FORMAL METHODS OF VALUE SHARING IN SUPPLY CHAINS

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FORMAL METHODS OF VALUE SHARING IN SUPPLY CHAINS

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To Mom, Dad, Özge, and Serhan

The four people who never failed to support me

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SUMMARY

We consider a decentralized, two-echelon supply chain where the upper echelon – the supplier– bears the inventory risk. To service the retailers, the supplier either keeps inventory reserved for each of her customers or else pools inventory to share among her customers. The common insight regarding inventory pooling is that it reduces costs and so increases profits for the supply chain party carrying inventory. However, it has recently been shown that inventory pooling may indeed reduce the *total* supply chain profits. We further show that inventory pooling may reduce supply chain profits even under traditional service contracts based on the frequently invoked measure of service, probability of stock-out.

We model the inventory transactions among the retailers and the supplier as a cooperative game. The players have the option of reserving inventory or forming inventory-pooling coalitions. The total profit of the coalitions is allotted to the players using a profit-sharing mechanism based on Shapley value. We analyze the properties of the proposed profit-sharing scheme in two steps. We first consider a stylized model with two retailers who are not necessarily identical. Then we extend the analysis to an arbitrary number of identical retailers. In both cases, we assume the demand across retailers is independent.

We find that the Shapley value allocations coordinate the supply chain and are individually rational. However for more than two retailers, they may not be in the core. Even when they satisfy all the stability properties, including membership in the core, they may be perceived unfair since a player's allocation can exceed his contribution to the total supply chain profit. In addition to analyzing the stability properties of the proposed allocation mechanism, we are also interested in the types of behavior the mechanism induces in the players. We find that the retailers prefer pooling partners with either very high or low service level requirements and the supplier prefers retailers with low service requirements since this gives her the ability to maximize her profit allocation. Finally, we analyze the effects of demand variance on the allocations and the profitability of strategic retailer coalitions.

CHAPTER I

INTRODUCTION

1.1 *Conflict of Interests in Supply Chains*

Supply chains, whether in the service sector or the manufacturing industry, are usually made up of players belonging to different companies. Even when one or more of the supply chain players belong to the same company, more often than not each player is a profit center and acts as an independent decision maker. One positive outcome of this type of decentralized management is that it allows each supply chain player to concentrate on his core competencies. The downside of decentralization however is that the authority to decide on supply-chain variables such as price, capacity and order quantity is also decentralized – each is decided on by a different supply chain player who is trying to maximize his profits. This individual profit maximization results in higher prices, lower capacity investments, and lower order quantities and is referred to as *double marginalization*. Spengler [72] characterizes double marginalization for the first time and shows that it leads to higher prices and lower profits.

One of the reasons behind this form of channel conflict is that each party benefits unilaterally from shifting the supply-chain risk to another party even though the overall supply chain performance may suffer as a result. Risk is inherent in each of the supply chain decisions such as capacity and inventory investments, wholesale and retail prices, order quantities, and so on. For example, the party with the autonomy (or authority) to make stocking level decisions can make those decisions in its own best interest, but must usually also accept the burden of any excess inventory. The focus of this research is the conflict of interests stemming from different inventory management policies in a two echelon supply chain (issues related to capacity investment and allocation are similar and most of the insights can be translated to that context). We restrict our attention to the transactions among the retailers and a single supplier in the supply chain.

Traditionally in multi-echelon supply chains, the echelon closest to the end consumer (we call the lowest echelon *the retailer*) owns and carries inventory. This is most often the case, for example, in the apparel industry where product life cycles are short and production lead times are long since manufacturing facilities are mostly outside of the United States. In this industry, retailers place orders prior to the selling season and bear the inventory risk. For example, Cachon [18] discusses the case of O'Neill Inc., a company selling water-sports apparel. However even in the apparel industry, retailers are increasingly forcing suppliers to share the risk through buy back of excess inventories or sharing the costs of discounts. On the other hand, supply chains in the electronics manufacturing service industry, supply chains where vendor managed inventory is employed, or e-supply chains where drop-shipping is the method of order fulfillment are examples where inventory is owned and managed by the upper echelon (which we call the *the supplier*).

In the electronics industry, a manufacturing trend since the 1990s is for the original equipment manufacturers (OEMs) to outsource production to contract manufacturers (CMs). There are only a small number of contract manufacturers and they provide services to OEMs in a wide range of industries such as telecommunications, personal computers, and medical equipment. The products the CMs manufacture for different companies often have common components. For example, the same cpu chip may be used to produce laptops for two competing companies or even products in different industries may have common components. An example is a Bluetooth module for wireless communication which can be used on a range of hosts such as personal computers, cellular phones, and even mobile medical equipment. In a recent survey Barnes et al. [6] discuss the supply chain management trends in the electronics manufacturing service industry and compare the practices of three major CMs – Flextronics, Solectron and JIT Holdings. Some of their observations are of particular interest to us and motivate this research.

- The CMs encourage their suppliers to participate in vendor managed inventory (VMI) programs. Suppliers who participate are required to stock a minimum level of inventory at a warehouse close to the CM facility. The warehouse is usually owned and managed by a third party logistics provider (3PL) and shared by a number of different suppliers. Under the VMI program, the suppliers retain ownership of the inventory until CM sees end customer demand and pulls inventory from the warehouse. Not all suppliers are willing to participate in VMI programs. For example, JIT Holdings reports problems in engaging some of its suppliers in the VMI program.

- Inventory levels are set in an ad hoc manner under the VMI program. The industry standard is to carry a minimum of two weeks' inventory for all parts.

- Suppliers work with many different customers. Considering the fact that the supplier needs to carry 2 weeks' inventory across many customers, the supplier has an opportunity of saving on inventory costs by pooling inventory; however the supplier needs to build a central distribution center rather than keep inventory separately at each of the CMs' warehouses.

- Since product life cycles are short (at the time the report was written the average life cycle of a mobile phone was six months) none of the supply chain partners want to own inventory. After shifting inventory ownership to the suppliers through VMI agreements, the CMs have incentive to inflate the minimum inventory level.

The survey by Barnes et al. alludes to some of the inventory-management related problems between the suppliers and the CMs. Our communications with a major CM suggest that similar problems exist between the OEMs and the CMs as well. Consider a contract manufacturer, who keeps inventory of cpu chips for two or more competing OEMs. The current inventory policy dictated by the OEMs is to keep each company's inventory physically separated. Is this the most profitable inventory policy for the CM? Furthermore, is the most profitable inventory policy for the CM also the most profitable for her customers?

Still another example where similar inventory-management concerns arise is e-supply chains where end consumer orders are satisfied via drop-shipping. In this order fulfillment strategy, the retailers (or rather e-tailers) create demand through their web sites and when demand occurs they pass this information on to the supplier who satisfies it if enough stock is available. The suppliers assume the inventory risk but since they carry the same item for a number of retailers they have the flexibility to pool inventory. Anupindi et al. [4] mention that the order fulfillment company Ingram Entertainment Services, Inc. works with many e-tailers including buy.com but the authors argue that the e-tailers may prefer to keep the inventory of "hot" items physically separated rather than allow Ingram to pool inventory. On the other hand, Netessine and Rudi [54] discuss the drop-shipping agreement between spun.com and Alliance Entertainment Corp. and conjecture that when the supplier works with more than one retailer, benefits of drop-shipping will be even higher (at least from the point of the supplier) due to the risk pooling effect.

Common characteristics of the supply chains discussed in the above examples include

- High inventory risk due to short product life cycles.
- A particular item is procured from a single or at most two suppliers. For example, for each sku, Flextronics tries to use only a single supplier as much as possible [6].
- The supplier works with multiple retailers and a subset of the skus in stock is common to more than one retailer that the supplier works with.
- The supplier can either keep each retailer's stock physically separated or pool the stocks and replenish each of the retailers from this common stock.
- The retailers prefer to keep their stock physically separated as well as shift the inventory risk to the supplier. They try to achieve both aims by engaging the supplier in VMI programs where the retailers determine the minimum acceptable inventory levels and require the supplier to keep the inventory at the retailer facilities which ensures inventory is physically separated.

