
indicate functional parameters. Thus $\phi(x)[y] = \phi(x,z)$ with

\[ z = \int f(\tilde{x}, y(\tilde{x}), y', y'', \ldots) \, d\tilde{x}, \]

where $f$ is some function of the
indicated parameter. That is, $\phi$ is a function (or distribution) in $x$
and a functional of $y$.

The variational derivative will indicate a derivative with respect to
a functional parameter. That is

\[
\frac{\delta \phi}{\delta y(s)}(x)[y] = \left. \frac{\partial \phi(x,z)}{\partial z} \right|_{y(x)} \int f(\tilde{x}, y(\tilde{x}), y', y'', \ldots) d\tilde{x}
\]

and

\[
\frac{\delta}{\delta y(s)} \int f(\tilde{x}, y(\tilde{x}), y', \ldots) \, dx = \int \frac{\partial}{\partial z_0} f(\tilde{x}, z_0, y', \ldots) \Bigg|_{z_0 = y(\tilde{x})} \frac{\delta y(x)}{\delta y(s)} \, d\tilde{x}
\]
\[ + \int \frac{\partial}{\partial z_1} (\bar{x}, y(\bar{x}), z_1 \ldots) \left| \frac{\delta}{\delta y(s)} y' \right|_{z_1 = y'} + \text{etc.} \]

and

\[ \frac{\delta y(\bar{x})}{\delta y(s)} = \delta(\bar{x} - s) \]

\[ \frac{\delta}{\delta y(s)} y' \Big|_{z = \bar{x} - s} = \frac{\partial}{\partial z} \delta(z) \]

\[ \frac{\delta}{\delta y(s)} y'' \Big|_{z = \bar{x} - s} = \frac{\partial^2}{\partial z^2} \delta(z) \]

\[ \text{etc.} \]

so that

\[ \frac{\delta \phi(x)[y]}{\delta y(s)} = \frac{\partial \phi(x, z)}{\partial z} \left| z = \int \bar{x} d\bar{x} \right| \sum_{0 \leq k} \left( \frac{\partial}{\partial z_k} \right)^k f(s, z_0, z_1, z_2, \ldots) \left| z_m = \int \frac{\partial}{\partial s} \right| z_m \]

Here \( \delta(x) \) is the Dirac delta distribution or generalized function.

While the round parens designate either function or distribution parameters in the indicated parameter, no distinction between function or distribution parameters will be made. However, it is understood that in any integration associated with a matrix multiplication one of the occurrences of the variable is a distribution and the other a fairly good function.

Distributions, the general theory of continuously infinite matrices, definitions, and theorems associated with these topics are given in the appendices. It should be noted that a functional parameter can also be a
function (or distribution) in some variable. In that case account must be taken in expressing the total and partial derivatives. Thus if 
\[ \dot{\phi} = \dot{\phi}(t)[y] \] where \( y = y(t) = y(x,t) \), then 
\[ \frac{\partial \phi}{\partial t} = \frac{\partial \phi(t)[y]}{\partial t} \bigg|_{z=y(t)} \]
and 
\[ \frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \int \frac{\delta \phi(t)[y]}{\delta y(x,t)} \frac{\partial y(x,t)}{\partial t} \, dx. \]

Here \( x \) and \( t \) are independent variables.

A. Lie's Theorem

A statement of Lie's theorem for use with partial differential equations is as follows:

"If the partial differential equation
\[ \frac{\partial}{\partial t} y(x,t) + Q(x,t)[y] = 0 \]
is invariant with respect to the transformations
\[ y(x,t) \leftarrow y(x,t) + \epsilon \alpha(\bar{x},x,t)[y] \]
for all relevant \( x \), \( \bar{x} \) and \( t \), and provided an \( \alpha^I(\bar{x},x,t)[y] \)
exists such that
\[ \int dx \alpha^I(\bar{x},x,t) \alpha(x,\bar{x},t) = \delta(\bar{x}-x) \]
then
\[ \alpha^{TI}(x,\bar{x},t) = \alpha^I(\bar{x},x,t) \] is an integrating factor of the partial differential equation. That is, there exists a \( \phi(x,t)[y] \) such that
\[ \frac{\delta \phi}{\delta y(x,t)} = \alpha^{TI}(x,\bar{x},t) \]
and
\[ \frac{\partial}{\partial t} Q(x,t)[y] = \int dx \alpha^{TI}(x,\bar{x},t) Q(\bar{x},t)[y]. " \]

The proof can be constructed along the lines of the discrete version given in the previous chapter. Such a proof depends on the construction of a satisfactory theory of continuously infinite matrices. Such a theory is outlined in the appendices and the reader will be referred to these for the necessary definitions and theorems as needed.

Proof:

The differential equation to be integrated,
\[ \frac{\partial y(x,t)}{\partial t} + Q(x,t)[y] = 0 \quad \text{III-1} \]
is to be invariant with respect to the infinitesimal transformation
\[ y(x,t) \leftarrow y(x,t) + \varepsilon \alpha(\bar{x},x,t)[y] \quad \text{III-2a} \]
\[ Q(x,t)[y] \leftarrow Q(x,t)[y+\varepsilon \alpha] \quad \text{III-2b} \]
to first order in \( \varepsilon \) for all relevant \( x \) and \( \bar{x} \). Making this transformation gives
\[ \frac{\partial y(x,t)}{\partial t} + Q(x,t)[y] + \varepsilon \frac{d}{dt} \alpha(\bar{x},x,t)[y] \]
\[ + \int dx \frac{\partial}{\partial y(\bar{x})} Q(x,t)[y] \alpha(\bar{x},\bar{x},t)[y] + \varepsilon^2 (\ldots) + = 0 \]
for the coefficient of \( \varepsilon \) to be zero
\[ \frac{d}{dt} \alpha(x,x,t)[y] = -\int dx \frac{\partial}{\partial y(\bar{x})} Q(x,t)[y] \alpha(\bar{x},\bar{x},t)[y]. \]
Considering $\alpha(\bar{x},x)$ as a matrix in the parameters $\bar{x}$ and $x$ and provided that the inverse of its transpose exists as specified in the statement of the theorem, then

$$\int \frac{d\alpha^T(x,\bar{x},t)}{dt} \alpha^{TI}(\bar{x},\bar{x},t) = -\frac{\delta}{\delta y(\bar{x})}Q(x,t)[y]$$

Here $\alpha^T(x,\bar{x}) = \alpha(\bar{x},x)$ and $\alpha^{TI}$ is the inverse of $\alpha^T$. From the properties of the matrices

$$\int \frac{d\alpha^T(x,\bar{x},t)}{dt} \alpha^{TI}(\bar{x},\bar{x},t) =$$

$$-\int \frac{d\alpha^T(x,\bar{x},t)}{dt} \alpha^{TI}(\bar{x},\bar{x},t)$$

then

$$\int \frac{d\alpha^T(x,\bar{x},t)}{dt} \alpha^{TI}(\bar{x},\bar{x},t) =$$

$$\frac{\delta}{\delta y(\bar{x})}Q(x,t) . \text{ III-3}$$

(The functional dependence on $y$ is understood here and will not be written where this would cause no confusion.)

The left side of equation III-3 is the right Volterra derivative of the matrix $\alpha^{TI}$. It is straightforward to show that if two matrices have the same Volterra derivative, they are proportional to each other through a nonsingular (matrix) constant. It is also clear that if a matrix is an integration factor, a constant matrix multiplied by the integration factor is also an integration factor. Thus it only needs to be proved that the integration factor $\lambda$, also satisfies the equation
\[ \int dx \, \lambda^I(x, \bar{x}, t) \frac{d}{dt} \lambda(x, \bar{x}, t) = \frac{\delta}{\delta y(\bar{x})} Q(x, t) \tag{III-4} \]

To show this we note that the integration factor to equation III-1 is defined so that

\[ \lambda(x, \bar{x}, t) = \frac{\delta}{\delta y(\bar{x})} \varphi(x, t)[y], \tag{III-5a} \]

and

\[ \int dx \, \lambda(x, \bar{x}, t) Q(x, t) = \frac{\partial}{\partial t} \varphi(x, t)[y], \tag{III-5b} \]

and \( \lambda(x, \bar{x}, t)[y] \) nonsingular.

Taking a partial derivative with respect to \( t \) of the first of these two equations and substituting \( \delta \bar{y}/\delta t \) from the second gives

\[
\frac{\partial}{\partial t} \lambda(x, \bar{x}, t) = \frac{\delta}{\delta y(\bar{x})} \int dx \, \lambda(x, \bar{x}, t) Q(x, t) \\
= \int dx \frac{\delta \lambda(x, \bar{x}, t)}{\delta y(\bar{x})} Q(x, t) + \int dx \, \lambda(x, \bar{x}, t) \frac{\delta}{\delta y(\bar{x})} Q(\bar{x}, t) ,
\]

and

\[
\frac{d}{dt} \lambda(x, \bar{x}, t)[y] = \frac{\partial}{\partial t} \lambda(x, \bar{x}, t) + \int dx \, \frac{\delta}{\delta y(\bar{x})} \lambda(x, \bar{x}, t) \frac{\partial y(\bar{x}, t)}{\partial t} \\
= \int dx \, \lambda(x, \bar{x}, t) \frac{\delta}{\delta y(\bar{x})} Q(\bar{x}, t). \tag{III-6} \]

Multiplying by the inverse of \( \lambda \) on both sides gives

\[
\int dx \, \lambda^I(x, \bar{x}, t) \frac{d}{dt} \lambda(x, \bar{x}, t) = \frac{\delta}{\delta y(\bar{x})} Q(\bar{x}, t) 
\]

showing that the integration factor and \( \alpha^{III} \) have the same Volterra derivative.
They then are proportional to each other through a nonsingular constant matrix and thus \( \alpha^{TT} \) is also an integration factor, completing the proof.

In comparing this version of Lie's Theorem with the discrete version we note that no mention is made here of obtaining a solution to the differential equation by performing the line integral in \( y, t \) space. The theorem for the continuous case only gives an integrating factor and not \( \phi \) directly. While a line integral in a discrete (even infinite) vector space is a straightforward concept, a line integral in a continuously infinite-dimensional vector space is not so readily achieved. In practice to perform a line integral in a continuously infinite vector space, one would discretize the problem, apply the line integral to the finite (or countable) dimensional vector space and then perform a limiting process.

In the absence of a solution by a line integral it appears that the theorem is not now nearly so powerful. In fact Lie's theorem now only allows one to change the partial differential equation into an equivalent variational equation. That is, the partial differential equation

\[
\frac{\partial}{\partial t} y(x,t) + Q(x,t)[y] = 0
\]

and the variational equation

\[
\frac{\partial}{\partial t} \hat{y}(x,t)[y] - \int dx \hat{Q}(x) \frac{\delta}{\delta y}(\hat{X}) = 0
\]

are equivalent to each other. It may or may not be more convenient to solve the variational equation by a "pseudo line integral" than to attach the original equation.
The following sections examine the heat flow equation and others
in view of the continuous form of the Lie theorem.

B. Examples

1. The One-dimensional Heat-Flow Equation

Here the heat flow equation will be analyzed again, this time
using the continuous form of the Lie theorem stated in the previous section
of this chapter.

The partial differential equation to be solved is

\[
\frac{\partial y(x,t)}{\partial t} - \frac{\partial^2 y(x,t)}{\partial x^2} = 0
\]  

(III-10)

with initial conditions

\[
y(x,0) = y^0(x)
\]

defined everywhere on the real \( x \) axis. (This is equivalent to setting

\[
Q(x,t)[y] = -\int dx \frac{1}{2} \frac{\partial}{\partial x} \delta(x-x) y(x,t).
\]

Equation III-10 is invariant with respect to the transformation

\[
y(x,t) \to y(x,t) + \epsilon y(\bar{x}+x,t)
\]

for all \( \bar{x} \) and \( x \). That is, \( \alpha \) for the transformation is given by

\[
\alpha(\bar{x},x,t) = y(\bar{x}+x,t).
\]

\( \alpha \) is symmetric (\( \alpha^T = \alpha \)) and is an anticirculant continuous matrix.
Its inverse is also a anticirculant. It is straightforward to show that if
\[ g(x,t) = \int_{-\infty}^{+\infty} dp \frac{e^{2\pi i xp}}{2\pi i x} \int_{-\infty}^{+\infty} d\tilde{x} e^{2\pi i \tilde{x}p} y(\tilde{x},t) \]

then
\[ \int_{-\infty}^{+\infty} d\tilde{x} \ g(\tilde{x}+x) \ y(\tilde{x}+\tilde{x}) = \delta(\tilde{x}-x). \]

Thus we have
\[ \frac{\partial \phi(x,t)}{\partial y(\tilde{x},t)} = \alpha T I(x,\tilde{x},t) = q(x+\tilde{x}) \]
\[ = \int_{-\infty}^{+\infty} dp \frac{e^{2\pi i (\tilde{x}+x)p}}{2\pi i \tilde{x}} \int_{-\infty}^{+\infty} d\tilde{x} e^{2\pi i \tilde{x}p} y(\tilde{x},t) \]

and
\[ \frac{\partial \phi(x,t)}{\partial t} = - \int_{-\infty}^{+\infty} d\tilde{x} \int_{-\infty}^{+\infty} d\tilde{x} \alpha T I(x,\tilde{x},t) \frac{\partial^2}{\partial \tilde{x}^2} \delta(\tilde{x}-\tilde{x}) \ y(\tilde{x},t) \]
\[ = - \frac{\partial^2}{\partial x^2} \delta(x). \]

The general solution to the partial differential equation

then is
\[ \phi(x,t)[y] = K(x) \]
where $K$ is an arbitrary "constant" vector. $(K(x)$ is a constant in the sense that

$$\frac{\partial}{\partial t} K(x) = 0 \text{ and } \frac{\partial}{\partial y} K(x) = 0.$$

Finding $\phi(x,t)[y]$ from $\delta \phi/\delta y$ and $\partial \phi/\partial t$ is something of a problem. In this particular case it is possible to look at the discrete version of this problem and figure out what $\phi$ ought to be in the continuous case. The discrete case can be solved by taking a line integral in a finite-dimensional space. In the continuous case one must essentially guess the integral and verify by substitution in III-12. While this may appear crude it is nevertheless the way all quadrature is done, the problem here being more complex in that it is the entire line integral that must be guessed rather than the individual components of a line integral.

By inspection of III-12 it is not difficult to see that

$$\phi(x,t)[y] = \int_{-\infty}^{+\infty} dp \ e^{2\pi ipx} \ln \left( \int_{-\infty}^{+\infty} e^{2\pi ipx} y(x,t) \right) - t \frac{\partial^2}{\partial x^2} \delta(x) \tag{III-13}$$

gives the correct $\delta \phi/\delta y$ and $\partial \phi/\partial t$. The general solution is $\phi(x,t) = K(x)$ where $K$ is independent of $y$ and $t$ but can depend on $x$. To relate this solution to the initial value problem where $y(x,0) = y^0(x)$, let

$$\ln(M(p)) = \int_{-\infty}^{+\infty} e^{-2\pi ipx} K(x).$$
and take the Fourier transform of both sides of
\[
\int_{-\infty}^{+\infty} dp \; e^{2\pi i px} \ln \left( \int_{-\infty}^{+\infty} dx \; e^{2\pi i px} y(x, t) \right) - t \frac{\partial^2}{\partial x^2} \delta(x) = K(x) \tag{III-14}
\]
giving
\[
\ln \left( \int_{-\infty}^{+\infty} dx \; e^{2\pi i px} y(x, t) \right) - t(2\pi p)^2 = \ln M(p). 
\]
Solving for \( y \):
\[
y(x, t) = \int_{-\infty}^{+\infty} dp \; e^{-2\pi i px - (2\pi p)^2 t} M(p).
\]

\( M(p) \) then is the Fourier transform of \( y(x) \)
\[
M(p) = \int_{-\infty}^{+\infty} dx \; e^{2\pi i px} y^0(x),
\]
so that \( y(x, t) \) in terms of \( y^0(x) \) is
\[
y(x, t) = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dx \; e^{-2\pi i (x-x')p - (2\pi p)^2 t} y^0(x').
\tag{III-15}
\]

Doing the \( p \) integration first gives the usual form of the solution to the initial value problem:
\[
y(x, t) = \int_{-\infty}^{+\infty} dx \; e^{-\frac{(x-x')^2}{4\sqrt{\pi t}}} y^0(x).
\tag{III-16}
\]

It will be noted here in comparing with the procedure for solving the discrete form of the heat flow equation that there is a one-to-one
correspondence between the steps in each. The discrete solution can be used as a model or guide in following the continuous case or vice versa. The continuous case is possibly easier to follow because of the absences of the discretization and limiting processes. One notes that all the discrete Fourier transforms are replaced by the corresponding continuous transformation and these are only introduced to relate the general solution \( \phi = K \) to the initial value type solution.

