THE VOLUME CONJECTURE, THE AJ CONJECTURES AND SKEIN MODULES

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SUMMARY

This dissertation studies quantum invariants of knots and links, particularly the colored Jones polynomials, and their relationships with classical invariants like the hyperbolic volume and the A-polynomial.

In Chapter 1, we study the volume conjecture of Kashaev-Murakami-Murakami that relates the Kashaev invariant, a specialization of the colored Jones polynomial at a specific root of unity, and the hyperbolic volume of a link. We establish the conjecture for $(m, 2)$-cables of the figure 8 knot, when m is odd. For $(m, 2)$-cables of general knots where m is even, we show that the limit in the volume conjecture depends on the parity of the color (of the Kashaev invariant). There are many cases when the volume conjecture for cables of the figure 8 knot is false if one considers all the colors, but holds true if one restricts the colors to a subset of the set of positive integers.

Chapter 2 studies the AJ conjecture of S. Garoufalidis that relates the linear recurrence relations of the colored Jones polynomials and the A-polynomial of a knot, using skein theory. We confirm the AJ conjecture for those hyperbolic knots satisfying certain conditions. In particular, we show that the conjecture holds true for some classes of two-bridge knots and pretzel knots. This extends the previous result of T. Le where he established the AJ conjecture for a large class of two-bridge knots, including all twist knots. Along the way, we explicitly calculate the universal character ring of the knot group of the $(-2, 3, 2n+1)$-pretzel knot and show that it is reduced, i.e. has no nilpotent elements, for all integer n.

Chapter 3 studies the Kauffman bracket skein module (KBSM) of the complement of all two-bridge links. For a two-bridge link, we show that the KBSM of its complement is free over the ring $\mathbb{C}[t^{\pm 1}]$ and when reducing $t = -1$, it is isomorphic to the ring of regular functions on the character variety of the link group.

In Chapter 4, we study a stronger version of the AJ conjecture, proposed by A. Sikora, and confirm it for all torus knots.

Results in chapters 1–3 of this dissertation are joint works with T. Le.

CHAPTER I

ON THE VOLUME CONJECTURE FOR CABLES OF KNOTS

1.1 Introduction

1.1.1 The colored Jones polynomial and the Kashaev invariant of a link

Suppose K is framed oriented link with m ordered components in S^3. To every m-tuple $(n_1, \dots, n_m)$ of positive integers one can associate a Laurent polynomial $J_K(n_1, \dots, n_m; q) \in \mathbb{Z}[q^{\pm 1/4}]$, called the colored Jones polynomial, with n_j being the color of the j-component of K. The polynomial $J_K(n_1, \dots, n_m; q)$ is the quantum link invariant, as defined by Reshetikhin and Turaev [50, 57], associated to the Lie algebra $sl_2(\mathbb{C})$, with the color n_j standing for the irreducible $sl_2(\mathbb{C})$-module V_{n_j} of dimension n_j. Here we use the functorial normalization, i.e. the one for which the colored Jones polynomial of the unknot colored by n is

$$[n] := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.$$

When all the colors are 2, the colored Jones polynomial is the usual Jones polynomial [28]. The colored Jones polynomials of higher colors are more or less the usual Jones polynomials of cables of the link.

Following [44], we define the Kashaev invariant of a link K as the sequence $\langle K \rangle_N$, $N = 1, 2, \dots$, by

$$\langle K \rangle_N := \frac{J_K(N, \dots, N; q)}{[N]}|_{q^{1/4}=\exp(\pi i/2N)}.$$

1.1.2 The volume conjecture for knots and links

According to Thurston theory, by cutting the link complement $S^3 \setminus K$ along appropriate disjoint tori one gets a collection of pieces, each is either Seifert fibered or hyperbolic; and $\mathrm{Vol}(K)$ is defined as the sum of the hyperbolic volume of the hyperbolic pieces. It is known that $\mathrm{Vol}(K) = v_3\,||S^3 \setminus K||$, where v_3 is the volume of the ideal regular tetrahedron, and $||S^3 \setminus K||$ is the Gromov norm. We can now formulate the volume conjecture of Kashaev-Murakami-Murakami [30, 44]:

Conjecture 1. *Suppose K is a knot in S^3, then*

$$\lim_{N\to\infty} \frac{\log |\langle K\rangle_N|}{N} = \frac{\mathrm{Vol}(K)}{2\pi}.$$

For a survey on the volume conjecture, see [43]. Already in [44] it was noted that the volume conjecture in the above form cannot be true for split links, since for split links the Kashaev invariant vanishes. There are a few cases of links (of more than one components) when the volume conjecture had been confirmed: in particular, the volume conjecture was established for the Borromean rings [20], the Whitehead link [60], and more general, for Whitehead chains [58].

When the Kashaev invariant vanishes, one might hope to remedy the conjecture by renormalizing the colored Jones polynomial. One of consequences of the present chapter is that the normalization alone is not good enough, we have also to distinguish between the cases N even and N odd.

1.1.3 Main results

For a knot K with framing 0, let $K^{(m,p)}$ be the (m, p)-cable of K, also with framing 0, see the precise definition in §1.2. Note that if m and p are co-prime, then $K^{(m,p)}$ is again a knot. The two-component link $K^{(0,2)}$ is called the *disconnected cable* of K. Note that we always have $\mathrm{Vol}(K^{(m,p)}) = \mathrm{Vol}(K)$.

In this chapter we study the volume conjecture for cables of a knot K. It turns out that the case N even and the case N odd are quite different.

Theorem 1. *Suppose that K is a knot and $K^{(0,2)}$ the disconnected cable of K. Then $\langle K^{(0,2)}\rangle_N = 0$ for every* even *N.*

The case of odd N is quite different, at least for the figure 8 knot:

Theorem 2. *Suppose $\mathcal{E}$ is the figure 8 knot and $\mathcal{E}^{(0,2)}$ its disconnected cable. Then the volume conjecture holds true for $\mathcal{E}^{(0,2)}$ if the colors are restricted to the set of odd numbers:*

$$\lim_{N\to\infty,\ N\ \text{odd}} \frac{\log|\langle \mathcal{E}^{(0,2)}\rangle_N|}{N} = \frac{\mathrm{Vol}(\mathcal{E}^{(0,2)})}{2\pi}.$$

Thus for figure 8 knot, the sequence of Kashaev invariant $|\langle \mathcal{E}^{(0,2)}\rangle_N|$ grows exponentially if $N \to \infty$ and N odd. While if N is even, then $|\langle \mathcal{E}^{(0,2)}\rangle_N| = 0$ (for any knot).

However, when $m \neq 0$, i.e. when the two components of $K^{(m,2)}$ do have non-trivial linking number, the volume conjecture might still hold true even for even N. For example, we have the following result.

Theorem 3. *Suppose $\mathcal{E}$ is the figure 8 knot and $m \equiv 2 \pmod 4$. Then the volume conjecture holds true for $\mathcal{E}^{(m,2)}$ if the colors are restricted to the set of numbers divisible by 4:*

$$\lim_{N\to\infty,\ N\equiv 0 \pmod 4} \frac{\log|\langle \mathcal{E}^{(m,2)}\rangle_N|}{N} = \frac{\mathrm{Vol}(\mathcal{E}^{(m,2)})}{2\pi}.$$

According to the survey [43], the volume conjecture has been so far established for the following knots:

- 4_1 (by Ekholm),
- $5_2, 6_1, 6_2$ (by Y. Yokota),
- torus knots (by Kashaev and Tirkkonen [31]), and
- Whitehead doubles of torus knots of type $(2, b)$ (by Zheng [60]).

We add to this the following result.

Theorem 4. *Suppose $\mathcal{E}$ is the figure 8 knot. Then the volume conjecture holds true for the knot $\mathcal{E}^{(m,2)}$ for every odd number m.*

Actually, we will prove some generalizations of Theorems 1-4.

Remark 1.1.1. We arrived at the theorems through the symmetry principle studied in [32, 35], although we will not use the symmetry here. One important tool in our proof is the Habiro expansion of the colored Jones polynomial [22], which has been instrumental in integrality of the Witten-Reshetikhin-Turaev invariant of 3-manifolds (see [2, 22, 37]) and in the proof of a generalization of the volume conjecture for small angles [20].

The odd colors correspond to the representations of the group $SO(3)$, or representations of sl_2 with highest weights in the root lattice.

1.1.4 Plan of the chapter

In Section 1.2 we exactly formulate the more general results that we want to prove. Sections 1.3 contains some elementary calculations involving the building blocks in the Habiro expansion. Sections 1.4 and 1.5 contain the proof of the main theorems.

1.2 Cables, the colored Jones polynomial, and results

1.2.1 Cables, the colored Jones polynomial

Suppose K is a knot with 0 framing and m, p are two integers with d their greatest common divisor. The (m, p)-cable $K^{(m,p)}$ of K is the link consisting of d parallel copies of the $(m/d, p/d)$-curve on the torus boundary of a tubular neighborhood of K. Here an $(m/d, p/d)$-curve is a curve that is homologically equal to m/d times the meridian and p/d times the longitude on the torus boundary. The cable $K^{(m,p)}$ inherits an orientation from K, and we assume that each component of $K^{(m,p)}$ has framing 0.

The colored Jones polynomial is a special case of tangle invariants defined using ribbon Hopf algebras and their modules [50]. The ribbon Hopf algebra in our case is the quantized enveloping algebra $U_h(sl_2)$, e.g. [46]. For each positive integer n, there is a unique irreducible $U_h(sl_2)$-module V_n of rank n. In [46] our $J_K(n_1, \ldots, n_m; q)$ is denoted by $Q^{sl_2;V_{n_1},\ldots,V_{n_m}}(K)$.

The calculation of $J_{K^{(m,2)}}(N;q)$ is standard: one decomposes $V_N \otimes V_N$ into irreducible components

$$V_N \otimes V_N = \oplus_{l=1}^{N} V_{2l-1}.$$

Since the R-matrix commutes with the actions of the quantized algebra, it acts on each component V_{2l-1} as a scalar μ_l times the identity. The value of μ_l is well-known:

$$\mu_l = (-1)^{N-l} q^{\frac{1-N^2}{2}} q^{\frac{l(l-1)}{2}}.$$

Hence from the theory of quantum invariants, we have

$$J_{K^{(m,2)}}(N;q) = \sum_{l=1}^{N} \mu_l^m J_K(2l-1;q).$$

The symmetry of quantum invariant at roots of unity [32, 35] prompts us to combine the color $N-j$ with $N+j$. So we rewrite the above formula as follows

$$J_{K^{(m,2)}}(N;q) = a_N^m \sum_{j=1-N, N-j+1 \text{ even}}^{N-1} t_{j,N}^m J_K(N+j;q), \tag{1}$$

where

$$t_{j,N} = i^{N-1-j} \, q^{\frac{(N+j)^2}{8}} \quad \text{and} \quad a_N = q^{(3-4N^2)/8}.$$

1.2.2 General knot case and even m

Here we relate the Kashaev invariant of $K^{(m,2)}$ and the colored Jones polynomial of $K_{m/2}$, which is the same knot K, only with framing $m/2$. Increasing the framing by 1 has the effect of multiplying the invariant by $q^{(N^2-1)/4}$, hence

$$J_{K_p}(N;q) = q^{p\frac{N^2-1}{4}} J_K(N;q).$$

Theorem 1.2.1. *Suppose one of the following:*

(i) $m \equiv 0 \pmod 4$ *and* N *is even.*

(ii) $m \equiv 2 \pmod 4$ *and* $N \equiv 2 \pmod 4$. *Then, with* $q^{1/4} = \exp(\pi i/2N)$, *one has*

$$\langle K^{(m,2)} \rangle_N = q^{m/2}(q^{1/2} - q^{-1/2})\frac{mN}{4}\sum_{j=1}^{N/2} J_{K_{m/2}}(2j-1;q).$$

In particular, if $m = 0$ *and* N *is even, then* $\langle K^{(m,2)} \rangle_N = 0$.

1.2.3 Figure 8 knot case

Let $\mathcal{E}$ be the figure 8 knot. We will show that the volume conjecture for $\mathcal{E}^{(m,2)}$ holds true under some restrictions.

For an integer m there are 4 possibilities, we list them here together with the definition of a set S_m:

(i) m is odd. Define $S_m = \mathbb{N}$, the set of positive integers.

(ii) $m \equiv 0 \pmod 8$. Define $S_m = \{N \in \mathbb{N}, N \equiv 1 \pmod 2\}$, the set of odd positive integers.

(iii) $m \equiv 2 \pmod 4$. Define $S_m = \{N \in \mathbb{N}, N \not\equiv 2 \pmod 4\}$.

(iv) $m \equiv 4 \pmod 8$. Define $S_m = \emptyset$.

Theorem 1.2.2. *Suppose* $\mathcal{E}$ *is the figure 8 knot and* m *is in one of first three cases (i)–(iii) listed above. Then the volume conjecture for* $\mathcal{E}^{(m,2)}$ *holds true if the colors are restricted to the corresponding set* S_m, *i.e. one has*

$$\lim_{N\to\infty, N\in S_m} \frac{\log |\langle \mathcal{E}^{(m,2)} \rangle_N|}{N} = \frac{\mathrm{Vol}(\mathcal{E}^{(m,2)})}{2\pi}.$$

The proof of the theorem will be given in section 1.5. Theorems 2, 3, and 4 are parts of this theorem. We still don't have any conclusion for the case (iv).

1.3 Some elementary calculations

1.3.1 Notations and conventions

We will work with the variable $q^{1/4}$. Let $v = q^{1/2}$. We will use the following notations. Here j, k, l, N are integers.

$$\{j\} := v^j - v^{-j}, \quad [j] := \{j\}/\{1\}, \quad S(k,l) := \prod_{k \le j \le l} \{j\}, \quad S'(k,l) := \prod_{k \le j \le l,\ j \notin \{0,N\}} \{j\}.$$

$$A(j,k) := \{j-k\}\{j+k\}, \quad {\prod_{i\in I}}' a_i := \sum_{i \in I} \prod_{j \in I \setminus \{i\}} a_j, \quad \text{and} \quad t_{j,N} := i^{N-1-j}\, q^{\frac{(N+j)^2}{8}}.$$

For f, g and h in $\mathbb{Z}[v^{\pm 1}]$, the equation $f \equiv g \pmod{h}$ means that $f - g$ is divisible by h.

1.3.2 The building blocks $A(N,k)$ and $S(k,l)$

The expressions $A(N,k)$ and $S(k,l)$ will be the building blocks in the Habiro expansion. We will prove here a few simple facts.

Lemma 1.3.1. *One has*

$$\begin{aligned} A(N-j,k) &\equiv A(N+j,k) + 2\{2j\}\{N\}\{N-1\}/\{1\} \pmod{\{N\}^2} \quad \textit{in } \mathbb{Z}[v^{\pm 1}], \\ \{N-j\} &= -\{N+j\} + \{N\}(v^j + v^{-j}), \\ t^m_{-j,N} &= t^m_{j,N} + \frac{mj}{2}\, t^m_{j,N}\, \{N\} + t^m_{j,N}(v^N+1)^2\, Q, \quad \textit{where } Q \in \mathbb{Q}[v^{\pm 1}]. \end{aligned}$$

Proof. The second equality follows directly from definition. For the first one, by noting that $\{-j\} = -\{j\}$ and $A(j,k) = q^j + q^{-j} - q^k - q^{-k}$ we have

$$\begin{aligned} A(N-j,k) - A(N+j,k) &= -\{2j\}\{2N\} \\ &= -\{2j\}\{N\}(v^N + v^{-N})(v - v^{-1})/\{1\} \\ &= -\{2j\}\{N\}[\{N\}(v+v^{-1}) - 2\{N-1\}]/\{1\} \\ &\equiv 2\{2j\}\{N\}\{N-1\}/\{1\} \pmod{\{N\}^2}. \end{aligned}$$

To prove the last one, we note that for a positive integer k there is a polynomial $P(a)$, whose coefficients depend on k only, such that $1-a^k = k(1-a)+(1-a)^2P(a)$. Hence if $mj \geq 0$ then

$$\begin{aligned} t^m_{-j,N} - t^m_{j,N} &= -t^m_{j,N}[1-(-v^{-N})^{mj}] \\ &= -t^m_{j,N}[(1+v^{-N})mj + (1+v^{-N})^2P(v^{-1})] \\ &= -t^m_{j,N}[\frac{mj}{2}(v^{-N}(1+v^N)^2 - \{N\}) + (1+v^{-N})^2P(v^{-1})] \\ &= \frac{mj}{2}\, t^m_{j,N}\,\{N\} + t^m_{j,N}(v^N+1)^2[-\frac{mj}{2}\, v^{-N} + v^{-2N}P(v^{-1})], \end{aligned}$$

which asserts the third equality. Otherwise, i.e. if $mj < 0$, we have

$$\begin{aligned} t^m_{-j,N} - t^m_{j,N} &= -t^m_{j,N}[1-(-v^{N})^{-mj}] \\ &= -t^m_{j,N}[-(1+v^{N})mj + (1+v^{N})^2P(v)] \\ &= -t^m_{j,N}[-\frac{mj}{2}(v^{-N}(1+v^N)^2 + \{N\}) + (1+v^{N})^2P(v)] \\ &= \frac{mj}{2}\, t^m_{j,N}\,\{N\} + t^m_{j,N}(v^N+1)^2[\frac{mj}{2}\, v^{-N} - P(v)]. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 1.3.2. *Suppose $v^N = -1$, then*

$$\frac{t^m_{j,N}[N+j](\prod_{k=1}^{l} A(N+j,k)) + t^m_{-j,N}[N-j]\prod_{k=1}^{l} A(N-j,k)}{[N]} = t^m_{j,N}D(j,l), \qquad (2)$$

where

$$D(j,l) = (v^j + v^{-j})\left((\prod_{k=1}^{l} A(j,k)) + 2\{j\}^2 \prod_{1\leq k\leq l}{}' A(j,k)\right) + \frac{mj}{2}\{j\}\prod_{k=1}^{l} A(j,k).$$

Proof. From lemma 1.3.1, we have

$$\begin{aligned} \prod_{k=1}^{l} A(N-j,k) &\equiv \prod_{k=1}^{l}[A(N+j,k) + 2\{2j\}\{N\}\{N-1\}/\{1\}] \pmod{\{N\}^2} \\ &\equiv \prod_{k=1}^{l} A(N+j,k) + \\ &+ 2\{2j\}\{N\}\{N-1\}/\{1\} \prod_{1\leq k\leq l}{}' A(N+j,k) \pmod{\{N\}^2}. \end{aligned}$$

This, together with the last two equalities in lemma 1.3.1, implies that when $v^N = -1$ the left hand side of (2) is equal to

$$
\begin{aligned}
&- \frac{mj}{2} t^m_{j,N}\{N+j\}(\prod_{k=1}^{l} A(N+j,k)) + t^m_{j,N}(v^j + v^{-j})(\prod_{k=1}^{l} A(N+j,k)) \\
&- 2\{2j\}\{N-1\}/\{1\} \prod_{1\leq k\leq l}{}' A(N+j,k) t^m_{j,N}\{N+j\}
\end{aligned}
$$

Hence (2) follows from the facts that $A(N+j,k) = A(j,k)$ and $\{N+j\} = -\{j\}$ if $v^N = -1$. □

Lemma 1.3.3. *One has $D(j,l) = D_1(j,l) + D_2(j,l)$, where*

$$
D_1(j,l) = \begin{cases} \left(\frac{mj}{2} + \frac{v^j+v^{-j}}{\{j\}} + 2\{2j\}\sum_{k=1}^{l}\frac{1}{A(j,k)}\right) S(j-l, j+l), & \textit{if } l < \min(j, N-j), \\ 0 & \textit{if } l \geq \min(j, N-j), \end{cases}
$$

and

$$
D_2(j,l) = \begin{cases} 2S'(j-l, j+l) & \textit{if } j \leq l < N-j, \\ -2S'(j-l, j+l) & \textit{if } N-j \leq l < j, \\ 0 & \textit{if } l < \min(j, N-j) \textit{ or } l \geq \max(j, N-j). \end{cases}
$$

Proof. This can be checked easily by direct calculations. □

Lemma 1.3.4. *For $j \leq l \leq N-j$, the sign of $S'(j-l, j+l)$ is $(-1)^j$. For $N-j \leq l \leq j$, the sign of $S'(j-l, j+l)$ is $(-1)^{N-j}$.*

Proof. If $j \leq l \leq N-j$ then the sign of $S'(j-l, j+l)$ is $i^{l+j}(-i)^{l-j} = (-1)^j$. For $N-j \leq l \leq j$, the proof is similar. □

Lemma 1.3.5. *Suppose $v^N = -1$. For $1 \leq j \leq N-1$ and $0 \leq l \leq N-1$ one has*

$$
D(N-j,l) + D(j,l) = \frac{mN}{2} S(j-l, j+l). \tag{3}
$$

Proof. Let

$$B(j,l) = (v^j + v^{-j}) \left((\prod_{k=1}^{l} A(j,k)) + 2\{j\}^2 \prod_{1\le k\le l}{}' A(j,k) \right).$$

Then by definition, $D(j,l) = B(j,l) + \frac{mj}{2}\{j\}\prod_{k=1}^{l} A(j,k)$. It is easy to see that if $v^N = -1$ then $B(N-j,l) + B(j,l) = 0$ and

$$\{N-j\}\prod_{k=1}^{l} A(N-j,k) = \{j\}\prod_{k=1}^{l} A(j,k) = S(j-l, j+l).$$

It implies that (3) holds true. □

1.4 *Proof of Theorem 1.2.1*

1.4.1 Habiro expansion

By a deep result of Habiro [22], there are *Laurent polynomials* $C_K(l;q) \in \mathbb{Z}[q^{\pm 1}]$, depending on the knot K, such that

$$J_K(N;q) = [N]\sum_{l=0}^{N-1} C_K(l;q)\prod_{k=1}^{l} A(N,k).$$

From now on let $q^{1/4} = \exp(\pi i/2N)$. Then $v^N = -1$. Using Eq. (2), we have for $0 \le j \le N-1$

$$\frac{t^m_{j,N} J_K(N+j;q) + t^m_{-j,N} J_K(N-j;q)}{[N]} = \sum_{l=0}^{N-1} C_K(l;q)\, t^m_{j,N}\, D(j,l).$$

Hence Eq. (1) implies that

$$\langle K^{(m,2)}\rangle_N = a^m_N \sum_{l=0}^{N-1} C_K(l;q) \left(\Big(\sum_{j=1, N-j+1 \text{ even}}^{N-1} t^m_{j,N}\, D(j,l)\Big) + \frac{1-(-1)^N}{4}\, t^m_{0,N} D(0,l) \right). \tag{4}$$

1.4.2 Proof of Theorem 1.2.1

We assume that m, N satisfy the conditions of Theorem 1.2.1, i.e. $m \equiv 0 \pmod 4$ and N is even; or $m \equiv 2 \pmod 4$ and $N \equiv 2 \pmod 4$.

The symmetry [32, 35] hints that we should combine j and $N-j$. Since N is even, both j and $N-j$ are odd, so they both appear in the sum (4). Hence we rewrite the expression in the big parenthesis in right hand side of Eq. (4) as follows

$$(\sum_{j=1, j \text{ odd}}^{N/2-1} t^m_{j,N} D(j,l) + t^m_{N-j,N} D(N-j,l)) + \frac{1-(-1)^{N/2}}{2} t^m_{N/2,N} D(N/2,l).$$

Under the assumption in the theorem on m and N, we can easily check that $t^m_{N-j,N} = t^m_{j,N}$. Therefore it follows from Eq. (3) that $\langle K^{(m,2)}\rangle_N$ is equal to $a^m_N \sum_{l=0}^{N-1} C_K(l;q)$ times

$$(\sum_{j=1, j \text{ odd}}^{N/2-1} t^m_{j,N} \frac{mN}{2} S(j-l, j+l)) + \frac{1-(-1)^{N/2}}{2} t^m_{N/2,N} \frac{mN}{4} S(N/2-l, N/2+l).$$

Now, by noting that $a^m_N = q^{m(3-4N^2)/8} = q^{3m/8}$ and

$$t^m_{N-j,N} = t^m_{j,N} = q^{mj^2/8}, \quad S(j-l,j+l) = S(N-j-l, N-j+l),$$

we obtain

$$\begin{aligned}
\langle K^{(m,2)}\rangle_N &= q^{3m/8} \frac{mN}{4} \sum_{l=0}^{N-1} C_K(l;q) \sum_{j=1, j \text{ odd}}^{N-1} q^{mj^2/8} S(j-l, j+l) \\
&= q^{m/2} \frac{mN}{4} \sum_{j=1, j \text{ odd}}^{N-1} q^{\frac{m}{2}\cdot\frac{j^2-1}{4}} \{1\} J_K(j;q) \\
&= q^{m/2}\{1\} \frac{mN}{4} \sum_{j=1, j \text{ odd}}^{N-1} J_{K_{m/2}}(j;q),
\end{aligned}$$

where $K_{m/2}$ is K with framing $m/2$. This proves theorem 1.2.1.

1.5 Proof of theorem 1.2.2

Let $\delta := \exp(\pi i/4)$. We will write t_j for $t_{j,N}$. Then, with $q^{1/4} = \exp(\pi i/2N)$ one has

$$t_j = \delta^{3N-2} q^{j^2/8}.$$

For the figure 8 knot $\mathcal{E}$, we know that $C_{\mathcal{E}}(l,q) = 1$, see [22]. From Eq. (4) we have

$$\begin{aligned}
\frac{\langle \mathcal{E}^{(m,2)}\rangle_N}{\delta^{(3N-2)m} a^m_N} &= (\sum_{l=0}^{N-1} \sum_{j=1, N-j+1 \text{ even}}^{N/2-1} q^{mj^2/8}\, D(j,l)) + (\sum_{l=0}^{N-1} \sum_{j=N/2, N-j+1 \text{ even}}^{N-1} q^{mj^2/8}\, D(j,l)) \\
&+ \frac{1-(-1)^N}{4} \sum_{l=0}^{N-1} D(0,l). \qquad (5)
\end{aligned}$$

1.5.1 The case $j < N/2$

We now consider the first sum in the right hand side of Eq. (5). By lemma 1.3.3,

$$D(j,l) = \begin{cases} D_1(j,l) = \left(\frac{mj}{2} + \frac{v^j+v^{-j}}{\{j\}} + 2\{2j\}\sum_{k=1}^{l}\frac{1}{A(j,k)}\right) S(j-l,j+l) & \text{if } l < j \\ D_2(j,l) = 2S'(j-l,j+l) & \text{if } j \le l < N-j \\ 0 & \text{if } l \ge N-j \end{cases}$$

We will consider two subcases when $D(j,l) \neq 0$: $j \le l < N-j$ and $j < l$.

