# ONE AND TWO WEIGHT THEORY IN HARMONIC ANALYSIS

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# ONE AND TWO WEIGHT THEORY IN HARMONIC ANALYSIS

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*To my parents.*

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# SUMMARY

This thesis studies several problems dealing with weighted inequalities and vector-valued operators. A weight is a nonnegative locally integrable function, and weighted inequalities refers to studying a given operator's continuity from $L^p(w)$ to $L^p(\sigma)$ (or $L^{p,\infty}(\sigma)$) with $1 < p < \infty$ and $w$ and $\sigma$ weights. The case where $\sigma = w$ is known as a one weight inequality and the case where $\sigma \neq w$ is called a two weight inequality. These types of inequalities appear naturally in harmonic analysis from attempts to extend classical results to function spaces where the underlying measure is not necessarily Lebesgue measure. For most operators from harmonic analysis, Muckenhoupt $A_p$ weights represent the class of weights for which a one weight inequality holds. Chapters II and III study questions involving these weights. In particular, Chapter II focuses on determining the sharp dependence of a vector-valued Calderón-Zygmund operator's norm on an $A_p$ weight's characteristic; we determine that the vector-valued operator recovers the scalar dependence. Chapter III presents material from a joint work with M. Lacey. Specifically, in this chapter we estimate the weak-type norms of a simple class of vector-valued operators, but are unable to obtain a sharp result. The final two chapters consider two weight inequalities. Chapter IV characterizes the two weight inequality for a subset of the vector-valued operators considered in Chapter III. The final chapter presents examples to argue there is no relationship between the Hilbert transform and the Hardy-Littlewood maximal operator in the two weight setting; the material is taken from a joint work with M. Reguera.

# CHAPTER I

# INTRODUCTION

## *1.1 Preliminaries*

This thesis studies a branch of harmonic analysis known as weighted inequalities. Our particular focus will be on vector-valued operators and Calderón-Zygmund operators. The theory of weighed inequalities is pertinent to a variety of subjects. There are deep ties between weights and the regularity of solutions to certain partial differential equations. Operator theory and spectral theory can be related to this subject through two weight inequalities for singular integrals. Additionally, weighted inequalities also find application in approximation theory and probability theory.

The present chapter will provide an overview of the area of weighted inequalities. Subsequent chapters detail recent advances in the subject. The material of Chapters II-V are drawn from [40], [37], [41], and [22].

We begin by introducing some basic terms and ideas. First, we refer to a locally integrable nonnegative function $w$ on $\mathbb{R}^n$ as a weight. In harmonic analysis, weighted theory or weighted norm inequalities refers to the study of a given operator's continuity properties when considered as acting on functions from $L^r(w)$ to $L^p(\sigma)$ (or $L^{p,\infty}(\sigma)$), where $\sigma$ and $w$ are fixed weights and $1 < p, r < \infty$. The terms one weight and two weight refer to the cases where $w = \sigma$ and $w \neq \sigma$.

We let $M$ denote the Hardy-Littlewood maximal operator defined as

**Definition 1.** *For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ let*

$$Mf(x) = \sup_{Q \ni x} \langle f \rangle_Q$$

*where the supremum is taken over all cubes $Q$ containing $x$ and $\langle f \rangle_Q = \frac{1}{|Q|} \int_Q f(y)dy$,* and by a Calderón-Zygmund operator, we will mean the following:

**Definition 2.** *We call a function $K$ a Calderón-Zygmund kernel if there is $0 < \alpha \leq 1$ such that $K$ satisfies the following:*

(i.) $|K(x,y)| \lesssim \frac{1}{|x-y|^n}$ *for $x, y \in \mathbb{R}^n$ such that $x \neq y$*

(ii.) $|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \lesssim \frac{|x-x'|^\alpha}{|x-y|^{n+\alpha}}$ *with $|x - x'| < \frac{|x-y|}{2}$.*

*We call an operator $T$ a Calderón-Zygmund operator if $T$ is bounded on $L^2(\mathbb{R}^n)$ and there is a Calderón-Zygmund kernel $K$ such that*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \quad x \notin \operatorname{supp}(f)$$

*for compactly supported $f \in L^2(\mathbb{R}^n)$.*

The canonical example of Calderón-Zygmund operator is of course the Hilbert transform $H$ which we define as

$$Hf(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)dy}{x-y}.$$

A notion closely related to Calderón-Zygmund operators is that of Haar functions and Haar shift operators.

**Definition 3.** *Let $\mathcal{D}$ be a dyadic grid and $Q \in \mathcal{D}$. Then $h_Q$ is a Haar function if $h_Q$ satisfies*

$$h_Q(x) = \sum_{Q' \in \mathcal{C}(Q)} c_{Q'} \mathbf{1}_{Q'}(x)$$

*where $\mathcal{C}(Q)$ is the collection of all dyadic children for $Q$.*

**Definition 4.** *For integers $(m,n) \in \mathbb{Z}^2$ we will call $\mathbb{S}$ a Haar shift operator of complexity $(m,n)$ if*

$$\mathbb{S}f(x) = \sum_{Q \in \mathcal{D}} \sum_{\substack{Q', R' \in \mathcal{D} \\ Q', R' \subset Q \\ \ell(Q') = \ell(Q)2^{-m}, \ \ell(R') = \ell(Q)2^{-n}}} \frac{\langle f, h_{R'}^{Q'} \rangle}{|Q|} k_{Q'}^{R'}(x).$$

*where for a given cube $I$, $\ell(I) = |I|^{\frac{1}{n}}$. The functions $h_{R'}^{Q'}$ and $k_{Q'}^{R'}$ represent generalized Haar functions. The complexity $\kappa$ of $\mathbb{S}$ is defined as* $\max\{m, n, 1\}$.

It is a well known but deep fact that a general Calderón-Zygmund operator $T$ can be recovered via suitable averaging of Haar shift operators, see [17].

Central to this thesis are vector-valued operators and functions; here, following [9], we make explicit the meaning behind these terms.

**Definition 5.** *Let $(X, \mu)$ be a $\sigma$-finite measure space and $B$ a Banach space. A function $F : X \to B$ is measurable if the following holds:*

(i.) *there is a separable subspace $B_0$ of $B$ such that $F(x) \in B_0$ for almost every $x \in X$.*

(ii.) *for each $b' \in B'$, $g(x) = \langle F(x), b' \rangle$ is measurable.*

*We let $L_B^p(\mu)$ be the space consisting of all measurable $F : X \to B$ such that*

$$\left( \int_X \|F\|_B^p d\mu \right)^{\frac{1}{p}} < \infty.$$

*and analogously we take $L_B^{p,\infty}(\mu)$ to be the collection of all $F : X \to B$ satisfying*

$$\|F\|_{L^{p,\infty}(w)} = \sup_{t>0} t\mu\left(\{x \in X : \|F(x)\| > t\}\right)^{\frac{1}{p}}.$$

Our focus is on the case when $B$ is a sequence space $\ell^r$ with $1 < r < \infty$; unless otherwise indicated, we use vector and vector-valued in reference to such an $\ell^r$ space.

Other classical operators we will be interested in are

**Definition 6.** *For $1 < r < \infty$ we define the vector-valued maximal operator $\mathbf{M}_r$ as*

$$\mathbf{M}_r(\mathbf{f})(x) = \left( \sum_{j=1}^{\infty} M f_j(x)^r \right)^{\frac{1}{r}}$$

*for $\mathbf{f} = \{f_j\}_{j=1}^{\infty}$ a sequence of locally integrable functions,*

**Definition 7.** *For $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, define the dyadic square function to be*

$$Sf(x) = \left(\sum_{I \in \mathcal{D}} \langle f, h_I \rangle_I^2 h_I(x)\right)^{\frac{1}{2}},$$

and as in [45] we define the intrinsic square function to be

**Definition 8.** *Let $C_\alpha$ be the collection of functions $\gamma$ supported in the unit ball with mean zero and such that $|\gamma(x) - \gamma(y)| \le |x - y|^\alpha$. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ let*

$$A_\alpha f(x,t) = \sup_{\gamma \in C_\alpha} |f * \gamma_t(x)|$$

*where $\gamma_t(x) = t^{-n}\gamma(xt^{-n})$ and take*

$$G_\alpha f(x) = \left(\int_{\Gamma(x)} A_\alpha f(y,t)^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

*where $\Gamma(x) := \left\{(y,t) \in \mathbb{R}^{n+1}_+ \ : \ |y| < t\right\}$ is the cone of aperture one in the upper-half plane. We call $G_\alpha$ the intrinsic square function.*

We make one last definition,

**Definition 9.** *We refer to a collection of cubes $\mathcal{Q} = \left\{Q^k_j\right\}_{j,k \in \mathbb{N}}$ as sparse if for fixed $j$, $Q^k_j \cap Q^l_j = \emptyset$ and if for $Q^k_j \in \mathcal{Q}$ we have*

$$|D(Q^k_j) \cap Q^k_j| \le 2^{-1}|Q^k_j|$$

*where $D(Q^k_j) = Q^k_j \setminus \bigcup_{\substack{Q^m_l \subset Q^k_j \\ Q^m_l \in \mathcal{Q}}} Q^m_l$.*

## *1.2 Main Results and Background*

### 1.2.1 One Weight Inequalities

The theory of one weight inequalities is well known and largely restricted to the study of Muckenhoupt $A_p$ weights:

**Definition 10.** *Let $w$ be a weight which is strictly positive almost everywhere. We say $w \in A_p$ for $1 < p < \infty$ provided the following quantity is finite:*

$$[w]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} \right)^{p-1},$$

*where the supremum is taken over all cubes in $\mathbb{R}^n$. When $p = 1$ we say $w \in A_1$ provided the following is finite:*

$$[w]_{A_1} = \left\| \frac{Mw}{w} \right\|_{L^\infty}.$$

For most classical operators from harmonic analysis, Muckenhoupt weights comprise the weights for which a one weight inequality holds. In [27] Muckenhoupt showed the $A_p$ condition was necessary and sufficient for the Hardy-Littlewood maximal operator to be a bounded operator from $L^p(w)$ into $L^p(w)$. Later, Muckenhoupt, Hunt and Wheeden [10] demonstrated the $A_p$ condition characterized the weighted continuity of the Hilbert transform, and Coifman-Fefferman [4] proved this for more general singular integrals.

Sharp one weight estimates were first studied by Buckley [1] when he obtained the sharp strong-type and weak-type bounds for the Hardy-Littlewood maximal function. Later, the subject was motivated by [34], where A. Volberg and S. Petrmichl use sharp weighted results to study solutions for the Beltrami equation; in particular, the authors prove a linear bound for the Beurling-Alfohrs transform $T$, i.e. if $w \in A_2$ then

$$\|T\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}$$

so that by extrapolation,

$$\|T\|_{L^p(\omega)\to L^p(\omega)} \lesssim [w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}$$

for $\omega \in A_p$ with $1 < p < \infty$. Following [34] a series of results appeared verifying the linear bound for singular integral operators and dyadic shift operators (see [18, 44]);

and, the question of whether the linear bound extended to all Calderón-Zygmund operators became the focus of intense research, eventually becoming known as the $A_2$ conjecture. The conjecture was finally solved in all generality by T. Hytönen [17].

The second chapter of this thesis considers the question of extending Hytönen's result to $\ell^r$ spaces. That is, Chapter II focuses on the following: given a Calderón-Zygmund operator $T$ and its $\ell^r$ extension $\mathbf{T}$, we want an estimate of the following type

$$\|\mathbf{T}\|_{L^p_{\ell^r}(w)\to L^p_{\ell^r}(w)} \lesssim \alpha_{p,r}([w]_{A_p})$$

for some function $\alpha_{p,r}(t)$ which is the best possible choice in the sense $\alpha_{p,r}(t)$ cannot be replaced by a function $\beta_{p,r}(t)$ which grows more slowly as $t\to\infty$. We are able to show that the best possible choice of $\alpha_{p,r}$ in the above inequality is $t^{\max\left\{1,\frac{1}{p-1}\right\}}$; in particular, we obtain the same dependence as in the scalar case. This type of dependence is somewhat unexpected, contrasting greatly with similar operators such as the dyadic square function and vector-valued maximal operator (see Chapter II and [8]). Additionally, we also note that the implied constants in our estimates do depend on $r$. Hence, Chapter II presents the unusual result that scalar valued Calderón-Zygmund operators are just as singular as $\ell^r$ extensions of Calderón-Zygmund operators.

An important remark is that Hytönen's proof relied on probabilistic techniques and the notion of Haar shift operators. Different methods of proof for this theorem were simultaneously and subsequently investigated. The most successful of these involved using A. Lerner's decomposition theorem [23]:

**Theorem 1.2.1.** *Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and let $Q$ be a fixed cube. Then there exists a collection of dyadic cubes $\left\{Q_j^k\right\}_{j,k\in\mathbb{N}}$ such that*

(i.) *for each $k,j\in\mathbb{N}$, we have $Q_j^k\subset Q$*

(ii.) *for almost every $x\in Q$,*

$$|f(x)-m_f(Q)|\le 4M^\sharp_{2^{-n-2};Q}f(x)+4\sum_k\sum_j \omega_{2^{-n-2}}(f;Q_j^k)\mathbf{1}_{Q_j^k}(x)$$

(iii.) *for fixed* $k$, $Q_j^k \cap Q_i^k = \emptyset$ *for* $i \neq j$

(iv.) *letting* $\Omega_k = \bigcup_j Q_j^k$, *we have* $|\Omega_k \cap Q_j^k| \leq 2^{-1}|Q_j^k|$ *and* $\Omega_{k+1} \subset \Omega_k$

and avoided averaging techniques altogether (see [13, 24]). Other ideas focused on reducing the strong-type inequality to a weak-type inequality; this was in fact the basis of [17]. With regard to this line of investigation, two conjectures received considerable attention, namely the $A_1$ conjecture, i.e.

**Conjecture 1.2.2** ($A_1$ Conjecture). *If* $T$ *is a Calderón-Zygmund operator and* $w \in A_1$, *then*

$$w\left(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right) \lesssim \frac{[w]_{A_1}}{\lambda} \int_{\mathbb{R}^n} |f(x)|w$$

*for* $f \in L^1(w)$,

and a related conjecture by Muckenhoupt and Wheeden

**Conjecture 1.2.3.** *Let* $T$ *be a Calderón-Zygmund operator and* $w$ *a weight. Then*

$$w\left(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right) \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)|Mw$$

*for* $f \in L^1(w)$.

Were either conjecture true, an extrapolation argument would imply the $A_2$ conjecture; however, both conjectures have recently been shown to fail [29, 36]. The best known bound for $A_1$ weights was obtained by Lerner-Perez-Ombrosi [26]:

**Theorem 1.2.4.** *Let* $T$ *be a Calderón-Zygmund operator and* $w \in A_1$. *Then we have*

$$w\left(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right) \lesssim \frac{[w]_{A_1}(1 + \log [w]_{A_1})}{\lambda} \int_{\mathbb{R}^n} |f(x)|w.$$

The addition of the logarithm to some power is necessary as shown by Nazarov-Reznikov-Vasyunin-Volberg [29]. The sharp dependence for the endpoint estimate remains an open question. In the range $1 < p < \infty$, the sharp result was obtained by [16]:

**Theorem 1.2.5.** *If $T$ is a Calderón-Zygmund operator and $w \in A_p$ then we have*

$$\|T\|_{L^{p,\infty}(w)\to L^{p,\infty}(w)} \lesssim [w]_{A_p}.$$

We can contrast the above behavior with the less singular square functions and vector-valued maximal function. Wilson [45] showed if $w$ is a weight and $G_\alpha$ is the intrinsic square function then

$$w\left(\{x \in \mathbb{R}^n : G_\alpha f(x) > \lambda\}\right) \lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n} |f(x)| Mw$$

and Perez [33] gave the same estimate with $G_\alpha$ replaced by the vector-valued maximal function $\mathbf{M}_r$ with exponent $1 < r < \infty$. As a result, both the vector-valued maximal function and square functions satisfy a linear $A_1$ bound.

Chapter III will consider a problem related to the endpoint estimates above. Namely, we consider the vector-valued operators $\mathcal{T}_{\mathcal{Q},r,\rho}$ defined by

**Definition 11.** *Let $\mathcal{Q}$ be a sparse collection of cubes, $1 < r < \infty$, and parameter $1 \le \rho < \infty$. Then for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we define*

$$\mathcal{T}_{\mathcal{Q},r,\rho} f(x) = \left(\sum_{I\in\mathcal{Q}} |\langle f\rangle_{\rho I}|^r \mathbf{1}_I(x)\right)^{\frac{1}{r}}$$

*where given a cube $Q \subset \mathbb{R}^n$ we let $\rho Q$ be the cube with the same center as $Q$ but with side length $\rho\ell(Q)$,*

and show

**Theorem 1.2.6.** *For $1 < p, r < \infty$, $1 \le \rho < \infty$, and $w \in A_p$ we have*

$$\|\mathcal{T}_{\mathcal{Q},r,\rho}\|_{L^{p,\infty}(w)} \lesssim \phi_{p,r}([w]_{A_p})$$

*where $\phi_{p,r}(x) = x^{\frac{1}{p}}$ for $1 < p < r$ and $\phi_{p,r}(x) = x^{\frac{1}{r}}(1 + \log x)$ for $r \le p$.*

We are interested in the operators $\mathcal{T}_{\mathcal{Q},r,\rho}$ because application of Theorem 1.2.1 reduces study of the intrinsic square function and the vector-valued maximal function with exponent $r$ to that of the maximal function and operators of the form $\mathcal{T}_{\mathcal{Q},2,\rho}$ and $\mathcal{T}_{\mathcal{Q},r,\rho}$ (see Chapter III for details). Hence, as a result of Theorem 1.2.6, we improve the implicit weak-type bounds for the intrinsic square function in the range $1 < p < 3$ and the vector-valued maximal operator in the range $1 < p < r+1$. The logarithm in our theorem can be compared with that in (1.2.4) for Calderón-Zygmund operators; however, we are unable to show that the addition of a logarithm is necessary.

### 1.2.2 Two Weight Inequalities

Two weight inequalities are more difficult and complicated than one weight inequalities. Due to the work of Eric Sawyer on the two weight inequality for fractional integrals [39] and the maximal function [38], there is a standard method for characterizing the two weight inequality via testing conditions. Explicitly, given an operator $T : L^r(w) \to L^p(\sigma)$, we test the following inequality

$$\|T(f)\|_{L^p(\sigma)} \lesssim \|f\|_{L^r(w)}$$

over all $f$ in some special, usually simpler, class of functions. For most integral operators with positive kernels, the above is an efficient method of characterization. The main ingredient used in the arguments for such results is the weighted Carleson embedding theorem:

**Theorem 1.2.7** (Weighted Carleson Embedding Theorem)**.** *Let $w$ be a weight on $\mathbb{R}^n$ and $\{\tau_J\}_{J\in\mathcal{D}}$ a collection of nonnegative numbers. Then we have*

$$\sup_I \frac{1}{w(I)} \sum_{J\subset I} \tau_J \lesssim 1$$

*if and only if*

$$\sup_{\substack{f\in L^p(w)\\ \|f\|_{L^p(w)}=1}} \sum_{J\in\mathcal{D}} (\langle f\rangle_J^w)^p \tau_J \lesssim 1, \tag{1.2.8}$$

*where $\langle f\rangle_J^w = \frac{1}{w(J)}\int_J f(x)w$ for a given interval $J$.*

Indeed, Theorem 1.2.7 can be used to prove E. Sawyer's two weight characterization for the maximal function and discrete positive operators, see [43]. We provide a generalization of Theorem 1.2.7 in Chapter IV, specifically showing

**Theorem 1.2.9.** *Suppose $w$ and $\sigma$ are weights and $1 < r, p < \infty$ with $\mathcal{Q}$ a collection of sparse cubes. Then for $\mathcal{T}_{\mathcal{Q},r,1} = \mathcal{T}_{\mathcal{Q}}$, we have $\|\mathcal{T}_{\mathcal{Q},r}(\cdot\sigma)\|_{L^p(\sigma)\to L^p(w)}$ if and only if there are $\mathcal{L}$ and $\mathcal{L}_*$ such that*

$$\sup_Q \int_Q \mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_Q\sigma)(x)^p w \leq \mathcal{L}\sigma(Q) \tag{1.2.10}$$

$$\sup_{\mathbf{a}} \sup_Q \int_Q \mathbf{U}_{\mathcal{Q}}(\mathbf{1}_Q\mathbf{a}w)(x)^{p'}\sigma \leq \mathcal{L}_* w(Q) \tag{1.2.11}$$

*where we define*

$$\mathbf{U}_{\mathcal{Q}}(\mathbf{g})(x) = \sum_{I\in\mathcal{Q}} \langle g_I\rangle_I \mathbf{1}_I(x)$$

*for $\mathbf{g} = \{g_I\}_{I\in\mathcal{Q}}$ a sequence of locally integrable functions and where the second supremum is taken over all sequences $\mathbf{a}$ of locally integrable functions satisfying $\|\mathbf{a}\|_{\ell^r} = 1$.*

Additionally, using A. Lerner's decomposition theorem, an immediate consequence of Theorem 1.2.9 is that we obtain sufficient conditions for a two weight inequality for the vector-valued maximal function and dyadic square function.

