Clique Immersion in Digraphs

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What is digraph immersion?

Given two digraphs D, F, we say “D has an F-immersion” if:

F can be obtained from D via deletion and splitting-off edges.

Equivalently: “D immerses F” or “F is immersed in D” or “D contains F as an immersion”
Equivalent definition:

\( D \) has an \( F \)-immersion if there exists an injective map \( \phi : V(F) \to V(D) \) and a collection of edge-disjoint directed paths in \( D \), one from \( \phi(u) \) to \( \phi(v) \) for every edge \( uv \) in \( F \).

- The vertices \( \{ \phi(v) : v \in V(F) \} \) are called *terminals*.

- The collection of directed paths may not be internally disjoint from the set of terminals. If they are, we have a *strong* immersion.
Today we ask:

When does a digraph $D$ have a $\vec{K}_t$-immersion?

Outline

1. Moving from graphs to digraphs

2. Two proofs of one theorem

3. Two theorems, one corollary

4. Conclusion
1 Moving from graphs to digraphs

Given two graphs $G$, $H$, we say “$G$ has an $H$-immersion” if:

$H$ can be obtained from $G$ via deletion and splitting-off edges.

**Theorem (DeVos, Dvořák, Fox, M., Mohar, Scheide)**

Every simple graph with minimum degree $\geq 200t$ has a (strong) $K_t$-immersion.

- $200t$ can be lowered to $t-1$ (best possible) when $t \leq 7$ (Lescure and Mayniel; DeVos, Kawarabayashi, Mohar, Okamura)
- $t-1$ does not suffice when $t \geq 10$ (Seymour)

Immersion is harder in digraphs...but is there an analogous result?
Theorem (DeVos, M., Mohar, Scheide)
For every $k \in \mathbb{Z}^+$, there exists a digraph with minimum in-degree and out-degree $\geq k$, but no $\overrightarrow{K}_3$-immersion.

**proof:**

Add an arc from each vertex to all those vertices between it and the root (and to the root)

What went wrong?

- To split a vertex completely: Eulerian.
- Graphs: $\delta \geq 2k \supset$ Eulerian, $\delta \geq k$ (Tutte, Nash-Williams).
- The above example is very far from being Eulerian.
Theorem (DeVos, M., Mohar, Scheide)

For every \( k \in \mathbb{Z}^+ \), there exists a digraph with minimum in-degree and out-degree \( \geq k \), but no \( \vec{K}_3 \)-immersion.

**proof:**

- Add an arc from each vertex to all those vertices between it and the root (and to the root).
- Add an arc from each vertex to all those vertices between \( D_3 \) it and the root (and to the root).

**What went wrong?**

- To split an vertex completely: Eulerian.
- Graphs: \( \delta \geq 2k \supseteq \text{Eulerian}, \delta \geq k \) (Tutte, Nash-Williams).
- The above example is very far from being Eulerian.
Theorem (DeVos, Dvořák, Fox, M., Mohar, Scheide).
Every simple graph with minimum degree at least $200t$ has a strong $K_t$-immersion.

Theorem (DeVos, M., Mohar, Scheide)
For every $k \in \mathbb{Z}^+$, there exists a digraph with minimum in-degree and out-degree $\geq k$, but no $\overrightarrow{K}_3$-immersion.

Theorem (DeVos, M., Mohar, Scheide).
Every simple Eulerian digraph with minimum degree at least $t(t-1)$ contains a $\overrightarrow{K}_t$-immersion.

Theorem (DeVos, M., Mohar, Scheide).
If $t \leq 4$, every simple Eulerian digraph with minimum degree at least $t - 1$ contains a $\overrightarrow{K}_t$-immersion (and this is best possible).
Two proofs of one theorem

**Theorem (DeVos, M., Mohar, Scheide).**
Every simple Eulerian digraph with minimum degree at least \( t(t-1) \) contains a \( \tilde{K}_t \)-immersion.

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<th>Proof Ingredients</th>
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<td>Edmonds’ Disjoint Arborescence Theorem</td>
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Arborscence w. root $v = \text{spanning tree, all edges away from } v$

**Edmonds’ Disjoint Arborescence Theorem.**
Let $v_1, \ldots, v_r$ be vertices in a digraph $D$ (not necessarily distinct). Then $\exists$ edge-disjoint arborescences $T_1, \ldots, T_r$ so that $T_i$ has root $v_i$ iff every $X \subset V(D)$ satisfies $d^+(X) \geq |\{i : v_i \in X, 1 \leq i \leq r\}|$.

**Corollary.** If a digraph is strongly $t(t-1)$-edge connected with at least $t$ vertices then it contains a $\overrightarrow{K_t}$-immersion.

**proof:**

- Let $v_1, \ldots, v_t$ be distinct, then choose each $t - 1$ times.

- We may apply Edmonds’ Thm to this set of $t(t-1)$ vertices.

