Computing Linkless and Flat Embeddings of Graphs in $\mathbb{R}^3$

Stephan Kreutzer
Technical University Berlin

based on joint work with
Ken-ichi Kawarabayashi, Bojan Mohar and Bruce Reed

Graph Theory @ Georgie Tech
Symposium in Honour of Robin Thomas
7 - 11 May 2012
**Drawing Graphs Nicely**

*Graph drawing.* The motivation for this work comes from graph drawing: we would like to draw a graph in a way that it is nicely represented.

*Drawing in two dimensions.* Usually we draw graphs on the plane or other surfaces.

*Planar graphs.* Draw a graph on the plane such that no edges cross. Plane embeddings of planar graphs can be computed in linear time.

![Graph](image-url)
Kuratowski Graphs

Of course not every graph is planar. The graphs $K_{3,3}$ and $K_5$ are examples of non-planar graphs.

**Theorem.** (Kuratowski)

Every non-planar graph contains a sub-division of $K_5$ or $K_{3,3}$ as sub-graph.

**Definition.**

- Kuratowski graph: subdivision of $K_5$ or $K_{3,3}$.
- $H$ Kuratowski-subgraph of $G$ if $H \subseteq G$ is isomorphic to a subdivision of $K_{3,3}$ or $K_5$
Drawings in $\mathbb{R}^3$

Non-planar graphs. One option is to draw them on surfaces of higher genus. Again such embeddings (if exist) can be computed in linear time (Mohar).

Drawings in $\mathbb{R}^3$. We are interested in drawing graphs nicely in $\mathbb{R}^3$.

Clearly, every graph can be drawn in $\mathbb{R}^3$ without crossing edges.

To define nice drawings one wants to generalise the property of planar graphs that disjoint cycles are not intertwined, i.e. that of any two disjoint cycles one can be contracted into a single point without cutting the other.
**Linklessly embeddable graphs**

**Definition.**

Let $C_1, C_2$ be two disjoint cycles embedded in $\mathbb{R}^3$.

$C_1, C_2$ are **un-linked** if there is a two-dimensional topological disk in $\mathbb{R}^3$ that contains one cycle but not the other.

$C_1, C_2$ are **linked** if they are not unlinked.

A graph $G$ is **linklessly embeddable** if there is an embedding into $\mathbb{R}^3$ so that no two disjoint cycles in $G$ are linked.
**Flat Embeddings of Graphs**

We want a slightly different form of embeddings called flat.

**Definition.** An embedding of a graph $G$ is flat if for every cycle in $G$ there exists a closed topological disk $D$ such that $D \cap G = \partial D = C$.

Clearly, every flat embedding is linkless but the converse is false.

**Theorem.** (Robertson, Seymour, Thomas)

A graph is linklessly embeddable if, and only if, it has a flat embedding.
Observation. Not every graph is linklessly embeddable, e.g. $K_6$.

Observation. The class of linklessly embeddable graphs is closed under taking minors.

Contracting an edge does not create a link where there was none before.

Consequence. There is a characterisation of linklessly embeddable graphs by a finite set of excluded minors.
To define the minors explicitly we need to define the following transformation.

\[ \Delta - Y \text{ Transformations.} \]

\[ H \text{ and } G \text{ are } Y\Delta\text{-equivalent if } H \text{ can be obtained from } G \text{ by a sequence of } Y\Delta\text{-transformations.} \]

**Theorem.**

If \( H \) and \( G \) are \( Y\Delta\)-equivalent then \( G \) is linklessly embeddable iff \( H \) is.
**Petersen Family**

**Definition.** The Petersen family is the family of 7 graphs obtained from $K_6$ by $Y\Delta$-transformations.
**Theorem.** (Robertson, Seymour, Thomas)

A graph is linklessly embeddable if, and only if, it does not contain any minor of the Petersen family of graphs.

\( \rightsquigarrow O(n^3) \) algorithm for testing whether a graph is linklessly embeddable.

But it does not yield an algorithm for constructing a flat embedding.

**Theorem.** (van der Holst)

There is a polynomial time \((O(n^5))\) algorithm which, given a graph \(G\) either computes a linkless embedding or witnesses that there is none.
Excluded Minors for Linklessly Embeddable Graphs

**Theorem.** (Robertson, Seymour, Thomas)

A graph is linklessly embeddable if, and only if, it does not contain any minor of the Petersen family of graphs.

