FORCING LARGE TRANSITIVE SUBTOURNAMENTS

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H is an induced subgraph of G if

\[ V(H) \subseteq V(G) \]

and

\[ u, v \in E(H) \iff u, v \in V(H) \land u, v \in E(G) \]

\[ G \]

\[ \text{induced S.G. of } G \]

\[ \text{not an induced S.G. of } G \]

\[ 0 \]
Excluding induced subgraphs → Some Theorems

**Thm** (C., Robertson, Seymour, Thomas)

G is perfect iff no induced subgraph of G is a cycle of odd length > 5 or the complement of one

But no global structural results known / conjectured
THE ERDÖS-HAJNAL CONJ:

\[ \forall \text{ graph } H \exists \delta(H) > 0 \text{ s.t. for every graph } G, \text{ if } H \text{ is not contained in } G \text{ as an induced subgraph, then } G \text{ contains a clique or a stable set of size } \]

\[ \geq |V(G)|^{\delta(H)}. \]

In a random graph, expected clique/stable set size is \[ \log n \].
A graph $H$ has the Erdős-Hajnal property if $\exists \delta(H) > 0$ s.t. for every graph $G$, if $H$ is not contained in $G$ as an induced subgraph, then $G$ contains a clique or a stable set of size

$$\geq \left\lfloor \frac{|V(G)|}{\delta(H)} \right\rfloor$$

The EH Conjecture:

Every graph has the Erdős-Hajnal property.
KNOWN:

- Graphs on $\leq 4$ vertices have the Eh property

- The Bull (c.f. Safra) has the Eh property

IF $H_1$, $H_2$ have the Eh property, then the graphs obtained from them by substitution have the Eh property (Alon, Pach, Solymosi)
A tournament is a complete graph with directions on edges.

A transitive tournament is a tournament with no directed cycles.

$L(T)$ is the max size of a transitive tournament in $T$. 
THE ERDŐS-HAJNAL CONJ

∀ TOURNAMENT S ∈ E(S)
S. T. IF A TOURNAMENT T DOES
NOT CONTAIN S AS A
SUBTOURNAMENT, THEN
\( \chi(T) \geq |V(T)| ) \in E(S).

(ALON, PACH, SOLYMOSI)
A tournament $S$ has the Erdős-Hajnal property if

$\exists \mathcal{E}(S)$ s.t. if a tournament $T$ does not contain $S$ as a subtournament, then

$\mathcal{Z}(T) \geq (\bigcup \mathcal{U}(T)).^{\mathcal{E}(S)}$
Every tournament on \( \leq 4 \) vertices has the EH property.

**Thm.** (Berger, Choromanski, C.)

If \( S \) is a tournament s.t. \( S \setminus v \) is transitive for some \( v \in V(S) \), then \( S \) has the EH property.

An infinite class of tournaments with the EH property, not obtained by substitution.
S is a galaxy tournament if

3 ordering $v_1, \ldots, v_n$ of $V(S)$ s.t.

- The graph of backedges is the union of disjoint left stars and right stars

- $\exists \ i < j < k \ s.t.
  v_i, v_j, v_k$ are leaves of a star
  $v_j$ is a center of another star
THM (BERGER, CHORDOMANSKI, C.)

EVERY GALAXY TOURNAMENT HAS THE EH PROPERTY

PROOF: REGULARITY LEMMA + THE TRICK FROM THE LAST THEOREM
All \( 5 \)-vertex tournaments are galaxy tournaments, except

\[ C_5 \]

Thm (Choromanski)\[ C_5 \] has the EH property.
1. For which tournaments $S \ni c(S)$, s.t.
   $S \not\subset T \Rightarrow \Delta(T) \geq c(S) |V(T)|$?
   (Trivial for graphs: $|V(S)| \leq 2$)

2. For which tournaments $S \ni c(S)$, s.t.
   $S \not\subset T \Rightarrow \chi(T) \leq d(S)$?

**The Chromatic Number of a Tournament $T$ ($\chi(T)$)**

Is the min # of transitive tournaments needed to cover $V(T)$.
S is a **HERO** if \( \exists \mathcal{d}(S) \) s.t.
\[
S \notin T \implies \mathcal{x}(T) \leq \mathcal{d}(S).
\]

S is a **CELEBRITY** if \( \exists \mathcal{c}(S) \) s.t.
\[
S \notin T \implies \mathcal{z}(T) \geq \mathcal{c}(S) |V(D)|.
\]

Describe all heroes (celebrities)

*Berg, Choromanski, C., Fox, Lovel, Scott, Seymour, Thomasse*
THEM NOT ALL TOURNAMENTS ARE CELEBRITIES (HEROES).

THEM IF $S$ IS A CELEBRITY, THEN $V(S)$ CAN BE ORDERED S.T. THE BACK EDGES ARE A FOREST

PF SUPPOSE NOT.
LET $G$ BE A GRAPH WITH $d(G) < \frac{1}{2}|V(G)|$
AND GIRTH $(G) \geq |V(S)|$
CONSTRUCT $T$ BY ORDERING $V(G)$.
THEN $S \neq T$, AND $d(T) \leq 2d(G)$.
Corollary 1.6

If $G$ is a celebrity, then $\chi(G) \leq 2$.