Among others, Eppen [26] shows that inventory pooling reduces costs for the supply chain party owning the inventory. Therefore it is only natural for the suppliers, as the bearer of inventory risk in all of the above examples, to prefer inventory-pooling. On the other hand, the main reason retailers may object to inventory pooling is the concern regarding inventory allocation in case of shortages. For example, the survey by Barnes et al. already indicates component shortages as one of the biggest problems of the electronics manufacturing service industry as of 2000. They report an incident where a supplier, due to limited product availability, placed its customers including Solectron on "allocation", limiting each customer to only a fixed percentage of the production capacity where the allocation is typically less than the amount each customer needs. The retailers may feel these types of allocation problems can only be elevated if inventory is pooled at the supplier.

Even without mathematical analysis, one observes that there is a misalignment of interests between the retailers and the supplier in the examples we discuss above. The retailers, whose profits are increasing in the level of stock the supplier carries, want the supplier to carry as much stock as possible. On the other hand, the supplier, as the bearer of inventory risk, has incentive to understock and prefers to pool inventory across the retailers. Due to double marginalization, the stock level the supplier prefers to carry is less than and the stock level the retailers want the supplier to carry is greater than the supply-chain-optimal inventory level. In this dissertation we aim to design channel coordinating profit-allocation mechanisms which induce the supplier to carry the supply-chain-optimal inventory level. An acceptable and implementable profit-allocation mechanism must be able to achieve this without making the retailers any worse off. In general we are interested in knowing whether a supplier should pool inventory held for her customers (the retailers). If so, what will be the benefits and how should they be shared over the supply chain in order to give each supply chain player incentive to participate in inventory pooling?

We explore such questions on a 2-echelon supply chain with a single supplier and multiple retailers where only the supplier carries inventory. Inventory is transferred to the retailers without significant transportation delay after the retailers receive end-customer demand. The supplier either keeps reserved inventory for each of her customers or replenishes all customers from a shared inventory. We assume unsatisfied demand is lost and analyze a single period, newsvendor-based model. In the electronics industry, most of the revenue on a product is made in a short selling period. Similarly in the drop-shipping supply chains the retailers, for example those working with Ingram Entertainment, are more concerned about the inventory management policies regarding "hot" products which are usually fashion items with short life cycles. Therefore we believe a single period model with lost sales will capture most of the managerial insights for the problems that motivated this research.

Inventory pooling (or other related types of risk pooling such as substitution or transshipment of stocks) has mostly been analyzed in a single echelon setting (see Section 2.1.2 for a review of the relevant literature). In this dissertation, we consider a two-echelon setting and our focus is on the conflicts of interest between the two-echelons. We model the interactions among the players in this supply chain as a cooperative game. As the name implies cooperative games are games where cooperation among the players are possible since they are assumed to negotiate effectively [53]. Cooperative game theory focuses on the value created by coalitions rather than the specific actions taken by the players (which is the focus in noncooperative game theory). A solution to a cooperative game is an allocation mechanism that distributes the total worth of the coalition to the players. The specific solution methodology we employ in this dissertation is called the Shapley value (see Section 4.4 for a discussion on the choice of Shapley value as the solution concept). Our aim is to assess the applicability of a profit-sharing mechanism based on Shapley value as a way of coordinating the supply chain. We also would like to analyze the types of behavior such a profit-sharing mechanism induces on the players.

1.2 Dissertation Outline and Research Contribution

The remainder of this dissertation is organized as follows. In Chapter 2 we review the relevant literature and position our model. In Chapter 3 we compare two inventory management policies – the reserved and shared inventory management policies – in terms of the total supply chain profit created and how this total pie is distributed to different players. We show that even under a service contract the supplier following a shared inventory policy may have incentive to understock which hurts the retailer profits. We find that the gap between optimal supply chain profit and the supply chain profit when inventory level is set to maximize supplier profit gets larger with increasing demand variance. On the other hand, the gap between optimal supplier profit and the supplier profit when inventory is set to maximize supply-chain profit also gets larger with increasing demand variance. This identifies a growing conflict of interest between the supplier and the supply chain as demand variance increases.

Since we find that service contracts may be inadequate in aligning the incentives of the supply chain members, we model the inventory transactions as a cooperative game between the supplier and the retailers and propose a value-sharing scheme based on Shapley value. In Chapter 4 we discuss some basic concepts from cooperative game theory and the properties of the Shapley value solution concept. We also note several desirable properties of cost (or profit) allocation mechanisms as suggested by previous work in this area and show that the Shapley value satisfies most of these properties.

There exist a wide range of work on contract design and incentive alignment in the context of supply chains. Most of this work is concerned with the equilibrium behavior and the stability properties of the proposed contracts. In our analysis, in addition to dwelling on stability properties, we also investigate the strategic behavior induced by our proposed value-sharing scheme. We are interested in questions such as: With whom is it more profitable to pool inventory? Even when theoretical stability conditions are satisfied, are some players receiving an "unfairly" high allocation? Do the retailers have incentive to form alliances against the supplier?

We carry out the analysis in two steps. In Chapter 5 we consider a stylized, two-retailer supply chain where demand across retailers are independent but not necessarily identical. We show that the retailers prefer pooling partners with either very high or low service requirements. If the supplier is powerful enough to change the terms of the contract to maximize her profits, the retailer with higher demand variance is more susceptible to losing some of his profits. Then in Chapter 6 we allow an arbitrary number of retailers in the supply chain but assume that they are identical. We find that while the supplier always prefers to pool inventory over as many retailers as possible, retailer profit allocations may drop as more retailers enter the pooling coalition. We show that increasing demand variance has opposite effects on the retailers' and the supplier's profit allocations. Surprisingly, the direction of change depends on the supply chain's cost and revenue parameters, especially the markup charged by the retailers. At high markup levels, the retailers' allocations decrease and the supplier's allocation increases with increasing demand variance. Therefore as demand variance increases, the supplier prefers to move to a higher markup environment while the retailers prefer to lower the markup. Additionally, we show that the retailers may actually lose profits by forming alliances against the supplier. At least for a special case where demand is normally distributed, whether the retailers will benefit from alliances or not depends again on the cost and revenue parameters. Increasing markup, which may be viewed as an indication of retailer bargaining power, increases the possibility that retailer alliances will increase retailer profit allocations.

Finally, in Chapter 7 we discuss the limitations of our model and future research directions. The main drawback of the proposed allocation mechanism is that it is a form of revenue sharing. Researchers [22, 12] have argued that revenue-sharing contracts are not prevalent in industry due to the high cost of monitoring revenues. One remedy offered in the literature is to design equivalent contracts that do not require revenue monitoring (for example [20] discusses an option contract). We plan to work on the design of an equivalent contract – possibly based on options – as the first avenue of future research.

CHAPTER II

LITERATURE REVIEW

The research problem we tackle in this dissertation is related to three different avenues of research in the operations research literature.

- Inventory centralization or inventory pooling
- Contract and incentive design for decentralized supply chains
- Cooperative game theory

We review the literature in the first two areas in this chapter and position our model. We discuss the related work in the cooperative game theory literature in Chapter 4 where we talk about our modelling methodology and solution approach. It is worth noting that most of the cost models analyzed up to now in the inventory centralization and supply chain contracts literature are extensions of the classical news vendor problem [47], for which Porteus [61] provides a review.

2.1 Inventory Centralization and Risk Pooling

Eppen [26] is the first to model and analyze the benefits of inventory centralization. He assumes normally distributed demand at each location and proves that inventory costs are reduced due to pooling. Recently Cherikh [21] shows that the same results hold when the objective is profit maximization rather than cost minimization. Eppen and Schrage [27] later extend [26] to the multi-echelon setting and derive optimal ordering policies for the central location. Many streams of research followed from the two seminal papers, [26] and [27], on inventory centralization; however the literature most relevant to our research can be classified under three headings.

- Benefits of inventory centralization and how to share them

- Inventory rationing/transshipment in single echelon supply chains
- Inventory and risk pooling in multi echelon supply chains

2.1.1 Benefits of Inventory Centralization and How to Share Them

An intriguing question within the inventory pooling context is the direction of change in the physical size of inventory. Several researchers attack this question for inventory systems with both common and product-specific components and where common-component inventories are pooled. Baker, Magazine, and Nuttle [5] consider a two product system with service level constraints in which the objective is to minimize total safety stock. They show that total safety stock (common and specialized) drops after pooling; however total stock of product-specific components increases. Gerchak, Magazine, and Gamble [33] extend these results to a profit maximization setting. Finally, Gerchak and Henig [32] extend these models to a multi-period setting and show that myopic policies are optimal for the infinite horizon models. Most of the work on component commonality assumes independent demands. A notable exception is Eynan [29].