$\Rightarrow O(n^3)$ algorithm for testing whether a graph is linklessly embeddable.

But it does not yield an algorithm for constructing a flat embedding.

**Theorem.** (van der Holst)

There is a polynomial time ($O(n^5)$) algorithm which, given a graph $G$ either computes a linkless embedding or witnesses that there is none.
Computing Flat Embeddings

**Theorem.** (Kawarabayashi, K., Mohar 2010)
There is an $O(n^2)$ algorithm which, given a graph $G$, decides whether $G$ is linklessly embeddable and if so constructs a flat embedding.

We can strengthen this result to linear time.

**Theorem.** (Kawarabayashi, K., Mohar, Reed (unpubl) 2012)
There is an $O(n)$ algorithm which, given a graph $G$, decides whether $G$ is linklessly embeddable and if so constructs a flat embedding.

This matches the running time for computing embeddings into surfaces.

**In this talk.** Present some of the ideas for the $O(n^2)$-algorithm.
Computing Flat Embeddings

**Theorem.** (Kawarabayashi, K., Mohar 2010)

There is an $O(n^2)$ algorithm which, given a graph $G$, decides whether $G$ is linklessly embeddable and if so constructs a flat embedding.

We can strengthen this result to linear time.

**Theorem.** (Kawarabayashi, K., Mohar, Reed (unpubl) 2012)

There is an $O(n)$ algorithm which, given a graph $G$, decides whether $G$ is linklessly embeddable and if so constructs a flat embedding.

This matches the running time for computing embeddings into surfaces.

**In this talk.** Present some of the ideas for the $O(n^2)$-algorithm.
**An $O(n^2)$ Algorithm for Computing Flat Embeddings**

**Theorem.** There is an $O(n^2)$ algorithm which, given a graph $G$, decides whether $G$ is linklessly embeddable and if so constructs a flat embedding.

**High level proof idea.** Analyse the tree-width of $G$.

If the tree-width is high:

1. Convert to graph of min. degree 4 using $Y\Delta$-transformations

2. By the weak structure theorem for $K_6$-free graphs we get
   2.1 A Petersen-family minor
   Output “not linklessly embeddable”

2.2 One of two specific forms of separations of order at most 4 embed the two parts individually and combine the embeddings.

2.3 An irrelevant vertex, more precisely, an apex $x \in V(G)$ and a cross-free and dividing wall $W$ in $G - x$ and an irrelevant vertex $v \in V(W)$
   - compute embedding of $G - v$
   - put the vertex $v$ back in.

Case 2.2 and 2.3 yield smaller graphs $\Rightarrow$ small tree-width
An $\mathcal{O}(n^2)$ Algorithm for Computing Flat Embeddings

**Theorem.** There is an $\mathcal{O}(n^2)$ algorithm which, given a graph $G$, decides whether $G$ is linklessly embeddable and if so constructs a flat embedding.

**High level proof idea.** Analyse the tree-width of $G$.

If the tree-width is high:

1. Convert to graph of min. degree 4 using $Y\Delta$-transformations
2. By the weak structure theorem for $K_6$-free graphs we get
   2.1 A Petersen-family minor
      Output “not linklessly embeddable”
   2.2 One of two specific forms of separations of order at most 4
      embed the two parts individually and combine the embeddings.
   2.3 An irrelevant vertex, more precisely, an apex $x \in V(G)$ and a cross-free and dividing wall $W$ in $G - x$ and an irrelevant vertex $v \in V(W)$
      - compute embedding of $G - v$
      - put the vertex $v$ back in.

Case 2.2 and 2.3 yield smaller graphs $\rightsquigarrow$ small tree-width
Now suppose the tree-width of $G$ is small.

**Step 1.** Test for Petersen-family minor
Output “not linklessly embeddable”

**Step 2.** Decompose into 4-connected components.

**Step 2.a: 1-separations.** Suppose $G$ has a 1-separation $(H_1, H_2)$.

1. Compute flat embedding of $H_1$
2. Compute flat embedding of $H_2$
3. Combine the embeddings
Flat Embeddings For Graphs of Small Tree-Width

Now suppose the tree-width of $G$ is small.

**Step 1.** Test for Petersen-family minor
Output “not linklessly embeddable”

**Step 2.** Decompose into 4-connected components.

**Step 2.a: 1-separations.** Suppose $G$ has a 1-separation $(H_1, H_2)$.

1. Compute flat embedding of $H_1$
2. Compute flat embedding of $H_2$
3. Combine the embeddings
2-separations. Suppose $G$ has a 2-separation $(H_1, H_2)$ but no cutvertex.

