RESEARCH REPORT OF YEN DO

This is a research report of Yen Do for the academic activities during the academic year 2010-2011 under an NSF Math Institute Postdoctoral Fellowship awarded by the Institute for Advanced Study (grant DMS-0635607002). The research work was done primarily at Georgia Institute of Technology, the host institution for the fellowship.

RESEARCH THEMES

My research work has been about the applications of real-variable harmonic analysis (in particular, time frequency analysis and oscillatory integrals theory) to nonlinear settings, such as integrable systems. Two main themes in my work are Riemann-Hilbert problems, and variational estimates for multilinear operators appearing in the Taylor expansion of nonlinear operators such as the scattering transform. I recently became interested in questions related to endpoint behaviors of Fourier series, from collaborating with my scientific mentor at Georgia Institute of Technology, Professor Michael Lacey.

TRANSITION IN RESEARCH PUBLICATIONS

The following paper, which I wrote and submitted prior to the academic year 2010-2011, was published online on September 15, 2010:


Abstract: We study the asymptotic behavior of oscillatory Riemann-Hilbert problems (RHPs) arising in the Ablowitz-Kaup-Newell-Segur hierarchy of integrable nonlinear partial differential equations [1]. Our method is based on the Deift-Zhou nonlinear steepest descent method [10] in which the given RHP localizes to small neighborhoods of stationary phase points. In their original work, Deift and Zhou only considered analytic phase functions. Subsequently, Varzugin [33] extended the Deift-Zhou method to a certain restricted class of nonanalytic phase functions. In this paper, we extend Varzugin’s method to a substantially more general class of nonanalytic phase functions. In our work, real variable methods play a key role.

In 2010-2011, together with collaborators I have written the following papers, one of these has been accepted and the other has been submitted:


Abstract: We generalize a family of variation norm estimates of Lépine [22] with endpoint estimates of Bourgain [4] and Pisier–Xu [32] to a family of variational estimates for paraproducts, both in the discrete and the continuous setting. This expands
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on work of Friz and Victoir [15], our focus being on the continuous case and an expanded range of variation exponents.


Abstract: We show that for function $f$ on $[0, 1]$, with $\int |f|(|f| + \log |f| + \log |f| + \log |f|) dx < \infty$, and lacunary subsequence of integers $\{n_j\}$, it holds that $S_{n_j} f \to f$ a.e., where $S_m f$ is the $m$-th Walsh-Fourier partial sum of $f$. According to a result of Konyagin [17, 18, 19], the sharp integrability condition would not have the triple-log term in it. The method of proof uses four ingredients, (1) analysis on the Walsh Phase Plane [21], (2) the new multi-frequency Calderón-Zygmund Decomposition of Nazarov-Oberlin-Thiele [29, 30], (3) a classical inequality of Zygmund [34], giving an improvement in the Hausdorff-Young inequality for lacunary subsequences of integers, and (4) the extrapolation method of Carro-Martín [6], which generalizes the work of Antonov [2] and Arias-de-Reyna [3].

Description of ongoing projects


Description: We prove the following strengthened version of previous results of Lacey-Thiele [20], Muscalu-Tao-Thiele [27], and Oberlin-Seeger-Tao-Thiele-Wright [31]. The estimates in Theorem 1 are also motivated by Terry Lyons’ theory of differential equations driven by rough signals [24].

Theorem 1. For any $2 < r < 4$ and any $r' < p_1, p_2 < r$, $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$, we have

$$\left\| \sup_{K, \xi_1 < \cdots < \xi_K} \left| \sum_{k=1}^K \int_{\xi_{k-1} < \xi_1 < \xi_2 < \xi_k} \hat{F}_1(\xi_1) \hat{F}_2(\xi_2) e^{i(\alpha_1 \xi_1 + \alpha_2 \xi_2) x} d\xi_1 d\xi_2 \right|^{r/2} \right\|_{L^p_{x\xi}} \leq C \|F_1\|_{p_1} \|F_2\|_{p_2}$$

as long as $\alpha_1, \alpha_2$ is not a degenerate pair in the sense that $\alpha_1 \alpha_2 (\alpha_1 + \alpha_2) \neq 0$.

The proof of Theorem 1 requires time-frequency analysis, in which the variational estimates for paraproducts that we proved in [13] are needed for the so-called tree estimates.

5. Y. Do, C. Muscalu, C. Thiele, A Carleson-type theorem for AKNS systems under a symmetric condition, in preparation.

Description: Consider the Ablowitz-Kaup-Newell-Segur system

$$(0.1) \quad \partial_x G(x, k) = e^{ikxJ}W(x) e^{-ikxJ} G(x, k), \quad G(0, k) = Id,$$

where $k \in \mathbb{R}$, $J$ is a constant diagonal matrix with distinct real entries, and the diagonal entries of $W$ are 0, and $Id$ is the identity matrix (c.f. [1]). We prove the following
Theorem, which extends a simpler result in Muscalu-Tao-Thiele [28] where $W$ is assumed triangular.

**Theorem 2.** If the entries of $W$ are in $L^2$ and satisfy the symmetric condition

\[
\forall i \neq j, \quad W_{ij} \equiv 0 \quad \text{or} \quad W_{ji} \equiv 0
\]

then for almost every $k$ we have

\[
\sup_x \|G(x, k)\| < \infty.
\]

For $p < 2$, the simpler $L^p$ variant of Theorem 2 follows from a well-known result of Christ-Kiselev [7], where the symmetric condition (0.2) is not required (c.f. Oberlin-Seeger-Tao-Thiele-Wright [31] for a strengthened version of Christ-Kiselev’s result).

