Planarity and Dimension for Graphs and Posets

William T. Trotter
trotter@math.gatech.edu
Robin Thomas and WTT - ???
WTT in Prague with Nešetřil and Růdl, 1983
Order Diagrams and Cover Graphs

Order Diagram

Cover Graph
Planar Posets

Definition: A poset $P$ is planar when it has an order diagram with no edge crossings.

Fact: If $P$ is planar, then it has an order diagram with straight line edges and no crossings.
A Non-planar Poset

This height 3 non-planar poset has a planar cover graph.
Definition of Dimension

The dimension of a poset $P$ is the least integer $n$ for which $P$ is a subposet of $\mathbb{R}^n$. This embedding shows that $\dim(P) \leq 3$. In fact,

$$\dim(P) = 3$$
Problem Let $f(k)$ be the maximum chromatic number of a graph $G$ with $\Delta(G) = k$.

Solution (Brooks Theorem) $f(k) = k + 1$.

Problem Let $f(k)$ be the maximum dimension of a poset $P$ with $\Delta(P) = k$.

Solution (Erdős, Kierstead and WTT; Füredi and Kahn)

$$ck \log k < f(k) < c'k \log^2 k$$
Standard Examples

Fact For $n \geq 2$, the standard example $S_n$ is a poset of dimension $n$. 
Planar Posets with Zero and One

**Theorem** (Baker, Fishburn and Roberts, 1971 + Folklore)

If $P$ has both a 0 and a 1, then $P$ is planar if and only if it is a lattice and has dimension at most 2.
Theorem (WTT and Moore, 1977) If $P$ has a $0$ and the diagram of $P$ is planar, then $\dim(P) \leq 3$. 
A 4-dimensional planar poset

**Fact** The standard example $S_4$ is planar!
**Theorem** (Kelly, 1981) For every $n \geq 5$, the standard example $S_n$ is non-planar, but it is a subposet of a planar poset.
The vertex-edge poset of a graph is also called the incidence poset of the graph.
Schnyder’s Theorem

**Theorem** (Schnyder + Babai and Duffus, 1989) A graph is planar if and only if the dimension of its vertex-edge poset is at most 3.

**Note** Testing graph planarity is linear in the number of edges while testing for dimension at most 3 is NP-complete!!!
Schnyder’s proof is a classic, elegant and rich in structure.

His principal motivation was to find an efficient layout of a planar graph on a small grid.

Recently, Haxell and Barrera-Cruz (2011) have found a direct - and very compact - proof, sans the structure, but the value of Schnyder’s original approach remains intact.
Planar Multigraphs
Planar Multigraphs and Dimension

**Theorem** (Brightwell and WTT, 1996): Let $D$ be a non-crossing drawing of a planar multigraph $G$, and let $P$ be the vertex-edge-face poset determined by $D$. Then $\dim(P) \leq 4$.

**Note** Inductive proof with planar 3-connected graphs as the base case. Done by GRB and WTT four years earlier.

**Fact** Different drawings may determine posets with different dimensions.
Bipartite Planar Graphs

**Theorem** (Felsner, Li, WTT, 2010) If $P$ has height 2 and the cover graph of $P$ is planar, then $\dim(P) \leq 4$. 
Conjecture (Felsner, Li and WTT, 2010) For every integer \( h \), there exists a constant \( c_h \) so that if \( P \) is a poset of height \( h \) and the cover graph of \( P \) is planar, then \( \text{dim}(P) \leq c_h \).

Observation The conjecture holds trivially for \( h = 1 \) and \( c_1 = 2 \). Although very non-trivial, the conjecture also holds for \( h = 2 \), and \( c_2 = 4 \).

Fact Kelly’s construction shows that \( c_h \) - if it exists - must be at least \( h + 1 \).
Conjecture Resolved

**Theorem (Streib and WTT, 2012)** For every integer $h$, there exists a constant $c_h$ so that if $P$ is a poset of height $h$ and the cover graph of $P$ is planar, then $\dim(P) \leq c_h$.

**Fact** A straightforward modification to Kelly’s construction shows that $c_h$ must be at least $h + 2$.

However, our proof uses Ramsey theory at several key places and the bound we obtain is very large in terms of $h$. 
**Fact** For every $h \geq 2$, the standard example $S_{h+2}$ is contained in a poset of height $h$ having a planar cover graph.
Planarity and Dimension

**Theorem** (Felsner, WTT and Wiechert, 2012)

Let $P$ be a poset.

1. If the comparability graph of $P$ is planar, then $\dim(P) \leq 4$.

2. If the cover graph of $P$ is outerplanar, then $\dim(P) \leq 4$.

3. If the cover graph of $P$ is outerplanar, and $P$ has height at most 3, then $\dim(P) \leq 3$. 
Some Open Questions

1. For each $t \geq 4$, what is the smallest planar poset having dimension $t$?
2. Improve the bounds for the constant $c_h$ in the Streib-WTT theorem.
3. What is the maximum dimension of a poset with a planar incomparability graph?
Robin Thomas is 50!!

Maple told me that

$$50! = 304140932017133780436126081660647\text{\ldots}$$

But when I asked for $50!!$, Maple replied

"Kernel connection has been lost."