Compute flat embedding of $H_1 + uv$ and $H_2 + uv$.

The cycle in $H_1 + uv$ bounds a disk into which we can embed $H_2 + uv$. 

```latex
\begin{align*}
\text{H}_1 + uv \subseteq G \\
\text{H}_2 + uv \subseteq G
\end{align*}
```
Reduction to 4-connected graphs

3-separations. Suppose $G$ has a 3-separation $(H_1, H_2)$ but no 2-separation.

Perform $Y\Delta$-transform on $H_1 + v$ and $H_2 + v$.

We join the drawings at the triangles in $H_1 + uv$ and $H_2 + uv$. 
Step 3. Flat Embeddings of 4-Connected Graphs

Lemma. (Robertson, Seymour, Thomas)

1. Any two flat embeddings of a planar graph are ambient isotopic.
2. $K_5$ and $K_{3,3}$ have exactly two non-ambient isotopic flat embeddings.
3. Let $\Phi_1, \Phi_2$ be flat embeddings of $G$ which are not ambient isotopic. Then there is a Kuratowski-subgraph $H \subseteq G$ for which $\Phi_1|_H$ and $\Phi_2|_H$ are not ambient isotopic.

Definition. $\Phi_1, \Phi_2$ are ambient isotopic, $\Phi_1 \cong_{a.i} \Phi$, if there is an orientation preserving homeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ mapping $\Phi_1$ to $\Phi_2$.

Consequence. If we draw all Kuratowski-subgraphs consistently, then there is a unique extension to an embedding of the whole graph.

Lemma. If $G$ is 4-connected and non-planar, then every edge is contained in a Kuratowski-subgraph.
Step 3. Flat Embeddings of 4-Connected Graphs

Lemma. (Robertson, Seymour, Thomas)

1. Any two flat embeddings of a planar graph are ambient isotopic.
2. $K_5$ and $K_{3,3}$ have exactly two non-ambient isotopic flat embeddings.
3. Let $\Phi_1, \Phi_2$ be flat embeddings of $G$ which are not ambient isotopic. Then there is a Kuratowski-subgraph $H \subseteq G$ for which $\Phi_1|_H$ and $\Phi_2|_H$ are not ambient isotopic.

Definition. $\Phi_1, \Phi_2$ are ambient isotopic, $\Phi_1 \cong_{a.i} \Phi$, if there is an orientation preserving homeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ mapping $\Phi_1$ to $\Phi_2$.

Consequence. If we draw all Kuratowski-subgraphs consistently, then there is a unique extension to an embedding of the whole graph.

Lemma. If $G$ is 4-connected and non-planar, then every edge is contained in a Kuratowski-subgraph.
Adjacent Kuratowski Graphs

1-adjacent. Let $H_1 \neq H_2 \subseteq G$ be Kuratowski-subgraphs.

$H_1, H_2$ are 1-adjacent if there is a path $P \subseteq G$ and $i \in \{1, 2\}$ s.t.

- $P$ has only its endpoints in common with $H_i$ and
- $H_{3-i} \subseteq H_i \cup P$

2-adjacent. $H_1, H_2$ are 2-adjacent if there are distinct $v_1, \ldots, v_7 \in V(G)$ and pairwise internally vertex disjoint paths $L_{i,j}$ for

$$1 \leq i \leq 4 \text{ and } 5 \leq j \leq 7 \text{ or } i = 3 \text{ and } j = 4$$

linking $v_i$ and $v_j$ such that

- $H_1 = \bigcup \{L_{i,j} : 2 \leq i \leq 4 \text{ and } 5 \leq j \leq 7\}$ and
- $H_2 = \bigcup \{L_{i,j} : i \in \{1, 3, 4\} \text{ and } 5 \leq j \leq 7\}$

The path $L_{3,4}$ is not used but required to exist.
Adjacent Kuratowski Graphs

1-adjacent. Let $H_1 \neq H_2 \subseteq G$ be Kuratowski-subgraphs.

$H_1, H_2$ are 1-adjacent if there is a path $P \subseteq G$ and $i \in \{1, 2\}$ s.t.