1.5.1.1 The subcase $j \le l < N-j$

By lemma 1.3.4, the sign of $S'(j-l,j+l)$ is $(-1)^j = (-1)^{N-1}$. Note that $\{k\} = 2i\sin\frac{k\pi}{N}$, hence $S'(j-l,j+l) = (-1)^{N-1}E(j,l)$, where

$$E(j,l) = (\prod_{r=1}^{l-j} 2\sin\frac{r\pi}{N})(\prod_{r=1}^{l+j} 2\sin\frac{r\pi}{N}).$$

We will see that $E(j,l)$ is maximized when $j=0$ and $l = 5N/6$. Moreover, we have the following result.

Proposition 1.5.1. *There exists a nonzero number C such that for any $\alpha \in (\frac{1}{2},\frac{2}{3})$, we have*

$$\sum_{j=1, N-j+1 \text{ even}}^{N/2-1} \sum_{l=j}^{N-j} q^{mj^2/8} D_2(j,l) = (-1)^{N-1} C\, E(0,\frac{5N}{6}) N(1+O(N^{3\alpha-2})).$$

Proof. By setting

$$s_n = -\sum_{j=1}^{n} \log|2\sin\frac{j\pi}{N}|,$$

we have $\log E(j,l) = -s_{l-j} - s_{l+j}$. Consider the Lobachevsky function

$$L(x) := -\int_0^x \log|2\sin u|du.$$

By a standard argument, see e.g. [20, 60], $s_n = \frac{N}{\pi}L(\frac{n\pi}{N}) + O(\log N)$. Hence

$$\log E(j,l) = -\frac{N}{\pi}L(\frac{(l-j)\pi}{N}) - \frac{N}{\pi}L(\frac{(l+j)\pi}{N}) + O(\log N) = \frac{N}{\pi}f(\frac{j\pi}{N},\frac{l\pi}{N}) + O(\log N),$$

where $f(x,y) = -L(-x+y) - L(x+y)$ for $\pi \geq y \geq x \geq 0$ and $\pi \geq x+y$.

It is easy to show that the function f attains its maximum at the unique point $(x,y) = (0, \frac{5\pi}{6})$. Moreover, the Taylor expansion of f around $(0, \frac{5\pi}{6})$ is

$$f(h, \frac{5\pi}{6} + k) = f(0, \frac{5\pi}{6}) - \sqrt{3}(h^2 + k^2) + O(|h|^3 + |k|^3).$$

By the same argument as in the proof of theorem 1.2 in [60], there exists $\epsilon > 0$ such that

$$\log E(j,l) \begin{cases} < \log E(0, \frac{5N}{6}) - \epsilon N^{2\alpha - 1} + O(1) & \text{if } j^2 + (l - \frac{5N}{6})^2 \geq N^{2\alpha}, \\ = \log E(0, \frac{5N}{6}) - \frac{\pi\sqrt{3}}{N}[j^2 + (l - \frac{5N}{6})^2] + O(N^{3\alpha-2}) & \text{otherwise} \end{cases}$$

Let

$$\begin{aligned} I_1 &= \{(j,l) : 1 \leq j < N/2, N-j+1 \text{ even}, j \leq l \leq N-j \text{ and } j^2 + (l - \frac{5N}{6})^2 \geq N^{2\alpha}\}, \\ I_2 &= \{(j,l) : 1 \leq j < N/2, N-j+1 \text{ even}, j \leq l \leq N-j \text{ and } j^2 + (l - \frac{5N}{6})^2 \leq N^{2\alpha}\}. \end{aligned}$$

Then we have

$$E(j,l) = \begin{cases} E(0, \frac{5N}{6}) \exp(-\epsilon N^{2\alpha-1}) O(1) & \text{if } (j,l) \in I_1, \\ E(0, \frac{5N}{6}) \exp(-\frac{\pi\sqrt{3}}{N}[j^2 + (l - \frac{5N}{6})^2])(1 + O(N^{3\alpha-2})) & \text{if } (j,l) \in I_2. \end{cases}$$

It implies that

$$\sum_{(j,l)\in I_1} q^{mj^2/8} E(j,l) = E(0, \frac{5N}{6}) N^2 \exp(-\epsilon N^{2\alpha-1}) O(1),$$

and $\sum_{(j,l)\in I_2} q^{mj^2/8} E(j,l) =$

$$\begin{aligned} &= \sum_{(j,l)\in I_2} \exp(\frac{\pi i m j^2}{4N}) E(0, \frac{5N}{6}) \exp(-\frac{\pi\sqrt{3}}{N}[j^2 + (l - \frac{5N}{6})^2])(1 + O(N^{3\alpha-2})) \\ &= \frac{N}{4} \int_{x^2+y^2<N^{2\alpha-1}} \exp(\frac{\pi i m}{4} x^2 - \pi\sqrt{3}(x^2+y^2)) dx dy E(0, \frac{5N}{6})(1 + O(N^{3\alpha-2})) \\ &= \frac{N}{4} \int_{\mathbb{R}^2} \exp(\frac{\pi i m}{4} x^2 - \pi\sqrt{3}(x^2+y^2)) dx dy E(0, \frac{5N}{6})(1 + O(N^{3\alpha-2})). \end{aligned}$$

Hence

$$\sum_{j=1, N-j+1 \text{ even}}^{N/2-1} \sum_{l=j}^{N-j} q^{mj^2/8} D_2(j,l) = 2(-1)^{N-1}\left[\sum_{(j,l)\in I_1} q^{mj^2/8}E(j,l) + \sum_{(j,l)\in I_2} q^{mj^2/8}E(j,l)\right]$$
$$= (-1)^{N-1} C E(0, \frac{5N}{6}) N(1+O(N^{3\alpha-2})),$$

where

$$C = \frac{1}{2}\int_{\mathbb{R}^2} \exp(\frac{\pi i m}{4}x^2 - \pi\sqrt{3}(x^2+y^2))dxdy.$$

We can easily check that C is a nonzero number. This completes the proof the proposition 1.5.1. □

From now on, we fix the number $\alpha \in (\frac{1}{2}, \frac{2}{3})$.

1.5.1.2 The subcase $l < j$

By lemma 1.3.4, the sign of $S(j-l, j+l)$ is $i^{l+j}/i^{j-l-1} = i(-1)^l$. Hence we get $S(j-l, j+l) = i(-1)^l F(j,l)$, where

$$F(j,l) = (\prod_{r=1}^{l+j} 2\sin\frac{r\pi}{N})/(\prod_{r=1}^{j-l-1} 2\sin\frac{r\pi}{N})$$

for $0 < l < j < N/2$. Note that $\log F(j,l) = s_{j-l-1} - s_{l+j}$ and roughly speaking, in this case $F(j,l)$ attains its maximum at $j = N/2$ and $l = N/3$. We claim that

Proposition 1.5.2. *One has*

$$\sum_{j=1, N-j+1 \text{ even}}^{N/2-1} \sum_{l<j} q^{mj^2/8} D_1(j,l) = N^{3\alpha} F(\frac{N}{2}, \frac{N}{3}) O(1). \tag{6}$$

Proof. Let

$$I_3 = \{(j,l) : 1 \le j < N/2, N-j+1 \text{ even}, l < j \text{ and } (j-\frac{N}{2})^2 + (l-\frac{N}{3})^2 \ge N^{2\alpha}\},$$
$$I_4 = \{(j,l) : 1 \le j < N/2, N-j+1 \text{ even}, l < j \text{ and } (j-\frac{N}{2})^2 + (l-\frac{N}{3})^2 \le N^{2\alpha}\}.$$

By the same argument as in the proof of the previous proposition, we have

$$F(j,l)\begin{cases} = F(\frac{N}{2},\frac{N}{3})\exp(-\epsilon N^{2\alpha-1})O(1) & \text{if } (j,l)\in I_3,\\ \le F(\frac{N}{2},\frac{N}{3}) & \text{if } (j,l)\in I_4.\end{cases}$$

To prove the proposition, we need the following three lemmas:

Lemma 1.5.3. *One has*

$$\sum_{(j,l)\in I_3} q^{mj^2/8}\left(\frac{mj}{2}+\frac{v^j+v^{-j}}{\{j\}}+2\{2j\}\sum_{k=1}^{l}\frac{1}{A(j,k)}\right)S(j-l,j+l)=$$

$$=N^5F(\frac{N}{2},\frac{N}{3})\exp(-\epsilon N^{2\alpha-1})O(1).$$

Proof. It suffices to show that if $(j,l)\in I_3$ then

$$\frac{mj}{2}+\frac{v^j+v^{-j}}{\{j\}}+2\{2j\}\sum_{k=1}^{l}\frac{1}{A(j,k)}=N^3O(1). \tag{7}$$

Indeed, since $\sin x\ge\frac{2}{\pi}x$ for $x\in[0,\frac{\pi}{2}]$, we have

$$|\frac{v^j+v^{-j}}{\{j\}}|=|\frac{\cos(j\pi/N)}{\sin(j\pi/N)}|\le\frac{\pi}{2}\cdot\frac{N}{j\pi}\le\frac{N}{2}.$$

Note that $k\le l\le j-1$, hence

$$|A(j,k)|=|\{j-k\}\{j+k\}|=4\sin\frac{(j-k)\pi}{N}\sin\frac{(j+k)\pi}{N}\ge 4\sin\frac{\pi}{N}\sin\frac{(2j-1)\pi}{N}.$$

It implies that

$$|\frac{\{2j\}}{A(j,k)}|\le\frac{\sin\frac{2j\pi}{N}}{2\sin\frac{\pi}{N}\sin\frac{(2j-1)\pi}{N}}.$$

If $j>\frac{N}{4}$ then $\sin\frac{(2j-1)\pi}{N}>\sin\frac{2j\pi}{N}>0$, therefore

$$|\frac{\{2j\}}{A(j,k)}|\le\frac{1}{2\sin\frac{\pi}{N}}\le\frac{N}{4}.$$

If $j\le\frac{N}{4}$ then

$$|\frac{\{2j\}}{A(j,k)}|\le\frac{1}{2\sin^2\frac{\pi}{N}}\le\frac{N^2}{8}.$$

Hence (7) holds true and then the claim of the lemma is proved. □

Lemma 1.5.4. *One has*

$$\sum_{(j,l)\in I_4} q^{mj^2/8}\left(\frac{m}{2}(j-\frac{N}{2})+\frac{v^j+v^{-j}}{\{j\}}+2\{2j\}\sum_{k=1}^{l}\frac{1}{A(j,k)}\right)S(j-l,j+l)=$$

$$=N^{3\alpha}F(\frac{N}{2},\frac{N}{3})O(1).$$

Proof. Since $(j-\frac{N}{2})^2+(l-\frac{N}{3})^2\leq N^{2\alpha}$, we have

$$|\frac{v^j+v^{-j}}{\{j\}}|=\cot\frac{j\pi}{N}=|\frac{\pi}{2}-\frac{j\pi}{N}|O(1)=N^{\alpha-1}O(1),$$

and

$$|\frac{\{2j\}}{A(j,k)}|\leq|\frac{\{2j\}}{A(j,l)}|=\frac{\sin\frac{2j\pi}{N}}{2\sin\frac{(j-l)\pi}{N}\sin\frac{(l+j)\pi}{N}}=N^{\alpha-1}O(1),$$

which proves the equality of the lemma. □

Lemma 1.5.5. *One has*

$$\sum_{(j,l)\in I_4} q^{mj^2/8}S(j-l,j+l)=N^{3\alpha-1}F(\frac{N}{2},\frac{N}{3})O(1).$$

Proof. Denote by L the left hand side of the equality. We first see that L is essentially equal to the following expression

$$L'=\frac{1}{2}\sum_{(j,l)\in I_4} q^{mj^2/8}[S(j-l,j+l)+S(j-l-1,j+l+1)].$$

Note that for $(j,l)\in I_4$, we have $S(j-l,j+l)+S(j-l-1,j+l+1)=$

$$\begin{aligned}
&= S(j-l,j+l)[1-4\sin\frac{(j-l-1)\pi}{N}\sin\frac{(j+l+1)\pi}{N}]\\
&= S(j-l,j+l)[(2+2\cos\frac{2j\pi}{N})-(1+2\cos\frac{(2l+1)\pi}{N})]\\
&= N^{\alpha-1}F(\frac{N}{2},\frac{N}{3})O(1).
\end{aligned}$$

This implies that $L'=N^{3\alpha-1}F(\frac{N}{2},\frac{N}{3})O(1)$. Hence to complete the proof of the lemma, we need to estimate the difference between L and L'.

Let $J = \{j : (j,l) \in I_4 \text{ for some } l\}$. For each $j \in J$, let $J_j = \{l : (j,l) \in I_4\}$. We have

$$L' - L = \frac{1}{2} \sum_{j \in J} q^{mj^2/8} \sum_{l \in J_j} S(j-l-1, j+l+1) - S(j-l, j+l). \tag{8}$$

For each j in J, it is easy to see that the set J_j is just a closed interval $[a_j, b_j]$. Hence

$$\sum_{l \in J_j} S(j-l-1, j+l+1) - S(j-l, j+l) = S(j-b_j-1, j+b_j+1) - S(j-a_j, j+a_j)$$

has absolute value less than or equal to $2F(\frac{N}{2}, \frac{N}{3})$. Eq. (8) then implies that $|L'-L| \leq N^{\alpha} 2F(\frac{N}{2}, \frac{N}{3})$. Since $\alpha < 3\alpha - 1$, the conclusion of the lemma follows. □

We now come back to the proof of proposition 1.5.2. It is easy to see that lemmas 1.5.4 and 1.5.5 imply

$$\sum_{(j,l) \in I_4} q^{mj^2/8} \left(\frac{mj}{2} + \frac{v^j + v^{-j}}{\{j\}} + 2\{2j\} \sum_{k=1}^{l} \frac{1}{A(j,k)} \right) S(j-l, j+l) = N^{3\alpha} F(\frac{N}{2}, \frac{N}{3}) O(1).$$

Hence, by combining this with lemma 1.5.3, we obtain the equality of the proposition.

□

We can now estimate the first sum in the right hand side of Eq. (5). To do that, we need one more lemma which allows us to express the right hand side of Eq. (6), and hence the sum in its left hand side, in terms of $E(0, \frac{5N}{6})$.

Lemma 1.5.6. *One has*

$$F(\frac{N}{2}, \frac{N}{3}) = \frac{1}{N} E(0, \frac{5N}{6}) O(1).$$

Proof. The follows from the fact that $\prod_{k=1}^{N-1} (2 \sin \frac{k\pi}{N}) = N$. □

From proposition 1.5.2 and lemma 1.5.6, we have

$$\sum_{j=1, N-j+1 \text{ even}}^{N/2-1} \sum_{l<j} q^{mj^2/8} D_1(j,l) = N^{3\alpha-1} E(0, \frac{5N}{6}) O(1).$$

This, together with proposition 1.5.1, implies that

$$\sum_{l=0}^{N-1} \sum_{j=1, N-j+1 \text{ even}}^{N/2-1} q^{mj^2/8} D(j,l) = (-1)^{N-1} C E(0, \frac{5N}{6}) N (1 + O(N^{3\alpha-2})). \tag{9}$$

1.5.2 The case $j \geq N/2$

Again, by lemma 1.3.3 we have

$$D(j,l) = \begin{cases} D_1(j,l) = \left(\frac{mj}{2} + \frac{v^j + v^{-j}}{\{j\}} + 2\{2j\} \sum_{k=1}^{l} \frac{1}{A(j,k)}\right) S(j-l, j+l) & \text{if } l < N-j \\ D_2(j,l) = -2S'(j-l, j+l) & \text{if } N-j \leq l < j \\ 0 & \text{if } l \geq j \end{cases}$$

In this case all the estimations we have done in section 1.5.1 also work for the second sum in the right hand side of Eq. (5) except two things. The first one is that for $N-j \leq l < j$, by lemma 1.3.4, the sign of $S'(j-l, j+l)$ is $(-1)^{N-j} = -1$ and the other is

$$q^{mj^2/8} = q^{m(N-j)^2/8} \exp(\pi i m(2j-N)/4),$$

where

$$\exp(\pi i m(2j-N)/4) = \begin{cases} \beta = \exp(\pi i m(N+2)/4), & \text{if } N-j+1 \equiv 0 \pmod 4, \\ \gamma = \exp(\pi i m(N-2)/4), & \text{if } N-j+1 \equiv 2 \pmod 4. \end{cases}$$

Therefore by similar arguments as in the proof of Eq. (9), we get

$$\sum_{l=0}^{N-1} \sum_{j=N/2, N-j+1 \text{ even}}^{N-1} q^{mj^2/8} D(j,l) = \frac{1}{2}(\beta+\gamma) C E(0, \frac{5N}{6}) N(1 + O(N^{3\alpha-2})). \tag{10}$$

1.5.3 Proof of theorem 1.2.2

From Equations (9) and (10), we have

$$\sum_{l=0}^{N-1} \sum_{j=1, N-j+1 \text{ even}}^{N-1} q^{mj^2/8} D(j,l) = \frac{1}{2}(\beta+\gamma+2(-1)^{N-1}) C E(0, \frac{5N}{6}) N(1 + O(N^{3\alpha-2})). \tag{11}$$

Moreover, it is easy to see that

$$\sum_{l=0}^{N-1} D(0,l) = N^{\alpha} E(0, \frac{5N}{6}) O(1). \tag{12}$$

Therefore, to complete the proof of theorem 1.2.2, we need the following lemma

Lemma 1.5.7. *We have $\beta+\gamma+2(-1)^{N-1}=0$ if and only if one of the following holds:*

(i) $m \equiv 0 \pmod 4$ and N is even.

(ii) $m \equiv 2 \pmod 4$ and $N \equiv 2 \pmod 4$.

(iii) $m \equiv 4 \pmod 8$ and N is odd.

Moreover, if $\beta+\gamma+2(-1)^{N-1} \neq 0$ then $|\beta+\gamma+2(-1)^{N-1}| \geq 2$.

Proof. Suppose that $\beta+\gamma+2(-1)^{N-1}=0$. If N is even then $\beta+\gamma=2$. Since $|\beta|=|\gamma|=1$, it implies that $\beta=\gamma=1$ which means that $m \equiv 0 \pmod 4$; or $m \equiv 2 \pmod 4$ and $N \equiv 2 \pmod 4$. If N is odd then similarly, we have $\beta=\gamma=-1$, which is equivalent to the condition that $m \equiv 4 \pmod 8$.

Note that if m is odd then $\beta+\gamma=0$, hence $|\beta+\gamma+2(-1)^{N-1}|=2$. Now let us consider the case m is even. Then $\beta=\gamma$ and $\beta+\gamma+2(-1)^{N-1}=2(\beta+(-1)^{N-1})$, so it is remaining to show that $|\beta+(-1)^{N-1}| \geq 1$. Since $\beta=i^{\frac{m}{2}(N+2)} \in \{\pm 1, \pm i\}$ and $\beta+(-1)^{N-1} \neq 0$, it implies that the angle between β and $(-1)^{N-1}$ is less than or equal to $\frac{\pi}{2}$. Hence

$$|\beta+(-1)^{N-1}|^2 \geq |\beta|^2+|(-1)^{N-1}|^2=2,$$

which completes the proof of the lemma. □

Under the assumption of theorem 1.2.2, by lemma 1.5.7, $|\beta+\gamma+2(-1)^{N-1}| \geq 2$. Hence from Equations (11) and (12) we get

$$\frac{\langle \mathcal{E}^{(m,2)} \rangle_N}{\delta^{(3N-2)m} a_N^m} = \frac{1}{2}(\beta+\gamma+2(-1)^{N-1}) C E(0, \frac{5N}{6}) N(1+O(N^{3\alpha-2})),$$

which, together with the simple fact that

$$\log E(0, \frac{5N}{6}) = -\frac{2N}{\pi} L(\frac{5\pi}{6}) + O(\log N) = \frac{2N}{\pi} L(\frac{\pi}{6}) + O(\log N),$$

confirms the volume conjecture for $\mathcal{E}^{(m,2)}$:

$$2\pi \lim_{N \to \infty, N \in S_m} \frac{\log |\langle \mathcal{E}^{(m,2)} \rangle_N|}{N} = 4L(\frac{\pi}{6}) = \mathrm{Vol}(\mathcal{E}^{(m,2)}).$$

CHAPTER II

ON THE AJ CONJECTURE FOR KNOTS

2.1 Introduction

2.1.1 The AJ conjecture

For a knot K in S^3, let $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$ be the colored Jones polynomial of K colored by the (unique) n-dimensional simple representation of sl_2 [28, 50], normalized so that for the unknot U,

$$J_U(n) = [n] := \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

The color n can be assumed to take negative integer values by setting $J_K(-n) = -J_K(n)$. In particular, $J_K(0) = 0$. It is known that $J_K(1) = 1$, and $J_K(2)$ is the ordinary Jones polynomial.

Define two linear operators L, M acting on the set of discrete functions $f : \mathbb{Z} \to \mathcal{R} := \mathbb{C}[t^{\pm 1}]$ by

$$(Lf)(n) = f(n+1), \qquad (Mf)(n) = t^{2n} f(n).$$

It is easy to see that $LM = t^2 ML$. Besides, the inverse operators L^{-1}, M^{-1} are well-defined. One can consider L, M as elements of the quantum torus

$$\mathcal{T} = \mathcal{R}\langle L^{\pm 1}, M^{\pm 1} \rangle / (LM - t^2 ML),$$

which is not commutative, but almost commutative.

Let

$$\mathcal{A}_K = \{P \in \mathcal{T} \mid PJ_K = 0\},$$

which is a left ideal of $\mathcal{T}$, called the *recurrence ideal* of K. It was proved in [19] that for every knot K, the recurrence ideal $\mathcal{A}_K$ is non-zero. Partial results were obtained earlier by Frohman, Gelca, and Lofaro through their theory of non-commutative A-ideal [15, 21]. An element in $\mathcal{A}_K$ is called a recurrence relation for the colored Jones polynomial of K.

The ring $\mathcal{T}$ is not a principal left-ideal domain, i.e. not every left-ideal of $\mathcal{T}$ is generated by one element. In [17], by adding the inverses of polynomials in t, M to $\mathcal{T}$ one gets a principal left-ideal domain $\tilde{\mathcal{T}}$, and a generator α_K of the extension $\tilde{\mathcal{A}}_K$ of $\mathcal{A}_K$. The element α_K can be presented in the form

$$\alpha_K(t; M, L) = \sum_{j=0}^{d} \alpha_{K,j}(t, M)\, L^j,$$

where the degree in L is assumed to be minimal and all the coefficients $\alpha_{K,j}(t, M) \in \mathbb{Z}[t^{\pm 1}, M]$ are assumed to be co-prime. The polynomial α_K is defined up to a polynomial in $\mathbb{Z}[t^{\pm 1}, M]$. Moreover, one can choose $\alpha_K \in \mathcal{A}_K$, i.e. it is a recurrence relation for the colored Jones polynomial. We call α_K the *recurrence polynomial* of K.

Garoufalidis [17] formulated the following conjecture (see also [15, 21]).

Conjecture 2. (AJ conjecture) *For every knot K, $\alpha_K|_{t=-1}$ is equal to the A-polynomial, up to a factor depending on M only.*

Here in the definition of the A-polynomial [10], we also allow the abelian component of the character variety, see Section 2.

2.1.2 Main results

Conjecture 2 was established for a large class of two-bridge knots, including all twist knots, by T. Le using skein theory [36]. Here we have the following stronger results.

Theorem 5. *Suppose K is a knot satisfying all the following conditions:*

(i) K is hyperbolic,

(ii) The SL_2-character variety of $\pi_1(S^3 \setminus K)$ consists of 2 irreducible components (one abelian and one non-abelian),

(iii) The universal SL_2-character ring of $\pi_1(S^3 \setminus K)$ is reduced,

(iv) The recurrence polynomial of K has L-degree greater than 1.

Then the AJ conjecture holds true for K.

Theorem 6. *The following knots satisfy all the conditions (i)–(iv) of Theorem 5 and hence the AJ conjecture holds true for them.*

(a) All pretzel knots of type $(-2, 3, 6n \pm 1)$, $n \in \mathbb{Z}$.

(b) All two-bridge knots for which the character variety has exactly 2 irreducible components; these includes all twist knots, double twist knots of the form $J(k, l)$ with $k \neq l$ in the notation of [24], all two-bridge knots $\mathfrak{b}(p, m)$ with p prime or $m = 3$. Here we use the notation $\mathfrak{b}(p, m)$ from [8].

Remark 2.1.1. Besides the infinitely many cases of two-bridge knots listed in Theorem 6, explicit calculation seems to confirm that "most two-bridge knots" satisfy the conditions of Theorem 5 and hence AJ conjecture holds for them. In fact, among 155 $\mathfrak{b}(p, m)$ with $p < 45$, only 9 knots do not satisfy the conditions of Theorem 5.

2.1.3 Other results

In our proof of Theorem 6, it is important to know whether the universal character ring of a knot group is reduced, i.e. whether its nil-radical is 0. Although it is difficult to construct a group whose universal character ring is not reduced (see [38]), so far there are a few groups for which the universal character ring is known to be reduced: free groups [52], surface groups [11], two-bridge knot groups [48], torus knot groups [39], two-bridge link groups (see Chapter 3). In the present chapter, we show that the universal character ring of the $(-2, 3, 2n+1)$-pretzel knot is reduced for all integer n.

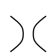
2.1.4 Plan of the chapter

We review skein modules and their relation with the colored Jones polynomial in Section 2.2. In Section 2.3 we prove some properties of the character variety and the A-polynomial a knot. We discuss the role of the quantum peripheral polynomial in the AJ conjecture and give the proof of Theorem 5 and Theorem 6 in Section 2.4. In Section 2.5, we prove the reducedness of the universal character ring of the $(-2, 3, 2n+1)$-pretzel knot for all integer n.

2.2 Skein Modules and the colored Jones polynomial

In this section we will review skein modules and their relation with the colored Jones polynomial. The theory of Kauffman bracket skein module (KBSM) was introduced by Przytycki [47] and Turaev [56] as a generalization of the Kauffman bracket [29] in S^3 to an arbitrary 3-manifold. The KBSM of a knot complement contains a lot, if not all, of information about the colored Jones polynomial.

