PREFACE

The recent widespread interest in single-sideband systems of intelligence transmission and reception has indicated the need for $90^\circ$ phase-difference circuits which operate over a considerable span of audio frequencies. The advantages of single-sideband behavior which prompted this interest are manifold; notable among these are the savings of spectrum space and lower power requirements per unit of transmitted intelligence.

The phase-difference systems described in this report stem from the writings of Professor M. A. Honnell of Georgia Institute of Technology, who pointed out the importance of the $90^\circ$ phase-difference systems, and Mr. R. B. Dome of the General Electric Company, who first published information regarding the semi-lattice networks of the type which the author prefers to class as special cases of one-pole networks. Numerous others have used the one-pole networks in single-sideband systems, and one investigator, Mr. D. G. C. Luck, of RCA, has advanced a unique analysis of one-pole network operation. However, to the author's knowledge, no published data is available on higher pole networks.

The $90^\circ$ phase-difference systems described in this report are by no means limited to single-sideband application. Many and varied uses appear from time to time for two equal voltages which are continuously different in phase by $90^\circ$ over a wide frequency range. Furthermore, the methods of this report may be adapted to produce networks of phase difference other than $90^\circ$.

Approved: 

Gerald A. Rosselot, Director
State Engineering Experiment Station

Respectfully submitted:

Daniel C. Fielder,
Project Director
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<tr>
<td>D</td>
<td>Error Calculation Term (appears in Appendix only)</td>
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<tr>
<td>D'</td>
<td>Phase-Shift Section Output-Input Ratio</td>
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<tr>
<td>F</td>
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<td>I</td>
<td>Image Subscript</td>
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<td>I</td>
<td>Effective Current</td>
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<td>K</td>
<td>Arbitrary Constant</td>
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<tr>
<td>L</td>
<td>Inductance</td>
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<tr>
<td>M</td>
<td>Mid-Frequency or Mid-Log Frequency Subscript</td>
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<tr>
<td>Q</td>
<td>Quality Factor $\omega L/R$ or $1/\omega RC$</td>
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<tr>
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<td>V</td>
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<td>x</td>
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<td>$\beta$</td>
<td>Phase Angle Variable</td>
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γ  Error Variable
δ  Center Spacing in Log₁₀ Units
ζ  Error Curve Shift Angle
η  Channel Spacing Factor
η₀  Channel Spacing in Log₁₀ Units
λ  Change in Channel Spacing Factor
Θᵣ  Image Transfer Constant
π  3.14159......
σ  Unit Channel Spacing in Log₁₀ Units, 6/2
ψ  Arbitrary Phase Constant
ω  Angular Velocity, 2πf
±  Is approximately equal to
≠  Is not equal to
>  Is greater than
<  Is less than
≥  Is equal to or greater than
≤  Is equal to or less than
»  Is much greater than
«  Is much less than
∑ₖ₌₁  Summation from 1 to k
±  Plus or minus
₀  Zero Frequency Subscript
∞  Infinity, Infinite Frequency Subscript
|x|  The Absolute Value of x
ABSTRACT

Various communication applications require the use of 90° phase-difference systems. Such systems may range from a simple R-C combination to a complex network configuration of many elements. As restrictions are placed on the operating characteristics, the networks obviously become more complicated in structure.

The systems described in this report are restricted to yield two audio voltages different in phase by 90°, equal in absolute value to each other and equal to a set fraction of a common input audio voltage. The audio frequency range over which the foregoing conditions prevail determines to a great extent the usefulness of such systems.

Although any practical system must eventually include dissipative elements as well as purely reactive elements, the mathematics involved in the solutions is extremely unwieldy. However, relatively simple mathematics suffice for purely reactive system solutions. Accordingly, it is possible to determine certain fundamental equations on a purely reactive basis and then force the dissipative element solution to duplicate the required equations. This method of approach is applied to phase-frequency formulas and specific solutions for one-, two- and three-pole systems are thereby found. By noting the trend in the one-, two- and three-pole cases, the general, or n-pole, solution is obtained.

It is important to have a means of ascertaining solutions for the higher network complexities, as the audio span for any given allowable error is a function of the network complexity. By means of the general, or n-pole, solution, networks or systems of any complexity may be determined merely by replacing n by the desired complexity number.
Considerable tabulated and graphical data are presented for two- and three-pole systems. In addition, a numerical example of a three-pole system is solved and the results arranged in a manner readily adaptable for extension to four- or higher pole systems.

The methods by which the various solutions are found are included in the report, either in the main body or in one of several appendices.

PURPOSE

A mathematical analysis of 90° phase-difference systems which will lead to simplified design data is to be conducted.

The analysis is to consist of a resume of solution methods for one- and two-pole systems and the development of the techniques for extension to three- and higher pole systems. Both the purely reactive and dissipative cases are to be considered.

The design data are to be presented in the form of formulas, tabulated results, curves, diagrams and a specific numerical example of a three-pole solution.
CHAPTER I
THEORY AND BACKGROUND MATERIAL

I-1 Introduction

Certain types of single-sideband transmitting and receiving systems utilize two audio voltages which, theoretically, should satisfy the following requirements:

(a) equality of magnitude over a specified range of frequencies,
(b) 90° difference in phase over the specified range of frequencies and
(c) constancy of magnitude over the specified range of frequencies.

The requirements may be illustrated by considering one voltage as \( V_A \) and the other as \( V_B \), where \( V_A \) is equal to \( K V_C \sin (2 \pi ft + \psi) \) volts, and \( V_B \) is equal to \( K V_C \sin (2 \pi ft + \psi + 90°) \) volts. The equality condition is satisfied since the magnitude, \( K V_C \), of \( V_A \) is the same as the magnitude, \( K V_C \), of \( V_B \). The term, \( 2 \pi ft \), is the radian measure associated with any frequency, \( f \), in the specified range. To render the discussion more general, an arbitrary phase constant, \( \psi \), is included. Thus, it is evident that the difference between the angles of \( V_A \) and \( V_B \) is \( \pm 90° \). Condition (c) implies that the magnitudes of voltages \( A \) and \( B \), while equal because of condition (a), must be constant multiples of the magnitude of the voltage, \( V_C \sin 2 \pi ft \). \( V_C \) may change, but \( K \) must not. If it were possible to satisfy all three conditions at the same time, some of the major problems of single-sideband transmission and reception would be simplified immeasurably.

---

While it is theoretically possible to fulfill two of the conditions simultaneously, there is no assurance that the remaining condition will be perfectly satisfied. As in many physical problems, there does exist a minimum state for the remaining condition in which the difference between the desired and the attainable is least.

I-2 The Problem in Terms of Purely Reactive Full-Lattice Systems

Before an analysis of the network configurations which tend to satisfy conditions (a) through (c) is made, a discussion of the problem in terms of purely reactive, full lattices is in order. The phase-frequency and amplitude characteristics of properly terminated lattices bear a distinct similitude to those of the 90° phase-difference networks later described herein.

A lattice-type network may be employed to yield an output voltage, the magnitude of which remains a constant multiple of the input voltage and the phase angle of which is a nearly linear function of the common logarithm \( \log_{10} \) of frequency over a wide frequency range. The circuit parameters and terminations must be properly designated in order to obtain the above type of operation. The parameters for two lattice networks may be chosen such that the central linear portions of the phase curves are separated by a predetermined number of log frequency units which, in turn, may be interpreted in terms of a frequency ratio. By connecting the two lattice networks to a common source, it is possible to obtain a reasonably constant difference in phase between the output voltages over a considerable portion of the range shared mutually by the linear sections of the two curves. With emphasis again on the choice of circuit parameters, it may be stated that an approximately 90° difference setting may be found.
among the infinite variety of possible phase-difference positions. The width of the frequency range over which the approximately $90^\circ$ difference prevails is governed by the complexity of the lattice impedance arms, while the location of the frequency range depends on certain frequencies associated with the impedance behavior groups of the circuit elements. Discussion of phase differences other than $90^\circ$ is omitted, not because of lack of interesting possibilities, but because the primary concern of this report is in $90^\circ$ systems.

In general, the mathematics required for the solution of purely reactive networks is less cumbersome than is that required for resistive-reactive networks. For this reason, lattice networks having reactive elements are considered in the earlier derivations of phase-frequency formulas. Before proceeding with the analysis of lattice-type networks, it is desirable to review a few concepts of impedance and lattice theory.

The circuit engineer has borrowed the terms, "poles" and "zeros" (of impedance and admittance), from complex variable theory in order to describe circuit behavior more eloquently. A pole of impedance at a certain frequency indicates that the impedance of the branch in question is infinite at that frequency. Likewise, a zero of impedance implies zero impedance at some particular frequency. Conversely, a zero of admittance implies zero admittance, and a pole of admittance implies infinite admittance. It may thus be seen that a pole of impedance is identical to a zero of admittance, and conversely. As examples, the anti-resonant frequency of a parallel L-C branch is the frequency at which a pole of impedance or zero of admittance occurs, and, in the converse sense, the reso-
nant frequency of a series L-C branch is the frequency at which a zero of impedance or pole of admittance occurs.

When several sets of parallel L-C branches are connected in series, poles of impedance appear at the anti-resonant frequencies of each branch. Zeros of impedance appear at frequencies between the frequencies for poles. The general formula for the impedance of the entire combination is

\[ Z = \frac{j \frac{f}{2\pi C}}{f} \frac{(f_2^2 - f^2)(f_4^2 - f^2)}{(f_1^2 - f^2)(f_3^2 - f^2)} \cdots \frac{(f_{m-1}^2 - f^2)}{(f_m^2 - f^2)} \]  

(1)

where the number in parentheses in the denominator denotes the number of poles of impedance and the frequency terms, \(f_1, f_3, \ldots, f_m\), specify the locations of the poles. Zeros of impedance occur at \(f\) equal to \(f_2, f_4, \ldots, f_{m-1}\). The constant, \(C\), is the value of capacitance which the impedance approximates as \(f\) approaches infinity. As \(f\) approaches zero, the impedance takes on the characteristics of an inductive reactance. For this reason, the impedance is generally designated an L-C type to indicate behavior at zero and infinite frequencies.

When several sets of series L-C branches are connected in parallel, zeros of impedance occur at the resonant frequencies of each set. This time, it is the frequencies for poles of impedance which appear between the frequencies for zeros. The corresponding formula for this type of impedance is

\[ Z = -j \frac{2\pi L}{f} \frac{(f_1^2 - f^2)(f_3^2 - f^2)}{(f_2^2 - f^2)(f_4^2 - f^2)} \cdots \frac{(f_{m-1}^2 - f^2)}{(f_m^2 - f^2)} \]  

(2)

where the number of parentheses in the numerator denotes the number of zeros of impedance and the frequency terms, \(f_1, f_3, \ldots, f_m\), specify the
location of the zeros. Poles of impedance occur at \( f = f_2, f_4, \ldots, f_{m-1} \). The constant, \( L \), is the value of the inductance which the impedance approximates at infinite frequencies. The designation C-L is applied to this type of impedance to specify behavior at zero and infinite frequencies.

Should an \( n \)-pole L-C or an \( n \)-zero C-L impedance be desired, the \( m \) of equations (1) and (2), respectively, may be obtained from the relation

\[
m = (2n - 1),
\]

where \( m \) is the total number of \( f_k \)'s. In specifying the number, \( n \), possible poles or zeros at zero and infinite frequencies are not included. It remains necessary to specify the constant, \( C \) or \( L \), as the case may be, and the frequencies, \( f_1, f_2, f_3, \ldots, f_m \), in order to obtain analytical expressions of impedance as a function of frequency. Typical reactance-frequency curves and circuit diagrams are shown in Figure 1 for the L-C and C-L types with \( n \) equal to 2.

A lattice-type filter section, when properly terminated, may be utilized to provide transmission characteristics such that the image impedance and image-transfer constant are specified independently of each other. A typical lattice section is shown in Figure 2. The image impedance, \( Z_I \), and image transfer constant, \( \Theta_I \), are, in terms of the lattice impedance arms,

\[
Z_I = \sqrt{Z_a Z_b}
\]

and

\[
\Theta_I = \alpha + j\beta = 2 \tanh^{-1} \sqrt{\frac{Z_a}{Z_b}}
\]

where \( \alpha \) equals the attenuation in nepers and \( \beta \) the phase shift in radians between input and output terminals. The advantages of manipulations with reactive elements are apparent when \( Z_a \) and \( Z_b \) are pure reactances of oppos-
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(a) L-C Network

(b) C-L Network

FIGURE 1.
Reactive Network Characteristics

FIGURE 2.
Lattice Structure And Ideal Transformer

Ideal Transformer
\[ R_s/R = \left(\frac{N_1}{N_2}\right)^2 \times (Turns \ Ratio)^2 \]
site sign. Under these circumstances, the attenuation is zero,

$$\beta = 2 \tan^{-1} \left( j \frac{Z_a}{Z_b} \right)^*$$  \hspace{1cm} (6)

and the image impedance is a pure resistance. Further, if $Z_a$ and $Z_b$ are chosen such that

$$Z_a Z_b = \pi^2$$  \hspace{1cm} (7)

at all frequencies, it is possible to secure a value, $R$, for the terminating resistance which will insure a reflectionless impedance match at all frequencies. (When restricted by a relationship such as (7), impedances are said to be reciprocal impedances.) Should the source impedance also equal $R$, proper matching will be accomplished at both ends of the lattice. If the source impedance does not equal $R$, proper matching may be accomplished by means of an ideal transformer** connected between the source and the lattice input terminals. This arrangement is illustrated in Figure 2.

The general criterion for reciprocation is the fact that in the pass band the poles of impedance of $Z_a$ must occur at the same frequencies as do the zeros of $Z_b$, and vice versa. Although a single inductance may be compared with a single capacitance for reciprocation, this case will not be discussed, since no poles or zeros exclusive of those at zero and infinite frequencies appear.

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* Equation (6) refers to the angle of $V_{3\frac{1}{2}}$ in order to be consistent with the later development of the angle expression for $V_{3\frac{1}{2}}$ of the semi-lattice network. The usual form of equation (6) is $\beta = 2 \tan^{-1} \left( -j \frac{Z_a}{Z_b} \right)$ which expresses the angle of $V_{34}$.

** An ideal transformer is a theoretical air-core transformer having infinite self inductance and zero resistance in the windings. The transformation ratio, $(N_1/N_2)^2$, should equal $R_s/R$ in the above instance, where $R_s$ is the source impedance. Strictly speaking, ideal transformers cannot be realized in a practical sense, but the concept is extremely valuable in theoretical circuit calculations.
When reciprocal networks of the types described form the arms of a suitably terminated lattice, the result is an all-pass structure having zero attenuation at all frequencies. The total phase shift obtained in passing through the entire positive frequency spectrum is an integral number of $-2\pi$ radians, depending on the complexity of the reciprocal networks. The term, \( n \)-pole, is applied to define a reciprocal lattice network having \( n \)-poles associated with the L-C arm and \( n \)-zeros associated with the C-L arm. Employing this terminology, it may be stated that a total phase shift of $-2\pi n$ radians between source and load occurs as the source frequency is varied from zero to infinity in a properly reciprocated and terminated "\( n \)-pole" circuit. The shape of the phase versus log\( _{10} f \) curve is dependent on the relative values of the constants, \( C \) and \( L \), and the \( f_k \)'s of equations (1) and (2). An unlimited variety of curve shapes is available. Judicious choice of \( C \), \( L \) and the \( f_k \)'s results in curves which show the phase shift to be approximately linear over a central log-frequency range.

Equation (6), when examined critically, gives rise to some interesting results. When equation (6) is rearranged, the phase angle, \( \beta \), for a properly terminated and reciprocated reactive lattice becomes

\[
\beta = \tan^{-1} \frac{2j \sqrt{Z_a/Z_b}}{1 + Z_a/Z_b} = \tan^{-1} \frac{2jR}{Z_a + Z_b}.
\]

(8)

Substitution of the expressions for \( Z_a \) from equation (1), and \( Z_b \) from equation (2) in the first relation of equation (8) yields

\[
\beta = \tan^{-1} \frac{2 \left[ \frac{\pi \sqrt{LC}}{2} \cdot f \left( f^2 - f_1^2 \right) \left( f^2 - f_2^2 \right) \ldots \ldots \left( f^2 - f_m^2 \right) \right]}{(f^2 - f_1^2)(f^2 - f_2^2) \ldots \ldots (f^2 - f_m^2) - \frac{1}{4\pi^2} \cdot \frac{1}{LC} \left( f^2 - f_k^2 \right) \left( f^2 - f_k^2 \right) \ldots \ldots (f^2 - f_m^2)^2}.
\]

(9)
The quantities, \( \frac{1}{2\pi\sqrt{\frac{1}{LC}}} \) and \( \frac{1}{4\pi^2\frac{1}{LC}} \), may be evaluated in terms of certain constants of the expressions for \( Z_a \) and \( Z_b \). By imposing the conditions that

1. \( Z_a \) and \( Z_b \) be reciprocal pure reactances; i.e., \( Z_aZ_b = R^2 \) (a constant) at all frequencies;
2. poles of impedance of \( Z_a \) occur at the same frequencies as do zeros of impedance of \( Z_b \), and conversely; and
3. \( f_n \), the frequency at which one-half of the total possible phase shift occurs, be the geometric mean between \( f_{n-q} \) and \( f_{n+q} \) where \( q \) is any positive integer from zero through \( n-1 \),

\[
\frac{1}{2\pi\sqrt{\frac{1}{LC}}} = \sqrt{\frac{L_{ao}}{L_{b\infty}}} \cdot f_n = \mathcal{S} f_n, \quad (10)
\]

where \( L_{ao} \) is the inductance of \( Z_a \) at zero frequency, \( L_{b\infty} \) is the inductance of \( Z_b \) at infinite frequencies and \( L \) and \( C \) have the same meanings as before.

The validity of equation (10) is demonstrated in Appendix I.

Equation (9), upon substitution of equation (10), becomes

\[
\beta = \tan^{-1} \left( \frac{2Sf f_n (f^2 - f_1^2)(f^2 - f_2^2)(f^2 - f_3^2)\cdots(f^2 - f_m^2)}{(f^2 - f_1^2)(f^2 - f_2^2)(f^2 - f_3^2)\cdots S f^2 f_1^2} \right), \quad (11)
\]

In particular, for \( n = 1 \),

\[
\beta = \tan^{-1} \left( \frac{2Sf f_1 (f^2 - f_1^2)}{(f^2 - f_1^2)^2 - S f^2 f_1^2} \right), \quad (12)
\]

and for \( n = 2 \),

\[
\beta = \tan^{-1} \left( \frac{2Sf f_2 (f^2 - f_1^2)(f^2 - f_2^2)(f^2 - f_3^2)}{(f^2 - f_1^2)(f^2 - f_2^2)(f^2 - f_3^2)^2 - S f^2 f_2^2 (f^2 - f_2^2)^2} \right). \quad (13)
\]
Curves of $\beta$ versus frequency and $\log_{10} f$ with $s$ equal to 4, 8, and 12 for the $n$ equal to 1 case and with $s$ equal to 22.345 for the $n$ equal to 2 case are shown in Figures 3 and 4, respectively.

The value of $s$ determines to a great extent the behavior of the phase-log frequency curves in the region about $f_n$. The statement may be verified qualitatively, at least, for the curves of Figure 3. Only in the region about $f_1$ do the curves for $s$ equal to 4, 8, and 12 differ appreciably. The curve for $s$ equal to 4 appears to be the most linear of the three in the central region about $f_1$. Using a value of $s$ equal to 4, a different value of $f_1$ yields a curve which is identical in shape to the $s$ equal to 4 curve shown in Figure 3, but which is shifted parallel to the log frequency axis and centered about the new value of $f_1$. By choosing appropriate values of the $f_1$'s for two, one-pole lattice networks with $s$ equal to 4, it is possible to obtain two curves, the vertical phase difference between which is approximately 90° over a wide frequency range.

The remaining sections of this report are devoted to the analytical discussion of methods for obtaining the constants of theoretical and practical circuits which duplicate the desired 90° difference in phase over a wide frequency range.

I-3 One-Pole Semi-Lattice Systems

The phase-frequency equations of the previous section, while of the desired form, were obtained from purely reactive, full-lattice networks requiring ideal transformers for impedance matching purposes. Although it is realized that a pure reactance, especially a pure, inductive reactance, is as much a physical impossibility as is the ideal transformer, it is feasible to derive a circuit which eliminates the pure inductive reactance...
FIGURE 3
Phase-Frequency Curves for One-Pole Lattice or Semi-Lattice for Various Values of $s$. 

$\delta$ in Degrees

$F$ in Log Frequency Units

$f$ in Cycles Per Second
FIGURE 4
Phase-Frequency Curve For Two-Pole Lattice Or Semi-Lattice
ance, while no such equivalent replacement for the ideal transformer may be found. For the present, the logical step forward is to examine circuits which eliminate the need for the ideal transformer. The "semi-lattice" network serves this purpose.

The semi-lattice network is similar to the full lattice but has one half the number of impedance arms. The source voltages for the semi-lattice make up the remaining arms of the lattice. These voltage sources are assumed to have zero internal impedance and are equal in magnitude to each other but are opposite in polarity. It will be demonstrated that the purely reactive, semi-lattice network when properly reciprocated and terminated has phase-frequency and amplitude characteristics identical to those of the full lattice. Several forms of the semi-lattice, which are electrically equivalent, are shown in Figure 5. A derivation of the output voltage, $V_{34}$, in terms of an input voltage, $V_{13}$, is included in Appendix II.

Three variations of semi-lattice networks which reproduce phase-frequency characteristics of one-pole reciprocated lattices have appeared in recent literature. Since the one-pole network forms a basic unit for practical phase-difference systems, the three, one-pole networks, which are shown in Figures 6, 7 and 8, respectively, are examined in considerable detail.

As a minor digression, it is interesting to note the manner in which the so-called zero impedance voltage sources for the semi-lattices are obtained. In each of the networks, the phase inverting properties of

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(a) Lattice Form

(b) Bridge Form

Note: \( V_{13} = -V_{23} \)
In All Forms

(c) Mesh Form

FIGURE 5
Equivalent Semi-Lattice Forms
FIGURE 6
Purely Reactive One-Pole Semi-Lattice

FIGURE 7
Resistive-Reactive One-Pole Semi-Lattice
with Isolation

FIGURE 8
Resistive-Reactive One-Pole Semi-Lattice
with no Isolation
equal plate and cathode resistors are used to obtain the required input voltages. If the resistances are sufficiently small compared with the impedances of the other elements in the semi-lattice section, the source impedance is negligible. In addition to being instrumental in obtaining source voltages, the vacuum tube circuit also serves to isolate the phase-shift section from other portions of the circuit. It is assumed that no phase shift occurs within the vacuum tube.

I-4 Purely Reactive One-Pole Semi-Lattice

The first network to be considered is shown in Figure 6. This network contains purely reactive elements in combination with a single resistance as the output load. It will be shown that the network of Figure 6 is the semi-lattice counterpart of the purely reactive full lattice and, like the purely reactive full lattice, exists only in a theoretical or mathematical state.

Appendix II illustrates that

\[ V_{34} = V_{13} \frac{(Z_1 - Z_2)}{Z_1 Z_2 + (Z_1 + Z_2)} \]  \hspace{1cm} (14)

In the network of Figure 6,

\[ Z_1 = \frac{j\omega L_{111}}{(1 - \omega^2 L_{111} C_{111})} \] \hspace{1cm} (15)

\[ Z_2 = \frac{(1 - \omega^2 L_{211} C_{211})}{j\omega C_{211}} \] \hspace{1cm} (16)

and

\[ Z_3 = R_{31} \] \hspace{1cm} (17)

where \( \omega \) equals \( 2\pi f \).
For further simplicity, if \( L_{111} C_{111} \) equals \( k_1 \), \( L_{211} C_{211} \) equals \( k_2 \) and \( L_{111} C_{211} \) equals \( k_3 \), equation (14) becomes

\[
V_{34} = V_{13} \frac{-\omega^4 k_1 k_2 - \omega^2 (k_3 - k_2 - k_1) - 1}{[\omega^4 k_1 k_2 - \omega^2 (k_1 + k_2 + k_3) + 1] - j \left[ \omega^3 k_2 \frac{L_{111}}{R_{31}} - \omega \frac{L_{111}}{R_{31}} \right]}. \tag{18}
\]

The phase angle between \( V_{34} \) and \( V_{13} \) is the angle of the coefficient of \( V_{13} \) in equation (18). The ratio of the magnitude of \( V_{34} \) to \( V_{13} \) is given by the magnitude of the coefficient. Thus, if the magnitude of the coefficient is equal to unity at all frequencies, the magnitude of \( V_{34} \) will always be equal to the magnitude of \( V_{13} \). Since the magnitude of a complex quantity is equal to the square root of the sum of the squares of the magnitudes of the real and imaginary parts, the restriction which must be imposed in order to make \( |V_{34}| \) equal to \( |V_{13}| \) is

\[
\frac{-\omega^4 k_1 k_2 - \omega^2 (k_3 - k_2 - k_1) - 1}{\sqrt{\left[ \omega^4 k_1 k_2 + \omega^2 (k_3 - k_2 - k_1) + 1 \right]^2 + \left[ \omega^3 k_2 \frac{L_{111}}{R_{31}} - \omega \frac{L_{111}}{R_{31}} \right]^2}} = 1. \tag{19}
\]

Squaring each side of equation (19), transposing terms, and arranging them in descending powers of \( \omega \) results in

\[
\omega^8 (k_1^2 k_2^2 - k_3^2) + \omega^6 \left( -4 k_1 k_2^2 - 4 k_1^2 k_2 + k_2^2 \frac{L_{111}}{R_{31}^2} \right) + \omega^4 \left( 4 k_1 k_2 + 4 k_1 k_3 + 4 k_2 k_3 - 2 k_2^2 \frac{L_{111}}{R_{31}^2} \right) + \omega^2 \left( -4 k_3 + \frac{L_{111}}{R_{31}^2} \right) = 0. \tag{20}
\]

For equation (20) to be satisfied regardless of frequency, each coefficient of powers of \( \omega \) must equal zero. This is obviously true for the coefficient of the \( \omega^8 \) term. Consideration of the coefficient of \( \omega^2 \) yields...
With the substitution of $\frac{L_{ll}^2}{C_{2ll}^2}$ for $k_3$, equation (21) reduces to the condition

$$\frac{L_{ll}^2}{C_{2ll}^2} = 4 R_{3l}^2.$$  \hspace{1cm} (22)

When the coefficient of $\omega^4$ is set equal to zero, it is seen that

$$4 \left( k_1 k_2 + k_2 k_3 + k_1 k_3 \right) = 2 k_2 \frac{L_{ll}^2}{R_{3l}^2}.$$  \hspace{1cm} (23)

and when values for $k_1$, $k_2$ and $k_3$ are substituted, the condition,

$$\frac{2 L_{ll}^2 L_{2ll}}{C_{ll} L_{2ll} + C_{2ll} L_{2ll} + C_{ll} L_{ll}} = 4 R_{3l}^2.$$  \hspace{1cm} (24)

is indicated.

Similarly, the coefficient of the $\omega^6$ term may be arranged as

$$4 k_1 k_2^2 + 4 k_1^2 k_2 = k_2^2 \frac{L_{ll}^2}{R_{3l}^2}.$$  \hspace{1cm} (25)

or the condition,

$$\frac{L_{ll} L_{2ll} C_{2ll}}{C_{ll} \left( L_{2ll} C_{2ll} + L_{ll} C_{ll} \right)} = 4 R_{3l}^2.$$  \hspace{1cm} (26)

When $4R_{3l}^2$ is eliminated between equations (24) and (26),

$$\frac{L_{ll}^2}{C_{2ll}} = \frac{2 L_{ll} L_{2ll}}{C_{ll} L_{2ll} + C_{2ll} L_{2ll} + C_{ll} L_{ll}}.$$  \hspace{1cm} (27)

and

$$L_{ll} C_{ll} = L_{2ll} C_{2ll} - L_{2ll} C_{ll}.$$  \hspace{1cm} (28)
Each side having been divided by \( C_{111} C_{211} \), equation (28) becomes

\[
\frac{L_{111}}{C_{211}} = \frac{L_{211}}{C_{111}} - \frac{L_{211}}{C_{211}} = \frac{L_{211}}{C_{111}} \left( 1 - \frac{C_{111}}{C_{211}} \right) .
\]  

(29)

With \( C_{111} \) and \( C_{211} \) chosen such that

\[
\frac{C_{111}}{C_{211}} < \left| \frac{C_{211}}{C_{111}} \right|, \quad (30)
\]

then

\[
\frac{L_{111}}{C_{211}} = \frac{L_{211}}{C_{111}} ; \quad L_{111} C_{111} = L_{211} C_{211} ,
\]

(31)

or, in a practical sense,

\[
L_{111} C_{111} = L_{211} C_{211} .
\]

(32)

Equations (22), (24) and (26) must be satisfied in order for the coefficient of \( V_{13} \) to be exactly unity at all frequencies. If condition (30) is met, the relationships expressed by equations (22) and (32) are sufficient to insure practically unit value for the coefficient of \( V_{13} \) at all values of frequency.

It remains necessary to determine the phase angle of the coefficient of \( V_{13} \). By letting \( \beta \) be the phase angle, \(-\beta\) is the phase angle of the reciprocal of the coefficient of \( V_{13} \). If follows that

\[
-\beta = \tan^{-1} \left[ \frac{-\omega^3 k_2 \frac{L_{111}}{R_{31}} + \omega \frac{L_{111}}{R_{31}}}{\omega^4 k_1 k_2 - \omega^2 (k_1 + k_2 + k_3) + 1} \right] ,
\]

(33)

and

\[
\beta = \tan^{-1} \left[ \frac{\omega^3 k_2 \frac{L_{111}}{R_{31}} - \omega \frac{L_{111}}{R_{31}}}{\omega^4 k_1 k_2 - \omega^2 (k_1 + k_2 + k_3) + 1} \right] .
\]

(34)
At a value of \( \omega \) equal to \( \omega_1 \), a pole of impedance occurs in \( Z_1 \) since \( f_1 \) equals \( \omega/2\pi \) and \( f_1 \) is the anti-resonant frequency of the parallel \( L-C \) combination which makes up \( Z_1 \). In terms of \( L_{111}C_{111} \),
\[
\frac{\omega_1}{2\pi} = f_1 = \frac{1}{2\pi} \sqrt{\frac{1}{L_{111}C_{111}}}.
\] (35)

Further, the resonant frequency of a single, series \( L-C \) combination is also equal to \( 1/2\pi \) multiplied by the square root of the reciprocal of the LC product. At the resonant frequency, a zero of impedance occurs for the series \( L-C \) combination. By choosing values of \( C_{111} \) and \( C_{211} \) such that condition (30) and equations (22) and (32) are satisfied for unity magnitude, the resonant frequency of \( Z_2 \) becomes
\[
\frac{\omega_1}{2\pi} = f_1 = \frac{1}{2\pi} \sqrt{\frac{1}{L_{211}C_{211}}}.
\] (36)

By letting \( C_{211}/C_{111} \) equal \( s^{2k} \) where \( s \) is a constant of the circuit, it is seen that for a sufficiently large value of \( s \), the difference between \( L_{111}C_{111} \) and \( L_{211}C_{211} \) is actually so slight that the above equalities are valid to an extent far in excess of accepted engineering requirements. As an example, the difference between \( L_{111}C_{111} \) and \( L_{211}C_{211} \) when \( s \) equals 10 is one per cent of \( L_{111}C_{111} \). This illustration corroborates equation (32).

