On Tractability of Minimum 0-Extension Problems

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Contents

- Minimum 0-extension problems (Karzanov 98)
- Known results and conjectures on tractability
- Main results
- Proof idea from *Discrete Convex Analysis* (Murota 96 ~)
- Concluding remarks

Notation

- **(semi) metric** \(d\) on \(X \iff d : X \times X \to \mathbb{R}_+\),

\[d(x, x) = 0, \quad d(x, y) = d(y, x), \quad d(x, z) + d(z, y) \geq d(x, y)\]

- For an undirected graph \(\Gamma = (V_\Gamma, E_\Gamma)\),

\[d_\Gamma \coloneqq \text{shortest path metric on } V_\Gamma\]
Minimum 0-Extension Problem (Karzanov 98)

$\Gamma$: simple connected undirected graph

Minimum 0-Extension Problem on $\Gamma$: $0$-$\text{Ext}[\Gamma]$

Given $X \supseteq V_\Gamma$ and $c : \binom{X}{2} \to \mathbb{Q}_+$,

minimize $\sum_{xy} c(xy)d(x, y)$

subject to $d$ is a metric on $X$,

$d$ extends $d_\Gamma$ (i.e., $d \Big|_{V_\Gamma} = d_\Gamma$),

$\forall x \in X, \exists s \in V_\Gamma, d(s, x) = 0$.

Such a metric is called a 0-extension of $d_\Gamma$.
Obs. $d$: 0-_extension of $d_\Gamma$
$\Leftrightarrow d(x, y) = d_\Gamma(\rho(x), \rho(y))$ for $\exists \rho : X \to V_\Gamma, \rho|_{V_\Gamma} = id$

0-Ext$: facility location form

Minimize $\sum_{xy} c(xy)d_\Gamma(\rho(x), \rho(y))$
subject to $\rho : X \to V_\Gamma, \rho|_{V_\Gamma} = id$.

Application: computer vision, clustering, learning theory, ...

c.f. Metric Labeling (Kleinberg-Tardos 98)
Obs. \( d \): 0-extension of \( d_\Gamma \)
\[ \Leftrightarrow d(x, y) = d_\Gamma(\rho(x), \rho(y)) \text{ for } \exists \rho : X \to V_\Gamma, \rho|_{V_\Gamma} = id \]

0-Ext[\( \Gamma \)]: facility location form

Minimize \( \sum_{xy} c(xy)d_\Gamma(\rho(x), \rho(y)) \)
subject to \( \rho : X \to V_\Gamma, \rho|_{V_\Gamma} = id. \)
Obs. $d$: 0-extension of $d_{\Gamma}$

$\iff d(x, y) = d_{\Gamma}(\rho(x), \rho(y))$ for $\exists \rho : X \to V_{\Gamma}, \rho|_{V_{\Gamma}} = id$

$0$-$\text{Ext}[\Gamma]$: facility location form

Minimize $\sum_{xy} c(xy) d_{\Gamma}(\rho(x), \rho(y))$

subject to $\rho : X \to V_{\Gamma}, \rho|_{V_{\Gamma}} = id$. 

$\Gamma = K_2$
Obs. $d$: 0-extension of $d_\Gamma$
\[
\Leftrightarrow d(x, y) = d_\Gamma(\rho(x), \rho(y)) \text{ for } \exists \rho : X \to V_\Gamma, \rho|_{V_\Gamma} = id
\]

0-Ext$[\Gamma]$: facility location form

Minimize \[
\sum_{xy} c(xy)d_\Gamma(\rho(x), \rho(y))
\]
subject to \[
\rho : X \to V_\Gamma, \rho|_{V_\Gamma} = id.
\]
Obs. $d$: 0-extension of $d_{\Gamma}$
$\iff d(x, y) = d_{\Gamma}(\rho(x), \rho(y))$ for $\exists \rho : X \to V_{\Gamma}, \rho|_{V_{\Gamma}} = id$

**0-Ext[$\Gamma$]: facility location form**

Minimize $\sum_{xy} c(xy) d_{\Gamma}(\rho(x), \rho(y))$

subject to $\rho : X \to V_{\Gamma}, \rho|_{V_{\Gamma}} = id$. 
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0-Ext[\Gamma]: facility location form

