Embeddability of infinite graphs

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May 7, 2012
Theorem (Kuratowski, 1930)

A **finite** graph $G$ is embeddable in the plane if and only if it does not contain a subgraph homeomorphic to the complete graph $K_5$ or the complete bipartite graph $K_{3,3}$. 
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**Theorem (Wagner, 1937)**

A **finite** graph $G$ is embeddable in the plane if and only if it contains neither $K_{3,3}$ nor $K_5$ as a minor.
Embeddability in the plane: Kuratowski, Wagner

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Every compact surface has a “Wagner’s Theorem”:

Theorem (Robertson and Seymour, 1990)

For every compact surface there is a finite list of graphs such that a graph $G$ is embeddable in this surface if and only if it does not contain any of these as a minor.
$g + 1$ disjoint Kuratowski graphs: natural obstacle for embedding in genus $g$

$K_{3,3}$ and $K_5$ are the Kuratowski graphs.

Torus (genus 1): can host one Kuratowski graph (and no more).

The compact surface of genus $g$ can host the disjoint union of $g$ Kuratowski graphs (and no more)
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$\implies$ having many disjoint (or “sufficiently” disjoint) Kuratowski graphs makes the genus grow

Is this the reason why the genus of a graph grows?
Why does the genus of a graph grow?

Robertson and Seymour (unpublished)

There is a function $f(g)$ tending to infinity so that, if a graph $G$ does not embed in any surface of Euler characteristic at least $2 - 2g$, then $G$ has one of the following graphs as a minor:

1. $f(g)$ disjoint copies of either $K_3$, $K_3$, or $K_5$;
2. $f(g)$ copies of either $K_3$, $K_3$, or $K_5$ that are disjoint except for a common vertex;
3. $f(g)$ copies of either $K_3$, $K_3$, or $K_5$ that are disjoint except for two common vertices; or
4. $K_3$, $f(g)$.

If we restrict ourselves to orientable surfaces, then we have to add the $f(g)$-projective grid to the list.
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Our problem

Which graphs do not embed into any surface of bounded genus?
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Plausible answer, in view of the Robertson-Seymour result

Those that contain as a minor either:

- Infinitely many “sufficiently disjoint” $K_{3,3}$’s or $K_5$’s.
- $K_{3,\infty}$ (yes, abuse of notation)
Which graphs do not embed into any surface of bounded genus?
Some previous fine-tuning

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- If an infinite graph embeds in a surface, then it has only countably many vertices of degree $\geq 3$ (Wagner, 1967). Thus it consists of a countable graph, plus possibly continuumly many cycles, paths, rays, and double rays.

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Thus, it all boils down to:

**Question**

Which countable graphs embed in some surface of bounded genus?
Our result: embeddability in bounded genus

Theorem (Christian, Richter, and S., 2011+)

A countable graph $G$ embeds in some (orientable) surface of bounded genus if and only if $G$ does not contain as a minor any of:

1. infinitely many disjoint copies of either $K_{3,3}$ or $K_5$;
2. infinitely many copies of either $K_{3,3}$ or $K_5$ that are disjoint except for a common vertex;
3. infinitely many copies of either $K_{3,3}$ or $K_5$ that are disjoint except for two common vertices; or
4. $K_{3,\aleph_0}$.

A (slightly surprising?) consequence

There is no distinction between embeddability in some orientable surface and embeddability in some surface. In other words, no graph can embed in some (non-orientable) surface and have arbitrarily large projective grids.
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The interesting direction

The “only if” part is easy: a graph with infinitely many (sufficiently disjoint) copies of $K_{3,3}$ or $K_5$, or with $K_{3,\aleph_0}$, cannot be embedded in any surface of bounded genus.

The “if” part is the interesting one.
For the rest of the talk, for brevity,

“surface” means *bounded genus, orientable* surface.
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A graph is **good** if it can be embedded in some surface.

Otherwise it is **bad**.
A bad $G$ has infinitely many disjoint copies of $K_{3,3}$ or $K_5$ or:

There is a $J \subseteq G$, and a vertex $u_1$ of $J$, such that $J$ is bad and $J - u_1$ is good.

Let $G_0 := G$, and as long as $G_i$ has a subgraph $H_{i+1}$ (may choose finite, if one exists) that contracts to $K_{3,3}$ or $K_5$, set $G_{i+1} := G_i - V(H_{i+1})$. 
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2. If for every positive $i$, $H_i$ exists, then we are done ($G$ contains as a minor infinitely many disjoint copies of either $K_{3,3}$ or $K_5$). Thus we may assume that for some $i$, $G_i$ has no Kuratowski minor; so $G_i$ is planar. Note that $G_i$ is obtained from $G$ by the deletion of finitely many vertices $v_1, v_2, \ldots, v_k$. 
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3. For $j = 1, 2, \ldots, k$, consider $G^j := G - \{v_{j+1}, v_{j+2}, \ldots, v_k\}$. There is a least $j$ so that $G^j$ does not embed in any surface. Set $J_0 := G^j$, and $u_1 := v_j$. Thus $J_0$ is a subgraph of $G$ that does not embed in any surface ($J_0$ is bad), yet $J_0 - u_1$ does ($J_0 - u_1$ is good).
REPEAT AND GET: Either $G$ has one of the listed minors or $\exists$: 

$M \subseteq G$

- $M$ is bad
- $M - u_i$ is good for each $i$
- $M - A$ has no subdivision of $K_{1,3}$ with $u_1, u_2, u_3$ as the degree 1 vertices.
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IF THIS HAPPENS: Every component of $M - (A \cup \{u_1, u_2, u_3\})$ attaches to at most two of $u_1, u_2, u_3$. Let $N_{j,k}$ be the subgraph of $M$ induced by the vertices in $A \cup \{u_j, u_k\}$ and all components of $M - (A \cup \{u_1, u_2, u_3\})$ that attach to at most $u_j$ and $u_k$. 

Gelasio Salazar
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Each of $N_{1,2}$, $N_{1,3}$, and $N_{2,3}$ embeds in some surface... combine these embeddings to obtain an embedding of $M$ (contradiction!).
Continuing in the countably infinite theme...

* A countably infinite number of men went into a pub. The first one ordered a pint. The second ordered a half-pint. The third ordered a quarter of a pint ... The barkeeper, with a face full of disgust, finally poured two pints and put them on the bar and said, “It’s good when people know their limits.”
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Thanks for your attention!