When the operator under consideration fails to be positive, the two weight problem typically requires more elaborate arguments. In particular, singular integrals such as the Hilbert transform have been notoriously difficult to characterize in the two weight setting. It is readily seen that a two weight $A_p$ condition is not sufficient for the Hilbert transform to be bounded, see [28]. An alternative condition for the Hilbert transform was suggested by D. Sarason:

**Conjecture 1.2.12** (Sarason's Conjecture)**.** *For two weights $w$ and $\sigma$ the Hilbert transform is bounded from $L^2(w)$ to $L^2(\sigma)$ if and only if*

$$\sup_{z\in\mathbb{C}_+} P_w(z)P_v(z) \lesssim 1$$

*where $P_w$ and $P_v$ are the Poisson extensions of $w$ and $v$ to the upper half plane $\mathbb{C}_+$.*

However, F. Nazarov [28] constructed counterexamples to show Sarason's conjecture is false. One important positive result in this direction was given by Nazarov-Treil-Volberg [31]:

**Theorem 1.2.13.** *Suppose $\sigma$ and $w$ are positive Borel measures such that $M(\cdot\sigma) : L^2(\sigma) \to L^2(w)$ and $M(\cdot w) : L^2(w) \to L^2(\sigma)$ both hold. Then $H(\cdot\sigma)$ is bounded from $L^2(\sigma)$ to $L^2(w)$ if and only if the following hold:*

(i.) $\|H(\mathbf{1}_I\sigma)\|_{L^2(w)} \lesssim \sigma(I)^{\frac{1}{2}}$

(ii.) $\|H(\mathbf{1}_I w)\|_{L^2(\sigma)} \lesssim w(I)^{\frac{1}{2}}$

(iii.) $\sup_{z\in\mathbb{C}} P_\sigma(z)P_w(z) \lesssim 1.$

Recently, [21] obtained a characterization in terms of weak-type inequalities. The problem is still open for more general singular integrals.

Chapter V considers examples which illustrate some of the difficulties presented by the two weight problem when the underlying operator is no longer positive. Using the constructions of [19, 35–37] as inspiration we construct weights to refute an old conjecture of Muckenhoupt and Wheeden

**Conjecture 1.2.14** ($L^p$ Muckenhoupt-Wheeden)**.** *Let $T$ be a Calderón-Zygmund operator and let $w$ and $v$ be weights on $\mathbb{R}^n$. Then*

$$M : \quad L^p(v) \mapsto L^p(w) \tag{1.2.15}$$

$$M : \quad L^{p'}(w^{1-p'}) \mapsto L^{p'}(v^{1-p'}) \tag{1.2.16}$$

*if and only if*

$$T : L^p(v) \mapsto L^p(w). \tag{1.2.17}$$

and a pair of measures which show the assumptions of Theorem 1.2.13 are distinct. We conclude that in the two weight setting, there is no relationship between the maximal function and the Hilbert transform.

# CHAPTER II

# SHARP ONE WEIGHT ESTIMATE FOR A VECTOR-VALUED CALDERÓN-ZYGMUND OPERATOR

## *2.1 Introduction*

The present chapter will focus on strong-type inequalities for $\ell^r$ extensions of singular integral operators on weighted spaces $L^p(w)$. Our goal is to give a quantitative estimate of these operators' norm in terms of a given weight's $A_p$ characteristic. The scalar version of this problem has recently been given a great deal of attention. In this context the sharp dependence can be extrapolated from the case $p = 2$ which gives a linear estimate, i.e. if $T$ is a Calderón-Zygmund operator and $w \in A_2$,

$$\|T\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}; \tag{2.1.18}$$

further, the above inequality is referred to as the $A_2$ Theorem. The authors of [32] reduced the proof of (2.1.18) to estimating Sawyer-type testing conditions for $w \in A_2$,

$$\|T\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2} + \|T\|_{L^2(w)\to L^{2,\infty}(w)} + \|T^*\|_{L^2(w^{-1})\to L^{2,\infty}(w^{-1})}. \tag{2.1.19}$$

Using probabilistic techniques, Hytönen [17] first proved (2.1.18) in all generality by demonstrating the weak-type norms in (2.1.19) satisfy a linear bound. Several subsequent proofs of (2.1.18) have also appeared, some of which appeal to averaging techniques [15, 23], and others avoiding this altogether [13, 24].

In the vector-valued setting, several different types of operators have been considered. In [8] the authors show that the dyadic square function $S$ and vector-valued maximal operator $\mathbf{M}_r$ with exponent $r$ satisfy

$$\|S\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}^{\max\left\{\frac{1}{2},\frac{1}{p-1}\right\}}$$
$$\|\mathbf{M}_r\|_{L^p_{\ell^r}(w)\to L^p(w)} \lesssim [w]_{A_p}^{\max\left\{\frac{1}{r},\frac{1}{p-1}\right\}}$$

where $1 < p < \infty$ and $w \in A_p$. Using similar methods, [25] gives sharp bounds for the intrinsic square function $G_\alpha$ on weighted $L^p(w)$ spaces, resolving a well-known conjecture [25]. We aim to generalize the forgoing types of results to vector-valued extensions of a Calderón-Zygmund operator. Based on the previous examples, it would be natural to expect the estimates of the $A_p$ characteristic to depend on the exponent $r$ associated with the $\ell^r$ extension of the given Calderón-Zygmund operator; however, in the estimates we obtain this does not occur. In particular, the main theorem of this chapter can be formulated as the following

**Theorem 2.1.20.** *Given a Calderón-Zygmund operator $T$ on $\mathbb{R}^n$, for $1 < r < \infty$ we denote by $\mathbf{T}$ the $\ell^r$ extension of $T$, i.e. $\mathbf{T}(\mathbf{f}) = \{T(f_j)(x)\}_{j=1}^{\infty}$ and*

$$\mathcal{T}_r(\mathbf{f})(x) = \left( \sum_{j=1}^{\infty} |T(f_j)(x)|^r \right)^{\frac{1}{r}}$$

*for $\mathbf{f} = \{f_j\}_{j=1}^{\infty}$ with $f_j \in \mathcal{S}(\mathbb{R}^n)$. Let $1 < p < \infty$ and $w \in A_p$. Given a Calderón-Zygmund operator $T$ we have the following bound*

$$\|\mathcal{T}_r\|_{L^p_{\ell^r}(w) \to L^p(w)} \lesssim [w]_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}}. \tag{2.1.21}$$

We stress that unexpectedly, the dependence on $[w]_{A_p}$ in Theorem 2.1.20 is the same as in Hyönen's original theorem, and this is in contrast to the implied constants which do depend on $r$. Hence, our theorem indicates that scalar and vector-valued Calderón-Zygmund operators can be equally singular. Additionally, the paper [11] considers more general Banach valued Calderón-Zygmund operators and achieves our Theorem 2.1.20 as a corollary using different proof methods.

Now we make a few remarks about the proof of Theorem 2.1.20. In the scalar case, the proof strategy is to reduce the study of $T$ to simpler operators, typically Haar shift operators of a fixed complexity. We follow this tract, reducing the study of a given $T$ to consideration of vector-valued Haar shift operators of a fixed complexity $\kappa$. Indeed, we show it will be enough to prove the following theorem

**Theorem 2.1.22.** *Given a vector-valued Haar shift operator $\mathcal{S}_r$ of complexity $\kappa$, we have*

$$\|\mathcal{S}_r\|_{L^p_{\ell^r}(w)\to L^p(w)} \lesssim \kappa^4 [w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}.$$

The chief difficulty in proving Theorem 2.1.22 will be maintaining a polynomial dependence on $\kappa$. To this end, we follow the argument outlined in [24] for the scalar case. We rely heavily on the application of Lerner's decomposition theorem, applying this inequality multiple times before obtaining our desired estimates. An alternative method of proof would be to verify the bounds in Theorem 2.1.22 via testing conditions; this is easily achieved in certain cases, such as when the Calderón-Zygmund operator has bounded complexity, but we were unable to achieve the result for general Calderón-Zygmund operators.

The outline of this chapter is as follows. In Section 2.1.1 we introduce our main theorems and Section 2.1.2 lists several results which will be used in our proofs. Subsequent sections refer to the proofs of specific theorems, beginning with arguments for our Lebesgue estimates and continuing with proofs of Theorem 2.1.22 and Theorem 2.1.20.

### 2.1.1 Preliminaries

In this section we fix notation and introduce our theorems. Let $1 < p, r < \infty$ and $w \in A_p$ weight with $\kappa \in \mathbb{N}$.

**Definition 12.** *For $u \in \{0, 3^{-1}\}^n$ we denote by $\mathcal{D}^u$ the dyadic grid defined by*

$$\mathcal{D}^u = \left\{2^{-k}([0,1)^n + m + (-1)^k u) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\right\}$$

*and note that this defines a collection of $2^n$ dyadic grids on $\mathbb{R}^n$. In the special case $u = \mathbf{0}$, we let $\mathcal{D}^u = \mathcal{D}$. Given a dyadic grid $\mathcal{D}^u$ and a cube $Q \in \mathcal{D}^u$, we use $Q^{(\kappa)}$ to denote the $\kappa$-fold parent of $Q$ from the dyadic grid.*

**Definition 13.** *Let* $\mathbf{S} = \{S^j\}_{j=1}^{\infty}$ *be a collection of generalized Haar shift operators of complexity* $\kappa$ *such that* $S^j f(x) = \sum_{I \in \mathcal{D}} \langle f, k_I^j \rangle h_I^j(x) = \sum_{I \in \mathcal{D}} S_I^j f(x)$ *for* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*. Take*

$$\mathcal{S}_r \mathbf{f}(x) = \left( \sum_{j=1}^{\infty} |S^j f_j(x)|^r \right)^{\frac{1}{r}}$$

*for* $\mathbf{f} = \{f_j\}$ *with* $f_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*. We call* $\mathcal{S}_r$ *a vector-valued Haar shift operator of complexity* $\kappa$*.*

**Definition 14.** *We define an operator* $\mathcal{P}_r$ *as follows. For each* $j$ *let* $\mathcal{Q}_j$ *be a sparse collection of dyadic cubes from the same dyadic system. For* $\mathbf{f} = \{f_j\}_{j=1}^{\infty}$*, define*

$$P^j(f_j)(x) = \sum_{Q \in \mathcal{Q}_j} \langle f_j \rangle_Q \mathbf{1}_{E_j(Q)}(x)$$

*where for each* $Q$*,* $E_j(Q)$ *is a union of sub-cubes of* $Q$ *satisfying* $2^{-\kappa}|Q| \leq |E_j(Q)|$ *and take*

$$\mathcal{P}_r(\mathbf{f})(x) = \left( \sum_{j=1}^{\infty} |P^j f_j(x)|^r \right)^{\frac{1}{r}}.$$

*We refer to operators of the above type as positive vector-valued Haar shift operators.*

**Definition 15.** *For given* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*,* $0 < \lambda < 1$*, and* $Q$ *we have*

$$\omega_\lambda(f; Q) = \inf_{c \in \mathbb{R}} \left( (f - c)\mathbf{1}_Q \right)^* (\lambda |Q|)$$
$$M^{\sharp}_{\lambda, Q} f(x) = \sup_{I \subset Q} \mathbf{1}_Q(x) \omega_\lambda(f, I)$$

*where for* $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*,* $g^*$ *represents the symmetric non-increasing rearrangement.*

Now we list the main theorems of this chapter:

**Theorem 2.1.23.** *The operator* $\mathcal{S}_r(\cdot)$ *satisfies* $\|\mathcal{S}_r\|_{L^1_{\ell^r} \to L^{1,\infty}} \lesssim \kappa^{1+\frac{1}{r'}}$*.*

**Theorem 2.1.24.** *For* $\mathcal{P}_r$ *as above, the following inequalities hold for Lebesgue measure*

$$\|\mathcal{P}_r\|_{L^p_{\ell^r} \to L^p} \lesssim \kappa^2 \kappa^{\max\{\frac{1}{r}, \frac{1}{r'}\}}. \tag{2.1.25}$$

**Theorem 2.1.26.** *With $w$ and $p$ as above we have*

$$\|\mathcal{S}_r\|_{L^p_{\ell^r}(w)\to L^p(w)} \lesssim \kappa^4 [w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}. \tag{2.1.27}$$

**Theorem 2.1.28.** *Let $T$ be a Calderón-Zygmund operator and $w \in A_p$ with $1 < p < \infty$. For $1 < r < \infty$,*

$$\|\mathcal{T}_r\|_{L^p_{\ell^r}(w)\to L^p(w)} \lesssim [w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}.$$

### 2.1.2 Technical Lemmas and Theorems

We begin by stating some known technical Lemmas and Theorems which will be used to initiate our proofs.

**Lemma 2.1.29.** [8] *Given a measurable function $f$ and $Q \in \mathcal{D}$, then for $0 < \lambda < 1$ and $0 < p < \infty$ we have*

$$(f\mathbf{1}_Q)^*(\lambda|Q|) \leq \frac{\|f\|_{L^{p,\infty}(Q,|Q|^{-1}dx)}}{\lambda^{\frac{1}{p}}}.$$

**Lemma 2.1.30.** [24] *Let $T$ be a Calderón-Zygmund operator and $Q \subset \mathbb{R}^n$ a cube. If $1 < p < \infty$ and $w \in A_p$ then for $f \in L^p(w)$*

$$\omega_\lambda(Tf;Q) \lesssim \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f(y)| dy \right)$$

**Lemma 2.1.31.** [12] *If $S$ is a generalized Haar shift operator of complexity $\kappa$ then we have*

$$\omega_\lambda(Sf;Q)(\lambda|Q|) \lesssim \frac{\kappa\langle|f|\rangle_Q}{\lambda} + \frac{1}{\lambda}\sum_{j=1}^{\kappa}\langle|f|\rangle_{Q^{(j)}}.$$

**Theorem 2.1.32.** [23] *Let $1 < q, p < \infty$, $0 < \lambda < 1$, and assume that $f$ and $g$ are functions such that for any cube $Q$ we have*

$$\omega_\lambda(|g|^q;Q) \lesssim \left(\frac{\langle|f|\rangle_Q}{\lambda}\right)^q$$

*for some constant independent of $Q$. Then we have*

$$\|g\|_{L^p}(w) \lesssim [w]_{A_p}^{\max\left\{\frac{1}{q},\frac{1}{p-1}\right\}} \|f\|_{L^p(w)}.$$

**Theorem 2.1.33.** [8] *For $1 < r, p < \infty$ and $w \in A_p$ we have the following bound*

$$\|\mathbf{M}_r\|_{L^p_{\ell^r}(w) \to L^p(w)} \lesssim [w]_{A_p}^{\max\left\{\frac{1}{r}, \frac{1}{p-1}\right\}}.$$

Additionally, we have the following estimate which we prove in the next section

**Lemma 2.1.34.** *If $Q \in \mathcal{D}$ then*

$$\omega_\lambda(\mathcal{S}_r \mathbf{f}; Q) \lesssim \kappa^{1+\frac{1}{r}} 2^\kappa \langle \|\mathbf{f}\|_{\ell^r} \rangle_{Q^{(\kappa)}}.$$

## 2.2 *The Lebesgue Estimates*

### 2.2.1 Proof of Theorem 2.1.23

We will perform a Calderón-Zygmund decomposition. Fix $\lambda > 0$ and let $\{Q_j\}_{j=1}^\infty$ be the maximal dyadic cubes such that $\frac{1}{|Q_j|} \int_{Q_j} \|\mathbf{f}\|_{\ell^r} dx \geq \lambda$. For each $j$ define $\mathbf{b}^j$ by

$$b^j_k(x) = \left( f_k - \frac{1}{|Q_j|} \int_{Q_j} f_k dx \right) \mathbf{1}_{Q_j}(x).$$

and let $\mathbf{b} = \sum_{j=1}^\infty \mathbf{b}^j$. Further, we let $\mathbf{g} = \mathbf{f} - \mathbf{b}$. Then we have the following:

(i.) $\|\mathbf{g}\|_{L^1_{\ell^r}} \lesssim \|\mathbf{f}\|_{L^1_{\ell^r}}$

(ii.) for each $j$, supp $b^j_k \subset Q_j$ all $k \in \mathbb{N}$

(iii.) $\sum_{j=1}^\infty \|\mathbf{b}^j\|_{L^1_{\ell^r}} \lesssim \|\mathbf{f}\|_{L^1_{\ell^r}}$

(iv.) for almost all $x \in \mathbb{R}^n$, $\|\mathbf{g}\|_{\ell^r} \lesssim \lambda \|\mathbf{f}\|_{\ell^r}$

(v.) $\sum_{j=1}^\infty |Q_j| \lesssim \frac{\|\mathbf{f}\|_{L^1_{\ell^r}}}{\lambda}$.

Notice

$$|\{x \in \mathbb{R}^n : \mathcal{S}_r \mathbf{f}(x) > \lambda\}| \leq \left| \left\{ x \in \mathbb{R}^n : \mathcal{S}_r \mathbf{g}(x) > \frac{\lambda}{2} \right\} \right| + \left| \left\{ x \in \mathbb{R}^n : \mathcal{S}_r \mathbf{b}(x) > \frac{\lambda}{2} \right\} \right|$$

and consider by Chebyshev's inequality,

$$\left|\left\{x \in \mathbb{R}^n : \mathcal{S}_r \mathbf{g}(x) > \frac{\lambda}{2}\right\}\right| \leq \frac{2^r}{\lambda^r} \int_{\mathbb{R}^n} \mathcal{S}_r \mathbf{g}(x)^r dx$$
$$\lesssim \frac{2^r}{\lambda^r} \int_{\mathbb{R}^n} \|\mathbf{g}\|_{L^r_{\ell^r}}.$$

By properties (i.) and (iv.) from above,

$$\int_{\mathbb{R}^n} \|\mathbf{g}\|^r_{\ell^r} dx \lesssim \lambda^{r-1} \int_{\mathbb{R}^n} \|\mathbf{f}\|_{\ell^r} dx$$

so that

$$\frac{2^r}{\lambda^r} \int_{\mathbb{R}^n} \|\mathbf{g}\|_{\ell^r} dx \lesssim \frac{2^r}{\lambda} \int_{\mathbb{R}^n} \|\mathbf{f}\|_{\ell^r} dx. \tag{2.2.35}$$

On the other hand,

$$\mathcal{S}_r \mathbf{b}(x) \leq \sum_{j=1}^{\infty} \mathcal{S}_r \mathbf{b}^j(x).$$

Further, for $Q_j^{(\kappa)} \subset I$, we have $\int_I b_k^j(x)dx = 0$ so that $S_I^k(b_k^j)(x) = 0$ for $Q_j^{(\kappa)} \subset I$. Hence, by standard computations

$$\begin{aligned}
\sum_{j=1}^{\infty} \mathcal{S}_r \mathbf{b}^j(x) &= \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} \left| S^k(b_k^j)(x) \right|^r \right)^{\frac{1}{r}} \\
&\leq \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} \left| \sum_{I \subseteq Q_j} S_I^k(b_k^j)(x) \right|^r \right)^{\frac{1}{r}} + \kappa^{\frac{1}{r'}} \sum_{j=1}^{\infty} \sum_{Q_j \subset I \subset Q_j^{(\kappa)}} \left( \sum_{k=1}^{\infty} \langle |b_k^j| \rangle_I^r \mathbf{1}_I(x) \right)^{\frac{1}{r}} \\
&= A(x) + B(x)
\end{aligned}$$

where

$$A(x) = \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} \left| \sum_{I \subseteq Q_j} S_I^k(b_k^j)(x) \right|^r \right)^{\frac{1}{r}}$$
$$B(x) = \kappa^{\frac{1}{r'}} \sum_{Q_j \subset I \subset Q_j^{(\kappa)}} \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} \left( \langle |b_k^j| \rangle_I \right)^r \mathbf{1}_I(x) \right)^{\frac{1}{r}}$$

so that

$$\left|\left\{x \in \mathbb{R}^n : \mathcal{S}_r \mathbf{b}(x) > \frac{\lambda}{2}\right\}\right| \leq \left|\left\{x \in \mathbb{R}^n : A(x) > \frac{\lambda}{4}\right\}\right| + \left|\left\{x \in \mathbb{R}^n : B(x) > \frac{\lambda}{4}\right\}\right|.$$

Notice that $A$ is supported on $\cup Q_j$ so that

$$\left|\left\{x \in \mathbb{R}^n : A(x) > \frac{\lambda}{4}\right\}\right| \leq \sum_{j=1}^{\infty} |Q_j| \lesssim \frac{\|\mathbf{f}\|_{L^1_{\ell^r}}}{\lambda}$$

and using Chebyshev's inequality we have

$$\left|\left\{x \in \mathbb{R}^n : B(x) > \frac{\lambda}{4}\right\}\right| \leq \frac{4}{\lambda}\kappa^{\frac{1}{r'}} \int \sum_{Q_j \subset I \subset Q_j^{(\kappa)}} \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} \left(\langle |b_k^j| \rangle_I\right)^r \mathbf{1}_I(x) \right)^{\frac{1}{r}} dx.$$

Applying Minkowskii's integral inequality to the inner sum of expectations in the above and continuing gives

$$\begin{aligned}
\frac{4\kappa^{\frac{1}{r'}}}{\lambda} \int \sum_{Q_j \subset I \subset Q_j^{(\kappa)}} \sum_{j=1}^{\infty} \langle (\|\mathbf{b}^j\|_{\ell^r}) \rangle_I \mathbf{1}_I(x) dx &\leq \frac{4\kappa^{\frac{1}{r'}}}{\lambda} \sum_{Q_j \subset I \subset Q_j^{(\kappa)}} \sum_{j=1}^{\infty} \|\mathbf{b}^j\|_{L^1_{\ell^r}} \\
&\lesssim \frac{4\kappa^{\frac{1}{r'}}}{\lambda} \sum_{Q_j \subset I \subset Q^{(\kappa)}} \|\mathbf{f}\|_{L^1_{\ell^r}} \\
&\leq \frac{4\kappa^{1+\frac{1}{r'}}}{\lambda} \|\mathbf{f}\|_{L^1_{\ell^r}}.
\end{aligned}$$

Combining the above estimates gives $\|\mathbf{S}\|_{L^1_{\ell^r} \to L^{1,\infty}} \lesssim \kappa^{1+\frac{1}{r'}}$.