Minimize $\sum_{xy} c(xy) d_{\Gamma}(\rho(x), \rho(y))$

subject to $\rho : X \to V_{\Gamma}, \rho|_{V_{\Gamma}} = id.$
Tractability Question (Karzanov 98, 04)

\[ \Gamma = K_2 \Rightarrow \text{minimum cut} \Rightarrow P \]
\[ \Gamma = K_n \Rightarrow \text{multiway cut} \Rightarrow \text{NP-hard} \quad (n \geq 3) \]

**Question**

*What is \( \Gamma \) for which \( 0\text{-Ext}[\Gamma] \) is in \( P \)?*
A classical result in location theory

Theorem (Picard-Ratliff 78)

If $\Gamma$ is a tree, then $\text{0-Ext}[\Gamma]$ is in $P$. 

\begin{align*}
\Gamma \\
\Gamma' \\
\Gamma \times \Gamma'
\end{align*}
A classical result in location theory

Theorem (Picard-Ratliff 78)

If $\Gamma$ is a tree, then $\mathbf{0}$-Ext[$\Gamma$] is in P.

Obs. $\Gamma \in \mathbb{P}$ and $\Gamma' \in \mathbb{P} \Rightarrow \Gamma \times \Gamma' \in \mathbb{P}$

($\times :=$ Cartesian product)
Median graph

- Median of $x_1, x_2, x_3 \iff v \in V_{\Gamma}$ satisfying
  \[ d_{\Gamma}(x_i, x_j) = d_{\Gamma}(x_i, v) + d_{\Gamma}(v, x_j) \quad (1 \leq i < j \leq 3) \]

- Median graph $\iff$ $\forall$ triple has a unique median.
Median of $x_1, x_2, x_3 \Leftrightarrow v \in V_\Gamma$ satisfying
\[
d_\Gamma(x_i, x_j) = d_\Gamma(x_i, v) + d_\Gamma(v, x_j) \quad (1 \leq i < j \leq 3)
\]
Median graph $\Leftrightarrow \forall$ triple has a unique median.
Median of $x_1, x_2, x_3 \iff v \in V_{\Gamma}$ satisfying

$$d_\Gamma(x_i, x_j) = d_\Gamma(x_i, v) + d_\Gamma(v, x_j) \quad (1 \leq i < j \leq 3)$$

Median graph $\iff \forall$ triple has a unique median.

Theorem (Chepoi 96)

If $\Gamma$ is a median graph, then $0$-$\text{Ext}[\Gamma]$ is in $\mathbb{P}$. 
**0-Ext**[\(\Gamma\)] : Minimize \(\sum_{xy} c(xy)d(x, y)\)

subject to

\(d\) is a metric on \(X\),

\(d\) extends \(d_\Gamma\) (i.e., \(d|_{V_\Gamma} = d_\Gamma\)),

\(\forall x \in X, \exists s \in V_\Gamma, d(s, x) = 0.\)
Metric relaxation (Karzanov 98)

\[ \text{0-Ext}[\Gamma] : \text{ Minimize } \sum_{xy} c(xy)d(x, y) \]
subject to \(d \) is a metric on \(X\),
\(d\) extends \(d_\Gamma\) (i.e., \(d|_{V_\Gamma} = d_\Gamma\)),
\(\forall x \in X, \exists s \in V_\Gamma, d(s, x) = 0.\)

Q. What is \(\Gamma\) for which \(\text{Ext}[\Gamma]\) is exact (for every \(X, c\))?

c.f. Multiflows & tight spans (Karzanov 98, H. 09~)
c.f. \(O(\log n)\)-integrality gap (Calinescu-Karloff-Rabani 04)
Frame = graph for which $\text{Ext}[\Gamma]$ is exact

$\Gamma$: frame $\iff$

- bipartite
- no isometric cycle of length $> 4$
- orientable $\iff \exists$ orientation: $\forall$ 4-cycle

Rem: frame is obtained by gluing $K_{2,m}$ and $K_2$ (in a certain way)
Frame = graph for which $\text{Ext}[\Gamma]$ is exact

$\Gamma$: frame $\iff$

- bipartite
- no isometric cycle of length $> 4$
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Rem: frame is obtained by gluing $K_{2,m}$ and $K_2$ (in a certain way)
Frame: a graph for which $\text{Ext}[\Gamma]$ is exact.

