A refinement of the Corradi-Hajnal Theorem

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joint work with H. Kierstead and E. Yeager

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Corrádi–Hajnal Theorem

Theorem 1 [Corrádi and Hajnal, 1963]: Let \( k \geq 1, n \geq 3k \) and let \( H \) be an \( n \)-vertex graph with \( \delta(H) \geq 2k \). Then \( H \) contains \( k \) vertex-disjoint cycles.

Corollary 2 [Corrádi and Hajnal]: Let \( n = 3k \) and let \( H \) be an \( n \)-vertex graph with \( \delta(H) \geq 2k \). Then \( H \) contains \( k \) vertex-disjoint triangles.

Both bounds are sharp
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Examples

Figure: Graphs with mindegree 5 with no 3 disjoint cycles.
Refinements

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**Theorem 3 [Enomoto, Wang]:** Let $k \geq 1$, $n \geq 3k$ and let $H$ be an $n$-vertex graph with $\Theta(H) \geq 4k - 1$. Then $H$ contains $k$ vertex-disjoint cycles.

**Theorem 4 [Aigner and Brandt, Alon and Fisher]:** Let $n \geq 3$ and $H$ be an $n$-vertex graph with $\delta(H) \geq 2n/3$. Then $H$ contains each 2-factor.

**Theorem 5 [A.K. and Yu]:** Let $n \geq 3$ and $H$ be an $n$-vertex graph with $\Theta(H) \geq 4n/3 - 1$. Then $H$ contains each 2-factor.

**Theorem 6 [Fan and Kierstead]:** Let $n \geq 3$ and $H$ be an $n$-vertex graph with $\delta(H) \geq 2n - 1/3$. Then $H$ contains the square of the $n$-vertex path.
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**Theorem 6 [Fan and Kierstead]:** Let $n \geq 3$ and $H$ be an $n$-vertex graph with $\delta(H) \geq \frac{2n-1}{3}$. Then $H$ contains the square of the $n$-vertex path.
Theorem 7 [Kierstead, A.K., and Yeager]: Let $k \geq 1$, $n \geq 3k + 1$ and let $H$ be an $n$-vertex graph with $\delta(H) \geq 2k - 1$. Then either $H$ contains $k$ vertex-disjoint cycles or $\alpha(G) = n - 2k + 1$. 
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Definitions

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**APPLICATIONS:**
1. Scheduling, partitioning, and load balancing problems.
2. Deviation bounds for sums of random variables with limited dependence [Alon-Füredi, Janson-Ruciński, Pemmaraju].
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Figure: An equitable 4-coloring of \( K_{7,7} \).
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Figure: An equitable 4-coloring of $K_{7,7}$.

To decide whether a graph has an equitable $k$-coloring is $NP$-complete even for $k = 3$. 
The difficult case

If $G$ has an equitable $k$-coloring, then $k \geq \omega(G)$. So, if $|V(G)| = n = ks - t$, where $t < k$, let $G^+ := G + K_t$.
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Figure: $G$ and $G^+$. 
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**Corollary 2′ [Corrádi and Hajnal]:** Let $n = 3k$ and $H$ be an $n$-vertex graph with $\Delta(H) \leq k - 1$. Then $H$ has an equitable $k$-coloring.
Hajnal–Szemerédi Theorem

Theorem 8 [Hajnal and Szemerédi]: If $\Delta(G) \leq r$, then $G$ is equitably $(r + 1)$-colorable.
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Figure: A graph $G$ with $\Delta(G) = 7$ and no equitable 7-coloring.
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3. Theorem 10 [H.K. and A.K.]: If $d(x) + d(y) \leq 2r + 1$ for every $xy \in E(G)$, then $G$ is equitably $(r + 1)$-colorable.