The above papers suggest that the size of pooled inventory decreases even though pooling may cause an increase in the non-pooled inventory levels of the complementary components. However, Gerchak and Mossman [34], Pasternack and Drezner [58], and Yang and Schrage [75] show that, contrary to intuition, physical size of *pooled* inventory does not always decrease. These papers all present inventory increase as an undesired outcome of pooling, but an increase in inventory also increases sales and we show that an increase in inventory may be beneficial for the supply chain by increasing total profits. Conversely, Tagaras [74] looks at a two retailer model and shows that if the total reserved safety stock for the two retailers is pooled and used to replenish both of the retailers from a central location, service levels at both of the retailers will increase.

Eppen [26] shows that the benefits of inventory pooling increase with increasing demand variance and decrease with increasing positive demand correlation under the assumption of normally distributed demand. Gerchak and He [31] relax the assumption of normal demand and tackle the question of the effect of demand variance on the benefits of inventory pooling. They show by example that, counter to intuition, benefits of inventory pooling may *decrease* with increasing demand variance. Raghunathan [62] is concerned with the benefits of information rather than inventory sharing but his research is still relevant since he analyzes the effect of demand correlation on the benefits of risk pooling. In a two echelon supply chain, where the benefits of information sharing are allotted to different players using Shapley value, Raghunathan finds that the demand correlation affects the allocations of different players in different directions. Even though Raghunathan uses Shapley value to allocate profits, he is not concerned with the types of behavior this mechanism induces on the players which is one of the main focuses of our research. Finally, most of the work on the benefits of inventory pooling considers benefits in terms of *expected* profits or costs. One notable exception is Benjaafar et al. [7] who prove the benefits of pooling based on a sample path argument.

Another question stemming from the centralization of inventory is how to allocate the cost (or equivalently the benefit) of keeping this centralized inventory to the otherwise decentralized supply chain members. The work up-to-day in this area concentrates mostly on allocation to members in the same echelon. The notable exception is the paper by Raghunathan where the emphasis is on benefits of information rather than inventory sharing. The problem of joint cost allocation has traditionally been viewed as a cost accounting problem (see Biddle and Steinberg [8] for a thorough discussion). It has been introduced to the operations research literature by Gerchak and Gupta [30] where the authors argue that allocating joint costs in proportion to stand-alone costs is the "fair" way. Robinson [63] proposes the analysis of the problem as a cooperative game and compares the allocations based on the Shapley value and the Louderback method with those proposed by Gerchak and Gupta. Hartman and Dror [41, 42] also approach the cost allocation in problem as a cooperative game. They identify efficiency, justifiability and membership in the core as the three properties of stable allocation rules and compare different cost allocation mechanisms with respect to these criteria. With respect to core allocations, Hartman et al. [43] show that the core of the inventory cost game is nonempty under some restricting assumptions on the demand distributions and the cost parameters. Muller et al. [52] extend their results and show that the core is nonempty in general. However, [63], [41], and [42] do not analyze the operational properties of the proposed allocation mechanisms. We point out several shortcomings of the core as a measure of stability and show that even when theoretical stability conditions are satisfied, some players may believe others are getting unfairly high allocations which may make the coalitions unstable in implementation.

2.1.2 Inventory Rationing in Single Echelon Supply Chains

Inventory rationing defines the rules of how to allocate total inventory to n different members of the same echelon of a supply chain in case of a shortage (shortage for all members or shortage for some and overage for others). This can either be done through transshipments among supply chain members carrying decentralized inventory or by defining rules to allocate inventory when it is centralized at a single location. This approach is different from our work in that it concentrates on one of the echelons only. Even though some results and insights extend to two-echelon systems, we show that optimizing a single echelon may lead to inefficiency for the overall supply chain.

There are two different approaches to the rationing problem in the literature. One approach is to consider a multi-period model and analyze the inventory replenishment and rationing policies using dynamic programming. The other approach is to consider a single-period model and analyze the transshipment problem in a game theoretic framework.

Jackson and Muckstadt [44] analyze a two-period model where the centralized inventory can be allocated to the retailers either only at the beginning of the first period or in installments over two periods. They propose approximation algorithms to find the policy parameters (optimal reserve stock levels at the central location) in both cases. Robinson [64] analyzes a similar multi-period, multi-location model and shows that significant savings can be achieved through transshipments even when approximating heuristics, rather than optimal policies, are used to find the order-up-to levels. Contrary to Robinson's model where each location carries its own inventory and transships stock when it has excess inventory while another location faces a stockout, Diks and de Kok [24] consider a model where a central depot carries the inventory and allocates it to the retailers using a rationing policy which they call *consistent appropriate share*. The aim of this rationing policy is to maintain target service levels at each of the retailers. Swaminathan and Srinivasan [73] take a similar service-level-based approach to the allocation of centralized inventory. They derive an optimal allocation policy which randomly allocates stock to the retailers such that the retailers' service level constraints are satisfied in the long run. However, as we show by examples, the shortcoming of the service-level-based allocation mechanisms is that they do not guarantee expected levels of profit for the retailers even when service constraints are met.

Another area where the rationing problem arises is when different demand points differentiated (or rather prioritized) by their service requirements are replenished from the same central inventory. In a series of papers, Ha [38, 40, 39] analyzes a production system modelled as a make-to-stock queue and derives optimal policies characterized by a monotone sequence of rationing levels. Deshpande et al. [23] consider a logistics system with two demand classes and compare the performance of a (Q, r, K) policy with the performance of the optimal priority clearing policy. By a (Q, r, K) policy, they mean that an order of size is Q is placed whenever inventory level drops to r and demand from both of the classes is satisfied on a first come first serve basis until the inventory level reaches K; after which only demand from the higher priority class is satisfied. Zhao et al. [77] prove that a base stock policy is optimal for a similar model with service-differentiated demand where inventory is transshipped between retailers rather than being kept at a central facility. They find that in the optimal policy there is a threshold level of inventory below which a retailer will reject a transshipment request and a second (lower) threshold level below which a retailer will prefer to send the customer to another retailer rather than satisfy the demand in house. These five papers share two common assumptions: 1. For analytical tractability in a multi-period setting, analysis is restricted to Markovian models (except [40]). 2. The papers implicitly assume inventory centralization or transshipment is optimal for the supply chain. They restrict themselves to a single echelon and ignore the possibility that the suppliers who supply the product whose inventory is centralized or the customers who buy the product may prefer decentralized inventories. Even more importantly, the overall supply chain performance may suffer if centralized inventory levels are optimized with respect to only a subset of the supply chain players.

Another stream of papers analyzes the inventory-sharing problem as a single-period transshipment problem among different players at the same echelon, possibly with positive transshipment costs. These papers are more closely related to our work in that they consider decentralized systems, but they differ from our work in that they concentrate on different players within the same echelon. Anupindi, Bassok, and Zemel analyze the problem in a cooperative game theoretic framework [4]. They propose a modified duality-based allocation mechanism that achieves the profit level of the centralized system. Granot and Sosic [36] extend their work by relaxing an assumption on the amount of residual inventory available for transshipments among the retailers. Rudi, Kapur, and Pyke [65] analyze a similar problem with only two retailers. Instead of fixing the transshipment prices like Anupindi et al. do, they let the transshipment prices be variable and try to derive the prices that would coordinate the supply chain. A related problem to the transshipment of stock is the substitution of stock. In this context, the problem is to determine the optimal level of inventory given that a predetermined percentage of the demand is substitutable. Netessine and Rudi [55] settle a long-standing question in the operations literature and derive the necessary (and sometimes sufficient) conditions for the optimal inventory levels for n substitutable products. The conditions are tractable enough to facilitate the comparison of centralized and decentralized inventory policies. The authors show the intuitive result that the total profit of the centralized supply chain is decreasing in demand correlation when demand is multivariate normal.

To our knowledge Parlar [56] provides the first game theoretic analysis of the inventory substitution problem. He proves the existence and uniqueness of a Nash equilibrium for a two-product game in addition to showing that both of the retailers are better off cooperating when compared to their Nash solutions. This work is an extension of Parlar and Goyal [57] where the authors derive the necessary conditions for the optimal levels of inventory for the two substitutable products. Lippman and McCardle [49] and Cachon and Lariviere [19] consider similar problems again in the game theoretic framework. Lippman and McCardle analyze a newsvendor-based model where the retailers compete for the market demand; in such a setting increasing inventory at one retailer stochastically reduces the sales at the other retailers. They show that a Nash equilibrium exists and under at least one of the four demand allocation rules they consider, it is also unique. In Cachon and Lariviere, retailers compete for limited capacity rather than fixed market demand. Cachon and Lariviere consider different capacity-allocation rules and compare them in terms of their equilibrium and truth inducing behavior. They find that all the allocation mechanisms but the uniform allocation rule (where all the retailers get equal capacity) induce the retailers to inflate their orders.