\(\Gamma\): frame \iff
- bipartite
- no isometric cycle of length \(\geq 4\)
- orientable \iff \exists\ orientation: \(\forall\ 4\)-cycle

**Theorem (Karzanov 98)**
\(\Gamma\) is a frame if and only if \(\text{Ext}[\Gamma] = 0\text{-Ext}[\Gamma]\).

**Corollary (Karzanov 98)**
If \(\Gamma\) is a frame, then \(0\text{-Ext}[\Gamma]\) is in \(\mathbb{P}\).
Rem. \{ frames \} is not closed under Cartesian product
Rem. frame \neq \text{median graph}

- \( \Gamma: \text{modular} \iff \forall \text{ triple has a median.} \)
- \( \Gamma: \text{orientable} \iff \exists \text{ orientation: } \forall \text{ 4-cycle} \)

Rem. frame = orientable \text{ hereditary modular graph (c.f. Bandelt 85)}
Rem. \{frames\} is not closed under Cartesian product
Rem. frame \neq \text{median graph}

- $\Gamma$: modular $\iff$ \forall triple has a median.
- $\Gamma$: orientable $\iff$ \exists orientation: \forall 4$-cycle

Rem. frame = orientable \text{ hereditary} modular graph (c.f. Bandelt 85)

\begin{itemize}
  \item \textbf{Theorem (Karzanov 98)}
  \item If $\Gamma$ is not modular or not orientable, then $0$-Ext[$\Gamma$] is NP-hard.
\end{itemize}
NP-hard

orientable modular

? 

median graph

P
frame

P

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On Tractability of Minimum 0-Extension Problems
Orbit & Orbit graph of $\Gamma$

- *Orbit* $Q \Leftrightarrow$ an equivalence class on $E_{\Gamma}$ of transitive closure of $e \sim e' \Leftrightarrow \exists$ 4-cycle

- *Orbit graph* $\Gamma_Q \Leftrightarrow$ graph obtained by contracting $E_{\Gamma} \setminus Q$. 

\[ e \quad e' \]
Orbit & Orbit graph of $\Gamma$

- **Orbit Q** $\Leftrightarrow$ an equivalence class on $E_\Gamma$ of transitive closure of $e \sim e' \Leftrightarrow \exists$ 4-cycle

- **Orbit graph $\Gamma_Q$** $\Leftrightarrow$ graph obtained by contracting $E_\Gamma \setminus Q$. 

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On Tractability of Minimum 0-Extension Problems
Orbit $Q$ $\iff$ an equivalence class on $E_\Gamma$ of transitive closure of $e \sim e' \iff \exists$ 4-cycle

Orbit graph $\Gamma_Q$ $\iff$ graph obtained by contracting $E_\Gamma \setminus Q$. 
Theorem (Karzanov 04)

If \( \Gamma \) is (orientable) modular s.t. \( \forall \) orbit \( Q \):
\[
\Gamma_Q = \text{frame}, \text{ and } \Gamma_Q = \text{a subgraph of } (\mathcal{V}_\Gamma, Q),
\]
then \( 0\text{-Ext}[\Gamma] \) is in \( \mathbf{P} \).

Rem: median graph = modular graph s.t. \( \forall \) orbit \( Q \): \( \Gamma_Q = K_2 \)
(c.f. Mulder-Schrijver 79)
**Theorem (Karzanov 04)**

If \( \Gamma \) is (orientable) modular s.t. \( \forall \) orbit \( Q \):

\[ \Gamma_Q = \text{frame}, \text{ and } \Gamma_Q = \text{a subgraph of } (V_\Gamma, Q), \]

then \( 0\text{-Ext}[\Gamma] \) is in \( P \).