### 2.2.2 Proof of Theorem 2.1.24

Fix $\mathbf{f} \in L^p_{\ell^r}$ and suppose first $p = r$. In this case we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{P}_r(\mathbf{f})(x)^p dx &= \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} |P^j(f_j)(x)|^r dx \\
&\lesssim \kappa^r \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} |f_j(x)|^r dx \\
&= \kappa^r \int_{\mathbb{R}^n} \|\mathbf{f}\|^p dx.
\end{aligned}$$

Now by Theorem 2.1.23 and the Marcinkiewicz Interpolation Theorem for vector-valued operators we have for $1 < p \leq r$,

$$\|\mathcal{P}_r\|_{L^p_{\ell^r} \to L^p} \lesssim \kappa^{2+\frac{1}{r'}}.$$

For the range $1 < r < p$ we notice there is a vector $\mathbf{h} \in L^{p'}_{\ell^{r'}}$ with $\|\mathbf{h}\|_{L^{p'}_{\ell^{r'}}} = 1$ such that

$$\int_{\mathbb{R}^n} \mathcal{P}_r(\mathbf{f})(x)^p dx = \int_{\mathbb{R}^n} \mathcal{P}_r(\mathbf{f}) \cdot \mathbf{h} dx$$
$$\leq \left(\int_{\mathbb{R}^n} \|\mathbf{f}\|^p_{\ell^r} dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \mathbf{U}(\mathbf{h})(x)^{p'} dx\right)^{\frac{1}{p'}}$$

where $\mathbf{U}$ represents a 'dual' operator for $\mathcal{P}_r$, i.e. if $(P^j)^*$ is the dual for each $P^j$ then

$$\mathbf{U}(\mathbf{g})(x) = \left(\sum_{j=1}^{\infty} |(P^j)^*(g_j)(x)|^{r'}\right)^{\frac{1}{r'}}$$

with $\mathbf{g} = \{g_j\}$ and $g_j \in L^1_{\text{loc}}(\mathbb{R}^n)$. Arguing as before with $\mathbf{U}$ in place of $\mathcal{P}_r$, we see

$$\left(\int_{\mathbb{R}^n} \mathbf{U}(\mathbf{h})(x)^{p'} dx\right)^{\frac{1}{p'}} \lesssim \kappa^{2+\frac{1}{r}}.$$

Hence, we have

$$\|\mathcal{P}_r\|_{L^p_{\ell^r} \to L^p} \lesssim \kappa^2 \max\left\{\kappa^{\frac{1}{r}}, \kappa^{\frac{1}{r'}}\right\}.$$

### 2.2.3 Proof of Lemma 2.1.34

By the triangle inequality we have,

$$\left|\mathbf{1}_Q(x)\mathcal{S}_r(\mathbf{f})(x) - \mathbf{1}_Q(x)\mathcal{S}_r(\mathbf{1}_{(Q^{(\kappa)})^c}\mathbf{f})(x)\right| \leq \mathbf{1}_Q(x)\mathcal{S}_r(\mathbf{f}\mathbf{1}_{Q^{(\kappa)}})(x).$$

Notice, $\mathcal{S}_r(\mathbf{f}\mathbf{1}_{(Q^{(\kappa)})^c})(x)\mathbf{1}_Q(x)$ is constant on $Q$. Define

$$C(Q, \mathbf{f}, \kappa) = C = \mathcal{S}_r\left(\mathbf{1}_{(Q^{(\kappa)})^c}\mathbf{f}\right)(x)\mathbf{1}_Q(x).$$

Now the above implies

$$\omega_\lambda(\mathcal{S}_r(\mathbf{f}); \lambda|Q|) \leq (\mathbf{1}_Q\mathcal{S}_r(\mathbf{f}\mathbf{1}_{Q^{(\kappa)}}))^*(\lambda|Q|).$$

Applying Lemma 2.1.29 gives

$$(\mathbf{1}_Q\mathcal{S}_r(\mathbf{f}\mathbf{1}_{Q^{(\kappa)}}))^*(\lambda|Q|) \lesssim \|\mathcal{S}_r(\mathbf{f}\mathbf{1}_{Q^{(\kappa)}})\|_{L^{1,\infty}(Q,|Q|^{-1}dx)}$$

and from the weak-(1,1) inequality for $\mathcal{S}_r$ we obtain

$$\|\mathcal{S}_r\left(\mathbf{f}\mathbf{1}_{Q^{(\kappa)}}\right)\|_{L^{1,\infty}(Q,|Q|^{-1}dx)} \lesssim \kappa^{1+\frac{1}{r}}2^{\kappa}\langle\|\mathbf{f}\|_{\ell^r}\rangle_{Q^{(\kappa)}}$$

Thus,

$$\omega_\lambda(\mathcal{S}_r(\mathbf{f});\lambda|Q|) \lesssim \kappa^{1+\frac{1}{r}}2^{\kappa}\langle\|\mathbf{f}\|_{\ell^r}\rangle_{Q^{(\kappa)}}$$

## *2.3 Proof of Theorem 2.1.26*

### 2.3.1 Proof of (2.1.27)

Let $\mathbf{f} \in L^p_{\ell^r}(w)$ be such that $\|\mathbf{f}\|_{\ell^r}$ has compact support. By applying Lerner's inequality to each component of $\mathcal{S}_r$ on a sufficiently large cube $J$, we obtain the bound

$$\mathcal{S}_r(\mathbf{f})(x) \lesssim \left(\sum_{j=1}^{\infty} M^{\sharp}_{\frac{1}{4};J}(S^j(f_j))(x)^r\right)^{\frac{1}{r}} + \left(\sum_{j=1}^{\infty}\left(\sum_{Q\in\mathcal{Q}_j}\mathbf{1}_Q(x)\omega_{2^{-n-2}}(S^j(f_j);Q)\right)^r\right)^{\frac{1}{r}}$$

where $\mathcal{Q}_j$ is the collection of cubes which results from applying Theorem 1.2.1 to $S^j(f_j)$. Using Lemma 2.1.31 as in [12] we obtain for each $j$,

$$M^{\sharp}_{\frac{1}{4};J}(S^j(f_j))(x)^r \lesssim \kappa^r Mf_j(x)^r$$

and

$$\left(\sum_{j=1}^{\infty} M^{\sharp}_{\frac{1}{4};J}(S^j(f_j))(x)^r\right)^{\frac{1}{r}} \lesssim \mathbf{M}_r(\mathbf{f})(x).$$

Now we consider the function

$$\left(\sum_{j=1}^{\infty}\left(\sum_{Q\in\mathcal{Q}_j}\mathbf{1}_Q(x)\omega_{2^{-n-2}}(S^j(f_j);Q)\right)^r\right)^{\frac{1}{r}}. \tag{2.3.36}$$

Applying Lemma 2.1.31 for each $j$ we obtain the following expression

$$\left(\sum_{j=1}^{\infty}\left(\sum_{Q\in\mathcal{Q}_j}\kappa\cdot\mathbf{1}_Q(x)\cdot\langle|f_j|\rangle_Q + \sum_{i=1}^{\kappa}\mathbf{1}_Q(x)\cdot\langle|f_j|\rangle_{Q^{(i)}}\right)^r\right)^{\frac{1}{r}}. \tag{2.3.37}$$

For each $j$ and $0 \le i \le \kappa$ define $E(Q)^i = \bigcup_{\substack{(I)^{(i)}=Q \\ I\in\mathcal{Q}_j}} I$ with the convention $E(Q)^0 = Q$.

Continuing, we have

$$(2.3.37) \lesssim \kappa \left( \sum_{j=1}^{\infty} \left( \sum_{Q\in\mathcal{Q}_j} \mathbf{1}_{E(Q)^0}(x) \cdot \langle |f_j| \rangle_Q \right)^r \right)^{\frac{1}{r}} +$$
$$\kappa \sum_{i=1}^{\kappa} \left( \sum_{j=1}^{\infty} \left( \sum_{Q\in\mathcal{Q}_j} \mathbf{1}_{E(Q)^i}(x) \cdot \langle |f_j| \rangle_{Q^{(i)}} \right)^r \right)^{\frac{1}{r}}.$$

For $0 \le i \le \kappa$ let

$$P^{j,i}(g)(x) = \sum_{Q\in\mathcal{Q}_j} \mathbf{1}_{E(Q)^i}(x) \langle g \rangle_Q$$

with $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathbf{P}_i$ be defined by

$$\mathbf{P}_i(\mathbf{g})(x) = \left( \sum_{j=1}^{\infty} |P^{j,i}(g)(x)|^r \right)^{\frac{1}{r}}$$

for $\mathbf{g} = \{g_j\}_{j=1}^{\infty}$ and $g_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Hence we have the following pointwise bound

$$\mathcal{S}_r(\mathbf{f})(x) \lesssim \mathbf{M}_r(\mathbf{f})(x) + \kappa \sum_{i=0}^{\kappa} \mathbf{P}_i(\mathbf{f})(x). \tag{2.3.38}$$

By Theorem 2.1.33,

$$\int_{\mathbb{R}^n} \mathbf{M}_r(\mathbf{f})(x)^p w \lesssim [w]_{A_p}^{\max\left\{\frac{p}{r}, \frac{p}{p-1}\right\}} \|\mathbf{f}\|^p_{L^p_{\ell^r}(w)}.$$

So from (2.3.38),

$$\int_{\mathbb{R}^n} \mathcal{S}_r(\mathbf{f})(x)^p w \lesssim \int_{\mathbb{R}^n} \mathbf{M}_r(\mathbf{f})(x)^p w + \kappa^{2p} \sum_{i=0}^{\kappa} \int_{\mathbb{R}^n} \mathbf{P}_i(\mathbf{f})(x)^p w$$
$$\lesssim [w]_{A_p}^{\max\left\{p, \frac{p}{r}\right\}} \int_{\mathbb{R}^n} \|\mathbf{f}\|^p_{\ell^r} w + \kappa^{2p} \sum_{i=0}^{\kappa} \int_{\mathbb{R}^n} \mathbf{P}_i(\mathbf{f})(x)^p w.$$

From duality, there is a vector $\mathbf{h} = \{h_j\} \in L^{p'}_{\ell^{r'}}(w)$ such that

$$\left( \int_{\mathbb{R}^n} \mathbf{P}_i(\mathbf{f})(x)^p w \right)^{\frac{1}{p}} = \int_{\mathbb{R}^n} \mathbf{P}_i(\mathbf{f})(x) \cdot \mathbf{h} w$$
$$\le \|\mathbf{f}\|_{L^p_{\ell^r}(w)} \left( \int_{\mathbb{R}^n} \mathbf{U}_i(\mathbf{h} w)(x)^{p'} \sigma \right)^{\frac{1}{p'}},$$

where $\sigma = w^{1-p'}$ and for each $i$

$$\mathbf{U}_i(\mathbf{g})(x) = \left(\sum_{j=1}^{\infty} |(P^{j,i})^*(g_j)(x)|^{r'}\right)^{\frac{1}{r'}}$$

with $\mathbf{g} = \{g_j\}$ and $g_j \in L^1_{\text{loc}}(\mathbb{R}^n)$. We apply Lerner's Theorem in each component of $\mathbf{U}_i$ to obtain the bound

$$\begin{aligned}
\mathbf{U}_i(\mathbf{h}w)(x) &\lesssim \mathbf{M}_{r'}(\mathbf{h}w)(x) + \kappa \left(\sum_{j=1}^{\infty} |L^{j,i}(h_j w)|^{r'}\right)^{\frac{1}{r'}} \\
&= \mathbf{M}_{r'}(\mathbf{h}w)(x) + \kappa \mathcal{L}_i(\mathbf{h}w)(x)
\end{aligned}$$

where for each $j$,

$$L^{j,i}(h_j w)(x) = \sum_{I \in \mathcal{N}_{j,i}} \langle h_j w \rangle_I \mathbf{1}_I(x).$$

Notice $\mathcal{L}_i$ is a vector-valued Haar shift operator of complexity 1 which is $L^2(\mathbb{R}^n)$ bounded; hence, by Lemma 2.1.34,

$$\omega_\lambda(\mathcal{L}_i(\mathbf{h}w))(\lambda|Q|) \lesssim \langle \|\mathbf{h}\|_{\ell^{r'}} w \rangle_Q$$

so that from another application of Lerner's Theorem we obtain a sparse collection of cubes $\mathcal{K}_i$,

$$\mathcal{L}_i(\mathbf{h}w)(x) \lesssim M(\|\mathbf{h}\|_{\ell^{r'}} w)(x) + \sum_{I \in \mathcal{K}_i} \langle \|\mathbf{h}\|_{\ell^{r'}} w \rangle_I \mathbf{1}_I(x).$$

Hence for each $i$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathbf{U}_i(\mathbf{h}w)(x)^{p'} \sigma &\lesssim \int_{\mathbb{R}^n} \mathbf{M}_{r'}(\mathbf{h}w)(x)^{p'} + \kappa^{p'} M(\|\mathbf{h}\|_{\ell^{r'}} w)^{p'} + \kappa^{p'} \mathcal{L}_i(\mathbf{h}w)(x)^{p'} \sigma \\
&\lesssim \kappa^{p'} [w]_{A_p}^{\max\left\{p', \frac{p'}{p-1}\right\}} \int_{\mathbb{R}^n} \|\mathbf{h}\|_{\ell^{r'}}^{p'} \sigma \\
&\lesssim \kappa^{p'} [w]_{A_p}^{\max\left\{p', \frac{p'}{p-1}\right\}}.
\end{aligned}$$

Now,

$$\|\mathcal{S}_r\|_{L^p_{\ell^r}(w) \to L^p(w)} \lesssim \kappa^4 [w]_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}}$$

giving the result.

### 2.3.2 An Example to Show Sharpness

The dependence on the $A_p$ characteristic from Theorem 2.1.26 is sharp by the scalar bound, but here we give an explicit example to show the dependence is sharp. For each $j$ let $I_j = [0, 2^{-j})$ and define

$$S(f)(x) = \sum_{j=1}^{\infty} \langle f \rangle_{I_j} \mathbf{1}_{I_j}(x).$$

Let $w(x) = |x|^{(\delta-1)(p-1)}$ and $f(x) = |x|^{\delta-1}\mathbf{1}_{[0,1)}(x)$. Then

$$\begin{aligned}
\|f\|_{L^p(w)}^p &= \int_{[0,1)} |x|^{\delta-1} dx \\
&= \frac{1}{\delta}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|S(f)\|_{L^p(w)}^p &= \int_{[0,1)} \left( \sum_{j=1}^{\infty} \langle |x|^{\delta-1} \rangle_{I_j} \mathbf{1}_{I_j}(x) \right)^p |x|^{(1-\delta)(p-1)} dx \\
&= \sum_{k=0}^{\infty} \int_{[2^{-k-1}, 2^{-k})} \left( \sum_{j=0}^{\infty} \langle |x|^{\delta-1} \rangle_{I_j} \mathbf{1}_{I_j}(x) \right)^p |x|^{(1-\delta)(p-1)} dx \\
&\sim \sum_{k=0}^{\infty} \int_{[2^{-k-1}, 2^{-k})} \delta^{-p} |x|^{(\delta-1)p} |x|^{(1-\delta)(p-1)} dx \\
&= \int_{[0,1)} \delta^{-p} |x|^{\delta-1} dx \\
&= \delta^{-p-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
[w]_{A_p}^{\frac{1}{p-1}} &\sim \delta^{-1} \\
&\lesssim \frac{\|S(f)\|_{L^p(w)}}{\|f\|_{L^p(w)}}.
\end{aligned}$$

As a consequence,

$$\begin{aligned}\left(\int S(f\sigma)(x)^p w\right)^{\frac{1}{p}} &= \sup_{\substack{h\in L^{p'}(w)\\ \|h\|_{L^{p'}(w)}=1}} \int S(f\sigma)(x)h(x)w \\ &= \sup_{\substack{h\in L^{p'}(w)\\ \|h\|_{L^{p'}(w)}=1}} \int f(x)S^*(hw)(x)\sigma \\ &= \sup_{\substack{h\in L^{p'}(w)\\ \|h\|_{L^{p'}(w)}=1}} \int f(x)S(hw)(x)\sigma \\ &\geq \int f(x)S(\mathbf{1}_{[0,1)}w)(x)w([0,1))^{\frac{-1}{p'}}\sigma\end{aligned}$$

so that

$$\begin{aligned}[w]_{A_p} &\lesssim w([0,1))^{\frac{-1}{p'}}\left(\int_{[0,1)} S(\mathbf{1}_{[0,1)}w)(x)^{p'}\sigma\right)^{\frac{1}{p'}} \\ &\lesssim \|S(\cdot\sigma)\|_{L^p(\sigma)\to L^p(w)} \\ &\sim \|S\|_{L^p(w)\to L^p(w)}.\end{aligned}$$

As a result, $[w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}} \lesssim \|S\|_{L^p(w)\to L^p(w)}$. Since $S$ is a positive operator, $S$ extends to a vector-valued operator $\mathcal{S}$ on $L^p_{\ell^r}(w)$ defined by

$$\mathcal{S}(\mathbf{f})(x) = \left(\sum_{j=1}^{\infty} |S(f_j)(x)|^r\right)^{\frac{1}{r}}$$

and $\|\mathcal{S}\|_{L^p_{\ell^r}(w)\to L^p(w)} \sim [w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}$.