2.2.1 Skein modules

Recall that $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$. A *framed link* in an oriented 3-manifold Y is a disjoint union of embedded circles, equipped with a non-zero normal vector field. Framed links are considered up to isotopy. Let $\mathcal{L}$ be the set of isotopy classes of framed links in the manifold Y, including the empty link. Consider the free $\mathcal{R}$-module with basis $\mathcal{L}$, and factor it by the smallest submodule containing all expressions of the form  $- t$  $- t^{-1}$  and  $+ (t^2 + t^{-2})\emptyset$, where the links in each expression are identical except in a ball in which they look like depicted. This quotient is denoted by $\mathcal{S}(Y)$ and is called the Kauffman bracket skein module, or just skein module, of Y.

For an oriented surface Σ we define $\mathcal{S}(\Sigma) = \mathcal{S}(Y)$, where $Y = \Sigma \times [0, 1]$, the cylinder over Σ. The skein module $\mathcal{S}(\Sigma)$ has an algebra structure induced by the operation of gluing one cylinder on top of the other. The operation of gluing the cylinder over ∂Y to Y induces a $\mathcal{S}(\partial Y)$-left module structure on $\mathcal{S}(Y)$.

2.2.2 The skein module of S^3 and the colored Jones polynomial

When $Y = S^3$, the skein module $\mathcal{S}(Y)$ is free over $\mathcal{R}$ of rank one, and is spanned by the empty link. Thus if ℓ is a framed link in S^3, then its value in the skein module $\mathcal{S}(S^3)$ is $\langle \ell \rangle$ times the empty link, where $\langle \ell \rangle \in \mathcal{R}$, known as the Kauffman bracket of ℓ [29], and is just the Jones polynomial of *framed links* in a suitable normalization.

Let $S_n(z)$ be the Chebychev polynomials defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_{n+1}(z) = zS_n(z) - S_{n-1}(z)$ for all $n \in \mathbb{Z}$. For a framed knot K in S^3 and an integer $n \geq 0$, we define the n-th power K^n as the link consisting of n parallel copies of K. Using these powers of a knot, $S_n(K)$ is defined as an element of $\mathcal{S}(S^3)$. We then define the colored Jones polynomial $J_K(n)$ by the equation

$$J_K(n+1) := (-1)^n \times \langle S_n(K) \rangle.$$

The $(-1)^n$ sign is added so that for the unknot U, $J_U(n) = [n]$. Then $J_K(1) = 1$, $J_K(2) = -\langle K \rangle$. We extend the definition for all integers n by $J_K(-n) = -J_K(n)$ and $J_K(0) = 0$. In the framework of quantum invariants, $J_K(n)$ is the sl_2-quantum invariant of K colored by the n-dimensional simple representation of sl_2.

2.2.3 The skein module of the torus

Let $\mathbb{T}^2$ be the torus with a fixed pair (μ, λ) of simple closed curves intersecting at exactly 1 point. For co-prime k and l, let $\lambda_{k,l}$ be a simple closed curve on the torus homologically equal to $k\mu + l\lambda$. It is not difficult to show that the skein algebra $\mathcal{S}(\mathbb{T}^2)$ of the torus is generated, as an $\mathcal{R}$-algebra, by all $\lambda_{k,l}$. In fact, Bullock and Przytycki [7] showed that $\mathcal{S}(\mathbb{T}^2)$ is generated over $\mathcal{R}$ by 3 elements μ, λ and $\lambda_{1,1}$, subject to some explicit relations.

Recall that $\mathcal{T} = \mathcal{R}\langle M^{\pm 1}, L^{\pm 1} \rangle / (LM - t^2 ML)$ is the quantum torus. Let $\sigma : \mathcal{T} \to \mathcal{T}$ be the involution defined by $\sigma(M^k L^l) = M^{-k} L^{-l}$. Frohman and Gelca [14] showed that there is an algebra isomorphism $\Upsilon : \mathcal{S}(\mathbb{T}^2) \to \mathcal{T}^\sigma$ given by

$$\Upsilon(\lambda_{k,l}) = (-1)^{k+l} t^{kl} (M^k L^l + M^{-k} L^{-l}).$$

The fact that $\mathcal{S}(\mathbb{T}^2)$ and $\mathcal{T}^\sigma$ are isomorphic algebras was also proved by Sallenave [51].

2.2.4 The orthogonal and peripheral ideals

Let $N(K)$ be a tubular neighborhood of an oriented knot K in S^3, and X the closure of $S^3 \setminus N(K)$. Then $\partial(N(K)) = \partial(X) = \mathbb{T}^2$. There is a standard choice of meridian μ and longitude λ on $\mathbb{T}^2$ such that the linking number between the longitude and the knot is 0. We use this pair (μ, λ) and the map Υ in the previous subsection to identify $\mathcal{S}(\mathbb{T}^2)$ with $\mathcal{T}^\sigma$.

The torus $\mathbb{T}^2 = \partial(N(K))$ cut S^3 into two parts: $N(K)$ and X. We can consider $\mathcal{S}(X)$ as a left $\mathcal{S}(\mathbb{T}^2)$-module and $\mathcal{S}(N(K))$ as a right $\mathcal{S}(\mathbb{T}^2)$-module. There is a bilinear bracket

$$\langle \cdot, \cdot \rangle : \mathcal{S}(N(K)) \otimes_{\mathcal{S}(\mathbb{T}^2)} \mathcal{S}(X) \to \mathcal{S}(S^3) \equiv \mathcal{R}$$

given by $\langle \ell', \ell'' \rangle = \langle \ell' \cup \ell'' \rangle$, where ℓ' and ℓ'' are links in respectively $N(K)$ and X. Note that if $\ell \in \mathcal{S}(\mathbb{T}^2)$, then

$$\langle \ell' \cdot \ell, \ell'' \rangle = \langle \ell', \ell \cdot \ell'' \rangle.$$

In general $\mathcal{S}(X)$ does not have an algebra structure, but it has a unit: The empty link serves as the unit. The map

$$\Theta : \mathcal{S}(\mathbb{T}^2) \to \mathcal{S}(X), \quad \Theta(\ell) = \ell \cdot \emptyset$$

is $\mathcal{S}(\mathbb{T}^2)$-linear. Its kernel $\mathcal{P} = \ker \Theta$ is called the *quantum peripheral ideal*, first introduced in [15]. In [15, 21], it was proved that every element in $\mathcal{P}$ gives rise to a recurrence relation for the colored Jones polynomial.

The *orthogonal ideal* $\mathcal{O}$ in [15] is defined by

$$\mathcal{O} := \{\ell \in \mathcal{S}(\partial X) \quad | \quad \langle \ell', \Theta(\ell)\rangle = 0 \quad \text{for every } \ell' \in \mathcal{S}(N(K))\}.$$

It is clear that $\mathcal{O}$ is a left ideal of $\mathcal{S}(\partial X) \equiv \mathcal{T}^\sigma$ and $\mathcal{P} \subset \mathcal{O}$. In [15], $\mathcal{O}$ was called the formal ideal. According to [36], if $\mathcal{P} = \mathcal{O}$ for all knots then the colored Jones polynomial distinguishes the unknot from other knots.

2.2.5 Relation between the recurrence ideal and the orthogonal ideal

As mentioned above, the skein algebra of the torus $\mathcal{S}(\mathbb{T}^2)$ can be identified with $\mathcal{T}^\sigma$ via the $\mathcal{R}$-algebra isomorphism Υ sending μ, λ and $\lambda_{1,1}$ to respectively $-(M + M^{-1}), -(L + L^{-1})$ and $t(ML + M^{-1}L^{-1})$.

Proposition 2.2.1. *One has*

$$(-1)^n \langle S_{n-1}(\lambda), \Theta(\ell)\rangle = \Upsilon(\ell) J_K(n)$$

for all $\ell \in \mathcal{S}(\mathbb{T}^2)$.

Proof. We know from the properties of the Jones-Wenzl idempotent (see e.g. [46]) that

$$\begin{aligned}
\langle S_{n-1}(\lambda) \cdot \mu, \emptyset\rangle &= (t^{2n} + t^{-2n})\langle S_{n-1}(\lambda), \emptyset\rangle \\
\langle S_{n-1}(\lambda) \cdot \lambda, \emptyset\rangle &= \langle S_n(\lambda) + S_{n-2}(\lambda), \emptyset\rangle \\
\langle S_{n-1}(\lambda) \cdot \lambda_{1,1}, \emptyset\rangle &= -\langle t^{2n+1} S_n(\lambda) + t^{-2n+1} S_{n-2}(\lambda), \emptyset\rangle.
\end{aligned}$$

By definition $J_K(n) = (-1)^{n-1}\langle S_{n-1}(\lambda), \emptyset\rangle$ and $(MJ_K)(n) = t^{2n} J_K(n)$, $(LJ_K)(n) = J_K(n+1)$. Hence the above equations can be rewritten as

$$\begin{aligned}
(-1)^n \langle S_{n-1}(\lambda), \Theta(\mu)\rangle &= -(M + M^{-1}) J_K(n) = \Upsilon(\mu) J_K(n), \\
(-1)^n \langle S_{n-1}(\lambda), \Theta(\lambda)\rangle &= -(L + L^{-1}) J_K(n) = \Upsilon(\lambda) J_K(n), \\
(-1)^n \langle S_{n-1}(\lambda), \Theta(\lambda_{1,1})\rangle &= t(ML + M^{-1}L^{-1}) J_K(n) = \Upsilon(\lambda_{1,1}) J(n).
\end{aligned}$$

Since $\mathcal{S}(\mathbb{T}^2)$ is generated by μ, λ and $\lambda_{1,1}$, we conclude that

$$(-1)^n \langle S_{n-1}(\lambda), \Theta(\ell) \rangle = \Upsilon(\ell) J_K(n)$$

for all $\ell \in \mathcal{S}(\mathbb{T}^2)$. □

Noting that $\{S_n(\lambda)\}_n$ generates the skein module $\mathcal{S}(N(K))$, we get

Corollary 2.2.2. *One has* $\mathcal{O} = \mathcal{A}_K \cap \mathcal{T}^\sigma$.

Remark 2.2.3. Corollary 2.2.2 was already obtained in [16] by another method. Our proof here uses the properties of the Jones-Wenzl idempotent only.

2.3 Character varieties and the A-polynomial

For non-zero $f, g \in \mathbb{C}[M, L]$, we say that f is *M-essentially equal to* g, and write $f \overset{M}{=} g$, if the quotient f/g does not depend on L. We say that f is *M-essentially divisible by* g if f is M-essentially equal to a polynomial divisible by g.

2.3.1 The character variety of a group

The set of representations of a finitely presented group G into $SL_2(\mathbb{C})$ is an algebraic set defined over $\mathbb{C}$, on which $SL_2(\mathbb{C})$ acts by conjugation. The naive quotient space, i.e. the set of orbits, does not have a good topology/geometry. Two representations in the same orbit (i.e. conjugate) have the same character, but the converse is not true in general. A better quotient, the algebro-geometric quotient denoted by $\chi(G)$ (see [12, 38]), has the structure of an algebraic set. There is a bijection between $\chi(G)$ and the set of all characters of representations of G into $SL_2(\mathbb{C})$, hence $\chi(G)$ is usually called the *character variety* of G. For a manifold Y we use $\chi(Y)$ also to denote $\chi(\pi_1(Y))$.

Suppose $G = \mathbb{Z}^2$, the free abelian group with 2 generators. Every pair of generators λ, μ will define an isomorphism between $\chi(G)$ and $(\mathbb{C}^*)^2/\tau$, where $(\mathbb{C}^*)^2$ is the set of non-zero complex pairs (L, M) and τ is the involution $\tau(M, L) = (M^{-1}, L^{-1})$, as follows: Every representation is conjugate to an upper diagonal one, with L and M being the upper left entry of λ and μ respectively. The isomorphism does not change if one replaces (λ, μ) with (λ^{-1}, μ^{-1}).

2.3.2 The universal character ring

For a finitely presented group G, the character variety $\chi(G)$ is determined by the traces of some fixed elements $g_1, \cdots, g_k$ in G. More precisely, one can find $g_1, \cdots, g_k$ in G such that for every element $g \in G$ there exists a polynomial $\mathbf{P}_g$ in k variables such that for any representation $r : G \to SL_2(\mathbb{C})$ we have $\mathrm{tr}(r(g)) = \mathbf{P}_g(x_1, \cdots, x_k)$ where $x_j := \mathrm{tr}(r(g_j))$. The universal character ring of G is then defined to be the quotient of the ring $\mathbb{C}[x_1, \cdots, x_k]$ by the ideal generated by all expressions of the form $\mathrm{tr}(r(v)) - \mathrm{tr}(r(w))$, where v and w are any two words in $g_1, \cdots, g_k$ which are equal in G. The universal character ring of G is actually independent of the choice of $g_1, \cdots, g_k$. The quotient of the universal character ring of G by its nil-radical is equal to the ring of regular functions on the character variety of G.

2.3.3 The A-polynomial

Let X be the closure of S^3 minus a tubular neighborhood $N(K)$ of a knot K. The boundary of X is a torus whose fundamental group is free abelian of rank 2. An orientation of K will define a unique pair of an oriented meridian and an oriented longitude such that the linking number between the longitude and the knot is 0, as in Subsection 2.2.4. The pair provides an identification of $\chi(\partial X)$ and $(\mathbb{C}^*)^2/\tau$ which actually does not depend on the orientation of K.

The inclusion $\partial X \hookrightarrow X$ induces the restriction map

$$\rho : \chi(X) \longmapsto \chi(\partial X) \equiv (\mathbb{C}^*)^2/\tau$$

Let Z be the image of ρ and $\hat{Z} \subset (\mathbb{C}^*)^2$ the lift of Z under the projection $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2/\tau$. The Zariski closure of $\hat{Z} \subset (\mathbb{C}^*)^2 \subset \mathbb{C}^2$ in $\mathbb{C}^2$ is an algebraic set consisting of components of dimension 0 or 1. The union of all the 1-dimension components is defined by a single polynomial $A_K \in \mathbb{Z}[M, L]$, whose coefficients are co-prime. Note that A_K is defined up to ± 1. We call A_K the *A-polynomial* of K. By definition, A_K does not have repeated factors. It is known that A_K is always divisible by $L - 1$. The A-polynomial here is actually equal to $L - 1$ times the A-polynomial defined in [10].

2.3.4 The B-polynomial

It is also instructive and convenient to see the dual picture in the construction of the A-polynomial. For an algebraic set V (over $\mathbb{C}$) let $\mathbb{C}[V]$ denote the ring of regular functions on V. For example, $\mathbb{C}[(\mathbb{C}^*)^2/\tau] = \mathfrak{t}^\sigma$, the σ-invariant subspace of $\mathfrak{t} := \mathbb{C}[L^{\pm 1}, M^{\pm 1}]$, where $\sigma(M^k L^l) = M^{-k} L^{-l}$.

The map ρ in the previous subsection induces an algebra homomorphism

$$\theta : \mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\sigma \longrightarrow \mathbb{C}[\chi(X)].$$

We will call the kernel $\mathfrak{p}$ of θ the *classical peripheral ideal*; it is an ideal of $\mathfrak{t}^\sigma$. Let $\hat{\mathfrak{p}} := \mathfrak{t}\,\mathfrak{p}$ be the ideal extension of $\mathfrak{p}$ in $\mathfrak{t}$. The set of zero points of $\hat{\mathfrak{p}}$ is the closure of $\hat{Z}$ in $\mathbb{C}^2$.

The ring $\mathfrak{t} = \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$ embeds naturally into the principal ideal domain $\tilde{\mathfrak{t}} := \mathbb{C}(M)[L^{\pm 1}]$, where $\mathbb{C}(M)$ is the fractional field of $\mathbb{C}[M]$. The ideal extension of $\hat{\mathfrak{p}}$ in $\tilde{\mathfrak{t}}$, which is $\tilde{\mathfrak{t}}\,\hat{\mathfrak{p}} = \tilde{\mathfrak{t}}\,\mathfrak{p}$, is thus generated by a single polynomial $B_K \in \mathbb{Z}[M, L]$ which has co-prime coefficients and is defined up to a factor $\pm M^k$ with $k \in \mathbb{Z}$. Again B_K can be chosen to have integer coefficients because everything can be defined over $\mathbb{Z}$. We will call B_K the *B-polynomial* of K.

2.3.5 Relation between the A-polynomial and the B-polynomial

From the definitions one has immediately that the polynomial B_K is M-essentially divisible by A_K. Moreover, their zero sets $\{B_K = 0\}$ and $\{A_K = 0\}$ are equal, up to some lines parallel to the L-axis in the LM-plane.

Proposition 2.3.1. *The B-polynomial B_K does not have repeated factors.*

Proof. We first note that the ring $\mathbb{C}[\chi(X)]$ has a $\mathfrak{t}^\sigma$-module structure via the algebra homomorphism $\theta : \mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\sigma \to \mathbb{C}[\chi(X)]$, hence a $\mathbb{C}[M^{\pm 1}]^\sigma$-module structure since $\mathbb{C}[M^{\pm 1}]^\sigma$ is a subring of $\mathfrak{t}^\sigma$.

The extension from $\mathbb{C}[M^{\pm 1}]^\sigma$ to $\mathbb{C}(M)$ can be done in two steps: The first one is from $\mathbb{C}[M^{\pm 1}]^\sigma$ to its field of fractions $\mathbb{C}(x)$ where $x := M + M^{-1}$; the second step is from $\mathbb{C}(x)$ to $\mathbb{C}(M)$. Each step is a flat extension. The extension from $\mathfrak{t}^\sigma$ to $\tilde{\mathfrak{t}}$ can be viewed as the extension from $\mathbb{C}[M^{\pm 1}]^\sigma$ to $\mathbb{C}(M)$: One can easily check that

$$\tilde{\mathfrak{t}} = \mathfrak{t}^\sigma \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M), \quad \tilde{\mathfrak{p}} = \mathfrak{p} \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M).$$

We want to show that $\tilde{\mathfrak{p}}$ is radical, i.e. $\sqrt{\tilde{\mathfrak{p}}} = \tilde{\mathfrak{p}}$. Here $\sqrt{\tilde{\mathfrak{p}}}$ denotes the radical of $\tilde{\mathfrak{p}}$.

Let

$$\bar{\mathfrak{t}} = \mathfrak{t}^\sigma \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(x), \quad \overline{\mathfrak{p}} = \mathfrak{p} \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(x).$$

Note that $\mathfrak{p}$, the kernel of $\theta : \mathfrak{t}^\sigma \to \mathbb{C}[\chi(X)]$, is radical since the ring $\mathbb{C}[\chi(X)]$ is reduced. We claim that $\overline{\mathfrak{p}}$ is also radical. Indeed, suppose $\gamma \in \bar{\mathfrak{t}}$ and $\gamma^2 \in \overline{\mathfrak{p}}$. Then $\gamma^2 = \delta/f$ for some $\delta \in \mathfrak{p}$ and $f \in \mathbb{C}[x]$. It implies that $(f\gamma)^2 = f\delta$ is in $\mathfrak{p}$. Hence $f\gamma \in \sqrt{\mathfrak{p}} = \mathfrak{p}$ which means $\gamma \in \overline{\mathfrak{p}}$.

Since $\bar{\mathfrak{t}} = \mathbb{C}(x)[L^{\pm 1}]$ is a principal ideal domain, the radical ideal $\overline{\mathfrak{p}}$ can be generated by one element, say $\gamma(L) \in \mathbb{C}(x)[L^{\pm 1}]$, which does not have repeated factors. Note that the polynomial $\gamma(L)$ and $\delta(L) := \gamma'(L)$, the derivative of $\gamma(L)$ with respect to L, are co-prime. Since $\mathbb{C}(x)[L^{\pm 1}]$ is an Euclidean domain, there are $f, g \in \mathbb{C}(x)$ such that $f\gamma + g\delta = 1$. It follows that $\gamma(L)$ and $\delta(L)$ are also co-prime in $\mathbb{C}(M)[L^{\pm 1}]$. Hence the ideal $\tilde{\mathfrak{p}} = \overline{\mathfrak{p}} \otimes_{\mathbb{C}(x)} \mathbb{C}(M)$ in $\mathbb{C}(M)[L^{\pm 1}]$ is radical. This means that the B-polynomial B_K does not have repeated factors. □

Corollary 2.3.2. *For every knot K one has*

$$B_K = \frac{A_K}{\text{its } M\text{-factor}}.$$

Here the M-factor of A_K is the maximal factor of A_K depending on M only; it is defined up to a non-zero complex number.

Remark 2.3.3. Corollary 2.3.2 implies that the A-polynomial and B-polynomial are equal for a knot K if the A-polynomial of K has trivial M-factor.

2.3.6 Small knots

A knot K is called *small* if its complement X does not contain closed essential surfaces. It is known that all 2-bridge knots and all 3-tangle pretzel knots are small [26, 45].

Proposition 2.3.4. *Suppose K is a small knot. Then the A-polynomial A_K has trivial M-factor. Hence the A-polynomial and B-polynomial of a small knot are equal.*

Proof. The A-polynomial A_K always contains the factor $L-1$ coming from characters of abelian representations [10]. Hence we write $A_K = (L-1)A_{nab}$ where A_{nab} is a polynomial in $\mathbb{C}[M, L]$.

Suppose the polynomial A_{nab} of a knot has non-trivial M-factor, then the Newton polygon of A_{nab} has the slope infinity. It is known that every slope of the Newton polygon of A_{nab} is a boundary slope of the knot complement [10]. Hence the knot complement has boundary slope infinity. The complement of a small knot does not have boundary slope infinity by a result in [9], hence its polynomial A_{nab} has trivial M-factor. □

Remark 2.3.5. There exists a non-small knot whose A-polynomial has non-trivial M-factor; it is the knot 9_{38} in the Rolfsen table, see [27].

A knot K is called SL_2*-small* if the character variety $\chi(X)$ has dimension 1. It is known that every small knot is SL_2-small, see e.g. [10].

Proposition 2.3.6. *Suppose K is SL_2-small. Then both $\mathbb{C}[\chi(X)]$ and the universal character ring have finite rank over $\mathbb{C}[M+M^{-1}]$.*

Proof. By [38], the universal character ring of K is the quotient of the polynomial ring $R := \mathbb{C}[x, x_1, \dots, x_k]$ by an ideal $\mathcal{I} \subset \mathbb{C}[x, x_1, \dots, x_k]$, where $x = M + M^{-1}$ is the trace of the meridian and $x_1, \dots, x_k$ are traces of some fixed elements in the knot group. Then the character ring of K is $\mathbb{C}[\chi(X)] = \mathbb{C}[x, x_1, \dots, x_k]/\sqrt{\mathcal{I}}$.

We first claim that the $\mathbb{C}[x]$-module $\sqrt{\mathcal{I}}/\mathcal{I}$ has finite rank. Indeed, since R is Noetherian $\sqrt{\mathcal{I}}$ can be generated by a finite number of elements f_j's. Choose N such that $f_j^N \in \mathcal{I}$ for all j. Then $(\sqrt{\mathcal{I}})^N \subset \mathcal{I}$ and hence the rank of the $\mathbb{C}[x]$-module $\sqrt{\mathcal{I}}/\mathcal{I}$ is less than or equal to N.

Consider the exact sequence of $\mathbb{C}[x]$-modules

$$0 \to \sqrt{\mathcal{I}}/\mathcal{I} \to R/\sqrt{\mathcal{I}} \to R/\mathcal{I} \to 0.$$

One has $\text{rank}_{\mathbb{C}[x]} R/\mathcal{I} = \text{rank}_{\mathbb{C}[x]} R/\sqrt{\mathcal{I}} + \text{rank}_{\mathbb{C}[x]} \sqrt{\mathcal{I}}/\mathcal{I}$. Hence to prove the theorem, it suffices to show that the $\mathbb{C}[x]$-module $\mathbb{C}[\chi(X)]$ has finite rank for a SL_2-small knot K.

Let V_j, $1 \le j \le l$, be irreducible components of $\chi(X)$. Since $\mathbb{C}[\chi(X)]$ embeds into the product $\mathbb{C}[V_1] \times \cdots \times \mathbb{C}[V_l]$, it suffices to show that each $\mathbb{C}[V_j]$ has finite rank, as a $\mathbb{C}[x]$-module.

Let V be an irreducible component of the character variety. Let $pr : \mathbb{C}^{k+1} \to \mathbb{C}$ be the projection onto the first coordinate axis. Then $pr(V)$ is either (i) one point, or (ii) the Zariski closure of $pr(V)$ is $\mathbb{C}$, i.e. $pr(V)$ is $\mathbb{C}$ minus a finite number of points.

In case (ii): the map pr is a finite map. Hence $\mathbb{C}[V]$ has finite rank over $\mathbb{C}[x]$.

In case (i): V is determined by the ideal $(J, x - x_0)$, where $J \subset \mathbb{C}[x_1, \dots, x_k]$ and $x_0 \in \mathbb{C}$. It follows that $\mathbb{C}[V]$ is a torsion $\mathbb{C}[x]$-module: it is annihilated by $x - x_0$. In this case $\mathbb{C}[V]$ has rank 0, although as a $\mathbb{C}[x]$-module it is infinitely generated. □

2.4 Skein modules and the AJ conjecture

Our proof of the main theorems is more or less based on the ideology that the KBSM is a quantization of the $SL_2(\mathbb{C})$-character variety [5, 48] which has been exploited in the work of Frohman, Gelca, and Lofaro [15] where they defined the non-commutative A-ideal. In this section we will discuss the quantum peripheral polynomial and its role in our approach to the AJ conjecture, and then prove Theorems 5 and 6.

2.4.1 Skein modules as quantizations of character varieties

Let ε be the map reducing $t = -1$. An important result [5, 48] in the theory of skein modules is that $\varepsilon(\mathcal{S}(Y))$ has a natural $\mathbb{C}$-algebra structure and is isomorphic to the universal SL_2-character algebra of the fundamental group of Y. The product of 2 links in $\varepsilon(\mathcal{S}(Y))$ is their union. Using the skein relation with $t = -1$, it is easy to see that the product is well-defined, and that the value of a knot in the skein module depends only on the homotopy class of the knot in Y. The isomorphism between $\varepsilon(\mathcal{S}(Y))$ and the universal SL_2-character algebra of $\pi_1(Y)$ is given by $K(r) = -\operatorname{tr} r(K)$, where K is a homotopy class of a knot in Y, represented by an element, also denoted by K, of $\pi_1(Y)$, and $r : \pi_1(Y) \to SL_2(\mathbb{C})$ is a representation of $\pi_1(Y)$. The quotient of $\varepsilon(\mathcal{S}(Y))$ by its nilradical is canonically isomorphic to $\mathbb{C}[\chi(Y)]$, the ring of regular functions on the SL_2-character variety of $\pi_1(Y)$.