The constants may be restated as:
\[
k_1 = L_{111}C_{111} = \frac{1}{\omega_1^2}, \quad k_2 = L_{211}C_{211} = \frac{1}{\omega_1^2}.
\] (37)

*Note: The ratio, \( L_{111}/L_{211} \), is also equal to \( s^2 \), and \( s \) is equal to the square root of the ratio of the inductance of \( Z_1 \) at zero frequency to the inductance of \( Z_2 \) at infinite frequencies. See Appendix I.*
Substituting the expressions given in (37) above in equation (34), rearranging the terms and simplifying reduce equation (34) to

\[ \beta = \tan^{-1} \left( \frac{2Sw_1(\omega^2 - \omega_1^2)}{(\omega^4 - 2\omega^2 \omega_1^2 + \omega_1^4) - s^2 \omega^2 \omega_1^2} \right) \]  

or, in terms of \( f \),

\[ \beta = \tan^{-1} \left( \frac{2sf_1 (f^2 - f_1^2)}{(f^4 - 2f^2 f_1^2 + f_1^4) - s^2 f^2 f_1^2} \right) \]

which is precisely the expression developed in equation (12) as the phase angle of a one-pole reciprocated full lattice. Identical curves describe the phase-frequency behavior of the one-pole reciprocated full lattice and the one-pole semi-lattice discussed above.

Further comparisons may be made by noting that \( Z_1 \) and \( Z_2 \) fulfill the condition for reciprocation, namely

\[ Z_1 Z_2 = (2R_{31})^2 \]
In addition, the expression for $\beta$ reduces to

$$\beta = \tan^{-1} \frac{2j(2R_s)}{Z_1 + Z_2},$$

(41)

which is identical in form to equation (8) for the full lattice. Should $Z_1$ equal $Z_a$, $Z_2$ equal $Z_b$, $R_{31}$ equal $R/2$ and $V_{13}$ (and $-V_{23}$) equal the input voltage for the properly terminated full lattice, then $V_{34}$ will be the same in both magnitude and phase as the corresponding output voltage of the full lattice. Thus, an equivalent full lattice may be determined for the semi-lattice and vice versa.

An alternate, and often more convenient, means of expressing network behavior may be found by the substitution of $10^F$ for $f$, where

$$10^F = F; f = 10^F.$$ 

(42)

The relationships expressed by equations (42) are used interchangeably throughout this report. In terms of the $F$'s, $\beta$ becomes

$$\beta = \tan^{-1} \frac{2s \cdot 10^{F+F} \left(10^{2F} - 10^{2F} \right)}{\left(10^{2F} - 10^{2F} \right)^2 - s^2 \left(10^{2F} + 2F \right)}.$$  

(43)

(It should be noted that the curves of Figures 3 and 4 are plotted against a logarithmic frequency scale and a linear $F$ scale. One of the obvious advantages of $F$ notation is that differences in $F$ may be specified independently of the location of the curves.)

At $F$ equal to $F_1$, $\beta$ equals $-180^\circ$; at $F$ equal to $-\infty$, $\beta$ equals $0^\circ$, and at $F$ equal to $+\infty$, $\beta$ equals $-360^\circ$ (or $-2\pi$ radians). Reference to Figure 3 discloses that a point symmetry exists in each curve at $F$ equal to $F_1$. For the particular example shown, $F_1$ equals 3 when $f$ equals 1000 c.p.s. For equal departures of $F$ from $F_1$ in either direction, the differences be-
tween the value of \( p \) at \( F_1 \) and the values of \( p \) at \((F_1 - F)\) and \((F_1 + F)\) are equal in magnitude, though opposite in sign. It is evident from equation (43) that a change in the value of \( F_1 \) merely determines a new curve centered on the new \( F_1 \). The fact that the curves may be shifted in this manner provides a very convenient means of plotting curves with equal values of \( s \) but with different values of \( F_1 \).

The two channels, A and B, briefly described in the previous section for full lattices, are comprised of two networks, each identical to that shown in Figure 6 except for the values of the circuit elements in the semi-lattice or phase-shift sections. As before, references to similar quantities in each channel are identified by the subscript A or B following the quantity in question. The \( F_1 \)'s of the two networks are thus \( F_{1A} \) and \( F_{1B} \), respectively.

When energized by a common voltage source of variable frequency, channels A and B each cause definite phase shifts between the common input voltage and the output voltages, \( V_{34A} \) and \( V_{34B} \). By proper selection and adjustment of the constants of the phase-shift sections, \( V_{34A} \) and \( V_{34B} \) retain a phase difference of approximately 90° over a wide range of frequencies, or on a logarithmic basis, a wide range of \( F \). Should the source voltage change in magnitude and/or phase, the same per cent magnitude change and the same phase shift is experienced by both \( V_{34A} \) and \( V_{34B} \).

The region over which the approximately 90° phase difference prevails is symmetrical about a value of \( F \), designated as \( F_M \), which is midway between \( F_{1A} \) and \( F_{1B} \). More exactly, \( F_M \) is the arithmetic mean between \( F_{1A} \) and \( F_{1B} \). In accordance with the definition of \( f \) in terms of \( F \), the value \( f_M \) is the geometric mean between \( f_{1A} \) and \( f_{1B} \).
From an inspection of Figure 3, it is apparent that networks with $s$ equal to $4$ present good possibilities for obtaining $90^\circ$ phase differences over a wide range. Although the curve for $s$ equal to $4$ appears very linear in a broad range about $F_1$, the scale to which the curves are drawn is such that variations up to $4$ degrees are not readily noticeable. The very nature of a tangent curve prohibits exact linearity. Questions naturally arise in regard to the best location of $F_{1A}$ (or $f_{1A}$) and $F_{1B}$ (or $f_{1B}$) for a given $F_M$ (or $f_m$) with an optimum value of $s$. The precise answers to such questions for higher pole networks are deferred for discussion in a later section; however, an illustration of the immediate problem for one-pole systems should suffice to point out the need for a wise selection of values for $F_{1A}$ (or $f_{1A}$) and $F_{1B}$ (or $f_{1B}$).

In the discussion of one-pole networks in recent literature, the values corresponding to $f_{1A}$ and $f_{1B}$ are chosen under the assumption that $\phi_A$ at $f_m$ should equal $-225^\circ$ and $\phi_B$ at $f_m$ should equal $-135^\circ$. At $f_m$, $(\phi_A - \phi_B)$ equals $-90^\circ$, which satisfies the phase difference requirement exactly at $f_m$. For other values of $f$ the phase difference may vary from $90^\circ$ by an amount specified as the "error." The calculated error curve based on the above assumption is shown in Figure 9 for $s$ equal to $4$. For identification purposes, this method of obtaining error curves is called the "45° Point Method."

An alternate means for determining the location of $f_{1A}$ and $f_{1B}$ depends on the slope of the line, tangent to the phase curve at $f_1$ (or $F_1$), in terms of degrees per unit of $F$. Since for $s$ equal to $4$ the phase curve appears quite linear with respect to $F$, the tangent line should nearly

\(^3\)Dome, R. B., loc. cit.
Comparison of Error Curves of One-Pole System Obtained by Two Methods.
coincide with the phase curve over a wide range. By considering tangent lines as idealized phase curves, two such tangent lines may be so spaced as to yield a vertical difference of 90°. The positions of $F_{1A}$ and $F_{1B}$ (and $f_{1A}$, $f_{1B}$) are thereby specified by the location of the tangent lines. Variations of the actual phase curves from the tangent lines lead to deviations from the exact 90° difference between the two tangent lines. This method of selecting $f_{1A}$ (or $F_{1A}$) and $f_{1B}$ (or $F_{1B}$) is designated the "Slope Method," and the error curve resulting from use of the Slope Method is compared to the error curve from the 45° Point Method in Figure 9.

Inspection of the curves of Figure 9 reveals that less error results from use of the 45° Point Method than from use of the Slope Method, the maximum errors being approximately 4° and 8°, respectively. Thus, for $s$ equal to 4, the calculated phase difference between channels A and B of a one-pole system is approximately $90 + 4°$ for the 45° Point Method over a range of 1.4 units of $F$ (frequency ratio 25.1:1) and approximately $90 + 8°$ for the Slope Method over a range of 1.65 units of $F$ (frequency ratio 43.7:1). Although the 45° Point Method is definitely superior to the Slope Method, it does not represent the optimum nor do the curves show that the chosen value for $s$ is the best. The error curves for both methods are similar to each other in shape but are displaced from the zero error line by different amounts. The optimum error curve is that curve which is centered on the zero error line with equal departures (in plus and minus directions) to the principle maximum error points in the vicinity of $F_M$. When it is realized that in this particular case the difference between $F_{1A}$ and $F_{1B}$ is 0.656 units of $F$ for the 45° Point Method and 0.682 units of $F$ for the Slope Method, it becomes evident that the selection of $F_{1A}$
and $F_{1B}$ must be undertaken with extreme care. The exact spacing is indicated as less than 0.656 units of $F$.

In a still more recent article a method for definitely determining the optimum values for a one-pole network is developed. Although the values of $s$ and spacing between $F_{1A}$ and $F_{1B}$ are not presented directly, it is a relatively simple matter to interpret the given parameters in the appropriate manner. The optimum values of $s$ and spacing between $F_{1A}$ and $F_{1B}$ are found to be 3.106 and 0.5273, respectively, which substantiates the reasoning in the previous paragraphs.

The error problems of two- and three-pole networks are much more complex than those of one-pole networks, and a simple mathematical treatment is not readily obtained. Therefore, graphical methods are employed in this phase of the report.

Should a purely reactive one-pole L-C system be desired, the following tabulation sets forth a suggested procedure for determining the circuit constants for $s$ equal to 3.106 and a spacing, in log units, of 0.5273 between $F_{1A}$ and $F_{1B}$.

**Determination of Constants for One-Pole L-C Phase Difference System**

1. Select a value for $f_M$ and determine $F_M = \log_{10} f_M$.
2. Add 0.2636 to $F_M$ to obtain $F_{1B}$; subtract 0.2636 from $F_M$ to obtain $F_{1A}$.
3. Determine $f_{1A}$ and $f_{1B}$ from relationship $10^F = f$.

For Channel A:

1. Select a suitable value for $C_{111A}$.

---

(5) Determine $C_{11A}$ from relationship $C_{21A} = s^2C_{11A}$ or $C_{21A} = 9.647C_{11A}$.

(6) Determine $L_{11A}$ and $L_{21A}$ such that $L_{11A} = \frac{1}{(C_{11A}(2\pi f_{1A})^2)}$ and $L_{21A} = \frac{1}{(C_{21A}(2\pi f_{1A})^2)}$.

(7) Determine $R_{31A}$ such that $R_{31A} = (1/2)\sqrt{L_{11A}/C_{21A}}$.

For Channel B:

(8) Repeat steps (4) through (7) for circuit components, using B subscripts in place of A.

The order of determining the circuit constants is immaterial. In some instances it might be more advantageous to determine the L's, or possibly the R's, first.

The above derivation of circuit parameters is based on the assumption that the reactive elements of the circuits are pure inductances and capacitances. However, it is not possible to obtain inductances and capacitances which do not have equivalent resistive components to some degree. The difference between ideal and actual conditions is, therefore, governed by the factor of merit, or $Q$, of the L-C circuits. Magnetic coupling between inductances is also a contributing factor to the departure of purely reactive network behavior from ideal conditions.

As was previously indicated, solutions of networks containing pure reactances are obtained more easily, in general, than solutions of networks having mixed resistances and reactances. The significant fact about equations (38), (39) and (43) is not that the equations describe the behavior of L-C networks but that the equations completely define a phase angle, $\beta$, in an expression involving only frequencies (or specified relationships of frequencies) and a constant, s. If the equations for $\beta$ apply for a
particular network, that network, regardless of the manner in which it is formed, has the same phase-frequency characteristics as derived for the L-C network.

It is shown in the two succeeding sections that R-C combinations, which are more practical than L-C combinations, may be defined in terms of equations (38), (39) and (43) by placing such restrictions on the relationships among the circuit elements as to force the network behavior to comply with equations (38), (39) and (43). The procedure of first determining the phase-frequency equations for a purely reactive system and then applying the equations to a practical system is one of the basic trends of reasoning in this report. The formulas for practical n-pole systems are derived from the direct application of this principle.

I-5 Resistive-Reactive One-Pole Semi-Lattice with Isolation

The second one-pole network to be analyzed is shown in Figure 7. Two vacuum tubes are required as compared with the single tube needed for the network of Figure 6 and, as will be shown later, the network of Figure 8. It may be seen that the network is composed of two meshes of the type considered in Appendix II, coupled through a vacuum tube. The vacuum tube serves the dual purpose of isolating one mesh from the other and providing a means of obtaining the voltage sources for the second mesh. It is assumed that constant gain (approximately 0.5) and negligible phase shift occur in the vacuum tube and associated plate and cathode resistors. Under these conditions, each of the input voltages supplied to the second mesh is a set fraction of the output voltage of the first mesh and is either of the same polarity as, or of opposite polarity to, the first mesh output voltage, depending on which voltage is considered.
Appendix II illustrates that

\[ V_{34} = V_{13} \frac{Z_1 - Z_2}{Z_1 Z_2 + (Z_1 + Z_2)} \]  

(44)

In the first mesh of Figure 7,

\[ Z_1 = R_{11} \]  

(45)

\[ Z_2 = C_{21} \]  

(46)

and

\[ Z_3 \equiv \infty \]  

(47)

In the second mesh of Figure 7,

\[ Z_1 = R_{11} \]  

(48)

\[ Z_2 = C_{21} \]  

(49)

and

\[ Z_3 \equiv \infty \]  

(50)

As applied to the first mesh, equation (44) becomes

\[ V_{34}' = V_{12}' \frac{(-1 - j\omega R_{11}' C_{21}')}{(1 - j\omega R_{11}' C_{21}')} \]  

(51)

The impedance \( Z_3 \) in the first mesh is the input impedance of the coupling vacuum tube which is considered very high compared to the other circuit impedances.

The voltages \( V_{13} \) and \( V_{23} \) which appear in the second mesh are thus equal to

\[ V_{13} = k_+ V_{13}' \frac{(-1 - j\omega R_{11}' C_{21}')}{(1 - j\omega R_{11}' C_{21}')} \]  

(52)
and

\[ V_{23} = -k_4 V_{13} \left( \frac{-1 - j \omega R'_{1II} C'_{2II}}{1 - j \omega R'_{1II} C'_{2II}} \right) \]  \quad (53)

respectively, \( k_4 \) being a fixed constant of the vacuum tube coupling. By again applying equation (44) in conjunction with equations (48), (49), (50) and (52), the voltage, \( V_{34} \), which appears across the output terminals of the second mesh is

\[ V_{34} = \left[ \frac{k_4 V_{13} (-1 - j \omega R'_{1II} C'_{2II})}{1 - j \omega R'_{1II} C'_{2II}} \right] - \left[ \frac{(-1 - j \omega R'_{1II} C'_{2II})}{1 + j \omega R'_{1II} C'_{2II}} \right] \]  \quad (54)

In a practical application, the infinite impedance required for \( Z_3 \) may be obtained by connecting the output of the second mesh to a vacuum tube amplifier stage having a very high input impedance.

When the terms of equation (54) are recombined as

\[ V_{34} = k_4 V_{13} \left[ \frac{(1 - \omega^2 R'_{1II} C'_{2II} R_{1II} C_{2II}) + j (\omega R'_{1II} C'_{2II} + \omega R_{1II} C_{2II})}{(1 - \omega^2 R'_{1II} C'_{2II} R_{1II} C_{2II}) - j (\omega R'_{1II} C'_{2II} + \omega R_{1II} C_{2II})} \right] \]  \quad (55)

it is seen that the bracketed expression is the quotient of a complex number and its conjugate for all values of \( \omega \). The magnitude of such an expression is unity. The amplitude of the output voltage thus remains a constant fixed fraction of the input voltage of the entire network; i.e.,

\[ |V_{34}| = k_5 |V_{in}| \]  \quad (56)

where \( k_5 \) is a constant which accounts for the attenuation in the vacuum tubes, and \( V_{in} \) is the total input voltage of the entire network. Equation (56) thus satisfies the requirements for constancy of output magnitude.
The phase angle is determined by rationalizing equation (55) and finding the arc tangent of the quotient formed by the imaginary component divided by the real component. Thus,

\[ V_{64} = K_4 V_9 \left[ \frac{(1-\omega^2 R''_{mm} C_2'' R''_{mm} C_2'')+\left(\omega R''_{mm} C_2'' + \omega R''_{mm} C_2'\right)^2}{(1-\omega^2 R''_{mm} C_2'' R''_{mm} C_2'')+\left(\omega R''_{mm} C_2'' + \omega R''_{mm} C_2'\right)^2} \right. \]

\[ \left. + \frac{j \frac{1}{2}\left(\omega R''_{mm} C_2'' + \omega R''_{mm} C_2'\right)(1-\omega^2 R''_{mm} C_2'' R''_{mm} C_2'')}{(1-\omega^2 R''_{mm} C_2'' R''_{mm} C_2'')+\left(\omega R''_{mm} C_2'' + \omega R''_{mm} C_2'\right)^2} \right] \]  

(57)

and

\[ \beta = \tan^{-1} \left[ \frac{2 \left( R''_{mm} C_2'' + \omega R''_{mm} C_2'\right)(1-\omega^2 R''_{mm} C_2'' R''_{mm} C_2'')+\omega}{(1-\omega^2 R''_{mm} C_2'' R''_{mm} C_2'')+\left(\omega R''_{mm} C_2'' + \omega R''_{mm} C_2'\right)^2} \right] \]  

(58)

By letting

\[ \omega_1^2 = \frac{j}{R''_{mm} C_2'' R''_{mm} C_2''} \]  

(59)

\[ \beta = \tan^{-1} \left[ \frac{2 \left( R''_{mm} C_2'' + \omega R''_{mm} C_2'\right)\omega_1 \left(\omega_1^2 - \omega_1^2\right)}{\left(\omega_1^2 - 2\omega_1^2 + \omega_1^2\right) - \omega_1^2 \left( R''_{mm} C_2'' + \omega R''_{mm} C_2'\right)} \right] \]  

(60)

In order for equation (60) to be of the desired form as typified by equation (38), it is evident that the following equality must apply:

\[ S = \frac{R''_{mm} C_2'' + R''_{mm} C_2'}{\sqrt{R''_{mm} C_2'' R''_{mm} C_2'\cdot R''_{mm} C_2')} \]  

(61)
It is apparent that, in general, $\frac{R_{11} C_{211}}{R_{111} C_{211}}$ should not equal $\frac{R_{11} C_{211}}{R_{111} C_{211}}$, since the value of $s$ would thereby be restricted to 2. However, by letting

$$R_{11}' = R_{11} \quad (62)$$

and

$$C_{211}' = k_6 C_{211} \quad (63)$$

where $k_6$ is a constant, $s$ becomes

$$s = \frac{k_6 R_{11} C_{211} + R_{111} C_{211}}{\sqrt{k_6 R_{11}^2 C_{211}^2}} = \frac{k_6 + 1}{\sqrt{k_6}} \quad (64)$$

$s^2$ becomes

$$s^2 = \frac{k_6^2 + 2k_6 + 1}{k_6} \quad (65)$$

and

$$k_6 = \frac{s^2 - 2 + \sqrt{s^4 - 4s^2}}{2} \quad (66)$$

To avoid imaginary components in equation (66), $s$ must be equal to, or greater than, 2. A smaller value of $k_6$ results when the minus sign is chosen in equation (66).

From equation (59) and the relationship, $f$ equal to $\frac{\omega}{2\pi}$,

$$f = \frac{1}{2\pi} \sqrt{R_{11} C_{211} R_{111}^2 C_{211}} = \frac{1}{2\pi R_{11} C_{211}} \sqrt{\frac{1}{k_6}} \quad (67)$$

With a predetermined value of $s$ and the required relationships expressed by equations (62), (63), (66) and (67), equation (60) reduces to the familiar form,
The equations for $\beta$ in terms of $\omega$ and $F$ may be determined directly.

Networks of the above type are more practical than L-C networks. The problem of inductances is avoided and commercial capacitors and resistors are sufficiently "pure" to be used in the phase-shift sections.

The following tabulation sets forth a suggested procedure for determining the circuit constants of a two-channel phase-difference system with $s$ equal to 3.106 and a log unit spacing of 0.5273 between $F_{1A}$ and $F_{1B}$.

Determination of Constants for a Typical One-Pole R-C Phase Difference System with Isolating Vacuum Tubes

1. Select a value for $f_M$ and determine $F_M = \log_{10} f_M$.
2. Add 0.2636 to $F_M$ to obtain $F_{1B}$; subtract 0.2636 from $F_M$ to obtain $F_{1A}$.
3. Determine $f_{1A}$ and $f_{1B}$ from relationship $10^F = f$.
4. Determine $k_6$ from relationship $k_6 = \frac{s^2 - 2 + \sqrt{s^4 - 4s^2}}{2}$.
5. Select a value $R_{111A} = R_{111A}^*$.
6. Determine $C_{211A}$ from relationship $C_{211A} = \frac{1}{2\pi f_{1A} R_{111A}} \sqrt{\frac{1}{k_6}}$.
7. Determine $C'_{211A}$ from relationship $C'_{211A} = k_6 C_{211A}$.

For Channel B:

8. Repeat steps (5) through (7) using B subscripts in place of A.

The computations are shortened if the same values for $R_{111}$ and $k_6$ are used throughout.
I-6 Resistive-Reactive One-Pole Semi-Lattice with No Isolation

The third one-pole network to be analyzed is shown in Figure 8. Although R-C elements are used exclusively, there is a marked resemblance between this network and the L-C network of Figure 6. Series and parallel R-C combinations make up the impedance arms of the phase-shift and semi-lattice section. Also, as in the case of the L-C network, no isolating vacuum tube is required. The source voltages for the phase-shift section are obtained in the same manner and under the same conditions as are the source voltages in Figures 6 and 7.

The use of the terms poles and zeros is avoided in direct description of the behavior of the impedance arms, since a series R-C combination has no zero of impedance at any finite frequency and has a pole of impedance only at zero frequency, and since a parallel R-C combination has no pole of impedance at any frequency and has a zero of impedance only at infinite frequencies. There are frequencies between zero and infinity, however, at which the voltage drops across the condenser and resistor of a series R-C combination are equal in magnitude and differ in phase by 90°. A similar situation exists with respect to currents in a parallel R-C combination. The frequencies at which the magnitude equality and 90° phase difference occurs are termed critical frequencies. In a general sense, the critical frequency of either a series or parallel R-C combination is given by

\[ f_c = \frac{1}{2\pi RC} \quad ; \quad \omega_c = \frac{1}{RC} \]  \hspace{1cm} (69)
From Appendix II,

\[ V_{34} = V_{19} \frac{(Z_1 - Z_2)}{Z_1 Z_2 + (Z_1 + Z_2)} \]  \hspace{1cm} (70)

In the network of Figure 8,

\[ Z_1 = -j \frac{R_{III}}{\omega} \cdot \frac{(j\omega + \frac{1}{R_{III} C_{III}})}{1} \]  \hspace{1cm} (71)

\[ Z_2 = \frac{1}{C_{2II}} \cdot \frac{1}{(j\omega + \frac{1}{R_{2II} C_{2II}})} \]  \hspace{1cm} (72)

and

\[ Z_3 = \frac{1}{C_{3II}} \cdot \frac{1}{(j\omega + \frac{1}{R_{3II} C_{3II}})} \]  \hspace{1cm} (73)

Equating the critical frequencies for \( Z_1 \), \( Z_2 \) and \( Z_3 \) discloses the following relationship:

\[ \frac{1}{\omega_1} = \frac{R_{III} C_{III}}{1} = \frac{R_{2II} C_{2II}}{1} = \frac{R_{3II} C_{3II}}{1} \]  \hspace{1cm} (74)

The ratio \( Z_2/Z_3 \) then becomes \( C_{3II}/C_{2II} \) or \( R_{2II}/R_{3II} \).

Equation (70) may be rewritten as

\[ V_{34} = V_{19} \frac{(Z_1 - Z_2)}{Z_1 \left( \frac{R_{2II}}{R_{3II}} + 1 \right) + Z_2} = \frac{(Z_1 - Z_2)}{Z_1 \left( \frac{R_{2II}}{R_{3II}} + 1 \right) + Z_2} \]  \hspace{1cm} (75)

After introducing a new constant, \( D' \), such that

\[ \frac{R_{3II}}{R_{2II} + R_{3II}} = D' \]  \hspace{1cm} (76)
equation (75) becomes

$$V_{34} = V_{13} D' \frac{(Z_1 - Z_2)}{(Z_1 + D'Z_2)} \quad (77)$$

With further restrictions placed on $D'$, the coefficient of $V_{13} D'$ in equation (77) is equal to unity in magnitude at all frequencies. The restrictions are such as to force the phase angle of the coefficient to be identical to the desired phase angle of a one-pole network. $D'$ becomes the multiplying factor by which the magnitude of $V_{13}$ is modified to yield the magnitude of $V_{34}$.

Substitution of the values for $Z_1$ and $Z_2$ in equation (77) results in

$$V_{34} = V_{13} D' \left[ \left( \frac{R_{11} - j\omega C_{11}}{\omega C_{11}} \right) - \left( \frac{R_{211} - j\omega R_{211} \theta_{211}}{1 + \omega^2 R_{211}^2 \theta_{211}^2} \right) \right] \quad (77a)$$

Equation (77a) may be simplified and reduced to

$$V_{34} = V_{13} D' \left\{ \frac{R_{11}^2 \left( \frac{\omega^2 + 1}{\omega^2} \right) + 2R_{11}R_{211} \left( \frac{D' - 1}{\omega^2 + 1} \right) + R_{211}^2 \left( -\frac{D'}{\omega^2 + 1} \right)}{\text{Denominator}} \right\} + \left\{ \frac{R_{11}R_{211} (1 + D') \left( \frac{1}{\omega \omega_i} - \frac{1}{\omega \left( \omega_i^2 + 1 \right)} \right)}{\text{Denominator}} \right\} \quad (77b)$$
The expression for the phase angle of equation (77b) is

\[
\beta = \tan^{-1} \left\{ \frac{R_{2ii}/R_{iii} (1+D') \omega_0 (\omega^2 - \omega_i^2)}{(\omega^4 - 2\omega^2 \omega_i^2 + \omega_i^4) - \omega_i^2 \left[ \frac{D'R_{2ii}^2}{R_{iii}^2} - 2(D'-1) \frac{R_{2ii}}{R_{iii}} - 4 \right]} \right\} \tag{78}
\]

In order for equation (78) to be of the required form typified by equation (68), the condition,

\[
\left[ \frac{R_{2ii}}{R_{iii}} \left( \frac{1+D'}{2} \right) \right]^2 = \left[ D' \frac{R_{2ii}^2}{R_{iii}^2} - 2(D'-1) \frac{R_{2ii}}{R_{iii}} - 4 \right], \tag{79}
\]

must be imposed, where the first bracketed expression must equal \( s \) and the second must equal \( s^2 \). After transposing terms and clearing fractions, it is possible to rearrange equation (79) as the quadratic equation in \( D' \):

\[
D'^2 + D' \left( -2 + 8 \frac{R_{2ii}}{R_{iii}} \right) + \left( 1 - 8 \frac{R_{2ii}}{R_{2ii}} + 16 \frac{R_{2ii}^2}{R_{2ii}^2} \right) = 0. \tag{80}
\]

The solution for \( D' \) is

\[
D' = 1 - 4 \frac{R_{2ii}}{R_{2ii}}. \tag{81}
\]

By letting the ratio \( R_{1ii}/R_{2ii} \) (or \( C_{2ii}/C_{1ii} \)) equal \( k_7 \), it follows that

\[
D' = 1 - 4k_7. \tag{82}
\]

From equation (79),

\[
s = \frac{R_{2ii}}{R_{iii}} \left( \frac{1+D'}{2} \right). \tag{83}
\]
With the application of the relationships expressed above,

\[ S = \frac{1 - 2k_7}{k_7} \]  
\[ k_7 = \frac{1}{s + 2} \]  
and

\[ D' = \frac{s - 2}{s + 2} \]  

From equations (76) and (82),

\[ 1 - 4k_7 = \frac{R_{311}}{R_{211} + R_{311}} \]  
\[ \frac{R_{311}}{R_{211}} = \frac{1 - 4k_7}{4k_7} \]  

Other relationships which may be developed from use of equation (74) are

\[ \frac{C_{311}}{C_{111}} = \frac{R_{111}}{R_{311}} = \frac{4k_7^2}{1 - 4k_7} \]  
\[ \frac{C_{211}}{C_{311}} = \frac{1 - 4k_7}{4k_7} \]  
and

\[ \frac{C_{211}}{C_{111}} = k_7 \]

It was stated, without proof, that a suitable value of \( D' \), independent of frequency, could be found which would make the coefficient of \( v_{13} D' \) in equation (77) equal to unity. This is a necessary condition. In addition, the value of \( D' \) must be consistent with the two values expressed by equations (76) and (79). Equation (77a) may be rearranged as
If the absolute value of the numerator of a complex fraction equals the absolute value of the denominator at all frequencies, the absolute value of the fraction is equal to unity at all frequencies. The value of \( D' \) (consistent with previously developed relationships) which satisfies this condition in the bracketed portion of equation (92) is the desired value.