## 2.4 Proof of Theorem 2.1.28

Let $T$ be as in the statement of Theorem 2.1.28. For each $j$ we apply Lerner's inequality to obtain the following bound

$$T(f_j)(x) \lesssim Mf_j(x) + \sum_{Q\in\mathcal{Q}_j} \mathbf{1}_Q(x)\omega_{2^{-n-1}}(T(f_j);Q).$$

For each $j$, we have by Lemma 2.1.30,

$$\omega_{2^{-n-1}}(T(f_j);Q) \lesssim \sum_{m=0}^{\infty} \frac{1}{2^{m\alpha}}\left(\frac{1}{|2^m Q|}\int_{2^m Q} |f_j(y)|dy\right).$$

Now we make an observation (see [14], [24], [13]), for any cube $Q \subset \mathbb{R}^n$ there is $u$ and $I \in \mathcal{D}^u$ such that $Q \subset I$ and $\ell(I) \leq 6\ell(Q)$. Hence for each $u \in \{0, 3^{-1}\}^n$ we may choose a collection of dyadic cubes $\mathcal{Q}_{j,u}$ in $\mathcal{D}^u$ such that

$$\begin{aligned}\sum_{Q\in\mathcal{Q}_j} \mathbf{1}_Q(x)\langle f_j\rangle_{2^mQ} &\lesssim \sum_{u\in\{0,3^{-1}\}} \sum_{Q\in\mathcal{Q}_{j,u}} \mathbf{1}_Q(x)\langle f_j\rangle_Q \\ &= \sum_{u\in\{0,3^{-1}\}} P_{j,m,u}(f_j)(x).\end{aligned}$$

Define

$$\mathbf{P}_{m,u}(\mathbf{f})(x) = \left(\sum_{j=1}^{\infty} |P_{j,m,u}(f_j)(x)|^r\right)^{\frac{1}{r}};$$

we have the following bound

$$\mathcal{T}_r(\mathbf{f})(x) \lesssim \mathbf{M}_r(\mathbf{f})(x) + \sum_{m=0}^{\infty} \frac{1}{2^{\alpha m}} \sum_{u\in\{0,3^{-1}\}^n} \mathbf{P}_{m,u}(\mathbf{f})(x). \tag{2.4.39}$$

By Theorem 2.1.33

$$\int_{\mathbb{R}^n} \mathbf{M}_r(\mathbf{f})(x)^p w \lesssim [w]_{A_p}^{\max\left\{\frac{p}{r}, \frac{p}{p-1}\right\}} \int_{\mathbb{R}^n} \|\mathbf{f}\|_{\ell^r}^p w.$$

For fixed $m$ and $u$, we may apply Theorem 2.1.26 to obtain

$$\int_{\mathbb{R}^n} \mathbf{P}_{m,u}(\mathbf{f})(x)^p w \lesssim m^{4p+\frac{p}{r}} [w]_{A_p}^{\max\left\{p, \frac{p}{p-1}\right\}} \int_{\mathbb{R}^n} \|\mathbf{f}\|_{\ell^r}^p w$$

so that

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{1}{2^{\alpha m}} \sum_{u\in\{0,3^{-1}\}^n} \int_{\mathbb{R}^n} \mathbf{P}_{m,u}(\mathbf{f})(x)^p w &\lesssim \left(\sum_{m=0}^{\infty} \frac{m^{4p+\frac{p}{r}}}{2^{\alpha m}}\right) [w]_{A_p}^{\max\left\{p, \frac{p}{p-1}\right\}} \int_{\mathbb{R}^n} \|\mathbf{f}\|_{\ell^r}^p w \\ &\lesssim [w]_{A_p}^{\max\left\{p, \frac{p}{p-1}\right\}} \int_{\mathbb{R}^n} \|\mathbf{f}\|_{\ell^r}^p w.\end{aligned}$$

As a result, from (2.4.39) we have

$$\left(\int_{\mathbb{R}^n} \mathcal{T}_r(\mathbf{f})(x)^p w\right)^{\frac{1}{p}} \lesssim [w]_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}} \left(\int_{\mathbb{R}^n} \|\mathbf{f}\|^p w\right)^{\frac{1}{p}}.$$

# CHAPTER III

# WEAK-TYPE ONE WEIGHT ESTIMATES FOR A VECTOR-VALUED OPERATOR

## 3.1 Introduction

This chapter is devoted to weak-type inequalities on weighted spaces $L^p(w)$ with $w \in A_p$ and $1 < p < \infty$. Our focus is on obtaining estimates for the vector-valued operators $\mathcal{T}_{\mathcal{Q},r,\rho}$; recall, for $\mathcal{Q}$ a sparse collection of cubes, $1 < r < \infty$, and $1 \leq \rho < \infty$ we define

$$\mathcal{T}_{\mathcal{Q},r,\rho} f(x) = \left( \sum_{I \in \mathcal{Q}} |\langle f \rangle_{\rho I}|^r \mathbf{1}_I(x) \right)^{\frac{1}{r}}$$

for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. The main theorem we present here is the following:

**Theorem 1.2.6.** *For $1 < p, r < \infty$, $1 \leq \rho < \infty$, and $w \in A_p$ we have*

$$\|\mathcal{T}_{\mathcal{Q},r,\rho}\|_{L^{p,\infty}(w)} \lesssim \phi_{p,r}([w]_{A_p})$$

*where $\phi_{p,r}(x) = x^{\frac{1}{p}}$ for $1 < p < r$ and $\phi_{p,r}(x) = x^{\frac{1}{r}}(1 + \log x)$ for $r \leq p$.*

In the final section of this chapter, we show our result is sharp for the range $1 < p < r$. Due to A. Lerner's decomposition theorem we obtain the following as a corollary to the above:

**Corollary 3.1.40.** *Recall, $\mathbf{M}_r$ denotes the vector-valued maximal function with exponent $r$ and $\mathcal{S}_r$ a vector-valued Haar shift operator of complexity $\kappa$ with exponent $r$. Further, let $T$ be any of the dyadic square function, area integral, or the intrinsic* *square function. Then*

$$\begin{aligned}
\|\mathcal{S}_r\|_{L^p_{\ell^r}(w)\to L^{p,\infty}(w)} &\lesssim [w]_{A_p} \\
\|\mathbf{M}_r\|_{L^p_{\ell^r}(w)\to L^{p,\infty}(w)} &\lesssim \phi_{p,r}([w]_{A_p}) \\
\|T\|_{L^p(w)\to L^{p,\infty}(w)} &\lesssim \phi_{p,2}([w]_{A_p})\|f\|_{L^p(w)}
\end{aligned}$$

*where $\phi_{p,r}$ is defined as before.*

Previously, the best bounds for the operators in Corollary 3.1.40 were those implied by their corresponding strong-type bounds; namely

$$\begin{aligned}
\|\mathcal{S}_r\|_{L^p_{\ell^r}(w)\to L^p_{\ell^r}(w)} &\lesssim [w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}} \\
\|\mathbf{M}_r\|_{L^p_{\ell^r}(w)\to L^p(w)} &\lesssim [w]_{A_p}^{\max\left\{\frac{1}{r},\frac{1}{p-1}\right\}} \\
\|T\|_{L^p(w)\to L^p(w)} &\lesssim [w]_{A_p}^{\max\left\{\frac{1}{2},\frac{1}{p-1}\right\}}.
\end{aligned}$$

Hence, Corollary 3.1.40 improves the known weak-type bound for $\mathcal{S}_r$ in the range $1 < p < 2$, $\mathbf{M}_r$ in the range $1 < p < r+1$ and those of the square functions in the range $1 < p < 3$.

In the literature, weak-type estimates for several classical operators have been considered. Buckley was first to quantify the dependence of the maximal function's norm on a weight's $A_p$ characteristic, proving

$$\|M\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\frac{1}{p}}$$

for $w \in A_p$ and $1 < p < \infty$. The authors of [26] were able to show for $p = 1$ and $T$ an $L^2(\mathbb{R}^n)$ bounded Calderón-Zygmund operator, we have

$$\|T\|_{L^1(w)\to L^{1,\infty}(w)} \lesssim [w]_{A_1}(\log[w]_{A_1} + 1).$$

Subsequently, [16] considered the remaining values of $p$, giving

$$\|T\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim [w]_{A_p}.$$

The authors of [2] established

$$w\left(\{x \in \mathbb{R}^n : Af(x) > \lambda\}\right) \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^n} f(x) Mw$$

where $A$ denotes the area integral and $w$ is a weight; a similar type of argument extends the result to more general square functions (see [45] and [46]).

The remainder of this chapter is outlined as follows. In the next two sections we consider the proofs of Theorem 1.2.6 and Corollary 3.1.40. The final section discusses an example to show Theorem 1.2.6 is sharp for $p < r$.

## *3.2 Proofs of Main Results*

### 3.2.1 Proof of the Theorem 1.2.6

Fix $\mathcal{Q}$, $\rho$, $r$, $p$, and let $f \in L^p(w)$ such that $f$ is nonnegative. We wish to show

$$w(\{x \in \mathbb{R}^n : \mathcal{T}_{\mathcal{Q},r,\rho} f(x) > \lambda\})\lambda^p \lesssim \phi_{p,r}([w]_{A_p})^p \|f\|^p_{L^p(w)},$$

and it will be enough to consider $\lambda = 1$. Let $\mathcal{Q}^1$ consist of all $I$ such that $\langle f \rangle_{\rho I} > 1$. Then if $I \in \mathcal{Q}^1$ we have $\rho I \subset \{x \in \mathbb{R}^n : Mf(x) > 1\}$ so that $I \subset \{x \in \mathbb{R}^n : Mf(x) > 1\}$. As a result,

$$\begin{aligned} w\left(\left\{x \in \mathbb{R}^n : \sum_{I \in \mathcal{Q}^1} \langle f \rangle_{\rho I}^r \mathbf{1}_I(x) > 1\right\}\right) &\leq w\left(\bigcup_{I \in \mathcal{Q}^1} I\right) \\ &\leq w\left(\{x \in \mathbb{R}^n : Mf(x) > 1\}\right) \\ &\lesssim [w]_{A_p} \|f\|^p_{L^p(w)}. \end{aligned}$$

We split the remaining cubes into disjoint collections setting

$$\mathcal{Q}_\ell := \left\{ I \in \mathcal{Q} \ : \ 2^{-\ell-1} < \langle f \rangle_{\rho I} \leq 2^{-\ell} \right\}, \qquad \ell = 0, 1, \ldots,$$

Now let $E(I) = \rho I \backslash \bigcup\{\rho I' : I' \subsetneq I, I \in \mathcal{Q}_\ell\}$ and $R(I) = \bigcup\{\rho I' : I' \subsetneq I, I \in \mathcal{Q}_\ell\}$. Notice, if $|R(I)| < \frac{|\rho I|}{8} = \frac{\rho^n |I|}{8}$ then we have

$$\begin{aligned}\langle f\mathbf{1}_{E(I)}\rangle_{\rho I} &= \langle f\rangle_{\rho I} - \langle f\mathbf{1}_{R(I)}\rangle_{\rho I} \\ &\geq \langle f\rangle_{\rho I} - 8^{-1}\langle f\rangle_{R(I)} \\ &\geq 2^{-\ell-1} - 8^{-1}2^{-\ell} \\ &\gtrsim 2^{-\ell}\end{aligned}$$

so that $\langle f\mathbf{1}_{E(I)}\rangle_{\rho I} \gtrsim 2^{-\ell}$. By possibly considering the dyadic descendants of a given cube $Q \in \mathcal{Q}_\ell$ we may assume without loss of generality $|R(I)| < \frac{|\rho I|}{8}$ and $\langle f\mathbf{1}_{E(I)}\rangle_{\rho I} \gtrsim 2^{-\ell}$. Finally, we need the following lemma:

**Lemma 3.2.41.** *Let $\mathcal{R}$ be a collection of cubes, $1 < p < \infty$, and $\{g_I\}_{I\in\mathcal{R}}$ a sequence of nonnegative functions.*

$$\left\|\left(\sum_{I\in\mathcal{R}}\langle g_I\rangle_{\rho I}^p \mathbf{1}_I\right)^{1/p}\right\|_{L^p(w)} \lesssim [w]_{A_p}^{1/p}\left\|\left(\sum_{I\in\mathcal{R}} g_I^p\right)^{1/p}\right\|_{L^p(w)}.$$

*Proof.* Consider, with $\sigma = w^{1-p'}$,

$$\begin{aligned}\int_{\mathbb{R}^n}\sum_{I\in\mathcal{R}}\langle g_I\rangle_{\rho I}^p\mathbf{1}_I(x)w &\leq \sum_{I\in\mathcal{R}}\left(\langle g_I\sigma^{-1}\rangle_{\rho I}^\sigma\right)^p\left(\frac{\sigma(\rho I)}{|\rho I|}\right)^p w(\rho I) \\ &\leq [w]_{A_p}\sum_{I\in\mathcal{R}}\sigma(I)\left(\langle g_I\sigma^{-1}\rangle_{\rho I}^\sigma\right)^p \\ &\leq [w]_{A_p}\sum_{I\in\mathcal{R}}\int_I M^\sigma(g_I)(x)^p\sigma \\ &\lesssim [w]_{A_p}\sum_{I\in\mathcal{R}}\int_{\mathbb{R}^n} g_I(x)^p\sigma(x)^{-p}\sigma \\ &= [w]_{A_p}\int_{\mathbb{R}^n}\sum_{I\in\mathcal{R}} g_I(x)^p w.\end{aligned}$$

□

*3.2.1.1 The Case of* $1 < p < r$

Given a collection $\mathcal{I} \subset \mathcal{Q}$ and a sequence of functions $\{a_I(x)\}_{I\in\mathcal{I}}$ indexed by $\mathcal{I}$, define

$$E\left(\mathcal{I}, \{a_I\}, \lambda\right) = E\left(\mathcal{I}, \{a_I\}_{I\in\mathcal{I}}, \lambda\right) = \left\{x \in \mathbb{R}^n : \sum_{I\in\mathcal{I}} |a_I(x)|^r > \lambda\right\}.$$

We let $k_\epsilon \simeq \epsilon^{-1}$ be a constant such that

$$\begin{aligned} w\left(E\left(\mathcal{Q}\backslash\mathcal{Q}^1, \{\langle f\rangle_{\rho I}\mathbf{1}_I(x)\}, k_\epsilon\right)\right) &\leq \sum_{\ell=0}^{\infty} w\left(E\left(\mathcal{Q}_\ell, \left\{2^{-r\ell}\mathbf{1}_I(x) - 2^{-\epsilon\ell}\right\}, 0\right)\right) \\ &= \sum_{\ell=0}^{\infty} w\left(E\left(I \in \mathcal{Q}_\ell, \left\{2^{-r\ell}\mathbf{1}_I(x)\right\}, 2^{-\epsilon\ell}\right)\right). \end{aligned}$$

For fixed $\ell$ we have

$$E\left(\mathcal{Q}_\ell, \left\{2^{-r\ell}\mathbf{1}_I(x)\right\}, 2^{-\epsilon\ell}\right) \subset E\left(\mathcal{Q}_\ell, \left\{\langle f\mathbf{1}_{E(I)}(x)\rangle_{\rho I}^p\mathbf{1}_I(x)\right\}, 2^{(r-p-\epsilon)\ell-5}\right)$$

and

$$w\left(E\left(\mathcal{Q}_\ell, \left\{\langle f\mathbf{1}_{E(I)}(x)\rangle_{\rho I}^p\mathbf{1}_I(x)\right\}, 2^{(r-p-\epsilon)\ell-5}\right)\right) \lesssim [w]_{A_p} 2^{-(r-p-\epsilon)\ell/r}\|f\|_{L^p(w)}^p$$

by Lemma 3.2.41. Choosing $\epsilon = (r-p)/2$ and summing over $\ell$ gives the result.

*3.2.1.2 The case of* $p = r$

We consider

$$w\left(E\left(\mathcal{Q}\backslash\mathcal{Q}^1, \left\{\langle f\rangle_{\rho I}^r\mathbf{1}_I(x)\right\}, 1\right)\right) \leq w(A) + w(B)$$

where

$$\begin{aligned} A &= \left\{x \in \mathbb{R}^n : \sum_{\ell=0}^{\ell_0-1}\sum_{I\in\mathcal{Q}_\ell} \langle f\rangle_{\rho I}^r\mathbf{1}_I(x) > \frac{1}{2}\right\} \\ B &= \left\{x \in \mathbb{R}^n : \sum_{\ell=\ell_0}^{\infty}\sum_{I\in\mathcal{Q}_\ell} \langle f\rangle_{\rho I}^r\mathbf{1}_I(x) > 2^{-\ell/8-1}\right\}. \end{aligned}$$

Note

$$w(B) \leq w\left(\left\{x \in \mathbb{R}^n : \sum_{\ell=\ell_0}^{\infty} \sum_{I \in \mathcal{Q}_\ell} \langle f \rangle_{\rho I}^r \mathbf{1}_I(x) - 2^{-\ell/8-1} > 0\right\}\right)$$
$$\leq \sum_{\ell=\ell_0}^{\infty} w\left(\left\{x \in \mathbb{R}^n : \sum_{I \in \mathcal{Q}_\ell} \langle f \rangle_{\rho I}^2 \mathbf{1}_I(x) - 2^{-\ell/8-1} > 0\right\}\right)$$
$$\leq \sum_{\ell=\ell_0}^{\infty} w\left(\left\{x \in \mathbb{R}^n : \sum_{I \in \mathcal{Q}_\ell} \langle f \rangle_{\rho I}^r \mathbf{1}_I(x) > 2^{-\ell/8-1}\right\}\right).$$

Using the $A_\infty$ estimate we have

$$\begin{aligned} w\left(E\left(\mathcal{Q}_\ell, \left\{\langle f \rangle_{\rho I}^r \mathbf{1}_I(x)\right\}, 2^{-\ell/8-1}\right)\right) &\leq w\left(E\left(\mathcal{Q}_\ell, \left\{\mathbf{1}_I(x)\right\}, 2^{(8r\ell-\ell)/8-1}\right)\right) \\ &\lesssim \exp\left((-c2^{r\ell/2})/[w]_{A_r}\right) w\left(\bigcup_{I \in \mathcal{Q}_\ell} I\right) \\ &\lesssim [w]_{A_2} 2^\ell \exp\left(-c2^{r\ell/2}/[w]_{A_r}\right) \|f\|_{L^r(w)}^r, \end{aligned}$$

where $0 < c < 1$ is a fixed constant. This is summable in $\ell \geq \ell_0$ to at most a constant. For the case of $0 \leq \ell < \ell_0$, we use the estimate of Lemma 3.2.41 to obtain

$$w(A) \leq \sum_{\ell=0}^{\ell_0-1} w\left(\left\{x \in \mathbb{R}^n : \sum_{I \in \mathcal{Q}_\ell} \langle f \mathbf{1}_{E(I)} \rangle_{\rho I}^2 \mathbf{1}_I(x) > \frac{1}{128\ell_0}\right\}\right)$$
$$\lesssim \ell_0^r [w]_{A_r} \|f\|_{L^r(w)}^r = [w]_{A_r}(1 + \log[w]_{A_r})^r \|f\|_{L^r(w)}^r$$

concluding the proof of this case.

#### *3.2.1.3 The case of $r < p < \infty$*

We have

$$\begin{aligned} w(E(\mathcal{Q}, \{\langle f \rangle_I^r \mathbf{1}_I(x)\}, 1))^{\frac{1}{p}} &= \left(w(E(\mathcal{Q}, \{\langle f \rangle_I^r \mathbf{1}_I(x)\}, 1))^{\frac{r}{p}}\right)^{\frac{1}{r}} \\ &= (hw(E(\mathcal{Q}, \{\langle f \rangle_I^r \mathbf{1}_I(x)\}, 1)))^{\frac{1}{r}} \end{aligned}$$

for $h \in L^{q'}(w)$ with norm 1, where $q = \frac{p}{r}$. Now by the Rubio de Francia algorithm there is a function $H$ such that the following hold:

(i.) $h \leq H$

(ii.) $\|H\|_{L^{q'}(w)} \lesssim \|h\|_{L^{q'}(w)}$

(iii.) $Hw \in A_1$

(iv.) $[Hw]_{A_1} \lesssim [w]_{A_p}$.

We can continue,

$$\begin{aligned} hw(E(\mathcal{Q}, \{\langle f\rangle_I^r \mathbf{1}_I(x)\}, 1)) &\leq Hw(E(\mathcal{Q}, \{\langle f\rangle_I^r \mathbf{1}_I(x)\}, 1)) \\ &\lesssim [Hw]_{A_r} \left(1 + \log [Hw]_{A_r}\right)^r \int_{\mathbb{R}^n} f(x)^r Hw. \end{aligned} \tag{3.2.42}$$

Using Hölder's inequality we obtain

$$\int_{\mathbb{R}^n} f(x)^r Hw \leq \|f\|_{L^p(w)}^r \|H\|_{L^{q'}(w)}^r \lesssim \|f\|_{L^p(w)}^r$$

so that

$$\begin{aligned} (3.2.42) &\lesssim [Hw]_{A_r} \left(1 + \log [Hw]_{A_r}\right)^r \|f\|_{L^p(w)}^r \\ &\lesssim [w]_{A_p} \left(1 + \log [w]_{A_p}\right)^r \|f\|_{L^p(w)}^r \end{aligned}$$

which implies the result.

### 3.2.2 Proof of Corollary 3.1.40

The following lemma is known (see [8] and [25]):

**Lemma 3.2.43.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathbf{g}$ be a sequence of $\ell^r$ summable locally integrable functions. For $\mathbf{M}_r$ the vector-valued maximal function with exponent $r$ and $T$ any of square functions in Corollary 3.1.40,*

$$\omega_\lambda(Tf^2, Q_0) \lesssim \lambda^{-1} \langle f\rangle_{\rho Q_0}^2$$
$$\omega_\lambda(\mathbf{M}_r(\mathbf{g})^r, Q_0) \lesssim \lambda^{-1} \langle \|\mathbf{g}\|_{\ell^r}\rangle_{Q_0}^r$$

*for some $\rho \geq 1$ which depends on the choice of $T$.*

By Lerner's decomposition theorem, for each cube $Q_N$ there is an appropriate $\rho$ and collections of sparse cubes $\mathcal{Q}_N$ and $\mathcal{I}_N$ such that

$$|Tf(x) - m_{Q_N}| \lesssim M^\sharp(f)(x) + \mathcal{T}_{\mathcal{I}_N,\rho,r}(f)(x)$$
$$|\mathbf{M}_r(\mathbf{g})(x) - m_{Q_N}| \lesssim M^\sharp(\|\mathbf{g}\|_{\ell^r})(x) + \mathcal{T}_{\mathcal{Q}_N,r}(\|\mathbf{g}\|_{\ell^r})(x);$$

the conclusion of the corollary for $\mathbf{M}_r$ and the operators represented by $T$ follow immediately.