In many cases $\varepsilon(\mathcal{S}(Y))$ is reduced, i.e. its nilradical is 0, and hence $\varepsilon(\mathcal{S}(Y))$ is exactly equal to the ring of regular functions on the character variety of $\pi_1(Y)$. For example, this is the case when Y is a torus, or when Y is the complement of a two-bridge knot/link (see [36, 48] and Chapter 3), or when Y is the complement of the $(-2, 3, 2n+1)$-pretzel knot for any integer n (see Section 2.5 below). We conjecture that

Conjecture 3. *For every knot K the universal character ring is reduced.*

2.4.2 The peripheral polynomial and its role in the AJ conjecture

Suppose for a knot K, the nilradical of $\varepsilon(\mathcal{S}(X)$ is trivial. One has the following commutative diagram

$$\begin{array}{ccc} \mathcal{S}(\partial X) \equiv \mathcal{T}^\sigma & \xrightarrow{\Theta} & \mathcal{S}(X) \\ \varepsilon \big\downarrow & & \varepsilon \big\downarrow \\ \mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\sigma & \xrightarrow{\theta} & \mathbb{C}[\chi(X)]. \end{array}$$

Recall that the classical peripheral ideal $\mathfrak{p}$ is the kernel of θ; it is an ideal of $\mathfrak{t}^\sigma$. The ring $\mathfrak{t}^\sigma$ embeds into the principal ideal domain $\widetilde{\mathfrak{t}} = \mathbb{C}(M)[L^{\pm 1}]$. The ideal extension $\widetilde{\mathfrak{p}} = \widetilde{\mathfrak{t}}\mathfrak{p}$ of $\mathfrak{p}$ in $\widetilde{\mathfrak{t}}$ is generated by the B-polynomial B_K.

Let us now adapt the construction of the polynomial B_K to the quantum setting. By definition the quantum peripheral ideal $\mathcal{P}$ is the kernel of Θ; it is a left-ideal of $\mathcal{T}^\sigma$. The ring $\mathcal{T}^\sigma$ embeds into the principal left-ideal domain $\widetilde{\mathcal{T}}$ which can be formally defined as follows. Let $\mathcal{R}(M)$ be the fractional field of the polynomial ring $\mathcal{R}[M]$. Let $\tilde{\mathcal{T}}$ be the set of all Laurent polynomials in the variable L with coefficients in $\mathcal{R}(M)$:

$$\tilde{\mathcal{T}} = \{\sum_{j\in\mathbb{Z}} f_j(M)L^j \mid \quad f_j(M) \in \mathcal{R}(M), \;\; f_j = 0 \quad \text{almost everywhere}\},$$

and define the product in $\tilde{\mathcal{T}}$ by $f(M)L^k \cdot g(M)L^l = f(M)\, g(t^{2k}M)L^{k+l}$.

The left-ideal extension $\widetilde{\mathcal{P}} := \widetilde{\mathcal{T}}\mathcal{P}$ of $\mathcal{P}$ in $\widetilde{\mathcal{T}}$ is then generated by a polynomial

$$\beta_K(t; M, L) = \sum_{j=0}^{d'} \beta_{K,j}(t, M)\, L^j,$$

where d' is assumed to be minimum and all the coefficients $\beta_{K,j}(t, M) \in \mathbb{Z}[t^{\pm 1}, M]$ are co-prime. Note that the polynomial β_K is defined up to $\pm t^k M^l$ with $k, l \in \mathbb{Z}$. We will call β_K the *peripheral polynomial* of K. The polynomial β_K was introduced in [36].

Proposition 2.4.1. *a) $\varepsilon(\beta_K)$ is M-essentially divisible by B_K, and hence is M-essentially divisible by A_K.*

b) β_K is divisible by the recurrence polynomial α_K in the sense that there are polynomials $g(t,M) \in \mathbb{Z}[t,M]$ and $\gamma(t,M,L) \in \mathcal{T}$ such that

$$\beta_K(t,M,L) = \frac{1}{g(t,M)}\,\gamma(t,M,L)\,\alpha_K(t,M,L). \tag{13}$$

Moreover $g(t,M)$ and $\gamma(t,M,L)$ can be chosen so that $\varepsilon(g) \neq 0$.

Proof. See [36]. Note that the first part follows from the fact that $\varepsilon(\mathcal{P}) \subset \mathfrak{p}$ which can be easily deduced from the above commutative diagram, while the second fact follows the fact that the peripheral ideal is contained in the recurrence ideal by Proposition 2.2.2. □

From Proposition 2.4.1, we see that both $\varepsilon(\alpha_K)$ and A_K divide $\varepsilon(\beta_K)$ for a knot K. This observation gives us some important information for studying the AJ conjecture for K. Indeed, we have the following.

Proposition 2.4.2. *Suppose K is a knot satisfying all the following conditions:*

(i) The L-degrees of the polynomials A_K and $\varepsilon(\beta_K)$ are equal,

(ii) The A-polynomial has exactly two irreducible factors,

(iii) The recurrence polynomial α_K has L-degree greater than 1.

Then the AJ conjecture holds for K.

Proof. Since $\varepsilon(\beta_K)$ is M-essentially divisible by A_K, conditon (i) implies that $\varepsilon(\beta_K) \overset{M}{=} A_K$. Combining this with (13), we get

$$\varepsilon(\gamma)\varepsilon(\alpha_K) \overset{M}{=} A_K. \tag{14}$$

It is known that A_K always contains the factor $L-1$ coming from characters of abelian representations [10]), and $\varepsilon(\alpha_K)$ is also divisible by $L-1$ [36, Proposition 2.3]. Hence we can rewrite (14) as follows:

$$\varepsilon(\gamma)\frac{\varepsilon(\alpha_K)}{L-1} \overset{M}{=} \frac{A_K}{L-1}. \tag{15}$$

By (ii), A_K has exactly two irreducible factors. One of them is $L-1$, hence the other factor $\frac{A_K}{L-1}$ is irreducible. Equation (15) then implies that

$$\frac{\varepsilon(\alpha_K)}{L-1} \overset{M}{=} 1, \quad \text{or} \quad \frac{\varepsilon(\alpha_K)}{L-1} \overset{M}{=} \frac{A_K}{L-1}.$$

If $\frac{\varepsilon(\alpha_K)}{L-1} \overset{M}{=} 1$, then, by Lemma 2.4.3 below, the recurrence polynomial α_K has L-degree 1. This contradicts (iii), hence we must have $\frac{\varepsilon(\alpha_K)}{L-1} \overset{M}{=} \frac{A_K}{L-1}$. In other words, the AJ conjecture holds true for K. □

Lemma 2.4.3. *The polynomial $\varepsilon(\alpha_K)$ is M-essentially equal to $L-1$ if and only if the L-degree of the recurrence polynomial α_K is* 1*.*

Proof. The backward direction is obvious since $\varepsilon(\alpha_K)$ is always divisible by $L-1$.

Now suppose the polynomial $\varepsilon(\alpha_K)$ is M-essentially equal to $L-1$, i.e. $\varepsilon(\alpha_K) = g(M)(L-1)$ for some non-zero $g(M) \in \mathbb{C}[M^{\pm 1}]$. Then we have

$$\alpha_K = g(M)(L-1) + (1+t)\sum_{j=0}^{d} f_j(M)L^j \tag{16}$$

where $f_j(M)$'s are Laurent polynomials in $\mathcal{R}[M^{\pm 1}]$ and d is the L-degree of α_K.

By a result in [16], the recurrence ideal $\mathcal{A}_K$ is invariant under the involution σ. Hence $\sigma(\alpha_K)$ is contained in $\mathcal{A}_K$. Since α_K is the generator, it follows that $\alpha_K = h(M)\sigma(\alpha_K)L^d$ for some $h(M) \in \mathcal{R}(M)$. Equation (16) implies that

$$g(M)(L-1) + (1+t)\sum_{j=0}^{d} f_j(M)L^j$$

$$= h(M)g(M^{-1})(L^{-1}-1)L^d + (1+t)\sum_{j=0}^{d} h(M)f_j(M^{-1})L^{n-j}.$$

If $d > 1$ then by comparing the coefficients of L^0 in both sides of the above equation, we get $-g(M) + (1+t)f_0(M) = (1+t)h(M)f_d(M^{-1})$, i.e.

$$g(M) = (1+t)\left(f_0(M) - h(M)f_d(M^{-1})\right) \tag{17}$$

Since $g(M)$ is a Laurent polynomial in M with coefficients in $\mathbb{C}$, equation (17) implies that $g(M)$ must be equal to 0. This is a contradiction. Hence we must have $d = 1$. □

2.4.3 The L-degree of the A-polynomial

Suppose K is a hyperbolic knot. Then it has discrete faithful $SL_2(\mathbb{C})$-representations. Let χ_0 denote an irreducible component of $\chi(X)$ containing the character of a discrete faithful representation. By a result of Thurston [54], χ_0 has dimension 1 since X has one boundary component.

Recall that the inclusion $\partial X \hookrightarrow X$ induces the restriction map $\rho : \chi(X) \to \chi(\partial X)$. Dunfield [13] showed that the map $\rho \mid_{\chi_0}: \chi_0 \to \chi(\partial X)$ is a birational isomorphism onto its image. Suppose K is hyperbolic and the character variety $\chi(X)$ consists of 2 irreducible conponents (one abelian and one non-abelian containing the characters of discrete faithful representations). Then the restriction map $\rho : \chi(X) \to \chi(\partial X)$ is a birational isomorphism onto its image. i.e. $\chi(X)$ and its image are equal up to adding a finite number of points, since $\chi(X)$ has dimension 1. It implies that the map

$$\tilde{\theta} : \mathbb{C}(M)[L^{\pm 1}] = \mathbb{C}[\chi(\partial X)] \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M) \to \mathbb{C}[\chi(X)] \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M),$$

induced by ρ, is surjective. Hence the polynomial B_K, the generator of the kernel of $\tilde{\theta}$, has L-degree equal to the dimension of the $\mathbb{C}(M)$-vector space $\mathbb{C}[\chi(X)] \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M)$, which is equal to the rank of the $\mathbb{C}[M + M^{-1}]$-module $\mathbb{C}[\chi(X)]$. Since the A-polynomial is M-essentially equal to the B-polynomial by Corollary 2.3.2, we have

Proposition 2.4.4. *Suppose K is a knot satisfying all the following conditions:*

(i) K is hyperbolic,

(ii) The SL_2-character variety of $\pi_1(S^3 \setminus K)$ consists of 2 irreducible components (one abelian and one non-abelian).

Then the L-degree of the A-polynomial of K is equal to the rank of the $\mathbb{C}[M + M^{-1}]$-module $\mathbb{C}[\chi(X)]$.

2.4.4 Localization and the L-degree of the peripheral polynomial

2.4.4.1 Localization

Recall that $\mathcal{T}$ is the quantum torus and ε is the map reducing $t=-1$. Let $D := \mathcal{R}[M^{\pm 1}]$ and consider the localization of D at the ideal $(1+t)$:

$$\overline{D} := \{\frac{f}{g} \mid f, g \in D,\ \varepsilon(g) \neq 0\}.$$

Note that $\overline{D}$, being a localization of D, is flat over D. The ring $D = \mathcal{R}[M^{\pm 1}]$ is flat over $\mathcal{R}[M^{\pm 1}]^\sigma$, since it is free over $\mathcal{R}[M^{\pm 1}]^\sigma$:

$$\mathcal{R}[M^{\pm 1}] = \mathcal{R}[M^{\pm 1}]^\sigma \oplus M\,\mathcal{R}[M^{\pm 1}]^\sigma.$$

Tensoring $\mathcal{T}$ with $\overline{D}$, we get

$$\overline{\mathcal{T}} := \{\sum_{j\in\mathbb{Z}} f_j(M)L^j \mid f_j(M) \in \overline{D},\ f_j = 0 \text{ almost everywhere}\}$$

with commutation rule: $f(M)L^k \cdot g(M)L^l = f(M)g(t^{2k}M)L^{k+l}$. Note that if in the definition of $\overline{\mathcal{T}}$ we allow $f_j(M)$ to be in the fractional field $\mathcal{R}(M)$ of D then we get $\widetilde{\mathcal{T}}$.

For a left-ideal I of $\mathcal{T}$ (or $\mathcal{T}^\sigma$) let $\overline{I}$ and $\widetilde{I}$ be its extensions in $\overline{\mathcal{T}}$ and $\widetilde{\mathcal{T}}$ respectively.

2.4.4.2 The L-degree of the peripheral polynomial

The involution σ acts on D. Let D^σ denote the σ-invariant part of D. There is a natural $D^\sigma = \mathcal{R}[\bar{x}]$-module structure on $\mathcal{S}(X)$, here $\bar{x} \equiv -(M + M^{-1})$ is a meridian thus belongs to the boundary of X. Hence by tensoring $\mathcal{S}(X)$ with $\overline{D}$, $\overline{\mathcal{S}(X)} := \mathcal{S}(X) \otimes_{D^\sigma} \overline{D}$ is an $\overline{D}$-module. Note that $\overline{\mathcal{S}(\mathbb{T}^2)} := \mathcal{S}(\mathbb{T}^2) \otimes_{D^\sigma} \overline{D}$ is a free $\overline{D}$-module with basis $\{L^j : j \in \mathbb{Z}\}$.

Consider the following exact sequence of D^σ-modules

$$0 \to \mathcal{P} \hookrightarrow \mathcal{T}^\sigma \xrightarrow{\Theta} \mathcal{S}(X)$$

Since $\overline{D}$ is a flat over D^σ, the following sequence is also exact

$$0 \to \overline{D} \otimes_{D^\sigma} \mathcal{P} \hookrightarrow \overline{D} \otimes_{D^\sigma} \mathcal{T}^\sigma \stackrel{id \otimes \Theta}{\longrightarrow} \overline{D} \otimes_{D^\sigma} \mathcal{S}(X)$$

i.e. the sequence of $\overline{D}$-modules

$$0 \to \overline{\mathcal{P}} \hookrightarrow \overline{\mathcal{T}} \stackrel{\overline{\Theta}}{\longrightarrow} \overline{\mathcal{S}(X)} \tag{18}$$

is exact. Note that $\overline{\mathcal{S}(X)}$ is a module over the principal ideal domain $\overline{D}$.

Suppose $\overline{\mathcal{S}(X)} \otimes_{\overline{D}} \mathbb{C}(M)$ is a *finite dimensional* vector space over $\mathbb{C}(M)$. Consider the exact sequence (18). The third module $\overline{\mathcal{S}(X)}$ is a $\overline{D}$-module of finite rank and hence

$$\overline{\mathcal{S}(X)} = \overline{D}^k \bigoplus \bigoplus_{j=1}^{l} \overline{D}/(f_j).$$

Here each f_j is a power of $(1+t)$. The middle module $\overline{\mathcal{T}}$ of (10) is free $\overline{D}$-module with basis $\{L^j : j \in \mathbb{Z}\}$. Hence the image of $k+l+1$ elements $1, L, L^2, \ldots, L^{k+l}$ are linearly dependent. Hence there must be a non-trivial element δ in the kernel $\overline{\mathcal{P}}$ of L-degree less than or equal to $k+l$. This element δ is in $\widetilde{\mathcal{P}}$, hence it divides β_K, the generator of $\widetilde{\mathcal{P}}$. From this, we conclude that β_K is non-trivial and has L-degree less that or equal to $k+l$, which is the dimension of the vector space $\overline{\mathcal{S}(X)} \otimes_{\overline{D}} \mathbb{C}(M)$ over $\mathbb{C}(M)$ since $\overline{\mathcal{S}(X)} \otimes_{\overline{D}} \mathbb{C}(M) = \mathbb{C}(M)^{k+l}$.

It is easy to check that $\overline{\mathcal{S}(X)} \otimes_{\overline{D}} \mathbb{C}(M) = \varepsilon(\mathcal{S}(X)) \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M)$. Since the dimension of the $\mathbb{C}(M)$-vector space $\varepsilon(\mathcal{S}(X)) \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M)$ is equal to the rank of the $\mathbb{C}[M+M^{-1}]$-module $\varepsilon(\mathcal{S}(X))$, we have the following.

Proposition 2.4.5. *Suppose for a knot K, $\varepsilon(\mathcal{S}(X))$ is a finite rank $\mathbb{C}[M+M^{-1}]$-module. Then the L-degree of the peripheral polynomial β_K is less than or equal the rank of the $\mathbb{C}[M+M^{-1}]$-module $\varepsilon(\mathcal{S}(X))$.*

2.4.5 Proof of Theorems 5 and 6

2.4.5.1 Proof of Theorem 5

Suppose the knot K satisfies all the conditions of the theorem. By assumptions (i), (ii) and Proposition 2.4.4, the L-degree of the A-polynomial A_K is equal to the rank of the $\mathbb{C}[M+M^{-1}]$-module $\mathbb{C}[\chi(X)]$. Since the universal SL_2-character variety is reduced, $\varepsilon(\mathcal{S}(X)) = \mathbb{C}[\chi(X)]$. Hence the L-degree of the A-polynomial A_K is also equal to the the rank of the $\mathbb{C}[M+M^{-1}]$-module $\varepsilon(\mathcal{S}(X))$. This, together with Proposition 2.4.5, implies that the L-degree of β_K is less than or equal to that of A_K. From this, one can easily check that the knot K satisfies all the conditions of Proposition 2.4.2 and hence the AJ conjecture holds true for K.

2.4.5.2 Proof of Theorem 6

It is known that two-bridge knots and $(-2,3,2n+1)$-pretzel knots, excluding torus knots, are hyperbolic. (Note that the AJ conjecture holds true for torus knots by [25] and Chapter 4). Their universal character rings are reduced by [36] and Theorem 2.5.6 below, respectively. The L-degrees of their recurrence polynomials are greater than 1 according to the results in [36, Proposition 2.2] and [18, Section 4.7] respectively. Double twist knots of the form $J(k,l)$ with $k \neq l$, two-bridge knots of the form $\mathfrak{b}(p,m)$ with p prime or $m = 3$, and $(-2,3,6n \pm 1)$-pretzel knots satisfy assumption (ii) of Theorem 5 by [42], [23] and [6], and [40] respectively. Hence the theorem follows.

2.5 The universal character ring of $(-2,3,2n+1)$-pretzel knots

In this section we explicitly calculate the universal character ring of the $(-2,3,2n+1)$-pretzel knot and prove its reducedness for all integer n.

2.5.1 The character variety

For the $(-2, 3, 2n+1)$-pretzel knot K_{2n+1}, we have

$$\pi_1(X) = \langle a, b, c \mid cacb = acba,\ ba(cb)^n = a(cb)^n c\rangle,$$

where $X = S^3 \setminus K_{2n+1}$ and a, b, c are meridians depicted in Figure 1.

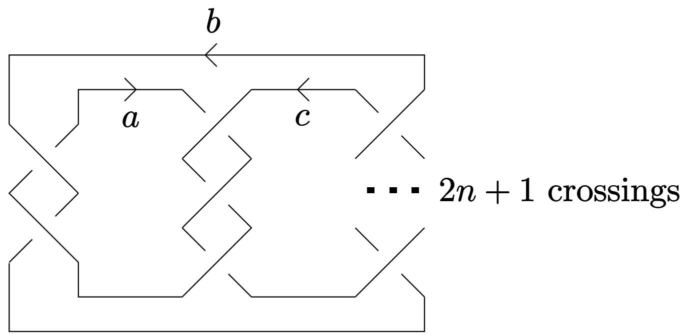


Figure 1: The $(-2, 3, 2n+1)$-pretzel knot

Let $w = cb$ then the first relation of $\pi_1(X)$ becomes $caw = awa$. It implies that $c = awaw^{-1}a^{-1}$ and $b = c^{-1}w = awa^{-1}w^{-1}a^{-1}w$. The second relation then has the form

$$awa^{-1}w^{-1}a^{-1}waw^n = aw^n awaw^{-1}a^{-1}$$

i.e.

$$w^n awa^{-1}w^{-1}a^{-1} = a^{-1}w^{-1}awaw^{-1}w^n.$$

Hence we obtain a presentation of $\pi_1(X)$ with two generators and one relation

$$\pi_1(X) = \langle a, w \mid w^n E = F w^n\rangle$$

where $E := awa^{-1}w^{-1}a^{-1}$ and $F := a^{-1}w^{-1}awaw^{-1}$.

The character variety of the free group $F_2 = \langle a, w\rangle$ in 2 letters a and w is isomorphic to $\mathbb{C}^3$ by the Fricke-Klein-Vogt theorem, see [38]. For every element $\omega \in F_2$ there is a unique polynomial $\mathbf{P}_\omega$ in 3 variables such that for any representation $r : F_2 \to SL_2(\mathbb{C})$ we have $\mathrm{tr}(r(\omega)) = \mathbf{P}_\omega(x, y, z)$ where $x := \mathrm{tr}(r(a)),\ y := \mathrm{tr}(r(w))$

and $z := \operatorname{tr}(r(aw))$. The polynomial $\mathbf{P}_\omega$ can be calculated inductively using the following identities for traces of matrices $A, B \in SL_2(\mathbb{C})$:

$$\operatorname{tr}(A) = \operatorname{tr}(A^{-1}), \quad \operatorname{tr}(AB) + \operatorname{tr}(AB^{-1}) = \operatorname{tr}(A)\operatorname{tr}(B). \tag{19}$$

Thus for every representation $r : \pi_1(X) \to SL_2(\mathbb{C})$, we consider x, y, and z as functions of r. The character variety of $\pi_1(X)$ is the zero locus of an ideal in $\mathbb{C}[x, y, z]$, which we describe explicitly in the next theorem.

Theorem 2.5.1. *The character variety of the pretzel knot K_{2n+1} is the zero locus of 2 polynomials $P := \mathbf{P}_E - \mathbf{P}_F$ and $Q_n := \mathbf{P}_{w^n Ea} - \mathbf{P}_{Fw^n a}$. Explicitly,*

$$P = x - xy + (-3 + x^2 + y^2)z - xyz^2 + z^3, \tag{20}$$

$$\begin{aligned} Q_n &= S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y) - S_{n-2}(y)\, x^2 \\ &\quad + \big(S_{n-1}(y) + S_{n-3}(y) + S_{n-4}(y)\big)\, xz - \big(S_{n-2}(y) + S_{n-3}(y)\big)\, z^2 \end{aligned} \tag{21}$$

where $S_n(y)$ are the Chebychev polynomials defined by $S_0(y) = 1$, $S_1(y) = y$ and $S_{n+1}(y) = yS_n(y) - S_{n-1}(y)$ for all integer n.

Proof. The explicit formulas (20) and (21) follow from an easy calculation of the trace polynomials using (19).

Because E and F are conjugate (by w^n) and $w^n Ea = Fw^n a$ in $\pi_1(X)$, we have $P = Q_n = 0$ for every representation $r : \pi_1(X) \to SL_2(\mathbb{C})$.

We will prove the converse: fix a solution (x, y, z) of $P = Q_n = 0$, we will find a representation $r : \pi_1(X) \to SL_2(\mathbb{C})$ such that $x = \operatorname{tr}(r(a))$, $y = \operatorname{tr}(r(w))$ and $z = \operatorname{tr}(r(aw))$.

We consider the following 3 cases:

<u>Case 1:</u> $y^2 \neq 4$. Then there exist $s, u, v \in \mathbb{C}$ such that $s + s^{-1} = y$, $u + v = x$, $su + s^{-1}v = z$. Since $S_k(y) = \frac{s^{k+1} - s^{-k-1}}{s - s^{-1}}$ for all integer k, we have

$$\begin{aligned} P &= s^{-3}(s-1)P', \\ Q_n &= s^{-3-n}\big((s^{2n}u - sv)P' - (1+s)(-1+uv)Q_n'\big), \end{aligned}$$

where

$$
\begin{aligned}
P' &= s^3u - s^4u - s^5u + v + sv - s^2v - s^2u^2v - s^3u^2v + s^4u^2v + s^5u^2v \\
&\quad - uv^2 - suv^2 + s^2uv^2 + s^3uv^2, \\
Q'_n &= s^5 + s^{2n} - s^{2+2n}u^2 + s^{4+2n}u^2 + s^3uv - s^5uv - s^{2n}uv + s^{2+2n}uv + sv^2 - s^3v^2.
\end{aligned}
$$

Since $s \neq \pm 1$, $P = Q_n = 0$ is equivalent to $P' = (-1 + uv)Q'_n = 0$. We consider the following 2 subcases:

Subcase 1.1: $Q'_n = 0$. Choose $r(a) = \begin{pmatrix} u & 1 \\ uv - 1 & v \end{pmatrix}$ and $r(w) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$. It is easy to check $x = \operatorname{tr}(r(a)),\ y = \operatorname{tr}(r(w)),\ z = \operatorname{tr}(r(aw))$ and the calculations in the following 2 lemmas.

Lemma 2.5.2. *One has*

$$
r(E) = \begin{pmatrix} s^{-2}H_{11} & -s^{-2}H_{12} \\ s^{-2}(-1+uv)H_{21} & -s^{-2}H_{22} \end{pmatrix},\ r(F) = \begin{pmatrix} -s^{-3}H_{22} & -s^{-1}H_{21} \\ s^{-3}(-1+uv)H_{12} & s^{-1}H_{11} \end{pmatrix}
$$

where

$$
\begin{aligned}
H_{11} &= s^2u - s^4u + v - s^2u^2v + s^4u^2v - uv^2 + s^2uv^2, \\
H_{12} &= 1 - s^2u^2 + s^4u^2 - uv + s^2uv, \\
H_{21} &= -s^4 - s^2uv + s^4uv - v^2 + s^2v^2, \\
H_{22} &= -s^4u + v - s^2v - s^2u^2v + s^4u^2v - uv^2 + s^2uv^2.
\end{aligned}
$$

Lemma 2.5.3. *One has*

$$
r(w^nE - Fw^n) = \begin{pmatrix} s^{-3+n}P' & -s^{-2-n}Q'_n \\ -s^{-3-n}(-1+uv)Q'_n & -s^{-2-n}P' \end{pmatrix}.
$$

Since $P' = Q'_n = 0$, Lemma 2.5.3 implies that $r(w^nE - Fw^n) = 0$, i.e. $r(w^nE) = r(Fw^n)$.