The one point in the continuous case that is possibly more complex than the discrete case is relating the gradients of \( \phi \) (with respect to \( y \) and \( t \)) to \( \phi \) itself. In the discrete case this can be done by a line integral in a finite-dimensional space; the analog in the continuous case would be a line integral in a continuously infinite-dimensional space --- a rather difficult concept. In any event Lie's theorem reduces the problem of integrating a differential equation to finding a quadrature or set of quadratures provided the appropriate invariance group can be found.

The wave equation example given in the preceding chapter can be worked out in a manner similar to the heat flow equation without recourse to discretizing. The two treatments are so similar though this will not be done here.

2. A Class of Linear Problems

From the heat-flow and wave equations it can be seen that there is a general class of first order linear initial value problems that can be solved by use of the same transformation. Consider the partial differential
equation of the form

\[ \frac{\partial}{\partial t} + f(t, \frac{\partial}{\partial x}) \cdot y(x, t) = 0 \]  \hspace{1cm} \text{III-20} \]

where \( f(t, x) \) is integrable in \( t \) and a fairly good function of \( x \). This equation is invariant with respect to the infinitesimal transformation

\[ y(x, t) \rightarrow y(x, t) + \epsilon y(x+x, t). \]

Thus

\[ \alpha(x, x, t) = y(x+x, t) \]

and

\[ \alpha^{TI}(x, x, t) = \int_{-\infty}^{+\infty} dp \ e^{2\pi i p (x+x)} \int_{-\infty}^{+\infty} ds \ e^{2\pi i p s} y(s, t). \]  \hspace{1cm} \text{III-21} \]

The functional \( \phi \) is given by

\[ \phi(x, t)[y] = \int_{-\infty}^{+\infty} dp \ e^{2\pi i x p} \ln \left( \int_{-\infty}^{+\infty} dx \ e^{2\pi i x p} y(x, t) \right) \]

\[ + \int_{0}^{t} dt \ f(t, \frac{\partial}{\partial x}) \delta(x) \]  \hspace{1cm} \text{III-22} \]

and the general solution to III-20 by

\[ \phi(x, t)[y] = K(x). \]

In terms of the initial conditions, \( y(x, t) \) is given by

\[ y(x, t) = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dx \ e^{2\pi i (x-x)p} \int_{0}^{t} dt \ f(t, 2\pi i p) y(x, 0). \]
3. Partial Differential Equations in More than Two Independent Variables

Lie's theorem is also applicable to partial differential equations that are first order in $t$ and have several independent variables $x_1, x_2, x_3 \ldots$.

In this case the statement of the theorem is modified so as to replace $x$ by the vector $\vec{x} = \{x_1, x_2, x_3 \ldots\}$, and $dx$ by the volume element in $x$ space, $d\vec{x} = dx_1 dx_2 dx_3 \ldots$.

The family of partial differential equations mentioned above in section 2 of this chapter can then be generalized to

$$\left\{ \frac{\partial}{\partial t} + f(t, \frac{\partial}{\partial \vec{x}}) \right\} y(\vec{x}, t) = 0 \quad \text{III-40}$$

where $\frac{\partial}{\partial x}$ is the gradient operation with respect to the component of $\vec{x}$.

This equation is invariant with respect to the transformation

$$y(\vec{x}, t) \leftarrow y(\vec{x}, t) + e y(\vec{x} + \vec{e}, t).$$

The integrating factor is

$$\alpha^{TI}(\vec{x}, \vec{e}, t) = \int_{-\infty}^{+\infty} \frac{e^{2\pi i \vec{p} \cdot \vec{x}}}{\int_{-\infty}^{+\infty} e^{2\pi i \vec{p} \cdot \vec{e}}} \int_{-\infty}^{+\infty} e^{2\pi i \vec{p} \cdot \vec{y}(\vec{s}, t)} ds \quad \text{III-41}$$

The functional $\phi$ is given by

$$\phi(x, t)[y] = \int_{-\infty}^{+\infty} e^{2\pi i \vec{x} \cdot \vec{p}} \ln \int_{-\infty}^{+\infty} e^{2\pi i \vec{p} \cdot \vec{y}(x, t)} dx + \int_{-\infty}^{+\infty} f(t, \frac{\partial}{\partial \vec{x}}) \delta(x)$$

and the solution to III-40 by
\[ \phi(\vec{x}, t)[y] = k(\vec{x}). \]

In terms of the initial conditions, \( y(\vec{x}, t) \) is given by

\[
y(\vec{x}, t) = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dx \ e^{2\pi i (\vec{x} \cdot \vec{p})} \int_0^t dt' f(t, 2\pi \vec{p}) y(\vec{x}, t) \quad \text{III-43}
\]

The notation used here is that \( dx, dp \) represent volume elements in \( \vec{x} \) and \( \vec{p} \) space respectively, \( \vec{x} \cdot \vec{p} \) is the scalar product, i.e. \( \vec{x} \cdot \vec{p} = x_1 p_1 + x_2 p_2 + \ldots \), and \( \delta(x) \) is the multidimensional delta (generalized) function

\[
\delta(x) = \delta(x_1) \delta(x_2) \delta(x_3) \ldots
\]

All of the results derived with a scalar \( x \) can be carried over to the case where \( x \) is a vector.
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 19, Project A-918
"Application of Dimensional Analysis and Group Theory to the
Solution of Ordinary and Partial Differential Equations"
Contract No. NASA-20286
Covering the Period from 1 December 1967 to 1 May 1968

Gentlemen:

During this reporting period the final report for the contract year
1967 was completed. The typing and proofreading of this report proved
to be exceptionally time consuming.

Work has also continued on the application of Lie's theorem to
non-linear partial differential equations.

Starting in June both R. H. Martin and C. N. Driskell will return
to work on this project for the summer months.

Continuation of this project has as its immediate goals consideration of
the following problems:

1) Application of Lie's theorem to non-linear partial differential
equations.

2) Extension of Lie's theorem to systems of partial differential
equations.

3) Extension of Lie's theorem to more general forms of partial
differential equations.
4) Investigation of the "two integrating factor" method of solution.

5) Perturbation theory approach to the use of Lie's theorem.

Respectfully submitted,

L. J. Gallaher
Project Director

LJG/hh
July 9, 1968

National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 20, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations" - Contract No. NAS8-20286
Covering the Period from 1 May to 1 July 1968

Gentlemen:

Work continues on the search for applications for Lie's theorem to non-linear partial differential equations. In the process of this investigation certain difficulties have arisen with the proof of Lie's theorem for systems of differential equations as given in the 1968 final report on this subject.

The difficulty with this proof is that an important assumption concerning the operators of the invariance groups is not stated in the theorem. The condition on these operators, $U_i$, is that they all commute with each other. This commutation property is not mentioned in the report, but is necessary for the validity of the theorem as stated.

The recognition of this condition on the $U_i$ clears up some of the difficulties with Lie's theorem, considerably simplifies the proof, and gives a basis for the connection with classical group theory.

The examples given in the 1968 final report are unaffected by this since the operators in all cases commute as required.

In the following reporting period work will continue on non-linear examples.

Respectfully submitted,

/\
L. J. Gallaher
Project Director

LJG/hh
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 21, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the Period from 1 July to 1 August 1968

Gentlemen:

During this reporting period some success was achieved in finding nonlinear partial differential equations that could be solved by the application of Lie's theorem. This was accomplished by working backwards from a transformation to the differential equation invariant under that transformation.

Consider the following system of differential equations:

\[ \frac{d}{dt} y_n(t) + Q_n(t,y) = 0 \quad n = 1, 2, 3, \ldots N. \]

Let this set of equations be invariant under the transformations

\[ y_n - y_n + e^{\sigma_{np}(t,y)} \quad n = 1, 2, \ldots N \quad \rho = 1, 2, \ldots N \]
where

\[
\alpha_{pn} = \left( \sum_{e} y_{e} e_{e} \right) \delta_{pn} - 2y_{p} y_{n},
\]

and

\[
\alpha_{km}^{I} = \left( \sum_{e} y_{e} e_{e} \right)^{-2} \alpha_{km}.
\]

These conditions will be satisfied if

\[
Q_{n}(t,y) = 2y_{n} \left( \sum_{e} h_{e}(t)y_{e} \right) - h_{n}(t) \left( \sum_{e} y_{e}^{2} \right)
\]

where \( h(t) \) is an arbitrary \( t \) dependent vector.

A solution to the nonlinear system of equations

\[
\frac{dy_{n}}{dt} + 2y_{n} \left( \sum_{e} h_{e}(t)y_{e} \right) - h_{n}(t) \left( \sum_{e} y_{e}^{2} \right) = 0
\]

is

\[
\phi_{i}(y,t) = \frac{y_{i}(t)}{\sum_{e} e_{e}} - \int_{0}^{t} h_{i}(t')dt' = C_{i} \quad i = 1, 2, \ldots, N
\]

where \( C \) is a constant vector. (In terms of initial conditions on \( y \),

\[
C_{i} = y_{i}(o) / \sum_{e} e_{e}(o) \quad i = 1, 2, \ldots, N.
\]
Extending these ideas to partial differential equations gives the following:

I) The partial differential-integral equation

\[
\frac{\partial}{\partial t} y(x, t) + 2y(x, t) \int_{-\infty}^{+\infty} h(x', t)y(x', t)dx' = \int_{-\infty}^{+\infty} h(x, t) \int_{-\infty}^{+\infty} y^2(x', t)dx' = 0
\]

is invariant with respect to the transformations

\[
y(x, t) \rightarrow y(x, t) + \epsilon \alpha(x, x', t)[y]
\]

where

\[
\alpha(x, x', t)[y] = \delta(x-x') \int_{-\infty}^{+\infty} y^2(x'', t)dx'' - 2y(x', t)y(x, t)
\]

II) A solution to this equation is

\[
\phi(x, t, y)[y] = \frac{y(x, t)}{+\infty} - \int_{0}^{t} h(x, t')dt' = C(x) \int_{-\infty}^{+\infty} y^2(x', t)dx'
\]

where \(C(x)\) is an arbitrary function of \(x\). (In terms of initial conditions on \(y(x, t)\),

\[
\phi(x, t, y)[y] = \frac{y(x, t)}{+\infty} - \int_{0}^{t} h(x, t')dt' = C(x) \int_{-\infty}^{+\infty} y^2(x', t)dx'
\]
\[ C(x) = y(x, o) \int_{-\infty}^{+\infty} y^2(x', o) dx' \] 

Solving for \( y \) this can be written as

\[ y(x, t) = \frac{C(x) + \int_0^t h(x, t') dt'}{\int_{-\infty}^{+\infty} \left( C(x') + \int_0^t h(x', t') dt' \right)^2 dx'} \]

There is a rather large class of nonlinear differential-integral equations that can be solved in this manner.

In the following reporting period, work will continue on the nonlinear problems.

Respectfully submitted,

L. J. Gallaher
Project Director
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Solution of Ordinary and Partial Differential Equations"  
Contract No. NAS8-20286  
Covering the Period from 1 August to 1 September 1968

Gentlemen:

During this reporting period work continued on the investigation of applications of Lie's theorem to systems of non-linear equations. The following has been established:

1) Application of Lie's theorem as a system of differential equations is equivalent to transforming the system into a system of linear equations; thus only those systems of differential equations that can be transformed into a linear system can be solved by application of Lie's theorem.

2) Every system of equations of the form

\[
\frac{dy_s}{dt} + Q_s(y;t) = 0 \quad s = 1, 2 \ldots
\]

where

\[
\frac{\partial Q_s}{\partial y_m} \quad \text{and} \quad \frac{\partial Q_s}{\partial t}
\]

are continuous can be transformed locally into a system of linear equations. Therefore every such system of differential equations has a Lie type solution, at least locally (i.e., valid in some finite region.)
These results suggest that most non-linear systems of differential equations of practical interest have local Lie type solutions.

Analogous theorems for non-linear partial differential equations have not yet been obtained. But the analogy is a practice quite useful for solving partial differential equations.

On August 23, Charles N. Driskell, Jr. was killed in an automobile accident. Mr. Driskell was a 2nd year graduate student in the mathematics department at Georgia Tech and made significant contributions to this project both last year and this.

In the following month work will continue on the investigation of non-linear equations.

Respectfully submitted,

L. J. Gallaher
Project Director
National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter 23, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NASA-20286
Covering the Period from 1 September to 1 October 1968

Gentlemen:

In connection with the solution of finite systems of ordinary differential equations the following theorem has been established:

"Let the matrix \( \Lambda(t, \vec{y}) \) be an integrating factor to the system of differential equations

\[
\frac{d}{dt} \vec{y}(t) - \Xi(t, \vec{y}(t)) = 0
\]

and let the system be invariant with respect to the infinitesimal transformation

\[
\vec{y} \rightarrow \vec{y} + \varepsilon \vec{a}(t, \vec{y}).
\]

Then \( \Xi(t, \vec{y}) = \Lambda(t, \vec{y}) \cdot \vec{a}(t, \vec{y}) = 0 \) where the Jacobian, \( \frac{\partial \Xi}{\partial \vec{y}}(t, \vec{y}) \), is invertible implicitly defines a solution to the system of differential equations."

This theorem is an extension of the result for a single differential equation that states that the ratio of two integrating factors set equal to a constant is a solution to the differential equation.
While the form that the analogous theorem would have for partial differential equations can be seen, its proof has not been established.

Work has begun on the outline and preparation of material for the final report.

Respectfully submitted,

L. J. Gallaher
Project Director

LJG/db
December 3, 1968

National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: Monthly Progress Letter, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the Period from 1 October to 1 December 1968

Gentlemen:

During this reporting period work continued on the final report. The first draft is completed and will be made available to you now if so desired.

Work has also continued on the preparation of tables of integration factors and solutions to some classes of partial differential equations for inclusion in the final report.

Respectfully submitted,

L. J. Gallaher
Project Director

LJG/al
May 20, 1966

National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attention: PR-EC

Subject: First Quarterly Report, Project A-918
"Application of Dimensional Analysis and Group Theory to the Solution of Ordinary and Partial Differential Equations"
Contract No. NAS8-20286
Covering the Period from February 1 to April 30, 1966

Gentlemen:

Enclosed is the first quarterly report on Contract No. NAS8-20286.

Respectfully submitted,

L. J. Gallaher
Project Director
QUARTERLY REPORT

THE APPLICATION OF DIMENSION ANALYSIS AND GROUP THEORY TO THE SOLUTION OF ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

L. J. Gallaher

NAS8-20286

1 February to 30 April 1966
During the first quarter a survey of the literature on group theory and its application to differential equations was begun. The survey indicated that an investigation of the use of Lie groups in partial differential equations would be a fruitful undertaking. The remainder of this report deals with a brief description of what was learned about the connection between Lie groups and a class of differential equations that commonly arise in physical problems.

Let us consider a class of partial differential equations that are derivable from a variational principle. These equations which arise in many physical problems are referred to as Euler-Lagrange equations.

The notation used here is the following:

x_\ell is a set of cartesian coordinates x_1, x_2, x_3 ..., referred to as the spatial coordinates;

t is referred to as the temporal coordinate;

dx is the differential volume element dx_1 \, dx_2 \, dx_3 ... ;

\psi is a function of the coordinates x_\ell and t, and will be referred to as the field variable;

definitions \psi_\ell = \frac{\partial \psi}{\partial x_\ell} and \dot{\psi} = \frac{\partial \psi}{\partial t} are used.

The integral I, then is defined as

I = \int_{t_1}^{t_2} \int_R \mathcal{L}(x_\ell | t) \, | \psi(x_\ell | t) | \, \psi_\ell | \dot{\psi} dx \, dt ,

where the integration is over some region R of the spatial coordinates and the
temporal interval \( t_1 \) to \( t_2 \). The summation is to be understood for repeated indices, i.e., \[ A_1 \mathcal{B}_1 \equiv A_1 B_1 + A_2 B_2 + A_3 B_3 + \ldots. \]

I then is a functional of \( \psi \) and the problem consists in finding \( \psi \) such that \( I \) is an extremum.