4. Question [H.K. and A.K.]: Is there a polynomial-time algorithm for an equitable $(r + 1)$-coloring of any $n$-vertex $G$ with $d(x) + d(y) \leq 2r + 1$ for every $xy \in E(G)$?
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Conjecture 1 [Chen, Lih and Wu]: Let $G$ be a connected graph with $\Delta(G) \leq r$. Then $G$ has no equitable $r$-coloring if and only if either (1) $G = K_{r+1}$, or (2) $r = 2$ and $G$ is an odd cycle, or (3) $r$ is odd and $G = K_{r,r}$. 
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Conjecture 1 was proved
1) For $r \leq 3$ [Chen-Lih-Wu],
2) For bipartite graphs [Lih-Wu],
3) For interval graphs [Chen-Lih-Yan],
4) For split graphs [Chen-Ko-Lih],
5) For outerplanar graphs [Yap-Zhang],
6) For planar graphs $G$ with $\Delta(G) \geq 13$ [Yap-Zhang],
7) For planar graphs $G$ with $\Delta(G) \geq 9$ [Nakprasit],
8) For graphs $G$ with $avdeg(G) \leq \Delta(G)/5$ [Kostochka-Nakprasit].
For odd $r \geq 3$, Conjecture 1 does not describe disconnected graphs with max.\,deg $r$ that are not equitably $r$-colorable. For example, for an odd $r$, $K_{r,r} \cup K_{r,r}$ is equitably $r$-colorable, but $K_{r,r} \cup K_r$ is not.
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**Observation 1:** If $r$ is odd and $G$ is the disjoint union of $K_{r,r}$ and an $r$-equitable graph, then $G$ has no equitable $r$-coloring.
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**Observation 2:** If a spanning subgraph of an $r$-colorable $G$ is the disjoint union of $r$-equitable graphs, then $G$ is $r$-equitable.
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Clearly, a graph $G$ can be $r$-equitable only for one $r$. Call $G$ equitable if it is $r$-equitable for some $r$. 
Basic equitable graphs

\begin{align*}
\mathcal{F}_1 & \quad \mathcal{F}_2 \\
\mathcal{F}_3 & \quad \mathcal{F}_4
\end{align*}
More basic equitable graphs
Decompositions

Together with $K_r$, the $r$-equitable graphs above are the $r$-basic graphs.
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**Conjecture 2 [H.K. and A.K.]:** Suppose that $r \geq 3$ and $G$ is an $r$-colorable graph with $\Delta(G) = r$. Then $G$ has no equitable $r$-coloring if and only if $r$ is odd and there exists $H \subseteq G$ such that $H = K_{r,r}$ and $G - H$ has an $r$-decomposition.
Theorem 11 [H.K. and A.K.]: Let $r \geq 3$. Let $G$ be an $r$-colorable graph with $\Delta(G) = r$ and $|V(G)|$ divisible by $r$. Then the following are equivalent:

(A) $G$ has an $r$-decomposition;
(B) $G$ is $r$-equitable;
(C) $G$ has an equitable $r$-coloring but does not have a nearly equitable $r$-coloring.

Corollary 12: For all positive integers $r$ and $n > r$, Conjecture 1 holds for all $r$-colorable graphs $G$ with $\Delta(G) \leq r$ and at most $n$ vertices if and only if Conjecture 2 holds for all such graphs.

Corollary 13: Conjecture 2 holds for $r = 3$. 
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**Theorem 14 [H.K. and A.K.]:** Conjecture 2 holds for $r \leq 4$.

A refinement of Corollary 2′ above by Corrádi and Hajnal is

**Theorem 15 [H.K. and A.K.]:** If $n \leq 4r$, then Conjecture 2 holds for all $n$-vertex graphs $G$ with $\Delta(G) = r$. 
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Theorem 15 [H.K. and A.K.]: If $n \leq 4r$, then Conjecture 2 holds for all $n$-vertex graphs $G$ with $\Delta(G) = r$.

Together, Theorems 7 and 15 refine Corrádi -Hajnal Theorem as follows.

Theorem 16 [H.K., A.K. and Yeager]: Let $k \geq 1$, $n \geq 3k$ and let $H$ be an $n$-vertex graph with $\delta(H) \geq 2k - 1$. Then either $H$ contains $k$ vertex-disjoint cycles or $\alpha(G) = n - 2k + 1$, or $n = 3k$, $k$ is odd and $G$ is the complement of $K_{k,k} \cup K_k$. 