By squaring the real and imaginary components of the numerator of the bracketed fraction of equation (92) and equating the result to the squared denominator, the following quadratic equation in \( D' \) is obtained:

\[
D'^2 R_{211}^2 + D'(2 R_{111} R_{211} + 2 \frac{R_{211}^2 C_{211}}{C_{111}}) + (2 R_{111} R_{211} + 2 \frac{R_{211}^2 C_{211}}{C_{111}} - R_{211}^2) = 0 \tag{93}
\]

The two roots of equation (93) are

\[
D' = -1 \tag{94}
\]

and

\[
D' = 1 - 2 \left( \frac{R_{111}}{R_{211}} + \frac{C_{211}}{C_{111}} \right) \tag{95}
\]

Obviously, the relationship of equation (94) is not the desired one. Since \( R_{111}/R_{211} \) equals \( C_{211}/C_{111} \) from equation (74), equation (95) reduces to

\[
D' = 1 - 4 \frac{R_{111}}{R_{211}} \tag{96}
\]
which is in complete accord with equation (79). It may now be stated definitely that $D'$ is the ratio by which the magnitude of $V_{13}$ is modified to produce the magnitude of $V_{34}$.

The effect of $s$ on the magnitude of the output voltage may be observed from equation (86). As $s$ increases in value, $D'$ increases. In any event, $s$ must be greater than 2, because the output is zero for $s$ equal to 2 and mathematically negative for $s$ less than 2.

Since the standard one-pole equation,

$$\beta = \tan^{-1} \left( \frac{2sff_{1}(f_{2}^{2} - f_{1}^{2})}{(f_{2}^{2} - f_{1}^{2})^{2} - s^{2}f_{2}^{2}f_{1}^{2}} \right)$$

applies, the complete network described in this section may be designated as a one-pole R-C network. As in the case of the two one-pole networks previously described, $f_{1}$ is the value of frequency at which $\beta$ equals $-180^\circ$ or $-\pi$ radians.

The following tabulation sets forth a suggested procedure for determining the circuit constants of a two-channel phase-difference system using one-pole R-C networks with $s$ equal to 3.106 and a log unit spacing of 0.5273 between $F_{LA}$ and $F_{LB}$.

**Determination of Constants for One-Pole R-C Phase Difference System (No Isolating Vacuum Tube)**

1. Select a value for $f_{m}$ and determine $F_{M} = \log_{10} f_{M}$.
2. Add 0.2636 to $F_{M}$ to obtain $F_{LB}$; subtract 0.2636 from $F_{M}$ to obtain $F_{LA}$.
3. Determine $f_{LA}$ and $f_{LB}$ from relationship $10^F = f$.

For Channel A:

4. Select a suitable value for $R_{11A}$. (Note: $R_{11A}$ partially determines the plate load of the vacuum tube.)
(5) Determine $R_{211A}$ from relationship $R_{211A} = \left( \frac{R_{111A}}{L_1} \right)$. 

(6) Determine $R_{311A}$ from relationship $R_{311A} = R_{211A} \left( \frac{1-iL_1}{4k_1} \right)$. 

(7) Determine $C_{111A}$, $C_{211A}$, and $C_{311A}$ from relationships $C_{111A} = \frac{1}{2\pi f_1 A R_{111A}}$, $C_{211A} = \frac{1}{2\pi f_1 A R_{211A}}$, and $C_{311A} = \frac{1}{2\pi f_1 A R_{311A}}$. 

For Channel B:

(8) Repeat steps (4) through (7) above, using B subscripts in place of A.

I-7 Network Extensions—Two-Pole Semi-Lattice Systems

In the previous sections three networks having identical phase-frequency characteristics were analyzed. All three networks satisfied the magnitude requirements. The phase-frequency equation of the one-pole L-C full-lattice and semi-lattice networks served as a guide for obtaining the more practical (R-C) types of networks. However, in order to extend the frequency range, more complex networks must be used. Logically, the two-pole network analysis follows that of the one-pole. Here too, the L-C network, while of questionable practical value as a physical network, serves as a means of determining the equations which a workable system must follow. It is shown in the following sections that a two-pole system may be described in terms of a single L-C phase-shift section or in terms of two, cascaded, one-pole phase-shift sections. Three network configurations which are discussed are shown in Figures 10, 11, and 12.

I-8 Purely Reactive Two-Pole Semi-Lattice

Equation (13) describes the phase-frequency characteristics of a two-pole, L-C full lattice. This equation, which is repeated below, must also
describe the phase-frequency relationships of a two-pole semi-lattice network.

\[
\beta = \tan^{-1} \frac{\sum \frac{f_{i_2} \left( f^2 - f_{i_1}^2 \right) \left( f^2 - f_{i_2}^2 \right) \left( f^2 - f_{i_3}^2 \right)}{\left( f^2 - f_{i_1}^2 \right)^2 \left( f^2 - f_{i_3}^2 \right)^2}}{S \frac{f_{i_2}^2 \left( f^2 - f_{i_3}^2 \right)}{\left( f^2 - f_{i_2}^2 \right)^2}}.
\] (98)

In accordance with a previously developed relationship, \( \beta \) varies from 0° to 720° as \( f \) varies from zero to infinite frequencies. Appendix I demonstrates that \( f_2 \) is the frequency at which \( \beta \) equals -360°. It may be shown that \( f_1 \) and \( f_3 \) are the frequencies at which \( \beta \) equals -180° and -540°, respectively.

The general equation of Appendix II applies for the phase-shift section of Figure 10 with the impedances, \( Z_1 \), \( Z_2 \) and \( Z_3 \), represented as

\[
Z_1 = j \frac{f}{2 \pi C_{i2}} \frac{\left( f_{i_2}^2 - f^2 \right)}{\left( f_{i_1}^2 - f^2 \right) \left( f_{i_3}^2 - f^2 \right)}.
\] (99)

\[
Z_2 = -j \frac{2 \pi L_{22}}{f} \frac{\left( f_{i_2}^2 - f^2 \right) \left( f_{i_3}^2 - f^2 \right)}{\left( f_{i_2}^2 - f^2 \right)}.
\] (100)

and

\[
Z_3 = R_{32}.
\] (101)

where, according to the discussion of equations (1) and (2),

\[
C_{i2} = \frac{C_{i21} C_{i23}}{C_{i21} + C_{i23}}
\] (102)

and

\[
L_{22} = \frac{L_{221} L_{223}}{L_{221} + L_{223}}.
\] (103)

In addition,

\[
L_{12} = L_{121} + L_{123}
\] (104)
-V_{13} = V_{23} In All Networks

Mesh Form

**FIGURE 10**
Purely Reactive Two-Pole Semi-Lattice

Mesh Form

**FIGURE 11**
Resistive-Reactive Two-Pole Semi-Lattice---Not Cascaded

Mesh Form

**FIGURE 12**
Resistive-Reactive Two-Pole Semi-Lattice---Cascaded
is the inductance which \( Z_1 \) approximates at zero frequency, and

\[
C_{22} = c_{221} + c_{223}
\]  

(105) is the capacitance which \( Z_2 \) approximates at zero frequency.

The frequencies, \( f_1 \) and \( f_3 \), at which poles of impedance occur in \( Z_1 \) are the same frequencies at which zeros of impedance occur in \( Z_2 \). The frequency, \( f_2 \), at which the zero of impedance occurs in \( Z_1 \) is likewise the same frequency at which the pole of impedance occurs in \( Z_2 \). The choice of the matching frequency values is predicated by the comparison between full-lattice and semi-lattice behavior.

Network analysis is frequently shortened by the use of the term \( p \) in place of \( j\omega \) or \( j2\pi f \) in the circuit equations. Historically, the term \( p \) is the differential operator, \( \frac{d}{dt} \), which reduces to \( j\omega \) when steady-state sinusoidal voltages and currents are considered. Equations (1) and (2) may be readily expressed in terms of \( p \). In fact, some authors prefer to use the \( p \) notation exclusively.\(^5\) Equations (99) and (100) become

\[
Z_1 = \frac{p}{C_{12}} \cdot \frac{(p^2 - p_2^2)}{(p^2 - p_1^2)(p^2 - p_3^2)}
\]  

(106) and

\[
Z_2 = \frac{L_{22}}{P} \cdot \frac{(p^2 - p_1^2)(p^2 - p_3^2)}{(p^2 - p_2^2)}
\]  

(107)

Substitution of the values of \( Z_1, Z_2 \) and \( Z_3 \) from equations (106), (107) and (101) yields

From equation (40), the conditions for reciprocation with $R_{32}$ as the load resistor are

$$Z_1 Z_2 = (2R_{32})^2; \quad \frac{L_{22}}{C_{12}} = 4R^2_{32} \quad (109)$$

from which

$$L_{22} = 4R^2_{32} C_{12} \quad (110)$$

Substitution of the value for $L_{22}$ in equation (108) yields

$$V_{34} = V_{13} \left[ \frac{\rho^2(p^2 - p_2^2) - L_{22} C_{12} (p^2 - p_1^2)(p^2 - p_3^2)}{4R^2_{32} C_{12} (p^2 - p_1^2)(p^2 - p_2^2)(p^2 - p_3^2) \rho + p^2(p^2 - p_2^2) + 4R^2_{32} C_{12} (p^2 - p_1^2)(p^2 - p_3^2)} \right] \quad (111)$$

By noting that the numerator of equation (111) contains no imaginary terms and that the first term of the denominator embraces all the imaginary terms, the phase angle of the coefficient of $V_{13}$ is obtained by determining the negative phase angle of the reciprocal. Such a procedure results in

$$\beta = \tan^{-1} \frac{-4 \omega R^2_{32} C_{12} (p^2 - p_2^2)(p^2 - p_1^2)(p^2 - p_3^2)}{4R^2_{32} C_{12} (p^2 - p_1^2)(p^2 - p_2^2)(p^2 - p_3^2) + p^2(p^2 - p_2^2)} \quad (112)$$

Substituting $j \omega$ for $p$, $j \omega$ for $p_1$, etc., in equation (112) and rearranging terms yield

$$\beta = \tan^{-1} \frac{-4 \omega R^2_{32} C_{12} (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)}{4R^2_{32} C_{12} (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2) - \omega^2(\omega^2 - \omega_3^2)} \quad (112a)$$
The numerator and denominator of equation (112a) may be divided by 

\[ 4R_{32}C_{12}^2 \]

and suitable simplifications may be made such that

\[
\beta = \tan^{-1} \frac{2 \frac{\omega \omega_z}{2R_{32}C_{12}} \left( (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2) \right)}{(\omega^2 - \omega_1^2) (\omega^2 - \omega_2^2) - \frac{\omega^2 \omega_z^2}{4R_{32}C_{12}^2 \omega_z^2} (\omega^2 - \omega_3^2)^2} \tag{112b}
\]

The expression for \( s \) is evidently

\[
s = \frac{1}{2R_{32}C_{12} \omega_z} \tag{113}
\]

leaving

\[
\beta = \tan^{-1} \frac{2s \omega \omega_z (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)}{(\omega^2 - \omega_1^2) (\omega^2 - \omega_2^2) - s^2 \omega^2 \omega_z^2 (\omega^2 - \omega_3^2)^2} \tag{114}
\]

Since \( \omega \) equals \( 2\pi f \),

\[
\beta = \tan^{-1} \frac{2s \frac{f_2 f_z}{(f^2 - f_1^2)(f^2 - f_2^2)(f^2 - f_3^2)}}{(f^2 - f_1^2) (f^2 - f_2^2) - s^2 f^2 f_z^2 (f^2 - f_3^2)^2} \tag{115}
\]

which is the desired form for a two-pole L-C network as given by equation (13). The expression in terms of \( F \) is

\[
\beta = \tan^{-1} \frac{2s \frac{10^F + f_z}{(10^F - 10^F_1)(10^F - 10^F_2)(10^F - 10^F_3)}}{(10^F - 10^F_1)(10^F - 10^F_2) - s^2 10^F f_z^2 (10^F - 10^F_3)^2} \tag{116}
\]

If equations (114) through (116) are to be useful, a suitable value of \( s \) and spacing, \( \delta \), between \( F_2, F_1 \) and \( F_3 \) must be found such that the

\footnote{Note: By suitable application of equation (38a) from Appendix I, it may be shown that \( s = \sqrt{L_{12}/L_{22}} \), the square root of the ratio of the inductance of \( Z_1 \) at zero frequency to the inductance of \( Z_2 \) at infinite frequencies.}
relatively linear portion of the $\beta$ versus $F$ curve extends over a wide range. According to previously developed relations, $2F_2$ is equal to $(F_1 + F_3)$. At $F$ equal to $-\infty$, $F_1$, $F_2$, $F_3$ or $+\infty$, $\beta$ equals $\tan^{-1} 0$.

As $F$ progresses from $-\infty$ to $+\infty$, $\beta$ varies from $0$ to $-720^\circ$ and $F_1$, $F_2$ and $F_3$ are those values of $F$ for which $\beta$ equals $-180^\circ$, $-360^\circ$ and $-540^\circ$, respectively. A straight line drawn between the points $(F_1, -180^\circ)$ and $(F_3, -540^\circ)$ passes through the points $(F_2, -360^\circ)$, $(\frac{F_1 + F_2}{2}, -270^\circ)$, and $(\frac{F_2 + F_3}{2}, -450^\circ)$. The $-180^\circ$, $-360^\circ$ and $-540^\circ$ points are automatically points of the phase curve. Should it be possible to determine $s$ such that the phase curve is forced to pass through the $-270^\circ$ and $-450^\circ$ points, at least five points of coincidence would be established between the straight line and the phase curve. The spacing between $F_1$ and $F_2$ and between $F_2$ and $F_3$ would be equal, thereby, to the spacing between $\frac{F_1 + F_2}{2}$ and $\frac{F_2 + F_3}{2}$.

In a later section the proper values of $s$ and $S$ are considered on a generalized basis and, therefore, are not discussed further at this point.

The question of magnitude may be dispensed with by noting that the pure L-C semi-lattice is a loss-free section. The output voltage is, therefore, equal to either of the input voltages in absolute value, as is demonstrated for the one-pole L-C network. In addition, the condition that

$$\frac{C_{12}}{C_{22}} \ll 1$$

must apply. (The corresponding condition for the one-pole L-C network is given in equation (30)).

Figures 13 and 14 show typical phase-frequency and error curves of a two-pole system with $s$ equal to $22.345$ and $S$ equal to $0.9$ units of $F$. 

-48-
FIGURE 13
Phase Curves Of Networks A And B--Two-Pole System.
Deviation from 90° difference between phase angles of channels A and B.

FIGURE 14
Error Curves Of Two-Pole System
Having established that the two-pole, semi-lattice network yields the required phase-frequency formula, it is now in order to proceed with the discussion of practical R-C systems which satisfy the phase-frequency and amplitude requirements. At first glance, it might appear that a network of the type shown in Figure 11 is a logical two-pole analog of the third one-pole network. The fact that, at most, one critical frequency exists at the external terminals of an R-C network (regardless of how complex may be the internal structure) precludes any attempt to utilize the network of Figure 11. It may be shown that no practical solution of this type exists. The pure L-C networks provide a segregation which cannot be obtained by nonisolated R-C networks. The next question which must be considered is, accordingly, one of obtaining isolation of operation with R-C networks.

The second one-pole network suggests a method for obtaining isolation within a two-pole system. By means of the coupling vacuum tube, two, one-pole networks may be cascaded to yield the phase-frequency characteristics of a two-pole L-C network. The one-pole networks may be any of the types previously described. As an example, two, one-pole R-C networks are shown connected in this manner in Figure 12.

The phase angle expression, equation (114), of the two-pole L-C network may be rearranged as

The total phase angle derived from two, one-pole networks, connected in a similar manner to that shown in Figure 12, is

\[ \beta = \beta_1 + \beta_2 = \tan^{-1} \left( \frac{2s_1 \omega_{o1} (\omega^2 - \omega_{o1}^2)}{(\omega^2 - \omega_{o1}^2)^2 - s_1^2 \omega^2 \omega_{o1}^2} \right) + \frac{2s_2 \omega_{o2} (\omega^2 - \omega_{o2}^2)}{(\omega^2 - \omega_{o2}^2)^2 - s_2^2 \omega^2 \omega_{o2}^2} \]  

where \( \beta_1, s_1 \) and \( \omega_{o1} \) are the values for \( \beta, s \) and \( \omega_1 \) of the first one-pole network and \( \beta_2, s_2 \) and \( \omega_{o2} \) are the values for \( \beta, s \) and \( \omega_1 \) of the second.

The sum of two arc tangents may be expressed as:

\[ \tan^{-1} \mu + \tan^{-1} \nu = \tan^{-1} \left( \frac{\mu + \nu}{1 - \mu \nu} \right) \]  

When equation (120) is applied to equation (119), the result is
If equation (121) is to be of the required form, represented by equation (118), the coefficients of $\omega$ and the constant terms of equation (121) must equal the corresponding coefficients and terms of equation (118). In equation (118), it should be noted that $\omega_2$ is the geometric mean between $\omega_1$ and $\omega_3$ ($\omega_2^2 = \omega_1 \omega_3$).

Comparison of the constant term, $\omega_{01} \omega_{02}$, of the denominator of equation (121) with the corresponding term, $\omega_2^8$, of equation (118), immediately fixes the condition

$$\omega_{01} \omega_{02} = \omega_2^2 \tag{122}$$

Equality between the $\omega^7$ coefficients of the numerators is expressed as

$$\sigma \omega_2 = s_1 \omega_{01} + s_2 \omega_{02} \tag{123}$$
Substituting condition (122) in the $\omega$ coefficient of equation (121) and equating the result to the coefficient of equation (118) result in

$$s\omega_2 = s\omega_{o1} + s\omega_{o2}$$

(124)

In order for equation (124) to be reconciled with equation (123), the relationships

$$s_1 = s_2$$

(125)

and

$$\frac{s\omega_2}{s_1} = (\omega_{o1} + \omega_{o2})$$

(126)

must apply.

One possible location of $\omega_{o1}$ and $\omega_{o2}$ with respect to $\omega_1$, $\omega_2$, and $\omega_3$ is at $\omega_2$, making $\omega_{o1}\omega_{o2}$ equal to $\omega_2^2$. Although this location satisfies condition (122) and the resulting phase angle progresses from $0^\circ$ to $-720^\circ$, the phase-frequency characteristic is not an improvement over that of a one-pole network. At any particular value of frequency, the phase angle is merely doubled, thereby indicating that $\omega_2$ is not a good location for $\omega_{o1}$ and $\omega_{o2}$.

At $\omega_1$ the phase angle of the output voltage of the two-pole L-C network is $180^\circ$. Since either $\beta_1$ or $\beta_2$ is greater than zero in the region under consideration and since the sum of $\beta_1$ and $\beta_2$ must equal $180^\circ$ at $\omega_1$, both $\beta_1$ and $\beta_2$ are less than $180^\circ$ at $\omega_1$. Therefore, the $\omega$'s at which $\beta_1$ and $\beta_2$ separately equal $180^\circ$ are greater than $\omega_1$. In a similar manner it may be deduced that the $\omega$'s at which $\beta_1$ and $\beta_2$ separately equal $180^\circ$ are less than $\omega_3$. This, together with the fact that $(\omega_{o1}\omega_{o2})$ equals $\omega_2^2$, places $\omega_{o1}$ between $\omega_1$ and $\omega_2$. The same reasoning leads to the conclusion that $\omega_{o2}$ should appear between $\omega_2$ and...
Also, $\omega_{o1}$ and $\omega_{o2}$ are symmetrically placed on a logarithmic basis with respect to $\omega_2$.

Noting that $s_1 s_2$ equals $s_1^2$ (and also $s_2^2$) and $\omega_{o1} \omega_{o2}$ equals $\omega_2^2$, it is evident that the last two terms in braces in the denominator of equation (121) may be regrouped as

$$
\omega^6 [s_i (\omega_{o1} + \omega_{o2})]^2 - 2 \omega^4 \omega_2^2 [s_i (\omega_{o1} + \omega_{o2})]^2 - \omega^2 \omega_2^4 [s_i (\omega_{o1} + \omega_{o2})]^2 \quad (127)
$$

By virtue of the equality of $s_1$ and $s_2$, equation (124) reduces to

$$
s_i [\omega_{o1} + \omega_{o2}] = \omega_2 s \quad ; \quad [\omega_{o1} + \omega_{o2}] = \frac{s \omega_2}{s_i} \quad (128)
$$

After substitution of the value for $(\omega_{o1} + \omega_{o2})$ from equation (128) in expression (127), expression (127) reduces to

$$
S^2 (\omega^6 \omega_2^2 - 2 \omega^4 \omega_2^4 + \omega^2 \omega_2^6) \quad (129)
$$

which is identical to the last term in the denominator of equation (118).

The coefficient of the remaining $\omega_6$ term in equation (121), when equated to the corresponding term of equation (118), yields

$$
\omega_{o1}^2 + \omega_{o2}^2 + s_i \omega_{o1} \omega_{o2} = \omega_i^2 + \omega_3^2 \quad , \quad (130)
$$

which is consistent with the relative magnitudes of $\omega_{o1}$ and $\omega_{o2}$ with respect to $\omega_1$ and $\omega_3$. By employing the relationship of equation (130) and with $\omega_2^2$ equal to $\omega_1 \omega_3$, the coefficient of the remaining $\omega_2^2$ term in the denominator of equation (121) reduces to

$$
2 \omega_i^4 \omega_3^2 + 2 \omega_i^2 \omega_3^4 \quad , \quad (131)
$$
which is the coefficient of the corresponding term in equation (118).

With rearrangement of terms and substitution of $s_1$ for $s_2$ and $\omega^2$ for $\omega_0 \omega_2$, the coefficient of the $\omega^5$ term in the numerator of equation (121) may be factored to yield

$$s_1 \left[ \omega_{01}^2 + \omega_{02}^2 \right] \left[ \omega_0^2 + \omega_{01}^2 + s_1 \omega_k^2 \right] + s_1 \left[ \omega_{01} + \omega_{02} \right] \omega_2^2 \quad (132)$$

Equation (128) and (130) reduce expression (132) to

$$\left( s \omega_2 \right) \omega_1^2 + \left( s \omega_2 \right) \omega_2^2 + \left( s \omega_2 \right) \omega_3^2 \quad (133)$$

which is identical to the corresponding term of equation (118).

The coefficients of the $\omega^3$ terms of the numerators of equations (118) and (121) are equal to the coefficients of the $\omega^5$ terms multiplied by $\omega_2^2$. Hence, expression (133) establishes correspondence for these terms, also.

Substituting $s_1$ for $s_2$ and $\omega_2^2$ for $(\omega_0 \omega_2)$ in the remaining $\omega^4$ coefficient in the denominator of equation (121) and rearranging terms determine the $\omega^4$ coefficient to be

$$\left[ \omega_{01}^2 + \omega_{02}^2 + s \omega_2^2 \right] + 2 \omega_2^4 \quad (134)$$

and, by virtue of equation (130), expression (134) equals

$$\left[ \omega_1^2 + \omega_3^2 \right]^2 + 2 \omega_2^4 \quad (135)$$

which is identical to the coefficient of the $\omega^4$ term in the denominator of equation (118).

All coefficients of $\omega$ and all constant terms in equation (121) are now equated to the corresponding terms of equation (118) under the following summarized conditions:
It is seen that two, suitably restricted, one-pole networks may be employed to yield the phase-frequency characteristics of a two-pole L-C network. It is stated again that any of the three, one-pole networks considered in the previous sections may be connected in tandem to yield the desired two-pole, L-C, phase-frequency curve. For a practical case, either the second or the third one-pole R-C network should be considered. The network of Figure 12 might just as well have been composed of two, one-pole R-C isolated-type networks. The one-pole network thus becomes a basic unit in developing the more complex networks.

The parameter, $s_1$, is an important factor in predicting the behavior of each one-pole network. The determination of $s_1$ for the two-pole network in terms of $\omega$ (or f and F) is a relatively simple task, while the general solution for the $s_1$ of three- and higher pole networks becomes increasingly complex. The calculations leading up to a solution for $s_1$ of the two-pole network are given at this point, based on the premise that the solution, while perfectly valid, represents a special, rather than the general, case.

\begin{align*}
\omega_{o1}, \omega_{o2} &= \omega_r^2 \quad , \\
S_1 &= S_2 \quad , \\
S_1 [\omega_{o1} + \omega_{o2}] &= S \omega_r \\
\text{and} \\
[\omega_{o1}^2 + \omega_{o2}^2] + S_1 \omega_{o1} \omega_{o2} &= \left[ \omega_1^2 + \omega_2^2 \right] .
\end{align*}

At $\omega_l$, 

\[ \beta = \beta_1 + \beta_2 = -180^\circ \quad , \]
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The expressions for $-\tan 1$ and $\tan 2$ at $\omega_1$, as determined from equation (119), may be equated in the following manner:

\[ \frac{2s \omega \omega_0 (\omega^2 - \omega_0^2)}{(\omega^2 - \omega_0^2)^2 - s \omega^2 \omega_0^2} \]

By cross multiplication, equation (143) becomes

\[ \omega_1^7(\omega_0 + \omega_0^2) - \omega_2^7(\omega_0^3 + \omega_0^3 + 2\omega_0^2 \omega_0^2 + 2\omega_0^2 \omega_0^2 + s \omega_0^2 \omega_0^2 + s \omega_0^2 \omega_0^2 + s \omega_0^2 \omega_0^2 - \omega_1(\omega_0^4 \omega_0^3 + \omega_0^3 \omega_0^3) = 0 \]

By direct expansion,

\[ (\omega_0 + \omega_0^2)^3 = (\omega_0^3 + 3\omega_0^2 \omega_0^2 + 3\omega_0^2 \omega_0^2 + \omega_0^3). \]

From equation (128),

\[ (\omega_0 + \omega_0^2)^3 = \left( \frac{s \omega_2}{s_1} \right)^3 \]

After combination of equations (145) and (146),

\[ (\omega_0^3 + \omega_0^3) = \left( \frac{s \omega_2}{s_1} \right)^3 \frac{s \omega_2}{s_1} - 3 \omega_2^2 (\omega_0 + \omega_0^2). \]

Substituting for $s \omega_2/s_1$ its value $\omega_0 \omega_0^2$, and rearranging terms reduce equation (147) to

\[ (\omega_0^3 + \omega_0^3) = \omega_2^2 (\omega_0 + \omega_0^2) \left( \frac{s_1^2}{s_1} - 3 \right). \]
Through the use of the relationship $\omega_{01} = \omega_{02} = \omega_2$, where desired, and the replacement of $(\omega_{01}^3 + \omega_{02}^3)$ by the relationship of equation (148), equation (144) is reduced to the following equation in $\omega_1$:

$$\omega_1^2 - \omega_1 \omega_2 (s_1 + \frac{s_2^2}{s_1^2} - 1) + \omega_2^2 (s_2^2 + \frac{s_2^2}{s_1^2} - 1) - \omega_1 \omega_2 = 0$$  (149)

Equation (149) may be further simplified to yield a quadratic equation in $s_1^2$, namely:

$$(s_1^2)(\omega_1^2 + \omega_2^2) - (s_1^2)(\omega_1^2 + \omega_2^2) + s_2^2(\omega_1^2 + \omega_2^2) = 0$$  (150)

Various forms of the solution for $s_1$ are given below.

$$s_1 = \sqrt{\frac{1}{2} \left( \frac{\omega_1^2 + \omega_2^2}{\omega_1 \omega_2} \right)^2 + \frac{1}{4} \left( \frac{\omega_1^2 + \omega_2^2}{\omega_1 \omega_2} \right)^4} - s_2^2$$  (151)

$$s_1 = \sqrt{\frac{1}{2} \left( \frac{f_1^2 + f_2^2}{f_1 f_2} \right)^2 + \frac{1}{4} \left( \frac{f_1^2 + f_2^2}{f_1 f_2} \right)^4} - s_2^2$$  (152)

and

$$s_1 = \sqrt{\frac{1}{2} \left( \frac{10^2 f_1 + 10^2 f_2}{10^2 f_1 + f_2} \right)^2 + \frac{1}{4} \left( \frac{10^2 f_1 + 10^2 f_2}{10^2 f_1 + f_2} \right)^4} - s_2^2$$  (153)

The $\pm$ sign in equations (151) through (153) allows a choice in the value of $s_1$. However, consideration of equation (86) indicates that a larger value of $s_1$ results in less attenuation per phase-shift section for the third type of one-pole network.

The solutions for $\omega_{01}$ and $\omega_{02}$ (also $f_{01}$ and $f_{02}$, $F_{01}$ and $F_{02}$) in terms of $s$ and $\omega$ (and $f$, $F$) follow. Combination of conditions (136) and (138) gives
\[ S \omega_2 = s, \frac{\omega_2^2}{\omega_{02}} + s, \omega_{02} \]  
\hspace{1cm} (154)

This may be rearranged as the quadratic equation in \( \omega_{02} \):
\[ \omega_{02}^2 (s) - \omega_{02} (s \omega_2) + s, \omega_2^2 = 0 \]  
\hspace{1cm} (155)

from which
\[ \omega_{02} = \frac{s \omega_2 \pm \sqrt{s^2 \omega_2^2 - 4s^2 \omega_2^2}}{2 s_1} \]  
\hspace{1cm} (156)

A similar solution for \( \omega_{01} \) yields the same equation, from which it may be deduced that
\[ \omega_{01} = \omega_2 \left( \frac{s - \sqrt{s^2 - 4s_1^2}^2}{2 s_1} \right) \]  
\hspace{1cm} (157)

and
\[ \omega_{02} = \omega_2 \left( \frac{s + \sqrt{s^2 - 4s_1^2}^2}{2 s_1} \right) \]  
\hspace{1cm} (158)

The corresponding formulas in \( f \) and \( F \) are
\[ f_{01} = f_2 \left( \frac{s - \sqrt{s^2 - 4s_1^2}}{2 s_1} \right) \]  
\hspace{1cm} (159)

\[ f_{02} = f_2 \left( \frac{s + \sqrt{s^2 - 4s_1^2}}{2 s_1} \right) \]  
\hspace{1cm} (160)

\[ F_{01} = F_2 + \log_{10} \left( \frac{s - \sqrt{s^2 - 4s_1^2}}{2 s_1} \right) \]  
\hspace{1cm} (161)
and

\[
F_{02} = F_2 + \log_{10}\left(\frac{s + \sqrt{s^2 - 4s_1^2}}{2s_1}\right).
\] (162)

Thus, it is demonstrated that the desired two-pole phase-frequency operation may be duplicated by means of two, one-pole networks connected in tandem through a coupling vacuum tube. The individual \(s_i\)'s and \(f_1\)'s of each one-pole network may be evaluated by means of the preceding equations for the special case of \(n\) equal to 2.