Via T. Lyons’ transfer principle [24], Theorem 2 is indeed an immediate consequence of Theorem 1 and a variational Carleson theorem of Oberlin-Seeger-Tao-Thiele-Wright [31]. Here, we actually prove a quantitative version of Theorem 2 without appealing to T. Lyons’ transfer principle. Removing the symmetric condition on the entries of $W$ would be relevant for the majority of AKNS systems appearing in scattering theory and is an area I will continue to study.


Description: In this project, we plan to extend our result in [12] to the Fourier case. The main task is to adapt the multi-frequency Calderón-Zygmund decomposition of Nazarov-Oberlin-Thiele [29] to the standard time-frequency analysis setting [21] (c.f. [5, 16] and [14]), and there are significant technical difficulties to overcome.


Description:

Let $p_n(x)$ be the normalized $n^{th}$ orthogonal polynomial with respect to $d\mu(x) = e^{-NV(x)}dx$, here $N \in \mathbb{N}$. For real analytic $V$ with sufficient growth, the Deift-Venakides-Zhou nonlinear steepest descent techniques [9] have been successfully used to compute the Plancherel-Rotach asymptotics of $p_n(x)$’s, which are then used to study the eigenvalues distribution of unitary ensembles of $N \times N$ random matrices in the limit $N \to \infty$ (c.f. [8]). McLaughlin and Miller [25, 26] use a $\bar{\partial}$-steepest descent extension of the Deift-Venakides-Zhou method to study these asymptotics of $p_n$’s for $V$ with two Lipschitz derivatives, modulo a set of nondegenerate conditions on the equilibrium measure of $V$. While the related universality results have been proved for non-analytic $V$ with very weak regularity assumptions by Levin and Lubinsky [23] using other approaches, it remains an interesting problem to understand large $n$ behavior of $p_n(x)$’s for rougher $V$’s.

In this project, assuming that $V$ has a non-degenerate equilibrium measure, I plan to extend McLaughlin and Miller’s results to allow for $V$ below two Lipschitz derivatives by adapting the techniques developed in [11]. Removing the nondegenerate assumptions on the equilibrium measure of $V$ is an area that I will continue to study.
**Dissemination**

**Seminars regularly attended.**
- Analysis Seminar at Georgia Institute of Technology.
- Math Physics Seminar at Georgia Institute of Technology.

**Conferences attended and forthcoming conferences.**
- Oberwolfach Meeting (July 2011);
- Completely integrable systems and applications, Erwin Schrödinger Institute (July 2011);
- NSF-CBMS Conference on Global Harmonic Analysis, University of Kentucky (June 2011);
- Analysis and Applications: A Conference in Honor of Elias M. Stein, Princeton (May 2011);
- 27th South Eastern Analysis Meeting and John Conway Day, University of Florida (Mar 2011);
- AMS Sectional Meeting, Statesboro, Georgia Southern University (Mar 2011);
- Workshop on Discrete Methods in Ergodic Theory, Northwestern University (Feb 2011);
- Ohio River Analysis Meeting, University of Cincinnati (Jan 2011);
- Mini Conference in Harmonic Analysis, Auburn University (Nov 2010);
- UCLA Summer/Fall School on weighted estimates for singular integrals, Lake Arrowhead (Oct 2010);
- AMS Sectional Meeting, UCLA (Oct 2010).

**Talks (presented at seminars/conferences).**
- *The lacunary Walsh-Fourier series and convergence near $L^1$:*
  - Oberwolfach Meeting (July 2011);
  - 27th South Eastern Analysis Meeting, Florida (May 2011);
  - Analysis Seminar, Georgia Institute of Technology (Feb 2011);
- *Nonlinear stationary phase for oscillatory Riemann-Hilbert problems:*
  - Completely Integrable Systems and Applications, Erwin Schrödinger Institute (July 2011, invited);
  - Analysis Seminar, University of West Georgia, (Nov 2010, invited);
  - Analysis and Math Physics seminar, Institute for Advanced Study (Oct 2010).
- *Variational estimates for paraproducts:*
  - AMS Sectional Meeting, Statesboro (Mar 2011, invited);
  - Ohio River Analysis Meeting (Jan 2011);
  - Harmonic Analysis Seminar, Louisiana State University (Nov 2010);
  - Mini Conference in Harmonic Analysis, Auburn University (Nov 2010);
  - Analysis Seminar, Cornell University (Nov 2010, invited);
  - Harmonic Analysis and PDE seminar, University of Illinois, Urbana Champaign (Nov 2010, invited);
  - AMS Sectional Meeting, UCLA (Oct 2010, invited);
– Analysis Seminar, Georgia Institute of Technology (Sept 2010).

**Broader impacts**

**Interaction with postdocs.** During my stay at Georgia Institute of Technology, I have had the pleasure of talking and discussing mathematics with other postdocs. I include an example below. After attending a talk at the Analysis Seminar at Georgia Tech, a complex analysis postdoc at Georgia Tech asked me a a real analysis question. He needed to understand this issue for a project that he and his postdoc mentor were working on. It was not straightforward but after thinking about it for some time I was able to answer his question, and thus saving him time on the project.

**Interaction with graduate students.** At Georgia Tech, I attended a course in Analytic Function Theory in the Fall Semester of 2010. Through making comments during lectures, and interacting with the students in and outside the class (these are mostly graduate students in Analysis), I think I contributed to the class environment and helped improve the class’s overall understanding of the materials. Apart from that, I also talk often with the graduate students that I know at Georgia Tech, and share with them any rewarding learning experience that I have.

**References**