Now we consider $\mathcal{S}_r$, where $\mathcal{S}_r$ is a vector-valued Haar shift operator of complexity $\kappa$. Let $\mathbf{f} \in L^p_{\ell^r}(w)$ such that $\|\mathbf{f}\|_{\ell^r}$ has compact support. For all cubes $Q$ which are sufficiently large, we have the following point-wise bound:

$$\mathcal{S}_r\mathbf{f}(x) \lesssim M^\sharp_{2^{-n-1};Q}(\mathcal{S}_r\mathbf{f})(x) + \sum_{I\in\mathcal{K}} \omega_{2^{-n-1}}(\mathcal{S}_r\mathbf{f}; I)\mathbf{1}_I(x),$$

where $\mathcal{K}$ is a sparse collection of cubes. By a Lemma 2.1.34 from Chapter II,

$$\begin{aligned} M^\sharp_{2^{-n-1};Q}(\mathcal{S}_r\mathbf{f})(x) &\lesssim M(\|\mathbf{f}\|_{\ell^r})(x) \\ \omega_{2^{-n-1}}(\mathcal{S}_r\mathbf{f}; I)\mathbf{1}_I(x) &\lesssim \langle\|\mathbf{f}\|_{\ell^r}\rangle_{I^{(\kappa)}}\mathbf{1}_I(x) \end{aligned}$$

so that

$$\mathcal{S}_r\mathbf{f}(x) \lesssim M(\|\mathbf{f}\|_{\ell^r})(x) + \sum_{I\in\mathcal{K}} 2^\kappa\langle\|\mathbf{f}\|_{\ell^r}\rangle_I\mathbf{1}_I(x) \tag{3.2.44}$$

and we are done.

### 3.3 An Example to Show Sharpness

Let $\mathcal{Q} = \{[0, 2^{-j}) : 0 \leq j\}$, $\rho = 1$, and $1 < r < \infty$. Take $w(x) = |x|^{(1-\delta)(p-1)}$ with $0 \leq \delta < 1$ and $f(x) = \mathbf{1}_{[0,1)}(x)|x|^{\delta-1}$. Then

$$\begin{aligned}
\mathcal{T}_{\mathcal{Q},r,\rho} f(x)^r &= \sum_{j=0}^{\infty} \mathbf{1}_{[0,2^{-j})}(x) \langle f \rangle_{[0,2^{-j})}^r \\
&= \sum_{j=0}^{\infty} \mathbf{1}_{[0,2^{-j})}(x) \delta^{-r} 2^{r(-j\delta+j)} \\
&= \sum_{j \leq \log|x|^{-1}} \mathbf{1}_{[0,2^{-j})}(x) \delta^{-r} 2^{r(-j\delta+j)} \\
&\sim \delta^{-r} |x|^{r(\delta-1)}
\end{aligned}$$

and so we need to consider

$$w\left(\left\{x \in [0,1) : (\lambda\delta)^{\frac{-1}{1-\delta}} > x\right\}\right) \lambda^p = w\left([0, (\lambda\delta)^{\frac{-1}{1-\delta}})\right) \lambda^p$$

for $0 < \lambda$. But

$$w\left([0, (\lambda\delta)^{\frac{-1}{1-\delta}})\right) = (\lambda\delta)^{\frac{-\epsilon}{1-\delta}}$$

with $\epsilon = (1-\delta)(p-1)$ so

$$\begin{aligned}
w\left([0, (\lambda\delta)^{\frac{-1}{1-\delta}})\right) \lambda^p &= (\lambda\delta)^{\frac{-\epsilon}{1-\delta}} \lambda^p \\
&\gtrsim \delta^{\frac{-\epsilon}{1-\delta}} \\
&= \delta^{-(p-1)} \\
&\sim [w]_{A_p}
\end{aligned}$$

and since $\|f\|_{L^p(w)} = 1$ we are done.

# CHAPTER IV

# TWO WEIGHT INEQUALITY FOR A VECTOR-VALUED OPERATOR

## *4.1 Introduction*

Our focus is on two weight inequalities. We study the simple vector-valued operator $\mathcal{T}_{\mathcal{Q},r}$ defined by a sparse collection of cubes $\mathcal{Q}$ and an exponent $1 \leq r < \infty$; recall, in this context we take

$$\mathcal{T}_{\mathcal{Q},r,1}f(x) = \mathcal{T}_{\mathcal{Q},r}f(x) = \left( \sum_{I \in \mathcal{Q}} |\langle f \rangle_I|^r \mathbf{1}_I(x) \right)^{\frac{1}{r}}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The aim of our efforts is to give a necessary and sufficient condition for the two weight inequality of $\mathcal{T}_{\mathcal{Q},r}$ to hold when $1 < r < \infty$. The main result of this chapter may be formulated as follows:

**Theorem 1.2.9.** *Suppose $w$ and $\sigma$ are weights and $1 < r, p < \infty$ with $\mathcal{Q}$ a sparse collection of cubes. Then we have $\|\mathcal{T}_{\mathcal{Q},r}(\cdot\sigma)\|_{L^p(\sigma)\to L^p(w)}$ if and only if there are $\mathcal{L}$ and $\mathcal{L}_*$ such that:*

$$\sup_Q \int_Q \mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_Q\sigma)(x)^p w \leq \mathcal{L}\sigma(Q) \tag{4.1.45}$$

$$\sup_{\mathbf{a}} \sup_Q \int_Q \mathbf{U}_{\mathcal{Q}}(\mathbf{1}_Q\mathbf{a}w)(x)^{p'}\sigma \leq \mathcal{L}_* w(Q) \tag{4.1.46}$$

*where $\mathbf{U}_{\mathcal{Q}}$ is an appropriate 'dual' operator (which we define later) and where the first supremum for $\mathbf{U}_{\mathcal{Q}}$ is taken over all sequences of functions $\mathbf{a}$ such that $\|\mathbf{a}\|_{\ell^r} = 1$.*

Special cases of our theorem have been considered before. Notably, when $p = r$ and $w = \sigma$ we obtain the weighted Carleson embedding theorem:

**Theorem 1.2.7** (Weighted Carleson Embedding Theorem)**.** *Let $w$ be a weight on $\mathbb{R}^n$ and $\{\tau_J\}_{J\in\mathcal{D}}$ a collection of nonnegative numbers. Then we have*

$$\sup_I \frac{1}{w(I)} \sum_{J\subset I} \tau_J \lesssim 1$$

*if and only if*

$$\sup_{\substack{f\in L^p(w)\\ \|f\|_{L^p(w)}=1}} \sum_{J\in\mathcal{D}} (\langle f\rangle_J^w)^p \tau_J \lesssim 1. \tag{4.1.47}$$

Theorem 1.2.7 is a fundamental result in two weight theory. For positive operators, the relationship between Theorem 1.2.7 and the corresponding two weight inequality is very strong. The two weight inequality for the maximal function is equivalent to Theorem 1.2.7 and the characterization of weighted inequalities for discrete positive operators can be reduced to Theorem 1.2.7, see [43]. The connection is less clear for operators without a positive kernel, but if $p = 2$ then Theorem 1.2.7 can be used to give the two weight inequality for the dyadic square function and Haar multipliers (see [30]). Our Theorem 1.2.9 generalizes Theorem 1.2.7, reducing to a special case of (4.1.47) when $p = r$.

Further, for $r = 1$ and $p = 2$, [30] gave a characterization of the operator $\mathcal{T}_{\mathcal{Q},r}$. This result was later extended to $p \neq 2$ by [20] (later a simplified argument was constructed by [43]). A crucial difference between [30] and [20] was that [30] used a Bellman function technique while [20] constructed a more flexible argument. We rely on the methods presented in [20], noting Theorem 1.2.9 follows largely from their argument but not directly from their results.

We mention the operators $\mathcal{T}_{\mathcal{Q},r}$ have also received attention with respect to one weight inequalities. The arguments of [8] imply the following:

**Theorem 4.1.48.** *Let $\mathcal{Q}$ be a sparse collection of cubes with $1 < r, p < \infty$ and $w \in A_p$. Then we have*

$$\|\mathcal{T}_{\mathcal{Q},r}\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{r},\frac{1}{p-1}\}}. \tag{4.1.49}$$

Using a decomposition theorem of A. Lerner in conjunction with (4.1.49) the authors of [8] were able to deduce sharp strong-type inequalities for the vector-valued maximal function and dyadic square function. Later, A. Lerner used a similar argument to extend the square function result to the intrinsic square function. Applying these type of arguments together with Theorem 1.2.9 and Sawyer's theorem for the maximal function we obtain the following

**Corollary 4.1.50.** *Suppose $w$ and $\sigma$ are two weights with $1 < p, r < \infty$. Assume the testing conditions (4.1.45) and (4.1.46) are satisfied with constants independent of the sparse collection $\mathcal{Q}$. Additionally, suppose $M(\cdot\sigma)$ satisfies*

$$\int_Q M(\mathbf{1}_Q\sigma)(x)^p w \lesssim \sigma(Q).$$

*Then $\mathbf{M}_r(\cdot\sigma)$ is bounded from $L^p(\sigma)$ to $L^p(w)$ and if $r = 2$, $S(\cdot\sigma)$ is bounded from $L^p(\sigma)$ to $L^p(w)$.*

The remainder of this chapter is structured as follows. In Section 2 we introduce certain definitions and theorems which will be useful for us. The subsequent section deals with several preliminary results and Section 4 contains the bulk of our argument for Theorem 1.2.9.

## *4.2 Initial Concepts*

Throughout the remainder of this chapter we assume $1 < r < \infty$. Recall, for $\mathcal{Q}$ a sparse collection of cubes and $\mathbf{g} = \{g_I\}_{I\in\mathcal{Q}}$ a collection of measurable functions we set

$$\mathbf{U}_{\mathcal{Q}}(\mathbf{g})(x) = \sum_{I\in\mathcal{Q}} \langle g_I \rangle_I \mathbf{1}_I(x).$$

We also consider an operator $\mathbf{T}_{\mathcal{Q},r}$ which allows us to overcome the non-linearity of $\mathcal{T}_{\mathcal{Q},r}$:

**Definition 16.** *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $1 < r < \infty$. We set*

$$\mathbf{T}_{\mathcal{Q},r}(f)(x) = \{\langle f\rangle_I \mathbf{1}_I(x)\}_{I\in\mathcal{Q}}\,.$$

Then we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{T}_{\mathcal{Q},r}(f\sigma)(x)^p w &= \int_{\mathbb{R}^n} \|\mathbf{T}_{\mathcal{Q},r}(f\sigma)\|_{\ell^r}^p w\\ &= \int_{\mathbb{R}^n} \langle \mathbf{T}_{\mathcal{Q},r}(f\sigma), \mathbf{a}w\rangle_{\ell^r} dx\\ &= \int_{\mathbb{R}^n} \langle f\sigma, \mathbf{U}_{\mathcal{Q}}(\mathbf{a}w)\rangle_{\ell^r} dx.\end{aligned}$$

Consequently, $\mathbf{U}_{\mathcal{Q}}$ can be loosely considered as the dual operator to $\mathcal{T}_{\mathcal{Q},r}$. Further, we define certain restrictions of $\mathcal{T}_{\mathcal{Q},r}$:

**Definition 17.** *Suppose $\mathcal{Q}$ is a sparse collection of cubes and $1 < r < \infty$. For $Q \subset \mathbb{R}^n$, we have*

$$\mathcal{T}^{\text{in}}_{\mathcal{Q},r,Q} f(x) = \left(\sum_{\substack{I\subseteq Q\\ I\in\mathcal{Q}}} |\langle f\rangle_I|^r \mathbf{1}_I(x)\right)^{\frac{1}{r}},$$

$$\mathcal{T}^{\text{out}}_{\mathcal{Q},r,Q}(f)(x) = \left(\sum_{\substack{Q\subset I\\ I\in\mathcal{Q}}} |\langle f\rangle_I|^r \mathbf{1}_I(x)\right)^{\frac{1}{r}}.$$

Now we consider a Whitney covering lemma whose statement we borrow from [20] and the universal maximal estimate:

**Lemma 4.2.51.** *For each $k$ there exists a collection $\mathcal{Q}_k$ of disjoint cubes satisfying:*

$$\Omega_k = \bigcup_{Q\in\mathcal{Q}_k} Q, \tag{4.2.52}$$

$$Q^{(1)} \subset \Omega_k\,,\ Q^{(2)} \cap \Omega_k^c \neq \emptyset, \tag{4.2.53}$$

$$\sum_{Q\in\mathcal{Q}_k} \mathbf{1}_{Q^{(1)}} \lesssim \mathbf{1}_{\Omega_k}, \tag{4.2.54}$$

$$\sup_{Q\in\mathcal{Q}_k} \sharp\left\{Q' \in \mathcal{Q}_k \,:\, Q'\cap Q^{(1)} \neq \emptyset\right\} \lesssim 1\,, \tag{4.2.55}$$

$$Q \in \mathcal{Q}_k\,,\ Q' \in \mathcal{Q}_l\,,\ Q \subsetneq Q' \qquad k > l\,. \qquad . \tag{4.2.56}$$

**Theorem 4.2.57.** *Let $\mu$ be a weight and $1 < s \leq \infty$. For $g \in L^s(\omega)$, define*

$$M^\mu g(x) = \sup_{\substack{Q \in \mathcal{D} \\ Q \ni x}} \langle |g| \rangle_Q^\mu.$$

*Then $M^\mu : L^s(\mu) \to L^s(\mu)$ is a bounded operator.*

The proofs of Lemma 4.2.51 and Theorem 4.2.57 are standard and we omit them, but relevant arguments can be found in [20] and [42].

**Definition 18.** *Let $\{\mathcal{Q}_k\}_{k\in\mathbb{Z}}$ be collections of cubes as in Lemma 4.2.51 and $R$ a dyadic cube. Provided there exists $k$ such that $R \in \mathcal{Q}_k$, define $C(R) = \sup\{k : R \in \mathcal{Q}_k\}$, $c(R) = \inf\{k : R \in \mathcal{Q}_k\}$ and $D(R) = C(R) - c(R)$; otherwise let $c(R) = C(R) = D(R) = 0$.*

## *4.3 Preliminary Results*

Here we formulate and prove some results which will be used in the argument for Theorem 1.2.9. We begin with the following weak-type estimate:

**Lemma 4.3.58.** *Assuming (4.1.45) and (4.1.46) hold, for $\mathbf{g} \in L^{p'}_{\ell^{r'}}(w)$ and $f \in L^p(\sigma)$, we have*

$$\|\mathbf{U}_{\mathcal{Q}}(\mathbf{g}w)\|_{L^{p',\infty}(\sigma)} \lesssim \mathcal{L}_*^{\frac{1}{p}} \|\mathbf{g}\|_{L^{p'}_{\ell^{r'}}(w)}, \tag{4.3.59}$$

$$\|\mathcal{T}_{\mathcal{Q},r}(f\sigma)\|_{L^{p,\infty}(w)} \lesssim \mathcal{L}^{\frac{1}{p'}} \|f\|_{L^p(\sigma)}. \tag{4.3.60}$$

A consequence of Lemma 4.3.58 is that we can make slight modifications to the testing conditions on $\mathcal{T}_{\mathcal{Q},r}$ and $\mathbf{U}_{\mathcal{Q}}$:

**Lemma 4.3.61.** *For each $Q \in \mathcal{D}$ and for any positive $\mathbf{a} = \{a_I\}_{I\in\mathcal{Q}}$ satisfying $\sum_{I\in\mathcal{Q}} |a_I(x)|^r = 1$ for almost all $x \in \mathbb{R}^n$, we have*

$$\int_{\mathbb{R}^n} \mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_Q\sigma)(x)^p w \lesssim \mathcal{L}\sigma(Q), \tag{4.3.62}$$

$$\int_{\mathbb{R}^n} \mathbf{U}_{\mathcal{Q}}\left(\mathbf{1}_Q \mathbf{a} w\right)(x)^{p'} \sigma \lesssim \mathcal{L}_* w(Q). \tag{4.3.63}$$

Now we consider the following the lemma:

**Lemma 4.3.64.** *Given collections of cubes $\{\mathcal{Q}_k\}_{k\in\mathbb{Z}}$ as in Lemma 4.2.51, for each $k$ and $Q \in \mathcal{Q}_k$ we have*

$$\max\left\{\mathcal{T}^{\text{out}}_{\mathcal{Q},r,Q^{(1)}}(\mathbf{1}_{Q^{(2)}}f\sigma)(x)\,,\,\mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_{(Q^{(2)})^c}f\sigma)(x)\right\} \le 2^k\,,$$

*with $x \in Q$.*

Further, Lemma 4.3.64 also implies the following maximum principle

**Lemma 4.3.65.** *For a given function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, let $\Omega_k = \{x \in \mathbb{R}^n : \mathcal{T}_{\mathcal{Q},r}f(x) > 2^k\}$. Denote by $\mathcal{Q}_k$ the corresponding Whitney cubes for the $\Omega_k$ and for a given cube $Q$ let*

$$E_k(Q) = Q \cap (\Omega_{k+2} - \Omega_{k+3})\,, \qquad Q \in \mathcal{Q}_k\,.$$

*Then for all $k$ and $x \in E_k(Q)$, we have*

$$2^k \le \mathcal{T}^{\text{in}}_{\mathcal{Q},r,Q^{(1)}}(\mathbf{1}_{Q^{(1)}}f)(x).$$

### 4.3.1 Proof of Lemma 4.3.58

We will argue the case for (4.3.59) first. Fix a sequence $\mathbf{g} \in L^{p'}_{\ell^{r'}}(w)$ and begin by defining $\Gamma_\alpha = \{x : \mathbf{U}(\mathbf{g}w)(x) > \alpha\}$ for $\alpha > 0$. $\mathbf{U}_{\mathcal{Q}}(\mathbf{g}w)(x)$ is lower semi-continuous and so $\Gamma_\alpha$ is open. Similar to Lemma 4.2.51, we will perform a Whitney-style decomposition; specifically, for fixed $\alpha$, let $\left\{L^\alpha_j\right\}_{j\in\mathbb{N}}$ be the dyadic cubes which are maximal with respect to the following two conditions: (i.) $L^\alpha_j \cap \Gamma_{2\alpha} \neq \emptyset$ and (ii.) $L^\alpha_j \subset \Gamma_\alpha$ for all $j \in \mathbb{N}$. First, we aim to put ourselves in a position to use the testing condition on $\mathcal{T}_{\mathcal{Q},r}$; for fixed $j$,

$$\begin{aligned}
\int_{L^\alpha_j} \mathbf{U}_{\mathcal{Q}}(\mathbf{g}w)(x)\sigma &= \int_{L^\alpha_j} \langle \mathbf{1}_{L^\alpha_j}\sigma, \mathbf{U}_{\mathcal{Q}}(\mathbf{g}w)\rangle_{\ell^r}dx \\
&= \int_{L^\alpha_j} \langle \mathbf{T}_{\mathcal{Q},r}(\mathbf{1}_{L^\alpha_j}\sigma), \mathbf{g}w\rangle_{\ell^r}dx \\
&\le \int_{L^\alpha_j} \mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_{L^\alpha_j}\sigma)(x)\|\mathbf{g}\|_{\ell^r}w.
\end{aligned}$$

Now as a result, we have

$$\begin{aligned}\left(\sigma(L_j^\alpha)^{-1}\int_{L_j^\alpha}\mathbf{U}_{\mathcal{Q}}(\mathbf{g}w)(x)\sigma\right)^{p'} &\leq \left(\sigma(L_j^\alpha)^{-1}\int_{L_j^\alpha}\mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_{L_j^\alpha}\sigma)(x)\|\mathbf{g}\|_{\ell^{r'}}w\right)^{p'}\\ &\leq \left(\int_{L_j^\alpha}\|\mathbf{g}\|_{\ell^{r'}}^{p'}w\right)\left(\int_{L_j^\alpha}\mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_{L_j^\alpha}\sigma)^p w\right)^{\frac{p'}{p}}\sigma(L_j^\alpha)^{-p'}\\ &\lesssim \mathcal{L}^{\frac{p'}{p}}\left(\int_{L_j^\alpha}\|\mathbf{g}\|_{\ell^{r'}}^{p'}w\right)\sigma(L_j^\alpha)^{\frac{p'}{p}-p'}\\ &= \mathcal{L}^{\frac{p'}{p}}\left(\int_{L_j^\alpha}\|\mathbf{g}\|_{\ell^{r'}}^{p'}w\right)\sigma(L_j^\alpha)^{-1}.\end{aligned}$$

As a consequence,

$$\left(\sigma(L_j^\alpha)^{-1}\int_{L_j^\alpha}\mathbf{U}_{\mathcal{Q}}(\mathbf{g}w)(x)\sigma\right)^{p'}\sigma(L_j^\alpha)\lesssim \mathcal{L}^{\frac{p'}{p}}\left(\int_{L_j^\alpha}\|\mathbf{g}\|_{\ell^{r'}}^{p'}w\right)$$

and summing over $j$ gives

$$\sum_{j\in\mathbb{N}}\left(\sigma(L_j^\alpha)^{-1}\int_{L_j^\alpha}\mathbf{U}(\mathbf{g}w)(x)\sigma\right)^{p'}\sigma(L_j^\alpha)\lesssim \mathcal{L}^{\frac{p'}{p}}\|\mathbf{g}\|_{L_{\ell^{r'}}^{p'}(w)}^{p'}. \tag{4.3.66}$$

At this point we will appeal to a 'good-lambda' trick. In particular, we fix $\alpha$ and $\epsilon = 2^{-p'-1} > 0$; further, we define $\mathcal{E} = \left\{j : \sigma(L_j^\alpha\cap\Gamma_{2\alpha}) < \epsilon\sigma(L_j^\alpha)\right\}$. So,

$$\begin{aligned}(2\alpha)^{p'}\sigma(\Gamma_{2\alpha}) &\lesssim \epsilon(2\alpha)^{p'}\sum_{j\in\mathcal{E}}\sigma(L_j^\alpha)+\epsilon^{-1}\sum_{j\notin\mathcal{E}}(2\alpha)^{p'}\sigma(L_j^\alpha)\\ &\leq \epsilon(2\alpha)^{p'}\sum_{j\in\mathcal{E}}\sigma(L_j^\alpha)+\sum_{j\notin\mathcal{E}}2^{-1}(\alpha\sigma(L_j^\alpha)\sigma(L_j^\alpha)^{-1})^{p'}\sigma(L_j^\alpha)\\ &\leq \epsilon(2\alpha)^{p'}\sum_{j\in\mathcal{E}}\sigma(L_j^\alpha)+\sum_{j\notin\mathcal{E}}2^{-1}\left(\sigma(L_j^\alpha)^{-1}\int_{L_j^\alpha}\mathbf{U}_{\mathcal{Q}}(\mathbf{g}w)(x)\sigma\right)^{p'}\sigma(L_j^\alpha)\\ &\lesssim \epsilon(2\alpha)^{p'}\sum_{j\in\mathcal{E}}\sigma(L_j^\alpha)+2^{-1}\mathcal{L}^{\frac{p'}{p}}\|\mathbf{g}\|_{L_{\ell^{r'}}^{p'}(w)}^{p'}\end{aligned}$$

where the final inequality follows from (4.3.66). Hence

$$\begin{aligned}(2\alpha)^{p'}\sigma(\Gamma_{2\alpha}) &\lesssim 2^{-1}(\alpha)^{p'}\sigma(\Gamma_\alpha)+2^{-1}\mathcal{L}^{\frac{p'}{p}}\|\mathbf{g}\|_{L_{\ell^{r'}}^{p'}(w)}^{p'}\\ &\leq 2^{-1}\|\mathbf{U}_{\mathcal{Q}}(\mathbf{g}w)\|_{L^{p',\infty}(\sigma)}^{p'}+2^{-1}\mathcal{L}^{\frac{p'}{p}}\|\mathbf{g}\|_{L_{\ell^{r'}}^{p'}(w)}^{p'}\end{aligned}$$

which gives (4.3.59).