The schematic diagram of an experimental, two-pole, two-channel system is shown in Figure 15. For this particular system, \(s\) is equal to 22.343, \(\theta\) is equal to 0.9, and the spacing between \(F_{2A}\) and \(F_{2B}\) is equal to 0.45. The range of operation is centered about 900 cycles per second, and the third type of one-pole network is employed. Values of the circuit components are given in Appendix III. In presenting the network of Figure 15, no claim is made that the values chosen for \(s, s_1, \text{ etc.}\), represent the optima. The experimental model was developed solely to demonstrate the validity of the theoretical calculations, toward which end it may be stated that the experimental results closely parallel the theoretical predictions.
FIGURE 15

Schematic Diagram Of Two-Pole 90° Phase-Difference System
CHAPTER II
GENERAL NOMENCLATURE

In the description of one- and two-pole networks and systems, certain terms are introduced which aid in describing the behavior of the systems. However, the terms apply specifically to the special cases where \( n \) is equal to one and two. In a broader sense, it is clear that some sort of generalized notation must be introduced in order to adequately and systematically depict \( n \)-pole behavior. This section explains the meaning of the terms used throughout the report.

II-1 Terms Applicable to All Types of Networks

The term \( n \) denotes the complexity of a network whether the network be a full or semi-lattice. The number of poles of impedance in the \( Z_a \) or \( Z_1 \) arm or the number of zeros of impedance in the \( Z_b \) or \( Z_2 \) arm is given by \( n \). The term \( n \) is also used as a subscript with quantities which apply for an \( n \)-pole network. When used as a superscript, \( n \) indicates that a term is to be raised to a power equal numerically to the network complexity.

The frequency variable is expressed in terms of \( f \), \( F \) or \( \omega \) where the relationships

\[
f = 10^F = \frac{\omega}{2\pi} \hspace{1cm} (163)
\]

\[
F = \log_{10} f = \log_{10} \frac{\omega}{2\pi} \hspace{1cm} (164)
\]

and

\[
\omega = 2\pi f = 2\pi \cdot 10^F \hspace{1cm} (165)
\]
apply. These may be used interchangeably. Particular frequencies are identified by means of the subscripts 1, 2, 3---used with the terms of relationships (163), (164) and (165). Terms such as $f_1$, $F_1$, etc., are thereby constants. Since the locations of poles and zeros appear alternately, the terms $f_1, f_3, f_5$, etc., specify pole locations if the terms $f_2, f_4, f_6$, etc., specify zero locations, and vice versa.

In an n-pole network, the total phase excursion is from 0 to $-2n\pi$ radians as $f$ varies from zero to infinity. At $f$ equal to $f_n$, $F$ equal to $F_n$, or $\omega$ equal to $\omega_n$, the phase shift equals $-n\pi$ radians or one-half the total excursion. For this reason, $f_n$ is designated the Center Frequency, $F_n$ the Center log Frequency and $\omega_n$ merely as $2\pi$ multiplied by the Center Frequency.

At $f_1$, $F_1$ or $\omega_1$, the phase shift equals $-180^\circ$; at $f_2$, $F_2$ or $\omega_2$, the phase shift equals $-360^\circ$, etc. Between $f_k$ and $f_{k+1}$, a phase shift of $-180^\circ$ or $-\pi$ radians occurs. Accordingly, $f_1$, $f_2$, $f_3$, etc., are known as $\pi$-Point Frequencies and $F_1$, $F_2$, $F_3$, etc., as $\pi$-Points. In an n-pole network, there exist $(2n-1)$ $\pi$-Points. This may be observed from consideration of equation (3) which relates $m$, the total number of $f_k$'s, to $n$.

The spacing between the $\pi$-Points is an important factor in determining network behavior. When expressed in terms of units of $F$, the spacing is designed as $\delta$, the Center Spacing. Accordingly

$$\delta = F_{k+1} - F_k$$

(166)

$$\delta = \log_{10} \frac{f_{k+1}}{f_k}$$

(167)
Manipulations in terms of $\delta$ (and with differences in the $F$'s, in general) allow the network behavior to be expressed independently of the frequency location.

Two other points of interest on the phase curve are those points at which the phase shift is $\pi/2$ radians greater than $-n\pi$ radians and $\pi/2$ radians less than $-n\pi$ radians, respectively. These are known as the 90 Degree Frequency Points. The phase shifts at these points are equal to $-n\pi/2$ radians and $-(n + 2)\pi/2$ radians, respectively. The subscripts $n - \frac{1}{2}$ and $n + \frac{1}{2}$ identify references to the 90 Degree Frequency Points.

The phase-frequency curves of this report are plotted against linear phase shift and logarithmic frequency scales. As a result, the curves are symmetrical about $\log f_n$ and $F_n$, since the log frequency scale may be interpreted in terms of a linear $F$ scale.

Another important factor in the phase-shift equation is the parameter $s$. The behavior of the phase-frequency curve in the neighborhood of $F_n$ is determined by $s$, a fact which is demonstrated for $n$ equal to one in Figure 13.

The difference in phase between two, $n$-pole networks may be represented on a common plot as the difference between the phase angles of the phase-frequency curves. In this case, the two networks may be designated as channels A and B, A being the channel with the smaller value of $F_n$. Each quantity must be identified with the proper channel by means of the subscript $A$ or $B$. For example, $f_n$ of the A channel is $f_{nA}$.

\[
10^\delta = \frac{f_{kr}}{f_k} \quad (166)
\]
There exists a value of $F$ midway between $F_{nA}$ and $F_{nB}$ which is centrally located with respect to the log frequency span over which a specified phase difference appears. This value of $F$ is designated as the Mid-Log Frequency and bears the subscript, $M$. The relationship

$$F_{nA} + F_{nB} = 2F_M$$

applies. The corresponding relationship in terms of the Mid-Frequency is

$$f_{nA} \cdot f_{nB} = f^2_M$$

The difference between $F_{nB}$ and $F_{nA}$ may be determined in terms of the parameter $\sigma$ as

$$F_{nB} - F_{nA} = \eta \sigma$$

The term $\sigma$ is equal to $\varepsilon/2$ and is designated as the Unit Channel Spacing. The multiplier $\eta$ serves to establish the spacing between $F_{nB}$ and $F_{nA}$ in log frequency units and the combination $\eta \sigma$ is thereby the Channel Spacing.

When the constant $\eta$ of an $n$-pole system is properly chosen and satisfactory values of $s$ and Center Spacing are used, the phase difference between the output voltages is equal to approximately 90° over a wide range of $F$. Variations from the 90° value are known as "errors" and are represented by the variable $Y$. A plot of $Y$ versus $F$ is symmetrical about $F_M$.

The circuit parameters in a phase-shift section are identified by the parameter symbol followed by the impedance location symbol, the network complexity, the frequency identification and the channel symbol. The frequency identification may be omitted in the event that a number of circuit elements are described as a group. The channel symbol may also be omitted when not necessary. For example, the symbol $L_{131}$ in Figure 16 indicates...
that the circuit element is an inductance belonging to $Z_1$. The "3" designates a three-pole network and the last "1" indicates that the parallel combination of $L_{131}$ and $C_{131}$ is antiresonant at the lowest of the three antiresonant frequencies assigned to $L_{131}C_{131}$, $L_{133}C_{133}$ and $L_{135}C_{135}$, respectively.

II-2 Terms Applicable to R-C Networks Only

While it is mathematically possible to develop n-pole, L-C networks in which all circuit elements are contained in a single, phase-shift section, the practical n-pole systems require the use of n, cascaded, one-pole networks coupled through vacuum tubes for isolation.

Since the Center Frequencies and Center Log Frequencies for the n, cascaded, one-pole networks do not necessarily coincide with the $\Pi$-Point Frequencies and the $\Pi$-Points, additional frequency and log frequency terms must be introduced. The Center Frequencies and Center Log Frequencies of the one-pole networks are, in order of ascending numerical value, $f_{01}$, $f_{02}$ up to $f_{on}$ and $F_{01}$, $F_{02}$ up to $F_{On}$, respectively.

For the special case of $n$ equal to 2, it is observed that the s's of the two, cascaded, one-pole networks must be equal to $s_1$. This concept is carried on for three-, four- and all higher pole networks. In addition, the $n$ equal to 2 case points out that $f_{01} + f_{02}$ is equal to $f_2^2$ and $F_{01} + F_{02}$ is equal to $2F_2$. The symmetry indicated for $n$ equal to 2 also prevails for all higher pole networks. Thus, the log frequencies $F_{01}$, $F_{02}$ up to $F_{On}$ are not only equally spaced, but the entire assembly of $F_{0j}$'s is symmetrically centered about $F_n$. 

-67-
FIGURE 16
Mesh Form Of Three-Pole L-C Network

FIGURE 17
Mesh Form Of n-Pole L-C Network
CHAPTER III

THE GENERALIZED NETWORK AND EQUATIONS

III-1 Purely Reactive Three-Pole Semi-Lattice

The three-pole, L-C, semi-lattice network follows the same general structural scheme as the one- and two-pole counterparts. Impedance arm $Z_1$ is an L-C network having three poles and two zeros of impedance, the impedance arm $Z_2$ is a C-L network having three zeros and two poles of impedance, and $Z_3$ is a pure resistance. The criteria for reciprocation and termination apply. The mesh form of the three-pole semi-lattice is shown in Figure 16.

From Appendix II,

$$V_{34} = V_{12} \frac{(Z_1 - Z_2)}{Z_1 Z_2 + (Z_1 + Z_2)} \quad (172)$$

which applies for the three-pole network with

$$Z_1 = \frac{P}{C_{13}} \cdot \frac{(p^2 - p_2^2)(p^2 - p_3^2)}{(p^2 - p_1^2)(p^2 - p_3^2)(p^2 - p_5^2)} \quad (173)$$

$$Z_2 = \frac{L_{23}}{P} \cdot \frac{(p^2 - p_2^2)(p^2 - p_3^2)(p^2 - p_5^2)}{(p^2 - p_2^2)(p^2 - p_5^2)} \quad (174)$$

and

$$Z_3 = R_{33} \quad (175)$$

where

$$C_{13} = \frac{1}{C_{131} + \frac{1}{C_{133}} + \frac{1}{C_{135}}} \quad (176)$$
and

$$L_{23} = \frac{1}{\frac{1}{L_{231}} + \frac{1}{L_{233}} + \frac{1}{L_{235}}}$$  \hspace{1cm} (176a)$$

In addition,

$$L_{13} = L_{131} + L_{133} + L_{135}$$  \hspace{1cm} (177)$$

and

$$C_{23} = C_{231} + C_{233} + C_{235}$$  \hspace{1cm} (178)$$

where $L_{13}$ is the inductance which $Z_1$ approximates at zero frequency and

$C_{23}$ is the capacitance which $Z_2$ approximates at zero frequency.

From equation (40), the condition for reciprocation results in

$$Z_1Z_2 = (2R_{33})^2; \quad \frac{L_{23}}{C_{13}} = 4R_{g3}^2$$  \hspace{1cm} (179)$$

Substitution of equations (173), (174), (175) and (179) in equation

(172) leads to

$$V_{34} = V_{13} \begin{bmatrix}
\frac{p^2(p^2-p_2^2)(p^2-p_3^2)^2-4R_{33}^2(p^2-p_2^2)(p^2-p_3^2)(p^2-p_4^2)(p^2-p_5^2)}{p^2(p^2-p_2^2)(p^2-p_4^2)^2+4R_{33}^2C_3(p^2-p_3^2)(p^2-p_3^2)(p^2-p_4^2)(p^2-p_5^2)+}
\end{bmatrix}$$  \hspace{1cm} (180)$$

from which

$$\beta = \tan^{-1}\left\{\frac{-4\omega R_{33}C_3(p^2-p_2^2)(p^2-p_2^2)(p^2-p_3^2)(p^2-p_4^2)(p^2-p_5^2)}{4R_{33}^2C_{13}^2(p^2-p_3^2)(p^2-p_3^2)(p^2-p_3^2)^2+\rho^2(p^2-p_2^2)^2(p^2-p_3^2)^2}\right\}$$  \hspace{1cm} (181)$$

*The reduction of equations in this section are analogous to the reduc-
tions performed in equations (108)-(116) for the two-pole L-C network.
Substituting $j\omega$ for $p$, $j\omega_1$ for $p_1$, etc., dividing numerator and denominator by $4R_{13}^2 + 3$ and arranging terms, reduce equation (181) to

$$\beta = \tan^{-1} \left\{ \frac{2\omega \omega_3 (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)}{(\omega^2 - \omega_1^2)^2(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)} \right\} \quad (182)$$

By letting

$$S = \frac{1}{2R_{13}C_{13}\omega_3} \quad (183)$$

in equation (182), there results

$$\beta = \tan^{-1} \left\{ \frac{2\omega \omega_3 (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)}{(\omega^2 - \omega_1^2)^2(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)} \right\} \quad (184)$$

Since $\omega$ equals $2\pi f$,

$$\beta = \tan^{-1} \left\{ \frac{2\pi f \omega_3 (f^2 - f_1^2)(f^2 - f_2^2)(f^2 - f_3^2)(f^2 - f_4^2)}{(f^2 - f_1^2)^2(f^2 - f_2^2)(f^2 - f_3^2)(f^2 - f_4^2)} \right\} \quad (185)$$

which is identical to equation (11) for $n$ equal to 3, the desired form for the three-pole phase formula. The expression for $\beta$ in terms of $F$ is

$$\beta = \tan^{-1} \left\{ \frac{2\pi f \omega_3 (f^2 - f_1^2)(f^2 - f_2^2)(f^2 - f_3^2)(f^2 - f_4^2)}{(f^2 - f_1^2)^2(f^2 - f_2^2)(f^2 - f_3^2)(f^2 - f_4^2)} \right\} \quad (186)$$

By suitable application of equation (184) from Appendix I, it may be shown that $s = \sqrt{L_{13}/L_{23}}$, the square root of the ratio of the inductance of $Z_1$ at zero frequency to the inductance of $Z_2$ at infinite frequencies.
III-2 The General Solution of the Phase Equation

It is evident that the above process may be repeated for determination of $\theta$ for $n$ equal to 4, 5, etc. The solution for $\theta$ may thus be obtained in general terms for the $n$-pole network as

$$\theta = \tan^{-1} \left\{ \frac{2s \omega_n (w^2 - w_1^2)(w^2 - w_2^2)(w^2 - w_3^2) (w^2 - w_m^2)}{(w^2 - w_1^2)(w^2 - w_2^2)(w^2 - w_3^2)(w^2 - w_m^2)(w_1^2 - w_2^2)(w_1^2 - w_3^2)(w_1^2 - w_m^2)} \right\}$$  \hspace{1cm} (187)

$$\beta = \tan^{-1} \left\{ \frac{2s + f_n (f^2 - f_1^2)(f^2 - f_2^2)(f^2 - f_3^2)(f^2 - f_m^2)}{(f^2 - f_1^2)(f^2 - f_2^2)(f^2 - f_3^2)(f^2 - f_m^2) - S^2 + f_n^2 + (f^2 - f_1^2)(f^2 - f_2^2)(f^2 - f_3^2)(f^2 - f_m^2)} \right\}$$  \hspace{1cm} (188)

and

$$\theta = \tan^{-1} \left\{ \frac{2s 10 \delta f_n (10^2 - 10^2)(10^2 - 10^2)(10^2 - 10^2) (10^2 - 10^2)}{(10^2 - 10^2)(10^2 - 10^2)(10^2 - 10^2)(10^2 - 10^2) - S^2 10^2 - 10^2 (10^2 - 10^2)(10^2 - 10^2)(10^2 - 10^2)} \right\}$$  \hspace{1cm} (189)

Equations (187), (188) and (189) must apply for any $n$-pole network. In particular, the mesh form of an $n$-pole L-C network is shown in Figure 17.

It should be noted that equation (188) is identical to equation (11), previously developed for the purely reactive full lattice.

III-3 The Parameters $s$ and $\delta$

On several previous occasions, the importance of $s$ was emphasized. The shape of the phase curve in the vicinity of $F_n$ is a direct reflection of the value of $s$. At $F$ equal to $F_n$, all phase curves, regardless of $s$, pass through the point $(F_n, -180^\circ)$. As $F$ approaches either zero or infinity, the phase curves tend to coincide. However, at values of $F$ near $F_n$, the values of $\theta$ differ considerably for different values of $s$. The slopes of
the phase curves at values of $F$ far removed from $F_n$ are essentially equal. This is decidedly not the case at $F_n$. As $s$ is increased in value, the slope of the corresponding phase curve, in terms of degrees (negative) per unit of $F$, is decreased. The above assertions may be verified qualitatively, at least, in Figure 3 for a one-pole network.

In this report, the parameter $s$ and the constant $\delta$ are interrelated to such an extent that one may not be specified independently of the other. A special interpretation must be placed on the term $\delta$ as applied to a one-pole network. As defined in previous sections, $\delta$ is equal to $F_k - F_{k-1}$, a relationship which does not exist explicitly for $n$ equal to 1. Fortunately, further work on the phase equations indicates that the 90 Degree Log Frequency Points may be placed midway between $F_n$, $F_{n-1}$ and $F_n$, $F_{n+1}$, respectively. The difference between the 90 Degree Log Frequency Points under these conditions is equal to $\delta$. Therefore, $\delta$ for the one-pole network may be considered as

$$\delta = F_{1.5} - F_{0.5}$$

(190)

The phase-frequency curve for an $n$-pole network must pass through all $\Pi$-Points, including the Center Frequency Point. In addition, equation (166), repeated below as

$$\delta = F_{k+1} - F_k$$

(191)

indicates that a straight line may be drawn through the $\Pi$-Points. The straight line is an ideal phase curve in the sense that complete coincidence of the actual phase curve with the straight line would result in an "errorless" system. However, at present only the $m, \Pi$-Points lie on both the phase curve and the straight line. It is possible to place additional
restrictions on the factors of the phase equation such that the 90 Degree Points are forced to lie on the straight line, thereby adding two points of coincidence. In an n-pole phase curve there are now \( m + 2 \) possible points, instead of \( m \) points, shared mutually by the phase curve and the straight line. The difference between the 90 Degree Log Frequency Points becomes

\[
\delta = F_{n+0.5} - F_{n-0.5}
\]

which substantiates the prediction made for the corresponding difference for \( n \) equal to 1 in equation (190).

The next step is to demonstrate the manner in which \( \delta \) may be specified in terms of \( s \) and, vice versa, under the above restricted conditions. Once a value of \( \delta \) has been chosen, an \( s \) must follow, and that \( s \) is the proper value for forcing the \( (m + 2) \) points of coincidence. In the same manner a value of \( \delta \) may be determined for a given \( s \). The required proofs for the \( n \)-pole case follow specific proofs for \( n \) equal to 2, 3 and 4. The proof for network complexities higher than 4 is immediately evident and may be established conclusively, if desired, by mathematical induction.

For the two-pole network \( F_n \) equals \( F_2 \), \( F_{n-1} \) equals \( F_1 \) and \( F_{n+1} \) equals \( F_3 \). At a 90 Degree Point, \( \theta \) equals \(-n \cdot 90^\circ \) or \(-(n + 2) \cdot 90^\circ \), depending on whether the upper or lower point is chosen. The value of \( F \) at the lower 90 Degree Point is, in accordance with the restrictions, equal to \((F_1 + F_2)/2 \).

Since the tangent of any integral multiple of 90° is equal to infinity, either the numerator of the expression for \( \theta \) must be infinite with the denominator finite, the denominator must be zero with the numerator finite, or the ratio of the numerator to the denominator must be an
indeterminate form, the limit of which is infinity. The first and third possibilities are discarded because the numerator is finite at F equal to \((F_1 + F_2)/2\). Hence, the denominator must equal zero and may be arranged in terms of \(F_1\) and \(F_2\) as

\[
\left(10^{F_1 + F_2 - 10F_i}\right) \left(10^{F_1 + F_2} - 10^{4F_2 - 2F_i}\right) = S \cdot 10^{F_1 + F_2 - 10F_i} \left(10^{F_1 + F_2 + 10F_2}\right)^* \tag{193}
\]

which reduces to

\[
10^{F_1} \left(10^{3F_2 - 2F_i} - 10^{F_i}\right) = S \cdot 10^{F_1 + \frac{3F_2}{2}} \tag{194}
\]

By squaring each side of equation (194) and rearranging terms, the following quadratic equation in \(10^{3F_2}\) is obtained:

\[
\left(10^{3F_2}\right)^2 - \left(2 + s^2\right)\left(10^{3F_2}\right) + 10^{6F_i} = 0 \tag{195}
\]

The solution of equation (195) leads to

\[
10^{3F_2} = 10^{3F_1} \left[\frac{(2 + s^2) \pm s \sqrt{4 + s^2}}{2}\right] \tag{196}
\]

or

\[
\frac{10^{3F_2}}{10^{3F_1}} = \frac{(2 + s^2) \pm s \sqrt{4 + s^2}}{2} \tag{197}
\]

where the plus sign is used to secure a positive ratio.

By taking the logarithm of each side and dividing the result by three, it follows that

\[
\delta = F_2 - F_1 = \frac{1}{3} \log_{10} \frac{(2 + s^2) \pm s \sqrt{4 + s^2}}{2} \tag{198}
\]

*In equation (193) use is made of relations \(2F_2 = F_1 + F_3\), \(F_3 = 2F_2 - F_1\).
The three-pole network solution for the value of $S$ follows the general trend of the two-pole solution. As before, it is convenient to establish $S$ as the difference between $F_n$ and $F_{n-1}$ or, specifically, $F_3 - F_2$. The value of $F$ at the lower 90 Degree Point is $(F_2 + F_3)/2$. Moreover, $F_1$, $F_4$ and $F_5$ are expressed in terms of $F_2$ and $F_3$ as

$$F_1 = F_3 - 2(F_3 - F_2) = 2F_2 - F_3$$  \hspace{1cm} (199)

$$F_4 = F_3 + (F_3 - F_2) = 2F_3 - F_2$$ \hspace{1cm} (200)

and

$$F_5 = F_3 + 2(F_3 - F_2) = 3F_3 - 2F_2$$ \hspace{1cm} (201)

The above relationships are evident upon consideration of the following simple sketch:

Substituting $(F_2 + F_3)/2$ for $F$ in the three-pole phase equation, setting the denominator equal to zero, and simplifying, as in the case of the two-pole equation, yield the following quadratic equation in $10^{5F_3}$.

$$(10^{5F_3})^2 - (2 + S^2)(10^{5F_2})(10^{5F_3}) + 10^{10F_2} = 0$$ \hspace{1cm} (202)

The solution of equation (202) is

$$10^{5F_3} = 10^{5F_2} \left[ \frac{(2 + S^2) \pm S \sqrt{4 + S^2}}{2} \right]$$ \hspace{1cm} (203)
or
\[
\frac{10^5 f_3}{10^5 f_2} = \frac{(2 + s^2) + s\sqrt{4 + s^2}}{2}.
\]  

Taking the logarithm of each side and dividing the result by five result in
\[
8 = f_3 - f_2 = \frac{1}{5} \log_{10} \frac{(2 + s^2) + s\sqrt{4 + s^2}}{2}.
\]

Again, the plus sign should be chosen in order to obtain a logical solution.

The four-pole phase equation may be considered in the same light, where \(8\) may be established as the difference between \(f_4\) and \(f_3\). The value of \(F\) at the lower 90 Degree Point is \((f_4 + f_3)/2\). The other significant values of \(F\) as expressed in terms of \(f_3\) and \(f_4\) are
\[
F_1 = f_4 - 3(f_4 - f_3) = 3f_3 - 2f_4,
\]
\[
F_2 = f_4 - 2(f_4 - f_3) = 2f_3 - f_4,
\]
\[
F_5 = f_4 + (f_4 - f_3) = 2f_4 - f_3,
\]
\[
F_6 = f_4 + 2(f_4 - f_3) = 3f_4 - 2f_3,
\]
and
\[
F_7 = f_4 + 3(f_4 - f_3) = 4f_4 - 3f_3.
\]

A sketch is again presented as a means of visualizing the above relationships.
By substituting \((F_3 + F_4)/2\) for \(F\) in the four-pole phase equation, setting the denominator equal to zero and simplifying, the following quadratic equation in \(10^{7F_4}\) results:

\[
\left(10^{7F_4}\right)^2 - (2 + S^2)(10^{7F_3})(10^{7F_4}) + 10^{14F_3} = 0.
\]  

(211)

The solution of equation (211) is

\[
10^{7F_4} = 10^{7F_3} \left[ \frac{(2 + S^2) \pm S\sqrt{4 + S^2}}{2} \right] \]  

(212)

or

\[
\frac{10^{7F_4}}{10^{7F_3}} = \left[ \frac{(2 + S^2) \pm S\sqrt{4 + S^2}}{2} \right].
\]  

(213)

when the plus sign is used to secure a positive ratio. Application of logarithms to both sides and division by seven yield

\[
\delta = F_4 - F_3 = \frac{1}{7} \log_{10} \frac{(2 + S^2) + S\sqrt{S^2 + 4}}{2}.
\]  

(214)

Without introducing further proofs for \(n\) equal to 5, 6, etc., it is evident that the equations

\[
\left(10^{(2n-1)F_n}\right)^2 - (2 + S^2)(10^{(2n-1)F_{n-1}})(10^{(2n-1)F_n}) + 10^{8(2n-1)F_{n-1}} = 0,
\]  

(215)

\[
\frac{10^{(2n-1)F_n}}{10^{(2n-1)F_{n-1}}} = \frac{(2 + S^2) + S\sqrt{S^2 + 4}}{2},
\]  

(216)
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and

\[ S = F_n - F_{n-1} = \frac{1}{2n-1} \log_{10} \left( \frac{(2+S^2) + S\sqrt{S^2+4}}{2} \right) \]  

(217)

apply for the general case.

The solution for \( s \) in terms of \( S \) may be obtained from equation (216), which is rearranged as

\[ 2(1 - 10^{(2n-1)S}) - S^2 \cdot 10^{(2n-1)S} = S \cdot 10^{(2n-1)S} \sqrt{4+S^2} \]  

(218)

By squaring each side of equation (218) and reducing to simpler terms,

\[ \left(10^{(2n-1)S} - 1\right)^2 = S^2 \cdot 10^{(2n-1)S} \]  

(219)

is obtained, from which the solution

\[ S = \frac{10^{(2n-1)S} - 1}{10^{(n-\frac{1}{2})S}} \]  

(220)

follows immediately. A plot of \( S \) versus \( s \) for various values of \( n \) is shown in Figure 18.

III-4 Investigation of Phase-Curve Crossover Points

A justifiable question now arises as to whether more than two points like the 90 Degree Points may be obtained in three- and higher pole networks. There are \( 2n-2 \) points (including the original 90 Degree Points) where the phase curve crosses the straight line drawn through the \( \Pi \)-Points. For \( n \) equal to two, the 90 Degree Points consume all possible choices; for \( n \) equal to one, the artificiality of designating 90 Degree Points is immediately apparent. Nevertheless, the possibility of obtaining more points of coincidence (thereby forcing the three- and higher pole curves to duplicate more nearly the ideal curve) is an interesting one which cer-
tainly should bear examination. The simplest curve which may be considered in this light is the three-pole phase curve. Success with the three-pole curve encourages further investigation; failure would doom all subsequent efforts.

The equation for $s$ of a three-pole network is given by

$$
S = \frac{10^{5s} - 1}{10^{5s}} = \frac{10^{5(F_3 - F_2)} - 1}{10^{5(F_3 - F_2)}}
$$

(221)

under the conditions of coincidence of the 90 Degree Points. Since established gains should not be relinquished, this value of $s$ must be consistent with any succeeding computations. If additional points like the original 90 Degree Points are to be established, the lower of these points must correspond to the point $((F_1 + F_2)/2, -270^\circ)$; i.e., at $F$ equal to $(F_1 + F_2)/2$, $\beta$ equals $-270^\circ$. The denominator of the phase equation should equal zero at $F$ equal to $(F_1 + F_2)/2$.

As in the proof for the 90 Degree Points, the denominator may be expressed in terms of $F_2$ and $F_3$ by means of equations (199), (200) and (201). From setting the denominator of the three-pole phase equation equal to zero at $F$ equal to $(F_1 + F_2)/2$, the equation

$$
S \cdot 10^{\frac{F_1 + F_2}{2}} \cdot 10^{F_3} \left(10^{F_1 + F_2} - 10^{2F_2}\right) = \left(10^{F_1 + F_2} - 10^{2F_1}\right)
$$

(10^{F_1 + F_2} - 10^{2F_3})

(222)

results. Substitution of the last equality of equation (221) and substitution of formulas (199) through (201) in equation (222), together with subsequent simplification and rearrangement of terms, reduces equation (222) to

$$
10^{-4F_2} - 2 \cdot 10^{-2F_2} + 2F_3 + 10^{-4F_3} = 0
$$

(223)
FIGURE 18
Variation of s with spacing δ between (n · 90°)° and [((n+2) · 90°)° Points.

Numerical Values of Parameters s

Spacing in Log Units (δ)

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from which
\[ 10^{2\beta_2} = 10^{2\beta_3} \]  
(224)

This is an obvious impossibility. Hence, there are, at most, \( m + 2 \) points of perfect coincidence between the \( n \)-pole phase curve and the idealized curve.

### III-5 Combination of Three One-Pole Networks to Duplicate Phase Characteristics of Three-Pole Purely Reactive Networks.

Considerable information has been amassed regarding the behavior of \( n \)-pole L-C networks. By means of the L-C analysis, not only are the overall phase-frequency equations for all types of \( n \)-pole networks established, but a wealth of information relative to the parameter relationships is established as well. Following the pattern of previous discussions, the logical procedure at this point is to examine the internal characteristics of practical three- and higher pole networks.

The phase-frequency formula which applies for any type of three-pole network is given by equation (185). The comparatively simple mathematics of purely reactive systems again becomes the tool by means of which the phase-frequency formula of practical three-pole networks may be obtained. For a practical network, three cascaded one-pole networks must be used. The method for obtaining the center frequencies of the one-pole networks is analogous to that employed for the practical two-pole network.

The total phase angle derived from three cascaded one-pole networks each having the same value of \( s_1 \) is, in terms of \( \omega \),

\[
\beta = \beta_1 + \beta_2 + \beta_3 = \tan^{-1} \left\{ \frac{2s_1 \omega \omega_{b_1} (\omega^2 - \omega_{b_1}^2)}{(\omega^2 - \omega_{b_1}^2)^2 - \omega^2 \omega_{b_1}^2} \right\} + \tan^{-1} \left\{ \frac{2s_2 \omega \omega_{b_2} (\omega^2 - \omega_{b_2}^2)}{(\omega^2 - \omega_{b_2}^2)^2 - \omega^2 \omega_{b_2}^2} \right\} + \tan^{-1} \left\{ \frac{2s_3 \omega \omega_{b_3} (\omega^2 - \omega_{b_3}^2)}{(\omega^2 - \omega_{b_3}^2)^2 - \omega^2 \omega_{b_3}^2} \right\},
\]  
(225)
where \( \theta_1, \theta_2, \theta_3 \) and \( \omega_{01}, \omega_{02}, \omega_{03} \) are the values for \( \theta \) and \( \omega \) of the first, second and third one-pole networks, respectively.