Let \( \psi \) be replaced by \( \psi + K \nu \), where \( K \) is in some sense small and \( \nu(x,t) \) an arbitrary function of \( x \) and \( t \) that vanishes on the boundary of one region of integration. Then letting

\[
\frac{dI(K)}{dK} \bigg|_{K=0} = 0
\]

one obtains in a straightforward manner the partial differential equation

\[
\frac{\partial \mathcal{L}}{\partial \psi} - \left( \frac{\partial \mathcal{L}}{\partial \psi}, \mathcal{L} \right), \mathcal{L} - \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \psi} \right) = 0.
\]

This is called the Euler-Lagrange equation or just the Lagrange equation. \( \mathcal{L} \) will be called the Lagrangian density and \( L = \int_R \mathcal{L} \, dx \), the Lagrangian. Further notations adopted here are that capital script letters will be used to indicate a density and the corresponding Latin capital will indicate the spatial integral of that density over the region \( R \), i.e.,

\[
A(t) = \int_R \mathcal{A}(x, |t|, \psi(t) |, \psi, \psi |, \psi \ldots) \, dx,
\]

\[
L = \int_R \mathcal{L} \, dx, \text{ etc.}
\]

Also define
\[ \frac{\delta A}{\delta \psi} = \frac{\delta A}{\delta \psi} = \frac{\partial A}{\partial \psi} - \left( \frac{\partial A}{\partial \psi}, \ell \right), \ell + \left( \frac{\partial A}{\partial \psi}, \ell, m \right), \ell, m \cdots \]

this will be called the functional derivative of \( A \) (or \( A' \)). Thus Lagrange's equation can be written

\[ \frac{\delta L}{\delta \psi} = \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \right) \quad \text{or} \quad \frac{\delta L}{\delta \psi} = \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \psi} \right). \]

The following definitions and terminology are introduced:

\[ \pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \quad \text{(conjugate momentum)} \]

\[ \mathcal{H} = \pi \dot{\psi} - \mathcal{L} \quad \text{(Hamiltonian or energy density)} \]

\[ \mathcal{J}_\ell = \dot{\psi} \frac{\partial \mathcal{L}}{\partial \psi}, \ell \quad \text{(energy flux density)} \]

\[ \mathcal{J}_\ell = \dot{\psi}, \ell \frac{\partial \mathcal{L}}{\partial \psi} \quad \text{(momentum density)} \]

\[ \mathcal{J}_{\ell m} = \dot{\psi}, m \frac{\partial \mathcal{L}}{\partial \psi}, \ell - \delta_{\ell m} \mathcal{L} \quad \text{(stress tensor)} \].

While these names are suggestive of certain physical quantities, they need not in fact correspond to the usual physical concept suggested and can be considered as merely convenient conventional names.

The following relationships exist:
\[ \frac{\delta H}{\delta \psi} = \frac{\delta H}{\delta \pi} = -\pi \quad \text{and} \quad \frac{\delta H}{\delta \pi} = \frac{\delta \mathcal{L}}{\delta \pi} = \dot{\pi} \] (Hamilton's Equations)

\[ \frac{\partial \mathcal{L}}{\partial t} = -\mathcal{J}_{,\ell} \ell + \frac{\partial \mathcal{L}}{\partial \pi} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial t} = -\mathcal{J}_{,m} m + \frac{\partial \mathcal{L}}{\partial x_{m}}. \]

The operation
\[ \frac{\partial \mathcal{L}}{\partial (t)} \]

is meant to indicate the derivative with respect to the explicit dependence of \( \mathcal{L} \) on \( t \), i.e.,

\[ \frac{\partial \mathcal{L}}{\partial t} = \frac{\partial \mathcal{L}}{\partial (t)} + \frac{\partial \mathcal{L}}{\partial \psi} \dot{\psi} + \frac{\partial \mathcal{L}}{\partial \pi} \dot{\pi} + \frac{\partial \mathcal{L}}{\partial x_{\ell}} \dot{x}_{\ell} \]

and similarly for \( \frac{\partial \mathcal{L}}{\partial (x_{m})} \). The function \( \dot{\psi} \) is considered to be a function of \( \psi, \pi, \text{and } \psi_{,\ell} \), i.e., \( \dot{\psi} = \dot{\psi}(\psi \pi \psi_{,\ell}) \).

The quantity
\[ \left\{ A, B \right\} = \int_{R} \left( \frac{\delta A}{\delta \psi} \frac{\delta B}{\delta \pi} - \frac{\delta B}{\delta \psi} \frac{\delta A}{\delta \pi} \right) dx \]

is called the Poisson bracket of \( A \) and \( B \). Spatial integrals of functions depending only on \( x, t, \psi, \text{and } \pi \) will be called dynamic variables of type I.

If \( A \) and \( B \) are type I dynamic variables, the Poisson bracket of \( A \) and \( B \) is also a type I dynamic variable. These type I dynamic variables form a Lie algebra under the Poisson bracket operation and this algebra has an associated Lie group. A vast of literature is available and much is known.
about Lie groups which can be applied to the study of the set of type I
dynamic variables of which \( \psi(x,t) \) itself is a member.

The Lie algebra of the complete set of type I dynamic variables, how-
ever, is not of great use but there are sub-algebras that are of interest.
For example, the conserved type I dynamic variables are of interest. (If
\( \frac{dA}{dt} = 0 \) then A is said to be conserved or a constant of the motion.) It can
be shown that, if \( H \) is a type I dynamic variable, then for conserved type I
dynamic variables A and B,

\[
\frac{d}{dt} \{A, B\} = \left\{ \frac{dA}{dt}, B \right\} + \left\{ A, \frac{dB}{dt} \right\} = 0
\]

so that the conserved type I dynamic variables form sub-algebras of the
algebra of the total set of type I dynamic variables.

Unfortunately, the type I dynamic variables alone are a rather
restricted set. If one defines a general dynamic variable as the spatial
integral of any function of \( x, t, \psi, \pi \), and any derivatives of \( \psi \) and \( \pi \),
these general dynamic variables are then also closed under the Poisson
bracket operation. But it is not at all evident that the Poisson bracket
operation forms a Lie algebra for the general dynamic variables. It is
hoped that further investigation will establish under what circumstances
the general dynamic variables either do or do not form Lie algebras.
FINAL REPORT

PROJECT A-918

APPLICATION OF DIMENSIONAL ANALYSIS AND GROUP THEORY TO THE SOLUTION OF ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

L. J. GALLAHER AND M. J. RUSSELL

Contract NAS8-20286

17 January 1966 to 16 January 1967

Performed for
George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama

Engineering Experiment Station
GEORGIA INSTITUTE OF TECHNOLOGY
Atlanta, Georgia
GEORGIA INSTITUTE OF TECHNOLOGY
Engineering Experiment Station
Atlanta, Georgia

FINAL REPORT
PROJECT A-918

APPLICATION OF DIMENSIONAL ANALYSIS AND
GROUP THEORY TO THE SOLUTION OF ORDINARY AND
PARTIAL DIFFERENTIAL EQUATIONS

By

L. J. GALLAHER and M. J. RUSSELL

CONTRACT NAS8-20286

17 January 1966 to 16 January 1967

Performed for
GEORGE C. MARSHALL SPACE FLIGHT CENTER
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
HUNTSVILLE, ALABAMA
ABSTRACT

This report gives a survey and analysis of the application of one parameter transformation groups to the solution of ordinary and partial differential equations.

The first part considers ordinary differential equations. Lie's method for finding an integrating factor for a single ordinary differential equation is discussed and examples given. It is then shown how Lie's method can be extended to total differential equations, and systems of total differential equations an extension thought to be new. Examples are given and the connection with dimensional analysis is pointed out.

The second part of this report deals with partial differential equations. Here Morgan's theorems for reducing the number of independent variables are discussed and applications given. It is shown that Morgan's theorems can also be applied to ordinary differential equations but are much less useful in this case.

A brief discussion is given of the connection between Hamiltonian, or Euler-Lagrange equations and Lie algebras and Lie groups, but no examples are given.

Finally, there are some recommendations for further study in this field.
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I. INTRODUCTION

Group theory has come to the forefront of applied mathematics in recent times. Best known or most publicized of these applications has been the study of Lie groups in connection with particle physics and symmetry groups in crystallography and chemistry. The applications of groups discussed in this report are generally less well known than those just mentioned and possibly less well developed. But in a historic perspective they are much older, the basic ideas going back to Sophus Lie (1872).

This report is concerned with the application of transformation groups to the solution of ordinary and partial differential equations. These groups are in fact "Lie groups" in the sense the term is generally used today but the application is such that no particular use is made of the usual properties associated with Lie groups. It is the transformation properties that are exploited and not the detailed structure of the group or associated algebra.

The first part of this report is concerned with ordinary differential equations and the application of one-parameter transformation groups to their solution. The principal theorem here, referred to as Lie's theorem, gives a method for finding an integration factor when an invariance group for the differential equation is known. This technique is then extended to total differential equations and to systems of total differential equations. The connection with dimensional analysis, in particular Brand's work, is pointed out, dimensional analysis being the study of the nonuniform magnification groups.

Partial differential equations are taken up in the second part of the report. Here it is Morgan's theorems that are most significant. Morgan
showed that if a system of partial differential equations was invariant with respect to a one-parameter transformation group, the number of independent variables can be reduced by one. Morgan's results are the most significant achieved so far in applying group theory to partial differential equations. The disadvantage of Morgan's method is that the transformed equations are not as general as the original set. Thus there is no assurance that the reduced equations have solutions obeying the original boundary conditions. Each problem must be considered individually, the boundary conditions together with the partial differential equations.

It is also shown here that Morgan's theorems can be applied to ordinary differential equations. The results are not however so interesting, giving particular solutions to the differential equations which are seldom those sought.

In a third section a brief outline is given of how Lie algebras and Lie groups are used in Hamiltonian theory. This is the area in which most of the present activity in particles physics takes place. Here it is the detailed structure of the individual groups that is important and the goals are not so much to solve the equations as to discover their structure from the symmetry considerations. This section is quite brief and no examples are given.

Appendices on the proof of some of the theorems are given. Also included as appendices are the definitions and some examples of groups, Lie groups, and Lie algebras.
II. ORDINARY DIFFERENTIAL EQUATIONS

A. Introduction

Lie introduced the theory of continuous groups into the study of differential equations* and thereby unified and illuminated in a striking way the earlier techniques for handling them. This section will give a short description of the application of the one-parameter transformation group to the solution of a single first order ordinary differential equation. Extension to systems of equations and higher order equations is given later in this chapter. Most of the material in this section is contained in Ince [15].

B. One-parameter transformation groups

Consider the aggregate of transformations included in the family

\[ \tilde{x} = \phi(x, y; a), \quad \tilde{y} = \psi(x, y; a) . \]

Here \( x \) and \( y \) are an initial set of coordinates and \( \tilde{x} \) and \( \tilde{y} \) are the transformed set, \( a \) is a parameter that characterizes the particular transformation. Now whenever two successive transformations of the family are equivalent to a single transformation of the family, then the transformations form a group.**

That is, if

\[ \phi(x, y; a_2) = \phi(\phi(x, y; a_1), \psi(x, y; a_1); a_2) \]
\[ \psi(x, y; a_2) = \psi(\phi(x, y; a_1), \psi(x, y; a_1); a_2) \]

such that the set of \( a \)'s are closed (every \( a_1, a_2 \) pair has an \( a_2' \) in the set)

* [16a], [16b]

** See Appendix I for definition of a group and some examples.
then the transformations form a one parameter group. Note that this means that the inverse of every transformation is present.

Examples of one-parameter transformation groups are the following.

1) The group of rotations about the origin:

\[ \bar{x} = x \cos a - y \sin a, \quad \bar{y} = x \sin a + y \cos a \]

Two successive rotations characterized by \( a_1 \) and \( a_2 \) are equivalent to a rotation characterized by \( a_3 \) where \( a_3 = a_1 + a_2 \), and the inverse of the rotation \( a \) is \(-a\).

2) The magnification group:

\[ \bar{x} = a^j x, \quad \bar{y} = a^k y \]

Here \( j \) and \( k \) are constants and if \( k = j \) this is called the uniform magnification group. The transformation determined by \( a_3 \) that is equivalent to the successive transformations determined by \( a_1 \) and \( a_2 \) is such that \( a_3 = a_1 a_2 \). The inverse of the transformation characterized by \( a \) is characterized by \( 1/a \).

1. Infinitesimal transformations

Let \( a_0 \) be the value of the parameter which characterizes the identity transformation of family so that

\[ x = \phi(x,y;a_0), \quad y = \psi(x,y;a_0) \]

Then if \( \varepsilon \) is small (an infinitesimal), the transformation

\[ \bar{x} = \phi(x,y;a_0+\varepsilon), \quad \bar{y} = \psi(x,y;a_0+\varepsilon) \]
will be such that \( \bar{x} \) and \( \bar{y} \) differ only infinitesimally from \( x \) and \( y \) or

\[
\bar{x} \approx x + \frac{\partial \phi (x, y; a_0)}{\partial a_0} \varepsilon = x + \alpha(x, y) \varepsilon
\]

\[
\bar{y} \approx y + \frac{\partial \psi (x, y; a_0)}{\partial a_0} \varepsilon = x + \beta(x, y) \varepsilon .
\]

This transformation is then said to be an infinitesimal transformation.

Now it can be proved* that every one-parameter transformation group contains one and only one infinitesimal transformation. Thus a group of transformations can be characterized either by the pair of functions \( \phi \) and \( \psi \) or by the pair of functions \( \alpha \) and \( \beta \) where

\[
\alpha(x, y) = \frac{\partial \phi (x, y; a)}{\partial a} \bigg|_{a = a_0}
\]

\[
\beta(x, y) = \frac{\partial \psi (x, y; a)}{\partial a} \bigg|_{a = a_0}
\]

and \( a_0 \) characterizes the identity transformation.

Some examples of infinitesimal transformations are the following:

1) The rotation group mentioned above is defined by \( \bar{x} = x \cos a - y \sin a \), \( \bar{y} = x \sin a + y \cos a \) and the infinitesimal rotation by

\[
\bar{x} = x - ye, \quad \bar{y} = y + xe
\]

since

\[
\frac{\partial}{\partial a} (x \cos a - y \sin a) \bigg|_{a=0} = -y
\]

and

\[
\frac{\partial}{\partial a} (x \sin a - y \cos a) \bigg|_{a=0} = x .
\]

---

2) The magnification group mentioned above is defined by

\[ \bar{x} = a^j x, \quad \bar{y} = a^k y \]

then

\[ \frac{\partial}{\partial a} (a^j x) \bigg|_{a=1} = jx \quad \text{and} \quad \frac{\partial}{\partial a} (a^k y) \bigg|_{a=1} = ky \]

so the infinitesimal transformation is

\[ \bar{x} = x(1 + j\varepsilon), \quad \bar{y} = y(1 + k\varepsilon). \]

Consider now the infinitesimal change in the function \( f(x,y) \) due to an infinitesimal transformation of \( x \) and \( y \)

\[ \bar{x} = x + \alpha(x,y)\varepsilon, \quad \bar{y} = y + \beta(x,y)\varepsilon, \]

\[ f(\bar{x},\bar{y}) \approx f(x,y) + \left( \alpha(x,y) \frac{\partial f}{\partial x} + \beta(x,y) \frac{\partial f}{\partial y} \right) \varepsilon \]

to first order in \( \varepsilon \). Thus the infinitesimal transformation (and hence the entire group) can be represented by the operator \( U \) where

\[ U \equiv \alpha(x,y) \frac{\partial}{\partial x} + \beta(x,y) \frac{\partial}{\partial y}. \]

\( U\varepsilon \) is the infinitesimal change in the function \( f(x,y) \) produced by the infinitesimal transformation of \( x \) and \( y \).