**Rem:** median graph = modular graph s.t. \( \forall \) orbit \( Q \): \( \Gamma_Q = K_2 \)

(c.f. Mulder-Schrijver 79)

**Conjecture 1 (Karzanov 04)**

If \( \Gamma \) is (orientable) modular s.t. \( \forall \) orbit \( Q : \Gamma_Q = \text{frame}, \)
then \( 0\text{-Ext}[\Gamma] \) is in \( P \).

**Conjecture 2 (Karzanov 04)**

If \( \Gamma \) is orientable modular s.t. \( \exists \) orbit \( Q : \Gamma_Q \neq \text{frame}, \)
then \( 0\text{-Ext}[\Gamma] \) is \( \text{NP-hard} \).
Conjecture 1 (Karzanov 04)

If $\mathcal{I}$ is orientable modular s.t. $\forall$ orbit $Q : \mathcal{I}_Q =$ frame, then $\mathbf{0-Ext}[\mathcal{I}]$ is in $\mathbf{P}$.
Our main results

Theorem 1 (H. 12)
If \( \Gamma \) is orientable modular s.t. \( \forall \) orbit \( Q : \Gamma_Q = \text{frame} \), then \( 0\text{-Ext}[\Gamma] \) is in \( P \).

Conjecture 2 (Karzanov 04)
If \( \Gamma \) is orientable modular s.t. \( \exists \) orbit \( Q : \Gamma_Q \neq \text{frame} \), then \( 0\text{-Ext}[\Gamma] \) is \( NP \)-hard.

Theorem 2 (H. 12)
There is \( s.t. \exists \) orbit \( Q : \Gamma_Q \neq \text{frame} \) and \( 0\text{-Ext}[\Gamma] \) has good characterization (i.e., optimality check of \( 0\text{-Ext}[\Gamma] \) is in \( NP \) \( \setminus \) \( co-NP \)).
Our main results

**Theorem 1 (H. 12)**

If $\Gamma$ is orientable modular s.t. $\forall$ orbit $Q : \Gamma_Q = \text{frame}$, then $0\text{-Ext}[\Gamma]$ is in $P$.

**Conjecture 2 (Karzanov 04)**

If $\Gamma$ is orientable modular s.t. $\exists$ orbit $Q : \Gamma_Q \neq \text{frame}$, then $0\text{-Ext}[\Gamma]$ is NP-hard.
**Our main results**

**Theorem 1 (H. 12)**

If \( \Gamma \) is orientable modular s.t. \( \forall \) orbit \( Q : \Gamma_Q = \text{frame} \), then \( \textbf{0-Ext}[^\Gamma] \) is in \( \text{P} \).

**Conjecture 2 (Karzanov 04)**

If \( \Gamma \) is orientable modular s.t. \( \exists \) orbit \( Q : \Gamma_Q \neq \text{frame} \), then \( \textbf{0-Ext}[^\Gamma] \) is \( \text{NP-hard} \).

**Theorem 2 (H. 12)**

There is \( \Gamma \) s.t. \( \exists \) orbit \( Q : \Gamma_Q \neq \text{frame} \) and \( \textbf{0-Ext}[^\Gamma] \) has \text{good characterization}.  

(i.e., optimality check of \( \textbf{0-Ext}[^\Gamma] \) is in \( \text{NP} \cap \text{co-NP} \))
Proof idea (of Thm 2) from *Discrete Convex Analysis*

- Discrete Convex Analysis (Murota 96 ~)
  \(\simeq\) A theory of “convex” functions on \(\mathbb{Z}^n\) for well-solvable combinatorial optimization problems including *network flows, matroids, and submodular functions.*

- *Our approach suggests “Discrete Convex Analysis on \(\Gamma\)”*
Obs. \( \text{0-Ext}[\Gamma] \simeq \)

\[
\text{Min. } \sum c_{ij}d_\Gamma(x_i, x_j) \text{ s.t. } (x_1, x_2, \ldots, x_n) \in V_{\Gamma \times \Gamma \times \cdots \times \Gamma}.
\]

Obs. If \( \Gamma = \text{path} \), then \( \Gamma \times \Gamma \times \cdots \times \Gamma \simeq \text{box } B \text{ in } \mathbb{Z}^n \),

\[
\text{Min. } \sum c_{ij}|x_i - x_j| \text{ s.t. } x \in B \subseteq \mathbb{Z}^n.
\]