The sum of three arc tangents may be expressed as

\[
tan^{-1} \mu + tan^{-1} \nu + tan^{-1} \omega = tan^{-1} \left( \frac{\mu + \nu + \omega - \mu \nu \omega}{1 - \mu \nu - \omega \mu \nu - \omega \nu} \right) \tag{226}
\]

By applying equation (226) to equation (225) and systematically arranging the result, the following expression for \( \theta \) is obtained:

\[
\theta = tan^{-1} \left[ \frac{2 \left( s \right) \left[ w_0 \left(w_0 + w_0 + w_0 \right) - w^2 \left(s \right) \left[ w_0 + w_0 + w_0 \right] + \right] - \left( s \right) \left[ 2 + s^2 \right] \left[ 2 w_0^2 \left(w_0 + w_0 + w_0 \right) + w_0^2 \left(w_0 + w_0 + w_0 \right) \right] + \right]}{1 - \left( s \right) \left[ 2 + s^2 \right] \left[ 2 w_0^2 \left(w_0 + w_0 + w_0 \right) + w_0^2 \left(w_0 + w_0 + w_0 \right) \right] + \left( s \right) \left[ 2 + s^2 \right] \left[ 2 w_0^2 \left(w_0 + w_0 + w_0 \right) + w_0^2 \left(w_0 + w_0 + w_0 \right) \right] + \right]}
\]

\[
\text{In equation (227), use is made of the relationships } \omega_0^2 = \omega_{01} \omega_{03} \text{ and } \omega_0^2 = \omega_{01} \omega_{02} \omega_{03}.
\]
Equation (184) may be expanded to yield

\[
\beta = \tan^{-1} \left( \frac{\omega^2 - \omega^6 (2[\omega_1^2 + \omega_2^2 + \omega_5^2] + s^2 \omega_3^2)}{\omega^8 ([\omega_1^4 + \omega_3^4 + \omega_5^4] + 4 \omega_5^2 [\omega_1^2 + \omega_3^2 + \omega_5^2] + 2 s^2 \omega_5^2 [\omega_2^2 + \omega_4^2]) - \omega^6 (2 \omega_5^2 [\omega_1^4 + \omega_5^4] + 4 \omega_5^2 [\omega_1^2 + \omega_3^2 + \omega_5^2] + 8 \omega_5^6 + s^2 \omega_3^2 [\omega_2^4 + \omega_4^4] + 4 s^2 \omega_5^6) + \omega^4 \omega_3^4 ([\omega_1^4 + \omega_3^4 + \omega_5^4] + 4 \omega_5^2 [\omega_1^2 + \omega_3^2 + \omega_5^2] + 2 s^2 \omega_3^2 [\omega_2^2 + \omega_4^2] - \omega^2 \omega_3^3 (2 [\omega_1^2 + \omega_5^2 + \omega_2^2] + s^2 \omega_5^2) + \omega_3^6 \right) \right)
\]

In order to obtain correspondence between the phase equations represented by equations (227) and (228), the two expressions must be identical, term by term. The conditions under which the identities are established are the conditions which must apply for the three-pole network made up of three cascaded one-pole networks.

The coefficient of the \(\omega^{11}\) term of equation (227) is equal to the coefficient of the \(\omega^{11}\) term of equation (228) under the condition that

\[
S \omega_3 = s_1 [\omega_{01} + \omega_{02} + \omega_{03}] \quad . \quad \quad (229)
\]

The \(\omega\) coefficients are immediately equal by virtue of equation (229) and the relationship

\[
\omega_3 = \omega_{02} \quad . \quad \quad (230)
\]
Since equality exists between the $\omega^{10}$ coefficients and between the $\omega^{2}$ coefficients,

$$S^{2}\omega^{2}_{3} + 2\left[\omega^{1}_{1} + \omega^{2}_{2} + \omega^{2}_{3}\right] = \left[2 + S^{2}_{1}\right]\left[\omega^{2}_{1} + \omega^{2}_{2} + \omega^{2}_{3}\right] +$$

$$4S^{2}_{1}\omega^{2}_{02}\left[\omega^{0}_{1} + \omega^{0}_{2} + \omega^{0}_{3}\right]$$

when equation (230) is considered. Since, after each side of condition (229) is squared,

$$S^{2}\omega^{2}_{3} = S^{2}_{1}\left(\omega^{2}_{1} + \omega^{2}_{2} + \omega^{2}_{3}\right) + 2S^{2}_{1}\omega^{2}_{02}\left(\omega^{0}_{1} + \omega^{0}_{2} + \omega^{0}_{3}\right)$$

subtraction of equation (232) from equation (231) leaves

$$\omega^{2}_{1} + \omega^{2}_{2} + \omega^{2}_{3} = \omega^{2}_{1} + \omega^{2}_{2} + \omega^{2}_{3} + 2S^{2}_{1}\omega^{2}_{02}\left[\omega^{0}_{1} + \omega^{0}_{2} + \omega^{0}_{3}\right],$$

from which

$$\omega^{2}_{1} + \omega^{2}_{2} + \omega^{2}_{3} = \omega^{2}_{1} + \omega^{2}_{2} + \omega^{2}_{3} + S^{2}_{1}\omega^{2}_{02}\left[\omega^{0}_{1} + \omega^{0}_{2} + \omega^{0}_{3}\right].$$

(234)

The coefficients of the $\omega^{8}$ and $\omega^{14}$ terms are next considered. By direct expansion,

$$\omega^{2}_{1} + \omega^{2}_{3} + \omega^{2}_{5} = \omega^{2}_{1} + \omega^{2}_{3} + \omega^{2}_{5} + 2\omega^{2}_{3}\left[\omega^{1}_{1} + \omega^{3}_{1} + \omega^{5}_{1}\right],$$

(235)

and

$$\omega^{2}_{1} + \omega^{2}_{3} + \omega^{2}_{5} = \omega^{2}_{1} + \omega^{2}_{3} + \omega^{2}_{5} + 2\omega^{2}_{3}\left[\omega^{1}_{1} + \omega^{3}_{1} + \omega^{5}_{1}\right].$$

(236)

After successive reductions,

$$\omega^{2}_{1} + \omega^{2}_{3} + \omega^{2}_{5} = \omega^{2}_{1} + \omega^{2}_{3} + \omega^{2}_{5} - 2\omega^{2}_{3}\left[\omega^{1}_{1} + \omega^{3}_{1} + \omega^{5}_{1}\right]$$

(237)

and

$$\omega^{4}_{1} + \omega^{4}_{3} + \omega^{4}_{5} = \omega^{4}_{1} + \omega^{4}_{3} + \omega^{4}_{5} - 2\omega^{2}_{3}\left[\omega^{2}_{1} + \omega^{2}_{3} + \omega^{2}_{5}\right]$$

(238)

By virtue of equation (234),

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When equation (233) is multiplied by \(2 \omega_3^2\), or the equivalent, \(2 \omega_{02}^2\),

\[
4 \omega_3^2 \left[ (\omega_2^2 + \omega_4^2 + \omega_5^2) = 4 \omega_{02}^2 \left[ (\omega_{01}^2 + \omega_{02}^2 + \omega_{03}^2) + 4 s_4 \omega_{02} \left[ (\omega_{01} + \omega_{02} + \omega_{03}) \right] \right] \right.
\]

By adding equations (239) and (240), the first two terms of the coefficient of \(\omega^8\) in equation (228) are obtained. The remaining term,

\(2 s_3^2 \omega_3^2 \left( \omega_2^2 + \omega_4^2 \right)\), must then equal the difference between the coefficient of equation (227) and the sum of the right-hand members of equations (239) and (240). When expanded and grouped, this difference is represented by

\[
2 s_3^2 \omega_3^2 \left[ (\omega_2^2 + \omega_4^2) = \left( 4 s_1 \omega_{02}^2 \left[ (\omega_{01}^2 + \omega_{02}^2 + \omega_{03}^2) + 2 s_1 \omega_{02} \left[ (\omega_{01} + \omega_{02} + \omega_{03}) \right] \right] \right) \right.
\]

By use of equation (229),

\[
[\omega_{01}^2 + \omega_{02}^2 + \omega_{03}^2] = \left[ (\omega_{01} + \omega_{02} + \omega_{03}) \right] - 2 \omega_{02} \left[ (\omega_{01} + \omega_{02} + \omega_{03}) \right].
\]

After division by \(2 s_3 \omega_3\), equation (244) becomes

\[
s s_3 \left[ (\omega_2^2 + \omega_4^2) = 2 s_3 \omega_{02} \left[ (\omega_{01} + \omega_{03}) \right] + s_3 \omega_{02} + s_4 \omega_{02} \left[ (\omega_{01}^2 + \omega_{03}^2) \right] \right.
\]

Equation (245) applies for the \(\omega^4\) coefficients, as well.

From the \(\omega^9\) and \(\omega^3\) coefficients,

\[
s s_3 \left[ (\omega_1^2 + \omega_3^2 + \omega_5^2) + s s_3 \left[ (\omega_2^2 + \omega_4^2) = s \left[ (\omega_{01}^3 + \omega_{02}^3 + \omega_{03}^3) + s \left[ (2 + s_1) \left[ 2 \omega_{02} \left( \omega_{01} + \omega_{03} \right) \right] + 4 s_3 \omega_{02} \left( \omega_{01}^2 + \omega_{02}^2 \right) \right] \right. \right.
\]

-86-
By dividing equation (233) by two, multiplying the left side by \( s \omega^3 \) and the right side by \( s_1 (\omega_{01} + \omega_{02} + \omega_{03}) \), the equation

\[
\frac{s \omega^3}{s_1(\omega_{01} + \omega_{02} + \omega_{03})} \left[ \omega^2 + \omega_{02}^2 + \omega_{03}^2 \right] = s_1 \left[ \omega_{01} + \omega_{02} + \omega_{03} \right] \left[ \omega_{01}^2 + \omega_{02}^2 + \omega_{03}^2 \right]
\]

is obtained. The sum of equations (245) and (243) yields equation (244), which thus establishes the desired correspondence between the appropriate \( \omega^9 \) and \( \omega^3 \) terms.

The coefficient of the \( \omega^7 \) term in equation (228) is equal to the sum of (a) the left side of equation (244), (b) the product of the left side of equation (234) multiplied by the left side of equation (243), divided by \( s \omega^3 \), and (c) the term \( \omega_4^3 \). The corresponding sums and products of the right sides yield identically the coefficient of the \( \omega^7 \) term of equation (227). The same procedure is followed in establishing correspondence between the \( \omega^5 \) terms.

By employing the previously established conditions, the expansion of the \( \omega^6 \) coefficients of equations (227) and (228) results in equality between the \( \omega^6 \) terms, which are the final terms to be accounted for.

Complete correspondence is established between equation (227) and (228) under the following summarized conditions:

\[
\omega_{01} \omega_{03} = \omega_{02}^2 = \omega_3^2 \quad \gamma
\]

\[
s_1 \left[ \omega_{01} + \omega_{02} + \omega_{03} \right] = s \omega^3 \quad \gamma
\]

\[
\left[ \omega_{01}^2 + \omega_{02}^2 + \omega_{03}^2 \right] + s_1^2 \left[ \omega_{01} \omega_{02} + \omega_{01} \omega_{03} + \omega_{02} \omega_{03} \right] = \left[ \omega_{1}^2 + \omega_{3}^2 + \omega_{5}^2 \right]
\]
and

\[
S_1 \omega_{02}^2 + 2S_2 \omega_{03} \left[ \omega_{01} + \omega_{03} \right] + S_3 \left[ \omega_{01}^2 + \omega_{03}^2 \right] = S \left[ \omega_2^2 + \omega_4^2 \right].
\]  

(249)

For a given \( s \) and \( \omega_3 \), the values for \( \omega_{02} \) and \( \omega_1 \) through \( \omega_5 \) may be determined. The values for \( \omega_{01}, \omega_{03} \) and \( s_1 \) must be chosen so that equations (246) through (249) are satisfied. Because of the restrictions placed on the locations of \( \omega_2, \omega_4, \omega_{01} \) and \( \omega_{03} \), it is possible to neglect equation (249) and obtain solutions which satisfy only equations (246), (247) and (248). The similarity between these equations and equations (135), (137) and (138) for the two-pole network should be noted. A direct solution for \( s_1, \omega_{01} \) and \( \omega_{03} \) is not as easily obtained as was the corresponding solution for \( s_1, \omega_{01} \) and \( \omega_{02} \) of the two-pole cascaded network.*

The three-pole case is considered in the general solution where \( n \) equals three.

III-6 The General Solution for Combining \( n \) One-Pole Networks to Duplicate Phase Characteristics of \( n \)-Pole Purely Reactive Networks.

The general method of solution which is applied to the two- and three-pole networks may be carried out successively on the four-, five- and higher pole networks in order to determine the conditions under which cascaded one-pole networks duplicate the phase characteristics of purely reactive networks. However, the very lengthy discussion required (as evidenced by the three-pole discussion just concluded) does not warrant inclusion of all the steps leading to the solutions of required conditions. It is sufficient to state that the method, as in the two- and three-pole cases, consists of determining the combined sum of \( n \) arc tangents and comparing the equa-

*It is emphasized in the two-pole case that the circuit values are obtained because the two-pole solution happens to be a special case in which the highest degree equations reduce to quadratics, and hence may be solved by relatively simple methods.
tion derived with the expanded n-pole phase equation. The desired conditions are then determined by finding the term by term correspondence. While the details might be cumbersome and wordy, the results are significant. For instance, in the four-pole case, the conditions corresponding to equations (136), (138) and (139) for the two-pole case and equations (246), (247) and (248) for the three-pole case are

\[\omega_{02}\omega_{03} = \omega_{01}\omega_{04} = \omega_4^2, \quad \text{(250)}\]

\[s_1 \left[\omega_{01} + \omega_{02} + \omega_{03} + \omega_{04}\right] = s\omega_4 \quad \text{(251)}\]

and

\[\left[\omega_{01}^2 + \omega_{02}^2 + \omega_{03}^2 + \omega_{04}^2\right] + s_1^2 \left[\omega_{01}\omega_{02} + \omega_{01}\omega_{03} + \omega_{01}\omega_{04} + \omega_{02}\omega_{03} + \omega_{02}\omega_{04} + \omega_{03}\omega_{04}\right] = \left[\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2\right]. \quad \text{(252)}\]

In addition, the conditions below follow by definition:

\[\omega_{01}\omega_{03} = \omega_{02} \quad \text{,} \quad \text{(253)}\]

\[\omega_{02}\omega_{04} = \omega_{03} \quad \text{,} \quad \text{(254)}\]

\[\omega_1\omega_7 = \omega_2\omega_6 = \omega_3\omega_5 = \omega_4^2 \quad \text{,} \quad \text{(255)}\]

\[\omega_1\omega_3 = \omega_2^2 \quad \text{,} \quad \text{(256)}\]

etc. For the five-pole network, the conditions are

\[\omega_{01}\omega_{05} = \omega_{02}\omega_{04} = \omega_{03}^2 = \omega_5^2 \quad \text{,} \quad \text{(257)}\]

\[s_1 \left[\omega_{01} + \omega_{02} + \omega_{03} + \omega_{04} + \omega_{05}\right] = s\omega_5 \quad \text{(258)}\]
and
\[
\begin{align*}
&\left[\omega_0^2 + \omega_0^2 + \omega_0^2 + \omega_0^2 + \omega_0^2\right] + \sum_i \left[\omega_i^2 + \omega_i^2 + \omega_i^2 + \omega_i^2 + \omega_i^2\right] + S_i \left[\omega_i^2 + \omega_i^2 + \omega_i^2 + \omega_i^2 + \omega_i^2\right] = \left[\omega_1^2 + \omega_1^2 + \omega_1^2 + \omega_1^2 + \omega_1^2\right].
\end{align*}
\]

The auxiliary conditions, similar to equations (253) through (256) apply.

The above process may be repeated indefinitely for successively higher pole networks. However, it is most desirable to generalize the notation at this point. In particular, it should be noted that the coefficient of \(s_1^2\) in equations of the type illustrated by (259) is a summation of all possible products of the \(\omega_1\)'s of the one-pole networks taken two at a time, with the exception that no \(\omega_1\) is multiplied by itself. Accordingly, the required conditions for the \(n\)-pole case may be stated as

\[
\omega_0^{(i+q)} \cdot \omega_0^{(n-q)} = \omega_0, \quad \omega_{on} = \omega_n^2, \quad (260)
\]

\[
S_i \left[\omega_i + \omega_{i+1} + \ldots + \omega_n\right] = S_i \sum_{k=1}^{n} \omega_{ok} = s \omega_n \quad (261)
\]

and

\[
\sum_{k=1}^{n} \omega_{ok}^2 + S_i \sum_{j=1}^{n} \sum_{k=1}^{n} \omega_{oj} \omega_{ok} = \sum_{k=1}^{n} \omega_{2k-1}^2 \quad (262)
\]

where \(j\) is not equal to \(k\).

In the \(F\) form, equations (260), (261) and (262) appear as

\[
10^{F_0(i+q)} \cdot 10^{F_0(n-q)} = 10^{2F_n} \quad (263)
\]

where \(q\) may equal 0, 1, 2, \ldots, \((2n-3) + (-1)^{n-1}\),

-90-
\[ S_i \sum_{k=1}^{n} 10^{F_{ok}} = S \cdot 10^{F_n} \quad (264) \]

and

\[ \sum_{k=1}^{n} 10^{2F_{ok}} + S_i^2 \sum_{j=1}^{n} \sum_{k=1}^{n} 10^{F_{oj}} \cdot 10^{F_{ok}} = \sum_{k=1}^{n} 10^{2F_{2k-1}} \quad (265) \]

where \( j \) is not equal to \( k \).

It is highly desirable to reduce the number of variables in equations (263), (264) and (265). Accordingly, the above equations may be expressed entirely in terms of \( 10^{F_n}, 10^{F_0} \) and \( \delta \) by determining the suitable replacements for the terms of the equations.

The solution of \( F_k \) in terms of \( F_n \) is deduced from the following procedure:

For \( n \) equal to one, \( F_n \) equals \( F_1 \) or, simply,

\[ F_1 = F_1 + O(\delta) \quad (266) \]

For \( n \) equal to two, a simple sketch illustrates the relationships.

\[ F_1 \quad F_2 \quad F_3 \]

Then

\[ F_1 = F_2 + (-1) \delta \quad (267) \]

\[ F_2 = F_2 + (0) \delta \quad (268) \]

and

\[ F_3 = F_2 + (1) \delta \quad (269) \]
For $n$ equal to three, the corresponding sketch is

\[
\begin{array}{cccccc}
F_1 & F_2 & F_3 & F_4 & F_5
\end{array}
\]

from which it is seen that

\[
F_1 = F_3 + (-2) \delta, \quad (270)
\]
\[
F_2 = F_3 + (-1) \delta, \quad (271)
\]
\[
F_3 = F_3 + (0) \delta, \quad (272)
\]
\[
F_4 = F_3 + (1) \delta, \quad (273)
\]

and

\[
F_5 = F_3 + (2) \delta. \quad (274)
\]

For $n$ equal to four, the sketch is

\[
\begin{array}{cccccccc}
F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7
\end{array}
\]

and

\[
F_1 = F_4 + (-3) \delta, \quad (275)
\]
\[
F_2 = F_4 + (-2) \delta, \quad (276)
\]
\[
F_3 = F_4 + (-1) \delta, \quad (277)
\]
\[
F_4 = F_4 + (0) \delta, \quad (278)
\]
\[
F_5 = F_4 + (1) \delta, \quad (279)
\]
\[
F_6 = F_4 + (2) \delta. \quad (280)
\]

and
\[ F_7 = F_4 + (3) \delta \quad \text{(281)} \]

From a comparison of the trends evidenced by equations (266) through (281), it may be deduced that, in general,

\[ F_k = F_n + (k-n) \delta \quad \text{(282)} \]

where \( k \) ranges from 1 to \( 2n-1 \). From equation (282), it is apparent that

\[ 10^F_k = 10^{F_n} \cdot 10^{(k-n)\delta} \quad \text{(283)} \]

The \( F_{0j}'s^* \) may be determined in terms of \( F_{01} \) and \( F_n \) in a somewhat similar manner. For \( n \) equal to two, the following sketch illustrates possible relative positions of the various \( F \)'s:

![Sketch for n=2]

It may be seen that

\[ F_{01} = (1)\left(\frac{F_{01}}{1}\right) + 0\left(\frac{F_2}{1}\right) \quad \text{(284)} \]

and

\[ F_{02} = (-1)\left(\frac{F_{01}}{1}\right) + 2\left(\frac{F_2}{1}\right) \quad \text{(285)} \]

For \( n \) equal to three, the following sketch applies:

![Sketch for n=3]

It is evident that

\[ F_{01} = (2)\left(\frac{F_{01}}{2}\right) + 0\left(\frac{F_3}{2}\right) \quad \text{(286)} \]

*The term \( j \) is used here to avoid confusion with the previous \( k \).
\[
F_{o2} = (0)\left(\frac{F_{o1}}{2}\right) + 2 \left(\frac{F_3}{2}\right) \quad (287)
\]

and

\[
F_{o3} = (-2)\left(\frac{F_{o1}}{2}\right) + 4 \left(\frac{F_3}{2}\right) \quad (288)
\]

For \(n\) equal to four, a sketch again shows the \(F\) positions.

\[
F_{o1} \quad F_{o2} \quad F_{o3} \quad F_{o4} \quad F_4 \quad F_5 \quad F_6 \quad F_7
\]

From the sketch, it is apparent that

\[
F_{o1} = (3)\left(\frac{F_{o1}}{3}\right) + 0 \left(\frac{F_4}{3}\right) \quad \ldots \quad (289)
\]

\[
F_{o2} = (1)\left(\frac{F_{o1}}{3}\right) + 2 \left(\frac{F_4}{3}\right) \quad \ldots \quad (290)
\]

\[
F_{o3} = (-1)\left(\frac{F_{o1}}{3}\right) + 4 \left(\frac{F_4}{3}\right) \quad \ldots \quad (291)
\]

and

\[
F_{o4} = (-3)\left(\frac{F_{o1}}{3}\right) + 6 \left(\frac{F_4}{3}\right) \quad \ldots \quad (292)
\]

The above process may be repeated for \(n\) equal to five, six, and so on ad infinitum. Inspection of the trends in equations (287) through (292), however, indicates that, in general,

\[
F_{o(j)} = (n-2j+1)\left(\frac{F_{o1}}{n-1}\right) + 2(j-1) \left(\frac{F_n}{n-1}\right) \quad \ldots \quad (293)
\]

from which

\[
10^{F_{o(j)}} = 10^{(n-2j+1)\left(\frac{F_{o1}}{n-1}\right)} \cdot 10^{2(j-1)\left(\frac{F_n}{n-1}\right)} \quad (294)
\]

and

\[
f_{o(j)} = f_{o1} \frac{n-2j+1}{n-1} \cdot f_n \frac{2(j-1)}{n-1} \quad \ldots \quad (295)
\]
In equations (293) through (295), \( j \) ranges from one to \( n \).

The next major step in the general solution is to apply the results of equations (263), (283) and (294) to the simultaneous solution of equations (236) through (265). Since use of equations (283) and (294) automatically accounts for any contribution of equation (263), the immediate problem reduces to the simultaneous solution of equations (264) and (265) as modified by equations (283) and (294). In this regard, it is stressed that \( n, s, b \) and \( F_n \) are known quantities, while \( s_1 \) and the \( F_{0k} \)'s are to be determined. The determination of \( s_1 \) and the \( 10F_{0k} \)'s is, in reality, the determination of the \( s \)'s and \( 10F_{01} \)'s of the \( n \) cascaded one-pole networks which make up the \( n \)-pole system.

Prior to the development of the desired solutions, several general expressions in terms of \( F_{01} \) and/or \( s_1 \) are required. One of these is

\[
\frac{S \cdot 10^{F_2}}{S_1} = \sum_{k=1}^{n} 10^{F_{0k}}
\]  

(296)

a rearrangement of equation (264). With the aid of equation (294), equation (296) reduces to

\[
\frac{S \cdot 10^{F_2}}{S_1} = \frac{(10F_{01})^2 + (10F_{02})^2}{(10F_{01})^2}
\]  

(297)

for \( n \) equal to two. The equivalent expression for \( n \) equal to three is

\[
\frac{S \cdot 10^{F_3}}{S_1} = \frac{(10F_{01})^4 + (10F_{02})^2(10F_{02})^2 + (10F_{02})^4}{(10F_{02})^2}
\]  

(298)

The general expression becomes

\[
\frac{S \cdot 10^{F_n}}{S_1} = \sum_{k=1}^{n} \frac{(10F_{01})^{2n-2k} \cdot (10F_{0n})^{2k-2}}{(10F_{01})^{n-1}}
\]  

(299)
The squares of equation (299) for n equal to two and three, respectively, are

\[
\left( \frac{S \cdot 10^{\frac{5}{2}}}{S_i} \right)^2 = \left( \frac{F_{0i}}{10} \right)^2 + 2 \left( \frac{F_{0i}}{10} \right)^2 \left( \frac{10^{\frac{5}{2}}}{10} \right)^2 + \left( \frac{10^{\frac{5}{2}}}{10} \right)^4
\] (300)

and

\[
\left( \frac{S \cdot 10^{\frac{5}{2}}}{S_i} \right)^2 = \left( \frac{F_{0i}}{10} \right)^2 + 2 \left( \frac{F_{0i}}{10} \right)^2 \left( \frac{10^{\frac{5}{2}}}{10} \right)^2 + 3 \left( \frac{F_{0i}}{10} \right)^2 \left( \frac{10^{\frac{5}{2}}}{10} \right)^2 + 2 \left( \frac{F_{0i}}{10} \right)^2 \left( \frac{10^{\frac{5}{2}}}{10} \right)^2 + \left( \frac{10^{\frac{5}{2}}}{10} \right)^4
\] (301)

In this case, the general term becomes

\[
\left( \frac{S \cdot 10^{\frac{5}{2}}}{S_i} \right)^2 = \left( \frac{F_{0i}}{10} \right)^2 + 2 \left( \frac{F_{0i}}{10} \right)^2 \left( \frac{10^{\frac{5}{2}}}{10} \right)^2 + 3 \left( \frac{F_{0i}}{10} \right)^2 \left( \frac{10^{\frac{5}{2}}}{10} \right)^2 + 2 \left( \frac{F_{0i}}{10} \right)^2 \left( \frac{10^{\frac{5}{2}}}{10} \right)^2 + \left( \frac{10^{\frac{5}{2}}}{10} \right)^4
\]

In the above form, expression (303) is not particularly attractive. However, a useful general expression may be developed. For n equal to two, expression (303) becomes, with the aid of previous equations,

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} 10^{F_{0j}} \cdot 10^{F_{0k}} , \quad j \neq k
\] (303)

Another term which warrants consideration is

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} 10^{F_{0j}} \cdot 10^{F_{0k}} \] (304)

In the above form, expression (303) is not particularly attractive. However, a useful general expression may be developed. For n equal to two, expression (303) becomes, with the aid of previous equations,
For $n$ equal to three,

$$
\sum_{j=1}^{3} \sum_{k=1}^{3} 10^{F_{ij}} \cdot 10^{F_{ok}} = (10^{\frac{F_{ij}}{3}})^2 \left[ \frac{(10^{\frac{F_{ij}}{3}^4} + (10^{\frac{F_{ij}}{3}})^2 (10^{\frac{F_{ij}}{3}})^2 + (10^{\frac{F_{ij}}{3}})^4)}{(10^{\frac{F_{ij}}{3}})^2} \right].
$$

For $n$ equal to four,

$$
\sum_{j=1}^{4} \sum_{k=1}^{4} 10^{F_{ij}} \cdot 10^{F_{ok}} = (10^{\frac{F_{ij}}{3}})^2 \left[ \frac{(10^{\frac{F_{ij}}{3}^8} + (10^{\frac{F_{ij}}{3}})^2 (10^{\frac{F_{ij}}{3}})^2 + (10^{\frac{F_{ij}}{3}})^4)}{(10^{\frac{F_{ij}}{3}})^4} \right].
$$

The expressions for $n$ equal to five, six, seven, etc., follow the same pattern. The general term for expression (303) is

$$
\sum_{j=1}^{n} \sum_{k=1}^{n} 10^{F_{ij}} \cdot 10^{F_{ok}} = \frac{\left\{ \left(10^{\frac{F_{ij}}{n^2}}\right)^{2n^4} \cdot \left(10^{\frac{F_{ij}}{n^2}}\right)^{2n^4-2k} \cdot \left(10^{\frac{F_{ij}}{n^2}}\right)^{2n^4-2k} \cdot \left(10^{\frac{F_{ij}}{n^2}}\right)^{2n^4-2k} \right\}}{(10^{\frac{F_{ij}}{n^2}})^{2n-4}}.
$$

The expression $\sum_{k=1}^{n} 10^2 F_{ok}$ which appears in the left side of equation (265) is

$$
\left[ 10^{2F_{10}} + 10^{2F_{02}} \right] = \left[ 10^{F_{10}} + 10^{F_{02}} \right]^2 - 2 \left[ 10^{F_{10}} \cdot 10^{F_{02}} \right]
$$

for $n$ equal to two. For $n$ equal to three,

$$
\left[ 10^{2F_{10}} + 10^{2F_{02}} + 10^{2F_{03}} \right] = \left[ 10^{F_{10}} + 10^{F_{02}} + 10^{F_{03}} \right]^2 - 2 \left[ 10^{F_{10}} \cdot 10^{F_{02}} + 10^{F_{10}} \cdot 10^{F_{03}} + 10^{F_{02}} \cdot 10^{F_{03}} \right].
$$
from which the general term is seen to be
\[
\sum_{k=1}^{n} 10^{F_{ok}} = \left( \sum_{k=1}^{n} 10^{F_{ok}} \right)^2 - 2 \sum_{j=1}^{n} \sum_{k=1}^{n} 10^{F_{oj}} \cdot 10^{F_{ok}}
\]  \tag{310}

where \( j \) is not equal to \( k \).