Subcase 1.2: $-1 + uv = 0$ then $v = u^{-1}$. In this case the equation $P' = 0$ becomes $s^2u^{-1}(s - u^2) = 0$ i.e. $s = u^2$. Let

$$r(a) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad r(w) = \begin{pmatrix} u^2 & 0 \\ 0 & u^{-2} \end{pmatrix}.$$

Then it is easy to check that $x = \operatorname{tr}(r(a))$, $y = \operatorname{tr}(r(w))$, $z = \operatorname{tr}(r(aw))$ and $r(Ew^n) = r(w^nF)$. (Note that $r(a)$ and $r(w)$ commute in this case).

Case 2: $y = 2$. Then $S_k(y) = k$ for all integer k. Hence

$$\begin{aligned} P &= (x - z)(-1 + xz - z^2), \\ Q_n &= 4 - (n-1)x^2 + (3n-5)xz - (2n-3)z^2. \end{aligned}$$

Hence $(x, z) = (-2, -2), (2, 2)$ or ($x = z + z^{-1}$ and $1 - n + (1 + n)z^2 - z^4 = 0$).

If $x = z = 2$ we choose

$$r(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r(w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $x = z = -2$ we choose

$$r(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad r(w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $x = z + z^{-1}$ and $1 - n + (1 + n)z^2 - z^4 = 0$ we choose

$$r(a) = \begin{pmatrix} z & 0 \\ -z^{-1} & z^{-1} \end{pmatrix}, \quad r(w) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.5.4. *One has*

$$r(w^nE - Fw^n) = \begin{pmatrix} 0 & z^{-1}(-1 + n - (1 + n)z^2 + z^4) \\ 0 & 0 \end{pmatrix}$$

Proof. By direct calculations we have $r(E) = \begin{pmatrix} z & -2z+z^3 \\ 0 & z^{-1} \end{pmatrix}$, $r(F) = \begin{pmatrix} z & z^{-1}-z \\ 0 & z^{-1} \end{pmatrix}$ and $r(w^n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. The lemma follows. □

Hence $x = \mathrm{tr}(r(a)),\ y = \mathrm{tr}(r(w)),\ z = \mathrm{tr}(r(aw))$ and $r(w^n E) = r(F w^n)$.

Case 3: $y = -2$. Then $S_k(y) = (-1)^k k$ for all integer k. Hence

$$\begin{aligned} P &= 3x + z + x^2 z + 2xz^2 + z^3, \\ Q_n &= (-1)^n \big(xP - (x+z)Q''_n\big)/2, \end{aligned}$$

where $Q''_n = x + 2nx + 2z + x^2 z + xz^2$.

Hence the system $P = Q_n = 0$ is equivalent to $P = (x+z)Q''_n = 0$. We consider the following 2 subcases:

Subcase 3.1: $x + z = 0$. Then it is easy to see that $P = 0$ is equivalent to $x = z = 0$. In this case we choose

$$r(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad r(w) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

where i is the imaginary number.

Subcase 3.2: $x + z \neq 0$. Then $Q''_n = 0$. Choose

$$r(a) = \begin{pmatrix} x/2 & (1 - x^2/4)/(x+z) \\ -x-z & x/2 \end{pmatrix}, \quad r(w) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Lemma 2.5.5. *One has*

$$r(w^n E - F w^n) = (-1)^n \begin{pmatrix} nP - Q''_n & Q''_n/2 \\ 0 & -(n-1)P + Q''_n \end{pmatrix}.$$

Proof. By direct calculations, we have $r(w^n) = (-1)^n \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and

$$r(E) = \begin{pmatrix} -(x+2z+x^2z+xz^2)/2 & -\frac{4+3x^2+4xz+x^3z+x^2z^2}{4(x+z)} \\ (x+z)(1+xz+z^2) & (3x+2z+x^2z+xz^2)/2 \end{pmatrix},$$

$$r(F) = \begin{pmatrix} (x+2z+x^2z+xz^2)/2 & -\frac{4+5x^2+10xz+3x^3z+4z^2+5x^2z^2+2xz^3}{4(x+z)} \\ (x+z)(1+xz+z^2) & (-5x-4z-3x^2z-5xz^2-2z^3)/2. \end{pmatrix}$$

The lemma follows. ∎

Hence $x = \mathrm{tr}(r(a)),\ y = \mathrm{tr}(r(w)),\ z = \mathrm{tr}(r(aw))$ and $r(w^nE) = r(Fw^n)$ in all cases. It implies that the character variety of the pretzel knot K_{2n+1} is exactly equal to the algebraic set $\{P = Q_n = 0\}$. ∎

2.5.2 The universal character ring

In this subsection, we will prove the following theorem.

Theorem 2.5.6. *The universal character ring of K_{2n+1} is reduced and is equal to the ring $\mathbb{C}[x, y, z]/(P, Q_n)$.*

Proof. Suppose we have shown that the ring $\mathbb{C}[x, y, z]/(P, Q_n)$ is reduced, then it is exactly the character ring $\mathbb{C}[\chi(X)]$ of K_{2n+1}.

Recall that $\pi_1(X) = \langle a, w \mid w^nE = Fw^n \rangle$ where $F_2 = \langle a, w \rangle$ is the free group on two generators a, w. It is known that the universal character ring of F_2 is the ring $\mathbb{C}[x, y, z]$ where $x = \mathrm{tr}(r(a)),\ y = \mathrm{tr}(r(w))$ and $z = \mathrm{tr}(r(aw))$ as above. The quotient map $h : F_2 \to \pi_1(X)$ induces the epimorphism $h_* : \mathbb{C}[x, y, z] \to \varepsilon(\mathcal{S}(X))$. Since P, Q_n come from traces, they are contained in $\ker h_*$.

Since $\mathbb{C}[\chi(X))]$ is the quotient of $\varepsilon(\mathcal{S}(X))$ by its nilradical, we have the quotient homomorphism $\phi : \varepsilon(\mathcal{S}(X)) \to \mathbb{C}[\chi(X))] = \mathbb{C}[x, y, z]/(P, Q_n)$. Then

$$\phi \circ h_* : \mathbb{C}[x, y, z] \to \varepsilon(\mathcal{S}(X)) \to \mathbb{C}[\chi(\pi)] = \mathbb{C}[x, y, z]/(P, Q_n)$$

is a homomorphism. It follows that $\ker h_* \subseteq (P, Q_n)$. Hence we must have $\ker h_* = (P, Q_n)$, which implies $\varepsilon(\mathcal{S}(X)) \cong \mathbb{C}[x,y,z]/(P,Q_n) \equiv \mathbb{C}[\chi(X)]$.

In the remaining part of this section we will show that the ring $\mathbb{C}[x,y,z]/(P,Q_n)$ is reduced, i.e. the ideal $I_n := (P, Q_n)$ is radical. The proof of this fact will be divided into several steps.

2.5.2.1 $\mathbb{C}[x,y,z]/I_n$ *is free over* $\mathbb{C}[x]$.

Lemma 2.5.7. *For every $x_0 \neq 0, \pm 2$, the polynomial $P\mid_{x=x_0}$ is irreducible in $\mathbb{C}[y,z]$.*

Proof. Assume that $P\mid_{x=x_0}$ can be decomposed as

$$z^3 - x_0 y z^2 + (y^2 + x_0^2 - 3)z + x_0(1-y) = \big(z + f_1\big)\big(z^2 - (x_0 y + f_1)z + f_2\big), \quad (22)$$

where $f_j \in \mathbb{C}[y]$. Equation (22) implies that $f_2 - f_1(x_0 y + f_1) = y^2 + x_0^2 - 3$ and $f_1 f_2 = x_0(1-y)$.

If f_1 is a constant then $f_2 = x_0(1-y)/f_1$ has y-degree 1. Hence $f_2 - f_1(x_0 y + f_1)$ has y-degree 1 also. It implies that $f_2 - f_1(x_0 y + f_1) \neq y^2 + x_0^2 - 3$.

If f_2 is a constant then $f_1 = x_0(1-y)/f_2$. Hence

$$\begin{aligned} f_2 - f_1(x_0 y + f_1) &= f_2 - \big(\frac{x_0}{f_2} - \frac{x_0}{f_2}y\big)\big(\frac{x_0}{f_2} - \frac{x_0}{f_2}y + x_0 y\big) \\ &= \frac{x_0^2}{f_2}\big(1 - \frac{1}{f_2}\big)y^2 - \frac{x_0^2}{f_2}\big(1 - \frac{2}{f_2}\big)y + \big(f_2 - \frac{x_0^2}{f_2^2}\big). \end{aligned}$$

Then since $f_2 - f_1(x_0 y + f_1) = y^2 + x_0^2 - 3$, we have $\frac{x_0^2}{f_2}\big(1 - \frac{1}{f_2}\big) = 1$, $\frac{x_0^2}{f_2}\big(1 - \frac{2}{f_2}\big) = 0$, and $f_2 - \frac{x_0^2}{f_2^2} = x_0^2 - 3$. This implies $x_0 = 0$ or $x_0 = \pm 2$. ☐

Lemma 2.5.8. *For every x_0, the polynomials $P\mid_{x=x_0}$ and $Q_n\mid_{x=x_0}$ are co-prime in $\mathbb{C}[y,z]$.*

Proof. If $x_0 \neq 0, \pm 2$ then, by Lemma 2.5.7, $P\mid_{x=x_0}$ is irreducible in $\mathbb{C}[y,z]$. Lemma 2.5.8 then follows since $P\mid_{x=x_0}$ and $Q_n\mid_{x=x_0}$ have z-degrees 3 and 2 respectively.

At $x_0 = 0$, we have $P = z(-3 + y^2 + z^2)$ and $Q_n = a_n + b_n z^2$ where

$$\begin{aligned} a_n &= S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y), \\ b_n &= -S_{n-2}(y) - S_{n-3}(y). \end{aligned}$$

In this case, it suffices to show that $Q_n \mid_{z^2=3-y^2} = a_n + b_n(3 - y^2) \neq 0$. This is true by Lemma 2.5.11 below.

At $x_0 = 2$, we have $P = \big(z + 1 - y\big)\big(z^2 - (1 + y)z + 2\big)$ and $Q_n = a'_n + b'_n z + c'_n z^2$ where $a'_n, b'_n, c'_n \in \mathbb{C}[y]$. When $z = y - 1$, we have $Q_0 = 1$ and $Q_1 = y - 1$ and $Q_{n+1} = yQ_n - Q_{n-1}$. It implies that $Q_n \mid_{z=y-1}$ is a polynomial of y-degree n and hence is not identically 0. It remains to show that $Q_n = a'_n + b'_n z + c'_n z^2 \neq c'_n(z^2 - (1+y)z + 2)$. It suffices to show that $b'_n \mid_{y=-1} \neq 0$. Indeed, when $x_0 = 2$ and $y = -1$ we have $b'_n = 2\big(S_{n-1}(-1) + S_{n-3}(-1) + S_{n-4}(-1)\big)$. It is easy to check that $S_k(-1) = 1$ if $k = 0 \pmod 3$, $S_k(-1) = -1$ if $k = 1 \pmod 3$ and $S_k(-1) = 0$ otherwise. Hence $b'_n = 2\big(S_{n-1}(-1) + S_{n-3}(-1) + S_{n-4}(-1)\big) \neq 0$.

The case $x_0 = -2$ is similar. □

Proposition 2.5.9. *$\mathbb{C}[x, y, z]/I_n$ is a free $\mathbb{C}[x]$-module.*

Proof. Since $\sqrt{I_n} = \sqrt{I}$, Proposition 2.3.6 implies that $\mathbb{C}[x, y, z]/I_n$ has finite rank over $\mathbb{C}[x]$. It suffices to show that $\mathbb{C}[x, y, z]/I_n$ is a torsion-free $\mathbb{C}[x]$-module. Suppose $S \in \mathbb{C}[x, y, z]$ and $(x - x_0)S \in I_n$ for some $x_0 \in \mathbb{C}$. We will show that $S \in I_n$.

Indeed, we have $(x - x_0)S = fP - gQ_n$ for some $f, g \in \mathbb{C}[x, y, z]$. Hence $(fP) \mid_{x=x_0} = (gQ_n) \mid_{x=x_0}$ which implies that $f \mid_{x=x_0}$ is divisible by $Q_n \mid_{x=x_0}$, since $P \mid_{x=x_0}$ and $Q_n \mid_{x=x_0}$ are co-prime in $\mathbb{C}[y, z]$ by Lemma 2.5.8. Hence $f_{x=x_0} = h\, Q_n \mid_{x=x_0}$ for some $h \in \mathbb{C}[y, z]$. From this, we may write $f = h\, Q_n + (x - x_0)Q$ for some $Q \in \mathbb{C}[x, y, z]$. Hence

$$(x - x_0)S = fP - gQ_n = \big(hQ_n + (x - x_0)Q\big)P - gQ_n = (x - x_0)QP + (hP - g)Q_n$$

which implies that $hP - g$ is divisible by $x - x_0$ and $S = QP + \frac{hP-g}{x-x_0} Q_n \in I_n$. □

2.5.2.2 Reduction to a special case

Proposition 2.5.10. *I_n is radical if $I_n\mid_{x=x_0}$ is radical for some $x_0 \in \mathbb{C}$.*

Proof. We first note that $I_n\mid_{x=x_0} = I_n \otimes_{\mathbb{C}[x]} \mathbb{C}$, where $\mathbb{C}$ is considered as an $\mathbb{C}[x]$-module by reducing $x = x_0$.

Let $R = \mathbb{C}[x, y, z]$. Consider the exact sequence of $\mathbb{C}[x]$-modules

$$0 \to \sqrt{I_n}/I_n \to R/\sqrt{I_n} \to R/I_n \to 0.$$

Since R/I_n is free by Proposition 2.5.9, the sequence splits and hence $\sqrt{I_n}/I_n$ is isomorphic to free sub-module of R/I_n. Let k be the rank of the $\mathbb{C}[x]$-module $\sqrt{I_n}/I_n$ then the rank of the $\mathbb{C}$-module $(\sqrt{I_n}/I_n)\mid_{x=x_0}$ is always k for every $x_0 \in \mathbb{C}$. Hence if $I_n\mid_{x=x_0}$ is radical for some $x_0 \in \mathbb{C}$ then $k = 0$ which implies that $\sqrt{I_n} = I_n$. □

2.5.2.3 $I_n\big|_{x=0}$ is radical

By Lemma 2.5.8, $P\big|_{x=0}$ and $Q_n\big|_{x=0}$ are co-prime. This means $I_n\big|_{x=0}$ is a zero-dimensional ideal of $\mathbb{C}[y, z]$. By Seidenberg's lemma (see [33, Proposition 3.7.15]), if there exist two non-zero free-square polynomials in $I_n\big|_{x=0} \cap \mathbb{C}[y]$ and $I_n\big|_{x=0} \cap \mathbb{C}[z]$ respectively, then $I_n\big|_{x=0}$ is radical.

From now on we fix $x = 0$. Then $P = z(-3 + y^2 + z^2)$ and $Q_n = a_n + b_n z^2$ where

$$\begin{aligned} a_n &= S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y), \\ b_n &= -S_{n-2}(y) - S_{n-3}(y). \end{aligned}$$

Let $U_n = a_n + b_n(3 - y^2)$. Then $U_0 = 1, U_1 = y + 1$ and $U_{n+1} = yU_n - U_{n-1}$. Hence

$$U_n = S_n(y) + S_{n-1}(y).$$

Lemma 2.5.11. *One has*

$$U_n = \prod_{j=1}^{n} (y - 2\cos\frac{j2\pi}{2n+1}).$$

Proof. It is easy to see that U_n is a polynomial of degree n in y. Note that if $y = s + s^{-1} \neq \pm 2$ then $S_k(y) = \frac{s^{k+1} - s^{-k-1}}{s - s^{-1}}$. We now take $y = e^{i\frac{j2\pi}{2n+1}} + e^{-i\frac{j2\pi}{2n+1}} = 2\cos\frac{j2\pi}{2n+1}$ where $1 \leq j \leq n$. Then

$$S_n(y) = \frac{\sin((n+1)\frac{j2\pi}{2n+1})}{\sin(\frac{j2\pi}{2n+1})} = -\frac{\sin(n\frac{j2\pi}{2n+1})}{\sin(\frac{j2\pi}{2n+1})} = -S_{n-1}(y).$$

The lemma follows. □

Lemma 2.5.12. *One has*

$$a_n = \prod_{k=0}^{n-3}(y - 2\cos\frac{(2k+1)\pi}{2n-5}).$$

Proof. The proof is similar to that of the previous lemma. □

Note that

$$b_n^2\, z\, P = b_n\, z^2\left((-3+y^2)b_n + b_n z^2\right) = (Q_n - a_n)(Q_n - U_n).$$

Hence $a_n\, U_n = a_n\, Q_n - Q_n^2 + Q_n\, U_n + b_n^2\, z\, P$ is contained in $I_n\mid_{x=0}$. But $a_n\, U_n$ is a polynomial in y, hence it is actually contained in $I_n\mid_{x=0} \cap \mathbb{C}[y]$. It is easy to see that $a_n\, U_n$ is square-free, i.e. does not have repeated factors.

Let

$$V_n = z\prod_{j=1}^{n}(-3 + 4\cos^2\frac{j2\pi}{2n+1} + z^2)\prod_{k=0}^{n-3}(-3 + 4\cos^2\frac{(2k+1)\pi}{2n-5} + z^2).$$

Then it is easy to show that $V_n \in \mathbb{C}[z]$ is square-free. Moreover, since

$$\begin{aligned} V_n &= z\prod_{j=1}^{n}(-3 + y^2 + z^2 + (4\cos^2\frac{j2\pi}{2n+1} - y^2)) \\ &\quad\times\prod_{k=0}^{n-3}(-3 + y^2 + z^2 + (4\cos^2\frac{(2k+1)\pi}{2n-5} - y^2)) \\ &\equiv z\prod_{j=1}^{n}(4\cos^2\frac{j2\pi}{2n+1} - y^2)\prod_{k=0}^{n-3}(4\cos^2\frac{(2k+1)\pi}{2n-5} - y^2) \pmod{P}, \\ &\equiv 0 \pmod{(P, a_nU_n)} \end{aligned}$$

it is contained in $I_n \mid_{x=0}$. Hence V_n is in $I_n \mid_{x=0} \cap \mathbb{C}[z]$ and is square-free.

Since both $a_n U_n \in I_n \mid_{x=0} \cap \mathbb{C}[y]$ and $V_n \in I_n \mid_{x=0} \cap \mathbb{C}[z]$ are square-free, $I_n \mid_{x=0}$ is a radical ideal by Seidenberg's lemma. Hence by Proposition 2.5.10, I_n is also radical. It implies that R/I_n is reduced. Hence the ring $\mathbb{C}[x, y, z]/(P, Q_n)$ is reduced and is equal to the universal character ring of K_{2n+1}. This completes the proof of Theorem 2.5.6. □

CHAPTER III

THE SKEIN MODULE OF TWO-BRIDGE LINKS

3.1 Introduction

The theory of Kauffman bracket skein module (KBSM) was introduced by Przytycki [47] and Turaev [56] as a generalization of the Kauffman bracket [29] in S^3 to an arbitrary 3-manifold. The KBSM of a knot complement contains a lot, if not all, of information about the colored Jones polynomial. It also contains a lot of information about classical geometric invariants such as the character variety, and has been instrumental in the study of the AJ conjecture which relates the colored Jones polynomial and the A-polynomial of a knot, see [15, 21, 17, 36] and Chapter 2. The calculation of the KBSM of a knot complement is a difficult task. At the moment, the KBSM has been calculated only for two-bridge knots [36] (with earlier work for twist knots [3]) and torus knots [39] (with earlier work for $(2, 2m+1)$-torus knots [4]). In this chapter, we calculate the KBSM of the complement of all two-bridge links (see Chapter 2 for a definition of the skein module of a 3-manifold). Applications to the theory of AJ conjecture for links will be discussed in a future work.

3.1.1 Main Results

A two-bridge link is a two-component link $L \subset S^3$ such that there is a 2-sphere $S^2 \subset S^3$ separating S^3 into 2 balls B_1 and B_2, and the intersection of L and each ball is isotopic to 2 trivial arcs in the ball. The branched double covering of S^3 along a two-bridge link is a lens space $L(2p, q)$, which is obtained by doing a $2p/q$ surgery on the unknot. Such a two-bridge link is denoted by $\mathfrak{b}(2p, q)$. Here $\gcd(q, 2p) = 1$, and one can always assume that $2p > q \geq 1$. It is known that $\mathfrak{b}(2p', q')$ is isotopic to $\mathfrak{b}(2p, q)$ if and only if $p' = p$ and $q' \equiv q^{\pm 1} \pmod{2p}$, see [8].

Assume the 3-ball B_1 is presented as a vertical cylinder $B_1 = D \times [0,1]$, where D is a 2-dimensional disk, and the two arcs of L inside B_1 are two vertical line segments $U \times [0,1]$ and $U' \times [0,1]$, where U and U' are 2 interior points of D. Let $D_{**} = D \setminus \{U, U'\}$, then $B_1 \setminus L = D_{**} \times [0,1]$. Hence $\mathcal{S}(B_1 \setminus L) = \mathcal{S}(D_{**})$ is an algebra. Let $x, x' \subset D_{**}$ are respectively small loops around U, U', and $y = \partial D \subset D_{**}$ is the boundary of D. We consider x, x', and y as elements of the algebra $\mathcal{S}(B_1 \setminus L)$. Using the embedding $(B_1 \setminus L) \subset (S^3 \setminus L)$ we will consider $x^a (x')^b y^c$ as an element of $\mathcal{S}(S^3 \setminus L)$.

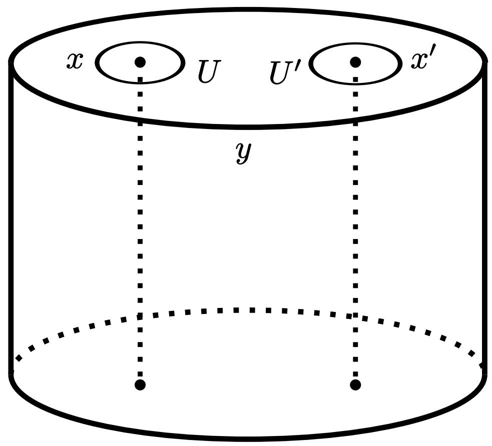


Figure 2: The loops x, x' and y

Theorem 7. *For the two-bridge link $L = \mathfrak{b}(2p, q)$, the skein module $\mathcal{S}(S^3 \setminus L)$ is free over $\mathbb{C}[t^{\pm 1}]$ with basis $\{x^a (x')^b y^c \mid 0 \le a, b,\ 0 \le c \le p\}$.*

3.1.2 The universal character ring

Let $\varepsilon(\mathcal{S}(Y))$ be the quotient of $\mathcal{S}(Y)$ by the relation $t = -1$. An important result [5, 48] in the theory of skein modules is that $\varepsilon(\mathcal{S}(Y))$ has a natural $\mathbb{C}$-algebra structure and is isomorphic to the universal SL_2-character algebra of the fundamental group of Y. For a definition of the universal character algebra, see [1, 38]. The product of 2 links in $\varepsilon(\mathcal{S}(Y))$ is their union. Using the skein relation with $t = -1$, it is easy to see that the product is well-defined, and that the value of a knot in the skein module depends only on the homotopy class of the knot in Y. The isomorphism between $\varepsilon(\mathcal{S}(Y))$ and the universal SL_2-character algebra of $\pi_1(Y)$ is given by $K(\rho) = -\mathrm{tr}\,\rho(K)$, where K is a homotopy class of a knot in Y, represented by an element, also denoted by K, of $\pi_1(Y)$, and $\rho : \pi_1(Y) \to SL_2(\mathbb{C})$ is a representation of $\pi_1(Y)$. The quotient of $\varepsilon(\mathcal{S}(Y))$ by its nilradical is canonically isomorphic to $\mathbb{C}[\chi(\pi_1(Y))]$, the ring of regular functions on the SL_2-character variety of $\pi_1(Y)$.

The above fact has been exploited in the work of Frohman, Gelca, and Lofaro [15] where they defined the non-commutative A-ideal of a knot, and in our proof of the AJ conjecture for some classes of two-bridge knots and pretzel knots in [36] and Chapter 2. In our work on the AJ conjecture, it is important to know whether the universal character algebra $\varepsilon(\mathcal{S}(Y))$ is reduced, i.e. whether its nilradical is 0. Although it is difficult to construct a group whose universal character algebra is not reduced (see [38]), so far there are a few groups for which the universal character algebra is known to be reduced: free groups [52], surface groups [11], two-bridge knot groups [48], torus knot groups [39], and some pretzel knot groups (see Chapter 2).

As a consequence of Theorem 7, we will show the following.

Proposition 1. *For a two-bridge link L, the universal SL_2-character algebra $\varepsilon(\mathcal{S}(S^3 \setminus L))$ is reduced, and hence $\varepsilon(\mathcal{S}(S^3 \setminus L))$ is canonically isomorphic to the ring of regular functions on the SL_2-character variety of $\pi_1(S^3 \setminus L)$.*

3.2 Proof of Theorem 7 and Proposition 1

We change the picture and will present the ball $B_1 \subset \mathbb{R}^3$ as the closed ball of radius $\sqrt{2}$ centered at the origin, i.e. $B_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2+x_2^2+x_3^2 \leq 2\}$. We suppose that the two-bridge link $L = \mathfrak{b}(2p, q)$ intersects the interior of B_1 in two straight intervals UV and $U'V'$ in the x_1x_2-plane, where $U = (-1, 1, 0)$, $U' = (1, 1, 0), V = (-1, -1, 0)$ and $V' = (1, -1, 0)$, see Figure 3. After an isotopy, we assume that the part of L outside the interior of B_1 are 2 non-intersecting arcs $\mathfrak{u}$ and $\mathfrak{u}'$ on the sphere $S = \partial B_1$, where $\mathfrak{u}$ connects U and V, and $\mathfrak{u}'$ connects U' and V'. If one cuts S along the arc

$\mathfrak{u}$, then one obtains a disk, hence the other arc $\mathfrak{u}'$, is uniquely determined by $\mathfrak{u}$, up to isotopy.