Now let the finite equations of a one parameter transformation group be

\[ \bar{x} = \phi(x,y; a_0 + t), \quad \bar{y} = \psi(x,y; a_0 + t) \]

where \( a_0 \) characterizes the identity transformation. Then

\[ f(\bar{x},\bar{y}) = f_0 + f'_0 t + \frac{1}{2!} f''_0 t^2 + \ldots \]
where

\[ f_0 = f(x, y) \bigg|_{t=0} = f(x, y) \]

\[ f'_0 = \frac{df}{dt} f(x, y) \bigg|_{t=0} = \left( \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right) \bigg|_{t=0} = \alpha(x, y) \frac{dx}{dt} + \beta(x, y) \frac{dy}{dt} = Uf \]

\[ f''_0 = \frac{d^2f}{dt^2} f(x, y) \bigg|_{t=0} = U^2f \quad \text{etc.} \]

Thus

\[ f(x, y) = f(x, y) + tUf + \frac{t^2}{2!} U^2f + \ldots \]

\[ = e^{tU} f(x, y) \]

where \( U^n f \) symbolizes the result of operating \( n \) times on \( f(x, y) \) and \( e^{tU} \) symbolizes the operator

\[ e^{tU} = 1 + tU + \frac{t^2}{2!} U^2 + \ldots \]

Thus the operator \( e^{tU} \) represents the finite transformation corresponding to the infinitesimal transformation

\[ U = \alpha(x, y) \frac{\partial}{\partial x} + \beta(x, y) \frac{\partial}{\partial y} \]

Some examples of obtaining the finite transformation from the infinitesimal are as follows:

1) Given the infinitesimal transformation

\[ U = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \]
Then
\[ \frac{\vec{x}}{x} = e^{tU} x \]
\[ = x + tUx + \frac{t^2}{2!} U^2 x + \ldots \]
\[ = x - yt - \frac{t^2}{2!} x + \frac{t^3}{3!} y + \frac{t^4}{4!} x - \ldots \]
\[ = x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots \right) - y\left(t - \frac{t^3}{3!} + \ldots \right) \]
\[ = x \cos t - y \sin t, \]
\[ \frac{\vec{y}}{y} = e^{tU} y \]
\[ = x \sin t + y \cos t. \]

This corresponds to the rotation group.

2) Let \( U = cx \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} \)
then
\[ \frac{\vec{x}}{x} = e^{tU} x = x + ctx + \frac{(ct)^2}{2!} x \ldots = xe^{ct} \]
\[ \frac{\vec{y}}{y} = e^{tU} y = y + bty + \frac{(by)^2}{2!} y \ldots = ye^{bt} \]

Letting \( \alpha = e^t \) it is seen that this is the magnification group
\[ \frac{\vec{x}}{x} = \alpha^c x, \quad \frac{\vec{y}}{y} = \alpha^b y. \]

If \( b = c \), it is the uniform magnification group.

2. Invariants

\( F(x,y) \) is said to be invariant if, when \( \vec{x} \) and \( \vec{y} \) are derived from \( x \) and \( y \) by a one-parameter group of transformations, one has
\[ F(\vec{x},\vec{y}) = F(x,y). \]
A necessary and sufficient condition for \( F(x,y) \) to be invariant is that

\[ UF = 0 \]

where \( U = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \) characterizes the group. Then \( F(x,y) \) is a solution to the partial differential equation

\[ \alpha \frac{\partial Z}{\partial x} + \beta \frac{\partial Z}{\partial y} = 0 \]

and

\[ F(x,y) = \text{constant} \]

is a solution of the equivalent ordinary differential equation

\[ \frac{dx}{\alpha} = \frac{dy}{\beta}. \]

This differential equation has only one solution depending on an arbitrary constant; thus every other invariant of the group can be expressed in terms of \( F \).

A family of curves is said to be invariant under a transformation group if

\[ F(x,y) = c \ (a \ constant) \]

and

\[ F(\tilde{x}, \tilde{y}) = \tilde{c} \ (another \ constant) \]

where \( \tilde{x}, \tilde{y} \) are derived from \( x, y \) by that transformation. A necessary and sufficient condition that \( F(x,y) = \text{const} \) represents a family invariant under the transformation group \( U \) is that \( UF \) be a function of \( F \), i.e., \( UF = g(F) \).
C. Integration of a differential equation using group properties

The principal theorem for use in the solution of ordinary differential equations and which will be referred to as Lie's theorem, is the following:

Let the differential equation be given by

\[ P(x,y)dx + Q(x,y)dy = 0. \]

Then if the family of solutions \( \phi(x,y) = \text{const} \) is invariant under the transformation \( U = \alpha(x,y) \frac{\partial}{\partial x} + \beta(x,y) \frac{\partial}{\partial y} \), the quantity \((P\alpha + Q\beta)^{-1}\) is an integration factor of the differential equation, provided \( P\alpha + Q\beta \) is not identically zero. That is, the solution can be reduced to a quadrature, and is

\[ \int \frac{Pdx + Qdy}{P\alpha + Q\beta} = K \]

where \( K \) is a constant. A proof will not be given here*, but a proof of a more general theorem that includes this as a special case is given in Appendix II.

Furthermore if the family of solutions is invariant under two distinct transformations \( U_1 \) and \( U_2 \)

\[ U_1 = \alpha_1 \frac{\partial}{\partial x} + \beta_1 \frac{\partial}{\partial y} \]

and

\[ U_2 = \alpha_2 \frac{\partial}{\partial x} + \beta_2 \frac{\partial}{\partial y} \]

then the solution is

\[ \frac{\alpha_1 P + \beta_1 Q}{\alpha_2 P + \beta_2 Q} = K \]

---

where K is a constant. This is just the application of a well-known result in differential equations* that if two distinct integration factors for a differential equation are known, say λ and µ, then provided that their ratio is not a constant, λ/µ = const is a general solution. But practically speaking, it is not always easy to find two distinct transformation groups for a differential equation.

Example 1: \(2xydy + (x - y^2) \, dx = 0\).

This is invariant under the transformation

\[
\frac{\dot{x}}{x} = a^j x, \quad \frac{\dot{y}}{y} = k \frac{y}{x}
\]

if

\[j + 2k = 2j \quad \text{or} \quad j = 2k.
\]

Thus

\[U = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}
\]

represents the invariance group and λ, the integration factor is

\[
\lambda = \frac{1}{xy^2 + (x - y^2)x}
\]

The quadrature problem becomes

\[
\int \frac{2xy \, dy + (x - y^2) \, dx}{xy^2 + (x - y^2)x} = K.
\]

Since this is a perfect differential the limits of integration can be chosen as those most convenient. Here we choose \(y = 0, x = 1\) to \(x, y\) along the path \(y = 0, x = 1\) to \(x\) and then along \(x = x, y = 0\) to \(y\): that is,

\[
\int_0^y \frac{2xy \, dy}{xy^2 + (x - y^2)x} + \int_1^x \frac{(x - y^2)}{xy^2 + (x - y^2)x} \, dx = K.
\]

or

\[ \int_0^y \frac{2y}{x} \, dy + \int_1^x \frac{dx}{x} = K \]

\[ \frac{y^2}{x} + \ln x = K \]

\[ y^2 = -x \ln cx \]

where \( c \) is an arbitrary constant.

**Example 2:** \( x \, dy - (y + x^m) \, dx = 0 \).

This is invariant under the transformation

\[ \bar{x} = a^k x, \quad \bar{y} = a^j y \]

if \( j = mk \). Thus \( U = x \frac{\partial}{\partial x} + my \frac{\partial}{\partial y} \) and \( \lambda = \frac{1}{-(y + x^m)x + xmy} \). The quadrature problem is

\[ \int x \, dy - (y + x^m) \, dx = K \]

\[-(y + x^m)x + xmy\]

Choose the path of integration as \( y = 0 \), \( x \) from 1 to \( x \) and then \( y = 0 \) to \( y \) or

\[ \int_{x=1}^{x} \frac{dx}{x} + \int_{y}^{y} \frac{dy}{(m-1)y - x^m} = K \]

giving

\[ \ln \left( \frac{y^{(m-1)}}{x} + \frac{x^m}{x} \right) = K \]

or

\[ y = (x^m - cx)^{\frac{1}{m-1}} \]

where \( c \) is an arbitrary constant. Both of these examples are of equations invariant under a nonuniform magnification and this treatment is equivalent to Brand's dimensional analysis [4] approach.
Example 3: \[ \frac{dy}{dt} - y^2 = 0 \]

This equation is invariant under a translation along the \(t\) axis, that is,

\[ \tilde{t} = t + a \quad \text{or} \quad U_1 = \frac{\partial}{\partial t} \]

and the nonuniform magnification

\[ \tilde{y} = ay \]

\[ \tilde{t} = a^{-1} t \]

for which \(U_2 = y \frac{\partial}{\partial y} - t \frac{\partial}{\partial t} \). Thus the two integration factors are \(\lambda = -y^{-2}\) and \(\mu = (y + ty^2)^{-1}\). Their ratio is

\[ \frac{\lambda}{\mu} = -(y + ty^2)/y^2 = \text{const}, \]

and the solution is \( \frac{1}{y} + t = c \). In this case the answer could have been easily obtained by direct integration.

**D. Total differential equations and transformation groups**

A total differential equation in \(n\) variables is a relation of the form

\[ \sum_{k=1}^{n} P_k \frac{dx_k}{dx} = P_1 \frac{dx_1}{dx} + P_2 \frac{dx_2}{dx} + \ldots + P_n \frac{dx_n}{dx} = 0. \quad (D-1) \]

Its solution, if it exists, consists of finding one of the \(x\)'s as a function of all the others. The \(P_k\) are, in general, functions of the \(x_k\). If a solution exists for \(D-1\), \(D-1\) is said to be integrable. Solutions to total differential equations are usually found by finding an integration factor \(\lambda(x_1, x_2, \ldots)\) such that \(d\phi = \sum_k \lambda P_k \frac{dx_k}{dx}\) is a perfect differential, that is,
finding a $\lambda$ such that

$$\frac{\partial \phi(x_1, x_2, \ldots)}{\partial x_k} = \lambda P_k, \quad k = 1, 2, \ldots n.$$ 

The general solution of the total differential equation is then given by the quadrature,

$$\phi(x_1, x_2, \ldots) = \int \sum_{k} \lambda P_k dx_k = \text{constant}.$$ 

It can be shown that $D-1$ has a solution and thus an integration factor if and only if

$$\text{Anti}_{[kms]}^{\frac{\partial P_s}{\partial x_m}} = 0$$

where the operator $\text{Anti}_{[kms]}$ means that the following term is antisymmetric with respect the indices $k, m, s$. (See Appendix III.)

Transformation groups are used in solving total differential equations through the following theorem:

If the total differential equation $\sum_{k=1}^{n} P_k dx_k = 0$ is invariant with respect to a transformation group characterized by

$$U = \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x_k} = \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \ldots + \alpha_n \frac{\partial}{\partial x_n}$$

then an integrating factor $\lambda$ is given by $\lambda = \left(\sum_{k=1}^{n} \alpha_k P_k\right)^{-1}$, provided this reciprocal is not identically zero. The $\alpha_k$ are in general functions of the $x$'s. Appendix II gives a proof of this.

This theorem is a generalization of the one given for ordinary differential equations (in paragraph C) and reduces to it in the case $n = 2$. It is also a generalization of Brand's theorem [4].

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Example 1 \([4]\): The total differential equation

\[2xyzdx + z(1 - yz^2)dy + y(3 - 2y^2)dz = 0\]

is invariant with respect to the transformation

\[
\begin{align*}
\bar{x} &= a^0 x \\
\bar{y} &= a^2 y \\
\bar{z} &= a^{-1} z
\end{align*}
\]

which is characterized by

\[U = 2y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z}.
\]

The integration factor is

\[
\lambda = \frac{1}{2yz(1 - yz^2) - zy(3 - 2yz^2)} = -\frac{1}{yz}.
\]

The quadrature problem then is

\[
\int \left\{2x \, dx + \left(\frac{1}{y} - z^2\right) \, dy + \left(\frac{3}{z} - 2yz\right) \, dz\right\} = \text{const}
\]

and can be integrated along the path \((0,1,1)\) to \((0,1,z)\) to \((0,y,z)\) to \((x,y,z)\) or

\[
\begin{align*}
\int_0^X 2t \, dt + \int_1^Y \left(\frac{1}{t} + z^2\right) \, dt + \int_1^Z \left(\frac{3}{t} - 2zy\right) \, dt.
\end{align*}
\]

Thus \(\ln(yz^3) + x^2 - yz^2 = \text{const}\) is the general solution.

Example 2: The total differential equation

\[-y \, dx + x \, dy + (x^2 + y^2) \, dz = 0\]

is invariant with respect to a rotation about the z axis, that is,
\[ \ddot{x} = x \cos a - y \sin a \]
\[ \ddot{y} = x \sin a + y \cos a \]
\[ \ddot{z} = z \]

which is characterized by
\[ U = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}. \]

The multiplier is
\[ \lambda = \frac{1}{x^2 + y^2}. \]

The quadrature problem is
\[ \int \left( \frac{-ydx}{x^2 + y^2} + \frac{x\partial y}{x^2 + y^2} + dz \right) = \text{const} \]
and is most easily integrated along the path
\[ (0,1,0) \text{ to } (0,1,z) \text{ to } (0,y,z) \text{ to } (x,y,z) \]
or
\[ \int_0^x \frac{-ydt}{t^2 + y^2} + \int_0^y \frac{(0)dt}{t^2} + \int_0^z dt = \text{const} \]
\[ = -\tan^{-1} \left( \frac{x}{y} \right) + z = \text{const}. \]

As with ordinary differential equations, if two integration factors can be found, say \( \lambda \) and \( \mu \), for a total differential equation then, provided that the ratio of \( \lambda \) to \( \mu \) is not a constant, the equation \( \lambda/\mu = \text{constant} \) is a solution to the total differential equation. Thus, if two distinct transformation groups can be found such that the total differential equation
is invariant under both, the solution can be given directly. But as mentioned in the section on ordinary differential equations, it seldom happens that two integration factors can easily be found from group considerations alone. The usual situation is that one integration factor can be found by transformation invariance while a second is found by other means.

Example 3 [4]: The equation

\[ x^2 \, dw + (2x + y^2 + 2xw - z) \, dx - 2xydy - xdz = 0 \]

is already an exact differential* so that \( \mu = 1 \) is an integration factor. It is also invariant with respect to the transformation

\[
\begin{align*}
\tilde{w} &= a^0 w \\
\tilde{x} &= a^2 x \\
\tilde{y} &= a^1 y \\
\tilde{z} &= a^2 z
\end{align*}
\]

so that

\[
\lambda = \frac{1}{4(x^2 + xy^2 + x^2w - xz)}
\]

is an integration factor. Therefore \( \mu/\lambda = \text{constant} \) is a solution or

\[ x^2 + xy^2 + x^2w - xz = \text{const} \]

is a solution.

*It is exact since \[ \frac{\partial P_r}{\partial x_m} - \frac{\partial P_m}{\partial x_r} = 0 \] for all \( r \) and \( m \).
E. Systems of total differential equations

The transformation group approach can also be applied to systems of total differential equations. Superficially it might appear that nothing new is necessary when dealing with systems of total differential equations, and that all that is necessary is to integrate each individual equation without regard to the others. This will not suffice since we are looking for solutions that have a common intersection and the individual solutions need not have a common intersection.

Consider the system of $J$ total differential equations

$$\sum_{m=1}^{M} \sum_{\beta}^{E} \alpha_{\beta m} \, dx_{m} = 0, \quad \beta = 1, 2 \ldots J(\leq M) \quad (E-1)$$

Here each of the $\alpha_{\beta m}$ can be a function of the $x_{m}$. By an integration factor to this system of equations we mean a matrix function, $\lambda_{\gamma}^{\beta} (x_{1}, x_{2}, \ldots)$ such that

$$\frac{\partial \phi_{\gamma}}{\partial x_{m}} = \sum_{\beta=1}^{J} \lambda_{\gamma}^{\beta} \alpha_{\beta m}, \quad \left\{ \begin{array}{l} \gamma = 1, 2, \ldots J, \\
\beta = 1, 2, \ldots M 
\end{array} \right.$$  

for some set of functions $\phi_{\gamma} (x_{1}, x_{2}, \ldots)$. The general solution to the system $E-1$ then is the system $\phi_{\gamma} (x_{1}, x_{2}, \ldots) = C_{\gamma}, \gamma = 1, 2 \ldots J,$ where the $C_{\gamma}$ are constants.

The principal theorem for use of transformation groups with systems of total differential equations is analogous to the theorems of sections C and D above, and is stated as follows:

If the system of total differential equations $\sum_{m=1}^{M} \sum_{\beta}^{E} \alpha_{\beta m} \, dx_{m} = 0,$ 
$\beta = 1, 2 \ldots J(\leq M)$ has solutions $\phi_{\gamma} (\gamma = 1, 2 \ldots J)$ which are invariant
as a family with respect to the transformations

\[ U^\beta = \sum_{m=1}^{M} \alpha^\beta_m \frac{\partial}{\partial x_m}, \beta = 1, 2 \ldots J \]

\[ \alpha^\beta_m = \alpha^\beta_m(x_1, x_2, \ldots) \]

then an integration factor \( \lambda^\beta_\gamma \) is given by

\[ \lambda^\beta_\gamma = \left( (R^T)_{-1} \right)^\beta_\gamma \]

where \( (R^T)_{-1} \) is the inverse of the matrix product \( R^T \), \( (R^T)^\gamma_\eta = \sum_m P^m_\eta \alpha^\gamma_m \).