The expression \( \sum_{k=1}^{n} 10^{F_{2k-1}} \) in equation (265) may be expressed conveniently as
\[
\sum_{k=1}^{n} 10^{F_{2k-1}} = 10^{F_{n}} \sum_{k=1}^{n} 10^{(-2n-2+4k)}
\]  \tag{311}

To illustrate this point, the development for \( n \) equal to two and three follows:
\[
\begin{align*}
\left[ 10^{F_{2}} + 10^{F_{3}} \right] &= \left[ 10^{F_{2}} + 10^{F_{3}} + 10^{F_{2}+26} \right] = 10^{F_{2}} \left[ 10^{F_{25}} + 10^{F_{26}} \right] \tag{312}
\end{align*}
\]
and
\[
\begin{align*}
\left[ 10^{F_{2}} + 10^{F_{3}} + 10^{F_{4}} \right] &= \left[ 10^{F_{2}} + 10^{F_{3}} + 10^{F_{4}+26} \right] = 10^{F_{2}} \left[ 10^{F_{25}} + 10^{F_{26}} \right] \tag{313}
\end{align*}
\]

As a consequence of equation (302), a useful value for \( s_1^2 \) may be found as
\[
S_1^2 = \frac{S^2 \left( \frac{F_{n}}{n-1} \right)^{2n-2} \left( \frac{F_{n}}{n-1} \right)^{2n-2}}{\sum_{k=1}^{n-1} \left( \frac{F_{o}}{n-1} \right)^{4n-2-2k} \left( \frac{F_{n}}{n-1} \right)^{2k-2} \cdot k} + \sum_{k=1}^{n-1} \left( \frac{F_{n}}{n-1} \right)^{2n-2} \left( \frac{F_{n}}{n-1} \right)^{2n-2} \cdot n} + \sum_{k=1}^{n-1} \left( \frac{F_{n}}{n-1} \right)^{2n-2+2k} \left( \frac{F_{n}}{n-1} \right)^{2n-2+2k} \cdot (n-k) \tag{314}
\]

All of the terms of equations (264) and (265) are reduced to expressions involving \( s, 10^{F_{n}} \) and \( s_1 \). Reference to equations (302), (307), (310) and (311) substantiates this statement. Sufficient conditions are established
to allow the development of the general equations of the n-pole network.

The procedure to be followed is much the same as before; that is, to verify
the validity of the various manipulations for several small values of n
and then to demonstrate the method of solution for the n-pole network.

For n equal to two, a rearrangement of equation (265) yields

\[
\left[10^2 F_0^2 + 10^2 F_0^2\right] - \left[10^2 F_1 + 10^2 F_2\right] + s^2 10^2 F_0 F_1 10^2 F_2 = 0. \quad (315)
\]

Successively substituting formulas (310), (296), (302), (307), (314) and
(311)* reduces equation (315) to

\[
\left(10^2 F_0^2\right)^6 + \left(10^2 F_1^2\right)^6 \left(10^2 F_2^2\right)^2 \left(10^2 F_1^2\right)^4 \left(10^2 F_2^2\right)^4 \left(10^2 F_1^2\right)^2 \left(10^2 F_2^2\right)^2 = 0, \quad (316)
\]

where

\[
A_2 = \left(10^{-26} + 10^{26}\right). \quad (317)
\]

From equations (137) and (297), the companion equation which must be solved
simultaneously with equation (316) is

\[
S_i = \frac{s \left(10^2 F_0^2\right) \left(10^2 F_1^2\right)}{\left(10^2 F_0^2\right)^2 + \left(10^2 F_1^2\right)^2}. \quad (318)
\]

The procedure to be followed is that of (a) finding the value of
\(10^2 F_0\) which satisfies equation (316) for a given \(10^2 F_0^2, A_2\) and \(s,\) and
(b) substituting this value in equation (318) for the solution of \(S_i.\) Since
the value for \(10^2 F_0,\) and hence for \(F_0,\) is known, use is made of formula
(285) to obtain \(F_0^2.\) From the value of \(F_0^2, 10^2 F_0^2\) may be determined. Thus,

*The formulas referred to above are general. The corresponding values for
n equal to two are, of course, used in this particular instance.
the value of \( s_1 \) and the equality between \( f_{0k} \) and \( 10^{F0k} \) establish the \( s_1 \)'s and \( f_1 \)'s of the two one-pole networks, which constitute the desired two-pole network. Any one of the three, original, one-pole networks satisfies the required conditions mathematically. However, a practical application demands that, of the networks available, the R-C types be used.

As a matter of general interest, attention is directed to a comparison of the solution obtained for \( s_1 \) in equation (153) and the solution for \( f_{01} \) (\( f_{01} = 10^{F01} \)) in equation (159) with solutions for these quantities as established by equations (318) and (316), respectively. It so happens that for \( n \) equal to two, equations (318) and (316) reduce to (153) and (159). For higher pole networks, unfortunately, no simple reductions are possible. At least an eighth degree equation must be solved. It should now be evident why the earlier solution for the two-pole network is referred to as a special case.

An identical procedure is followed to obtain the three-pole solution. The formula corresponding to equation (315) is

\[
\left[ \frac{2F_3}{10 + 10 + 10} \right] - \left[ \frac{2F_2}{10 + 10 - 10} \right] + \left[ \frac{2F_1}{10 + 10 + 10} \right] = 0 \tag{319}
\]

Repeated application of equations (310), (296), (302), (307), (314) and (311) with \( n \) equal to three reduces equation (319) to a form somewhat similar to equation (316) for the two-pole network. The expression is

\[
\begin{align*}
(10_{\frac{F_3}{2}})^{16} & + \left(10_{\frac{F_2}{2}}\right)^{10} \left(10_{\frac{F_1}{2}}\right)^{2} (2) + \left(10_{\frac{F_1}{2}}\right)^{12} \left(10_{\frac{F_2}{2}}\right)^{4} (4-A_3) + \\
(10_{\frac{F_2}{2}})^{10} \left(10_{\frac{F_3}{2}}\right)^{6} (4-2A_3 + S^2) + \left(10_{\frac{F_1}{2}}\right)^{8} \left(10_{\frac{F_2}{2}}\right)^{6} (5-3A_3 + S^2) + \\
(10_{\frac{F_1}{2}})^{6} \left(10_{\frac{F_3}{2}}\right)^{10} (4-2A_3 + S^2) + \left(10_{\frac{F_2}{2}}\right)^{4} \left(10_{\frac{F_3}{2}}\right)^{12} (4-A_3) + \\
(10_{\frac{F_3}{2}})^{2} \left(10_{\frac{F_2}{2}}\right)^{10} (2) + \left(10_{\frac{F_2}{2}}\right)^{16} &= 0
\end{align*}
\] (320)
where
\[
A_3 = \left( 10^{-45} + 1 + 10^{45} \right) \quad .
\]

The companion equation is
\[
S_i = \frac{S \left( 10^{F_01} \right)^2 \left( 10^{F_{3i}} \right)^2}{\left( 10^{F_{2i}} \right)^4 + \left( 10^{F_{2i}} \right)^2 \left( 10^{F_{3i}} \right)^2 + \left( 10^{F_{2i}} \right)^4} \quad .
\]

For \( n \) equal to four, the formula corresponding to equation (315) is
\[
S_i^2 \left[ 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 \right] = 0,
\]
which is reduced by a method similar to that used for the two- and three-pole cases to
\[
\left( 10^{F_01} \right)^{24} \left( 10^{F_{2i}} \right)^{22} \left( 10^{F_{3i}} \right)^{2} \left( 2 \right) + \left( 10^{F_01} \right)^{20} \left( 10^{F_{2i}} \right)^{18} \left( 10^{F_{3i}} \right)^{16} \left( 6 - A_4 \right) + \left( 10^{F_01} \right)^{16} \left( 10^{F_{2i}} \right)^{14} \left( 10^{F_{3i}} \right)^{12} \left( 7 - 2 A_4 + S^2 \right) + \left( 10^{F_01} \right)^{12} \left( 10^{F_{2i}} \right)^{10} \left( 10^{F_{3i}} \right)^{8} \left( 8 - 3 A_4 + S^2 \right) + \left( 10^{F_01} \right)^{8} \left( 10^{F_{2i}} \right)^{6} \left( 10^{F_{3i}} \right)^{4} \left( 8 - 3 A_4 + S^2 \right) + \left( 10^{F_01} \right)^{4} \left( 10^{F_{2i}} \right)^{2} \left( 10^{F_{3i}} \right)^{0} \left( 6 - A_4 \right) + \left( 10^{F_01} \right)^{0} \left( 10^{F_{2i}} \right)^{0} \left( 10^{F_{3i}} \right)^{0} \left( 2 \right) + \left( 10^{F_{2i}} \right)^{18} = 0,
\]
where
\[
A_4 = \left( 10^{-66} + 10^{-66} + 10^{-66} \right) \quad .
\]

In addition, the companion equation is
\[
S_i = \frac{S \left( 10^{F_01} \right)^3 \left( 10^{F_{2i}} \right)^3}{\left( 10^{F_{2i}} \right)^6 + \left( 10^{F_{2i}} \right)^4 \left( 10^{F_{3i}} \right)^2 + \left( 10^{F_{2i}} \right)^2 \left( 10^{F_{3i}} \right)^4 + \left( 10^{F_{2i}} \right)^6} \quad .
\]

As a final demonstration in which numerical values are employed, the development of the five-pole equation follows. The formulas corresponding to the formulas given above for the two-, three- and four-pole cases are
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\[
\begin{bmatrix}
2F_{o1} & 2F_{o2} & 2F_{o3} & 2F_{o4} & 2F_{o5} \\
10 & 10 & 10 & 10 & 10
\end{bmatrix}
- \begin{bmatrix}
2F_{l1} & 2F_{l2} & 2F_{l3} & 2F_{l4} & 2F_{l5} \\
10 & 10 & 10 & 10 & 10
\end{bmatrix}
\]

\[+ S^2 \begin{bmatrix}
F_{o1} & F_{o2} & F_{o3} & F_{o4} & F_{o5} \\
10 & 10 & 10 & 10 & 10
\end{bmatrix}
+ 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 + 10 \cdot 10 = 0 \]

(327)

\[A_5 = \left(10^{\frac{-86}{4}} + 10^{\frac{-48}{4}} + 1 + 10^{\frac{45}{4}} + 10^{\frac{95}{4}}\right)^2\]

(329)

and

\[S_i = \frac{S \left(\frac{F_{o1}}{10^{\frac{4}}}\right)^4 \left(\frac{F_{l1}}{10^{\frac{4}}}\right)^4}{\left(\frac{F_{o1}}{10^{\frac{4}}}ight)^3 + \left(\frac{F_{o2}}{10^{\frac{4}}}ight)^3 + \left(\frac{F_{o3}}{10^{\frac{4}}}ight)^3 + \left(\frac{F_{o4}}{10^{\frac{4}}}ight)^3 + \left(\frac{F_{o5}}{10^{\frac{4}}}ight)^3 + \left(\frac{F_{l1}}{10^{\frac{4}}}ight)^3 + \left(\frac{F_{l2}}{10^{\frac{4}}}ight)^3 + \left(\frac{F_{l3}}{10^{\frac{4}}}ight)^3 + \left(\frac{F_{l4}}{10^{\frac{4}}}ight)^3 + \left(\frac{F_{l5}}{10^{\frac{4}}}\right)^3} \]

(330)

With a normal span of audio frequencies, it is doubted very much that networks of complexity greater than five need ever be computed. However, for the sake of completeness and to accommodate any possibility, the general forms of the four equations typified by expressions (327) through (330) are given below:

\[\left[ \sum_{k=1}^{n} 10^{2F_{o(k)}} - \sum_{k=1}^{n} 10^{2F_{l(k-1)}} + S^2 \sum_{j=1}^{n} \sum_{k=1}^{n} F_{o(k)} \cdot 10^{F_{o(k)}} \right] = 0 \]

(331)
where \( j \) is not equal to \( k \),

\[
\left\langle \sum_{k=1}^{n-1} \left( \frac{F_n}{\lambda} \right)^{6n-6-2k} \left( \frac{F_n}{\lambda} \right)^{2k+2} \left( \frac{[2+k]!}{[k-1]! 3!} + k [i-k] \right) + \left( \frac{F_n}{\lambda} \right)^{6n-6} \left( \frac{F_n}{\lambda} \right)^{2n-2} \left( \frac{[n+2]!}{[n-1]! 3!} - \frac{1}{2} [2n-5+(-1)^n] - A_n \right) + \sum_{k=1}^{n-2} \left( \frac{F_n}{\lambda} \right)^{6n-6-2k} \left( \frac{F_n}{\lambda} \right)^{2n-2+2k} \left( \frac{[n+2+k]!}{[n-1+k]! 3!} - 4 \frac{[2+k]!}{[k-1]! 3!} \right) - 2 [n^2-3] [n-2][n+3-k] - \frac{1}{2} [2n-5+(-1)^n][k+1] - [k+] A_n + S^2 \right) \rightangle + \left\langle \sum_{k=1}^{n-2} \left( \frac{F_n}{\lambda} \right)^{4n-4-2k} \left( \frac{F_n}{\lambda} \right)^{4n-4+2k} \left( \frac{[2n-k+1]!}{[2n-2-k]! 3!} \right) - 4 \frac{[n-k+1]!}{[n-2-k]! 3!} - 2 [n^2-3] [n-3][k+1] - \frac{1}{2} [2n-5+(-1)^n][k-1] - [k-1] A_n + S^2 \right) \right\rangle = 0
\]

where

\[
A_n = \sum_{k=1}^{n} 10^{(-2n-2+4k)}
\]

and

\[
S_i = \sum_{k=1}^{n} \frac{S \left( \frac{F_n}{\lambda} \right)^{n-1} \left( \frac{F_n}{\lambda} \right)^{n-1}}{\left( \frac{F_n}{\lambda} \right)^{2n-2k} \left( \frac{F_n}{\lambda} \right)^{2k-2}}
\]

(334)
After \(10^{F_{01}}\) and \(s\) are found, equation (294), repeated below as

\[
10^{F_{0k}} = \left(10^{\frac{F_{01}}{n-1}}\right)^{j-1} \left(10^{\frac{F_{01}}{n-1}}\right)^{j-n+1}
\]

may be used to determine the individual \(10^{F_{0k}}\)'s of the \(n\) cascaded one-pole networks.

The development of equations (332), (334) and (335) represents the attainment of a major objective of this report. It is now possible to determine the individual \(s\)'s and \(f_{1}\)'s of any of the one-pole networks which make up the given \(n\)-pole network.

### III-7 Selection of Circuit Parameters

While it is unfortunate that equation (332) has no simple solution, nevertheless, approximate solutions do exist which yield results accurate to any desired degree. Of the several methods available, two typical means are Horner’s Method\(^7\) and Graeffe’s Root-Squaring Method.\(^8\) A third method for solving the general equation of the type exemplified by the three-pole version of equation (332) is demonstrated in Appendix IV.

Since a multidegree equation contains as many roots as the degree of the equation, the possibility exists that several values of \(F_{01}\) satisfy equation (332). It may be noted that the degree of equation (332) is equal to one-half the exponent of the first \(10^{F_{01}/n-1}\) term, which allows the first term of equation (332) to be written as \((10^{F_{01}/n-1})^2(1n-4)\). The degree of the \((10^{F_{01}/n-1})^2\) term is thus \((1n-4)\). Rapid verification of this fact may be made by reference to equations (316), (320)


(321) and (328), these equations having specific values for \( \eta \) equal to two, three, four and five, respectively. In each case, the maximum possible number of positive roots of equation (332) is equal to four, while the maximum possible number of negative roots is equal to \((4n-3)\). The probability of imaginary roots is also present. However, actual calculations of several equations do indicate the definite presence of four positive roots. Since only positive roots need be considered, the question of imaginary and negative roots may be summarily dispensed with, without further investigation as to their exact nature. Of the four positive roots, two apply to \( F_{0n} \); hence, only two roots apply to \( F_{01} \). Thus, there are but two positions for \( F_{01} \) and, consequently, but two values of \( s_1 \) from which to choose.

The choice of the upper or lower values of \( 10^F_{01} \) depends to a great extent on the type of network to be used. For a fixed value of \( s \), it may be demonstrated that \( 10^F_{01} \) increases with increasing \( s_1 \).* Hence, if the larger value of \( s_1 \) may be tolerated, the larger value of \( 10^F_{01} \) should be chosen. If it is necessary to keep \( s_1 \) as low as possible, the lower value of \( 10^F_{01} \) should be employed. In any event, the two choices of \( 10^F_{01} \) lie between \( 10^F_1 \) and \( 10^F_n \).

In selecting the appropriate value of \( 10^F_{01} \), consideration should be given to both the gain per phase-shift section and the physical size of circuit elements. Figure 19 shows the gain and the ratio between the elements for the three one-pole networks plotted versus \( s_1 \).

For the L-C network, the gain is constant at unity and, hence, the choice of element ratio may be made without regard to gain. Obviously, 

* See equations (318), (320), (326), (330) and (334).
FIGURE 19
Gain Per Phase-Shift Sections And Ratio Of Circuit Elements For One-Pole Networks.

NOTE: Actual gain is dependent on gain in coupling tube.
the lower $10^{F01}$ (and smaller $s_1$) should be used, since a smaller ratio between circuit elements results therefrom.

For the R-C network with internal isolation, also, the gain is constant and does not depend on element ratios. Here, too, the lower $10^{F01}$ value is the most advantageous choice.

The R-C network with no internal isolation presents a somewhat different problem. The gain is not constant but depends on $s_1$ and, of course, $10^{F01}$. It may also be shown that the gain is always less than unity.* To secure the highest possible gain, it is necessary that high circuit element ratios be used. The choice of the higher or lower value of $10^{F01}$ must be based on the particular requirements. If the element ratio demanded by the higher value of $10^{F01}$ is not excessive, the higher value should be chosen. On the other hand, should the higher value of $10^{F01}$ require an entirely disproportionate element ratio, the only recourse is to choose the lower value of $10^{F01}$ and accept the inevitably lower gain for the sake of preserving a reasonable element ratio.

*See equation (86).
CHAPTER IV
ERRORS AND ERROR CALCULATIONS

IV-1 Description of Errors

For 90° phase-difference systems, the desired difference in phase between channels A and B is 90°. The angle by which the actual phase difference fails to meet the desired 90° value is called the error and is designated by \( \gamma \) (in degrees). Since \( \beta_B \) is always less negative (and, hence, more positive) than \( \beta_A \), the difference equation

\[
\beta_B - \beta_A - 90° = \gamma
\]  

expresses the error at any particular value of \( f \) or \( F \). In Figures 9 and 11, typical error curves are shown.

Errors arise in n-pole systems because the phase angles of each channel are not linear functions of log frequency. This may be attributed directly to the nature of the functions which describe the phase angle, \( \beta \). It may be noted that the general equation for \( \beta \) such as equation (11) is of the form

\[
\beta = \tan^{-1} [g(f)]
\]  

Arc tangent functions of the above type are never exactly linear over a logarithmic span of the independent variable \( f \). When two such equations are combined to describe the behavior of an n-pole system, the inherent nonlinearity of each equation contributes to the resultant error.

IV-2 Factors Which Influence Errors

While it is true, as stated above, that equation (337) can never be exactly linear and that an n-pole system can never be perfectly free of
error, judicious choice of the parameters and network complexity reduces the nonlinearity to any desired amount and the error below any preassigned value. The factors $s$ and $b$ are available for controlling the linearity, while the factor $\eta$ may be used to adjust the resultant error. To obtain the least possible error for a given $s$ and $b$, $\eta$ must be chosen with extreme care.

By interrelating $s$ and $b$ such that the phase curves of each network pass through the 90 Degree Log Frequency Points, the greatest number of points of coincidence between actual phase curves and idealized phase curves (straight lines) is established as $(m + 2)$. Equation (220) specifies the coincidence at the 90 Degree Points and is repeated below as

$$
S = \frac{10^{(2n-1)b}}{10^{(n-\frac{1}{2})b} - 1}.
$$

The above equation, in effect, reduces the degree of linearity freedom to one.

Once $s$ (or $b$) has been established, the variation from linearity cannot be changed; however, the factor $\eta$ determines the spacing between the two phase curves and in turn the manner in which the individual variations combine to yield the error of a system.

Figure 20 shows exaggerated phase and error curves for a three-pole network. The variations from linearity are clearly recognizable. By decreasing the value of $\eta$, it is seen that the error at $F_M$ is increased in a negative sense. The symmetry of the error curve about $F_M$ should be noted. Actual calculations of errors need be made over one half of the
FIGURE 20
Exaggerated Typical Phase And Error Curves For Three-Pole System with $\pi$ Equal To One ($\pi, \sigma=5/2$)
total log frequency span because of this symmetry. For convenience, the Π-Points, 90 Degree Log Frequency Points and δ are also shown in Figure 20.

IV-3 Methods of Error Calculation

It is highly impractical by the method of analysis employed in this report to attempt a direct, analytical solution for errors resulting from various combinations of Channel Spacing, ησ, and Center Spacing, δ. Such a procedure requires the determination of values of F where relative maxima and minima occur in the error formulas. The standard calculus approach is that of finding the values of F where dY/dF equals zero. The evaluation of dY/dF is extremely laborious and tedious, even for small values of n, and becomes increasingly difficult as n increases. The solution for the values of F where dY/dF equals zero involves as the worst possible case a (4n-5)th order equation, which for n greater than two must be solved by methods of successive approximations.

The use of graphical solutions presents a very attractive means of obtaining error data. For systems where n is greater than, or equal to, two there are at most (4n-5) points at which the slope of the error versus log frequency curve is zero. The relative maxima and minima are located at these points. A wealth of information may be gleaned from the maximum and minimum points of error curves for various values of η and δ.

For a specified Center Spacing, the phase angle, θA, of channel A may be determined at the Π-Points and at equally spaced intervals between the Π-Points. As an example, the phase angles at F equal to F1A, F2A and F3A are the angles for the three Π-Points of a two-pole network. By considering F1.5A and F2.5A as the values of F midway between the
Points, the following tabulated values of $\theta$ may be determined for a two-pole network:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{1.0}$</td>
<td>$F_{1.0}$</td>
</tr>
<tr>
<td>$\theta_{1.5}$</td>
<td>$F_{1.5}$</td>
</tr>
<tr>
<td>$\theta_{2.0}$</td>
<td>$F_{2.0}$</td>
</tr>
<tr>
<td>$\theta_{2.5}$</td>
<td>$F_{2.5}$</td>
</tr>
<tr>
<td>$\theta_{3.0}$</td>
<td>$F_{3.0}$</td>
</tr>
</tbody>
</table>

The values obtained for $\theta$ apply equally well for $F_{1B}, F_{2B},$ etc., of channel B. If $\eta$ is fixed at unity, several values of $\theta$ may be chosen and a tabulation of $\theta$ and $F$ for each $\theta$ recorded.

Upon evaluation of the $\theta$'s for one channel alone, the difference in phase between the two channels may be ascertained by the difference in the appropriate $\theta$'s of channel A. There is necessity for additional calculations for the B channel $\theta$'s. For $\eta$ equal to one (or $\eta \sigma$ equal to the Unit Channel Spacing), the phase angle of channel A at $F$ equal to $F_{1A}$ is $\theta_{1.0}$. At the same value of $F$, the phase angle of channel B is $\theta_{0.5}$. At $F$ equal to $F_{1.5A}$, $\theta$ for channel A is $\theta_{1.5}$ while $\theta$ for channel B is $\theta_{1.0}$. This comparison process is valid for all $\theta$'s with the stipulation that the difference in the indices is 0.5, the index of A being 0.5 greater than that of B. A complete set of $\theta$'s from $\theta_{0.3}$ to $\theta_{3.7}$ is sufficient to determine the error values for a two-pole system with $\eta$ equal to one.

Obviously, the increments chosen for illustrative purposes are not sufficiently small to determine a very accurate error curve. However, if increments of 0.1 instead of 0.5 are chosen for the $\theta$'s, thirty-five values of $\theta$ are found in the range from $\theta_{0.3}$ to $\theta_{3.7}$. Because of the
symmetry of the error curves about $F_{2M}$, it is possible to obtain the error data from twenty-one values of $\beta$; i.e., $\beta_{0.3}$ to $\beta_{2.3}$. The following sketch and tabulation indicates the method by which an error curve may be obtained for a two-pole system with $Q$ equal to one. Any actual error calculations must involve $\beta$ increments smaller than shown below.

The $F$ terms with $M$ subscripts indicate the position of the calculated point with respect to $F_{2M}$, the Mid-Log Frequency. The unit of spacing for error points is equal to $6$; i.e., $(F_{2M} - F_{1M})$ is equal to $6$, etc.

Up to this point, $Q$ has been restricted to one. For $Q$ equal to one, the error data for several values of $6$ may be plotted in a manner similar to that shown in Figure 20. The error curves obtained in this manner are fairly well centered about the zero error axis, but, nevertheless, the maximum positive excursion is somewhat greater than the maximum negative excursion. A more equitable distribution of error may be obtained if the curve...
is shifted downward. A decrease in $\eta$ causes the entire curve to be shifted downward and at the same time decreases slightly the magnitude of the oscillation of the error curve. The first named effect is by far the greater, and the change in oscillation may be disregarded entirely for a small change in $\eta$.

The magnitude of the curve displacement caused by changes in $\eta$ is of extreme interest. A change of 0.2 in $\eta$ results in a displacement of $18^\circ$. Using the twenty values of $\theta$ described previously, the minimum $\eta$ changes which may be obtained are in increments of 0.2, yielding displacement increments of $18^\circ$. This figure is obviously much too large. Accordingly, if data is to be derived from this method, the displacement and $\eta$ increments must be much smaller.

The process of obtaining families of curves with displacement increments of $18^\circ$ involves $\eta$ values of $1 + 0.2k^*$ and $\theta$ values, the indices of which differ not by 0.5 but by 0.5$q$. To secure displacement increments of $0.18^\circ$, the difference in index between successive $\theta$'s must be 0.005. Two thousand $\theta$ calculations would be required. The law of diminishing returns precludes the $\eta$ increment method of finding the optimum $\eta$ and points to the need for a simpler method.

With reference to Figure 20, it may be seen that a downward shift of the error curve is needed for equalizing the maximum positive and negative errors. The curve must be shifted by the angle $\xi$, where

$$\xi = \frac{\text{algebraic value of most negative error}}{\text{algebraic value of most positive error}} \; (339)$$

$k = 0, 1, 2, 3, \ldots$
Neglecting the very slight changes in oscillation due to a small change in $\eta$, the ratio

$$\frac{S}{18} = \frac{\lambda}{0.2}$$

(340)

determines $\lambda$, the change in $\eta$, as

$$\lambda = \frac{S}{90}$$

(341)

When based on the curve for $\eta$ equal to one, the desired $\eta$ becomes

$$\eta' = 1 + \lambda$$

(342)

and the maximum error is

$$\gamma_{\text{max}} = \pm \frac{1}{2} \text{ (algebraic value of most positive error)} - \text{ (algebraic value of most negative error)}$$

(343)

As an illustration, let it be assumed that the maximum positive error is $5^\circ$ and the maximum negative error is $-1^\circ$ for a curve of some particular $\delta$ and of $\eta$ equal to one. Without further adjustment, the phase difference is $90 \pm 5^\circ$ over the frequency span. However, $\delta$ is equal to $-2^\circ$, and $\eta'$ (the desired value of $\eta$ for error equalization) is 0.9778. $\gamma_{\text{max}}$ becomes $\pm 3^\circ$, thus fixing the phase difference at $90 \pm 3^\circ$ over the frequency span.

Appendix V is devoted to a graphical study of errors in two- and three-pole systems. For each network complexity, $\delta$ values of 0.1, 0.6, 0.8 and 1.0 are chosen, and error curves based on $\eta$ equal to one are obtained. Index increments of 0.1 are selected for the $\theta$'s. The optimum values of $\eta$ are obtained by means of the last described method.
IV - 4 Interpretation of Graphical Results

The results of Appendix V are conveniently summarized by the curves of Figures 21 and 22, in which the maximum positive and maximum negative error excursions are plotted against $\delta$. The optimum value of $\eta$ (for each $\delta$ value) and the resultant minimum error are included in each figure.

In Figure 23, the maximum over-all span in terms of $\delta$ and the frequency ratio is shown for the two- and three-pole systems.

Thus, from consideration of Figures 21, 22 and 23, it is possible to determine either the maximum frequency span, $\delta$ or $\eta$ for a two- or three-pole network directly from curve values. The graphical design data in conjunction with the eighth degree equation solution of Appendix IV form the basis of practical three-pole system design. A specific design is considered in Chapter V.
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FIGURE 21
Error Tabulation And Optimum $n$ Curves For Two-Pole System
Error Tabulation And Optimum \( \alpha \) Curves For Three-Pole System
FIGURE 23
Maximum Frequency Ratio Curves
For Minimum Error Using Optimum
n -- Two-And Three-Pole Systems.
V-1 Choice of Constants

The determination of data for any practical system must include those facts pertinent to actual requirements. The error should be sufficiently small to insure operation within specified limits. For example, a single-sideband transmitting system requiring an attenuation of 40 db on the unwanted side band places limitations on the error of the phase-difference networks. Figure 24 shows the variation of unwanted side-band attenuation versus error in the 90° phase-difference system. The 40 db condition demands an error of not more than one degree.

From Figure 22, it is evident that a three-pole system with \( g \) equal to 0.725 and \( \theta \) equal to 0.997 yields a phase difference of 90° plus or minus less than one degree over a certain frequency span. The graphical results of Figure 23 indicate that the frequency span ratio is at least 370:1.

The two simultaneous equations which apply for a three-pole network are, in general form,*

\[
\begin{align*}
&\left(10^{\frac{f_1}{2}}\right)^6 + \left(10^{\frac{f_2}{2}}\right)^{14}\left(10^{\frac{f_3}{2}}\right)^2(2) + \left(10^{\frac{f_2}{2}}\right)^{2}(10^{\frac{f_3}{2}})^6(4-A_3) + \left(10^{\frac{f_2}{2}}\right)^{10}(10^{\frac{f_3}{2}})^6(6) \\
&\left(4-2A_3 + s^2\right) + \left(10^{\frac{f_2}{2}}\right)^{8}\left(10^{\frac{f_3}{2}}\right)^{8}(5-3A_3 + s^2) + \left(10^{\frac{f_2}{2}}\right)^{10}(10^{\frac{f_3}{2}})^6(10) \\
&(4-2A_3 + s^2) + \left(10^{\frac{f_2}{2}}\right)^{10}\left(10^{\frac{f_3}{2}}\right)^{12}(4-A_3) + \left(10^{\frac{f_2}{2}}\right)^{2}(10^{\frac{f_3}{2}})^4(2) + \\
&(10^{\frac{f_2}{2}})^{6} = 0
\end{align*}
\]

*See equations (332) and (333).
Attenuation Of Unwanted Sideband Versus The Error In 90° Phase-Difference Systems.