\( \rightarrow \) \text{This is L-convex function minimization in DCA}
Obs. **0-Ext**[$\Gamma$] $\simeq$

\[
\text{Min. } \sum c_{ij}d_{\Gamma}(x_i, x_j) \text{ s.t. } (x_1, x_2, \ldots, x_n) \in V_{\Gamma \times \Gamma \times \cdots \times \Gamma}.
\]

Obs. If $\Gamma$ = path, then $\Gamma \times \Gamma \times \cdots \times \Gamma \simeq \text{box } B \text{ in } \mathbb{Z}^n$,

\[
\text{Min. } \sum c_{ij}|x_i - x_j| \text{ s.t. } x \in B \subseteq \mathbb{Z}^n.
\]

$\rightarrow$ *This is L-convex function minimization in DCA*

**Fact.** L-convex functions have many nice properties:
- *Local* optimality $\Rightarrow$ *Global* optimality
- Checking Local optimality $\simeq$ Submodular Function Minimization
- Descend algorithm by successive SFM.
We define "$L$-convex functions" on orientable modular graphs:

- Local optimality $\Rightarrow$ Global optimality
- Checking local optimality $\simeq$ "Submodular" Function Minimization on modular semilattice (? $\in P$ or $\notin P$ ?)
- Descend algorithm by successive SFM.
- $0$-Ext$[\Gamma]$ is "$L$-convex function minimization" on $\Gamma \times \cdots \times \Gamma$

C.f. Modular semilattice (Bandelt-Van De Vel-Verheul 93)
We define “$L$-convex functions” on orientable modular graphs:
- $Local$ optimality $\Rightarrow$ $Global$ optimality
- Checking local optimality $\simeq$ “Submodular” Function Minimization on $modular$ $semilattice$ ($? \in P$ or $\not\in P$ ?)
- Descend algorithm by successive SFM.
- $0$-$\text{Ext}[\Gamma]$ is “$L$-convex function minimization” on $\Gamma \times \cdots \times \Gamma$

c.f. Modular semilattice (Bandelt-Van De Vel-Verheul 93)

There is $\Gamma$ s.t. $\exists Q : \Gamma_Q \neq$ frame and optimality checking $\simeq$ SFM on

\[ \times \quad \times \quad \times \cdots \times \]
We define "L-convex functions" on orientable modular graphs:
- Local optimality $\Rightarrow$ Global optimality
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- Descend algorithm by successive SFM.
- $0\text{-Ext}[\Gamma]$ is “L-convex function minimization” on $\Gamma \times \cdots \times \Gamma$

c.f. Modular semilattice (Bandelt-Van De Vel-Verheul 93)

There is $\Gamma$ s.t. $\exists Q : \Gamma_Q \neq$ frame and optimality checking $\simeq$ SFM on

Theorem (Kuivinen 09)
SFM on product of diamonds has good characterization
How to define $L$-convex function $g$ on $\Gamma, o$

- Modular complex $\Delta(\Gamma, o)$ of $(\Gamma, o)$

- Lovász extension $\bar{g} : \Delta(\Gamma, o) \to \mathbb{R}_+$ of $g : V_\Gamma \to \mathbb{R}$.

- Neighborhood semilattice (← modular semilattice)
Summary

Question

What is $\Gamma$ for which $0\text{-Ext}[\Gamma]$ is in $P$?

NP-hard

orientable modular

$?$

good char.

each orbit graph = frame

median graph

frame

$P$

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Summary

Question

What is $\Gamma$ for which $0\text{-Ext}[\Gamma]$ is in $P$?

Conjecture

If $\Gamma$ is orientable modular, then $0\text{-Ext}[\Gamma]$ is in $P$. 

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On Tractability of Minimum 0-Extension Problems
Many other interesting aspects:

- Combinatorial min-max theorems in multiflows
- Tight spans of metric spaces
- Modular lattices, and modular semilattices
- Connection to CAT(0)-complexes
- Toward *Discrete Convex Analysis* for well-solvable 0-extension problems and related multiflows ...
Many other interesting aspects:

- Combinatorial min-max theorems in multiflows
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*Thank you for your attention!*