Corollary

Is not a celebrity (and therefore not a hero)
Another property of Heroes

Thm S is a strongly conn. hero $\Rightarrow$

\[ V(S) = A \cup B \cup \{s\} \text{ s.t.} \]

PF Suppose not. Let $T$ be a tournament, $S \nsubseteq T$, $\chi(T)$ max. Then

\[ T' = \text{Diagram} \]

$S \nsubseteq T'$

$\chi(T') > \chi(T)$
Cor: $S$ is a strongly conn. hero $\Rightarrow$

$V(S) = A \cup B \cup \{s\}$ s.t.

one of $A, B$

transitive

PF

is not a hero
MAKING A HERO

THM 1: IF S IS A HERO, THEN S AND S ARE HEROES
Is this a hero?

\[ \begin{align*}
\text{THM (Loebl, Thomasse)} & \quad \text{YES}
\end{align*} \]
THM 2 CHAINING HEROES MAKES A HERO
THM 3 If $S$ is a hero, then $S + A$ handle is a hero.
THM:'

G is a hero iff G is obtained from \( S, T \) by

1) CHAINING

\[ S, T \sim S \neq T \]

AND

2) ADDING HANDLES

\[ \sim \rightarrow \]
Let $S$ be a $\xi$.

- If $S$ is not strongly connected, obtained by chaining.

- If $S$ is strongly connected, obtained by adding handles.
MINIMAL NON-HEROES
WHAT ABOUT CELEBRITIES?

THM EVERY HERO IS A CELEBRITY, AND EVERY CELEBRITY IS A HERO.

PROOF:

1. IF $G$ IS NOT STRONGLY CONNECTED, INDUCTION

2. $G$ IS OF THE FORM
IS NOT A CELEBRITY

A PROBABILISTIC CONSTRUCTION SHOWS THAT

IS NOT A CELEBRITY
S IS A PSEUDO-CELEBRITY IF

- For every $c > 0$ there exists $T$ s.t. $S \not \subseteq T$, and $d(T) < c|V(T)|$

- For every $0 < \varepsilon < 1$ there exists $c > 0$ s.t. for every $T$ s.t. $S \not \subseteq T$
  $d(T) > c|V(T)|^{1-\varepsilon}$

THM (CHOROMANSKI, C, SEYMOUR)

\[ \Delta(2, 2, 2) \]

IS A PSEUDO-CELEBRITY
THM (CHOROMANSKI, C, SEYMOUR)

$\Delta(z, k, k)$ is a PSEUDO-CELEBRITY
THM (CHOROMANSKI, C, SEYMOUR)

G is a pseudo-celebrity iff G is obtained from $\circ$ and $\Delta(2,k,k)$ by

1. CHAINING

$$\circ \sim T \not\Rightarrow \circ \not\Rightarrow T$$

AND

2. ADDING HANDLES

$$\sim$$
UNDIRECTED HEROES

F FAMILY OF GRAPHS

G is \( F \)-FREE if \( \forall \alpha \in \mathcal{F}, \alpha \not\in G \)

\( \mathcal{F} \) is a HERO if \( \exists d(\mathcal{F}) \) s.t.

Every \( \mathcal{F} \)-FREE \( G \) can be covered with \( d(\mathcal{F}) \) CLIQUES & STABLE SETS.

\( \mathcal{F} \) is a CELEBRITY if \( \exists c(\mathcal{F}) \) s.t.

\( \forall \mathcal{F} \)-FREE \( G \)

\( \chi(G) \geq c(\mathcal{F}) |V(G)| \)

or

\( \omega(\mathcal{F}) \geq c(\mathcal{F}) |V(G)| \)
DIFFICULT GRAPHS:

- Small cliques repeated
  \[ \sqrt{n} \]

- Stable set as subgraphs
  \[ \sqrt{n} \]

- LARGE GIRTH, NO LINEAR SIZE STABLE SET

- LARGE GIRTH IN THE COMPLEMENT, NO LINEAR SIZE CLIQUE
If \( \text{CELEBRITY} \Rightarrow \) 
\( \text{CONTAINS} \quad \text{A FOREST} \)
\( \text{A} \quad \text{(FOREST)}^c \)
\( k \underbrace{\leq \cdots \leq} k \)
\( S \leq S \cdots \leq \sigma \)
\begin{boxed}{red}\text{FULL}\end{boxed}

\text{CONJ 1} (c., Norin, Reed, Seymour).
\( \neg \text{HERO} \Rightarrow \neg \text{FULL} \)

\text{CONJ 2} (Gyarfas)
\( \forall \text{FOREST } T \exists f_T s.t. 
\ G \{T}\text{-FREE} \Rightarrow \chi(G) \leq f_T(w(G)) \)

\text{CONJ 1} \Rightarrow \text{CONJ 2}
THM (C., NORIN, REED, SEYMOUR)

\[ G \left\{ \begin{array}{c}
K \quad K \quad K \\
\begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ
\end{array}
\end{array} \right\} \quad b \quad c \quad d \quad e \quad f \quad g \quad h \quad i \quad j
\]

- FREE

\[ \Rightarrow \quad G \text{ is } k\text{-}SPLIT \]

\[ \begin{array}{c}
\circ \sim \circ \\
\circ \sim \circ \\
\circ \sim \circ
\end{array} \]

\[ 2 \leq k \quad w \leq k \]

so \quad \text{CONJ 2} \Rightarrow \text{CONJ 1}
THIS IS IT,
THANK YOU!