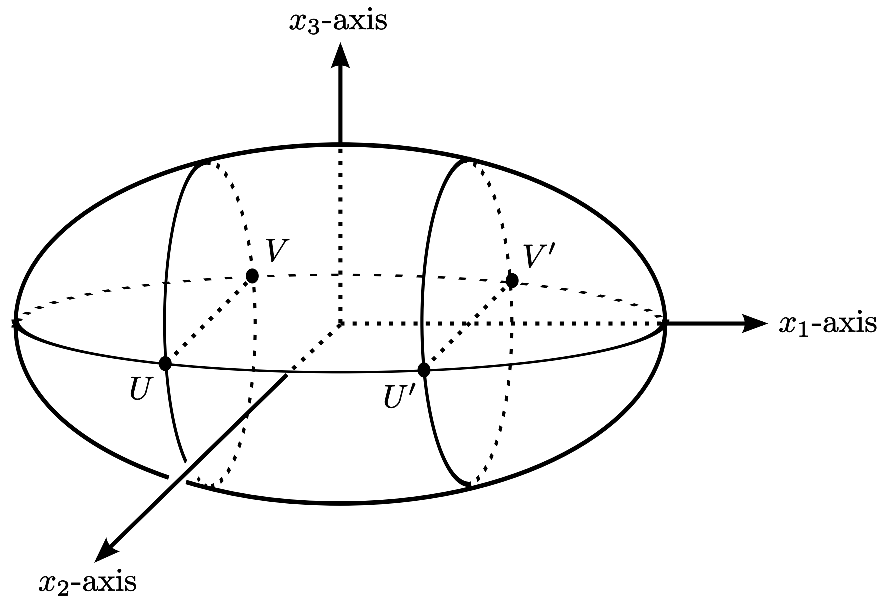


Figure 3: The ball B_1

For a set $Z \subset \mathbb{R}^3$ let $Z[\alpha, \beta]$ be the part of Z in the strip $\{\alpha \leq x_1 \leq \beta\}$, i.e. $Z[\alpha, \beta] := Z \cap \{(x_1, x_2, x_3) \mid \alpha \leq x_1 \leq \beta\}$.

Let $\tilde{S}$ be the 2-fold covering of S branched along the 4 points U, U', V, V'. Note that $\tilde{S}$ is a torus, with the following preferred meridian and longitude. The plane passing through U, U', V, V' (i.e. the x_1x_2 plane) intersects $S[-\sqrt{2}, -1]$ in an arc $\mathfrak{m}$ that connects U and V. In other words, $\mathfrak{m}$ is the shortest arc on the sphere S connecting U and V, see Figure 4. The total lift $\tilde{\mathfrak{m}}$ of $\mathfrak{m}$ is a closed curve on the the torus $\tilde{S}$ which will serve as the meridian, see Figure 5. Let $\mathfrak{l}$ be the shortest arc on S connecting U and U'. The total lift $\tilde{\mathfrak{l}}$ of $\mathfrak{l}$ is a closed curve serving as the longitude. It is easy to see that $\tilde{\mathfrak{m}}$ and $\tilde{\mathfrak{l}}$ form a basis of $H_1(\tilde{S}, \mathbb{Z})$.

According to [8, Chapter 12], the isotopy class of the pair of arcs $(\mathfrak{u}, \mathfrak{u}')$ in the ball B_2 is uniquely determined by the homology class of the total lift $\tilde{\mathfrak{u}}$ of the curve $\mathfrak{u}$ in $H_1(\tilde{S}, \mathbb{Z})$. Moreover, the homology class of $\tilde{\mathfrak{u}}$ is equal to $2p\,\tilde{\mathfrak{m}} + q'\tilde{\mathfrak{l}}$ for some $q' \in \mathbb{Z}$ satisfying the condition $q' \equiv q^{\pm 1} \pmod{2p}$. We will describe explicitly the arc $\mathfrak{u}$ in

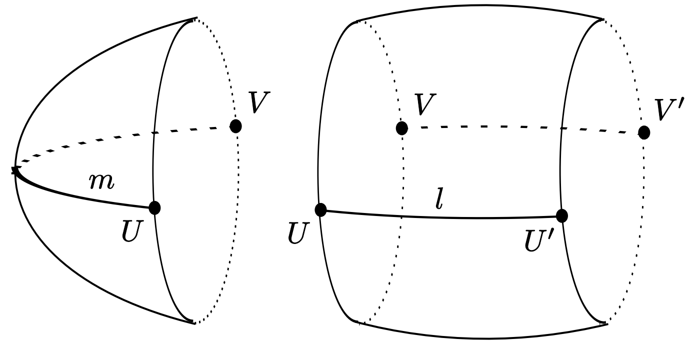


Figure 4: The curves $\mathfrak{m}$ and $\mathfrak{l}$.

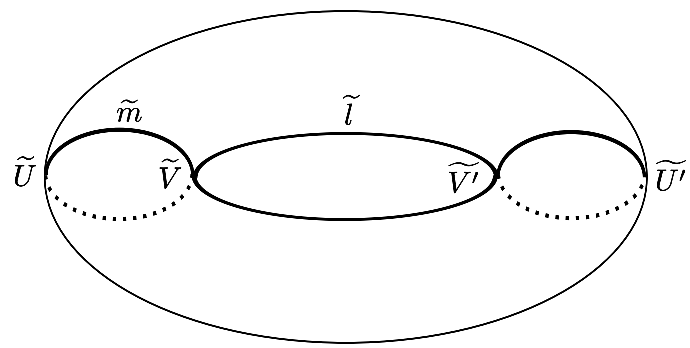


Figure 5: The total lifts $\tilde{\mathfrak{m}}, \tilde{\mathfrak{l}}$ of $\mathfrak{m}$ and $\mathfrak{l}$ on the torus $\tilde{S}$ respectively.

the next subsection.

3.2.1 Description of $\mathfrak{u}$

We will present $\mathfrak{u}$ by describing 3 parts of it: the left part $\mathfrak{u}_l$, the middle part $\mathfrak{u}_m$, and the right part $\mathfrak{u}_r$, which are respectively the intersection of $\mathfrak{u}$ with $S_l := S[-\sqrt{2}, -1]$, $S_m := S[-1, 1]$, and $S_r := S[1, \sqrt{2}]$. For two non-antipodal points A, B on the sphere S let $\gamma(AB)$ be the shortest geodesic on S connecting A and B.

The boundary $C_l := \partial S_l$ is a circle containing U and V. On the circle C mark $2p$ points $A_0 = V, A_1, \dots, A_{2p-1}$ which are: (i) counter-clockwise in that order if viewing from the origin of the coordinate system, and (ii) uniformly distributed on the circle C.

Then $A_p = U$, and for $1 \leq j \leq p-1$, the segment $A_{p-j}A_{p+j}$ is parallel to the x_3-axis. The shortest geodesic $\gamma(A_{p-j}A_{p+j})$ lies in S_l. Let $\mathfrak{u}_{0,l}$ be the union of all the disjoint $\gamma(A_{p-j}A_{p+j})$, $1 \le j \le p-1$. See Figure 6.

Let E_j be the midpoint of the arc A_jA_{j+1} on the circle C (indices are taken modulo $2p$). In other words, E_j is the image of A_j under the rotation by $2\pi/4p$ about the x_1-axis, counter-clockwise if viewing from the origin.

Let E'_j be the reflection of E_j through the x_2x_3-plane. Note that all the points E'_j are on the circle $C' := \partial S_r$. The p geodesics $\gamma(E'_{p-j}E'_{p+j-1})$, $j = 1, \ldots, p$, are disjoint and are in S_r. Let $\mathfrak{u}_{0,r}$ be the union of the p geodesics $\gamma(E'_{p-j}E'_{p+j-1})$, $j = 1, \ldots, p$.

On S_m let $\mathfrak{u}_{0,m}$ be the union of $2p$ geodesics $\gamma(A_jE'_{j+(q-1)/2})$, $j = 0, 1, \ldots, 2p-1$ (indices taken modulo $2p$). Note that the $2p$ components of $\mathfrak{u}_{0,m}$ are obtained from each other by rotations by $2j\pi/2p$ about the x_1-axis.

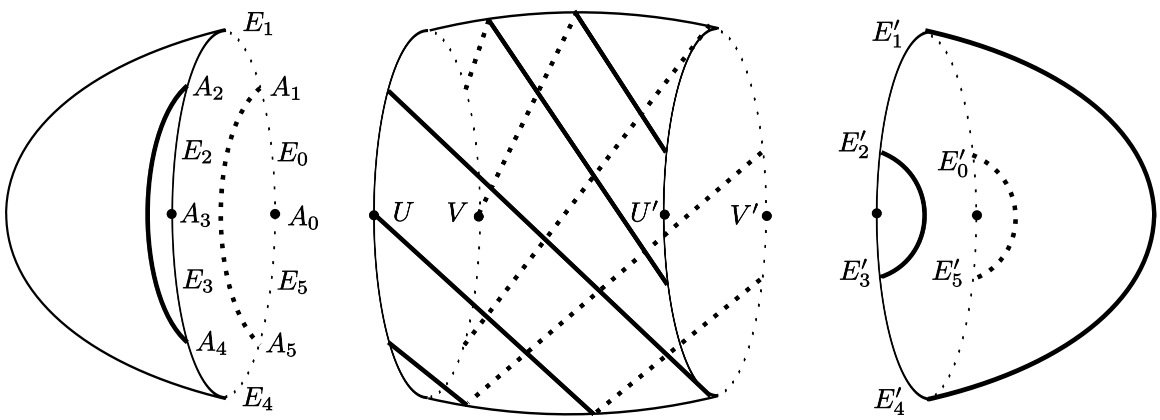


Figure 6: $\mathfrak{u}_{0,l}, \mathfrak{u}_{0,m}, \mathfrak{u}_{0,r}$ for $2p = 6$ and $q' = 5$

Let $\mathfrak{u}_0$ be the arc on S obtained by combining $\mathfrak{u}_{0,l}, \mathfrak{u}_{0,m}$ and $\mathfrak{u}_{0,r}$; it connects U and V. Up to isotopy there is a unique arc $\mathfrak{u}'_0$ on S connecting V and V', and disjoint from $\mathfrak{u}_0$.

Lemma 3.2.1. *The pair* $(\mathfrak{u}_0, \mathfrak{u}'_0)$ *is isotopic, relative endpoints, to* $(\mathfrak{u}, \mathfrak{u}')$ *in the ball* B_2.

Proof. It is easy that the homology class of the total lift of the arc $\mathfrak{u}_0$ in $H_1(\tilde{S}^2, \mathbb{Z})$ is equal to $2p\,\tilde{\mathfrak{m}} + q'\tilde{\mathfrak{l}}$, which is exactly equal to the homology class of the total lift of the arc $\mathfrak{u}$ in $H_1(\tilde{S}^2, \mathbb{Z})$. According to [8, Chapter 12], $(\mathfrak{u}_0, \mathfrak{u}'_0)$ is isotopic, relative endpoints, to $(\mathfrak{u}, \mathfrak{u}')$ in the ball B_2. □

From now on we identify $(\mathfrak{u}, \mathfrak{u}')$ and $(\mathfrak{u}_0, \mathfrak{u}'_0)$. Without loss of generality we also assume that $q' = q$.

3.2.2 The link complement

Let ω be the boundary curve of a small normal neighborhood of the arc $\mathfrak{u}$ in $S = \partial B_1$. Let $X_1 := B_1 \setminus (UV \cup U'V')$, which is homeomorphic to the cylinder over a two-punctured disk D_{**} in Subsection 3.1.1. Then the complement X of the link L is obtained from X_1 by gluing a 2-handle to X_1 along ω. Up to isotopy, ω can be described as follows.

On the circle $C = \partial S_l$ mark $4p$ points F_0, $F_1, \ldots, F_{4p-1}$ which are: (i) counterclockwise in that order if viewing from the origin of the coordinate system, and (ii) uniformly distributed on the circle C, and such that (iii) V is the midpoint of the arc $F_{4p-1}F_0$ on C. We can also say that F_{2j} is the midpoint of the arc A_jE_j, and F_{2j+1} is the midpoint of the arc E_jA_{j+1} on C, see Figures 7 and 8.

For $1 \leq j \leq 2p$, the segment $F_{2p-j}F_{2p+j-1}$ is parallel to the x_3-axis. The shortest geodesic $\gamma(F_{2p-j}F_{2p+j-1})$ lies in S_l. Let ω_l be the union of all the disjoint $\gamma(F_{2p-j}F_{2p+j-1})$, $1 \leq j \leq 2p$.

Let F'_j be the reflection of F_j through the x_2x_3-plane. We can also say that F'_{2j} is the midpoint of the arc $A'_jE'_j$, and F'_{2j+1} is the midpoint of the arc $E'_jA'_{j+1}$ on C, where A'_j is the reflection of A_j through the x_2x_3-plane. Note that all the points F'_j are on the circle $C' = \partial S_r$. For $1 \leq j \leq 2p$, the segment $F'_{2p-j}F'_{2p+j-1}$ is parallel to the x_3-axis. The shortest geodesic $\gamma(F'_{2p-j}F'_{2p+j-1})$ lies in S_r. Let ω_r be the union of all the disjoint $\gamma(F'_{2p-j}F'_{2p+j-1})$, $1 \leq j \leq 2p$.

Note that we can also say that ω_r is the reflection of ω_l through the x_2x_3-plane.

On S_m, let ω_m is the union of $4p$ geodesics $\gamma(F_jF'_{q+j})$, $j = 0, 1, \ldots, 4p-1$ (indices taken modulo $4p$). Note that the $4p$ components of ω_m are obtained from each other by rotations by $2j\pi/4p$ about the x_1-axis.

Then, up to isotopy, ω is obtained by combining ω_l, ω_m and ω_r.

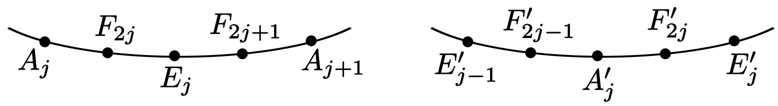


Figure 7: The distribution of the points A_j, F_j, E_j and A'_j, F'_j, E'_j on the circles C and C' respectively

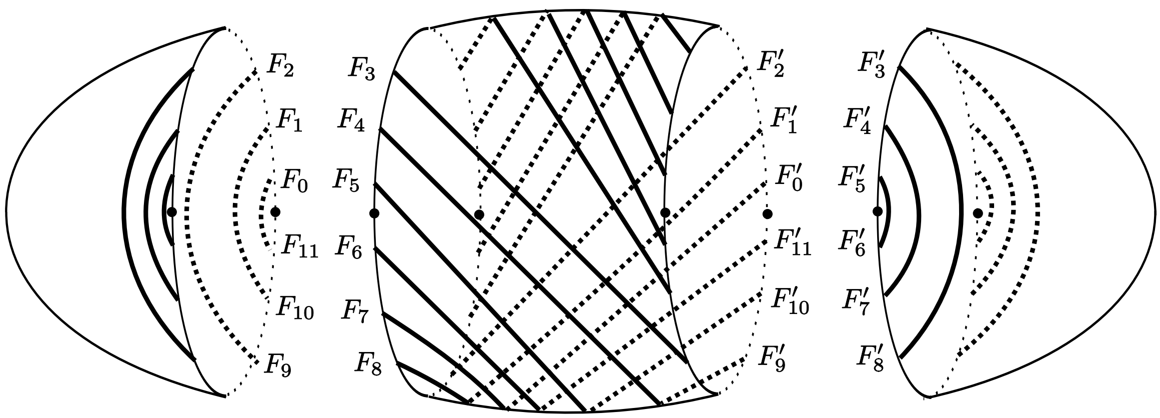


Figure 8: $\omega_l, \omega_m, \omega_r$ for $2p = 6$ and $q = 5$

Let ψ be the rotation by 180^o about the x_1-axis. One has $\psi(B_1) = B_1$. Up to isotopy, we can assume that $\psi(\omega) = \omega$.

Let $P = F_{p+(q+1)/2}$, $P' = F_{3p+(q+1)/2}$ and $Q = F'_{p+1-(q+1)/2}$, $Q' = F'_{3p+1-(q+1)/2}$. See Figure 9. Note that $\psi(P) = P'$ and $\psi(Q) = Q'$.

3.2.3 Relative skein modules

Let us recall the definition of the relative skein module $\mathcal{S}(X_1; P, Q')$ (see [3, 36]). A *type 1 tangle* is the disjoint union of a framed link and a framed arc in X_1 such that the parts of the arc near the two end points are on the boundary ∂X_1, and the framing on these parts are given by vectors normal to ∂X_1. Type 1 tangles are considered up to isotopy relative the endpoints. Then $\mathcal{S}(X_1; P, Q')$ is the $\mathbb{C}[t^{\pm 1}]$-module generated by type 1 tangles with endpoints at P, Q' modulo the usual skein relations, like in the definition of $\mathcal{S}(X)$. One defines in a similar way the relative Kauffman bracket skein module $\mathcal{S}(\partial X_1; P, Q') := \mathcal{S}(\partial X_1 \times [0,1]; P, Q')$, where we identify $\partial X_1 \times [0,1]$ with a collar of ∂X_1 in X_1.

There is a natural bilinear map $\mathcal{S}(\partial X_1; P, Q') \otimes \mathcal{S}(X_1) \to \mathcal{S}(X_1; P, Q')$, where $\ell \otimes \ell' \to \ell \star \ell'$, which is the disjoint union of ℓ and ℓ'.

The pair P, Q divide ω into two arcs, the one that is fully drawn in Figure 9 (and that goes around the points U and V' exactly once) is denoted by ω_s. Similarly, the pair P', Q' divide ω into two arcs, the one that is fully drawn in Figure 9 (and that goes around the points U' and V exactly once) is denoted by ω'_s.

Let $P_c = F_{3p+1-(q+1)/2}, P'_c = F_{p+1-(q+1)/2}$ and $Q_c = F'_{3p+(q+1)/2}, Q'_c = F'_{p+(q+1)/2}$. Then ω_s consists of 3 parts: the left part is an arc on S_l connecting P and P_c, the middle part is an arc on S_m connecting P_c and Q_c, and the right part is an arc on S_r connecting Q_c and Q. Similarly, ω'_s also consists of 3 parts: the left part is an arc on S_l connecting P' and P'_c, the middle part is an arc on S_m connecting P'_c and Q'_c, and the right part is an arc on S_r connecting Q'_c and Q'.

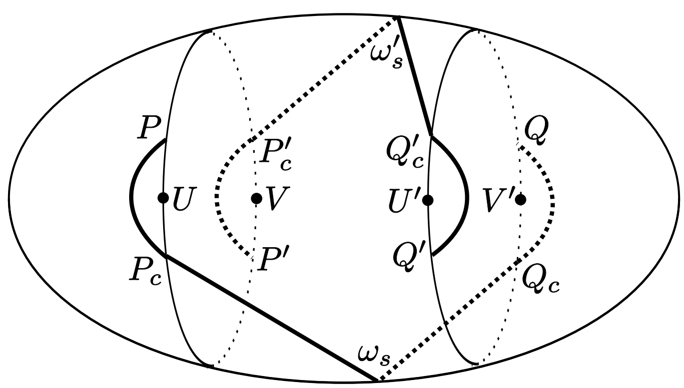


Figure 9: ω_s connects P, Q, and ω'_s connects P', Q'

Let $\gamma_{\text{in}}(PQ'), \gamma_{\text{in}}(P'Q)$ be respectively the shortest arcs on the surface $S = \partial B_1$ connecting P and Q', P' and Q, whose interiors are slightly pushed inside the interior of B_1 (to avoid intersections with other arcs on S) and whose framings are given by vectors normal to S.

Let $\mathfrak{d}_{\text{in}}(PP'), \mathfrak{d}_{\text{in}}(QQ')$ be respectively the straight intervals connecting P and P', Q' and Q, whose interiors are slightly pushed into the interior of $B_1[-\sqrt{2}, -1]$ and the interior of $B_1[1, \sqrt{2}]$ respectively (to avoid intersections with the straight lines UV and $U'V'$ respectively).

Let $\mathfrak{a}_1$ be $\gamma_{\text{in}}(PQ')$; $\mathfrak{a}_2$ be ω_s followed by $\mathfrak{d}_{\text{in}}(QQ')$; $\mathfrak{a}_3$ be $\mathfrak{d}_{\text{in}}(PP')$ followed by ω_s'; and $\mathfrak{a}_4$ be ω_s followed by $\gamma_{\text{in}}(QP')$ then followed by ω_s'.

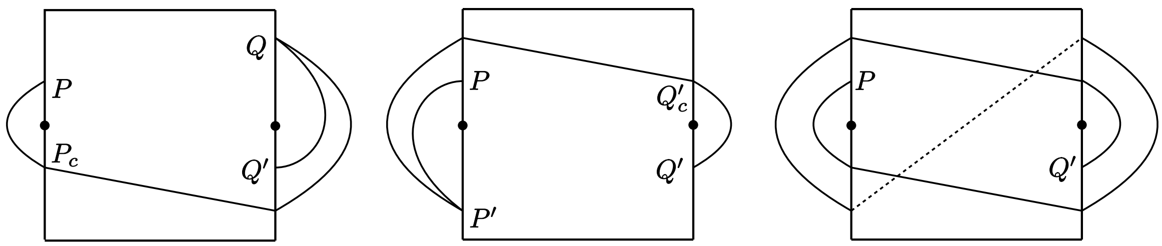


Figure 10: The projections of $\mathfrak{a}_2, \mathfrak{a}_3$ and $\mathfrak{a}_4$ onto the x_1x_3-plane

Lemma 3.2.2. *The relative skein* $\mathcal{S}(X_1; P, Q')$ *is equal to the union* $\cup_{i=1}^4(\mathfrak{a}_i \star \mathcal{S}(X_1))$.

Proof. Let $\mathfrak{a}_1' = \mathfrak{a}_1$. Let $\mathfrak{a}_2'$ be $\gamma(PP_c)$ folowed by $\gamma_{\text{in}}(P_cQ')$; $\mathfrak{a}_3'$ be $\gamma_{\text{in}}(PQ_c')$ followed by $\gamma(Q_c'Q')$; and $\mathfrak{a}_4'$ be $\gamma(PP_c)$ followed by $\gamma_{\text{in}}(P_cQ_c')$ then followed by $\gamma(Q_c'Q')$. Here $\gamma_{\text{in}}(P_cQ')$, $\gamma_{\text{in}}(PQ_c')$, $\gamma_{\text{in}}(P_cQ_c')$ are respectively the shortest arcs on S connecting P_c and Q', P and Q_c', P_c and Q_c', whose interiors are slightly pushed inside the interior of B_1 (to avoid intersections with other arcs on S) and whose framings are given by vectors normal to S.

Using the skein relations one can simplify the arc part of elements in $\mathcal{S}(X_1; P, Q')$, showing that the arc part is one of the four $\mathfrak{a}_i', i = 1, 2, 3, 4$. More precisely, by similar arguments as in the proof of [3, Lemma 3.1] one can show that the relative skein $\mathcal{S}(X_1; P, Q')$ is equal to the union $\cup_{i=1}^4(\mathfrak{a}_i' \star \mathcal{S}(X_1))$.

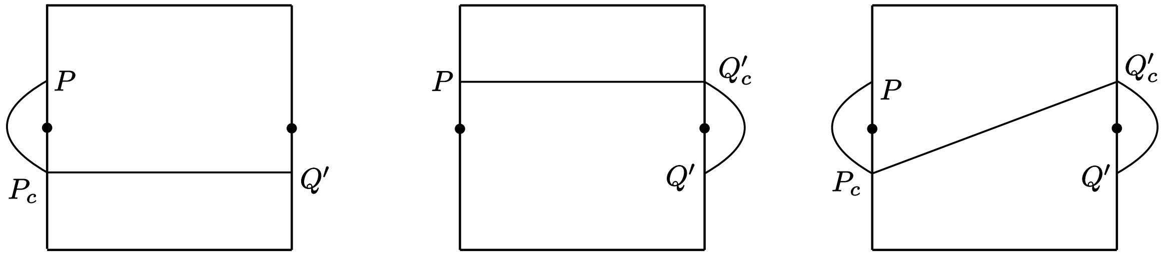


Figure 11: The projection of $\mathfrak{a}'_2, \mathfrak{a}'_3$ and $\mathfrak{a}'_4$ onto the x_1x_3-plane

It is easy to see that $\mathfrak{a}_i$ is isotopic to $\mathfrak{a}'_i$ for all $1 \le i \le 3$. Using the skein relations to resolve all the crossings of $\mathfrak{a}_4$ one can easily show that the set $\mathfrak{a}_4 \star \mathcal{S}(X_1)$ is equal to $\mathfrak{a}'_4 \star \mathcal{S}(X_1)$, modulo the union $\cup_{i=1}^{3}(\mathfrak{a}'_i \star \mathcal{S}(X_1))$. The lemma follows. $\square$

3.2.4 From $\mathcal{S}(X_1)$ to $\mathcal{S}(X)$ through sliding

Recall that X is obtained from X_1 by attaching a 2-handle along the curve ω. Note that $\mathcal{S}(X_1) = \mathcal{S}(D_{**})$ is isomorphic to the commutative algebra $\mathcal{R}[x, x', y]$, see [47]. The embedding of X_1 into X gives rise to a linear map from $\mathcal{S}(X_1) \equiv \mathcal{R}[x, x', y]$ to $\mathcal{S}(X)$. It is known that the map is surjective, and its kernel $\mathcal{K}$, see [47, 3], can be described through slides as follows.

Suppose $\mathfrak{a}$ is a type 1 tangle whose 2 endpoints are on ω such that outside a small neighborhood of the 2 endpoints $\mathfrak{a}$ is in the interior of X_1 and in a small neighborhood of the endpoints $\mathfrak{a}$ is on the boundary $S = \partial B_1$. The two end points of $\mathfrak{a}$ divide ω into 2 arcs ω_1 and ω_2. The loop ω partitions S, which is a sphere, into 2 parts; the one not containing U, U' is called the *outside one*. Let us isotope $\mathfrak{a}$ (relatively to the endpoints) to $\mathfrak{a}'$ so that in a small neighborhood of the endpoints, $\mathfrak{a}'$ is in the outside part of ω.