In this study we focus on a single period-problem and use a game theoretic framework to analyze it. Therefore our research is closer to the second set of papers reviewed in this section. However, by optimizing the inventory policy at only a single echelon, all the above work either ignores the effect of inventory centralization on the rest of the supply chain or implicitly assumes it will be beneficial to the supply chain as a whole. Our focus in this research will be to optimize the inventory management policies for the supply chain as a whole and at the same time give adequate incentives to all the players such that they will participate in the optimal inventory management policy. One important observation from the last two papers reviewed is that different allocation mechanisms induce different strategic behaviors – some more favorable than the others – on the supply chain players. However in the operations research literature, the focus is usually to derive the *optimal* policies ignoring the negative behavior they may induce. Our aim is, in addition to designing profit-maximizing allocation mechanisms, to investigate the strategic behavior our proposed mechanisms induce.

2.1.3 Inventory Pooling in Multi Echelon Supply Chains

Lee and Whang [48] provide one of the first analyses of inventory management in decentralized, multi-echelon supply chains. They consider a two-echelon supply chain with one player at each echelon and design a performance measurement scheme based on transfer prices, consignment, reimbursement, and a backlog penalty to coordinate the supply chain.

Cachon and Zipkin [15] consider a similar serial supply chain and show that when each echelon makes the inventory decision to maximize his expected profit (the Nash equilibrium solution), the total supply chain profit is less than optimal. They propose a linear payment scheme based on on-hand inventory and each stage's backorders to align the incentives of the two echelons. Cachon [16] analyzes a similar model but instead of backordering he assumes unsatisfied demand is lost. He finds that a contract based on lost sales transfer payment and inventory holding cost sharing will coordinate the supply chain.

Cachon [17] extends [15] and [16] by allowing multiple retailers in the second echelon who compete for the limited inventory at the supplier. He finds that competition in this decentralized supply chain does not necessarily cause inefficiency but for the cases it does proposes three coordination strategies. Of these three, the one Cachon calls "change control" is of particular interest to us. Change control is a vendor managed inventory (VMI) strategy where the inventory control is delegated to the supplier after all the players agree to fixed transfer payments through which they share the benefits of VMI. This strategy of delegating control to one member of the supply chain after a benefit-allocation mechanism is negotiated is similar to the approach we take in this study. Finally, Cachon [18] classifies different inventory management contracts with respect to their allocation of risk – in a push contract the retailer bears the inventory risk whereas in a pull contract the supplier does. He shows that whether a type of contract (for example a wholesale price contract) coordinates the supply chain depends also on the mode (push or pull) the supply chain operates in.

Work on capacity allocation in the multi-echelon setting is also relevant to this study. Cachon and Lariviere [10] are one of the first to tackle this problem. They analyze an allocation mechanism called "turn-and-earn" which is used by automobile dealerships and allocates capacity using past-sales information. Cachon and Lariviere show that turn-and-earn does not coordinate the system in general and may lead to poor supply chain performance depending on how tight the capacity is. Cachon and Lariviere [11] is a complementing paper to [10] where the authors compare different allocation mechanisms for their truth-inducing behavior. By truth-inducing, the authors refer to allocation mechanisms under which the retailers do not inflate their orders in order to receive larger allocations.

Surprisingly, they find that overall supply chain profits may be lower under truth-inducing allocation mechanisms.

In a recent paper, Plambeck and Taylor [60] discuss the consequences of outsourcing manufacturing to a contract manufacturer as a means of capacity pooling. They analyze the cooperative game between the contract manufacturer and the two original equipment manufacturers and find that the original equipment manufacturers benefit from outsourcing if they have strong bargaining power when compared to the contract manufacturer. On the other hand, another means of capacity pooling – through forming capacity-pooling joint ventures – always increases supply chain profits. This latter capacity pooling arrangement is similar to pooling inventory at a central location and similar to Plambeck and Taylor we show that inventory pooling benefits all members of the supply chain under the right contract.

Of the existing literature, the work that is closest to our work is that of Anupindi and Bassok [2]. They consider a two level supply chain with a single manufacturer and two retailers. Unlike our model, the inventory decision is made by the retailers without constraining service levels and the retailers bear all the inventory risk. They model a supply chain where only a fraction of the customers are willing to wait for a delivery from another retailer. They show that under this setting, the manufacturer may not always benefit from inventory pooling because total sales may drop. They discuss the possibility of optimizing wholesale prices or introducing holding cost subsidies as methods for coordinating the supply chain. Dong and Rudi [25] extend the model of Anupindi et al. [4] to a two echelon supply chain. Similar to our objective, they explore whether transshipments, which are beneficial for the retailers, are also beneficial for the upstream manufacturer. However, in their model the manufacturer does not hold inventory and the retailers make the transshipment decisions. As in our work, Netessine and Rudi [54] consider a model where the supplier bears all the inventory risk. Although they also consider a two-echelon system, the second echelon consists of a single retailer. In their model, the retailer is merely an intermediary between the end customer and the supplier and functions only to expand the customer base through marketing effort. The authors conjecture that the risk-pooling effect that will be observed in the case of multiple retailers will make this kind of business model even more profitable. However, we will show that a supplier who carries out inventory pooling in order to maximize her own profit may actually reduce the total supply chain profit.

2.2 Supply Chain Coordination and Revenue Sharing Contracts

This study may also be considered to lie within the literature on supply chain coordinating contracts, of which the chapter by Cachon [9] provides an excellent review (see especially the second section). The idea of channel coordination was first introduced in the marketing literature by Jeuland and Shugan [45]. In this section we will briefly review the literature on revenue-sharing contracts since the allocation mechanism we propose is a form of revenue sharing.

Dana and Spier [22] provide one of the first analyses of revenue-sharing contracts in the operations research literature. They show that revenue-sharing contracts coordinate the supply chain and are especially valuable when demand is stochastic and inventory levels are chosen before demand is realized. Cachon and Lariviere [12] show in addition that revenue-sharing contracts coordinate the supply chain in situations when other contracts such as buy-back contracts and quantity-flexibility contracts cannot. They also discuss some limitations and implementation costs of revenue-sharing contracts. Gerchak and Wang [35] discuss the benefits of revenue sharing contracts in an assembly setting and show that revenue-sharing contracts will not degrade performance of the supply chain unlike wholesale-price contracts which may. All three of the papers point out that revenue-sharing contracts can coordinate a wider range of supply chain situations when compared to other frequently used types of contracts; however they require costly revenue monitoring and thus are not as prevalent in industry.

Erkoc and Wu [28] and Jin and Wu [46] consider profit (or cost in the case of the former) sharing contracts in the context of capacity expansion and supply auctions respectively. Both papers show that profit-sharing contracts coordinate the respective supply chains. Erkoc and Wu argue that an important drawback of cost-sharing contracts is that cost information at both echelons must be transparent to the complete supply chain. Finally, Granot and Sosic [37] discuss a contract based on side payments which for their model on Internet-based exchanges is equivalent to profit-sharing.

Due to these implementation problems regarding the otherwise desirable revenue-sharing contracts, it is of interest to design equivalent contracts which will create the same share of revenue among the players and thus coordinate the supply chain. Cheng et al. [20] propose an options-based contract which is equivalent to a profit-sharing contract among the supply chain partners and thus coordinates the supply chain. One desirable property of tradeable option contracts as shown by Plambeck and Taylor [59] is that they are designed to anticipate renegotiation and thus lead to higher supply chain profits.

CHAPTER III

CENTRALIZED VS. DECENTRALIZED INVENTORY MANAGEMENT IN MULTI-ECHELON SUPPLY CHAINS

In a multi-echelon supply chain both the total supply chain profit (sum of the individual profits of the players) and the profits of the individual players depend on the inventory management policies followed. Determining the inventory management policy that would maximize the total supply chain profit is an easier task when all the players belong to the same company (an integrated supply chain) than when all the players are profit maximizers belonging to different companies. Due to double marginalization discussed in Chapter 1, when each player makes decisions to maximize its own profit, total supply chain profit falls short of the total profit of the integrated supply chain. We are interested in analyzing the different inventory management policies and the corresponding inefficiencies for this type of decentralized supply chain.