Proof of this theorem is given in Appendix IV.

Before giving examples of the use of this theorem it will be noted that to find one-parameter transformations \( U^\beta \) such that every one of the equations is invariant with respect to all the transformations is usually rather difficult. But often it happens that it is possible to transform the original equations to a new set that has the same solution. That is, if the original equations

\[ \sum_m P^m_\gamma dx_m = 0, \gamma = 1 \ldots J \]

have solutions \( \phi^\beta = C^\beta \) then if one introduces \( F'^m_\beta = \sum_{\gamma=1}^{J} T^\gamma_\beta P^m_\gamma \) where \( T^\gamma_\beta = T^\gamma_\beta(x_1, x_2, \ldots) \) then

\[ \sum_m F'^\gamma_\gamma dx_m = 0 \]

has solutions \( \phi'^\beta = C'^\beta \). While the \( \phi^\beta \) and \( \phi'^\beta \) are different, their intersection is the same, and it is the intersection that is the "solution" to the system.
Example 1: Consider the two total differential equations

\[ x \frac{dx}{dx} + y \frac{dy}{dy} - z \frac{dz}{dz} = 0 \]
\[ x \frac{dx}{dx} + y \frac{dx}{dy} + z \frac{dz}{dz} = 0 \]

In matrix form:

\[
\begin{bmatrix}
  x & y & -z \\
  x & y & +z
\end{bmatrix}
\begin{bmatrix}
  \frac{dx}{dx} \\
  \frac{dy}{dy} \\
  \frac{dz}{dz}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

This can be transformed to

\[
\begin{bmatrix}
  x & y & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  \frac{dx}{dx} \\
  \frac{dy}{dy} \\
  \frac{dz}{dz}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

by the matrix transformation \( T \),

\[
T = \begin{bmatrix}
  1 & 0 \\
  0 & 1/z
\end{bmatrix} \cdot \begin{bmatrix}
  1 & 0 \\
  -1 & 1
\end{bmatrix} \cdot \begin{bmatrix}
  1/2 & 1/2 \\
  0 & 1
\end{bmatrix}
\]

These two transformed equations are invariant with respect to the two transformations

\[ U^1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \quad \text{(uniform magnification along} \ x, \ y \ \text{and} \ z \ \text{axes)} \]

and

\[ U^2 = \frac{\partial}{\partial z} \quad \text{(translation along} \ z \ \text{axis)} \]

so that \( \alpha = \begin{bmatrix} x & y & z \\ 0 & 0 & 1 \end{bmatrix} \).

The product \( R \alpha \cdot T \) is

\[
R \alpha \cdot T = \begin{bmatrix}
  x & y & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x & 0 \\
  y & 0 \\
  z & 1
\end{bmatrix} = \begin{bmatrix}
  x^2 + y^2 & 0 \\
  z & 1
\end{bmatrix}
\]
and

$$\left( P \alpha^T \right)^{-1} = \frac{1}{(x^2 + y^2)} \begin{bmatrix} 1 & 0 \\ -z & x^2 + y^2 \end{bmatrix}.$$  

The integration factor $\lambda$ in matrix form is

$$\lambda = \begin{bmatrix} \frac{1}{x^2 + y^2} & 0 \\ -\frac{z}{x^2 + y^2} & 1 \end{bmatrix}$$

and multiplying by $\lambda$ gives the system of perfect differentials

$$\frac{x dx}{x^2 + y^2} + \frac{y dy}{x^2 + y^2} = 0$$

$$-\frac{z x dx}{x^2 + y^2} - \frac{z y dy}{x^2 + y^2} + dz = 0.$$  

Integrating along the path

$$(0,1,0) \to (0,1,2) \to (0,y,z) \to (x,y,z)$$

gives

$$\ln(x^2 + y^2) = C_1$$

$$-z \ln(x^2 + y^2) + z = C_2$$

or

$$x^2 + y^2 = C_1'$$

$$z = C_2'$$
whose intersection is a family of circles in planes parallel to the x, y plane with origin on the z axis.

Example 2: Consider the equations
\[-y \, dx + x \, dy + (x^2 + y^2) \, dz = 0\]

\[\frac{dx}{y} + \frac{dy}{x} = 0\,.

Each of these equations is invariant with respect to a rotation about the z axis \((U^1 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y})\) and a uniform magnification along the x and y axis \((U^2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})\).

Thus

\[P\alpha^T = \begin{bmatrix} -y & x & x^2 + z^2 \\ \frac{1}{y} & \frac{1}{x} & 0 \end{bmatrix} \begin{bmatrix} -y & x \\ x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x^2 + y^2 & 0 \\ 0 & y^2 + x^2 \end{bmatrix}\]

and

\[\lambda = \begin{bmatrix} \frac{1}{x^2 + y^2} & 0 \\ 0 & \left(\frac{y}{x} + \frac{x}{y}\right)^{-1} \end{bmatrix}.

Multiplying by \(\lambda\) gives the two equations

\[-\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy + dz = 0\]

\[\frac{x}{x^2 + y^2} \, dx + \frac{y}{x^2 + y^2} \, dy = 0\,.

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Integrating along the path

\[(0,1,0) \text{ to } (0,1,z) \text{ to } (0,y,z) \text{ to } (x,y,z)\]

gives the results

\[x^2 + y^2 = C_1\]

\[-\tan^{-1}(x/y) + z = C_2\]

The intersection of these two surfaces is a $45^\circ$ helix whose axis coincides with the $z$ axis.
III. PARTIAL DIFFERENTIAL EQUATIONS

In 1948 Birkhoff discussed the application of dimensional analysis to the solution of partial differential equations. He showed how it was possible to reduce by one the number of independent variables in a partial differential equation if that equation was invariant with respect to one of the transformation groups of dimensional analysis (magnifications). Later Morgan generalized this procedure to include all one-parameter transformation groups. Morgan's theorems represent the most progress to date in the application of groups to partial differential equations, and it is this work that will be discussed next.

A. Morgan's Theorems

We are concerned here with the sets of variables $x_1, x_2 \ldots x_m$, $y_1, y_2 \ldots y_n$ and the one-parameter group of transformations

$$\tilde{x}_k = f_k(x_1, x_2 \ldots x_m, a), \quad k = 1, 2 \ldots m$$

$$\tilde{y}_\beta = f_\beta(y_1, y_2 \ldots y_m, a), \quad \beta = 1, 2 \ldots n$$

where the functions $f_k$ and $f_\beta$ are differentiable with respect to the parameter $a$. The $y$'s in turn are considered to be differentiable (to any required order) functions of the $x$'s. If the transformations of the partial derivatives of the $y$'s with respect to the $x$'s are appended to the above transformations the resulting set of transformations also form a continuous one-parameter group called the enlargements of the group or the extended group. When considered as a function of the $m + n$ independent variables $x_1, x_2 \ldots x_m$, $y_1, y_2, \ldots y_n$, the group has $m + n - 1$ functionally independent absolute invariants.* Call the absolute invariants $\eta_1, \eta_2, \ldots \eta_{m-1}$ and $g_1, g_2 \ldots$

*See L. P. Eisenhart [8].
where $\eta_k = \eta_k(x_1, x_2 \ldots x_m)$ and $g_\beta = g_\beta(y_1, y_2 \ldots y_n, x_1, x_2 \ldots x_m)$.

Morgan's first theorem then states the following:

If the $y$ and $\tilde{y}$ are defined explicitly as functions of the $x$'s and $\tilde{x}$'s respectively by the relations

$$z_\beta(x_1, x_2 \ldots x_m) = g_\beta(y_1, y_2 \ldots y_n, x_1, x_2 \ldots x_m)$$

$$\tilde{z}_\beta(\tilde{x}_1, \tilde{x}_2 \ldots \tilde{x}_m) = g_\delta(\tilde{y}_1, \tilde{y}_2 \ldots \tilde{y}_n, \tilde{x}_1, \tilde{x}_2 \ldots \tilde{x}_m)$$

then a necessary and sufficient condition that the $y$ be exactly the same functions of the $x$'s as the $\tilde{y}$ are of the $\tilde{x}$'s is that

$$z_\beta(x_1, x_2 \ldots x_m) = \tilde{z}_\beta(\tilde{x}_1, \tilde{x}_2 \ldots \tilde{x}_m)$$

$$= z_\beta(\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_m) = F_\beta(\eta_1, \eta_2 \ldots \eta_{m-1}).$$

The $\eta$'s are the invariants of the subgroup

$$\tilde{x}_k = f_\beta(x_1, x_2 \ldots x_m, a).$$

This theorem will not be proved here; the reader is referred to Morgan [20].

Several definitions are as follows:

By an invariant solution of a system of partial differential equations is meant that class of solutions which has the property that the $y_\beta$ are exactly the same functions of the $x_k$ as the $\tilde{y}_\beta$ are of the $\tilde{x}_k$, where the $x$'s and $y$'s are related by some one-parameter transformation group.

By a differential form of the $k$-th order in $m$ independent variables is meant a function of the form

$$\phi(x_1, x_2 \ldots x_m, y_1, y_2 \ldots y_n, \frac{\partial y_1}{\partial (x_1)^k}, \ldots, \frac{\partial y_n}{\partial (x_m)^k}).$$
It has as arguments the x's, the y's and all partial derivatives of y's with respect to x's up to order k. The partial derivatives of $\phi$ with respect to all its arguments are assumed to exist.

A differential form $\phi$ is said to be **conformally invariant** under a one-parameter transformation group if under that group it satisfies

$$\phi(z_1, z_2, \ldots, z_p) = f(z_1, z_2, \ldots, z_p, a) \phi(z_1, z_2, \ldots, z_p).$$

If $f$ is a function of $a$ only, then $\phi$ is said to be constant conformally invariant and if $f = 1$ then absolutely invariant.

Morgan's second theorem then states the following:

A necessary and sufficient condition for $\phi$ to be conformally invariant under a continuous one-parameter group is that

$$U\phi = \omega(z_1, z_2, \ldots, z_p) \phi(z_1, z_2, \ldots, z_p)$$

for some $\omega(z_1, z_2, \ldots, z_p)$. Here $U$ is the operator characterizing the infinitesimal transformation

$$U = \alpha_1 \frac{\partial}{\partial z_1} + \alpha_2 \frac{\partial}{\partial z_2} \ldots + \alpha_p \frac{\partial}{\partial z_p}$$

$\alpha = \alpha(z_1, z_2, \ldots, z_p)$ (see earlier chapters of this report). Again the reader is referred to Morgan for a proof of this theorem [20].

A system of k-th order partial differential equations $\phi_{\beta} = 0$ is said to be invariant under a continuous one-parameter group of transformations if each of the $\phi_{\beta}$ is conformally invariant under the enlargements of that group.

Morgan's principal theorem then is the following statement:
If each of the differential forms \( \xi_\beta \) of the form

\[
\xi_\beta(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n, \ldots, \frac{\partial^k y_1}{\partial (x_1)^k}, \ldots, \frac{\partial^k y_n}{\partial (x_m)^k}) = 0
\]

is conformally invariant under the k-th enlargement of a transformation group, then the invariant solutions can be expressed as the system

\[
A_\beta(\eta_1, \eta_2, \ldots, \eta_{m-1}, F_1, F_2, \ldots, F_n, \ldots, \frac{\partial^k F_n}{\partial (\eta_{m-1})^k}) = 0
\]

a system of k-th order partial differential equations in m-1 independent variables. Here the \( \eta \) are the absolute invariants of the (sub) group of transformations on the x's, and the F's are the other invariants,

\[
F_\beta(\eta_1, \eta_2, \ldots, \eta_{m-1}) = g_\beta(y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_m).
\]

The proof of this theorem is given in Morgan's paper [20].

This theorem is exceedingly powerful and useful. The reduction of the number of independent variables by one in a system of differential equations can greatly aid in obtaining a solution. A partial differential equation in two variables will be reduced to an ordinary differential equation which can be much more quickly solved by numerical methods than the original equations. In the case of equations of three or more independent variables it may be possible to apply Morgan's theorem several times in succession, reducing the number of variables by one each time.

On the other hand, no account is taken in Morgan's theorem of the boundary conditions associated with a specific problem. The invariant solution found may or may not comply with the boundary conditions. The invariant solutions
are a smaller set than the total set of solutions. The solutions of the reduced equations are not as general as the original equations. In this sense Morgan's prescription does not give general solutions to the differential equation.

B. Applications of Morgan's Theorems

Example 1:

Consider the partial differential equation of the one dimensional homogeneous heat flow equation:

\[ \frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = 0 \]  \hspace{1cm} (B-1)

This equation is constant conformally invariant with respect to the non-uniform magnification transformation

\[ \tilde{y} = ay \]
\[ \tilde{x} = ax \]
\[ \tilde{t} = at \]

if \( k-m = k-2s \). One possibility is \( s = \frac{1}{2} \), \( m = 1 \), \( k = 0 \). For this transformation the invariant independent variable is

\[ \eta = x/t^{1/2} \]

and the invariant dependent variable is

\[ y = g = g(\eta) \].

Working out the partial differential operations in terms of the new variable we have
\[
\frac{\partial y}{\partial t} = \frac{dg}{d\eta} \frac{\partial \eta}{\partial t} = -\frac{1}{2} \frac{x}{t^{3/2}} \frac{dg}{d\eta} = -\frac{1}{2} \frac{\eta}{t} \frac{dg}{d\eta}
\]

\[
\frac{\partial y}{\partial x} = \frac{dg}{d\eta} \frac{\partial \eta}{\partial x} = \frac{1}{t^{3/2}} \frac{dg}{d\eta}
\]

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{t} \frac{d^2 g}{d\eta^2}.
\]

The reduced equation is then the ordinary differential equation

\[
\frac{d^2 g}{d\eta^2} + \frac{\eta}{2} \frac{dg}{d\eta} = 0. \quad (B-2)
\]

A general solution to this particular ordinary differential equation is

\[
g(\eta) = A \int_{0}^{\eta} e^{-\eta'^2/4} \, d\eta' + B
\]

or

\[
y(x,t) = A \int_{0}^{x/t^{1/2}} e^{-\eta'^2/4} \, d\eta' + B \quad (B-3)
\]

where A and B are the constants of integration.

It is noted that the solution B-3 may or may not be compatible with the boundary conditions of the original equation, B-1.

Also we note that this is not the only reduction possible. Equation B-1 is also invariant with respect to the transformation

\[
\tilde{y} = y + \ln a \\
\tilde{x} = a^{1/2} x \\
\tilde{t} = at
\]
so that invariants

g = y - \ln t

\eta = x/\sqrt{t}

are possible. The partial derivatives are

\frac{\partial y}{\partial t} = \frac{1}{t} - \frac{1}{2} \; \frac{\eta}{t} \frac{\partial g}{\partial \eta}

\frac{\partial y}{\partial x} = \frac{1}{t^{\frac{3}{2}}} \frac{\partial g}{\partial \eta}

\frac{\partial^2 y}{\partial x^2} = \frac{1}{t} \frac{\partial^2 g}{\partial \eta^2}

and the ordinary differential equation for \( g \) in \( \eta \) becomes

\frac{d^2}{d\eta^2} g + \frac{\eta}{2} \frac{\partial^2 g}{\partial \eta} - 1 = 0

which has a different solution from B-2.

Equation B-1 is invariant under a very large variety of transformations leading to different ordinary differential equations. Each set of boundary conditions must be considered separately and transformations compatible with the boundary conditions sought.