Now we consider (4.3.60). The argument will be similar to that for (4.3.59). Fix a positive function $f \in L^p(\sigma)$ and let $\Psi_\alpha = \{x : \mathcal{T}_{\mathcal{Q},r}(f\sigma)(x) > \alpha\}$ for $\alpha > 0$. Again, we perform a Whitney-style decomposition; explicitly, let $\{P_j^\alpha\}_{j\in\mathbb{N}}$ be the dyadic cubes which are maximal with respect to: (i.) $P_j^\alpha \cap \Psi_{2\alpha} \neq \emptyset$ and (ii.) $P_j^\alpha \subset \Psi_\alpha$ for all $j \in \mathbb{N}$. We define $\mathbf{a} = \mathbf{T}_{\mathcal{Q},r}(f\sigma)^{r-1}(\mathcal{T}_{\mathcal{Q},r}(f\sigma))^{-1}$ and attempt to place ourselves in a position where we may use the testing condition on $\mathbf{U}_{\mathcal{Q}}$; using duality as before, for each $j$ we see the expression

$$\left(w(P_j^\alpha)^{-1}\int_{P_j^\alpha} \mathcal{T}_{\mathcal{Q},r}(f\sigma)(x)w\right)^p w(P_j^\alpha) \tag{4.3.67}$$

is equivalent to

$$\left(w(P_j^\alpha)^{-1}\int_{P_j^\alpha} \mathbf{U}_{\mathcal{Q}}(\mathbf{1}_{P_j^\alpha} w\mathbf{a})(x)f(x)\sigma\right)^p w(P_j^\alpha). \tag{4.3.68}$$

Using Hölder's inequality,

$$\begin{aligned}(4.3.68) &\leq \left(\int_{P_j^\alpha} \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{P_j^\alpha} w)(x)^{p'}\sigma\right)^{\frac{p}{p'}} \left(\int_{P_j^\alpha} f(x)^p\sigma\right) w(P_j^\alpha)^{1-p} \\ &\lesssim \mathcal{L}_*^{\frac{p}{p'}} \left(\int_{P_j^\alpha} f(x)^p\sigma\right)\end{aligned}$$

and summing gives

$$\sum_{j\in\mathbb{N}} \left(w(P_j^\alpha)^{-1}\int_{P_j^\alpha} \mathcal{T}_{\mathcal{Q},r}(f\sigma)(x)w\right)^p w(P_j^\alpha) \lesssim \mathcal{L}_*^{\frac{p}{p'}} \|f\|_{L^p(\sigma)}^p.$$

As before we use a 'good-lambda' trick; we fix $\alpha$ and $\epsilon = 2^{-p-1}$. Further, define $\Upsilon = \{j : w(P_j^\alpha \cap \Psi_{2\alpha}) < \epsilon w(P_j^\alpha)\}$. So

$$\begin{aligned}(2\alpha)^p w(\Psi_{2\alpha}) &\lesssim \epsilon(2\alpha)^p \sum_{j\in\Upsilon} w(P_j^\alpha) + \epsilon^{-1}\sum_{j\notin\Upsilon}(2\alpha)^p w(P_j^\alpha) \\ &\lesssim \epsilon(2\alpha)^p \sum_{j\in\Upsilon} w(P_j^\alpha) + 2^{-1}\sum_{j\notin\Upsilon}(\alpha w(P_j^\alpha)w(P_j^\alpha)^{-1})^p w(P_j^\alpha) \\ &\lesssim \epsilon(2\alpha)^p \sum_{j\in\Upsilon} w(P_j^\alpha) + 2^{-1}\sum_{j\notin\Upsilon}\left(w(P_j^\alpha)^{-1}\int_{P_j^\alpha}\mathcal{T}_{\mathcal{Q},r}(f\sigma)(x)w\right)^p w(P_j^\alpha) \\ &\lesssim \epsilon(2\alpha)^p \sum_{j\in\Upsilon} w(P_j^\alpha) + 2^{-1}\mathcal{L}_*^{\frac{p}{p'}}\|f\|_{L^p(\sigma)}^p \\ &\le \epsilon(2\alpha)^p w(\Psi_\alpha) + 2^{-1}\mathcal{L}_*^{\frac{p}{p'}}\|f\|_{L^p(\sigma)}^p.\end{aligned}$$

Now we have

$$\begin{aligned}(2\alpha)^p w(\Psi_{2\alpha}) &\lesssim 2^{-1}\alpha^p w(\Psi_\alpha) + 2^{-1}\mathcal{L}_*^{\frac{p}{p'}}\|f\|_{L^p(\sigma)}^p \\ &\le 2^{-1}\|\mathcal{T}_{\mathcal{Q},r}(f\sigma)\|_{L^{p,\infty}(w)}^p + \mathcal{L}_*^{\frac{p}{p'}}\|f\|_{L^p(\sigma)}^p\end{aligned}$$

and this gives (4.3.60).

### 4.3.2 Proof of Lemma 4.3.61

First, we will show the case for (4.3.62). By (4.3.59) and duality, we have for each $f \in L^{p,1}(\sigma)$,

$$\|\mathcal{T}_{\mathcal{Q},r}(f\sigma)\|_{L^p(w)} \lesssim \mathcal{L}^{\frac{1}{p}}\|f\|_{L^{p,1}(\sigma)}.$$

Since for any cube $Q$, $\mathbf{1}_Q \in L^{p,1}(\sigma)$ and $\|\mathbf{1}_Q\|_{L^{p,1}(\sigma)} = \sigma(Q)^{\frac{1}{p}}$, we have

$$\|\mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_Q\sigma)\|_{L^p(w)} \lesssim \mathcal{L}^{\frac{1}{p}}\sigma(Q)^{\frac{1}{p}}$$

which gives the desired result.

We conclude by verifying (4.3.63) holds. Consider, for $\mathbf{a} = \mathbf{T}_{\mathcal{Q},r}(f\sigma)\mathcal{T}_{\mathcal{Q},r}(f\sigma)^{-1}$ and $Q$ fixed,

$$\left(\int_{\mathbb{R}^n}\mathbf{U}_{\mathcal{Q}}(\mathbf{1}_Q\mathbf{a}w)(x)^{p'}\sigma\right)^{\frac{1}{p'}} = \int_{\mathbb{R}^n}\mathbf{U}_{\mathcal{Q}}(\mathbf{1}_Q\mathbf{a}w)(x)h(x)\sigma$$

for some $h \in L^p(\sigma)$. Then using duality and Hölder's inequality in $\ell^r - \ell^{r'}$ we have

$$\int_{\mathbb{R}^n} \mathbf{U}_{\mathcal{Q}}(\mathbf{1}_Q \mathbf{a} w)(x)h(x)\sigma = \int_{\mathbb{R}^n} \langle \mathbf{1}_Q \mathbf{a} w, \mathbf{T}_{\mathcal{Q},r}(h\sigma)\rangle_{\ell^r} dx$$
$$\leq \int_Q \mathcal{T}_{\mathcal{Q},r}(h\sigma)(x)w. \tag{4.3.69}$$

Recall, by $(\mathcal{T}_{\mathcal{Q},r}(h\sigma)(x))^*$ and $(\mathbf{1}_Q)(x))^*$, we mean the symmetric decreasing rearrangements of $\mathcal{T}_{\mathcal{Q},r}(h\sigma)(x)$ and $\mathbf{1}_Q(x)$ with respect to $w$. We continue from (4.3.69) by applying Hölder's inequality and using (4.3.60) to obtain

$$\begin{aligned}
(4.3.69) &\leq \int_{\mathbb{R}} (\mathcal{T}_{\mathcal{Q},r}(h\sigma)(x))^* \, (\mathbf{1}_Q(x))^* \, w \\
&\leq \|\mathcal{T}_{\mathcal{Q},r}(h\sigma)\|_{L^{p,\infty}(w)} w(Q)^{\frac{1}{p'}} \\
&\leq \|\mathcal{T}_{\mathcal{Q},r}(\cdot\sigma)\|_{L^p(\sigma)\to L^{p,\infty}(w)} w(Q)^{\frac{1}{p'}} \\
&\lesssim \mathcal{L}_*^{\frac{1}{p'}} w(Q)^{\frac{1}{p'}}.
\end{aligned}$$

The foregoing inequalities yield

$$\int_{\mathbb{R}^n} \mathbf{U}_{\mathcal{Q}}(\mathbf{1}_Q \mathbf{a} w)(x)^{p'} \sigma \leq \mathcal{L}_* w(Q)^{\frac{p'}{p'}}$$

and we are done.

### 4.3.3 Proof of Lemma 4.3.64 and Lemma 4.3.65

*4.3.3.1 Proof of Lemma 4.3.64*

By Lemma 4.2.51, there is $z \in Q^{(2)} \cap \Omega_k^c$. Thus for $x \in Q$ we have

$$\mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_{(Q^{(2)})^c} f\sigma)(x) = \mathcal{T}^{\text{out}}_{\mathcal{Q},r,Q^{(1)}}(\mathbf{1}_{(Q^{(2)})^c} f\sigma)(x) \leq \mathcal{T}_{\mathcal{Q},r}(f\sigma)(z) \leq 2^k$$

and we are done.

*4.3.3.2 Proof of Lemma 4.3.65*

By Lemma 4.3.64 and the sub-linearity of $\mathcal{T}_{\mathcal{Q}}$, we have for $x \in E_k(Q)$

$$\begin{aligned}
2^{k+2} - 2^{k+1} &\leq \mathcal{T}_{\mathcal{Q},r}(f)(x) - \mathcal{T}^{\text{out}}_{\mathcal{Q},r,Q^{(1)}}(\mathbf{1}_{Q^{(1)}} f)(x) - \mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_{(Q^{(1)})^c} f\sigma)(x) \\
&\leq \mathcal{T}^{\text{in}}_{\mathcal{Q},r,Q^{(1)}}(\mathbf{1}_{Q^{(1)}} f)(x).
\end{aligned}$$

Noting $2^{k+2} - 2^{k+1} \geq 2^k$, we obtain $2^k \leq \mathcal{T}^{\text{in}}_{\mathcal{Q},r,Q^{(1)}}(\mathbf{1}_{Q^{(1)}} f)(x)$.

*4.3.3.3 Proof of Corollary 4.1.50*

Assuming Theorem 1.2.9 and recalling Sawyer's two weight theorem for the maximal function, the corollary follows from Lerner's decomposition theorem and arguments similar to those used for Corollary 3.1.40 in Chapter III.

### 4.3.4 Proof of Theorem 1.2.9: Necessity

Here we prove the necessity of the testing conditions. We suppose that $\mathcal{T}_{\mathcal{Q},r}$ is a bounded operator. The necessity of (4.1.45) is immediate by taking $f = \mathbf{1}_Q$ for an arbitrary cube, so we only need to verify the necessity of the conditions on $\mathbf{U}_{\mathcal{Q}}$. Fix a cube $Q$ and a sequence $\mathbf{a}$ such that $\|\mathbf{a}\|_{\ell^r} = 1$. Without loss of generality we assume $h$ and $\mathbf{a}$ are positive. Then,

$$\left(\int_Q \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_Q w)(x)^{p'}\sigma\right)^{\frac{1}{p'}} = \int_Q \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_Q w)(x) h\sigma$$

where $h$ is an appropriate function from $L^p(\sigma)$ satisfying $\|h\|_{L^p(\sigma)} = 1$. Now we use duality and apply Hölder's inequality in $\ell^r - \ell^{r'}$ and obtain

$$\begin{aligned}\int_Q \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_Q w)(x) h\sigma &= \int_{\mathbb{R}^n} \langle \mathbf{T}_{\mathcal{Q},r}(h\mathbf{1}_Q\sigma), \mathbf{a}\mathbf{1}_Q w\rangle_{\ell^r} dx \\ &\le \int_Q \mathcal{T}_{\mathcal{Q},r}(h\mathbf{1}_Q\sigma)(x) w \\ &\le \|\mathcal{T}_{\mathcal{Q},r}(h\mathbf{1}_Q\sigma)\|_{L^p(w)} w(Q)^{\frac{1}{p'}} \\ &\le \|\mathcal{T}_{\mathcal{Q},r}(\cdot\sigma)\|_{L^p(\sigma)\to L^p(w)} w(Q)^{\frac{1}{p'}}.\end{aligned}$$

Hence,

$$\int_Q \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_Q w)(x)^{p'}\sigma \le \|\mathcal{T}_{\mathcal{Q},r}(\cdot\sigma)\|^{p'}_{L^p(\sigma)\to L^p(w)} w(Q)$$

where $\mathbf{a}$ is arbitrary. Taking supremums we have

$$\sup_{\mathbf{a}} \sup_Q w(Q)^{-1} \int_Q \mathbf{U}_{\mathcal{Q}}(\mathbf{1}_Q \mathbf{a} w)(x)^{p'}\sigma \le \|\mathcal{T}_{\mathcal{Q},r}(\cdot\sigma)\|^{p'}_{L^p(\sigma)\to L^p(w)}$$

which gives the result.

## 4.4 *Proof of Theorem 1.2.9: Sufficiency*

We apply Lemma 4.2.51 to obtain a collection of cubes $\mathcal{Q}_k$ for each $k$ such that $\Omega_k = \left\{x \in \mathbb{R}^n : \mathcal{T}_{\mathcal{Q},r}(f\sigma)(x) > 2^k\right\} = \cup_{Q\in\mathcal{Q}_k} Q$. For $Q \in \mathcal{Q}_k$, define $E_k(Q) = (\Omega_k \backslash \Omega_{k+2}) \cap Q$. Then we have the following:

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{T}_{\mathcal{Q},r}(f\sigma)(x)^p w &\lesssim \sum_{k\in\mathbb{Z}} w(\left\{x \in \mathbb{R}^n : \mathcal{T}_{\mathcal{Q},r}(f\sigma)(x) > 2^k\right\})2^{kp} \\ &\lesssim \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k} w(E_k(Q))2^{kp}.\end{aligned}$$

By Lemma 4.3.65,

$$\begin{aligned}w(E_k(Q))2^k &\lesssim \int_{E_k(Q)} \mathcal{T}_{\mathcal{Q},r}(f\sigma\mathbf{1}_{Q^{(1)}})(x)w \\ &= \int_{Q^{(1)}} \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)}w)(x)f(x)\sigma;\end{aligned}$$

we split the above integral into two pieces so that

$$\int_{Q^{(1)}} \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)}w)(x)f(x)\sigma = S_{1,k}(Q) + S_{2,k}(Q)$$

with

$$\begin{aligned}S_{1,k}(Q) &= \int_{Q^{(1)}\backslash\Omega_{k+m}} \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)}w)(x)f(x)\sigma \\ S_{2,k}(Q) &= \int_{Q^{(1)}\cap\Omega_{k+m}} \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)})(x)f(x)\sigma.\end{aligned}$$

For each $k$, we partition $\mathcal{Q}_k$ into two collections:

$$\begin{aligned}\mathcal{Q}_{1,k} &= \{Q \in \mathcal{Q}_k : w(E_k(Q)) \le \eta w(Q)\} \\ \mathcal{Q}_{2,k} &= \{Q \in \mathcal{Q}_k : w(E_k(Q)) > \eta w(Q)\}\end{aligned}$$

where $0 < \eta < 1$ is a fixed parameter that will be defined later in the proof; further divide $\mathcal{Q}_{2,k}$ into:

$$\begin{aligned}\mathcal{Q}_k^2 &= \{Q \in \mathcal{Q}_{2,k} : S_{2,k}(Q) \le S_{1,k}(Q)\} \\ \mathcal{Q}_k^3 &= \{Q \in \mathcal{Q}_{2,k} : S_{2,k}(Q) > S_{1,k}(Q)\}.\end{aligned}$$

The sum $\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k} w(E_k(Q))2^{kp}$ is split into pieces corresponding to the collections above:

$$\begin{aligned}
I_1 &= \sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_{1,k}} w(E_k(Q))2^{kp}\\
I_2 &= \sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k^2} w(E_k(Q))2^{kp}\\
I_3 &= \sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k^3} w(E_k(Q))2^{kp}.
\end{aligned}$$

Trivially, we have

$$\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k} w(E_k(Q))2^{kp} = I_1 + I_2 + I_3$$

so that it suffices to estimate each $I_j$.

### 4.4.1 Estimating $I_1$

Consider,

$$\begin{aligned}
I_1 &\lesssim \eta \sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k} \eta^{-1} w(Q)2^{kp}\\
&\lesssim \eta \int_{\mathbb{R}^n} \mathcal{T}_{\mathcal{Q},r}(f\sigma)(x)^p w;
\end{aligned}$$

as $0<\eta<1$, we may absorb the term $I_1$ into $\|\mathcal{T}_{\mathcal{Q},r}(f\sigma)\|_{L^p(\sigma)}$.

### 4.4.2 Estimating $I_2$

Here, notice

$$\begin{aligned}
\eta 2^k w(Q) &\le \int_{Q^{(1)}} \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)} w)(x) f(x)\sigma\\
&\lesssim \int_{Q^{(1)}\setminus\Omega_{k+m}} \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)} w)(x) f(x)\sigma\\
&\le \left(\int_{Q^{(1)}\setminus\Omega_{k+m}} \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_Q w)(x)^{p'}\sigma\right)^{\frac{1}{p'}} \left(\int_{Q^{(1)}\setminus\Omega_{k+m}} f(x)^p\sigma\right)^{\frac{1}{p}}\\
&\lesssim \mathcal{L}_*^{\frac{1}{p'}} w(Q)^{\frac{1}{p'}} \left(\int_{Q^{(1)}\setminus\Omega_{k+m}} f(x)^p\sigma\right)^{\frac{1}{p}}
\end{aligned}$$

so that for fixed $Q$ and $k$,

$$\begin{aligned}
w(E_k(Q))2^{kp} &\lesssim \eta^{-p} w(E_k(Q)) \left( \int_{Q^{(1)}\setminus\Omega_{k+m}} \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)f(x)\sigma \right)^p \\
&\lesssim \eta^{-p} w(E_k(Q)) \mathcal{L}_*^{\frac{p}{p'}} w(Q) \int_{Q^{(1)}\setminus\Omega_{k+m}} f(x)^p\sigma \\
&= \eta^{-p} \mathcal{L}_*^{\frac{p}{p'}} \frac{w(E_k(Q))}{w(Q)} \int_{Q^{(1)}\setminus\Omega_{k+m}} f(x)^p\sigma \\
&\leq \eta^{-p} \mathcal{L}_*^{\frac{p}{p'}} \int_{Q^{(1)}\setminus\Omega_{k+m}} f(x)^p\sigma.
\end{aligned}$$

Summing, we have from (4.2.55)

$$\eta^{-p}\mathcal{L}_*^{\frac{p}{p'}} \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k^2} \int_{Q^{(1)}\setminus\Omega_{k+m}} f(x)^p\sigma \lesssim \eta^{-p}\mathcal{L}_*^{\frac{p}{p'}} \int_{\mathbb{R}^n} f(x)^p\sigma;$$

recalling

$$\begin{aligned}
I_2 &= \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k^2} w(E_k(Q))2^{kp} \\
&\lesssim \eta^{-p} \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k^2} \int_{Q^{(1)}\setminus\Omega_{k+m}} f(x)^p\sigma,
\end{aligned}$$

implies the result.