Let $sl(\mathfrak{a})$ be $\mathfrak{a}' \cdot \omega_1 - \mathfrak{a}' \cdot \omega_2$, considered as an element of the skein module $\mathcal{S}(X_1)$. Here $\mathfrak{a}' \cdot \omega_1$ is the framed link obtained by combining $\mathfrak{a}'$ and ω_1. Note that $sl(\mathfrak{a})$ is defined up to a factor $\pm t^{3n}, n \in \mathbb{Z}$. The exchange $\omega_1 \leftrightarrow \omega_2$ changes the sign, and isotopies in neighborhoods of the endpoints change the framing, which results in a factor equal to a power of $(-t^3)$.

It is clear that as framed links in X, $\mathfrak{a}' \cdot \omega_1$ is isotopic to $\mathfrak{a}' \cdot \omega_2$, since one is obtained from the other by sliding over the 2-handle attached to the curve ω. Hence we always have $sl(\mathfrak{a}) \in \mathcal{K}$. It was known that $\mathcal{K}$ is spanned by all possible $sl(\mathfrak{a})$, where $\mathfrak{a}$ can be chosen among all type 1 tangles with pre-given two endpoints on ω.

From the description of $\mathcal{S}(X_1; P, Q')$ in Lemma 3.2.2 we have

Lemma 3.2.3. *The kernel $\mathcal{K}$ is equal to the $\mathbb{C}[t^{\pm 1}]$-span of $\{sl(\mathfrak{a}_i) \star \mathcal{S}(X_1), i = 1, 2, 3, 4\}$.*

Lemma 3.2.4. *One has*

$$\begin{aligned} sl(\mathfrak{a}_1) &= sl(\gamma_{\text{in}}(PQ')), \\ sl(\mathfrak{a}_2) &= sl(\mathfrak{d}_{\text{in}}(PP')), \\ sl(\mathfrak{a}_3) &= sl(\mathfrak{d}_{\text{in}}(QQ')), \\ sl(\mathfrak{a}_4) &= sl(\gamma_{\text{in}}(P'Q)). \end{aligned}$$

Proof. The first identity is a tautology. The last three follows from trivially a simple isotopy of the links involved. □

Lemma 3.2.5. *For every $\ell \in \mathcal{S}(X_1)$, one has $\psi(\ell) = \ell$.*

Proof. This is because x, x' and y are invariant under the rotation ψ. □

Lemma 3.2.6. *One has*

$$\begin{aligned} sl(\gamma_{\text{in}}(PQ')) \star \mathcal{S}(X_1) &= sl(\gamma_{\text{in}}(P'Q)) \star \mathcal{S}(X_1), \\ sl(\mathfrak{d}_{\text{in}}(PP')) \star \mathcal{S}(X_1) &= 0, \\ sl(\mathfrak{d}_{\text{in}}(QQ')) \star \mathcal{S}(X_1) &= 0. \end{aligned}$$

Proof. Since $\psi(P) = P'$ and $\psi(Q) = Q'$, we have $\psi(sl(\gamma_{\text{in}}(PQ'))) = sl(\gamma_{\text{in}}(P'Q))$. Hence $sl(\gamma_{\text{in}}(PQ')) \star \mathcal{S}(X_1) = sl(\gamma_{\text{in}}(P'Q)) \star \mathcal{S}(X_1)$ by Lemma 3.2.5.

Since both $\mathfrak{d}_{\text{in}}(PP')$ and ω is invariant under ψ, we have $\psi(\mathfrak{d}_{\text{in}}(PP') \cdot \omega_1(P, P')) = \mathfrak{d}_{\text{in}}(PP') \cdot \omega_2(P, P')$, where $\omega_1(P, P')$ and $\omega_2(P, P')$ are the two arcs of ω obtained by dividing ω using the two points P, P'. It implies that

$$sl(\mathfrak{d}_{\text{in}}(PP')) \star \mathcal{S}(X_1) = (\mathfrak{d}_{\text{in}}(PP') \cdot \omega_1(P, P') - \mathfrak{d}_{\text{in}}(PP') \cdot \omega_2(P, P')) \star \mathcal{S}(X_1) = 0.$$

This completes the proof of the lemma. □

3.2.5 Proof of Theorem 7

Let $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$. We have $\mathcal{S}(X) = \mathcal{R}[x, x', y]/\mathcal{K}$, where $\mathcal{K}$ is the $\mathcal{R}$-span of $sl(\mathfrak{a}_1) \star \mathcal{R}[x, x', y]$, by Lemmas 3.2.3, 3.2.4 and 3.2.6. Note that there is a natural $\mathcal{R}[x, x']$-module structure on $\mathcal{S}(X)$: Here x, x' are meridians, thus belong to the boundary of X. Over $\mathcal{R}[x, x']$, $\mathcal{R}[x, x', y]$ is spanned by $1, y, y^2, \ldots$. Hence $\mathcal{K}$, as an $\mathcal{R}[x, x']$-module, is spanned by $sl(\mathfrak{a}_1) \star y^k = (\mathfrak{a}_1 \cdot \omega_1 - \mathfrak{a}_1 \cdot \omega_2) \star y^k, k = 0, 1, 2, \ldots$.

Note that $\mathfrak{a}_1 \cdot \omega_1$ is the closure in the sense of [36, Section 1.5] of a braid on $(2p+2)$ strands, while $\mathfrak{a}_1 \cdot \omega_2$ is the closure of a braid on $(2p-2)$ strands. Moreover, $(\mathfrak{a}_1 \cdot \omega_1) \star y^k$ is the closure of a braid on $(2p+2) + 2k$ strands, while $(\mathfrak{a}_1 \cdot \omega_2) \star y^k$ is the closure of of a braid on $(2p-2) + 2k$ strands. Lemma 1.1 in [36] then shows that $(\mathfrak{a}_1 \cdot \omega_1 - \mathfrak{a}_1 \cdot \omega_2) \star y^k$, as an element of $\mathcal{R}[x, x', y]$, has y-degree $(p+1) + k$, with highest coefficient invertible and of the form a power of t. Hence when we factor out $\mathcal{R}[x, x', y]$ by $\mathcal{K}$, we get a free $\mathcal{R}[x, x']$-module with representatives $y^l, l = 0, 1, 2, \ldots, p$ as a basis.

3.2.6 Proof of Proposition 1

From Theorem 7 it follows that $\varepsilon(\mathcal{S}(X))$ is the quotient of the ring $\mathbb{C}[\bar{x}, \bar{x'}, \bar{y}]$ by the ideal I generated by $\varepsilon(sl(\mathfrak{a}_1)) \star \mathbb{C}[\bar{x}, \bar{x'}, \bar{y}]$, where $\bar{z}$ denotes the negative of the trace of the loop z.

Note that $\varepsilon(\mathcal{S}(X))$ has a natural $\mathbb{C}$-algebra structure and $\star$ is just the multiplication of this algebra. It implies that I can be generated by only one element which is $\varepsilon(sl(\mathfrak{a}_1))$. Hence $\varepsilon(\mathcal{S}(X)) = \mathbb{C}[\bar{x}, \bar{x'}, \bar{y}]/(\varphi)$, where $\varphi = \varepsilon(sl(\mathfrak{a}_1)) \in \mathbb{C}[\bar{x}, \bar{x'}, \bar{y}]$. Note that φ is a polynomial of $\bar{y}$-degree $p+1$ with leading coefficient ± 1.

We claim that φ has no repeated factors. Since φ is a polynomial of $\bar{y}$-degree $p+1$ with leading coefficient ± 1, it suffices to show that $\varphi(0,0,\bar{y})$ has no repeated factors.

Lemma 3.2.7. *One has*

$$\varphi(0,0,\bar{y}) = \pm(\bar{y}^2 - 4)S_{p-1}(\bar{y}),$$

where $S_n(\bar{y})$ are the Chebyshev polynomials defined by $S_0(\bar{y}) = 1, S_1(\bar{y}) = \bar{y}$ and $S_{n+1}(\bar{y}) = \bar{y}S_n(\bar{y}) - S_{n-1}(\bar{y})$ for all integer n.

Proof. By [8], the fundamental group of the two-bridge link $L = \mathfrak{b}(2p, q)$ is

$$\pi_1(L) = \langle \tilde{x}, \tilde{x'} \mid \tilde{x}w = w\tilde{x} \rangle,$$

where $w = (\tilde{x'})^{\varepsilon_1}(\tilde{x})^{\varepsilon_2} \cdots (\tilde{x})^{\varepsilon_{2p-2}}(\tilde{x'})^{\varepsilon_{2p-1}}$ and $\varepsilon_k = (-1)^{\lfloor \frac{kq}{2p} \rfloor}$. Here $\tilde{x}, \tilde{x'}$ are meridians of the link L, and are conjugate to x, x' respectively.

The character variety of the free group in 2 letters $\tilde{x}$ and $\tilde{x'}$ is isomorphic to $\mathbb{C}^3$, by the Fricke-Klein-Vogt theorem. For every word z, the trace of z is a polynomial in 3 variables $\operatorname{tr}\tilde{x} = -\bar{x}, \operatorname{tr}\tilde{x'} = -\bar{x'}$ and $\operatorname{tr}(\tilde{x}\tilde{x'}) = -\bar{y}$.

Note that the traces of the words $(\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1}$ and $w(\tilde{x'})^{-1}$ are equal. Hence

$$\eta = \operatorname{tr}\left((\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1} - w(\tilde{x'})^{-1}\right)$$

is divisible by φ in $\mathbb{C}[\bar{x}, \bar{x'}, \bar{y}]$.

Suppose from now on $\bar{x} = \bar{x'} = 0$. We have $(\tilde{x})^{-1} + \tilde{x} = \operatorname{tr}\tilde{x} = -\bar{x} = 0$, i.e. $(\tilde{x})^{-1} = -\tilde{x}$, by the Cayley-Hamilton theorem applying for matrices in $SL_2(\mathbb{C})$. Here we identify $\tilde{x}$ with its representation matrix in $SL_2(\mathbb{C})$. Similarly, $(\tilde{x'})^{-1} = -\tilde{x'}$.

Let k be the the number of times the power -1 appears in the word $w(\tilde{x'})^{-1} = (\tilde{x'})^{\varepsilon_1}(\tilde{x})^{\varepsilon_2}\cdots(\tilde{x})^{\varepsilon_{2p-2}}$. Then it is easy to see that the number of times the power -1 appears in the word $(\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1} = (\tilde{x})^{-1}(\tilde{x'})^{\varepsilon_1}(\tilde{x})^{\varepsilon_2}\cdots(\tilde{x})^{\varepsilon_{2p-2}}(\tilde{x'})^{\varepsilon_{2p-1}}\tilde{x}(\tilde{x'})^{-1}$ is $k+2$. If we replace $(\tilde{x})^{-1}$ and $(\tilde{x'})^{-1}$ in $w(\tilde{x'})^{-1}$ by $\tilde{x}$ and $\tilde{x'}$ respectively then we pick up the sign $(-1)^k$, i.e. we have $w(\tilde{x'})^{-1} = (-1)^k(\tilde{x'}\tilde{x})^{p-1}$. Similarly, $(\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1} = (-1)^{k+2}(\tilde{x}\tilde{x'})^{p+1}$. It implies that $\eta(0,0,\bar{y}) = (-1)^k \operatorname{tr}\big((\tilde{x}\tilde{x'})^{p+1} - (\tilde{x'}\tilde{x})^{p-1}\big)$.

Let $\delta_n = \operatorname{tr}\big((\tilde{x}\tilde{x'})^{n+1} - (\tilde{x'}\tilde{x})^{n-1}\big)$. By the Cayley-Hamiton, $\tilde{x}\tilde{x'}+(\tilde{x}\tilde{x'})^{-1} = \operatorname{tr}(\tilde{x}\tilde{x'}) = -\bar{y}$. This implies that $\delta_{n+1} = -\bar{y}\delta_n - \delta_{n-1}$. It is easy to check that $\delta_1 = \bar{y}^2 - 4$, $\delta_2 = -(\bar{y}^2-4)\bar{y}$. Hence $\delta_n = (-1)^{n-1}(\bar{y}^2-4)S_{n-1}(\bar{y})$ where $S_n(\bar{y})$ are the Chebyshev polynomials defined by $S_0(\bar{y}) = 1, S_1(\bar{y}) = \bar{y}$ and $S_{n+1}(\bar{y}) = \bar{y}S_n(\bar{y}) - S_{n-1}(\bar{y})$ for all integer n.

We have $\eta(0,0,\bar{y}) = (-1)^k\delta_p = (-1)^{k+p-1}(\bar{y}^2-4)S_{p-1}(\bar{y})$, which is a polynomial of degree $p+1$ in $\bar{y}$ with leading coefficient $(-1)^{k+p-1}$. Since η is divisible by φ, and φ is also a polynomial of $\bar{y}$-degree $p+1$ with leading coefficient ±1, we must have $\varphi(0,0,\bar{y}) = \pm(\bar{y}^2-4)S_{p-1}(\bar{y})$ as desired. □

It is known that $S_{p-1}(\bar{y}) = \prod_{j=1}^{p-1}(\bar{y} - 2\cos\frac{\pi j}{p})$ and hence $(\bar{y}^2-4)S_{p-1}(\bar{y})$ has no repeated factors. By Lemma 3.2.7, it follows that φ has no repeated factors either. Hence the nil-radical of $\varepsilon(\mathcal{S}(X))$ is zero, which means that $\varepsilon(\mathcal{S}(X))$ is exactly equal to $\mathbb{C}[\chi(\pi_1(X))]$. This completes the proof of Proposition 1.

Corollary 3.2.8. *The character ring of the two-bridge link $\mathfrak{b}(2p,q)$ is the quotient of the ring $\mathbb{C}[\bar{x},\bar{x'},\bar{y}]$ by the ideal generated by the polynomial $\eta = \operatorname{tr}((\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1}) - \operatorname{tr}(w(\tilde{x'})^{-1})$, where $\bar{x}$, $\bar{x'}$, $\bar{y}$, $\tilde{x}$, $\tilde{x'}$ and w are defined as in the proof of Lemma 3.2.7.*

Proof. We still use the notations in the proof of Lemma 3.2.7.

Since $w = (\tilde{x'})^{\varepsilon_1}(\tilde{x})^{\varepsilon_2}\cdots(\tilde{x})^{\varepsilon_{2p-2}}(\tilde{x'})^{\varepsilon_{2p-1}}$ and $\varepsilon_k = (-1)^{\lfloor\frac{kq}{2p}\rfloor} = \pm1$, it is easy to show that the traces of the words $(\tilde{x})^{-1}w\tilde{x}(\tilde{x'})^{-1}$ and $w(\tilde{x'})^{-1}$ have $\bar{y}$-degrees equal to $p+1$ and $p-1$ respectively, with leading coefficients ±1. It implies that the polynomial η has $\bar{y}$-degree $p+1$ with leading coefficient ± 1. Since η is divisible by φ, we must have $\eta = \pm\varphi$. Hence, by Proposition 1, the character ring of $\mathfrak{b}(2p,q)$ is equal to the quotient of the ring $\mathbb{C}[\bar{x}, \bar{x'}, \bar{y}]$ by the ideal generated by the polynomial $\eta = \mathrm{tr}((\tilde{x})^{-1} w \tilde{x} (\tilde{x'})^{-1}) - \mathrm{tr}(w(\tilde{x'})^{-1})$. □

Remark 3.2.9. Corollary 3.2.8 was already obtained in [49] although it was not completely written in form of traces. The proof we present here essentially follows directly from Theorem 7.

One can easily show that the characters of abelian representations (into $SL_2(\mathbb{C})$) of the two-bridge link $\mathfrak{b}(2p,q)$ is determined by the polynomial

$$\eta_{ab} = \mathrm{tr}(\tilde{x}\tilde{x'}(\tilde{x})^{-1}(\tilde{x'})^{-1}) = \bar{y}^2 + \bar{x}^2 + \bar{x'}^2 + \bar{y}\bar{x}\bar{x'} - 4.$$

Hence, by Corollary 3.2.8, the characters of non-abelian representations of $\mathfrak{b}(2p,q)$ is determined by the polynomial $\eta_{nab} = \eta/\eta_{ab}$. The polynomial η_{nab} has $\bar{y}$-degree $p-1$ with leading coefficient ± 1. After a suitable change of variables, it is exactly the polynomial $\Phi_{\pi L}$ in [49, Lemma 2], up to ± 1.

CHAPTER IV

PROOF OF A STRONGER VERSION OF THE AJ CONJECTURE FOR TORUS KNOTS

4.1 Introduction

The AJ Conjecture was verified for the trefoil and figure 8 knots by Garoufalidis [17], and was partially checked for all torus knots by Hikami [25]. It was established for some classes of two-bridge knots and pretzel knots, including all twist knots and $(-2, 3, 6n \pm 1)$-pretzel knots, see [36] and Chapter 2. In this chapter we provide a full proof of the AJ conjecture for all torus knots. Moreover, we show that a stronger version of the conjecture, due to A. Sikora, holds true for all torus knots.

4.1.1 Main results

Recall from Chapter 2 that $\mathcal{A}$ is the recurrence ideal and $\mathfrak{p}$ is the classical peripheral ideal of a knot K. The involution σ acts on the quantum torus $\mathcal{T}$ by $\sigma(M^k L^l) = M^{-k} L^{-l}$. Let $\mathcal{A}^\sigma$ be the σ-invariant part of the recurrence ideal $\mathcal{A}$; it is an ideal of $\mathcal{T}^\sigma$. Let ε is the map reducing the quantum parameter t to -1. A. Sikora [52] proposed the following conjecture.

Conjecture 4. *Suppose K is a knot. Then $\sqrt{\varepsilon(\mathcal{A}^\sigma)} = \mathfrak{p}$.*

Here $\sqrt{\varepsilon(\mathcal{A}^\sigma)}$ denotes the radical of the ideal $\varepsilon(\mathcal{A}^\sigma)$ in the ring $\mathfrak{t}^\sigma = \varepsilon(\mathcal{T}^\sigma)$.

It is easy to see that Conjecture 4 implies the AJ conjecture. Conjecture 4 was verified for the unknot and the trefoil knot by Sikora [52]. In the present chapter we confirm it for all torus knots.

Theorem 8. *Conjecture 4 holds true for all torus knots.*

4.1.2 Plan of the chapter

We provide a full proof of the AJ conjecture for all torus knots in Section 4.2 and prove Theorem 8 in Section 4.3.

4.2 Proof of the AJ conjecture for torus knots

We will always assume that knots have framings 0.

We consider the two cases: $a, b > 2$ and $a = 2$ separately. Lemmas 4.2.1 and 4.2.5 below were first proved in [25] using formulas for the colored Jones polynomial and Alexander polynomial of torus knots given in [41]. We present here direct proofs.

4.2.1 The case $a, b > 2$

Lemma 4.2.1. *For the (a,b)-torus knot, we have*

$$J(n+2) = t^{-4ab(n+1)} J(n) + t^{-2ab(n+1)} \frac{t^2 \lambda_{(a+b)(n+1)} - t^{-2} \lambda_{(a-b)(n+1)}}{t^2 - t^{-2}},$$

where $\lambda_k := t^{2k} + t^{-2k}$.

Proof. For the (a,b)-torus knot, by [41], we have

$$J(n) = t^{-ab(n^2-1)} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4bj(aj+1)} [2aj+1]. \tag{23}$$

Hence

$$\begin{aligned} J(n+2) &= t^{-ab((n+2)^2-1)} \sum_{j=-\frac{n+1}{2}}^{\frac{n+1}{2}} t^{4bj(aj+1)} [2aj+1] \\ &= t^{-ab((n+2)^2-1)} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4bj(aj+1)} [2aj+1] + t^{-ab((n+2)^2-1)} \\ &\quad \times \Big(t^{b(n+1)(a(n+1)+2)} [a(n+1)+1] - t^{b(n+1)(a(n+1)-2)} [a(n+1)-1] \Big) \\ &= t^{-4ab(n+1)} J(n) + t^{-2ab(n+1)} \frac{t^2 \lambda_{(a+b)(n+1)} - t^{-2} \lambda_{(a-b)(n+1)}}{t^2 - t^{-2}}, \end{aligned}$$

since $[k] = (t^{2k} - t^{-2k})/(t^2 - t^{-2})$. ∎

Lemma 4.2.2. *The colored Jones polynomial of the (a,b)-torus knot is annihilated by the operator $\alpha_{a,b} = c_3L^3 + c_2L^2 + c_1L + c_0$ where*

$$\begin{aligned}
c_3 &= t^2\Big(t^{2(a+b)}M^{a+b} + t^{-2(a+b)}M^{-(a+b)}\Big) - t^{-2}\Big(t^{2(a-b)}M^{a-b} + t^{-2(a-b)}M^{-(a-b)}\Big),\\
c_2 &= -t^{-2ab}\Big(t^2\big(t^{4(a+b)}M^{a+b} + t^{-4(a+b)}M^{-(a+b)}\big) + t^{-2}\big(t^{4(a-b)}M^{a-b} + t^{-4(a-b)}M^{-(a-b)}\big)\Big),\\
c_1 &= -t^{-8ab}M^{-2ab}c_3,\\
c_0 &= -t^{-4ab}M^{-2ab}c_2.
\end{aligned}$$

Proof. It is easy to check that $c_3t^{-4ab(n+2)} + c_1 = c_2t^{-4ab(n+1)} + c_0 = 0$ and

$$c_3\big(t^2\lambda_{(a+b)(n+2)} - t^{-2}\lambda_{(a-b)(n+2)}\big) + c_2t^{2ab}\big(t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}\big) \;=\; 0.$$

Hence, by Lemma 4.2.1, $\alpha_{a,b}J(n)$ is equal to

$$\begin{aligned}
& c_3J(n+3) + c_2J(n+2) + c_1J(n+1) + c_0J(n)\\
=\;& c_3\Big(t^{-4ab(n+2)}J(n+1) + t^{-2ab(n+2)}\frac{t^2\lambda_{(a+b)(n+2)} - t^{-2}\lambda_{(a-b)(n+2)}}{t^2 - t^{-2}}\Big)\\
& + c_2\Big(t^{-4ab(n+1)}J(n) + t^{-2ab(n+1)}\frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}}\Big)\\
& + c_1J(n+1) + c_0J(n)\\
=\;& (c_3t^{-4ab(n+2)} + c_1)J(n+1) + (c_2t^{-4ab(n+1)} + c_0)J(n)\\
& + t^{-2ab(n+1)}\Big(c_3\frac{t^2\lambda_{(a+b)(n+2)} - t^{-2}\lambda_{(a-b)(n+2)}}{t^2 - t^{-2}} + c_2t^{2ab}\frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}}\Big)\\
=\;& 0.
\end{aligned}$$

This proves Lemma 4.2.2. □

Let $\mathcal{A}_{a,b}$ and $\widetilde{\mathcal{A}}_{a,b}$ denote the recurrence ideal of the (a,b)-torus knot and its extension in $\widetilde{\mathcal{T}}$ respectively.

Proposition 4.2.3. *For the (a,b)-torus knot, with $a, b > 2$, we have $\widetilde{\mathcal{A}}_{a,b} = \langle\alpha_{a,b}\rangle$.*

Proof. By Lemma 4.2.2 it suffices to show that if an operator $P = P_2L^2 + P_1L + P_0$, where P_j's are polynomials in $\mathbb{C}[t^{\pm 1}, M]$, annihilates the colored Jones polynomial then $P = 0$.

Indeed, suppose $PJ(n) = 0$. Then, by Lemma 4.2.1,

$$\begin{aligned}
0 &= P_2J(n+2) + P_1J(n+1) + P_0J(n) \\
&= P_2\Big(t^{-4ab(n+1)}J(n) + t^{-2ab(n+1)}\frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}}\Big) \\
&\quad + P_1J(n+1) + P_0J(n) \\
&= (t^{-4ab(n+1)}P_2 + P_0)J(n) + P_1J(n+1) + P_2t^{-2ab(n+1)}\frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}}.
\end{aligned}$$

Let $P_2' = t^{-4ab(n+1)}P_2 + P_0$ and $P_0' = P_2t^{-2ab(n+1)}\frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}}$. Then

$$P_2'J(n) + P_1J(n+1) + P_0' = 0. \tag{24}$$

Note that P_2' and P_0' are polynomials in $\mathbb{C}[t^{\pm 1}, M]$.

Lemma 4.2.4. *The lowest degree in t of $J(n)$ is*

$$l_n = -abn^2 + ab + \frac{1}{2}(1 - (-1)^{n-1})(a-2)(b-2).$$

Proof. From (23), it follows easily that $l_n = -abn^2 + ab$ if n is odd, and $l_n = (-abn^2 + ab) + (ab - 2b - 2a + 4)$ if n is even. ☐

Suppose $P_2', P_1 \neq 0$. Let r_n and s_n be the lowest degrees (in t) of P_2' and P_1 respectively. Note that, when n is large enough, r_n and s_n are polynomials in n of degrees at most 1. Equation (24) then implies that $r_n + l_n = s_n + l_{n+1}$, i.e.

$$r_n - s_n = l_{n+1} - l_n = -ab(2n+1) - (-1)^n(a-2)(b-2).$$

This cannot happen since the LHS is a polynomial in n, when n is large enough, while the RHS is not (since $(a-2)(b-2) > 0$). Hence $P_2' = P_1 = P_0' = 0$, which means $P = 0$. ☐

It is easy to see that $\varepsilon(\alpha_{a,b}) = M^{-2ab}(M^a - M^{-a})(M^b - M^{-b})A_{a,b}$ where $A_{a,b} = (L-1)(L^2M^{2ab} - 1)$ is the A-polynomial of the (a,b)-torus knot when $a, b > 2$. This means the AJ conjecture holds true for the (a,b)-torus knot when $a, b > 2$.