FIGURE 24
and
\[ S_3 = \frac{s(10^{\frac{2\alpha}{3}})^2(10^{\frac{3\alpha}{2}})^2}{(10^{\frac{2\alpha}{3}})^4 + (10^{\frac{3\alpha}{2}})^2 + (10^{\frac{3\alpha}{2}})^4} \]  
(345)

where
\[ A_3 = \left(10^{-\frac{4\alpha}{3}} + 1 + 10^{\frac{4\alpha}{3}}\right) \]  
(346)

and
\[ S^2 = \frac{10^{10^5} - 2 \cdot 10^{5^5} + 1}{10^{5^5}}; \quad S = \frac{10^{5^5}}{10^{5^5}} \]  
(347)

V-2 Compilation of Design Data

Since the shape and spacing of the phase-log frequency curves for a particular system are the same regardless of the location (on a log frequency of F basis) of the 90° phase-difference band, calculations may be simplified by assuming the value zero for F_3. This reduces the complexity of equation (344) without restricting its usefulness in determining the relative position for F_01. Further, if the general variable \( v \) replaces 10^F_01, equation (344) reduces to an equation of the form
\[ v^n + Gv^7 + Hv^6 + Jv^5 + Kv^4 + Jv^3 + Hv^2 + Gv + 1 = 0 \]  
(348)

For a \( b \) of 0.725,
\[ S^2 = + 4.2149563 \times 10^3 \]  
\[ A_3 = + 7.9532944 \times 10^2 \]  
\[ 4 - A_3 = - 7.9132944 \times 10^2 \]  
\[ 4 - 2A_3 + S^2 = + 2.6283064 \times 10^3 \]  
(349)

and
\[ 5 - 3A_3 + S^2 = + 1.8339770 \times 10^3 \]  

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With the values of expressions (3.49) substituted appropriately in equation (3.48), equation (3.48) assumes the numerical representation

\[ V^8 + 2V^7 - (7.9132944 \times 10^2)V^6 + (2.6283064 \times 10^3)V^5 + (1.8339770 \times 10^3)V^4 + (2.6283064 \times 10^3)V^3 - (7.9132944 \times 10^2)V^2 \]

\[ + 2V + 1 = 0 \]  

The eight roots of equation (3.50), as determined by the methods of Appendix VI, are:

\[
\begin{align*}
V_1 &= +4.18825 \\
V_2 &= +0.23879 \\
V_3 &= +25.16949 \\
V_4 &= +0.03973 \\
V_5 &= -30.60308 \\
V_6 &= -0.03268 \\
V_7 &= -0.500003 + j0.866007 \\
V_8 &= -0.500003 - j0.866007
\end{align*}
\]  

The numerical values of expressions (3.51) substantiate the predictions of section III-5, wherein it was stated that a maximum of four positive roots might be found. The existence of the two negative and two conjugate complex roots is of academic interest only. It may be noted that the real roots occur in reciprocal pairs; i.e., \( v_1 \) is equal to the reciprocal of \( v_2 \), etc. The smaller value of each positive reciprocal pair may be associated with \( F_{01} \), while the larger value may be associated with the corresponding \( F_{On} \). It is emphasized that these relationships apply only for the positive roots. For illustrative purposes, let it be assumed that the larger \( F_{01} \)
is desired. Therefore, since $v$ equals $10^{F_{01}}$, which, in turn, is evaluated as 0.23879, $F_{01}$ becomes

$$F_{01} = \log_{10} 0.23879 = -0.6219839 \quad (352)$$

From the specific data of formulas (287) through (289) and a "rounding off" to four figures,

$$F_{01} = -0.6220$$
$$F_{02} = 0.0000$$
and
$$F_{03} = +0.6220 \quad (353)$$

This fixes the value 0.6220 units of $F$ as the difference between adjacent $F_{0j}$'s for three-pole networks with $S$ equal to 0.725, regardless of the value of $F_3$.

The numerical value for $s_1$ may be readily calculated from equation (345) as

$$s_1 = 11.9639 \quad (354)$$

The $s_1$ given above is the $s$ for each of the one-pole networks which make up the complete three-pole system.

The data of Figure 22 indicate that a value of $\zeta$ equal to 0.997 should be used. The Channel Spacing equation, expression (171), and the discussion immediately following expression (171) indicate that

$$\zeta = \frac{(0.997)(0.725)}{2} = 0.3614 \quad (355)$$

*If the three one-pole networks for each composite three-pole network are to be of the R-C type with no internal isolation, less attenuation per phase-shift section occurs for the larger value of $F_{01}$. However, the ratio between the size of the element values is greater than that required by the smaller value of $F_{01}$. 

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Sufficient data are now available for an actual computation of network values based on the desired location of the 90° phase-difference band. Let it be assumed that the desired Mid-Frequency, \( f_M \), is 1000 c.p.s. \( f_M \) is thereby equal to three. Accordingly,

\[
F_{3A} = 3 - \frac{0.3614}{2} = 2.8193
\]  
(356)

and

\[
F_{3B} = 3 + \frac{0.3614}{2} = 3.1807
\]  
(357)

Since the difference is the adjacent \( F_0 \)'s is 0.6220 and, for a three-pole network, \( F_0 \) equals \( F_3 \), the following expressions may be determined:

\[
\begin{align*}
F_{01A} &= 2.8193 - 0.6620 = 2.1573 \\
F_{02A} &= 2.8193 + 0.0000 = 2.8193 \\
F_{03A} &= 2.8193 + 0.6220 = 3.4413 \\
F_{01B} &= 3.1807 - 0.6620 = 2.5587 \\
F_{02B} &= 3.1807 + 0.0000 = 3.1807 \\
F_{03B} &= 3.1807 + 0.6220 = 3.8027
\end{align*}
\]  
(358)

The Center Frequencies for each one-pole network may be found directly from the relationship \( 10^{F_3} \) equals \( f_3 \) as

\[
\begin{align*}
f_{01A} &= 10^{2.1573} = 157.507 \text{ c.p.s.} \\
f_{02A} &= 10^{2.8193} = 659.629 \text{ c.p.s.} \\
f_{03A} &= 10^{3.4413} = 2762.485 \text{ c.p.s.} \\
f_{01B} &= 10^{2.5587} = 361.993 \text{ c.p.s.}
\end{align*}
\]  
(359)
The network parameters for the three one-pole networks of channels A and B may be determined by application of the methods of section I-6 arranged in the following tabulation:

For Channel A:

First Phase-Shift Section

1. Select a suitable value for \( R_{131A} \).

2. Determine \( R_{231A} \) from relationship 
   \[
   R_{231A} = \frac{R_{131A}}{k_7}
   \]
   where 
   \[
   k_7 = \frac{1}{(s_1 + 2)}
   \]

3. Determine \( R_{331A} \) from relationship 
   \[
   R_{331A} = R_{231A} \left( \frac{1-k_7}{4k_7} \right)
   \]

4. Determine \( C_{131A} \), \( C_{231A} \) and \( C_{331A} \) from relationships 
   \[
   C_{131A} = \frac{R_{131A}}{2\pi f_{01A}}, \quad C_{231A} = \frac{R_{231A}}{2\pi f_{01A}}, \quad C_{331A} = \frac{R_{331A}}{2\pi f_{01A}}
   \]

Second Phase-Shift Section

5. Select a suitable value for \( R_{132A} \).

6. Determine \( R_{232A} \) from relationship 
   \[
   R_{232A} = \frac{R_{132A}}{k_7}
   \]

7. Determine \( R_{332A} \) from relationship 
   \[
   R_{332A} = R_{232A} \left( \frac{1-k_7}{4k_7} \right)
   \]

8. Determine \( C_{132A} \), \( C_{232A} \) and \( C_{332A} \) from relationships 
   \[
   C_{132A} = \frac{R_{132A}}{2\pi f_{02A}}, \quad C_{232A} = \frac{R_{232A}}{2\pi f_{02A}}, \quad C_{332A} = \frac{R_{332A}}{2\pi f_{02A}}
   \]

*As indicated in section I-6, \( R_{131A} \) partially determines the plate load impedance of the associated vacuum tube.
Third Phase-Shift Section

(9) Select a suitable value for $R_{133A}$.

(10) Determine $R_{233A}$ from relationship $R_{233A} = \frac{R_{133A}}{k7}$.

(11) Determine $R_{333A}$ from relationship $R_{333A} = R_{233A} \left(1 - \frac{1}{4k7}\right)$.

(12) Determine $C_{133A}$, $C_{233A}$ and $C_{333A}$ from relationships

$$C_{133A} = \frac{R_{133A}}{2\pi f_{03A}}, \hspace{1em} C_{233A} = \frac{R_{233A}}{2\pi f_{03A}} \hspace{1em} C_{333A} = \frac{R_{333A}}{2\pi f_{03A}}.$$

For Channel B:

(13) Repeat steps (1) through (12), using B subscripts in place of A.

It is suggested that the same value of resistance be chosen for $R_{131A}$, $R_{132A}$, $R_{133A}$, $R_{131B}$, $R_{132B}$ and $R_{133B}$. The differences in phase-shift section components then occur only in the capacitor values.

The location of the circuit parameters is shown in the suggested schematic diagram of Figure 25. The similarity between this diagram and Figure 15 should be noted.
VI-1 Review of Results Accomplished

General equations, applicable for any value of \( n \), have been developed, which describe the \( n \)-pole networks and systems. From the specific design data on two- and three-pole systems, the proper constants for the general equations may be selected. Solution of the general equations then leads directly to determination of parameters \( s_l \) and \( f_l \) of each of the individual one-pole networks. In the practical solution given in Chapter V, the design data and general equations are used to determine the appropriate constants of a three-pole system having, theoretically, an error of less than one degree and a frequency-span ratio in the order of 370:1.

In addition to the immediate problem, solution methods for certain eighth degree equations, special aspects of reactive circuit behavior and tabulated error calculation methods are presented in the form of appendices.

VI-2 Recommendations

The networks discussed throughout the report are of three types, L-C, R-C with isolation and R-C without isolation. It is recommended that other types of network configuration such as the parallel-T and bridged-T be investigated.

The purely reactive L-C systems of this report are presented as guides toward obtaining phase-frequency data for practical R-C systems. However, it is possible that finite Q, L-C combinations might be satisfactorily employed in phase-shift sections, provided suitable restrictions are placed on the parameter values. This aspect merits examination.

So-called zero impedance voltage sources are employed in each phase-
shift section. While the calculations are based on the assumption that the source impedances are small enough to be negligible when compared with the other impedances of the system, the effect of the source impedance must be evident to some extent in every practical phase-difference system. It is recommended that the effect of source impedance be investigated.

The error curves and calculation of constants for the one-pole networks are based on a value of the parameter $s$, which is so interrelated with $s$ as to cause the error curve to pass through fixed 90 Degree Points. It is recommended that error calculations be made with values of $s$ which allow a variation of the crossover points.

From previous experience with experimental models of phase-difference systems, it was found that less than unity gain could be expected in the coupling vacuum tube circuits if proper linearity considerations were adhered to. Accordingly, it is recommended that improved coupling systems between phase-shift sections be investigated.

Although this study is mainly theoretical, consideration should be given to the construction of practical working systems based on the theoretical predictions. Accordingly, it is recommended that a thorough investigation be made regarding the accuracy, stability, shock proofness, aging qualities and moisture resistance of the elements which make up the phase-shift sections. Again, with the working system in mind, it is recommended that a suitable program be conducted to design an accurate, compact and dependable $90^\circ$ phase-difference meter.
APPENDIX I
DETERMINATION OF $f_n$ IN TERMS OF CIRCUIT
BEHAVIOR AT ZERO AND INFINITE FREQUENCIES

Equations (1) and (2) describe impedances which may have many degrees of freedom. For the purpose of this report, the number of degrees of freedom are intentionally diminished in order to attain a specific objective. By a suitable choice of the circuit elements, $Z_a$ and $Z_b$ become subject to the following conditions:

1. $Z_a$ and $Z_b$ are reciprocal pure reactances; i.e., $Z_a Z_b = R^2$ (a constant) at all frequencies.

2. Poles of impedance of $Z_a$ occur at the same frequencies as do zeros of impedance of $Z_b$, and conversely.

3. The geometric mean between $f_{n-q}$ and $f_{n+q}$ is $f_n$, where $q$ is any positive integer from zero through $n-1$; i.e., $f_n^2 = f_{n-q} \cdot f_{n+q}$.

Under the above conditions, it will be demonstrated that

$$f_n = \frac{1}{2\pi} \sqrt{\frac{1}{L_{ao} C_{a\infty}}}$$

and

$$\frac{1}{2\pi} \sqrt{\frac{1}{LC}} = \sqrt{\frac{L_{ao}}{L_{b\infty}}} \cdot \frac{1}{2\pi} \sqrt{\frac{1}{L_{ao} C_{a\infty}}} = S f_n,$$

where the first subscript indicates the impedance $Z_a$ or $Z_b$ to which the circuit element $L$ or $C$ pertains, and the second subscript indicates specific impedance behavior at zero or infinite frequencies. $L$ without subscripts is the inductance value of $Z_b$ at zero frequency, and $C$ without subscripts is the capacitance value of $Z_b$ at infinite frequency.

*Note: $Z_a$ and $Z_b$ represent the form of the impedances. This discussion applies equally well to $Z_1$ and $Z_2$ in the L-C semi-lattice networks.
subscripts is the capacitance value of $Z_a$ at infinite frequencies. The constant $s$ is a circuit parameter which is equal to the square root of the ratio of the inductance of $Z_a$ at zero frequency to the inductance value which is approached in $Z_b$ as the frequency becomes infinite.

Case I—$n$ equal to 1

From condition (1),

$$Z_a Z_b = \frac{L_{a1}}{C_{b1}} = \frac{L_{b1}}{C_{a1}} = \frac{L}{C} = R^2. \quad (362)$$

From condition (2),

$$f_1 = \frac{1}{2\pi} \sqrt{\frac{1}{L_{a1} C_{a1}}} = \frac{1}{2\pi} \sqrt{\frac{1}{L_{b1} C_{b1}}} \quad (363)$$

Since for $n$ equal to 1, $L$ equals $L_{b1}$ and $C$ equals $C_{a1}$,

$$\frac{1}{2\pi} \sqrt{\frac{1}{L C}} = \frac{1}{2\pi} \sqrt{\frac{1}{L_{b1} C_{a1}}} = \sqrt{\frac{L_{a1}}{L_{b1}}} \cdot \frac{1}{2\pi} \sqrt{\frac{1}{L_{a1} C_{b1}}} \quad (364)$$

In the above networks, $C_{a0\infty}$ equals $C_{a1}$, $L_{a0}$ equals $L_{a1}$, and $L_{b\infty}$ equals $L_{b1}$.
Thus,

\[ f_1 = \frac{1}{2\pi} \sqrt{\frac{1}{L_{ao} C_{a\infty}}} \]  (365)

and

\[ \frac{1}{2\pi} \sqrt{\frac{1}{L C}} = \sqrt{\frac{L_{ao}}{L_{b\infty}}} \cdot \frac{1}{2\pi} \sqrt{\frac{1}{L_{ao} C_{a\infty}}} = S f_1 \]  (366)

Case II—\( n \) equal to 2

From condition (1),

\[ Z_a Z_b = \frac{L_{a1}}{C_{b1}} = \frac{L_{b1}}{C_{a1}} = \frac{L_{a3}}{C_{b3}} = \frac{L_{b3}}{C_{a3}} = \frac{L}{C} = \mathbb{R} \]  (367)
From condition (2),
\[ f_1 = \frac{1}{2\pi} \sqrt{\frac{1}{L_{a1}C_{a1}}} = \frac{1}{2\pi} \sqrt{\frac{1}{L_{b1}C_{b1}}} \]  
(368)

and
\[ f_3 = \frac{1}{2\pi} \sqrt{\frac{1}{L_{a3}C_{a3}}} = \frac{1}{2\pi} \sqrt{\frac{1}{L_{b3}C_{b3}}} \]  
(369)

At \( f \) equal to \( f_2 \), a zero of impedance of \( Z_a \) occurs. Then
\[ \frac{L_{a1}}{(2\pi f_2)^2 L_{a1}C_{a1} - 1} + \frac{L_{a3}}{(2\pi f_2)^2 L_{a3}C_{a3} - 1} = 0 \]  
(370)

from which
\[ (2\pi f_2)^2 L_{a1}L_{a3}(C_{a1} + C_{a3}) = (L_{a1} + L_{a3}) \]  
(371)

and
\[ f_2 = \frac{1}{2\pi} \sqrt{\frac{1}{\left(\frac{L_{a1}L_{a3}}{L_{a1} + L_{a3}}\right)(C_{a1} + C_{a3})}} \]  
(372)

Similarly, at \( f \) equal to \( f_2 \), a zero of admittance of \( Z_b \) occurs. Then
\[ \frac{C_{b1}}{(2\pi f_2)^2 L_{b1}C_{b1} - 1} + \frac{C_{b3}}{(2\pi f_2)^2 L_{b3}C_{b3} - 1} = 0 \]  
(373)

from which
\[ (2\pi f_2)^2 C_{b1}C_{b3}(L_{b1} + L_{b3}) = (C_{b1} + C_{b3}) \]  
(374)

and
\[ f_2 = \frac{1}{2\pi} \sqrt{\frac{1}{\left(\frac{C_{b1}C_{b3}}{C_{b1} + C_{b3}}\right)(L_{b1} + L_{b3})}} \]  
(375)
From condition (3), whereby \( f_1 f_3 \) equals \( f_2^2 \), the expressions for \( f_1 \) and \( f_3 \), given in equations (368) and (369), may be multiplied to yield

\[
f_1 f_3 = f_2^2 = \frac{1}{(2\pi)^2} \sqrt{\frac{1}{L_{a1} C_{a1} L_{a3} C_{a3}}} \quad (376)
\]

Dividing equation (376) by equation (372), results in

\[
f_2 = \frac{1}{2\pi} \sqrt{\frac{1}{C_{a1} C_{a3}} \left( \frac{L_{a1}}{C_{a1}} + \frac{L_{a3}}{C_{a3}} \right)} \quad (377)
\]

Since for \( n \) equal to 2, \( \frac{1}{L} \) equals \( \frac{1}{L_{b1}} + \frac{1}{L_{b3}} \), and \( \frac{1}{C} \) equals \( \frac{1}{C_{a1}} + \frac{1}{C_{a3}} \),

\[
\frac{1}{2\pi} \sqrt{\frac{1}{L C}} = \frac{1}{2\pi} \sqrt{\left( \frac{1}{L_{b1}} + \frac{1}{L_{b3}} \right) \left( \frac{1}{C_{a1}} + \frac{1}{C_{a3}} \right)} = \frac{1}{2\pi} \sqrt{\left( \frac{L_{b1} + L_{b3}}{L_{b1} L_{b3}} \right) \left( \frac{C_{a1} + C_{a3}}{C_{a1} C_{a3}} \right)} \quad (378)
\]

By multiplying equation (378) by \( \sqrt{\frac{L_{a1} + L_{a3}}{L_{b1} + L_{b3}}} \) and rearranging terms, the following expression is obtained:

\[
\frac{1}{2\pi} \sqrt{\frac{1}{L C}} = \frac{1}{2\pi} \sqrt{\left( \frac{L_{a1} + L_{a3}}{L_{b1} L_{b3}} \right) \left( \frac{C_{a1} C_{a3}}{C_{a1} + C_{a3}} \right)} \quad (379)
\]

For \( n \) equal to 2, \( C_{a\infty} \) equals \( \frac{C_{a1} C_{a3}}{C_{a1} + C_{a3}} \), \( L_{a0} \) equals \( L_{a1} + L_{a3} \), and \( L_{b\infty} \)

\[ \text{equals} \quad \frac{L_{b1} L_{b3}}{L_{b1} + L_{b3}} \]. Thus,
\[ f_2 = \frac{1}{2\pi} \sqrt{\frac{1}{L_{ao} C_{a\infty}}} \quad (380) \]

and

\[ \frac{1}{2\pi} \sqrt{\frac{1}{LC}} = \sqrt{\frac{L_{ao}}{L_{b\infty}}} \cdot \frac{1}{2\pi} \sqrt{\frac{1}{L_{ao} C_{a\infty}}} = Sf_2. \quad (381) \]

By a continuation of the same procedure for \( n \) equal to 3, 4, etc., it may be shown that, in general,

\[ f_n = \frac{1}{2\pi} \sqrt{\frac{1}{L_{ao} C_{a\infty}}} \quad (382) \]

and

\[ \frac{1}{2\pi} \sqrt{\frac{1}{LC}} = \sqrt{\frac{L_{ao}}{L_{b\infty}}} \cdot \frac{1}{2\pi} \sqrt{\frac{1}{L_{ao} C_{a\infty}}} = Sf_n \quad (383) \]

when the networks are restricted in accordance with conditions (1), (2) and (3).
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APPENDIX II
The standard mesh equations for the solution of currents \( I_1 \) and \( I_2 \) are

\[
V_{i3} = I_1 (Z_1 + Z_3) + I_2 (-Z_3) \tag{384}
\]

and

\[
-V_{23} = V_{i3} = I_1 (-Z_3) + I_2 (Z_2 + Z_3) \tag{385}
\]

From the simultaneous solution of equations (384) and (385), by determinant methods,

\[
I_1 = \frac{V_{i3} \begin{vmatrix} -Z_3 & Z_3 \\ Z_2 + Z_3 & -Z_3 \end{vmatrix}}{Z_1 + Z_3} = \frac{V_{i3} (Z_2 + 2Z_3)}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} \tag{386}
\]
\[ I_2 = \begin{vmatrix} Z_1 + Z_3 & V_{13} \\ -Z_3 & V_{13} \\ Z_1 + Z_3 & -Z_3 \\ -Z_3 & Z_2 + Z_3 \end{vmatrix} = \frac{V_{13}(Z_1 + 2Z_3)}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}, \] (387)

\[ I_1 - I_2 = \frac{V_{13}(Z_2 - Z_1)}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}, \] (388)

\[ V_{43} = \frac{V_{13}(Z_2 - Z_1) Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} = \frac{V_{13}(Z_2 - Z_1)}{Z_1 Z_2 \frac{Z_2}{Z_3} + (Z_1 + Z_2)} \] (389)

and

\[ V_{34} = -V_{43} = \frac{V_{13}(Z_1 - Z_2)}{Z_1 Z_2 \frac{Z_2}{Z_3} + (Z_1 + Z_2)} \] (390)
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## CIRCUIT CONSTANTS FOR EXPERIMENTAL TWO-POLE 90° PHASE-DIFFERENCE SYSTEM

<table>
<thead>
<tr>
<th>Circuit Constant</th>
<th>Value</th>
<th>Tolerance</th>
</tr>
</thead>
<tbody>
<tr>
<td>R5, R13, R22, R30</td>
<td>100, lw.</td>
<td>10%</td>
</tr>
<tr>
<td>R7, R12, R15, R20, R24, R29, R32, R37</td>
<td>220, lw.</td>
<td>10%</td>
</tr>
<tr>
<td>R9, R10, R17, R18, R25, R26, R27, R34, R35</td>
<td>10,000, lw.</td>
<td>10%</td>
</tr>
<tr>
<td>R8, R16, R25, R33</td>
<td>5,000, 2w.</td>
<td>10%</td>
</tr>
<tr>
<td>R11, R19, R26, R36</td>
<td>10,000, lw.</td>
<td>10%</td>
</tr>
<tr>
<td>R1, R21</td>
<td>50,000, lw. pot.</td>
<td>10%</td>
</tr>
<tr>
<td>R6, R14, R23, R31</td>
<td>5,000, lw. pot.</td>
<td>10%</td>
</tr>
<tr>
<td>C4, C5, C6, C7</td>
<td>8 uf., 350v.</td>
<td>10%</td>
</tr>
<tr>
<td>C8</td>
<td>50 uf., 350v.</td>
<td>10%</td>
</tr>
</tbody>
</table>

**Channel A:**

<table>
<thead>
<tr>
<th>Circuit Constant</th>
<th>Value</th>
<th>Tolerance</th>
</tr>
</thead>
<tbody>
<tr>
<td>R121A, R122A*</td>
<td>20,000</td>
<td>1%</td>
</tr>
<tr>
<td>R221A, R222A*</td>
<td>189,180</td>
<td>1%</td>
</tr>
<tr>
<td>R321A, R322A*</td>
<td>260,072</td>
<td>1%</td>
</tr>
<tr>
<td>C121A**</td>
<td>.038532 uf.</td>
<td>1%</td>
</tr>
<tr>
<td>C221A**</td>
<td>.004074 uf.</td>
<td>1%</td>
</tr>
<tr>
<td>C321A**</td>
<td>.002963 uf.</td>
<td>1%</td>
</tr>
<tr>
<td>C122A**</td>
<td>.005724 uf.</td>
<td>1%</td>
</tr>
<tr>
<td>C222A**</td>
<td>.00605 uf.</td>
<td>1%</td>
</tr>
<tr>
<td>C322A**</td>
<td>.000440 uf.</td>
<td>1%</td>
</tr>
</tbody>
</table>

**Channel B:**

<table>
<thead>
<tr>
<th>Circuit Constant</th>
<th>Value</th>
<th>Tolerance</th>
</tr>
</thead>
<tbody>
<tr>
<td>R121B, R122B*</td>
<td>20,000</td>
<td>1%</td>
</tr>
<tr>
<td>R221B, R222B*</td>
<td>189,180</td>
<td>1%</td>
</tr>
<tr>
<td>R321B, R322B*</td>
<td>260,072</td>
<td>1%</td>
</tr>
<tr>
<td>C121B**</td>
<td>.013672 uf.</td>
<td>1%</td>
</tr>
<tr>
<td>C221B**</td>
<td>.001445 uf.</td>
<td>1%</td>
</tr>
<tr>
<td>C321B**</td>
<td>.002031 uf.</td>
<td>1%</td>
</tr>
<tr>
<td>C122B**</td>
<td>.000215 uf.</td>
<td>1%</td>
</tr>
<tr>
<td>C222B**</td>
<td>.000156 uf.</td>
<td>1%</td>
</tr>
</tbody>
</table>

*Resistors for channels A and B were adjusted at d-c values on a Wheatstone Bridge of + 1% accuracy.  
**Condensers for Channels A and B were adjusted to have voltage drops equal to voltage drops across the corresponding resistors at proper center frequency for each one-pole network.
Equations of the form

\[ x^8 + Gx^7 + Hx^6 + Jx^5 + Kx^4 + Jx^3 + Hx^2 + Gx + 1 = 0 \quad (391) \]

are members of a class known as reciprocal equations, since the real roots occur in reciprocal pairs. The equations typified by expression (332) are of this type. In particular, equation (391) may be rewritten as the product of two fourth degree reciprocal expressions of the form

\[
\begin{align*}
(x^4 + ax^3 + bx^2 + ax + 1) \cdot (x^4 + cx^3 + dx^2 + cx + 1) &= x^8 + (a+c)x^7 + (b+ac+d)x^6 + (c+ad+bc+a)x^5 + \\
(2ac+bd)x^4 + (c+ad+bc+a)x^3 + \\
(b+ac+d)x^2 + (a+c)x + 1 &= 0 \quad (392)
\end{align*}
\]

provided, from equations (391) and (392), that

\[
\begin{align*}
(a+c) &= G, \quad (393) \\
(b+ac+d) &= H, \quad (394) \\
(c+ad+bc+a) &= J; \quad (ad+bc) = J-G \quad (395)
\end{align*}
\]

and

\[
(2ac+bd) = K \quad (396)
\]
G, H, J and K are the known quantities, while a, b, c, and d are to be determined. Direct simultaneous solution for the unknowns results in expressions of degree greater than four, for which no exact determination exists. However a successive approximation method, accurate to as many decimal places as desired, may be employed.

From equations (394) and (396), b may be determined as

\[ b = \frac{K - 2H - 2 + 2d}{d - 2} \]  

(397)

If a value for d greater than two is chosen at random and substituted in equation (397), an arbitrary value for b is found. Using the values for b and d in equation (395), an expression in a and c is found which may, in turn, be compared with equation (393) in order to find values for a and c. The product (ac) together with the assumed values for b and d may be substituted in

\[ b + ac + d - H \]  

(398)

which may be recognized as a rearrangement of equation (394). Should the choice of d be exactly correct, expression (398) is identically zero. It is probable that the first choice of d is not such as to make equation (398) zero but may result, instead, in making expression (398) either a positive or negative number. An increase in d makes expression (398) more negative (or less positive). Should the first choice of d make expression (398) positive, it is evident that the d is less than the exact value, and conversely. Repetitions of the above process with successively larger values
of d make expression (398) smaller and smaller until a point is reached where (398) is equal to zero within the range of the desired accuracy.