- We get $t(t-1)$ edge-disjoint arborescences, $t-1$ of which are rooted at $v_i$.

- These $t - 1$ arborescences give edge-disjoint paths from $v_i$. 
Theorem (DeVos, M., Mohar, Scheide).
Every simple Eulerian digraph $D$ with minimum degree at least $t(t-1)$ contains a $\vec{K}_t$-immersion.

Proof Ingredients

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Proofs 1 & 2:

It suffices to show that $D$ immerses a strongly $t(t-1)$-edge-connected digraph on at least $t$ vertices.
Proof 1 (sketch): Let \( t(t - 1) = r \). \( D \) simple, Eul., min. deg. \( \geq r \). Show \( D \) immerses strongly \( r \)-edge-connected digraph, \( \geq t \) vertices.

- We are able to find \( X \subseteq D \) such that all pairs of vertices in \( X \) are sufficiently connected (but perhaps through all of \( D \)), and...

- ...we are able to immerse the following Eulerian digraph in \( D \) (maintaining connectivity between pairs):

\[
\leq 2(r - 1)
\]

Mader’s Directed Splitting Theorem.
Given an Eulerian digraph and a non-isolated vertex \( w \), there is a pair of edges that can be split off of \( w \) so the size of the smallest edge-cut between any other pair of vertices doesn’t change.

- Use Mader’s Theorem to split \( v \) completely.

- At most \( r - 1 \) parallel edges. Coupled with the min. deg. condition, this says we have \( r \geq t \) vertices. \( \square \)
Theorem (DeVos, M., Mohar, Scheide).
Every simple Eulerian digraph $D$ with minimum degree at least $t(t-1)$ contains a $\vec{K}_t$-immersion.

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Proofs 1 & 2:

It suffices to show that $D$ immerses a strongly $t(t - 1)$-edge-connected digraph on a least $t$ vertices.
Proof 2: $D$ simple, Eulerian, min. deg. $\geq t(t-1)$. Show $D$ immerses a strongly $t(t-1)$-edge-connected digraph on $\geq t$ vertices.

The Gomory-Hu Theorem.
For every multigraph $G$ there exists a tree $F$ with vertex set $V(G)$ and a function $\mu : E(F) \to \mathbb{Z}$ such that:

- $\lambda_G(u, v) = \min \{ \mu(e) : e \in uFv \} \ \forall u, v \in V(G)$
- $\mu(e) = |\text{edge cut of } G \text{ associated with } e| \ \forall e \in E(F)$

Apply GH to the underlying multigraph.

The family of edge-cuts of $D$ associated with \{ $e \in E(F) : \mu(e) < 2t(t-1)$ \} induces a partition of $V(D)$.

Blocks of the partition must have size $\geq t$ by simple, min. degree.

Choose $t$ distinct vertices in one block and apply Mader’s Theorem to split all other vertices completely. □
Theorem (DeVos, M., Mohar, Scheide).
Every simple Eulerian digraph $D$ with minimum degree at least $t(t-1)$ contains a $\vec{K}_t$-immersion.

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Proofs 1 & 2:
It suffices to show that $D$ immerses a strongly $t(t - 1)$-edge-connected digraph on a least $t$ vertices...

...but in both cases we proved something better.
3 Two Theorems, One Corollary

Theorem (DeVos, Mohar, M., Scheide).
Every simple Eulerian digraph with minimum degree $\geq r$ immerses a strongly $r$-edge-connected digraph on at least $r$ vertices.

Theorem (DeVos, Mohar, M., Scheide).
If $D$ is an Eulerian digraph with no $\vec{K}_t$-immersion then it has a laminar family of edge cuts, each with size $< 2t(t - 1)$, so that every block of the resulting partition has size less than $t$.

Corollary.
Every simple Eulerian digraph $D$ with minimum degree at least $t(t-1)$ contains a $\vec{K}_t$-immersion.
Theorem (DeVos, Mohar, M., Scheide). If $D$ is an Eulerian digraph with no $\vec{K}_t$-immersion then it has a laminar family of edge cuts, each with size $< 2t(t - 1)$, so that every block of the resulting partition has size less than $t$.

Rough structure for $\vec{K}_t$-immersion.

(Backwards: no $\vec{K}_t^2$, $K_t^2$)

Rough structure for $K_t$-immersion.

Theorem (Seymour, Wollan). If $G$ is graph with no $K_t$-immersion then it has a laminar family of edge cuts, each with size $< (t - 1)^2$, so that every block of the resulting partition has size less than $t$. 
Immersing a clique is harder in digraphs than it is in graphs:

- We need to take Eulerian as an assumption.
- Can we lower min. deg. $\geq t(t - 1)$ to linear?

Along with the minimum degree result, we get:

- Simple Eulerian digraph with min. deg. $\geq r$ immerses a strongly $r$-edge-connected digraph with $\geq r$ vertices.
- Rough structure theorem for $\vec{K}_t$-immersion.
Thank-you

More questions?

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