We identify four different inventory management policies for a supply chain with N retailers. These four policies are depicted in Figure 1. In the first policy, which we call a *reserved* inventory management policy, inventory is kept separate for each of the retailers either at the supplier's facility or at the retailers' facilities. Under the *shared or pooled* inventory management policy, all the retailers are replenished from a single, large pool of inventory. Hybrid policies are also possible – two of which we characterize. In the first one (case (c) in the figure), the retailers first draw from their physically separated inventories. However when these run out they can draw from a shared inventory. In the second one (case (d) in the figure) the retailers first draw from a shared inventory and in the case this runs out they can draw from their emergency stocks which are physically separated. These emergency stocks may even be virtual and represent the retailer's option to buy additional units from the supplier. In this chapter we will concentrate on the first two policies and analyze how profits change depending on the inventory management policy followed and also on who makes the inventory-level decision. We first consider supply chain members

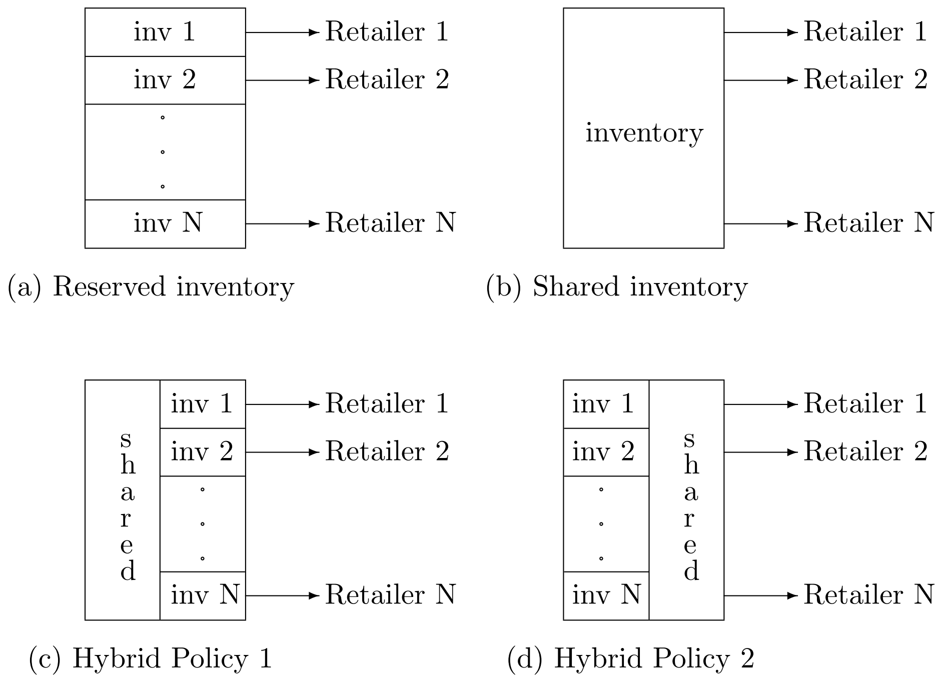


Figure 1: Four different types of inventory policies for the decentralized supply chain

with varying degrees of power, where we take power to be the ability to dictate a strategy of pooling or no pooling. We show that the supply-chain-optimal inventory level cannot be attained under powerful retailers who preclude pooling or a powerful supplier who pools inventory to maximize her profits. Furthermore, retailers may lose profits (compared to the case without inventory pooling) when the supplier pools inventory subject to the retailers' service constraints. We conclude that the frequently used service measure, probability of no stock-out, does not induce supply-chain-optimal inventory levels in the system. We show that another service measure, fill rate, will always coordinate the supply chain but at the expense of increasing computational complexity as the number of retailers increases. Furthermore we find that service contracts may create a conflict of interest among retailers since one retailer may end up subsidizing the service of another.

We also analyze supply chains where the supplier is powerful enough to pool inventory across retailers and set the inventory level as is optimal for herself regardless of retailer concern. We look at the gap between supply chain profit under this scenario and the optimal supply chain profit. We find that the supplier hurts the supply chain more by carrying suboptimal levels of stock when demand is more variable. The common insight (one that is not true in general [31]) about pooling is that it is more valuable when demand is more variable. Our result adds to this intuition that setting the level of pooled inventory optimally is also more valuable when demand is more variable.

3.1 Inventory Pooling: Definitions and Preliminary Results

Consider a supply chain with a single supplier and N retailers (such a supply chain with two retailers is depicted in Figure 2). The retailers require a minimum service level from the supplier and the service level is defined as the probability of no stock-out. For this single period model, probability of no stock-out is the probability that the supplier meets the retailer's demand. Each retailer observes local demand, places an order with the supplier, pays a per-unit-price, and receives the inventory immediately (zero lead-time). The supplier manufactures or buys the product and holds it in inventory at her expense until an order is placed from the retailer(s). The objective of each is to maximize her single period profit. Profit maximization for retailers is equivalent to maximizing their sales since they do not hold inventory.

Let p be the wholesale price the supplier charges to the retailers, c be the procurement/

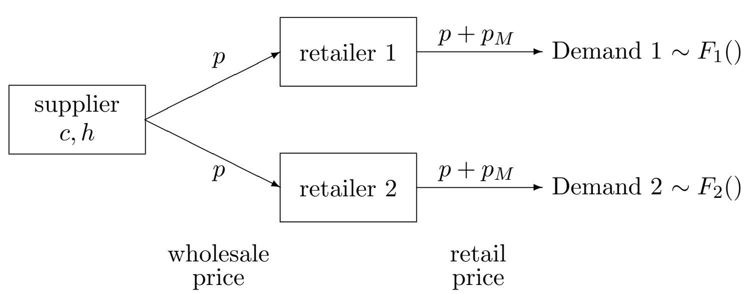


Figure 2: A decentralized 2-echelon supply chain and relevant cost and revenue parameters

manufacturing cost per unit, h be the holding cost per unit (for the single period model we can think of h as the disposal cost), and p_M be the markup on wholesale price the retailers charge. End customer demand is independent at the retailers and we assume the probability distributions of the demand functions are known. Let D_i be the random variable representing the demand at retailers i and $F_i(\cdot)$ denote the cumulative demand distribution for retailer i ($i = 1, 2, \ldots, N$). We assume $F_i(\cdot)$ is strictly increasing and differentiable (with pdf $f_i(\cdot)$ over the interval $[0, \beta)$ where $\beta = \inf\{y : F(y) = 1\}$ (β can be ∞).

We look at the single period inventory holding problem among the supplier and the retailers in two different perspectives: the supplier holds *reserved* inventory separately for each of the retailers or inventory at the supplier is pooled and is *shareable* by the retailers. The total supply chain profit and its allocation among supply chain partners depend on who owns the supplier and the retailers and who makes the pooling decision. We consider the following scenarios:

- When powerful retailers forbid pooling
- When a powerful supplier pools inventory
- When a centralized supply chain makes globally optimal pooling decisions
- When a weak supplier pools inventory subject to a service contract

3.1.1 Powerful Retailers – No Inventory Pooling

In this scenario the retailers are powerful enough to prevent inventory pooling at the supplier. Retailers may insist on a reserved-inventory policy if the product in question is scarce (like Intel chips) and there is ambiguity about how the scarce product would be allocated or if they fear they may be underwriting the service level of a competitor. The objective of the supplier is the maximization of expected profit, which is defined as expected revenue less the expected holding (or disposal) cost and the procurement (or manufacturing) cost subject to the service level constraints. Let x_i be the stock level kept for retailer i, S_i be the expected sales at retailer i given the stock level is x_i (i.e. $S_i = \min[x_i, D_i]$), and H_i ($H_i = \max[x_i - D_i, 0]$) be the corresponding excess stock in retailer i's stock (for notational ease we suppress this dependency on x_i in writing S_i and H_i). For each retailer, the supplier sets inventory levels to maximize profit by solving the problem as stated in Expression 1.

$$\begin{aligned} &\max\ pS_i - hH_i - cx_i \\ &\text{s.t. } F_i(x_i) \geq \underline{\rho}_i \end{aligned} \tag{1}$$

where $\underline{\rho}_i$ = minimum acceptable probability of no-stockout for retailer i (or "service level").

