Induced subgraphs and subtournaments

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joint with Maria Chudnovsky
Theorem

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- (Robertson, S., Thomas 1994; Diestel, Gorbunov, Jensen, Thomassen 1999) For every simple planar graph \( H \) with \( n \) vertices, taking \( k = 2^{O(n^5)} \) works.
- (Leaf, S., 2012) For every simple planar graph \( H \) with \( n \) vertices, taking \( k = 2^{O(n \log(n))} \) works.
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For every non-planar graph $H$ there is no such $k$.

Equivalently:

Theorem

A minor ideal has bounded treewidth if and only if some planar graph is not in it.
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A minor ideal has bounded treewidth if and only if it does not include the ideal of all planar graphs.
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A minor ideal has bounded pathwidth if and only if it does not include the ideal of all forests.
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If $H$ is planar, a minor ideal has bounded $f_1$ if and only if it does not include $I(H)$ ($f_1$ is the minimum number of vertices whose deletion leaves a graph with no $H$ minor)
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A minor ideal has bounded $f_2$ if and only if it does not include $I(K_5)$ or $I(K_{3,3})$ ($f_2$ is the minimum $k$ such that deleting some $k$ vertices leaves a graph of genus at most $k$).
Theorem

A minor ideal has

- bounded $f_3$ iff it does not include the ideal of all graphs that are subgraphs of a path
- bounded $f_4$ iff it does not include the ideal of all stars
- bounded $f_5$ iff it does not include the ideal of all graphs with crossing number at most one
- bounded $f_6$ iff it does not include the ideal of all graphs.
What about induced subgraphs?
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**Theorem**

An induced subgraph ideal has bounded clique number iff it does not include \{cliques\}. 
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**Theorem (Ramsey’s theorem)**

An induced subgraph ideal is finite iff it includes neither of \{cliques\}, \{graphs with no edges\}.
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**Theorem**

An induced subgraph ideal has bounded maximum degree iff [fill in the rest of the theorem]
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Are there any more?
Tournaments

Possible containment relations:

- subtournament (not wqo)
- topological containment (ie subdivision) (not wqo)
- immersion (wqo)
- butterfly minor (wqo?? – open)
- strong minor (wqo - Kim).
Tournament $G$ has cutwidth at most $c$ if $V(G)$ can be ordered $\{v_1, \ldots, v_n\}$ such that for each $i$, there are at most $c$ edges from $\{v_i+1, \ldots, v_n\}$ to $\{v_1, \ldots, v_i\}$.
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**Theorem**

*For every tournament $H$ there exists $c$ such that every tournament in which $H$ cannot be immersed has cutwidth at most $c$.***
Tournament $G$ has *cutwidth* at most $c$ if $V(G)$ can be ordered \{\(v_1, \ldots, v_n\)\} such that for each $i$, there are at most $c$ edges from \{\(v_{i+1}, \ldots, v_n\)\} to \{\(v_1, \ldots, v_i\)\}.

**Theorem**

*For every tournament $H$ there exists $c$ such that every tournament in which $H$ cannot be immersed has cutwidth at most $c$.***

**Equivalently:**

**Theorem**

*An immersion ideal of tournaments has bounded cutwidth iff it does not include \{tournaments\}.***
Theorem

For every tournament $H$ of type $A_0$, there exists $c$ such that every tournament $G$ not containing $H$ as a subtournament can be ordered such that the maximum backdegree is at most $c$. 
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Equivalently:

Theorem
A subtournament ideal has bounded backdegree iff it does not include $\{\text{type } A_0\}$. 
TYPES B AND B'
Theorem

A subtournament ideal has bounded cutwidth iff it does not include \{type A_0\}, \{type B\} or \{type B'\}.
Theorem (Fradkin, S.)

For every tournament $H$ there exists $c$ such that every tournament that does not contain $H$ topologically has pathwidth at most $c$. 
Theorem (Fradkin, S.)

For every tournament $H$ there exists $c$ such that every tournament that does not contain $H$ topologically has pathwidth at most $c$.

Equivalently:

Theorem

A topological containment ideal of tournaments has bounded pathwidth iff it does not include \{tournaments\}. 
Theorem (Fradkin, S.)

For every tournament $H$ there exists $c$ such that every tournament that does not contain $H$ topologically has pathwidth at most $c$.

Equivalently:

Theorem

A topological containment ideal of tournaments has bounded pathwidth iff it does not include \{tournaments\}.

Theorem (Kim, S.)

A strong minor ideal of tournaments has bounded pathwidth iff it does not include \{tournaments\}.
TYPE A
Theorem

A subtournament ideal has bounded pathwidth iff it does not include \{type A\}, \{type B\} or \{type B’\}.
Tournaments under subtournament containment
Theorem

A subtournament ideal has bounded length backedges iff it does not include \{type C\}. 
Circular interval tournaments
Circular interval tournaments

Theorem

A tournament is a circular interval tournament iff it contains neither of the tournaments above.
Theorem (Gaku Liu)

A subtournament ideal consists of blowups of circular interval tournaments by tournaments with bounded length backedges iff it does not include \{type D\}, \{type D'\}.
Theorem

A subtournament ideal consists of tournaments orderable such that each component of long backedges is a path iff it does not include \{type E\}, \{type E'\}.
Theorem

A subtournament ideal consists of boundedly fuzzy circular interval tournaments iff it does not include \{type $G$\}, \{type $G'$\}. 
Problem

A subtournament ideal consists of ??? iff it does not include \{\text{type F}\}.
Induced subgraphs revisited

A finite set of graphs is heroic if there exists $c$ such that every graph with no induced subgraph in the set has cochromatic number at most $c$.

**Theorem**

*Every heroic set contains*

- a disjoint union of cliques
- a complete multipartite graph
- a forest
- a graph whose complement is a forest.
Induced subgraphs revisited

A finite set of graphs is heroic if there exists \( c \) such that every graph with no induced subgraph in the set has cochromatic number at most \( c \).

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**Conjecture: Gyárfás’ 1975; Sumner 1981**

For every clique \( K \) and forest \( F \), there is a constant \( c \) such that every graph containing neither of \( K, F \) as an induced subgraph has chromatic number at most \( c \).
G has splitness at most $k$ if $V(G)$ can be partitioned into $X, Y$, where $G|X$ has no clique of size $k + 1$ and $G|Y$ has no stable set of size $k + 1$. 
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**Theorem**

An induced subgraph ideal has bounded splitness iff it does not include \{disjoint unions of cliques\}, \{complete multipartite graphs\}. 
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**Theorem**

An induced subgraph ideal has bounded splitness iff it does not include \{disjoint unions of cliques\}, \{complete multipartite graphs\}.

**Corollary**

If Gyárfás’ conjecture is true, then every set of graphs containing one of each of the four types is heroic.
Theorem

For every star $H_1$ and star complement $H_2$, there exists $c$ such that if $G$ contains neither of $H_1, H_2$ as an induced subgraph then one of $G, \overline{G}$ has maximum degree at most $c$. 
Theorem (Reed)

For every one-edge graph $H_1$ and clique $H_2$, there exists $c$ such that if $G$ contains neither of $H_1, H_2$ as an induced subgraph then $G$ is “almost complete multipartite with at most $c$ parts”.
Theorem (Reed)

For every one-edge graph $H_1$ and clique $H_2$, there exists $c$ such that if $G$ contains neither of $H_1, H_2$ as an induced subgraph then $G$ is “almost complete multipartite with at most $c$ parts”.

Theorem (Norine, Reed)

For every semistar $H_1$ and semistar complement $H_2$, there exists a structure theorem.