Example 2 \#*: The system of partial differential equations of the classical boundary-layer theory of Blasius is

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - b \frac{\partial^2 u}{\partial y^2} = 0 \]

\#* Taken from Morgan [20].
These are constant conformally invariant under the transformation

\[ \begin{align*}
\tilde{u} &= u \\
\tilde{v} &= a^{-1} v \\
\tilde{y} &= ay \\
\tilde{x} &= ax^2,
\end{align*} \]

a nonuniform magnification. A set of absolute invariants then are

\[ \begin{align*}
g_1 &= u \\
g_2 &= v x^{1/2} \\
\eta &= y/x^{1/2}.
\end{align*} \]

The derivatives are

\[ \begin{align*}
\frac{\partial v}{\partial y} &= \frac{1}{x} \frac{dg_2}{d\eta} \\
\frac{\partial u}{\partial x} &= -\frac{\eta}{2x} \frac{dg_1}{d\eta} \\
\frac{\partial u}{\partial y} &= \frac{1}{x^{3/2}} \frac{dg_1}{d\eta} \\
\frac{\partial^2 u}{\partial y^2} &= \frac{1}{x} \frac{d^2 g_1}{d\eta^2}
\end{align*} \]

giving the pair of ordinary differential equations

\[ \begin{align*}
-\frac{1}{2} \eta \frac{dg_1}{d\eta} + \frac{dg_2}{d\eta} &= 0 \\
-\frac{1}{2} \eta \ g_1 \frac{dg_1}{d\eta} + g_2 \frac{dg_1}{d\eta} - b \frac{d^2 g_1}{d\eta^2} &= 0.
\end{align*} \]

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Example 3: The wave equation in one dimension is

\[ \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0. \]  

(3-4)

This equation is absolutely invariant with respect to the transformation

\[ \tilde{y} = y, \]
\[ \tilde{x} = x + a, \]
\[ \tilde{t} = t + a, \]

a shift in the origin of \( x \) and \( t \) coordinates. The invariants are \( \xi = y, \)
\( \eta = x - t. \) The derivatives are

\[ \frac{\partial y}{\partial t} = -\xi', \]
\[ \frac{\partial^2 y}{\partial t^2} = \xi'' \]
\[ \frac{\partial y}{\partial x} = \xi', \]
\[ \frac{\partial^2 y}{\partial x^2} = \xi'' \]

where the primes indicate derivatives with respect to \( \eta. \) The original equation becomes then the identity

\[ 0 = 0 \]

and indicates that there are no restrictions on the function \( g(\eta). \) That is every function of \( \eta, \) (at least every twice differentiable function) is a solution to the wave equation, or

\[ g(\eta) = y(x-t) \]
is a solution for every \( y \). This of course is easily verified, and it is well
known that a pulse of arbitrary shape propagates with uniform velocity up
(or down) the \( x \) axis without changing shape if the medium is governed by
equation B-4.

C. Linear Equations

In the case of linear equations it is possible to use the transformation
groups to derive kernels for use in closed form integral solutions to initial
value problems and for deriving Green's functions. As an example let us look
again at the one dimensional linear heat flow equation:

\[
\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = 0. \tag{C-1}
\]

It is of interest here to find solutions that satisfy the boundary conditions

\[ y = 0 \text{ at } t = 0 \text{ and } x \neq 0 \]

and

\[ \lim_{t \to -\infty} \int_{-\infty}^{\infty} y(x,t) \, dx = 1. \]

These conditions are sometimes stated as

\[ y = \delta(x) \text{ at } t = 0 \]

where \( \delta(x) \) is the Dirac delta, defined to be zero if \( x \neq 0 \) but \( \int \delta(x)f(x) \, dx = f(0) \)
if the range of integration contains the origin and \( f(x) \) is reasonably well
behaved.

The equation C-1 is constant conformally invariant with respect to the
transformation group

\[ y = ay \]
\[ x = a^{-1} x \]
\[ t = a^{-2} t \]

for which invariant coordinates are \( g = y\sqrt{t}, \eta = x/\sqrt{t} \). The boundary conditions in terms of the invariant coordinates become

\[ g = 0 \text{ at } \eta = \pm \infty, \]

and

\[ \int_{-\infty}^{+\infty} g (\eta) \, d\eta = 1 \]

while the differential equation for \( g (\eta) \) is

\[ g'' + \frac{\eta}{2} g' + \frac{g}{2} = 0. \]

The solution satisfying both the equation and boundary conditions is

\[ g (\eta) = \frac{e^{-\eta^2/4}}{2\sqrt{\pi}}. \]

In terms of \( x, t \) and \( y \) this gives

\[ y (x, t) = \frac{e^{-(x^2/4t)}}{2\sqrt{mt}}, \quad t > 0. \]

It is easily verified that this \( y (x, t) \) is the desired solution by back substitution into both the original equation and the boundary conditions.

One now notes that the original differential equation is invariant with respect to arbitrary translations along the \( x \) and \( t \) axes so that another
solution to the equation C-1 is

$$y_1(x, t) = \frac{e^{-\frac{(x-x_1)^2}{4(t-t_1)}}}{2\sqrt{\pi(t-t_1)}}, \quad t \geq t_1$$

for all $x_1$ and $t_1$. $y_1$ does not satisfy the original boundary conditions but satisfies the boundary condition

$$y_1 = 0 \text{ at } t = t_1 \text{ and } x \neq x_1$$

and

$$\lim_{t-t_1} \int_{-\infty}^{+\infty} y_1(x, t) \, dx = 1.$$

Now since the original equation C-1 is linear, any linear superposition of solutions like $y_1(x, t)$ is also a solution. Thus

$$y_A(x, t) = \int_{-\infty}^{+\infty} A(x_1) \frac{e^{-\frac{(x-x_1)^2}{4(t-t_1)}}}{2\sqrt{\pi(t-t_1)}} \, dx_1, \quad t \geq t_1$$

is a solution for arbitrary $A(x_1)$ (provided this integral exists).

The quantity

$$K(x-x_1, t-t_1) = \frac{e^{-\frac{(x-x_1)^2}{4(t-t_1)}}}{2\sqrt{\pi(t-t_1)}}$$

is called the kernel (or propagator) of the operator $\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ and has, as noted, the property

$$K(x-x_1, t-t_1) = 0 \text{ at } t = t_1, \ x \neq x_1$$

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and

\[ \lim_{t \rightarrow t_1} \int_{-\infty}^{+\infty} K(x-x_1, t-t_1) \, dx_1 = 1 \]

that is,

\[ K(x-x_1, 0) = \delta(x-x_1). \]

This allows one to solve for \( A(x) \) giving

\[ y_A(x, t_1) = A(x). \]

A solution to the initial value problem (that is, given \( y_0(x) \) at \( t_0 \) and the differential equation C-1, find \( y(x, t) \) for \( t \geq t_0 \)) then is

\[ y(x, t) = \int_{-\infty}^{+\infty} y(x, t_0) \frac{e^{-(x-x_0)\sqrt{t-t_0}}}{\sqrt{2\pi(t-t_0)}} \, dx_0 \quad t \geq t_0. \]

Equation C-2 is well-known and is derived in most elementary text books on heat flow or applied mathematics. Here the derivation of the kernel is given by use of Morgan's theorem and the transformation group of nonuniform magnifications.

By a similar technique kernels can be derived for other linear partial differential operators such as the wave operator, \( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \), and the operator describing transverse vibrations in a rod* \( \frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4} \). In each case an ordinary linear differential equation is obtained whose solution can be used to give the kernel.

One can use the kernels further to obtain Green's functions for use in solving the inhomogeneous equation but this will not be covered here.

*See page 64 of Hansen [12]
D. Morgan's theorem for ordinary differential equations

In giving his proof of the basic theorems for reducing the number of independent variables by one, Morgan was careful to specify that there must be at least two independent variables. But in examining his proof it is clear that no use is made of this condition except in the terminology. This suggests that Morgan's process can be applied to ordinary differential equations and that in doing so a solution is obtained. Since the reduction by one of the number of independent variables in an ordinary differential equation gives no independent variable, the reduction is to an ordinary equation in the remaining absolute invariants.

The solutions so obtained have no arbitrary constants and are particular solutions to the system of differential equations. As such, they are not as general or as useful as the solutions obtained by the methods in Chapter II of this report. On the other hand, there are ordinary differential equations for which the methods of Chapter II do not work but for which Morgan's procedure will give particular solutions.

Example 1: In the example in the boundary layer problem, the partial differential equations

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{D-1a}
\]

\[
u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - \frac{b \partial^2 u}{\partial y^2} = 0 \tag{D-1b}
\]

were reduced to the pair of ordinary differential equation

\[-\frac{1}{2} \eta \frac{d g_1}{d \eta} + \frac{d g_2}{d \eta} = 0 \tag{D-2a}\]

\[-\frac{1}{2} \eta g_1 \frac{d g_1}{d \eta} + g_2 \frac{d g_1}{d \eta} - \frac{b d^2 g_1}{d \eta^2} = 0 \tag{D-2b}\]
by the transformations

\[ \eta = \frac{y}{\sqrt{x}} \]

\[ g_1 = u, \quad g_2 = v \sqrt{x}. \]

The equations D-2 are invariant with respect to the transformations

\[ \tilde{\eta} = a^{-1} \eta \]
\[ \tilde{g}_1 = a^2 g_1 \]
\[ \tilde{g}_2 = a g_2. \]

Two new invariants \( G_1 \) and \( G_2 \) can be introduced as

\[ G_1 = g_1 \eta^2 \]
\[ G_2 = g_2 \eta \]

where \( G_1 \) and \( G_2 \) now are independent of \( \eta \), that is, constants. Substituting for \( g_1 \) and \( g_2 \) in equations D-2 gives

\[ -\frac{1}{3} \eta (\frac{\partial}{\partial \eta}) (\frac{\partial}{\partial g_1} (-2g_1 \eta^{-3}) + \frac{\partial G_2}{\partial \eta}) = 0 \]

and

\[ -\frac{1}{3} \eta g_1 \eta^{-2} (-2g_1 \eta^{-3}) + g_2 \eta^{-1} (\frac{\partial G_1}{\partial \eta}) - b (6g_1 \eta^{-4}) = 0 \]

or

\[ G_1 - G_2 = 0 \text{ and } g_1^2 - 2G_1 G_2 - 6bG_1 = 0 \]

or

\[ G_1 = G_2 = -6b. \]
This gives the particular solutions

\[ g_1 = \frac{-6b}{\eta^2}, \quad g_2 = \frac{-6b}{\eta} \]

for equation D-2 or

\[ u = \frac{-6bx}{y^2}, \quad v = \frac{-6b}{y} \]

as solutions to equations D-1, which is easily verified by back substitution.

This solution is, however, of doubtful value. It contains no arbitrary constants and can therefore be made to satisfy only very special boundary conditions. It cannot, for example, satisfy the usual boundary conditions associated with the boundary layer flow over a flat plate.*

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* See [12] page 12.
IV. HAMILTONIAN AND EULER-LAGRANGE EQUATIONS

Most of the ordinary and partial differential equations arising in physical problems can be derived from a variational principle. Group theory can be applied to the study of these equations and their solutions through the study of the Lie algebras and corresponding Lie groups*. This section gives a brief outline of how this connection arises.

Consider a class of partial differential equations that are derivable from the variation of an action integral. These equations, which arise in many physical problems, are referred to as Euler-Lagrange equations.

(The notation used here is the following:

\( x_\ell \) is a set of Cartesian coordinates \( x_1, x_2, x_3 \ldots \), referred to as the spatial coordinates;

\( t \) is referred to as the temporal coordinate;

\( dx \) is the differential volume element \( dx_1 \, dx_2 \, dx_3 \ldots \);

\( \psi \) is a function of the coordinates \( x_\ell \) and \( t \), and will be referred to as the field variable;

the summation convention is used for repeated indices, i.e., \( A_\ell B_\ell \equiv A_1 B_1 + A_2 B_2 + A_3 B_3 + \ldots \);

definitions \( \psi, \ell = \frac{\partial \psi}{\partial x_\ell} \) and \( \dot{\psi} = \frac{\partial \psi}{\partial t} \) are used.)

The integral \( I \), called the action, is defined as

\[
I = \int_{t_1}^{t_2} \int_{R} \mathcal{L}(x_\ell|t|\psi(x_\ell|t)|\psi,\ell|\dot{\psi}) \, dx \, dt ,
\]

where the integration is over some region \( R \) of the spatial coordinates and the

* See Appendix V for definition of a Lie algebra and Lie group.
temporal interval \( t_1 \) to \( t_2 \). I then is a functional of \( \psi \) and the problem consists in finding \( \psi \) such that I is an extremum. By letting \( \psi \) be replaced by \( \psi + Kv \), where \( K \) is in some sense small and \( v(x,t) \) an arbitrary function of \( x \) and \( t \) that vanishes on the boundary of the region of integration, and setting

\[
\frac{dI(K)}{dK} \bigg|_{K=0} = 0
\]

one obtains in a lengthy but straightforward manner the partial differential equation

\[
\frac{\partial L}{\partial \psi} - \left( \frac{\partial L}{\partial \psi},_t \right),_t - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \psi},_t \right) = 0.
\]

This is called the Euler-Lagrange equation or just the Lagrange equation. \( L \) will be called the Lagrangian density and \( L = \int \mathcal{L} \, dx \), the Lagrangian. Further notations adopted here are that capital script letters will be used to indicate a density and the corresponding Latin capital will indicate the spatial integral of that density over the region \( R \), i.e.,

\[
A(t) \equiv \int_R \mathcal{L}(x,t | \psi(x,t) \mid \psi,_,_t \mid \ldots) \, dx,
\]

\[
L \equiv \int_R \mathcal{L} \, dx, \text{ etc.}
\]

Also define

\[
\frac{\delta A}{\delta \psi} = \frac{\delta A}{\delta \psi} = \frac{\delta A}{\delta \psi} - \left( \frac{\delta A}{\delta \psi},_t \right),_t - \left( \frac{\delta A}{\delta \psi},_t,_,m \right),,_t,_,m - \ldots;
\]

this will be called the functional derivative of \( A \) (or \( A \)). Thus Lagrange's
The equation can be written

\[ \frac{\delta L}{\delta \psi} = \frac{\partial}{\partial t} \left( \frac{\delta L}{\delta \psi} \right) \quad \text{or} \quad \frac{\delta L}{\delta \psi} = \frac{\partial}{\partial t} \left( \frac{\delta L}{\delta \psi} \right). \]

The following definitions and terminology are introduced:

\[ \pi \equiv \frac{\partial L}{\partial \psi} \]  

(conjugate momentum)

\[ H \equiv \psi \frac{\partial L}{\partial \psi} - L \]  

(Hamiltonian or energy density)

\[ I_\psi \equiv \psi \frac{\partial L}{\partial \psi} \]  

(energy flux density)

\[ I_m \equiv \psi, \psi \frac{\partial L}{\partial \psi} \]  

(momentum density)

\[ T_{\psi\psi} \equiv \psi, m \frac{\partial L}{\partial \psi, \psi} - \delta_{\psi\psi} L \]  

(stress tensor).

While these names are suggestive of certain physical quantities, they need not in fact correspond to the usual physical concepts suggested and can be considered as merely convenient conventional names.

The following relationships exist:

\[ \frac{\delta H}{\delta \psi} = \frac{\delta H}{\delta \psi} = -\pi \quad \text{and} \quad \frac{\delta H}{\delta \pi} = \frac{\delta H}{\delta \pi} = \frac{\partial H}{\partial \pi} = \pi \]  

(Hamilton's Equations)

\[ \frac{\partial H}{\partial t} = -I_{\psi, \psi} + \frac{\partial L}{\partial (t)} \quad \text{and} \quad \frac{\partial H}{\partial t} = -T_{\psi\psi, \psi} + \frac{\partial L}{\partial (x^m)}. \]

The operation

\[ \frac{\partial L}{\partial (t)} \]

is meant to indicate the derivative with respect to the explicit dependence
of $\mathcal{L}$ on $t$, i.e.,

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial \mathcal{L}}{\partial (t)} + \frac{\partial \mathcal{L}}{\partial \psi} \frac{\partial \psi}{\partial t} + \frac{\partial \mathcal{L}}{\partial \psi} \frac{\partial \psi}{\partial t}$$

and similarly for $\frac{\partial \mathcal{L}}{\partial (x_m)}$. The function $\psi$ is considered to be a function of $\psi$, $\pi$, and $\psi, t$, i.e., $\psi = \psi(\psi, \pi, t)$. 

Spatial integrals of functions depending on $x$, $t$, $\psi$, $\pi$, and any derivatives of $\psi$ and $\pi$ will be called dynamic variables. The quantity

$$[A, B] = \int_R \left( \frac{\delta A}{\delta \psi} \frac{\delta B}{\delta \pi} - \frac{\delta B}{\delta \psi} \frac{\delta A}{\delta \pi} \right) dx$$

is called the Poisson bracket of $A$ and $B$ and is also a dynamic variable. The dynamic variables form a Lie algebra under the Poisson bracket operation, and this algebra has an associated Lie group. A vast amount of literature is available, and much is known about Lie groups * which can be applied to the study of the set of dynamic variables of which $\psi(x, t)$ itself is a member.

The Lie algebra of the complete set of dynamic variables, however, is not of great use, but there are subalgebras that are of interest. For example, the conserved dynamic variables are of interest. (If $\frac{dA}{dt} = 0$ then $A$ is said to be conserved or a constant of the motion.) It can be shown that for conserved dynamic variables $A$ and $B$,

$$\frac{d}{dt} [A, B] = [\frac{dA}{dt}, B] + [A, \frac{dB}{dt}] = 0$$

so that the conserved dynamic variables form subalgebras of the algebra of the total set of dynamic variables.