### 4.4.3 Estimating $I_3$

Assume $N$ is some fixed positive integer and $0 \leq n < m$; we split the remaining cubes into collections modulo $m$ and intend to show

$$\sum_{\substack{k>-N \\ k\equiv n \mod m}} \sum_{Q\in\mathcal{Q}_k^3} w(E_k(Q))2^{kp} \lesssim \int_{\mathbb{R}^n} f(x)^p\sigma$$

with implied constants independent of $n$ and $N$. The monotone convergence theorem combined with summing over $n$ will yield

$$\sum_{Q\in\mathcal{Q}_k^3} w(E_k(Q))2^{kp} \lesssim \int_{\mathbb{R}^n} f(x)^p\sigma$$

To this end, we use a stopping time argument. Namely, set $\mathcal{P}(N,n,1)$ to be the collection of maximal cubes within $P_{N,n} = \cup_{\substack{j\equiv n \mod m \\ j\geq -N}} \cup_{Q\in\mathcal{Q}_j^3} Q$. For $j > 1$ define $\mathcal{P}(N,n,j)$ to be the collection of all cubes $I$ in $P_{N,n}$ which satisfy the following:

(i.) there is $I' \in \mathcal{P}(N,n,j-1)$ such that $I \subsetneq I'$

(ii.) $\langle f\rangle_I^\sigma > 2\langle f\rangle_{I'}^\sigma$

(iii.) $I$ is maximal with respect to properties (i.) and (ii.)

Denote by $\mathcal{P}(N,n) = \cup_{j=1}^\infty \mathcal{P}(N,n,j)$.

We define for $Q \in \mathcal{Q}_k^3$

$$\mathcal{N}(k,m,N,n,Q) = \left\{I \in \mathcal{Q}_{k+m}, k \equiv n \mod m : I \cap Q^{(1)} \neq \emptyset\right\}$$

$$\mathcal{N}(k,m,N,n) = \cup_{\substack{Q\in\mathcal{Q}_k \\ k\equiv n \mod m}} \mathcal{N}(k,m,N,n,Q)$$

and note that $Q^{(1)} \cap \Omega_{k+m} = \cup_{I\in\mathcal{N}(k,m,N,n,Q)} I$. Further, for each $I \in \mathcal{N}(k,m,N,n)$ there is $I_{k,m,N,n} \in \mathcal{Q}_k$ such that $I \subset I_{k,m,N,n}$. Since $k \equiv n \mod m$ we have $I \in \mathcal{P}$ or $\Gamma(I) = \Gamma(I_{k,m,N,n})$; as a consequence, we may split the sum

$$\int_{Q^{(1)}\cap\Omega_{k+m}} \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)f(x)\sigma = \sum_{I\in\mathcal{N}(k,m,N,n,Q)} \int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)f(x)\sigma$$

into two pieces:

$$A_1(k,m,N,n,Q) = \sum_{\substack{I\in\mathcal{N}(k,m,N,n,Q) \\ I\in\mathcal{P}(N,n)}} \int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)f(x)\sigma$$

$$A_2(k,m,N,n,Q) = \sum_{\substack{I\in\mathcal{N}(k,m,N,n,Q) \\ \Gamma(I)=\Gamma(I_{k,m,N,n})}} \int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)f(x)\sigma.$$

For the remainder of the proof, we will assume $k \equiv n \mod m$ and suppress the notational dependence on $N$ and $n$ (e.g. we will write $A_1(k,m,Q)$ for $A_1(k,m,N,n,Q)$). Continuing, from the defining properties of $\mathcal{Q}_k^3$,

$$\begin{aligned} 2^k w(Q) &\lesssim \eta^{-1} \int_{Q^{(1)}\cap\Omega_{k+m}} \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)f(x)\sigma \\ &\lesssim \eta^{-1}A_1(k,m,Q) + \eta^{-1}A_2(k,m,Q) \end{aligned}$$

so that

$$2^{kp}w(E_k(Q)) \lesssim \frac{w(E_k(Q))}{\eta^p w(Q)^p} A_1(k,m,Q)^p + \frac{w(E_k(Q))}{\eta^p w(Q)^p} A_2(k,m,Q)^p.$$

Recalling

$$I_3 = \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k^3} w(E_k(Q))2^{kp}$$

we see it is enough to estimate $I_{3,j} = \sum_{Q\in\mathcal{Q}_k^3} I_{3,j}(Q)$ for $j \in \{1,2\}$ and

$$\begin{aligned} I_{3,1}(Q) &= \frac{w(E_k(Q))}{w(Q)^p} A_1(k,m,Q)^p \\ I_{3,2}(Q) &= \frac{w(E_k(Q))}{w(Q)^p} A_2(k,m,Q)^p \end{aligned}$$

with $Q \in \mathcal{Q}_k^3$.

### 4.4.3.1 *Estimating* $I_{3,1}$

For a fixed cube $Q$ and $I \in \mathcal{N}(k,m,Q)$ we may write

$$\int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)f(x)\sigma = \int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)\langle f\rangle_I^\sigma \sigma$$

since the expression $\mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)$ is constant for $x \in I$. Continuing, for $G \in \mathcal{P}$,

$$\begin{aligned} \sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ \Gamma(I)=\Gamma(I_{k,m})}} \int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)f(x)\sigma &= \sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ \Gamma(I)=\Gamma(I_{k,m})}} \int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)f(x)\sigma \\ &= \sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ \Gamma(I)=\Gamma(I_{k,m})}} \int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)\langle f\rangle_I^\sigma \sigma \\ &\lesssim \langle f\rangle_G^\sigma \sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ \Gamma(I)=\Gamma(I_{k,m})}} \int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)\sigma. \end{aligned}$$

So for fixed $G \in \mathcal{P}$, using duality and Hölder's inequality we have

$$\begin{aligned} \frac{w(E_k(Q))}{w(Q)^p} A_1(k,m,Q)^p &\lesssim \frac{w(E_k(Q))}{w(Q)^p} (\langle f\rangle_G^\sigma)^p \left( \sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ \Gamma(I)=\Gamma(I_{k,m})=G}} \int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a1}_{E_k(Q)}w)(x)\sigma \right)^p \\ &\leq \frac{w(E_k(Q))}{w(Q)^p} (\langle f\rangle_G^\sigma)^p \left( \sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ \Gamma(I)=\Gamma(I_{k,m})=G}} \int_G \mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_G\sigma)(x)w \right)^p \\ &\leq (\langle (f)\rangle_G^\sigma)^p w(E_k(Q)) M_w(\mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_G\sigma))(x)^p. \end{aligned}$$

By the universal maximal estimate and the modified testing condition Lemma 4.3.61,

$$\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k^3} w(E_k(Q))M_w(\mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_G\sigma))(x)^p \lesssim \int_{\mathbb{R}^n} M_w(\mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_G\sigma))(x)^p w$$

$$\lesssim \int_{\mathbb{R}^n} \mathcal{T}_{\mathcal{Q},r}(\mathbf{1}_G\sigma)(x)^p w$$

$$\lesssim \mathcal{L}\sigma(G).$$

Hence,

$$\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k^3} \frac{w(E_k(Q))}{w(Q)^p}\left(\sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ \Gamma(I)=\Gamma(I_{k,m})}}\int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)}w)(x)f(x)\sigma\right)^p \lesssim \mathcal{L}\sum_{G\in\mathcal{P}}(\langle f\rangle_G^\sigma)^p\sigma(G)$$

$$\lesssim \mathcal{L}\int_{\mathbb{R}^n} f(x)^p\sigma$$

where in the last line we have used the Carleson embedding theorem.

#### *4.4.3.2 Estimating $I_{3,2}$*

We begin by noticing for fixed $Q$,

$$\frac{w(E_k(Q))}{w(Q)^p}A_2(k,m,Q)^p = \frac{w(E_k(Q))}{w(Q)^p}\left(\sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ I\in\mathcal{P}}}\frac{\sigma(I)^{\frac{1}{p}}}{\sigma(I)^{\frac{1}{p}}}\int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)}w)(x)f(x)\sigma\right)^p$$

$$\leq I_{4,1}(k,m,Q)I_{4,2}(k,m,Q)$$

where we define

$$I_{4,1}(k,m,Q) = \frac{w(E_k(Q))}{w(Q)^p}\left(\sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ I\in\mathcal{P}}}\sigma(I)^{\frac{-p'}{p}}\left(\int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)}w)(x)\sigma\right)^{p'}\right)^{\frac{p}{p'}}$$

$$I_{4,2}(k,m,Q) = \sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ I\in\mathcal{P}}}\sigma(I)(\langle f\rangle_G^\sigma)^p.$$

Notice for each $Q$ by Hölder's inequality,

$$\sigma(I)^{\frac{-p'}{p}}\left(\int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)}w)(x)\sigma\right)^{p'} \leq \sigma(I)^{\frac{-p'}{p}+\frac{p'}{p}}\int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)}w)(x)^{p'}\sigma,$$

so that

$$\sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ I\in\mathcal{P}}} \sigma(I)^{\frac{-p'}{p}} \left(\int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)}w)(x)\sigma\right)^{p'} \leq \sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ I\in\mathcal{P}}} \int_I \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)}w)(x)^{p'}\sigma$$

$$\lesssim \int_{\mathbb{R}^n} \mathbf{U}_{\mathcal{Q}}(\mathbf{a}\mathbf{1}_{E_k(Q)}w)(x)^{p'}\sigma$$

$$\lesssim \mathcal{L}_* w(Q);$$

since $\frac{w(E_k(Q))}{w(Q)^p} \leq w(Q)^{1-p}$ we obtain $I_{4,1}(k,m,Q) \lesssim \mathcal{L}_*^{\frac{p}{p'}} w(Q)^{\frac{p}{p'}-p+1} = \mathcal{L}_*^p$; as a result we need only consider the sum

$$\sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k^3} \sum_{\substack{I\in\mathcal{N}(k,m,Q)\\ I\in\mathcal{P}}} \sigma(R)(\langle f\rangle_R^\sigma)^p.$$

To finish the proof, we need a uniform bound on the number of times a cube $R$ may appear in the above sum. Consider the following lemma, whose proof we momentarily postpone.

**Lemma 4.4.70.** *Fix a cube $R$ which satisfies $R \in \mathcal{Q}_j$ for some integer $j$, and for $1 \leq l \leq D(R)$ suppose*

(i.) *there is an integer $k_l$ and $Q_l \in \mathcal{Q}_{k_l}^3$ with $R \in \mathcal{R}_{k_l}(Q)$,*

(ii.) *the pairs $(Q_l, k_l)$ are distinct.*

*We then have that $D(R) \lesssim 1$, with the implied constant depending upon the dimension, and $\eta$, the small constant previously mentioned.*

Using Lemma 4.4.70 and the Carleson embedding theorem, we may estimate

$$\sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k^3} \sum_{\substack{R\in N(k,m,Q)\\ R\in\mathcal{P}}} \sigma(R)(\langle f\rangle_R^\sigma)^p \lesssim \int_{\mathbb{R}^n} f(x)^p\sigma$$

to complete the proof modulo Lemma 4.4.70.

*4.4.3.3 Proof of Lemma 4.4.70*

Fix $R \in \mathcal{D}$ such that there exists $k_1, \cdots, k_{D(R)} \in \mathbb{Z}$ and cubes $Q_1, \cdots, Q_{D(R)}$ so that $R \in \mathcal{R}_{k_j}(Q_j)$ for all $1 \leq j \leq D(R)$ and the pairs $(Q_j, k_j)$ are distinct. We argue by contradiction that $D(R) \lesssim 1$. The dyadic structure of $\mathcal{D}$ immediately implies that by possibly reordering we must have the following

$$Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_{D(R)}. \tag{4.4.71}$$

Then, we have $R \subset Q_j^{(1)}$ for each $j$ by (4.2.53). At this point we consider two cases; namely

(a.) $Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q_{D(R)}$

(b.) $Q_1 = \cdots = Q_{D(R)}$ .

First we want to inspect case (a.). We may assume that $k_1 > \cdots > k_{D(R)}$ by (4.2.53) (Whitney condition); also it is clear that case (a.) implies

$$R \subset Q_1^{(1)} \subset \cdots \subset Q_{D(R)}^{(1)}.$$

Hence, by the above and the definition of $\mathcal{R}_{k_1}$ and $\mathcal{R}_{k_{D(R)}}$, $R \in \mathcal{Q}_{k_1+3}$ and $R \in \mathcal{Q}_{k_{D(R)}+3}$. We conclude $R \in \mathcal{Q}_l$ for $k_{D(R)} + 3 \leq l \leq k_1 + 3$. Since we are assuming that $D(R) \lesssim 1$ fails, without loss of generality we may take $D(R) = 7$. Then we have $R, Q_7 \in Q_{k_7}$ :

$$R \subset Q_1^{(1)} \subsetneq \cdots \subsetneq Q_7^{(1)} \implies$$
$$R^{(2)} \subset Q_7^{(1)}$$

and this contradicts (4.2.53). Hence, there is a uniform bound on the number of strict inequalities in (4.4.71), and so we only need to consider (b.).

If (b.) holds then by definition we have $w(E_{k_j}(Q_1)) > \eta w(Q_1)$ for all $1 \leq j \leq$

$D(R)$. We can without loss of generality assume the $k_i$ are distinct. Then the $E_{k_j}(Q_1)$ are also distinct and

$$w(Q_1) = \sum_{j \in \mathbb{Z}} w(E_j(Q_1)) \geq \sum_{j=1}^{D(R)} w(E_{k_j}(Q_1)) > \sum_{j=1}^{D(R)} w(Q_1)\eta$$

so that it must be $D(R) \leq \eta^{-1}$ and we are done.

# CHAPTER V

# JOINT ESTIMATES FOR THE HILBERT TRANSFORM AND MAXIMAL FUNCTION

## *5.1 Introduction*

In this chapter, our particular focus is on the relationship between the Hilbert transform $H$ and the Hardy-Littlewood maximal operator $M$ in the two weight setting. Links between the two operators in this context have been considered previously. The authors of [31] establish

**Theorem 1.2.13.** *Suppose $\sigma$ and $w$ are two positive Borel measures such that $M(\cdot\sigma) : L^2(\sigma) \to L^2(w)$ and $M(\cdot w) : L^2(w) \to L^2(\sigma)$ both hold. Then $H(\cdot\sigma)$ is bounded from $L^2(\sigma)$ to $L^2(w)$ if and only if the following hold:*

(i.) $\|H(\mathbf{1}_I\sigma)\|_{L^2(w)} \lesssim \sigma(I)^{\frac{1}{2}}$

(ii.) $\|H(\mathbf{1}_I w)\|_{L^2(\sigma)} \lesssim w(I)^{\frac{1}{2}}$

(iii.) $\sup_{z\in\mathbb{C}} P_\sigma(z)P_w(z) \lesssim 1,$

*where $P_w$ and $P_\sigma$ are the Poisson extensions of $w$ and $\sigma$,*

and suggest the boundedness of $M(\cdot\sigma)$ and $M(\cdot w)$ in Theorem 1.2.13 may be unnecessary. An old conjecture of Muckenhoupt and Wheeden stated in [5] implies the continuity of $H(\cdot\sigma)$ is equivalent to that of $M(\cdot\sigma)$ and $M(\cdot w)$:

**Conjecture 1.2.14** ($L^p$ Muckenhoupt-Wheeden)**.** *Let $M$ be the Hardy-Littlewood maximal operator, $T$ be a Calderón-Zygmund operator and let $w$ and $v$ be weights on*

$\mathbb{R}^d$. *Then*

$$M : \quad L^p(v) \mapsto L^p(w) \tag{5.1.72}$$

$$M : \quad L^{p'}(w^{1-p'}) \mapsto L^{p'}(v^{1-p'}) \tag{5.1.73}$$

*if and only if*

$$T : L^p(v) \mapsto L^p(w). \tag{5.1.74}$$

We will show that within the context of Theorem 1.2.13, boundedness of the Hilbert transform does not imply that of the maximal function; further, we will construct weights $w$ and $v$ which violate Conjecture 1.2.14. As a consequence, we conclude there is no a priori association between the operators in the two weight setting. The main results of this chapter may be formulated as follows:

**Theorem 5.1.75.** *Let $1 < p < \infty$ and let $p'$ be the dual exponent, $\frac{1}{p} + \frac{1}{p'} = 1$. There exist nontrivial weights $w$ and $v = \left(\frac{Mw}{w}\right)^p w$ for which the Hardy-Littlewood maximal operator satisfies*

$$M : \quad L^p(\left(\tfrac{Mw}{w}\right)^p w) \mapsto L^p(w) \tag{5.1.76}$$

$$M : \quad L^{p'}(w^{1-p'}) \mapsto L^{p'}\left(\frac{w}{(Mw)^{p'}}\right) \tag{5.1.77}$$

*but the Hilbert transform $H$ is unbounded from $L^p(\left(\frac{Mw}{w}\right)^p w)$ to $L^p(w)$.*

**Theorem 5.1.78.** *There exist measures $\gamma$ and $\lambda$ such that*

$$M(\cdot\gamma) : \quad L^2(\gamma) \not\to L^2(\lambda) \tag{5.1.79}$$

$$H(\cdot\gamma) : \quad L^2(\gamma) \to L^2(\lambda). \tag{5.1.80}$$

The examples we present here rely heavily on the Cantor-like constructions found in [21,35–37]. The authors of [35] and [36] were interested in showing certain endpoint estimates for Calderón-Zygmund operators failed and to this end built weights $\sigma$ and

$w$ for which $H(\cdot\sigma)$ failed to map $L^2(\sigma)$ into $L^{2,\infty}(w)$. We modify their weights slightly for the purposes of obtaining strong-type $L^p$ estimates and verify Theorem 5.1.75 holds. The weights considered for Theorem 5.1.78 were constructed in [21] to show a particular testing condition was not necessary for the two weight inequality of the Hilbert transform. To obtain the conclusion of Theorem 5.1.78 we verify the maximal function is unbounded for this pair of measures.

There are two important remarks about our results which should be made. The weights described above are allowed to take the value 0 on sets of non-zero Lebesgue measure. This feature is important for the weights' construction and is useful for the proofs of Theorem 5.1.75 and Theorem 5.1.78. Additionally, we consider the operators $M$ and $H$ as maps from one weighted $L^p$ space to another $L^q$ space with $p = q$; the assumption $p = q$ is necessary (see [6] for a proof conjecture 1.2.14 holds for $p < q$). The remainder of the chapter is structured as follows. In the next section we review some basic theorems which will be useful for us. The third and fourth sections focus on the proofs of Theorem 5.1.75 and Theorem 5.1.78.

## 5.2 Preliminaries

Here we introduce some key definitions and theorems which we refer to throughout the remainder of the chapter. First we introduce the concept of a triadic interval:

**Definition 19.** *We refer to an interval of the type $[3^j k, 3^j(k+1))$ with $j, k \in \mathbb{Z}$ as a triadic interval and use $\mathcal{T}$ to denote the corresponding triadic grid consisting of all triadic intervals. For a given triadic interval $I$ let $I^m$ be the triadic child which contains the midpoint (center) $c(I)$ of $I$.*

The following is a convenient dualized formulation of a two weight inequality due to Eric Sawyer (see [38] and [39]):

**Theorem 5.2.81.** *Let $w$ and $v$ be weights and $T$ a sublinear operator with $1 < p < \infty$. If $\sigma = \mathbf{1}_{\mathrm{supp} w} w^{1-p'}$ then the following are equivalent:*

(i.) $\|Tf\|_{L^p(v)}\lesssim \|f\|_{L^p(w)}$

(ii.) $\|T(f\sigma)\|_{L^p(v)}\lesssim \|f\|_{L^p(\sigma)}$.