Assuming that satisfactory values for a, b, c and d are found, it is necessary to solve for the roots of each of the fourth degree expressions of (392). Since these equations are of a degree not greater than four, an analytical solution may be obtained. Upon examination of

\[ x^4 + ax^3 + bx^2 + cx + d = 0 \]  

it is seen that the product of two reciprocal quadratic expressions of the form

\[(x^2 + ex + f)(x^2 + fx + g) = \]

\[ x^4 + (e+f)x^3 + (2+ef)x^2 + (e+f)x + f = 0 \]

may represent expression (399) if the values of e and f are determined under the following conditions:

\[ e + f = a \]  

(401)

and

\[ 2 + ef = b \]  

(402)

The conditions are found to be

\[ e = \frac{a + \sqrt{a^2 - 4b + 8}}{2} \]  

(403)
and

\[ f = \frac{a - \sqrt{a^2 - 4b + 8}}{2} \]  \hspace{1cm} (404)

Similarly, for the determination of the second fourth degree expression of (392)

\[ (x^2 + gx + 1)(x^2 + hx + 1) = x^4 + (g + h)x^3 + (2 + gh)x^2 + (g + h)x + 1 \]  \hspace{1cm} (405)

where

\[ g = \frac{c + \sqrt{c^2 - 4d + 8}}{2} \]  \hspace{1cm} (406)

and

\[ h = \frac{c - \sqrt{c^2 - 4d + 8}}{2} \]  \hspace{1cm} (407)

The values for e, f, g and h may be substituted in the individual quad-

ratic expression of

\[ (x^2 + ex + 1)(x^2 + fx + 1)(x^2 + gx + 1)(x^2 + hx + 1) = 0, \]  \hspace{1cm} (408)

from which the eight roots of equations (391) and (408) are

\[ x_i = \frac{-a - \sqrt{a^2 - 4b + 8}}{2} + \frac{\sqrt{2a^2 - 4b - 8 + 2a\sqrt{a^2 - 4b + 8}}}{4} \]  \hspace{1cm} (409),

\[ -147- \]
\[ \chi_2 = \frac{-a - \sqrt{a^2 - 4b + 8} - \sqrt{2a^2 - 4b - 8 + 2a \sqrt{a^2 - 4b + 8}}}{4} \]  
(110)

\[ \chi_3 = \frac{-a + \sqrt{a^2 - 4b + 8} + \sqrt{2a^2 - 4b - 8 - 2a \sqrt{a^2 - 4b + 8}}}{4} \]  
(111)

\[ \chi_4 = \frac{-a + \sqrt{a^2 - 4b + 8} - \sqrt{2a^2 - 4b - 8 - 2a \sqrt{a^2 - 4b + 8}}}{4} \]  
(112)

\[ \chi_5 = \frac{-c - \sqrt{c^2 - 4d + 8} + \sqrt{2c^2 - 4d - 8 + 2c \sqrt{c^2 - 4d + 8}}}{4} \]  
(113)

\[ \chi_6 = \frac{-c - \sqrt{c^2 - 4d + 8} - \sqrt{2c^2 - 4d - 8 + 2c \sqrt{c^2 - 4d + 8}}}{4} \]  
(114)

\[ \chi_7 = \frac{-c + \sqrt{c^2 - 4d + 8} + \sqrt{2c^2 - 4d - 8 - 2c \sqrt{c^2 - 4d + 8}}}{4} \]  
(115)

\[ \chi_8 = \frac{-c + \sqrt{c^2 - 4d + 8} - \sqrt{2c^2 - 4d - 8 - 2c \sqrt{c^2 - 4d + 8}}}{4} \]  
(116)

and

\[ \chi_9 = \frac{-c + \sqrt{c^2 - 4d + 8} - \sqrt{2c^2 - 4d - 8 - 2c \sqrt{c^2 - 4d + 8}}}{4} \]  
(117)
In order to graphically determine the variation of error, \( Y \), with \( \varepsilon \) and to ascertain optimum \( \eta \) values for two- and three-pole systems, it is necessary to calculate the phase angle, \( \beta \), of the networks over considerable spans of log frequency units (units of F). After selecting an arbitrary \( F_n \) (\( F_2 \) for a two-pole and \( F_3 \) for a three-pole network) and designating the \( \beta \) as \( \beta_{2,0} \) or \( \beta_{3,0} \), respectively, sufficient data for plotting error curves may be obtained by calculating the \( \beta \) values from \( \beta_{0,3} \) to \( \beta_{2,3} \) for the two-pole network and from \( \beta_{0,3} \) to \( \beta_{3,3} \) for the three-pole network. An increment of 0.1 in the indices of the \( \beta \)'s is sufficiently small to yield good results consistent with a relatively small number of calculated points. Section IV-3 of the report covers the theoretical approach to this method of selecting error data.

The values of \( \varepsilon \) to be used are 0.4, 0.6, 0.8 and 1.0. The determination of the \( \beta \) values for each \( \varepsilon \) in the two- and three-pole networks represents a major calculation project. Once these values are found, simple arithmetic computations yield error data for each \( \varepsilon \), as explained in detail in section IV-3.

The general equation for determining \( \beta_{2,3} \) for the two-pole network is

\[
\beta_{2,3} = \tan^{-1} \frac{25 \times 10^{.35} (10^{-1})}{(10^{-.65} - 10^{-28})(10^{-.65} + 10^{-28})} - \frac{S^2 \times 10^{.65} (10^{-1})^2}{(10^{-.65} - 10^{-28})(10^{-.65} + 10^{-28})}
\]  

(117)
which may be recognized as a form of the standard two-pole phase formula expressed in terms of $S$. For $\beta_{2.2}$ the formula is

$$\beta_{2.2} = \tan^{-1} \left( \frac{2S \cdot 10^{0.25} \cdot 10^{0.45} \cdot 10^{-1}}{(10^{-10} \cdot 10^{25})(10^{45} - 1)} \right)$$

and for $\beta_{2.1}$, the formula is

$$\beta_{2.1} = \tan^{-1} \left( \frac{2S \cdot 10^{-1.6} \cdot 10^{26} \cdot 10^{-1}}{(10^{-10} \cdot 10^{26})(10^{12} - 1)} \right)$$

Inspection of equations (417), (418) and (419) reveals that the exponent of the first ten in the numerator of equation (417) is $0.36$ and that it decreases by $0.16$ as the index of $S$ decreases by $0.1$. The exponent of the second ten of the numerator is twice the exponent of the first ten and is equal to the exponent of the first and third tens of the denominator. Likewise, the exponents of the first and second tens of the numerator of the fraction in the denominator and the first and third tens of the denominator of this fraction are twice that of the first ten of the main numerator. The unity term appears in the numerator for all values of $S$ as does the $-2S$ and $+2S$ exponents of the second and fourth tens of the denominator. The unity term appears again in the numerator of the denominator fraction, and the $-2S$ and $+2S$ exponents always appear in the denominator of this fraction. In addition, the constant multiplier two and the parameter $s$ are to be found in the numerator of the complete expression.

With reference again to equations (417), (418) and (419), it is seen that these equations may be expressed in the form

$$-151-$$
if $A$ and $D$ represent expressions of the type

$$A = 2s 10^{k \delta} \left( 10^{2(k \delta)} - 1 \right) \quad (421)$$

and

$$D = \left( 10^{2(k \delta)} - 10 \right) \left( 10^{2(k \delta)} - 10^{2 \delta} \right) \quad (422)$$

Special forms titled "Error Curve Calculation Sheets--Two-Pole Network" are used to systematically arrange the computations for each $\phi$. A reproduction of a typical completed sheet appears as Figure 26. It should be noted that the $A$ and $D$ simplifications are employed in obtaining the $\phi$'s.

The value for $s$ in the above equations is determined from equation (220) for $n$ equal to two as

$$s = \frac{10^{3\delta} - 1}{10^{3/2 \delta}} \quad (423)$$

It is to be noted that the $\phi$'s obtained for the networks are negative angles ranging from zero to $-720^\circ$ for the two-pole case. If values for $\phi$ are calculated from zero frequency with frequency increasing, $\phi$ takes on negatively increasing values corresponding to a clockwise rotation of the phase angle vector. $\phi$'s equal to $\phi_0, \phi_1, \phi_2$ correspond to $0^\circ$, $-180^\circ$ and $-360^\circ$, respectively. Since the numerical computations can only indicate either positive or negative values of the tangent of an acute angle, it is necessary to further determine not only the quadrant position but the multiples of $-360^\circ$ included in the actual phase angle. For example, a
Error Curve Calculation Sheet - 2 Pole Network

\[
\begin{array}{c|c|c|c|c}
\delta & \delta' & 10^{-3} & 10^{-7} & S \\
\hline
1.9 & 0.8 & 0.08 & 15.785472 & \frac{10^{36} - 1}{10^{36} \delta} \\
\end{array}
\]

\( I = \frac{(s)}{(s') - \frac{0.08}{10} - \frac{0.16}{10} - 1} = \)

\[
\begin{array}{c}
(15.785472 \times 10^{-7})(8.3176377 - 3.0816905) = -4.0461931 \\
\end{array}
\]

\( 2A = -8.0923862 \) \hspace{1cm} (B)

\( A^2 = +16.3716786 \) \hspace{1cm} (C)

\( (10^{-0.16} - 10^{-0.16} - 10^{+0.16}) = \)

\[
\begin{array}{c}
(6.9183095 - 2.5418285)(6.3176377 - 3.9810716) = -25.141601 \\
\end{array}
\]

\( C = +16.3716786 \)

\( D = -26.411601 \) \hspace{1cm} (E)

\( D - E = -26.411601 - -0.6262797 = -25.5148804 \) \hspace{1cm} (F)

\( B = -8.0923862 \)

\( F = -25.5148804 \) \hspace{1cm} (G)

\( G = \tan^{-1}\beta \)

\( \beta = 342.4029 \) \hspace{1cm} (H)

Computations:

FIGURE 26

Two-Pole Calculation Sheet
negative tangent value may refer to an angle located in either the second or fourth quadrant, while a positive tangent value may refer to an angle located in either the first or third quadrant. In neither case will the multiples of $-360^\circ$ be directly indicated. By establishing check points such as $\theta_0 = 0^\circ$, $\theta_1 = -180^\circ$, $\theta_2 = -360^\circ$, etc., it is possible to calculate progressively the $\theta$ values and retain the proper orientation. A change in sign of the tangent indicates that the angle vector has changed quadrants in a clockwise direction. An axis system is provided on the error curve calculation form for the express purpose of approximating the angle vector position, thereby minimizing the possibility of erroneous results.

After the $\theta$'s are determined for each $\mathbf{E}$, the values obtained are recorded on the "Error Calculation Tabulation Sheet" forms. One table is provided for each value of $\mathbf{E}$ with $\eta$ equal to one. Indicated additions and subtractions are performed for each set of values. As an example, for $\eta$ equal to one, the error value at $F_{0.3A}$ is equal to

$$\theta_{a.3} - \theta_{a.8} - 90^\circ = \gamma \quad \ldots \ (124)$$

A reproduction of a typical completed tabulation sheet is shown as Figure 27.

The calculations for a three-pole system are identical to those performed for the two-pole. The general equation for $\theta_{3,3}$ of a three-pole system is

$$\theta_{3,3} = \tan^{-1} \left( \frac{2S \cdot 10^{.35} (10 - 10^{.76}) (10^{.65} - 10^{.26})}{(10^{.76} - 10^{.84})(10^{.76} + 10^{.45}) - S^{2} (10^{.65} - 10^{.26})^2 (10^{.65} - 10^{.26})^2} \right) \quad \ldots \ (125)$$

-154-
Error Calculation Tabulation Sheet - 2 Pole Network

<table>
<thead>
<tr>
<th>Pole</th>
<th>θ</th>
<th>Error°</th>
<th>Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-17.3173°</td>
<td>-90°</td>
<td>F, 55 M</td>
</tr>
<tr>
<td>2</td>
<td>-162.9000°</td>
<td>-17.2180°</td>
<td>F, 65 M</td>
</tr>
<tr>
<td>3</td>
<td>-180.0000°</td>
<td>-90°</td>
<td>F, 75 M</td>
</tr>
<tr>
<td>4</td>
<td>-215.2283°</td>
<td>-90°</td>
<td>F, 85 M</td>
</tr>
<tr>
<td>5</td>
<td>-233.2940°</td>
<td>-90°</td>
<td>F, 95 M</td>
</tr>
<tr>
<td>6</td>
<td>-251.5702°</td>
<td>-90°</td>
<td>F, 105 M</td>
</tr>
<tr>
<td>7</td>
<td>-270.0000°</td>
<td>-90°</td>
<td>F, 115 M</td>
</tr>
<tr>
<td>8</td>
<td>-288.3811°</td>
<td>-90°</td>
<td>F, 125 M</td>
</tr>
<tr>
<td>9</td>
<td>-306.6038°</td>
<td>-90°</td>
<td>F, 135 M</td>
</tr>
<tr>
<td>10</td>
<td>-324.5934°</td>
<td>-90°</td>
<td>F, 145 M</td>
</tr>
<tr>
<td>11</td>
<td>-342.4029°</td>
<td>-90°</td>
<td>F, 155 M</td>
</tr>
<tr>
<td>12</td>
<td>-360.0000°</td>
<td>-90°</td>
<td>F, 165 M</td>
</tr>
<tr>
<td>13</td>
<td>-377.5971°</td>
<td>-90°</td>
<td>F, 175 M</td>
</tr>
<tr>
<td>14</td>
<td>-395.4066°</td>
<td>-90°</td>
<td>F, 185 M</td>
</tr>
<tr>
<td>15</td>
<td>-413.222°</td>
<td>-90°</td>
<td>F, 195 M</td>
</tr>
<tr>
<td>16</td>
<td>-430.0000°</td>
<td>-90°</td>
<td>F, 205 M</td>
</tr>
<tr>
<td>17</td>
<td>-447.818°</td>
<td>-90°</td>
<td>F, 215 M</td>
</tr>
</tbody>
</table>

FIGURE 27
Two-Pole Tabulation Sheet
Similar types of tabulation and error forms are used to obtain the errors for a three-pole system. The burden of calculation may be reduced somewhat by noting that the difference terms for a specific calculation are duplicates of terms needed for the same $\delta$ calculation of a two-pole network. Figures 28 and 29 show typical, completed three-pole calculation and tabulation forms.

It is imperative that values of $10^x$ be determined very accurately. A table of common logarithms having at least seven places and furnishing direct computation of logarithms from 1 to 100,000 is a practical necessity for this work.9

The trigonometric tables used in ascertaining the values of the $\beta$'s should provide direct computation to the nearest hundredth of a degree and interpolation to the nearest thousandth. A suitable table of trigonometric functions may be found in the reference listed below.10

Tables I through XVI list the results of the $\beta$ calculations for the various values of $\delta$ for the two- and three-pole networks. Directly following the $\beta$ calculations are the tabulated error calculations for $\zeta$ equal to one for all values of $\delta$.

The data of Tables I through XVI are shown graphically in Figures 30 through 39. Figures 30 and 31 illustrate the variation of $\beta$ from an idealized phase curve (straight line) for two- and three-pole networks, respectively, for the various chosen values of $\delta$. In Figures 32 through 39 the error curves for the two- and three-pole systems are plotted for $\zeta$ equal to


Error Curve Calculation Sheet - 3 Pole Network

\[ \theta = 2.9 \quad s = 0.8 \quad \frac{s}{10} = 0.08 \quad s = 99.99 \quad s = \frac{10^{5s} - 1}{10^{5/2}} \]

(A) \[ (s)(0.08) - (10 - 0.16 - 10 - 1.6)(0 - 0.16 - 10 + 1.6) = \]
\[ (-99.99)(8.317637 \times 10^{-1})(6.9183096 - 25118285)(8.3176377 - 3.980716) = \]
\[ (-99.99)(8.317637 \times 10^{-1})(6.916489 \times 10^{-4})(-3.8978952 \times 10^{-1}) = \]
\[ -0.174110 \times 10^{-4} (A) \]

(B) \[ 2A = -4.3482220 \times 10^{-7} (B) \]

(C) \[ A^2 = +4.7267587 \times 10^{-6} (C) \]

(D) \[ (10 - 0.16 - 10 - 0.16 - 10 - 0)(0 - 0.16 - 10 - 1.6) = \]
\[ (6.9183096 - 6.3045759)(6.9183096 - 1.000000)(6.9183096 - 1.584936) = \]
\[ (6.911999 \times 10^{-7})(-3.0816905 \times 10^{-7})(-1.5842013 \times 10^{-7}) = +0.3374450 \times 10^{-3} (D) \]

(E) \[ C = \frac{4.7267587 \times 10^{-6}}{0.3374450 \times 10^{-3}} = +14.0074936 \times 10^{-3} (E) \]

(F) \[ D - E = 0.3374450 \times 10^{-3} - 14.0074936 \times 10^{-3} = -13.6700486 \times 10^{-3} (F) \]

(G) \[ B = \frac{-4.3482220 \times 10^{-7}}{-13.6700486 \times 10^{-3}} = +0.3180839 (G) \]

(H) \[ G = \tan \theta \quad \text{Acute angle for } \theta = 2.9 = +17.6450^\circ \]

[Diagram with \( \theta = 2.9 \) and a compass needle pointing right]

\[ \theta = 2.9 = -522.3550^\circ \]

Computations:

FIGURE 28

Three-Pole Calculation Sheet
#### Error Calculation Tabulation Sheet - 3 Pole Network

<table>
<thead>
<tr>
<th>Pole</th>
<th>$\Delta$</th>
<th>$n$</th>
<th>$\theta$</th>
<th>$\Delta$</th>
<th>$\theta$</th>
<th>Error</th>
<th>Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>1.5</td>
<td>360.0000</td>
<td>-90°</td>
<td>+0.2174</td>
</tr>
<tr>
<td>2</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>1.6</td>
<td>377.6894</td>
<td>-90°</td>
<td>-0.4898</td>
</tr>
<tr>
<td>3</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>1.7</td>
<td>395.4926</td>
<td>-90°</td>
<td>-0.9420</td>
</tr>
<tr>
<td>4</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>1.8</td>
<td>413.4912</td>
<td>-90°</td>
<td>-0.9834</td>
</tr>
<tr>
<td>5</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>1.9</td>
<td>431.6842</td>
<td>-90°</td>
<td>-0.5540</td>
</tr>
<tr>
<td>6</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>2.0</td>
<td>450.0000</td>
<td>-90°</td>
<td>0.0000</td>
</tr>
<tr>
<td>7</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>2.1</td>
<td>468.2763</td>
<td>-90°</td>
<td>+0.5869</td>
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<tr>
<td>8</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>2.2</td>
<td>486.5023</td>
<td>-90°</td>
<td>+1.0097</td>
</tr>
<tr>
<td>9</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>2.3</td>
<td>504.5107</td>
<td>-90°</td>
<td>+1.0195</td>
</tr>
<tr>
<td>10</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>2.4</td>
<td>522.3560</td>
<td>-90°</td>
<td>+0.6708</td>
</tr>
<tr>
<td>11</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>2.5</td>
<td>540.0000</td>
<td>-90°</td>
<td>0.0000</td>
</tr>
<tr>
<td>12</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>2.6</td>
<td>557.6865</td>
<td>-90°</td>
<td>-0.5698</td>
</tr>
<tr>
<td>13</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>2.7</td>
<td>575.4899</td>
<td>-90°</td>
<td>-1.0124</td>
</tr>
<tr>
<td>14</td>
<td>0.8</td>
<td>1.0</td>
<td>90°</td>
<td>2.8</td>
<td>593.2934</td>
<td>-90°</td>
<td>-1.0124</td>
</tr>
</tbody>
</table>

**FIGURE 29**

Three-Pole Tabulation Sheet
one. Also, the derived axis for optimum $\eta$ is shown in each figure. The optimum value of $\eta$ and the location of the optimum $\eta$ axis are determined by the methods presented in Section IV-3.

From the graphical data of Figures 32 through 39, sufficient information is obtained for the construction of the curves of Figures 21, 22 and 23 which were presented earlier as guides in the selection of circuit constants.
TABLES OF APPENDIX V
TABLE I

TWO-POLE NETWORK -- $\beta$ VALUES FOR $\epsilon = 0.4$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Angle in Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>92.0104</td>
</tr>
<tr>
<td>0.4</td>
<td>101.3888</td>
</tr>
<tr>
<td>0.5</td>
<td>111.6587</td>
</tr>
<tr>
<td>0.6</td>
<td>123.0338</td>
</tr>
<tr>
<td>0.7</td>
<td>135.5345</td>
</tr>
<tr>
<td>0.8</td>
<td>149.2080</td>
</tr>
<tr>
<td>0.9</td>
<td>166.5892</td>
</tr>
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<td>180.0000</td>
</tr>
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<td>196.9313</td>
</tr>
<tr>
<td>1.2</td>
<td>214.6136</td>
</tr>
<tr>
<td>1.3</td>
<td>232.8988</td>
</tr>
<tr>
<td>1.4</td>
<td>251.4284</td>
</tr>
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</tr>
<tr>
<td>1.6</td>
<td>288.4103</td>
</tr>
<tr>
<td>1.7</td>
<td>306.6478</td>
</tr>
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<td>1.8</td>
<td>324.6050</td>
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<td>342.3592</td>
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</tr>
<tr>
<td>2.1</td>
<td>377.6408</td>
</tr>
<tr>
<td>2.2</td>
<td>395.3950</td>
</tr>
<tr>
<td>2.3</td>
<td>413.3533</td>
</tr>
</tbody>
</table>
TABLE II

TWO-POLE NETWORK—β VALUES FOR $S = 0.6$

<table>
<thead>
<tr>
<th>β</th>
<th>Angle in Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>-81.7351</td>
</tr>
<tr>
<td>0.4</td>
<td>-92.6305</td>
</tr>
<tr>
<td>0.5</td>
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<td>0.6</td>
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<td>-131.9401</td>
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<td>0.8</td>
<td>-147.1192</td>
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<tr>
<td>0.9</td>
<td>-163.1840</td>
</tr>
<tr>
<td>1.0</td>
<td>-180.0000</td>
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<tr>
<td>1.1</td>
<td>-197.1155</td>
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<td>-215.2717</td>
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<td>-233.4128</td>
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</tr>
<tr>
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<td>-413.6495</td>
</tr>
</tbody>
</table>
TABLE III

TWO-POLE NETWORK—\( \beta \) VALUES FOR \( S = 0.8 \)

<table>
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<tr>
<th>( \beta )</th>
<th>Angle in Degrees</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-73.3173</td>
</tr>
<tr>
<td>0.4</td>
<td>-85.7215</td>
</tr>
<tr>
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<td>-99.3915</td>
</tr>
<tr>
<td>0.6</td>
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</tr>
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<td>-129.8044</td>
</tr>
<tr>
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<td>-146.0993</td>
</tr>
<tr>
<td>0.9</td>
<td>-162.9000</td>
</tr>
<tr>
<td>1.0</td>
<td>-180.0000</td>
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<tr>
<td>1.1</td>
<td>-197.4564</td>
</tr>
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<td>-233.2940</td>
</tr>
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<td>-251.5902</td>
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<tr>
<td>1.5</td>
<td>-270.0000</td>
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<td>-288.3811</td>
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<td>1.7</td>
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<td>-324.5934</td>
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<td>1.9</td>
<td>-342.4029</td>
</tr>
<tr>
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<td>-377.5971</td>
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<td>-395.1966</td>
</tr>
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<td>2.3</td>
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</tr>
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</table>
**TABLE IV**

**TWO-POLE NETWORK—— β VALUES FOR δ = 1.0**

<table>
<thead>
<tr>
<th>β</th>
<th>Angle in Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
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</tr>
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<tr>
<td>1.5</td>
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<td>1.6</td>
<td>-288.9562</td>
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<tr>
<td>2.3</td>
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</table>
### TABLE V

THREE-POLE NETWORK — $\beta$ VALUES FOR $\delta = 0.1$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Angle in Degrees</th>
</tr>
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<tbody>
<tr>
<td>0.3</td>
<td>-95.1113</td>
</tr>
<tr>
<td>0.4</td>
<td>-104.3649</td>
</tr>
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THREE-POLE NETWORK——β VALUES FOR $\delta = 0.8$

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### TABLE VIII

**THREE-POLE NETWORK — β VALUES FOR S = 1.0**

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### TABLE IX

TWO-POLE SYSTEM--ERROR DATA FOR $\delta = 0.4, \zeta = 1.0$

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TABLE X

TWO-POLE SYSTEM--ERROR DATA FOR $\delta = 0.6$, $\eta = 1.0$

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**Final Report, Project No. 142-67**

**TABLE XI**

**TWO-POLE SYSTEM—ERROR DATA FOR $s = 0.8$, $\tau = 1.0$**

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TABLE XII
TWO-POLE SYSTEM--ERROR DATA FOR $\varepsilon = 1.0, \zeta = 1.0$

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TABLE XIII

THREE-POLE SYSTEM--ERROR DATA FOR \( \delta = 0.4, \, \eta = 1 \)

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TABLE XIV

THREE-POLE SYSTEM--ERROR DATA FOR $\delta = 0.6$, $\zeta = 1$

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<td>+ 0.0454</td>
</tr>
<tr>
<td>2.15</td>
<td>- 0.0208</td>
</tr>
<tr>
<td>2.25</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.35</td>
<td>+ 0.0621</td>
</tr>
<tr>
<td>2.45</td>
<td>+ 0.1159</td>
</tr>
<tr>
<td>2.55</td>
<td>+ 0.1258</td>
</tr>
<tr>
<td>2.65</td>
<td>+ 0.0819</td>
</tr>
<tr>
<td>2.75</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.85</td>
<td>- 0.0863</td>
</tr>
<tr>
<td>2.95</td>
<td>- 0.1104</td>
</tr>
</tbody>
</table>
### TABLE XV

THREE-POLE SYSTEM—ERROR DATA FOR \( \delta = 0.8, \ \zeta = 1 \)

<table>
<thead>
<tr>
<th>Position, ( F )</th>
<th>Error in Degrees ( (\gamma) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.55</td>
<td>-17.3087</td>
</tr>
<tr>
<td>0.65</td>
<td>-13.0109</td>
</tr>
<tr>
<td>0.75</td>
<td>-9.6113</td>
</tr>
<tr>
<td>0.85</td>
<td>-6.9713</td>
</tr>
<tr>
<td>0.95</td>
<td>-4.8719</td>
</tr>
<tr>
<td>1.05</td>
<td>-3.1035</td>
</tr>
<tr>
<td>1.15</td>
<td>-1.5488</td>
</tr>
<tr>
<td>1.25</td>
<td>0.2174</td>
</tr>
<tr>
<td>1.35</td>
<td>0.7879</td>
</tr>
<tr>
<td>1.45</td>
<td>1.3321</td>
</tr>
<tr>
<td>1.55</td>
<td>1.3573</td>
</tr>
<tr>
<td>1.65</td>
<td>0.8558</td>
</tr>
<tr>
<td>1.75</td>
<td>0.2174</td>
</tr>
<tr>
<td>1.85</td>
<td>0.1898</td>
</tr>
<tr>
<td>1.95</td>
<td>0.9420</td>
</tr>
<tr>
<td>2.05</td>
<td>0.9834</td>
</tr>
<tr>
<td>2.15</td>
<td>0.5540</td>
</tr>
<tr>
<td>2.25</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.35</td>
<td>0.5869</td>
</tr>
<tr>
<td>2.45</td>
<td>1.0097</td>
</tr>
<tr>
<td>2.55</td>
<td>1.0195</td>
</tr>
<tr>
<td>2.65</td>
<td>0.6708</td>
</tr>
<tr>
<td>2.75</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.85</td>
<td>0.5698</td>
</tr>
<tr>
<td>2.95</td>
<td>1.0124</td>
</tr>
</tbody>
</table>
### TABLE XVI
THREE-POLE SYSTEM--ERROR DATA FOR $\varepsilon = 1.0, \eta = 1$

<table>
<thead>
<tr>
<th>Position, $F$</th>
<th>Error in Degrees ($\gamma$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.55</td>
<td>-10.2967</td>
</tr>
<tr>
<td>0.65</td>
<td>-7.3780</td>
</tr>
<tr>
<td>0.75</td>
<td>-5.9685</td>
</tr>
<tr>
<td>0.85</td>
<td>-5.4167</td>
</tr>
<tr>
<td>0.95</td>
<td>-4.9309</td>
</tr>
<tr>
<td>1.05</td>
<td>-3.9191</td>
</tr>
<tr>
<td>1.15</td>
<td>-2.1915</td>
</tr>
<tr>
<td>1.25</td>
<td>-0.0562</td>
</tr>
<tr>
<td>1.35</td>
<td>+1.9242</td>
</tr>
<tr>
<td>1.45</td>
<td>+2.9887</td>
</tr>
<tr>
<td>1.55</td>
<td>+3.1021</td>
</tr>
<tr>
<td>1.65</td>
<td>+1.9350</td>
</tr>
<tr>
<td>1.75</td>
<td>+0.0562</td>
</tr>
<tr>
<td>1.85</td>
<td>-1.8189</td>
</tr>
<tr>
<td>1.95</td>
<td>-2.8600</td>
</tr>
<tr>
<td>2.05</td>
<td>-2.9888</td>
</tr>
<tr>
<td>2.15</td>
<td>-1.8500</td>
</tr>
<tr>
<td>2.25</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.35</td>
<td>+1.8519</td>
</tr>
<tr>
<td>2.45</td>
<td>+2.9963</td>
</tr>
<tr>
<td>2.55</td>
<td>+2.9967</td>
</tr>
<tr>
<td>2.65</td>
<td>+1.9100</td>
</tr>
<tr>
<td>2.75</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.85</td>
<td>-1.8521</td>
</tr>
<tr>
<td>2.95</td>
<td>-2.9971</td>
</tr>
</tbody>
</table>
Variation of Two-Pole Network Phase Curves From Idealized Phase Curve For Various Values of $\delta$. 

Note: Curves are symmetrical about $F_2$ in sense indicated.
Variation Of Three-Pole Network Phase Curves From Idealized Phase Curve For Various Values Of $\delta$. 
Maximum Span For +1.86° Error As Based On Zero Axis For \( \varepsilon \) Equal To 0.9978. Span Equal To 1.65 units Of \( \delta \) Or, For \( \delta \) Equal To 0.4, 0.66 units Of \( \varepsilon \). Frequency Ratio Corresponding To This Span 10.66 : 10 Or 4.58 : 1.

FIGURE 32

Error Curve For Two-Pole System With \( \delta \) Equal To 0.4
Error Curve For Two-Pole System With $\delta$ Equal To 0.6
Maximum Span For \( \Delta \geq 1.37^\circ \) Error As Based On Zero Axis For \( \omega \) Equal To 0.999. Span Equal To 1.68 Units Of \( \delta \), Or, For \( \delta \) Equal To 0.8, 1.344 Units Of Frequency Ratio Corresponding To This Span 10\(^1\).344:10\(^0\) Or 22.6:1.

FIGURE 34
Error Curve For Two-Pole System With \( \delta \) Equal To 0.8
Maximum span for a 3.21° error as based on zero axis for \( \zeta \) equal to 0.99945. Span equal to 1.82 units of \( \zeta \), or, for \( \zeta \) equal to 1.0, 1.82 units of \( \zeta \). Frequency ratio corresponding to this span is 10^1.82:10^0 or 66.1:1.

**FIGURE 35**

Error curve for two-pole system with \( \zeta \) equal to 1.0
Maximum Span For $\pm 1.305^\circ$ Error As Based On Zero Axis For $\eta$ Equal To 0.9854. Span Equal To 3.32 Units Of $\delta$. Or, For $\delta$ Equal To 0.4, 1.328 Units Of $F$. Frequency Ratio Corresponding To This Span $10^3.328 \times 10^0$ Or 21.3:1.

FIGURE 36
Error Curve For Three-Pole System With $\delta$ Equal To 0.4
Maximum Span For \( \delta \): 1.20\% Error As Based On Zero Axis For

\( \delta \) Equal To 0.99956. Span Equal To 3.8 Units Of \( \delta \), Or,

For \( \delta \) Equal To One, 3.8 Units Of F. Frequency Ratio Corresponding To This Span \( 10^{3.8} = 10^{0} \) Or 6320:1.

FIGURE 39

Error Curve For Three-Pole System With \( \delta \) Equal To 1.0