The present day uses of Lie groups in physical theories, especially in particle physics, however, are not for solving the differential equations, * See reference [1] for an excellent review of Lie groups and applications.
but rather for discovering the form of the Hamiltonian when the symmetries of the system are known. Lipkin's book [18] is suggested as one of the best elementary introductions to this use of Lie groups, and the subject will not be pursued further here.
V. RESULTS AND CONCLUSIONS

A. Results

1. Introduction

The results of this investigation can be divided into two categories. The first are those theorems and methods in transformation groups useful in solving ordinary and partial differential equations which are generally known by mathematicians working in the field but not generally known or in common use by physicists, engineers, or others who are concerned with practical problems. This would include Lie's basic results and Morgan's theorems. They are quite powerful yet not well known or as widely exploited as they might be, probably because most people have so little background in group theory. This is unfortunate since the group theory needed to exploit the results of Lie and Morgan in practical problems is quite simple, considerably less than what is needed to derive their results.

The other category of results in this report are those results which appear to be original as far as can be seen from the literature survey. The first of these is the application of the one-parameter transformation groups to finding integration factors for a total differential equation. This was a simple extension of Lie's basic theorem. A further extension that appears to be original here is to systems of total differential equations. An integration factor can be found for a system of total differential equations if a sufficient number of independent invariance groups can be found for the system.

In this report it was also possible to show that Morgan's theorem for partial differential equations can also be extended "backwards" to ordinary
differential equations. But in this case it is not nearly so powerful or useful, giving only particular solutions.

2. Summary of results

The basic results of group theory useful in solving differential equations then are as follows:

a) Lie's theorem. If a differential equation of the form

\[ P(x,y) \, dx + Q(x,y) \, dy = 0 \]

has solutions that are invariant as a family with respect to the transformation

\[ U = \alpha(x,y) \frac{\partial}{\partial x} + \beta(x,y) \frac{\partial}{\partial y} \]

then an integration factor is

\[ \lambda = \frac{1}{(\alpha P + \beta Q)} \]

provided the denominator is not zero.

b) Extension of Lie's theorem to total differential equations.

If a total differential equation of the form

\[ \sum_{k=1}^{n} P_k(x_1, x_2, \ldots, x_n) dx_k = 0 \]

has solutions that are invariant as a family with respect to the transformation

\[ U = \sum_{k=1}^{n} \alpha_k(x_1, x_2, \ldots, x_n) \frac{\partial}{\partial x_k} \]

then an integration factor is

\[ \lambda = \frac{1}{\left( \sum_{k=1}^{n} \alpha_k P_k \right)} \]

provided the sum in the denominator is not zero.
c) Extension of Lie's theorem to systems of total differential equations. If a system of total differential equations of the form
\[
\sum_{k=1}^{n} P^k_{\gamma} (x_1, x_2, \ldots, x_n) \frac{\partial}{\partial x_k} = 0, \quad \gamma = 1, 2, \ldots, M
\]
has solutions which as a family are each invariant with respect to all of the transformations
\[
U^\beta = \sum_{k=1}^{n} \alpha^\beta_k (x_1, x_2, \ldots, x_n) \frac{\partial}{\partial x_k}, \quad \beta = 1, 2, \ldots, M,
\]
then an integration matrix is given by
\[
\lambda^\beta_{\gamma} = \left( (P \alpha^T)^{-1} \right)^{-1}_{\gamma}.
\]
Here $P \alpha^T$ stands for the matrix product of $P$ with $\alpha^T$ (i.e., $\sum_k \alpha^\beta_k \alpha_k^\gamma$) and $(P \alpha^T)^{-1}$ is the inverse of this matrix. $\lambda^\beta_{\gamma}$ is an integration factor only if the matrix $P \alpha^T$ has an inverse.

d) Morgan's theorem. If each of a set of partial differential equations of the form
\[
\Phi_{\gamma} (x_1, \ldots, x_m, y_1, \ldots, y_n, \ldots, \frac{\partial y_1}{\partial x_1}, \ldots, \frac{\partial y_n}{\partial x_1}, \ldots, \frac{\partial y_1}{\partial x_m}, \ldots, \frac{\partial y_n}{\partial x_m}) = 0, \quad \gamma = 1, 2, \ldots
\]
is conformally invariant with respect to some one-parameter group of continuous transformations then the set of equations can be reduced to a new set of the form
\[
\Phi_{\gamma} (\eta_1, \ldots, \eta_{m-1}, F_1, \ldots, F_n, \ldots, \frac{\partial F_1}{\partial \eta_1}, \ldots, \frac{\partial F_n}{\partial \eta_1}, \ldots, \frac{\partial F_1}{\partial \eta_{m-1}}, \ldots, \frac{\partial F_n}{\partial \eta_{m-1}}) = 0
\]
where the \( \eta \)'s and \( F \)'s are the invariants of the transformation group. Note this reduces by one the number of independent variables in the system of partial differential equations.

f) Morgan's theorem for ordinary differential equations. Morgan's theorem can be applied even when there is only one independent variable. In this case it reduces a system of ordinary differential equations to ordinary equations in the invariant \( F \)'s.

B. Conclusions

Some conclusions can be drawn from a practical application of the above techniques for solving differential equations.

Lie's original theorem for partial differential equations and its extension to total differential equations is quite effective and practical in solving ordinary and total differential equations. But it depends on finding an invariance group without giving a straightforward prescription as to how to look for such a group. Thus it is a trial and error method that depends for its effectiveness on the skill of the user and is not a straightforward prescription for solving all equations.

Lie's theorem extended to systems of total differential equations is much more difficult to use in practice. It requires finding many invariance groups such that each and every one of the total differential equations is invariant with respect to every group. Furthermore, the groups must be independent. In the present form this theorem is of doubtful practical use.

Morgan's theorem is extremely powerful and useful in practice. Its use however has the same drawback as Lie's theorem, namely that an invariance group must be found and the theorems give no hint about how to search for
such groups. But a skilled user can often use physical reasoning to great advantage in practical problems and quickly discover the needed group.

A more serious difficulty with Morgan's results is that no considerations of the boundary conditions enter the theorems. Thus while one may discover an invariance group for the differential equations, this transformation may not be compatible with the boundary conditions and thus be of no use. Since the boundary conditions are different for different problems, each individual problem must be attacked separately.

C. Recommendations for further study

The treatment of systems of total differential equations by group theory outlined in this report appears to be useful. However, modification of this technique may be possible and desirable to make it more flexible. In the present form, if there are M equations then M different invariance groups are needed to find an integration factor (matrix), and this is a rather stringent requirement. Hopefully further investigation could show that these conditions could be relaxed to a single invariance group.

A second extension that would be useful is to extend the technique used for systems of equations to partial differential equations. Such an extension would not be easy but ought to be possible at least in principle since a partial differential equation can be viewed as a continuously infinite system of ordinary differential equations.

Another extension would be to try to apply the method for systems of differential equations to the discrete approximation of a partial differential equation. This would give only approximate solutions to the partial differential equation. But these approximations could be arbitrarily
close to the exact solutions, or exact solutions might be obtained by a limiting process. The limiting process might also be effective in obtaining the needed theorems mentioned in the previous paragraph.

Respectfully submitted,

L. J. Gallaher
Project Director
APPENDIX I

Definition and Examples of a Group

A set of elements are said to form a group under an associative operation (called product) if the following conditions are satisfied:

1. The product of any two elements in the group is in the group.

2. There is a unique identity element in the group such that its product with every element leaves that element unchanged.

3. A unique inverse of every element is in the group such that the product of the element with its inverse is the identity.

Some examples of groups are the following:

The numbers +1 and -1 form a two element group under multiplication.

The positive and negative integers with zero (as the identity) form a group under the operation of addition.

The positive rational numbers form a group under multiplication with 1 as the identity.

The complex numbers of unit magnitude form a group under multiplication.

The real numbers form a group under addition.

The set of all one-to-one transformations on any space is a group.
APPENDIX II

Lie's Theorem for Total Differential Equations

In this appendix it is shown that if the solution of a total differential equation is invariant as a family under a one-parameter group of transformations, an integrating factor can be given.

Consider the total differential equation in n variables (n \geq 2):

\[ \frac{P_1}{n} dx_1 + \frac{dx_2}{n} + \ldots + \frac{P_n}{n} dx_n = \sum_{k=1}^{n} P_k (x_1, x_2, \ldots) \, dx_k = 0 \quad \text{A II-1} \]

and let

\[ \phi (x_1, x_2, \ldots) = c \quad \text{(a constant)} \]

be the family of solutions. That is,

\[ \frac{\partial \phi}{\partial x_k} = P_k \quad \text{or} \]

\[ \frac{\partial \phi}{\partial x_k} = \lambda (x_1, x_2, \ldots) P_k, \quad k = 1, 2, \ldots n. \]

where \( \lambda \) is independent of \( k \). Then \( \phi \) is a solution to the set of partial differential equations

\[ \frac{1}{P_k} \frac{\partial f}{\partial x_k} - \frac{1}{P_{k+1}} \frac{\partial f}{\partial x_{k+1}} = 0, \quad k = 1, 2 \ldots n-1 \]

(provided none of the \( P_k \) are identically zero).

Assume that as a family, \( \phi = c \) is invariant under the group \( U \)

\[ U = \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3} \ldots + \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x_k} \]

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(the \( \alpha \)'s can be functions of the \( x \)'s), so that

\[
U \phi = g(\phi).
\]

Let

\[
\hat{\phi} = \int \frac{d\phi}{g(\phi)}
\]

so that \( \hat{\phi} = C \) is identical with the family \( \phi = c \). Then

\[
U \hat{\phi} = U \phi \frac{d\hat{\phi}}{d\phi} = 1.
\]

and \( \hat{\phi} \) is a solution of the partial differential equations

\[
\frac{1}{P_k} \frac{\partial \hat{\phi}}{\partial x_k} - \frac{1}{P_{k+1}} \frac{\partial \hat{\phi}}{\partial x_{k+1}} = 0 \quad k = 1, 2 \ldots n-1
\]

and

\[
\sum_k \alpha_k \frac{\partial \hat{\phi}}{\partial x_k} = 1.
\]

This system of \( n \) linear equations can be solved and gives

\[
\frac{\partial \hat{\phi}}{\partial x_k} = \frac{P_k}{\sum_s \alpha_s^s P_s}
\]

(provided the denominator is not identically zero). Then \( d\hat{\phi} \) is a perfect differential and

\[
d\hat{\phi} = \sum_k \frac{\partial \hat{\phi}}{\partial x_k} dx_k
\]

\[
= \sum_k \frac{P_k dx_k}{\sum_s \alpha_s^s P_s}.
\]
Thus if $\sum \alpha_s P_s$ is not identically zero,

$$\lambda = \frac{1}{\sum \alpha_s P_s}$$

is an integrating factor of the total differential equation

$$\sum_k P_k \, dx_k = 0$$

and

$$\int \frac{\sum_k P_k \, dx_k}{\sum \alpha_s P_s} = K$$

is the quadrature solution, where $K$ is a constant, and the path of integration in $x_k$ space can be chosen for convenience.

An alternate point of view to the above reasoning can also be given. The equation A II-1 is invariant under the group $U$, if it preserves its form under an infinitesimal transformation. That is, if

$$\sum_k P_k (\tilde{x}_1, \tilde{x}_2, \ldots) \, d\tilde{x}_k = 0 \quad \text{A II-2}$$

where

$$\tilde{x}_k = x_k + \alpha_k \delta t;$$

$\delta t$ is a small parameter. To first order in $\delta t$ we have
\[ \sum_k P_k(x_1, x_2 \ldots) \, dx_k = \]

\[ \sum_k \left( P_k(\tilde{x}) + \sum_s \frac{\partial P_k(\tilde{x})}{\partial x_s} \, \tilde{\alpha}_s \, \delta t \right) \, (dx_k + \sum_s \frac{\partial \alpha_k}{\partial x_s} \, dx_s \, \delta t) = \]

\[ \sum_k \left( P_k(\tilde{x}) + \delta t \sum_s \left( \frac{\partial P_k}{\partial x_s} \, \tilde{\alpha}_s + \frac{\partial \alpha_k}{\partial x_k} \, P_s \right) \right) \, dx_k \]

(here \( \tilde{\alpha}_k(\tilde{x}) = -\alpha_k(x) \) to lowest order in \( \delta t \)).

If A II-2 is to hold, then

\[ \sum_s \left( \frac{\partial P_k}{\partial x_s} \, \tilde{\alpha}_s + \frac{\partial \alpha_k}{\partial x_k} \, P_s \right) = w(\tilde{x}) \, P_k(\tilde{x}) \quad \text{A II-3} \]

where \( w(\tilde{x}) \) does not depend on the index \( k \) but may depend on the \( \tilde{x} \)'s. If A II-3 holds, then

\[ P_m \sum_s \left( \frac{\partial P_k}{\partial x_s} \, \tilde{\alpha}_s + \frac{\partial \alpha_k}{\partial x_k} \, P_s \right) - P_k \sum_s \left( \frac{\partial P_m}{\partial x_s} \, \tilde{\alpha}_s + \frac{\partial \alpha_s}{\partial x_m} \, P_s \right) = 0 \]

or (dropping the bars for convenience)

\[ \sum_s \alpha_s \left( P_m \frac{\partial P_k}{\partial x_s} - P_k \frac{\partial P_m}{\partial x_s} \right) + \sum_s P_s \left( P_m \frac{\partial \alpha_k}{\partial x_m} - P_k \frac{\partial \alpha_k}{\partial x_m} \right) \alpha_s = 0 \]

or

\[ \sum_s \alpha_s \left( P_m \frac{\partial P_k}{\partial x_s} - P_m \frac{\partial P_m}{\partial x_s} - P_m \frac{\partial P_m}{\partial x_k} + P_k \frac{\partial P_m}{\partial x_m} \right) \]

\[ - \left( P_k \frac{\partial P_m}{\partial x_k} - P_m \frac{\partial P_m}{\partial x_k} \right) \sum_s P_s \alpha_s = 0 \quad \text{A II-4} \]

Now in Appendix III it is shown that a necessary condition for the existence of an integrating factor for A II-1 is that
$\text{Anti}_{[s m k]} P_s \frac{\partial}{\partial x_m} P_k = 0.$

Thus if an integrating factor does exist, A II-4 can be written

$$\sum_{s} \alpha_s \left( P_s \frac{\partial}{\partial x_m} P_k - P_s \frac{\partial}{\partial x_k} P_m \right) - \left( P_k \frac{\partial}{\partial x_m} - P_m \frac{\partial}{\partial x_k} \right) \sum_{s} P_s \alpha_s = 0.$$ 

Provided $\sum_{s} P_s \alpha_s$ is not identically zero this gives

$$\frac{\partial}{\partial x_m} \left( \frac{P_k}{\sum_{s} P_s \alpha_s} \right) - \frac{\partial}{\partial x_k} \left( \frac{P_m}{\sum_{s} P_s \alpha_s} \right) = 0.$$  \hspace{1cm} \text{A II-5}

Now expression A II-5 states that the n dimensional "curl" of the vector $P_k/\sum_{s} P_s \alpha_s$ is zero, and it is a well-known theorem that, if the curl of the vector is zero, this is a necessary and sufficient condition that the vector be expressable as the gradient of some scalar function. Thus it is shown that

$$\frac{P_k}{\sum_{s} \alpha_s P_s} = \frac{\partial \phi}{\partial x_k}$$

for some $\phi$ or that $1/\sum_{s} \alpha_s P_s$ is an integrating factor for the original equation A II-1.

This second form of the proof is informative since it is more closely related to the test for invariance actually used in solving practical problems.
APPENDIX III

The Necessary Conditions for the Existence of an Integration Factor

It is shown here that an integration factor to the differential equation

$$\sum_{k=l}^{n} P_k \frac{dx_k}{x_k} = 0 \quad \text{A III-1}$$

exists only if

$$\text{Anti}_{[kms]} P_k \frac{\partial P_s}{\partial x_m} = 0. \quad \text{A III-2}$$

(Here the notation $\text{Anti}_{[kms]}$ means that the term following is to be anti-symmetric in the indices $k$, $m$ and $s$.)