Since the retailers do not hold inventory, their expected revenue is the markup times expected sales. Each retailer's profit is as given in Expression 2.

$$p_M S_i \tag{2}$$

Expression 1, without the service level constraint, is the news vendor problem [71]. It is well-known that the profit-maximizing stocking level for the supplier facing demand with distribution $F(\cdot)$ is $F^{-1}(\frac{p-c}{p+h})$. The optimal stocking level corresponds to a service level of $(\frac{p-c}{p+h})$, which we call the critical ratio. The critical ratio corresponds to the probability of no stock-out, also known as Type-1 service measure. In this paper, unless otherwise specified, service level always denotes Type-1 service level.

The optimal stocking level is $F^{-1}\left(\max\left(\underline{\rho}, \frac{p-c}{p+h}\right)\right)$ when service level constraints are present and the total stock supplier must hold is $\sum_i F_i^{-1}\left(\max\left(\underline{\rho}_i, \frac{p-c}{p+h}\right)\right)$. This means that if the required service level is higher than the critical ratio then the inventory level is found such that the service constraint is binding. Service level is an increasing function of inventory and expected profit is a concave function of inventory. Therefore, whenever the required service level is higher than the critical ratio, the supplier ends up with less than optimum profit. If the service level requirements of the retailers are in the range $\left(0, \frac{p-c}{p+h}\right)$ then it is optimal for the supplier to provide higher than required service. However, beyond $\frac{p-c}{p+h}$, the supplier loses money if she provides higher service to the retailers.

Examining the structure of the optimal decision, one may observe the following:

- When the profit margin of the supplier $(p - c)$ is small or when the holding cost h is large relative to the price p, it is costlier for the supplier to provide higher-than-required service to the retailers. Therefore, utilizing the "optimum" method of pooling becomes more important.

- Like service level, expected sales is an increasing function of total stock level. Therefore, in the region, $\rho \in (0, \frac{p-c}{p+h})$, the retailers' expected sales are greater than or equal to what their service level guarantees them. Beyond $\frac{p-c}{p+h}$, however, they get exactly what they ask for because higher stock levels are not optimal for the supplier.

3.1.2 Inventory Pooling by a Powerful Supplier

When the supplier pools inventory to be shared by the retailers, she effectively makes the inventory decision based on the cumulative demand $F_c(\cdot) = F_1(\cdot) * F_2(\cdot) * \cdots * F_N(\cdot)$. Let S_c be expected cumulative sales, H_c be the expected cumulative excess stock, and x_c be the stock level. The supplier's problem is

$$\max\ pS_c - hH_c - cx_c \tag{3}$$

which has the same news vendor structure as the no-pooling case. The optimum stock level the supplier will carry is $F_c^{-1}(\frac{p-c}{p+h})$.

3.1.3 Centralized Supply Chain Makes Pooling Decision

If both the retailers and the supplier were owned by the same company, the resulting centralized problem would be

$$\max\ (p_M + p)S_c - hH_c - cx_c \tag{4}$$

The centralized system revenue on each unit sold is $p + p_M$. Expression 4 has the form of a news vendor problem and so the optimal stock level is $F_c^{-1}(\frac{p+p_M-c}{p+p_M+h})$. The following observation relates the total stock in the centralized system to the total stock in the decentralized system where the supplier decides on the size of pooled inventory.

Observation 3.1.1 *In a decentralized system, the supplier always stocks less than the system-optimum stock level.*

Comparison of the critical ratio for the centralized system, $\frac{p+p_M-c}{p+p_M+h}$, with the critical ratio for the supplier, $\frac{p-c}{p+h}$, yields that $\frac{p+p_M-c}{p+p_M+h} > \frac{p-c}{p+h}$, which is equivalent to Observation 3.1.1. This is not surprising since it is the supplier who incurs the procurement and holding costs and thus has incentive to understock. This observation also indicates that the decentralized system will not reach its total sales capacity. On the other hand if the stock level is set to that of the centralized system, the supplier will profit less than she would in a decentralized system, where she can set inventory levels in order to maximize her expected profit.

Another important point is that $F_c^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)$ maximizes total supply chain profit profit but may not satisfy the service level requirements for the retailers. This means that enforcement of service level requirements may decrease total system profit. We explore this observation in the next section.

3.1.4 Weak Supplier, Weak Retailers: Inventory Pooling Subject to Service Constraints

Consider a supply chain where the supplier is too weak to make the pooling decision by herself and the retailers are too weak to preclude pooling. Instead, the retailers allow the supplier to pool inventory subject to the service level constraints they set.

Because of competition, retailers may be willing to share some but not all inventory. Thus we may consider the total stock to be broken up into four partitions. The supplier holds two types of inventory for each retailer: shareable and reserved. Shareable inventory may be used to satisfy the other retailers' demands once the demand of the primary inventory owner is satisfied; whereas reserved inventory cannot. For example, if the stock kept for retailer 1 runs out and there is stock available only in the reserved section of the inventory for retailer 2 then this cannot be used to satisfy the unsatisfied demand of retailer 1. Let us define the notation:

$$x_i^r = \text{amount of reserved stock for retailer } i$$

$$x_i^s = \text{amount of shareable stock for retailer } i$$

Total expected sales after pooling and total expected left-over inventory are simply the sum of the individual expected sales and expected left-over inventory figures. Let ρ_i be the Type-1 service level observed at retailer i. The problem of maximizing total profit may be formalized as

$$\max\ pS_c - hH_c - c \sum_{i=1}^{N} (\ x_i^r + x_i^s\ )$$
$$\text{subject to}$$
$$\rho_i \geq \underline{\rho}_i,\ i \in \{1, \ldots, N\}$$

If cost structures are symmetric and there are no extra incentives/costs regarding inventory sharing, we can make the following observation.

Observation 3.1.2 *To maximize total expected profit, one need never hold reserved inventory.*

This result is easy to see since the supplier's profit when $x_1^r = \ldots = x_N^r = 0$ is at least as much as her profit when $x_i^r > 0$ for at least one $i \in \{1 \ldots N\}$. A model similar to our 4-partition model allows only a fraction, $f \leq 1$, of a retailer's demand to be met at another retailer (or in our case using his stocks). This restriction may be due to transshipment delays or a fraction of customers not willing to wait. This differs from our model in that if the total extra demand at the remaining retailers is large enough, regardless of how small f is, the spill-over demand can deplete all extra inventory at retailer j with positive probability. In our model if $x_i^r > 0$ then whether it would be depleted or not depends only on the magnitude of demand at retailer i.

We drop the superscript notation differentiating between reserved and shareable inventory since by Observation 3.1.2 reserved inventory is zero in an optimal solution.

In the remainder of this section, we concentrate on calculating stocking levels after pooling. We first analyze the supplier's problem and ignore the effects on the retailers. It is known that expected profit increases due to pooling. We would also expect total stock level to decrease. However, Gerchak and Mossman [34] give a simple example in which total inventory level after pooling is *higher* than the total inventory level before pooling.

When stocking levels increase, the expected service level provided to the retailers and their expected sales also increase. If the required service level exceeds the critical ratio, the supplier loses money by providing a higher service level. Therefore it is important to calculate the stock levels so that the service level constraints are binding whenever the service level requirements exceed the critical ratio. When we calculate stock levels in this way, we can show that stock levels after pooling do not exceed those before pooling as formalized in Lemma 3.1.1. Under the complete pooling scheme, the probability of no stock-out at retailer i is

$$\rho_i = P(D_i \leq x_i) + P(x_i \leq D_i \leq x_i + x_j - D_j) \tag{5}$$

Lemma 3.1.1 *The after-pooling stock level does not exceed the total before-pooling stock level if the probability of no-stockout after pooling is equal to the probability of no-stockout before pooling for each retailer.*

Proof We find the before-pooling inventory levels x_1 and x_2 as solutions to

$$\rho_i = F_i(x_i) \qquad i = 1, 2 \tag{6}$$

Then defining x'_1 and x'_2 as the after pooling inventory levels and using Equation 5, we obtain the following two equations.

$$\begin{aligned}
\rho_1 &= F_1(x'_1) + P(x'_1 \leq D_1 \leq x'_1 + x'_2 - D_2 \text{ and } D_2 \leq x'_2) \\
&= F_1(x'_1) + \int_0^{x'_2} \int_{x'_1}^{x'_1+x'_2-y_2} f_1(y_1)\, f_2(y_2)\, dy_2\, dy_1
\end{aligned}$$

$$\begin{aligned}
\rho_2 &= F_2(x'_2) + P(x'_1 \leq D_2 \leq x'_1 + x'_2 - D_1 \text{ and } D_1 \leq x'_1) \\
&= F_2(x'_2) + \int_0^{x'_1} \int_{x'_2}^{x'_2+x'_1-y_1} f_2(y_2)\, f_1(y_1)\, dy_1\, dy_2
\end{aligned}$$