Finally, we also recall E. Sawyer's characterization of the two weight inequality for the maximal function:

**Theorem 5.2.82.** *Let $w$ and $v$ be weights with $1 < p < \infty$ and define $\sigma = v^{1-p'}$. Then $M$ is bounded from $L^p(v)$ to $L^p(w)$ if and only if*

$$\int_Q |M(\sigma 1_Q)(x)|^p\, v \lesssim \sigma(Q) \quad \textit{for all } Q \textit{ cubes.} \tag{5.2.83}$$

## 5.3 Proof of Theorem 5.1.75

### 5.3.1 Weight Construction

Here we will construct a sequence $\{w_k\}_{k=1}^{\infty}$ of weights which will be used to define the weight $w$ in Theorem 5.1.75. Fix $k$ and let $w_k^0$ be the uniform measure on $[0,1]$; define $\mathbf{J}_k^1 = \{[1/3,2/3]\}$ to be the middle triadic child of $[0,1]$ and $\mathbf{K}_k^1$ to be all triadic descendants of $[1/3,2/3]$ having length $3^{-k}$. Inductively, we set $\mathbf{J}_k^l = \left\{K^m : K \in \mathbf{K}_k^{l-1}\right\}$ and take $\mathbf{K}_k^l$ to be the collection of all triadic intervals which are contained in $\cup_{J\in \mathbf{J}_k^l} J$ and have length $|K|/3^{lk}$; define $\mathbf{J}_k = \left\{J : J \in \mathbf{J}_k^l \text{ some } l\right\}$, $\mathbf{K}_k = \left\{K : K \in \mathbf{K}_k^l \text{ some } l\right\}$ and $S_k = \cup_{\substack{K\in\mathbf{K}_k \\ J=K^m}} I(J)^m$. With each interval $J$ we associate a sign $\epsilon(J) \in \{-1,1\}$ whose choice we will describe momentarily and an interval $I(J)$; $I(J)$ will be the triadic interval of length $|J|/3^k$ which has as its right endpoint the left endpoint of $J$ if $\epsilon(J) = 1$ and $I(J)$ will be the triadic interval of length $|J|/3^k$ which has as its left endpoint the right endpoint of $J$. For each $l$ we take $w_k^l$ to be the measure which is equal to $w_k^{l-1}$ outside the intervals in $K_l$ and which is measure preserving on $K^m \cup I(K^m)$. Finally, for a given interval $J = K^m$ for some $K$, we choose $\epsilon(J)$ so that the following is satisfied for each $x \in I(J)$

$$\operatorname{sgn}\int_J \frac{w_k^l(y)dy}{y-x} = \operatorname{sgn}\int_{(\cup_{\mathbf{K}_k^l} K')^c} \frac{w_k^{l-1}(y)dy}{y-x} + \sum_{K'\in\mathbf{K}_k^l\setminus K} \int_{K'} \frac{w_k^{l-1}(y)dy}{c(K')-c(J)}.$$

Define $w_k$ to be the weak limit of the sequence $\{w_k^i\}_{i=0}^{\infty}$ and $w(x) = \sum_{k=0}^{\infty} w_k(x-3^k)$. Given the sequence of weights $\{w_k\}_{k=1}^{\infty}$ the following lemma holds

**Lemma 5.3.84.** [36,37] *For $K \in \mathbf{K}_k$ and $J = I^m$ with $k > 3000$,*

$$
\begin{aligned}
|H(w_k)(x)| &\gtrsim \frac{k}{3} w_k(x) \qquad x \in I(J)^m \\
M w_k(x) &\lesssim w_k(x) \qquad x \in I(J).
\end{aligned}
$$

*5.3.1.1 Unboundedness of H*

Let $\epsilon$ satisfy $1/p' < \epsilon < 1$ and set

$$
\begin{aligned}
f(x) &= \sum_{k=1}^{\infty} \frac{1}{k^{\epsilon}} \mathbf{1}_{[3^k, 3^k+1)}(x) \\
\sigma(x) &= \frac{w(x)}{Mw(x)^{p'}} \mathbf{1}_{\text{supp } w}(x).
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_{\mathbb{R}} f(x)^{p'} w &= \int_{\mathbb{R}} \sum_{k=1}^{\infty} \frac{1}{k^{p'\epsilon}} \mathbf{1}_{[3^k, 3^k+1)}(x) w \\
&= \sum_{k=1}^{\infty} \frac{1}{k^{p'\epsilon}}
\end{aligned}
$$

and $f \in L^{p'}(w)$. Additionally, we have

$$
\begin{aligned}
\int_{\mathbb{R}} |H(f)(x)|^{p'} \sigma &= \sum_{k=1}^{\infty} \int_{3^k}^{3^k+1} \left| \sum_{n=1}^{\infty} n^{-\epsilon} H(w_n(\cdot - 3^n))(x) \right|^{p'} \frac{w_k(x-3^k)}{Mw(x)^{p'}} \\
&\gtrsim \sum_{k=1}^{\infty} \int_{3^k}^{3^k+1} \left| \sum_{n=1}^{\infty} n^{-\epsilon} H(w_n(\cdot - 3^n))(x) \right|^{p'} \frac{w_k(x-3^k)}{M(w_k(\cdot - 3^k))(x)^{p'}}
\end{aligned}
\tag{5.3.85}
$$

and for fixed $x$,

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{-\epsilon} H(w_n(\cdot - 3^n))(x) &= \left| H(w_k(\cdot - 3^k))(x) k^{-\epsilon} + \sum_{n \neq k}^{\infty} n^{-\epsilon} H(w_n(\cdot - 3^n))(x) \right|^{p'} \\
&\geq \left| \left| H(w_k(\cdot - 3^k))(x) k^{-\epsilon} \right| - \left| \sum_{n \neq k}^{\infty} n^{-\epsilon} H(w_n(\cdot - 3^n))(x) \right| \right|^{p'}.
\end{aligned}
$$

We claim for $x \in \operatorname{supp} w_k(\cdot - 3^k)$

$$\begin{aligned}\left|\sum_{n\neq k}^{\infty} n^{-\epsilon} H(w_n(\cdot - 3^n))(x)\right| &\leq \mathcal{H}_1(x) + \mathcal{H}_2(x) \\ &\leq 4\end{aligned}$$

where

$$\begin{aligned}\mathcal{H}_1(x) &= \left|\sum_{n=1}^{k-1} n^{-\epsilon} H(w_n(\cdot - 3^n))(x)\right| \\ \mathcal{H}_2(x) &= \left|\sum_{n=k+1}^{\infty} n^{-\epsilon} H(w_n(\cdot - 3^n))(x)\right|.\end{aligned}$$

Consider, if $x \in [3^k, 3^k + 1)$, then for $n \neq k$

$$\begin{aligned}H(w_n(\cdot - 3^n))(x) &= \int_{3^n}^{3^n+1} \frac{w_n(y - 3^n)dy}{x - y} \\ &= \int_0^1 \frac{w_n(y)dy}{x - y - 3^n} \\ &= \int_0^1 \frac{w_n(y)dy}{x' - y + 3^k - 3^n} \qquad (5.3.86)\end{aligned}$$

for some $x' \in [0, 1)$. Provided $n < k$ (5.3.86) is nonnegative and $|x' - y| \leq 2^{-1}(3^k - 3^n)$ so for $y \in [0, 1)$

$$\begin{aligned}\int_0^1 \frac{w_n(y)dy}{x' - y + 3^k - 3^n} &\leq \int_0^1 \frac{2w_n(y)dy}{3^k - 3^n} \\ &\leq \frac{2}{3^k - 3^n} \\ &= \frac{2}{3^n(3^{k-n} - 1)} \\ &\leq \frac{2}{3^{k-n} - 1}\end{aligned}$$

and

$$\left|\sum_{n=1}^{k-1} n^{-\epsilon} H(w_n(\cdot - 3^n))(x)\right| \leq \sum_{n=1}^{k-1} \frac{2n^{-\epsilon}}{3^{k-n}-1}$$
$$\leq \sum_{n=1}^{k-1} \frac{2n^{-\epsilon}}{3^{k-n}-1}$$
$$\leq \sum_{n=1}^{k-1} \frac{4n^{-\epsilon}}{3^{k-n}}$$
$$\leq 4\sum_{n=1}^{\infty} 3^{-n}$$
$$= 2;$$

similarly, for $n > k$ we obtain

$$\left|\sum_{n=k+1}^{\infty} H(w_n(\cdot - 3^n))(x)\right| \leq 2$$

to give the claim. By Lemma 5.3.84, for $x \in S_k$, we have

$$\left|\sum_{n=1}^{\infty} n^{-\epsilon} H(w_n(\cdot - 3^n))(x)\right|^{p'} \frac{w_k(x-3^k)}{M_k(x)^{p'}} \gtrsim |H(w_k(\cdot - 3^k))(x)|^{p'} \frac{w_k(x-3^k)}{M_k(x)^{p'}}$$

where we use the abbreviation $M_k(x) = Mw_k(\cdot - 3^k))(x)$. Consequently,

$$(5.3.85) \gtrsim \sum_{k=1}^{\infty} \int_{S_k} |H(w_k(\cdot - 3^k))(x)|^{p'} \frac{w_k(x-3^k)}{M(w_k(\cdot - 3^k))(x)^{p'}}$$
$$\gtrsim \sum_{k=1}^{\infty} k^{p'-\epsilon p'}$$
$$= \infty.$$

Hence, $H(\cdot w)$ is unbounded as an operator from $L^{p'}(w)$ to $L^{p'}(\sigma)$. As a result of duality, $H$ is also an unbounded operator from $L^p(v)$ to $L^p(w)$.

#### *5.3.1.2 The Boundedness of M*

By the preceding argument for the Hilbert transform, to obtain Theorem 5.1.75 it will suffice to prove the following proposition:

**Proposition 5.3.87.** *For $1 < p < \infty$ and $\omega$ a weight, we have*

$$M : L^p(v) \mapsto L^p(w) \tag{5.3.88}$$

$$M : L^{p'}(w^{1-p'}) \mapsto L^{p'}(\nu), \tag{5.3.89}$$

*with $v = \left(\frac{M\omega}{\omega}\right)^p w$ and $\nu = v^{1-p'}$.*

Prior to proving Proposition 5.3.87, we recall a well-known lemma. Let $\mathcal{D}^{\frac{1}{3}}$ denote the shifted dyadic grid of Michael Christ, i.e.

$$\mathcal{D}^{\frac{1}{3}} = \left\{2^j\left([n, n+1) + (-1)^j 3^{-1}\right) : n, j \in \mathbb{Z}\right\}$$

and for $f \in L^1_{\text{loc}}(\mathbb{R})$ define

$$M^{\text{d}} f(x) = \sup_{I \in \mathcal{D}} \frac{1_I}{|I|} \int_I |f(y)| dy.$$

Equivalently we can define $M^{\text{d},\frac{1}{3}} f$, where the supremum is taken over intervals in $\mathcal{D}^{\frac{1}{3}}$. Then we have from [3],

**Lemma 5.3.90.** *For any finite interval $I$, there exists an interval $I_d \subset \mathcal{D} \cup \mathcal{D}^{\frac{1}{3}}$ such that $I \subset I_d$ and $|I| \approx |I_d|$. As a consequence, for a function $f \in L^1_{loc}(\mathbb{R})$, the following inequality holds:*

$$Mf(x) \lesssim M^{\text{d}} f(x) + M^{\text{d},\frac{1}{3}} f(x). \tag{5.3.91}$$

With Lemma 5.3.90 in hand, we now proceed to the proof of Proposition 5.3.87.

*Proof of Proposition 5.3.87.* The proof of (5.3.88) follows from an extrapolation argument of D. Cruz-Uribe and C. Pérez [7], so we only need to consider (5.3.89). Instead of proving (5.3.89) directly, by (5.2.81), we may verify the following equivalent expression

$$M(\cdot\omega) : L^{p'}(\omega) \to L^{p'}(\nu), \tag{5.3.92}$$

holds. Consideration of Lemma 5.3.90 implies it is sufficient to demonstrate (5.3.92) for an arbitrary dyadic linearization of the maximal function, i.e. we need to show

$$L(\cdot\omega) : L^{p'}(\omega) \mapsto L^{p'}(\nu) \tag{5.3.93}$$

with $L$ a linearization of the maximal function. To this end, let

$$L(f\omega)(x) = \sum_{I\in\mathcal{G}} \langle f\omega\rangle_I 1_{E(I)}(x) \tag{5.3.94}$$

where $\mathcal{G} = \mathcal{D}$ or $\mathcal{D}^{\frac{1}{3}}$ and each $E(I)$ satisfies $E(I) \subset I$ and $E(I) \cap E(\tilde{I}) = \emptyset$ if $I \neq \tilde{I}$. Before doing any computations, we invoke Theorem 5.2.82 which reduces proving (5.3.93) to showing

$$\|1_Q L(1_Q\omega)\|_{L^{p'}(\nu)} \lesssim w(Q)^{\frac{1}{p'}}$$

for $Q$ a dyadic subinterval of $\mathbb{R}$. Now we fix an interval $Q$ and notice that since $E(I) \cap E(Q) = \emptyset$ for $I \neq Q$,

$$\begin{aligned}
\|L(1_Q\omega)\|_{L^{p'}(\nu)}^{p'} &= \int_Q L(1_Q\omega)^{p'}(x)\nu(x) \\
&= \int_Q \left(\sum_{I\in\mathcal{G}} \langle 1_Q\omega\rangle_I 1_{E(I)}(x)\right)^{p'} \nu(x) \\
&= \sum_{I\in\mathcal{G}} \langle 1_Q\omega\rangle_I^{p'} \nu(E(I)\cap Q) \\
&= \sum_{I\in\mathcal{G}} \left(\frac{\omega(I\cap Q)}{|I|}\right)^{p'} \nu(E(I)\cap Q) \\
&= \sum_{\substack{I\in\mathcal{G}\\ I\subset Q}} \left(\frac{\omega(I\cap Q)}{|I|}\right)^{p'} \nu(E(I)\cap Q) + \sum_{\substack{I\in\mathcal{G}\\ Q\subset I}} \left(\frac{\omega(I\cap Q)}{|I|}\right)^{p'} \nu(E(I)\cap Q).
\end{aligned}$$

As $\nu = v^{1-p'} = \left(\frac{1}{M\omega(x)}\right)^{p'} \omega(x)$,

$$\nu(E(I)\cap Q) \;\; \leq \;\; \omega(E(I)\cap Q)\cdot \min\left\{\left(\frac{|I|}{\omega(I)}\right)^{p'}, \left(\frac{|Q|}{\omega(Q)}\right)^{p'}\right\}.$$

Consequently,

$$\begin{aligned}
\sum_{\substack{I\in\mathcal{G}\\ I\subset Q}}\left(\frac{\omega(I\cap Q)}{|I|}\right)^{p'}\nu(E(I)\cap Q) &\le \sum_{\substack{I\in\mathcal{G}\\ I\subset Q}}\left(\frac{\omega(I)}{|I|}\right)^{p'}\left(\frac{|I|}{\omega(I)}\right)^{p'}\omega(E(I)\cap Q)\\
&= \sum_{\substack{I\in\mathcal{G}\\ I\subset Q}}\omega(E(I)\cap Q)\\
&\le w(Q),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\substack{I\in\mathcal{G}\\ Q\subset I}}\left(\frac{\omega(I\cap Q)}{|I|}\right)^{p'}\nu(E(I)\cap Q) &\le \sum_{\substack{I\in\mathcal{G}\\ Q\subset I}}\left(\frac{\omega(Q)}{|I|}\right)^{p'}\left(\frac{|Q|}{\omega(Q)}\right)^{p'}\omega(E(I)\cap Q)\\
&\le \omega(Q)|Q|^{p'}\sum_{\substack{I\in\mathcal{G}\\ Q\subset I}}\frac{1}{|I|^{p'}}\\
&\le 2\omega(Q),
\end{aligned}$$

Thus,

$$\int_Q L(1_Q\omega)^{p'}(x)\nu(x) \le 3w(Q)$$

which implies the desired result and completes the proof of Proposition 5.3.87. □

## 5.4 Proof of Theorem 5.1.78

### 5.4.1 Weight Construction

In this subsection, we emphasize the disparity between the Hilbert transform and the maximal function by presenting a pair of measures $\lambda$ and $\gamma$ for which the Hilbert transform acts continuously while the maximal function is unbounded. The measures which we will use are due to Lacey, Sawyer, and Uriarte-Tuero (see [21]) and we begin by briefly describing their construction. In the interest of clarity we introduce $\gamma$ and some attendant notation by describing the Cantor set's construction. We let $I_1^0 = [0,1]$ and for $1 \le r$ we let $\{I_l^r\}_{l=1}^{2^r}$ denote the $2^r$ closed intervals (ordered left to right) which remain during the $r^{\text{th}}$ stage of the Cantor set's construction; in particular, we have $I_1^1 = [0, \frac{1}{3}]$ and $I_2^1 = [\frac{2}{3}, 1]$, $I_1^2 = [0, \frac{1}{9}]$, $I_2^2 = [\frac{2}{9}, \frac{1}{3}]$, $I_3^2 = [\frac{2}{3}, \frac{7}{9}]$, $I_4^2 = [\frac{8}{9}, 1]$ etc.

For each $I_l^r$, the corresponding open middle third interval which is removed during the $r+1$ stage of construction will be denoted by $G_l^r = (a_l^r, b_l^r)$; so, we have $G_1^0 = (\frac{1}{3}, \frac{2}{3})$, $G_1^1 = (\frac{1}{9}, \frac{2}{9})$, $G_2^1 = (\frac{7}{9}, \frac{8}{9})$ etc. Further, we denote the Cantor set by $E = \cap_{r=1}^{\infty} \cup_{j=1}^{2^r} I_j^r$. The measure $\gamma$ is the Cantor measure, the unique probability measure on $[0, 1]$ which satisfies $\gamma(I_l^r) = 2^{-r}$ for all $r \geq 0$ and $1 \leq l \leq 2^r$.

At this point, we would like to describe the measure $\lambda$. However, prior to doing so, we introduce a lemma which lists important properties of $H(\gamma)$ discussed in [21]:

**Lemma 5.4.95.** *For any $l, r \in \mathbb{N}$, we have the following:*

(i.) $H(\gamma)(x)$ *is decreasing monotonically on* $G_l^r$.

(ii.) $H(\gamma)(x)$ *approaches infinity as* $x$ *approaches* $a_l^r$.

(iii.) $H(\gamma)(x)$ *approaches negative infinity as* $x$ *approaches* $b_l^r$.

By Lemma 5.4.95, for each $r \in \mathbb{N}$ and $1 \leq l \leq 2^r$, there is a point $\zeta_l^r \in G_l^r$ which satisfies $H(\gamma)(\zeta_l^r) = 0$. We define

$$\lambda(x) = \sum_{r=0}^{\infty} \sum_{l=1}^{2^r} \delta_{\zeta_l^r}(x) p_l^r$$

where $p_l^r = \left(\frac{2}{9}\right)^r$ for $r \in \mathbb{N}$ and $1 \leq l \leq 2^r$. With $\lambda$ and $\gamma$ defined, we may now proceed to the proof of Theorem 5.1.78.

### 5.4.2 Verifying $M$ is Unbounded

The verification of (5.1.80) is shown in [21] so to finish the proof of Theorem 5.1.78 we need only consider (5.1.79). We will show for $r \in \mathbb{N}$ and $l = 1$ that $\displaystyle\int_{I_l^r} M(1_{I_l^r}\gamma)(x)^2 d\lambda$ is unbounded. Fix $r \in \mathbb{N}$ and define a collection of sets $\{\mathcal{G}_t\}_{t \in \mathbb{N}}$ in the following way:

$\mathcal{G}_0 = G_1^r$ and $\mathcal{G}_t = \bigcup_{s=1}^{2^{4t}} G_s^{r+4t}$ for $1 \leq t$. Then we have

$$\begin{aligned}\int_{I_1^r} M(1_{I_1^r}\gamma)(x)^2 d\lambda(x) &\gtrsim \sum_{i=0}^{\infty} \int_{\mathcal{G}_i} M(1_{I_1^r}\gamma)(x)^2 d\lambda(x) \\ &= \sum_{i=0}^{\infty} \sum_{s=1}^{2^{4i}} \int_{G_s^{r+4i}} M(1_{I_1^r}\gamma)(x)^2 d\lambda(x). \end{aligned} \tag{5.4.96}$$

But, by inspection

$$M(1_{I_1^r}\gamma)(\zeta_s^{r+4t}) \geq \left(\frac{3}{2}\right)^{r+4t}$$

for $t \in \mathbb{N}$ and $1 \leq s \leq 2^{4t}$. Now, continuing from the above, we obtain

$$\begin{aligned}(5.4.96) &\geq \sum_{i=0}^{\infty} \sum_{s=0}^{2^{4i}} \int_{G_s^{r+4i}} \left(\frac{3}{2}\right)^{2r+8i} d\lambda(x) \\ &\geq \sum_{i=0}^{\infty} \sum_{s=1}^{2^{4i}} p_s^{r+4i} \left(\frac{3}{2}\right)^{2r+8i} \\ &= \sum_{i=0}^{\infty} \sum_{s=1}^{2^{4i}} \left(\frac{2}{9}\right)^{r+4i} \left(\frac{3}{2}\right)^{2r+8i} \\ &= \sum_{i=1}^{\infty} 2^{-r} \\ &= \infty. \end{aligned}$$

Immediately, we have $\int_{I_1^r} M(1_{I_1^r}\gamma)(x)^2 d\lambda(x)$ is unbounded, which completes the proof.

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