Assume then that there exists a $\lambda \neq 0$, such that

$$\frac{\partial}{\partial x_m} \lambda P_k - \frac{\partial}{\partial x_k} \lambda P_m = 0 \quad \text{A III-3}$$

then

$$P_k \frac{\partial}{\partial x_m} \lambda + \lambda P_k \frac{\partial P_k}{\partial x_m} - P_m \frac{\partial \lambda}{\partial x_k} - \lambda \frac{\partial P_m}{\partial x_k} = 0$$

or

$$\left( P_k \frac{\partial}{\partial x_m} - P_m \frac{\partial}{\partial x_k} \right) \ln \lambda + \frac{\partial P_k}{\partial x_m} - \frac{\partial P_m}{\partial x_k} = 0.$$
\[ P_s \left( \frac{\partial}{\partial x_m} - \frac{\partial}{\partial x_k} \right) \ln \lambda + P_s \left( \frac{\partial}{\partial x_m} \frac{P_k}{P_m} - \frac{\partial P_m}{\partial x_k} \right) \]

\[ - P_k \left( \frac{\partial}{\partial x_m} - \frac{\partial}{\partial x_s} \right) \ln \lambda - P_k \left( \frac{\partial P_s}{\partial x_m} - \frac{\partial P_m}{\partial x_s} \right) \]

\[ - P_m \left( \frac{\partial}{\partial x_s} - \frac{\partial}{\partial x_k} \right) \ln \lambda - P_m \left( \frac{\partial P_k}{\partial x_s} - \frac{\partial P_s}{\partial x_k} \right) = 0 \]

The operator operating on \( \ln \lambda \) is identically zero. The remaining terms are \( \text{Anti}^{[\text{smk}]} \frac{\partial P_s}{\partial x_m} \). Thus if an integrating factor for AIII-1 exists AIII-2 must hold.

We note here that AIII-2 is a necessary condition for the existence of an integrating factor; it is also a sufficient condition, but that was not demonstrated here.
APPENDIX IV

Lie's Theorem for Systems of Total Differential Equations

In this appendix it is shown how one-parameter transformation groups are used to find integrating factors for systems of total differential equations. This is Lie's theorem extended to systems of total differential equations.

Consider the system of $M$ total differential equations in $n$ variables ($n \geq M \geq 2$).

$$\sum_{k=1}^{M} p_{\gamma}^{k} (x_{1}, x_{2}, \ldots) \, dx_{k} = 0 \ , \gamma = 1, 2, \ldots M.$$

A IV-1

The convention will be used here that the Latin indices run from 1 to $n$ and the Greek indices run from 1 to $M$. Let

$$\phi_{\rho} (x_{1}, x_{2}, \ldots) = c_{\rho} (\text{constants}), \rho = 1, 2, \ldots M$$

be the family of solutions to A IV-1. That is

$$\frac{\partial \phi_{\rho}}{\partial x_{k}} = \sum_{\beta} \lambda_{\rho}^{\beta} \frac{\partial p_{\beta}}{\partial x_{k}}$$

A IV-2

where the $\lambda_{\rho}^{\beta}$ may be functions of the $x$'s but are independent of the index $k$. ($\lambda$ is an integration factor and will be an $M \times M$ matrix here.)

Assume that as a family $\phi_{\rho} = c_{\rho}$ are invariant under the groups $U_{\gamma}^{\rho}$

$$U_{\gamma}^{\rho} = \sum_{k=1}^{n} a_{k}^{\gamma} (x_{1}, x_{2}, \ldots) \frac{\partial}{\partial x_{k}}, \gamma = 1, 2, \ldots M$$

so that
\[ U^{\gamma} \phi_x = g^{\gamma}_{\phi_x} . \]  

A IV-3

Introduce \( \phi_{\eta} \) defined as

\[ \phi_{\eta} = \sum_{\beta} \int (g^{-1})_{\eta}^{\beta} d\phi_{\beta} \]  

A IV-4

so that \( \phi_{\eta} = C_{\eta} \) (a constant) is identical with the family, \( \phi_\rho = c_\rho \). The notation here is that \( (g^{-1})_{\eta}^{\beta} \) is the \( \beta, \eta \) component of the inverse of the matrix \( g_{\eta}^{\beta} \) (it being assumed here that the inverse of \( g \) exists).

Then

\[ U^{\gamma} \phi_{\eta} = \sum_{\beta} U^{\gamma} \phi_{\beta} \frac{\partial \phi_{\eta}}{\partial \phi_{\beta}} = \sum_{\beta} g^{\gamma}_{\beta} (g^{-1})_{\eta}^{\beta} \delta_{\eta}^{\gamma} \]  

A IV-5

where \( \delta_{\eta}^{\gamma} = 1 \) if \( \gamma = \eta \), or 0 if \( \gamma \neq \eta \).

We note also that

\[ \frac{\partial \phi_{\eta}}{\partial x_k} = \sum_{\beta} \frac{\partial \phi_{\beta}}{\partial x_k} \frac{\partial \phi_{\eta}}{\partial \phi_{\beta}} \]

\[ = \sum_{\beta, \gamma} (g^{-1})_{\eta}^{\beta} \lambda_{\beta}^{\gamma} P_k^{\gamma} \]  

A IV-6

\[ = \sum_{\gamma} \Lambda_{\eta}^{\gamma} P_k^{\gamma} \]

where \( \Lambda_{\eta}^{\gamma} = \sum_{\beta} (g^{-1})_{\eta}^{\beta} \lambda_{\beta}^{\gamma}. \) (\( \Lambda \) may be a function of the \( x \)'s but does not depend on the index \( k \).)

Thus if

Thus if
\[
\frac{d\psi_\eta}{\delta x_k} = \sum_k \frac{\partial \psi_\eta}{\partial x_k} \, dx_k
\]

\[= \sum_{\gamma k} \lambda^\gamma_\eta \, F^{k}_\gamma \, dx_k \]

is to be a perfect differential, \(\lambda^\gamma_\eta\) is also an integration factor for the original equation A IV-1.

But in order that A IV-5 (equivalent to A III-3) be satisfied it is necessary that

\[\sum_{\gamma k} \lambda^\gamma_\eta \, F^{k}_\gamma \, \alpha^\beta_k = \delta^\beta_\eta \]

or that \(\lambda^\gamma_\eta = \left((P_\alpha^T)^{-1}\right)^\gamma_\eta\). Here \((P_\alpha^T)^{-1}\) is the inverse of the matrix product \(P_\alpha^T\); that is, \(\left((P_\alpha^T)^{-1}\right)^\gamma_\eta\) is the \(\gamma, \eta\) component of the inverse of \(\sum_k F^{k}_\eta \, \alpha^\gamma_k\).

(Again one makes the assumption that the needed inverse does in fact exist.)

The quadrature solution of the system A IV-1 is then given by

\[\int \sum_{k\gamma} \left((P_\alpha^T)^{-1}\right)^\gamma_\eta \, F^{k}_\gamma \, dx_k = K_\eta\]

where the \(K_\eta\) are the \(M\) arbitrary constants of the system and the path of integration is chosen for convenience.
APPENDIX V

Definition of Lie Algebras and Lie Groups

A Lie algebra and corresponding Lie group are defined as follows. A set of vectors a, b, c . . . is said to form a Lie algebra under an operation (denoted by [:,]) if the following conditions are satisfied:

1) the result of the operation [a, b] is a member of the set for all a and b in the set (closed).
2) \([a + c, b] = [a, b] + [c, b]\) (linearity).
3) \([a, b] + [b, a] = 0\) (antisymmetric).
4) \([a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0\) (Jacobi identity).

(The entities a, b, c . . . are vectors in some vector space in the sense that multiplication by a constant and vector addition are defined in the usual way. A norm may or may not be defined.)

Associated with every Lie algebra will be a Lie group. A group is obtained by putting the elements of the algebra a, b, c . . . in a one-to-one correspondence with a set of operators A, B, C . . . such that for all a, b, c, A, B, C, if \(c = [a, b]\) then \(C = [A, B]\) where \([A, B] = AB - BA\) is the commutator of A and B. The operators are to have quantities, \(\psi\), to operate on, and AB means operate first with B and then with A. The operators \(e^A, e^B, e^C . . .\) are then transformations on the \(\psi\) that form a corresponding Lie group, where

\[ e^A = I + A + AA/2! + AAA/3! + . . . \] and I is the identity operator.

Since a Lie algebra or group is defined for a set of vectors, it will have a set of basis vectors, \(x_m\), such that any member of the set can be given as a linear combination of the \(x_m\). (The index m may take on discrete values, either finite or infinite, or a continuous set of values, or have both a discrete range and a continuous range of values.) In terms of the basis vectors, condition (1) above can then be replaced by the condition
\[ [x_k, x_j] = \sum_m c_{k,j}^m x_m \]

where the \( c_{k,j}^m \) are constants and \( \sum_m \) indicates a sum over the discrete plus an integral over the continuous range of \( m \). These \( c_{k,j}^m \) are the structure constants of the Lie algebra or group and completely define it.

The most familiar example of a Lie algebra is formed by the vector cross product operation in three dimensional space. Here we define the unit vectors \( \hat{i}, \hat{j}, \text{ and } \hat{k} \), and their cross products, so that

\[ [\hat{i}, \hat{j}] = \hat{k} \]
\[ [\hat{j}, \hat{k}] = \hat{i} \]
\[ [\hat{k}, \hat{i}] = \hat{j} \]

The cross product is also antisymmetric and satisfies the Jacobi identity so this system is the basis of a Lie algebra. The corresponding Lie group is formed by putting the rotation operators in a one-to-one correspondence with \( \hat{i}, \hat{j}, \text{ and } \hat{k} \) and then letting the commutator correspond to the cross product

\[ \hat{i} \mapsto L_x = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \]
\[ \hat{j} \mapsto L_y = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \]
\[ \hat{k} \mapsto L_z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \]

Then

\[ [L_x, L_y] = (L_x L_y - L_y L_x) = L_z \] etc.
The transformation operators $e^{L_x}$, $e^{L_y}$, $e^{L_z}$ form the basis of the corresponding Lie group. This is the proper orthogonal or rotation group in three dimensions, usually called for short $SO(3)$. 
APPENDIX VI

A Discussion of Morgan's Theorems for Systems of Ordinary Differential Equations

In this appendix, the outline of a proof of Morgan's theorems for ordinary differential equations is given. The notation used here is the same as that in Morgan's paper [20] to which the reader is referred.

If \( G_1 \) is a continuous one-parameter group of transformations of the independent variable \( x \) and the dependent variables \( y_1, \ldots, y_n \) of a system of differential equations of the form \( \dot{y}_\delta = 0, \delta = 1, \ldots, n \), then a transformation in \( G_1 \) is of the form

\[
\begin{align*}
    \bar{x} &= f_0(x; a) \\
    \dot{\bar{y}}_\delta &= f_\delta(y_\delta; a), \delta = 1, \ldots, n
\end{align*}
\]

where \( a \) is a numerical parameter and the transformations \( x \rightarrow \bar{x} \) form a subgroup \( S_1 \) of \( G_1 \). Let \( G_1^{\infty} \) denote the enlargement of \( G_1 \) formed by adding successively to \( G_1 \) the transformations among the first, second, \ldots, and \( k \)-th derivatives of the \( y_\delta \).

Now consider \( x, y_1, \ldots, y_n \) to be independent variables. As Morgan indicated, \( G_1 \) has \( n \) functionally independent absolute invariants

\( g_1(y_1, \ldots, y_n, x), \ldots, g_n(y_1, \ldots, y_n, x) \) such that \( \frac{\partial (g_1, \ldots, g_n)}{\partial (y_1, \ldots, y_n)} \neq 0 \).

Now, consider the \( \bar{y}_\delta \) and \( \bar{y}_\delta \) to be implicitly defined as functions of \( x \) and \( \bar{x} \), respectively, by the equations

\[
Z_\delta(x) = g_\delta(y_1, \ldots, y_n, x)
\]

and

\[
Z_\delta(\bar{x}) = g_\delta(\bar{y}_1, \ldots, \bar{y}_n, \bar{x})
\]

where the \( g_\delta \) are the absolute invariants of \( G_1 \).
Then, with \( m = 1 \), Morgan's Th 1 takes the form:

**Th 1:** A necessary and sufficient condition for the \( y_0 \), implicitly defined as functions of \( x \) by the equations \( Z_0(x) = g_0(y_1, \ldots, y_n, x) \) to be exactly the same functions of \( x \) as the \( \bar{y}_0 \), implicitly defined as functions of \( \bar{x} \) by \( Z_0(\bar{x}) = g_0(\bar{y}_1, \ldots, \bar{y}_n, \bar{x}) \), are of \( \bar{x} \) is that

\[
Z(x) = Z(\bar{x}) = Z(\bar{x})
\]

or, equivalently, that \( Z \) is a constant function.

The proof is analogous to that given by Morgan.

Then, considering \( x \) and the \( y_0 \) to be the independent variable and the dependent variables, respectively, of a system of differential equations, we define:

**Def 1:** By *invariant solutions of a system of differential equations* is meant that class of solutions of a system of differential equations which have the property that the \( y_0 \) are exactly the same functions of \( x \) as the \( \bar{y}_0 \) are of \( \bar{x} \).

Theorem 1 makes it possible to reduce the problem of finding invariant solutions of a system of differential equations to one of finding solutions which satisfy relations of the form

\[
Z_0(x) = g_0(y_1, \ldots, y_n, x)
\]

where

\[
Z_0(x) \text{ is constant.}
\]

Then, since the conditions of the implicit function theorem are satisfied, the \( y_0 \) may be written in terms of \( x \) and these constants.

In the case where there is one independent variable, Morgan's Def 2 takes the form:
Def 2: By a differential form of the $k$-th order is meant a function of the form,

$$\phi(x, y_1, \ldots, y_n, \frac{dy_1}{dx}, \ldots, \frac{dy_n}{dx}, \ldots, \frac{d^k y_1}{dx^k}, \ldots, \frac{d^k y_n}{dx^k})$$

whose arguments are the independent variable $x$, the functions $y_1, \ldots, y_n$ dependent on $x$ and the derivatives of the $y_\delta$ up to the $k$-th order.

If each of these arguments transforms under the transformation laws of a continuous one-parameter group with symbol $V$ and numerical parameter $a$, then the arguments may be considered as independent variables of the group with symbol $V$ and called $Z_1, Z_2, \ldots, Z_p$, where $p = (k + 1)n + 1$.

Def 3: A differential form $\phi$ will be said to be conformally invariant under a one-parameter group $G_\lambda$ if, under the transformations of the group, it satisfies the relation.

$$\bar{\phi}(\bar{Z}_1, \ldots, \bar{Z}_p) = \Phi(Z_1, \ldots, Z_p; a) \phi(Z_1, \ldots, Z_p),$$

where $\bar{\phi}$ is exactly the same function of the $Z$'s as it is of the $\bar{Z}$'s and $\Phi$ is some function of the $x$'s and the parameter $a$.

If $\phi$ satisfies the above relation with $\Phi$ a function of $a$ only, $\phi$ is said to be constant conformally invariant; if the relation is satisfied with $\Phi$ identically equal to one, then $\phi$ is said to be absolutely invariant.

Th 2: If $\phi$ is a differential form of the $k$-th order and is at least in class $C^{(1)}$ with respect to each of its arguments, then a necessary and sufficient condition for $\phi$ to be conformally invariant under a one-parameter group of transformations with symbol $V$ is that

$$V\phi = \omega(Z_1, \ldots, Z_p) \phi(Z_1, \ldots, Z_p),$$

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for some $w(Z_1, \ldots, Z_p)$, or equivalently, that

$$\Psi(Z_1, \ldots, Z_p) = e^\zeta(Z_1, \ldots, Z_p)$$

$$\Psi_0(Z_1, \ldots, Z_p),$$

where $\Psi_0$ is a general absolute invariant of $V$ and $\zeta(Z_1, \ldots, Z_p)$ is a determinable function of $Z_1, \ldots, Z_p$.

The proof of Th 2 is as indicated by Morgan.

Def 4: It is said that a \textit{system of k-th order differential equations} $\Psi_0 = 0$ is \textit{invariant under a continuous one-parameter group of transformations} $G_1$ if each of the differential forms $\Psi_0$ is conformally invariant under the transformations of $G_1$.  

Th 3: If each of the k-th order differential equations $\Psi_1, \ldots, \Psi_n$ in a system of differential equations, with independent variable $x$ and dependent variables $y_1, \ldots, y_n$, is conformally invariant under the k-th enlargement of a continuous one-parameter group $G_1$ of transformations, then the invariant solutions of the system can be expressed in terms of $x$ and the constants $Z$ where $Z_0(x) = g_0(y_1, \ldots, y_n, k)$, the $g_0$ being functionally independent absolute invariants of $G_1$ (considering $x, y_1, \ldots, y_n$ as the independent variables).

The proof follows directly from theorem 1 and definition 1.
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(Continued)
VI. REFERENCES AND BIBLIOGRAPHY (Continued)