• $P$ has only its endpoints in common with $H_i$ and
• $H_{3-i} \subseteq H_i \cup P$

2-adjacent. $H_1, H_2$ are 2-adjacent if there are distinct $v_1, \ldots, v_7 \in V(G)$ and pairwise internally vertex disjoint paths $L_{i,j}$ for

$$1 \leq i \leq 4 \text{ and } 5 \leq j \leq 7 \text{ or } i = 3 \text{ and } j = 4$$

linking $v_i$ and $v_j$ such that

• $H_1 = \bigcup \{L_{i,j} : 2 \leq i \leq 4 \text{ and } 5 \leq j \leq 7\}$ and
• $H_2 = \bigcup \{L_{i,j} : i \in \{1, 3, 4\} \text{ and } 5 \leq j \leq 7\}$

The path $L_{3,4}$ is not used but required to exist.
Communicating Kuratowski-Subgraphs

Definition. Let $H_1, H_2 \subseteq G$ be Kuratowski-subgraphs.

1. $H_1, H_2$ are adjacent, if they are 1- or 2-adjacent.
2. $H_1, H_2$ communicate if there is a sequence

   \[ H_1 = U_1 \ldots U_k = H_2 \]

   of Kuratowski-subgraphs $U_i$ s.t. $U_i U_{i+1}$ are adjacent.

Lemma. (Robertson, Seymour, Thomas)

1. Let $\Phi_1, \Phi_2$ be a flat embedding of $G$ and $H, H'$ adjacent Kuratowski-subgraphs. If $\Phi_1|_H \cong_{a.i.} \Phi_2|_H$ then $\Phi_1|_{H'} \cong_{a.i.} \Phi_2|_{H'}$.
2. If $G$ is 4-connected then all pairs of Kuratowski-subgraphs communicate.
3. If $\Phi_1, \Phi_2$ are flat embeddings of $G$. Then $\Phi_1 \cong_{a.i.} \Phi_2$ or $\Phi_1 \cong_{a.i.} -\Phi_2$.

   ($-\Phi_2$: comp. with the antipodal map)
Algorithm on graphs of small tree-width

Computing a flat embedding of a graph $G$ of small tree-width.

1. Compute an optimal tree-decomposition.
2. Test whether $G$ contains any member of the Petersen family of graphs as minor. If so, reject.
3. Search for 1-separations to decompose into 2-connected components.
4. Search for 2-separations to decompose into 3-connected components.
5. Search for 3-separations to decompose into 4-connected components.
6. If $G'$ is planar compute the unique flat embedding. Stop.
7. Otherwise choose Kuratowski sub-graph and compute its embedding.
8. While there still is a Kuratowski-subgraph $H$ which is not embedded, choose $H$ so that it is adjacent to an already embedded Kuratowski-subgraph and extend the embedding.
An $O(n^2)$-Algorithm for Computing Flat Embeddings

**Theorem.** There is an $O(n^2)$ algorithm which, given a graph $G$, decides whether $G$ is linklessly embeddable and if so constructs a flat embedding.

**High level proof idea.** We analyse the tree-width of the graph.

1. If it is very high we can reduce the problem to a smaller graph
   
   More formally,
   
   • we can delete a vertex
   • recursively compute the flat embedding
   • put the vertex back in

2. If the tree-width is small:
   
   • Test for Petersen-family minors.
   • Decompose the graph into its 4-connected components.
   • Embed Kuratowski-subgraphs one by one.

**Theorem.** (Kawarabayashi, K., Mohar, Reed)

There is an $O(n)$ algorithm which, given a graph $G$, decides whether $G$ is linklessly embeddable and if so constructs a flat embedding.
An $O(n^2)$-Algorithm for Computing Flat Embeddings

**Theorem.** There is an $O(n^2)$ algorithm which, given a graph $G$, decides whether $G$ is linklessly embeddable and if so constructs a flat embedding.

**High level proof idea.** We analyse the tree-width of the graph.

1. If it is very high we can reduce the problem to a smaller graph

   More formally,
   - we can delete a vertex
   - recursively compute the flat embedding
   - put the vertex back in

2. If the tree-width is small:

   - Test for Petersen-family minors.
   - Decompose the graph into its 4-connected components.
   - Embed Kuratowski-subgraphs one by one.

**Theorem.** (Kawarabayashi, K., Mohar, Reed)

There is an $O(n)$ algorithm which, given a graph $G$, decides whether $G$ is linklessly embeddable and if so constructs a flat embedding.
Conclusion

In August, I’ll wish you a very happy birthday, Robin.