For each of these equations, the second term is clearly greater than or equal to zero. By Expression 6 and the fact that $F_1(\cdot)$ and $F_2(\cdot)$ are non-decreasing functions of inventory level $x'_1 \leq x_1$ and $x'_2 \leq x_2$, which proves the claim. □

3.1.4.1 Supplier-Optimal Pooled Stock Size

To avoid excessive inventory costs the supplier should provide no more than the contracted service level when service level requirements are higher than the critical ratio. We make use of this fact to characterize the optimal solution for the supplier in case of pooling subject

Table 1: Optimum pooled inventory level depending on service levels ρ_1 and ρ_2

	$\rho_2 \leq \rho_2^l$	$\rho_2^l < \rho_2 \leq \rho_2^u$	$\rho_1 > \rho_2^u$
$\rho_1 \leq \rho_1^l$	x_c^*	x_c^*	$F_2(x_2^*) = \rho_2, x_1^* = 0$
$\rho_1^l < \rho_1 \leq \rho_1^u$	x_c^*	Requires analysis (1)	Requires analysis (2)
$\rho_1 > \rho_1^u$	$F_1(x_1^*) = \rho_1, x_2^* = 0$	Requires analysis (2)	Solve service level equations

to service constraints. Again we calculate the inventory levels for a 2-retailer supply chain to demonstrate the solution methodology and point out some characteristics of the optimal solution. The characterization also determines the sizes of x_1 and x_2, shareable stock over which retailers 1 and 2 have priority respectively after pooling. The methodology extends to the general $N-retailer$ problem; however calculations become cumbersome as N grows.

Define $x_c^* = F_c^{-1}(\frac{p-c}{p+h})$, the optimum pooled inventory in the absence of service level constraints. Even though we assume complete sharing of available stock by the two retailers, we still distinguish the levels, x_1 and x_2, over which retailers 1 and 2 have priority in case of a stockout, because these levels determine the respective service levels observed at the retailers. By letting $x_1 = x_c^*$ or $x_2 = x_c^*$, we can obtain the boundary values on service level at the two retailers. Further define for $i, j \in \{1, 2\}$

$$\begin{aligned} \rho_i^l &= \text{service level at retailer } i \text{ when } x_i = 0 \text{ and } x_j = x_c^* \\ \rho_i^u &= \text{service level at retailer } i \text{ when } x_i = x_c^* \text{ and } x_j = 0 \end{aligned}$$

With respect to these boundary values, the required service level pair (ρ_1, ρ_2) will fall in one of the nine regions depicted in Table 1. For three of the nine combinations, x_c^* is also a feasible total stocking level given the service level requirements. For the two cases, in which one requirement is below its corresponding lower bound and the other is above its corresponding upper bound, the optimal stocking level is found by solving the service level constraint for the higher service level and setting the other stocking level to zero. In this case, the retailer with the lower service level has no stock over which he has priority. The stock kept for the retailer with the higher service level is used to cover the other retailer. This situation, although optimal for the supplier, may create a conflict of interest between the retailers and therefore may be unacceptable because the retailer with the higher service level requirement is underwriting the service level of the other retailer. For the case where both service level requirements exceed their corresponding upper bounds, the stocking level is found by solving both of the service level constraints as equalities. The solution is optimal because it provides the least stock to satisfy both of the equations.

If the service level pair falls in the region marked by (1) in Table 1, the situation is more complicated: If $F_c^{-1}(\frac{p-c}{p+h})$ can be partitioned such that both of the constraints are satisfied then it is obviously the optimal stock level. This can be checked simply by finding the partition that would still satisfy the service level constraint at the retailer with the higher requirement and then verifying whether the same partition satisfies the service level constraint of the other retailer. If so, then $F_c^{-1}(\frac{p-c}{p+h})$ is the optimal stocking level. If not, the second step is to set $x_i^* = 0$ where i is the retailer with the lower service level requirement and find x_j^* that satisfies retailer j's service level constraint. If x_j^* also satisfies retailer i's constraint, then it is optimal, as established in Lemma 3.1.2. Otherwise, one needs to solve for x_i^* and x_j^* by setting the two service level constraints as equalities. Clearly, providing more service (higher stock levels) is suboptimal.

Lemma 3.1.2 *When $F_c^{-1}(\frac{p-c}{p+h})$ is not feasible, $x_c^* = x_j^* = F_j^{-1}(\rho_j)$, where j is the retailer with the higher service level, is optimal when it is feasible.*

Proof Since the supplier profit is maximized beyond $\frac{p-c}{p+h}$ for the smallest stock level that satisfies the service level constraints, all we need to show is that $x_j^* = F_j^{-1}(\rho_j), x_i^* = 0$ gives a smaller inventory level than having both $x_j^* > 0, x_i^* > 0$. Consider two cases. In the following proof, we use the additional 1 or 2 in the subscript to denote the inventory levels under cases 1 and 2 respectively.

Case 1: Let $x_{i1} = 0$. The inventory level pair (x_{i1}, x_{j1}) are set so as to satisfy the service level constraints. The service level expressions are:

$$\begin{aligned} F_j(x_{j1}) &= \rho_2 \\ \int_0^{x_{j1}} F_i(x_{j1} - y_j)\, f_j(y_j)\, dy_j &\geq \rho_1 \end{aligned} \tag{7}$$

Case 2: Let $x_{i2} > 0$. The corresponding service level expressions are:

$$\begin{aligned} F_j(x_{j2}) + \int_0^{x_{i2}} F_j(x_{i2} + x_{j2} - y_i)\, f_i(y_i)\, dy_i - F_i(x_{i2})F_j(x_{j2}) &= \rho_2 \\ F_i(x_{i2}) + \int_0^{x_{j2}} F_i(x_{i2} + x_{j2} - y_j)\, f_j(y_j)\, dy_j - F_i(x_{i2})F_j(x_{j2}) &\geq \rho_1 \end{aligned} \tag{8}$$

The assumption $x_{i2} \geq 0$ implies $\int_0^{x_{i2}} F_j(x_{i2} + x_{j2} - y_i)\, f_i(y_i)\, dy_i - F_i(x_{i2})F_j(x_{j2}) \geq 0$. Therefore $x_{j1} \geq x_{j2}$. Now let $x_{i2} = x_{j1} - x_{j2}$ and compare the left hand sides of Equations 7 and 8.

$$\begin{aligned} & F_j(x_{j2}) + \int_0^{x_{j1}-x_{j2}} F_j(x_{j1} - y_i)\, f_i(y_i)\, dy_i - F_i(x_{j1} - x_{j2})F_j(x_{j2}) \\ \leq\ & F_j(x_{j2}) + (F_j(x_{j1}) - F_j(x_{j2}))\, F_i(x_{j1} - x_{j2}) \\ =\ & F_j(x_{j1})\ F_i(x_{j1} - x_{j2}) + F_j(x_{j2})\ (1 - F_i(x_{j1} - x_{j2})) \\ \leq\ & F_j(x_{j1}) \end{aligned}$$

which implies that $x_{i2} \geq x_{j1} - x_{j2}$ and thus proves our claim. □

If the the service level pair falls in the region marked by (2), then first find $x_j^* = F_j^{-1}(\rho_j)$ where j again is the retailer requiring the higher service level. If x_j^* is also feasible for retailer i, then $x_j^* = x_c^*$ is the optimal stock level. If not, one needs to solve for x_i^* and x_j^* by setting the two service level constraints as equalities as in the case of (1).

3.1.4.2 Retailer Profits under Pooling

Retailer profits may decrease due to pooling because the total inventory in the supply chain decreases. This phenomenon was first observed by Anupindi and Bassok [2] in a different setting, where the retailers pay the holding cost and it is their decision whether to pool inventory or not. We show by example that this loss cannot be prevented even with the introduction of Type-1 service measure constraints.