4.2.2 The case $a = 2$

Lemma 4.2.5. *For the $(2,b)$-torus knot, we have*

$$J(n+1) \quad = \quad -t^{-(4n+2)b}J(n) + t^{-2nb}[2n+1].$$

Proof. For the $(2,b)$-torus knot, by (23), we have

$$J(n) = t^{-2b(n^2-1)} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4bj(2j+1)}[4j+1].$$

Hence

$$J(n+1) = t^{-2b((n+1)^2-1)} \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} t^{4bk(2k+1)}[4k+1]$$

Set $k = -(j+\frac{1}{2})$. Then

$$\begin{aligned} J(n+1) &= t^{-2b((n+1)^2-1)} \sum_{j=\frac{n-1}{2}}^{-\frac{n+1}{2}} t^{4bj(2j+1)}[-(4j+1)] \\ &= t^{-2b((n+1)^2-1)}\Big(- \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4bj(2j+1)}[4j+1] + t^{2bn(n+1)}[2n+1]\Big) \\ &= -t^{-(4n+2)b}J(n) + t^{-2nb}[2n+1]. \end{aligned}$$

This proves Lemma 4.2.5. □

Lemma 4.2.6. *The colored Jones polynomial of the $(2,b)$-torus knot is annihilated by the operator $\alpha_{2,b} = d_2L^2 + d_1L + d_0$ where*

$$\begin{aligned} d_2 &= t^2M^2 - t^{-2}M^{-2}, \\ d_1 &= t^{-2b}\Big(t^{-4b}M^{-2b}(t^2M^2 - t^{-2}M^{-2}) - (t^6M^2 - t^{-6}M^{-2})\Big), \\ d_0 &= -t^{-4b}M^{-2b}(t^6M^2 - t^{-6}M^{-2}). \end{aligned}$$

Proof. From Lemma 4.2.5 we have

$$\begin{aligned} J(n+1) &= -t^{-(4n+2)b}J(n) + t^{-2nb}[2n+1], \\ J(n+2) &= t^{-8(n+1)b}J(n) - t^{-6(n+1)b}[2n+1] + t^{-2(n+1)b}[2n+3]. \end{aligned}$$

It is easy to check that

$$\begin{aligned} t^{-8(n+1)b}d_2 - t^{-(4n+2)b}d_1 + d_0 &= 0, \\ d_2\Big(-t^{-6(n+1)b}[2n+1] + t^{-2(n+1)b}[2n+3]\Big) + d_1 t^{-2nb}[2n+1] &= 0. \end{aligned}$$

Hence

$$\begin{aligned} \alpha_{2,b}J(n) &= d_2 J(n+2) + d_1 J(n+1) + d_0 J(n) \\ &= d_2\Big(t^{-8(n+1)b}J(n) - t^{-6(n+1)b}[2n+1] + t^{-2(n+1)b}[2n+3]\Big) \\ &\quad + d_1\Big(-t^{-(4n+2)b}J(n) + t^{-2nb}[2n+1]\Big) + d_0 J(n) \\ &= \Big(t^{-8(n+1)b}d_2 - t^{-(4n+2)b}d_1 + d_0\Big)J(n) \\ &\quad + d_2\Big(-t^{-6(n+1)b}[2n+1] + t^{-2(n+1)b}[2n+3]\Big) + d_1 t^{-2nb}[2n+1] \\ &= 0. \end{aligned}$$

This proves Lemma 4.2.6. □

Proposition 4.2.7. *For the $(2,b)$-torus knot, we have $\widetilde{\mathcal{A}}_{2,b} = \langle \alpha_{2,b} \rangle$.*

Proof. By Lemma 4.2.6, it suffices to show that if an operator $P = P_1 L + P_0$, where P_j's are polynomials in $\mathbb{C}[t^{\pm 1}, M]$, annihilates the colored Jones polynomial then $P = 0$.

Indeed, suppose $PJ(n) = 0$. Then

$$\begin{aligned} 0 &= P_1 J(n+1) + P_0 J(n) \\ &= P_1\Big(-t^{-(4n+2)b}J(n) + t^{-2nb}[2n+1]\Big) + P_0 J(n) \\ &= \big(-t^{-(4n+2)b}P_1 + P_0\big)J(n) + t^{-2nb}[2n+1]P_1. \end{aligned}$$

Let $P_1' = -t^{-(4n+2)b}P_1 + P_0$ and $P_0' = t^{-2nb}[2n+1]P_1$. Then P_1', P_0' are polynomials in $\mathbb{C}[t^{\pm 1}, M]$ and $P_1'J(n) + P_0' = 0$. This implies that $P_1' = P_0' = 0$ since the lowest degree in t of $J(n)$ is $-2bn^2 + 2b$, which is quadratic in n, by Lemma 4.2.4. Hence $P = 0$. □

It is easy to see that $\varepsilon(\alpha_{2,b}) = M^{-2b}(M^2 - M^{-2})A_{2,b}$ where $A_{2,b} = (L-1)(LM^{2b}+1)$ is the A-polynomial of the $(2,b)$-torus knot. This means the AJ conjecture holds true for the $(2,b)$-torus knot.

4.3 Proof of Theorem 8

As in the previous section, we consider the two cases: $a, b > 2$ and $a = 2$ separately.

4.3.1 The case $a, b > 2$

We claim that

Proposition 4.3.1. *The colored Jones polynomial of the (a,b)-torus knot is annihilated by the operator PQ where*

$$\begin{aligned} P &= t^{-10ab}(L^3M^{2ab} + L^{-3}M^{-2ab}) - (t^{2(a-b)} + t^{2(b-a)})t^{-4ab}(L^2M^{2ab} + L^{-2}M^{-2ab}) \\ &\quad + t^{2ab}(LM^{2ab} + L^{-1}M^{-2ab}) - (t^{2ab} + t^{-2ab})(L + L^{-1}) \\ &\quad + (t^{2(a-b)} + t^{2(b-a)})(t^{4ab} + t^{-4ab}), \\ Q &= t^{-6ab}(L^3M^{2ab} + L^{-3}M^{-2ab}) - (t^{2(a+b)} + t^{-2(a+b)})q^{-ab}(L^2M^{2ab} + L^{-2}M^{-2ab}) \\ &\quad + t^{-2ab}(LM^{2ab} + L^{-1}M^{-2ab}) - (t^{2ab} + t^{-2ab})(L + L^{-1}) + 2(t^{2(a+b)} + t^{-2(a+b)}). \end{aligned}$$

Proof. We first prove the following two lemmas.

Lemma 4.3.2. *For the (a,b)-torus knot, we have*

$$QJ(n) = t^{4ab-2}(\lambda_{a+b} - \lambda_{a-b})\frac{t^{2abn}\lambda_{(a-b)(n+1)} - t^{-2abn}\lambda_{(a-b)(n-1)}}{t^2 - t^{-2}}.$$

Proof. Let

$$g(n) = t^{-2abn}\frac{t^2\lambda_{(a+b)n} - t^{-2}\lambda_{(a-b)n}}{t^2 - t^{-2}}.$$

Then, by Lemma 4.2.1, $J(n+2) = t^{-4ab(n+1)}J(n) + g(n+1)$. Hence $QJ(n)$ is equal to

$$\begin{aligned}
& t^{-6ab}\Big(t^{4ab(n+3)}J(n+3) + t^{-4ab(n-3)}J(n-3)\Big) \\
& -(t^{2(a+b)} + t^{-2(a+b)})t^{-4ab}\Big(t^{4ab(n+2)}J(n+2) + t^{-4ab(n-2)}J(n-2)\Big) \\
& + t^{-2ab}\Big(t^{4ab(n+1)}J(n+1) + t^{-4ab(n-1)}J(n-1)\Big) \\
& -(t^{2ab} + t^{-2ab})\Big(J(n+1) + J(n-1)\Big) + 2(t^{2(a+b)} + t^{-2(a+b)})J(n) \\
= \; & t^{-6ab}\Big(t^{4ab}\big(J(n+1) + J(n-1)\big) + t^{2ab(n+5)}g(n+2) - t^{-2ab(n-5)}g(n-2)\Big) \\
& -(t^{2(a+b)} + t^{-2(a+b)})t^{-4ab}\Big(2t^{4ab}J(n) + t^{2ab(n+4)}g(n+1) - t^{-2ab(n-4)}g(n-1)\Big) \\
& + t^{-2ab}\Big(t^{4ab}\left(J(n-1) + J(n+1)\right) + \left(t^{2ab(n+3)} - t^{-2ab(n-3)}\right)g(n)\Big) \\
& -(t^{2ab} + t^{-2ab})\Big(J(n+1) + J(n-1)\Big) + 2(t^{2(a+b)} + t^{-2(a+b)})J(n) \\
= \; & t^{-6ab}\Big(t^{2ab(n+5)}g(n+2) - t^{-2ab(n-5)}g(n-2)\Big) \\
& -\left(t^{2(a+b)} + t^{-2(a+b)}\right)t^{-4ab}\Big(t^{2ab(n+4)}g(n+1) - t^{-2ab(n-4)}g(n-1)\Big) \\
& + t^{-2ab}\left(t^{2ab(n+3)} - t^{-2ab(n-3)}\right)g(n).
\end{aligned}$$

Using the definition of $g(n)$, we can rewrite

$$\begin{aligned}
QJ(n) \;=\; & t^{4ab}\Big(t^{2abn}\frac{t^2\lambda_{(a+b)(n+2)} - t^{-2}\lambda_{(a-b)(n+2)}}{t^2 - t^{-2}} - t^{-2abn}\frac{t^2\lambda_{(a+b)(n-2)} - t^{-2}\lambda_{(a-b)(n-2)}}{t^2 - t^{-2}}\Big) \\
& -(t^{2(a+b)} + t^{-2(a+b)})t^{4ab} \times \\
& \Big(t^{2abn}\frac{t^2\lambda_{(a+b)(n+1)} - t^{-2}\lambda_{(a-b)(n+1)}}{t^2 - t^{-2}} - t^{-2abn}\frac{t^2\lambda_{(a+b)(n-1)} - t^{-2}\lambda_{(a-b)(n-1)}}{t^2 - t^{-2}}\Big) \\
& + t^{4ab}(t^{2abn} - t^{-2abn})\frac{t^2\lambda_{(a+b)n} - t^{-2}\lambda_{(a-b)n}}{t^2 - t^{-2}}.
\end{aligned}$$

Now applying the equality $\lambda_{k+l} + \lambda_{k-l} = \lambda_k\lambda_l$, we then obtain

$$QJ(n) = t^{4ab-2}(\lambda_{a+b} - \lambda_{a-b})\frac{t^{2abn}\lambda_{(a-b)(n+1)} - t^{-2abn}\lambda_{(a-b)(n-1)}}{t^2 - t^{-2}}.$$

This proves Lemma 4.3.2. □

Let $h(n) = t^{2abn}\lambda_{(a-b)(n+1)} - t^{-2abn}\lambda_{(a-b)(n-1)}$.

Lemma 4.3.3. *The function $h(n)$ is annihilated by the operator P, i.e. $Ph(n) = 0$.*

Proof. Let $c = a - b$. Then $Ph(n)$ is equal to

$$\begin{aligned}
& t^{-10ab}\Big(t^{4ab(n+3)}h(n+3) + t^{-4ab(n-3)}h(n-3)\Big) \\
& - (t^{2(a-b)} + t^{2(b-a)})t^{-4ab}\Big(t^{4ab(n+2)}h(n+2) + t^{-4ab(n-2)}h(n-2)\Big) \\
& + t^{2ab}\Big(t^{4ab(n+1)}h(n+1) + t^{-4ab(n-1)}h(n-1)\Big) \\
& - (t^{2ab} + t^{-2ab})\Big(h(n+1) + h(n-1)\Big) + (t^{2(a-b)} + t^{2(b-a)})(t^{4ab} + t^{-4ab})h(n) \\
= \ & \Big(t^{2ab(3n+4)}\lambda_{c(n+4)} - t^{2ab(n-2)}\lambda_{c(n+2)} + t^{-2ab(n+2)}\lambda_{c(n-2)} - t^{-2ab(3n-4)}\lambda_{c(n-4)}\Big) \\
& - \lambda_c\Big(t^{2ab(3n+4)}\lambda_{c(n+3)} - t^{2abn}\lambda_{c(n+1)} + t^{-2abn}\lambda_{c(n-1)} - t^{-2ab(3n-4)}\lambda_{c(n-3)}\Big) \\
& + \Big(t^{2ab(3n+4)}\lambda_{c(n+2)} - t^{2ab(n+2)}\lambda_{cn} + t^{-2ab(n-2)}\lambda_{cn} - t^{-2ab(3n-4)}\lambda_{c(n-2)}\Big) \\
& - (t^{2ab} + t^{-2ab})\Big(t^{2ab(n+1)}\lambda_{c(n+2)} - t^{-2ab(n+1)}\lambda_{cn} + t^{2ab(n-1)}\lambda_{cn} - t^{-2ab(n-1)}\lambda_{c(n-2)}\Big) \\
& + \lambda_c(t^{4ab} + t^{-4ab})\Big(t^{2abn}\lambda_{c(n+1)} - t^{-2abn}\lambda_{c(n-1)}\Big).
\end{aligned}$$

Note that $\lambda_{k+l} + \lambda_{k-l} = \lambda_k\lambda_l$. Hence $Ph(n)$ is equal to

$$\begin{aligned}
& \Big(-t^{2ab(n-2)}\lambda_{c(n+2)} + t^{-2ab(n+2)}\lambda_{c(n-2)}\Big) - \lambda_c\Big(-t^{2abn}\lambda_{c(n+1)} + t^{-2abn}\lambda_{c(n-1)}\Big) \\
& + \Big(-t^{2ab(n+2)}\lambda_{cn} + t^{-2ab(n-2)}\lambda_{cn}\Big) \\
& - (t^{2ab} + t^{-2ab})\Big(t^{2ab(n+1)}\lambda_{c(n+2)} - t^{-2ab(n+1)}\lambda_{cn} + t^{2ab(n-1)}\lambda_{cn} - t^{-2ab(n-1)}\lambda_{c(n-2)}\Big) \\
& + \lambda_c(t^{4ab} + t^{-4ab})\Big(t^{2abn}\lambda_{c(n+1)} - t^{-2abn}\lambda_{c(n-1)}\Big) \\
= \ & -(t^{4ab} + t^{-4ab} + 1)t^{2abn}\lambda_{c(n+2)} + (t^{4ab} + t^{-4ab} + 1)t^{-2abn}\lambda_{c(n-2)} \\
& - (t^{4ab} + t^{-4ab} + 1)(t^{2abn} - t^{-2abn})\lambda_{cn} \\
& + \lambda_c(t^{4ab} + t^{-4ab} + 1)\Big(t^{2abn}\lambda_{c(n+1)} - t^{-2abn}\lambda_{c(n-1)}\Big) \\
= \ & -(t^{4ab} + t^{-4ab} + 1)t^{2abn}\Big(\lambda_{c(n+2)} + \lambda_{cn} - \lambda_c\lambda_{c(n+1)}\Big) \\
& + (t^{4ab} + t^{-4ab} + 1)t^{-2abn}\Big(\lambda_{c(n-2)} + \lambda_{cn} - \lambda_c\lambda_{c(n-1)}\Big) \\
= \ & 0.
\end{aligned}$$

This proves Lemma 4.3.3. □

Proposition 4.3.1 follows directly from Lemmas 4.3.2 and 4.3.3. □

4.3.2 The case $a = 2$

We claim that

Proposition 4.3.4. *The colored Jones polynomial of the* $(2, b)$*-torus knot is annihilated by the operator*

$$\begin{aligned} R &= t^{-4b}(L^2M^{2b} + L^{-2}M^{-2b}) + (t^{2b} + t^{-2b})(L + L^{-1}) \\ &\quad -(t^4 + t^{-4})t^{-2b}(LM^{2b} + L^{-1}M^{-2b}) + (M^{2b} + M^{-2b}) - 2(t^4 + t^{-4}). \end{aligned}$$

Proof. From Lemma 4.2.5 we have

$$\begin{aligned} J(n+1) &= -t^{-(4n+2)b}J(n) + t^{-2nb}[2n+1], \\ J(n+2) &= t^{-8(n+1)b}J(n) - t^{-6(n+1)b}[2n+1] + t^{-2(n+1)b}[2n+3], \\ J(n-1) &= -t^{(4n-2)b}J(n) + t^{2nb}[2n-1], \\ J(n-2) &= t^{8(n-1)b}J(n) - t^{6(n-1)b}[2n-1] + t^{2(n-1)b}[2n-3]. \end{aligned}$$

Hence $RJ(n)$ is equal to

$$
\begin{aligned}
& t^{-4b}\Big(t^{4(n+2)b}J(n+2)+t^{-4(n-2)b}J(n-2)\Big)+(t^{2b}+t^{-2b})\Big(J(n+1)+J(n-1)\Big) \\
& -(t^4+t^{-4})t^{-2b}\Big(t^{4(n+1)b}J(n+1)+t^{-4(n-1)b}J(n-1)\Big) \\
& +\Big((t^{4nb}+t^{-4nb})-2(t+t^{-4})\Big)J(n) \\
= \;& t^{-4b}\Big(-t^{-2(n-1)b}[2n+1]+t^{2(n+3)b}[2n+3]-t^{2(n+1)b}[2n-1]+t^{-2(n-3)b}[2n-3]\Big) \\
& +(t^{2b}+t^{-2b})\Big(t^{-2nb}[2n+1]+t^{2nb}[2n-1]\Big) \\
& -(t^4+t^{-4})t^{-2b}\Big(t^{(2n+4)b}[2n+1]+t^{(-2n+4)b}[2n-1]\Big) \\
& -(t+t^{-4})t^{2b}\Big(t^{2nb}[2n+1]+t^{-2nb}[2n-1]\Big) \\
= \;& t^{2b}t^{2nb}\Big([2n+3]+[2n-1]-(t^4+t^{-4})[2n+1]\Big) \\
& +t^{2b}t^{-2nb}\Big([2n-3]+[2n+1]-(t^4+t^{-4})[2n-1]\Big) \\
= \;& 0,
\end{aligned}
$$

since $[k+l]+[k-l]=(t^{2l}+t^{-2l})[k]$. ☐

4.3.3 Proof of Theorem 8

We first note that the A-ideal $\mathfrak{p}$, the kernel of $\theta : \mathfrak{t}^\sigma \longrightarrow \mathbb{C}[\chi(X)]$, is radical i.e. $\sqrt{\mathfrak{p}} = \mathfrak{p}$. This is because the character ring $\mathbb{C}[\chi(X)]$ is reduced, i.e. has nil-radical 0, by definition.

Lemma 4.3.5. *Suppose $\delta(t, M, L) \in \mathcal{A}_K$. Then there are polynomials $g(t, M) \in \mathbb{C}[t^{\pm 1}, M]$ and $\gamma(t, M, L) \in \mathcal{T}$ such that*

$$\delta(t, M, L) = \frac{1}{g(t, M)}\, \gamma(t, M, L)\, \alpha_K(t, M, L). \tag{25}$$

Moreover, $g(t, M)$ and $\gamma(t, M, L)$ can be chosen so that $\varepsilon g \neq 0$.

Proof. By definition α_K is a generator of $\widetilde{\mathcal{A}}_K$, the extension of $\mathcal{A}_K$ in the principal left-ideal domain $\widetilde{\mathcal{T}}$. Since $\delta \in \mathcal{A}_K$, it is divisible by α_K in $\widetilde{\mathcal{T}}$. Hence (25) follows.

We can assume that $t+1$ does not divide both $g(t, M)$ and $\gamma(t, M, L)$ simultaneously. If $\varepsilon(g) = 0$ then g is divisible by $t+1$, and hence γ is not. But then from the equality $g\delta = \gamma\alpha_K$, it follows that α_K is divisible by $t+1$, which is impossible, since all the coefficients of powers of L in α_K are supposed to be co-prime. □

Showing $\sqrt{\varepsilon(\mathcal{A}^\sigma)} \subset \mathfrak{p}$. For torus knots, by Section 4.2, we have $\varepsilon(\alpha_K) = f(M)A_K$, where $f(M) \in \mathbb{C}[M^{\pm 1}]$. For every $\delta \in \mathcal{A}_K$, by Lemma 3.2.7, there exist $g(t, M) \in \mathbb{C}[t^{\pm 1}, M]$ and $\gamma \in \mathcal{T}$ such that $\delta = \frac{1}{g(t,M)}\gamma\,\alpha_K$ and $\varepsilon g \neq 0$. It implies that

$$\varepsilon(\gamma) = \frac{1}{\varepsilon g(M)}\,\varepsilon(\gamma)\,\varepsilon(\alpha_K) = \frac{1}{\varepsilon g(M)}\,\varepsilon(\gamma)\,f(M)A_K. \tag{26}$$

The A-polynomial of a torus knot does not contain any non-trivial factor depending on M only. Since $\varepsilon(\gamma) \in \mathfrak{t} = \mathbb{C}[L^{\pm 1}, M^{\pm 1}]$, equation (26) implies that $h := \frac{1}{\varepsilon g(M)}\,\varepsilon(\gamma)\,f(M)$ is an element of $\mathfrak{t}$. Hence $\varepsilon(\gamma) \in A_K \cdot \mathfrak{t}$, the ideal of $\mathfrak{t}$ generated by A_K. It follows that $\varepsilon(\mathcal{A}_K) \subset A_K \cdot \mathfrak{t}$ and thus $\varepsilon(\mathcal{A}^\sigma) \subset (A_K \cdot \mathfrak{t})^\sigma = \mathfrak{p}$. Hence $\sqrt{\varepsilon(\mathcal{A}^\sigma)} \subset \sqrt{\mathfrak{p}} = \mathfrak{p}$.

Showing $\mathfrak{p} \subset \sqrt{\varepsilon(\mathcal{A}^\sigma)}$. For $a, b > 2$, by Proposition 4.3.1 the colored Jones polynomial of the (a, b)-torus knot is annihilated by the operator PQ. Note that

$$\begin{aligned}\varepsilon(PQ) &= (L + L^{-1} - 2)^2(L^2M^{2ab} + L^{-2}M^{-2ab} - 2)^2 \\ &= L^{-2}\big(L^{-1}M^{-ab}(L-1)(L^2M^{2ab} - 1)\big)^4.\end{aligned}$$

If $u \in \mathfrak{p}$ then $u = vA'_{a,b}$, where $A'_{a,b} = L^{-1}M^{-ab}(L-1)(L^2M^{2ab} - 1) = L^{-1}M^{-ab}A_{a,b}$ and $v \in \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$. It is easy to see that $\sigma(v) = Lv$ since $\sigma(u) = u$ and $\sigma(A'_{a,b}) = L^{-1}A'_{a,b}$. This implies that $\sigma(v^2L) = \sigma(v)^2L^{-1} = v^2L$. We then have

$$u^4 = v^4A'^4_{a,b} = \varepsilon(v^4L^2PQ) \in \varepsilon(\mathcal{A}^\sigma),$$

hence $u \in \sqrt{\varepsilon(\mathcal{A}^\sigma)}$.

For $a = 2$, by Proposition 4.3.4 the colored Jones polynomial of the $(2, b)$-torus knot is annihilated by the operator R. Note that $\sigma(R) = R$ and

$$\begin{aligned}\varepsilon(R) &= (L + L^{-1} - 2)(LM^{2b} + L^{-1}M^{-2b} + 2) \\ &= \big(L^{-1}M^{-b}(L-1)(LM^{2b} + 1)\big)^2.\end{aligned}$$

If $u \in \mathfrak{p}$ then $u = vA'_{2,b}$, where $A'_{2,b} = L^{-1}M^{-b}(L-1)(LM^{2b}+1) = L^{-1}M^{-b}A_{2,b}$ and $v \in \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$. It is easy to see that $\sigma(v) = -v$ and hence $\sigma(v^2) = \sigma(v)\sigma(v) = v^2$. We then have

$$u^2 = v^2 A'^2_{2,b} = \varepsilon(v^2 R) \in \varepsilon(\mathcal{A}^\sigma),$$

hence $u \in \sqrt{\varepsilon(\mathcal{A}^\sigma)}$.

In both cases $\mathfrak{p} \subset \sqrt{\varepsilon(\mathcal{A}^\sigma)}$. Hence $\sqrt{\varepsilon(\mathcal{A}^\sigma)} = \mathfrak{p}$ for all torus knots.

CHAPTER V

FUTURE PLANS

There are several problems we want to study in the future:

Problem 1. *Study the volume conjecture for all cables of a knot.*

The colored Jones polynomials of cables of a knot are available in [59], so we hope to apply the method in Chapter 1 to study the volume conjecture for them, especially to extend the results of Theorems 1, 2, 3 and 4 to all cables of a knot.

Problem 2. *Study the AJ conjecture for those knots whose character varieties have at least 3 irreducible components.*

All the knots for which the AJ conjecture has been so far confirmed have character varieties consisting of 2 irreducible components. Hence we want to investigate the conjecture for those knots whose character varieties have 3 or more irreducible components. One way to study this is to understand the action of the skein algebra of the boundary torus on the skein module of the knot complement.

Problem 3. *Study the AJ conjecture for links.*

By the result of Chapter 3, for a two-bridge link when reducing $t = -1$, the skein module is isomorphic to the ring of regular functions on the character variety. This is the starting point for considering the AJ conjecture for links. In a joint work in progress with T. Le, we have formulated the AJ conjecture for links and confirmed it for $(2, 2m)$-torus links. The next step we plan to do is to extend Theorems 5, 6 and 2.5.6 to the case of hyperbolic links, in particular to some classes of two-bridge links and pretzel links.

Problem 4. *Study the $\mathcal{A}^\sigma = \mathcal{P}$ conjecture.*

Although the recurrence ideal $\mathcal{A}_K$ is defined algebraically, it has a beautiful geometric meaning relating to skein modules. Let $N(K)$ be a tubular neighborhood and X be the complement of a knot K. There is a bilinear pairing

$$\mathcal{S}(N(K)) \otimes \mathcal{S}(X) \to \mathcal{S}(S^3) = \mathbb{C}[t^{\pm 1}]$$

defined by $l \otimes l' = \langle l, l' \rangle := \langle l \cup l' \rangle$, where l and l' are framed links in $N(K)$ and X respectively. The inclusion $\partial X \hookrightarrow X$ induces a map $\Theta : \mathcal{S}(\partial X) \to \mathcal{S}(X)$. The kernel $\mathcal{P}$ of Θ is the *non-commutative A-ideal* of K. The *orthogonal ideal* $\mathcal{O}$ of K is

$$\mathcal{O} = \{l' \in \mathcal{S}(\partial X) \mid \langle l, \Theta(l') \rangle = 0 \quad \text{for all } l \in \mathcal{S}(N(K))\}.$$

It is known that $\mathcal{O} = \mathcal{A}^\sigma$, the σ-invariant part of $\mathcal{A}_K$ under the involution $\sigma(M^a L^b) = M^{-a} L^{-b}$, see [21, 16] and Chapter 2. It implies that $\mathcal{P}$ is contained in $\mathcal{A}^\sigma = \mathcal{O}$. If $\mathcal{P} = \mathcal{A}^\sigma$ for all knots then according to [36] the colored Jones polynomial distinguishes the unknot from other knots.

Problem 5. *Study the stronger version of the AJ conjecture.*

In the future we hope to confirm the conjecture for all the knots for which the AJ conjecture has been established.

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