Example 3.1.1 *Consider a supply chain with two retailers. Let both demand distributions be $U(0,1)$ and the critical ratio be $\frac{p-c}{p+h} = 0.9$. Then the optimal before-pooling stocking levels are $x_1 = x_2 = 0.9$ with total expected sales at 0.99. The before-pooling service levels at the retailers are each 0.9. We set the Type I service level constraints to 0.9. The* *optimum after-pooling stock level corresponding to* $F_c^{-1}(\frac{p-c}{p+h})$ *is 1.55279. We can assume each retailer has priority over half of the stock since the retailers are identical. This stock level corresponds to a service level of approximately 0.92 (calculated using Equation 5) at each of the retailers, which means service level constraints are more than satisfied. However, the total expected sales is 0.985. The expected profits of the retailers drop even though the service level constraints are satisfied.*

Thus a simple contract between the retailers and the supplier, where the retailers only enforce their expected Type I service levels, is not adequate to protect the retailers from losing sales when the supplier has the power to pool inventory.

3.1.5 Service Contracts and Fill Rate as an Alternative Service Measure

A service contract based on probability of no stock-out does not always guarantee profits for the retailers. Although a more sophisticated service contract based on *fill rate* can achieve this, fill rate has weaknesses that render it less attractive as a basis for a service contract. Fill rate β is defined as the fraction of demand routinely satisfied from shelf:

$$\beta \quad = \quad 1 - \frac{\mathrm{E[shortage]}}{\mathrm{E[demand]}}$$

We also differentiate between fill rate observed at retailer i before and after pooling, β_i^b and β_i^a respectively. Define $E[x_{ji}]$, the expected size of retailer j's shareable stock used by retailer i. Expected before and after-pooling fill rates are

$$\begin{aligned}
\beta_i^b &= 1 - \frac{\int_{x_i}^{\infty}(y_i - x_i) f_i(y_i) dy_i}{\mu_i} \\
&= \frac{S_i^b}{\mu_i} \\
\beta_i^a &= 1 - \frac{\int_{x_i}^{\infty}\int_{x_j}^{\infty}(y_i - x_i) f_i(y_i) dy_i f_j(y_j) dy_j + \int_0^{x_j}\int_{x_i+x_j-y_i}^{\infty}(y_j - (x_i + x_j - y_i)) f_i(y_i) dy_i f_j(y_j) dy_j}{\mu_i} \\
&= \frac{S_i^b + \int_0^{x_j}(1 - F_i(x_i + x_j - y_i)) F_j(y_i) dy_i}{\mu_i} \\
&= \frac{S_i^b + E[x_{ji}]}{\mu_i} \\
&= \frac{S_i^a}{\mu_i}
\end{aligned}$$

Expected fill rate is a function of expected sales when unsatisfied demand is lost. Therefore contracting to assure a minimum expected fill rate guarantees a minimum expected sales level, and thus a minimum profit level, for the retailers. In addition, after pooling, fill rate at retailer i increases by the expected size of retailer j's shareable stock used by retailer i scaled by expected demand.

Fill rate, but not probability of no stock-out, ensures minimum expected sales because fill rate takes into account the size of a shortage when it happens, whereas probability of no stock-out does not. The size of shortages becomes important in evaluating expected sales when sales are lost in case of a stock-out. In addition, the magnitude of probability of stock-out is not a good estimate of the ratio of unsatisfied demand to expected demand [61].

Even though a service contract based on fill rate guarantees a minimum profit level for the retailers in case the supplier pools inventory, it does not necessarily induce the supplier to hold the supply-chain-optimal level of inventory. In addition, due to the dependencies in service levels after pooling, calculations become complicated especially as the number of retailers in the supply chain increases. Another problem with any service contract is you may be underwriting the service of your competitor if he chooses to work at a significantly lower service level. We discuss this problem and the approach we take to circumvent it in the last section of this chapter.

3.2 Supply Chain and Supplier Profits – Inventory Decision by Powerful Supplier vs. Coordinated Inventory Decision

In section 3.1.2 we show that it is optimal for the supplier to pool inventory across retailers and set the total pooled-inventory level at $F_c^{-1}(\frac{p-c}{p+h})$. On the other hand, the optimal inventory policy for the supply chain is also to pool inventory but the optimum inventory level is $F_c^{-1}(\frac{p+p_M-c}{p+p_M+h})$. In supply chains we analyze where a single supplier sells the same product to multiple retailers, the supplier may have monopolistic powers especially if the product is a specialty item. In this case, due to the higher bargaining power of the supplier, she may pool inventory and set the level at $F_c^{-1}(\frac{p-c}{p+h})$. The total supply chain profit falls short of the optimum supply chain profit. We would like to answer the question "How large is the gap between the optimum supply chain profit and the supply chain profit when the supplier sets the pooled-inventory level at $F_c^{-1}(\frac{p-c}{p+h})$?" Even more interesting would be to be able to identify the conditions under which the gap is larger or smaller. This would give insights as to when supply-chain-coordination incentives are more valuable.

3.2.1 Supply Chain Profit as Demand Variance Changes

In this section we concentrate our analysis on the properties of the convolution of the cumulative demand functions, namely $F_c(\cdot)$ since the pooled inventory level is determined with respect to the sum of the demands across the retailers. We drop the c in the subscript since all demand distributions we consider are convolutions. Let us first consider an example where the cumulative demand follows a gamma(α, β) distribution where $(\alpha, \beta) \in \{(1, 16), (2, 8), (4, 4), (9, 16/9), (16, 1)\}$. In the five combinations we consider the mean demand is always 16 and the standard deviation changes from 16 to 4. We keep the mean constant to distinguish the effect of demand variance. We consider two scenarios. In the first, we vary the ratio $\frac{p+p_M-c}{p+p_M+h}$ (which is the supply chain service level across all retailers) on the range $[0.6, 0.9]$ and keep the ratio $\frac{p-c}{p+h}$ (supplier's critical ratio) constant at 0.59. In the second scenario, we vary the ratio $\frac{p+p_M-c}{p+p_M+h}$ on the same range and vary the ratio $\frac{p-c}{p+h}$ on the range $[0.2, 0.8]$. On Figure 3.2.1 we plot the size of the gap between the optimum supply chain profit and the supply chain profit when the supplier sets the inventory level as is optimal for herself under scenario 1. On Figure 3.2.1 we plot the corresponding gap under scenario 2.

We make one common and one conflicting observation on these two graphs. In both of the graphs, the gap increases as coefficient of variation increases for all service levels. At high variability one may expect any sort of pooling (regardless of whether the stock size is determined optimally or not) to benefit the supply chain. However this example demonstrates that when demand is highly variable, the supply chain profit falls short of optimal by a larger amount if inventory level is not set optimally. On the other hand, when demand has low variability the profit loss for the supply chain is relatively smaller even when the supplier sets the inventory level at what is optimal for herself. We establish by Theorem 3.2.1 that this observation is always true under a mild condition regarding the demand functions. The conflicting observation is that in Figure 3.2.1, the gap is the largest when $\frac{p+p_M-c}{p+p_M+h} = 0.9$ and in Figure 3.2.1 the gap is the largest when $\frac{p+p_M-c}{p+p_M+h} = 0.6$. The reason behind this conflicting behavior is that not the absolute size of $\frac{p+p_M-c}{p+p_M+h}$ but rather how different it is from $\frac{p-c}{p+h}$ affects the size of the gap. In both of the graphs the highest curve (representing the largest gap) corresponds to the largest difference between these two critical ratios. This is an intuitive result since two critical ratios that are close together imply that the supplier-optimal inventory level is close to the supply-chain-optimal inventory level.

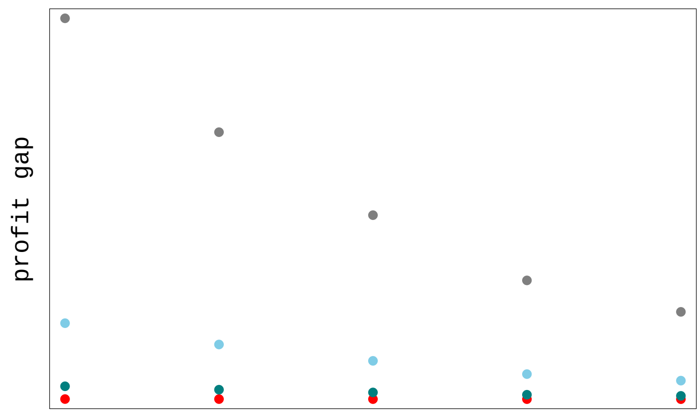


Figure 3: Gap Between Optimal Supply Chain Profit and Supply Chain Profit under Powerful Supplier as Coefficient of Variation of Demand Increases: Case 1 – Supplier Critical Ratio Fixed, Supply Chain Critical Ratio Changing

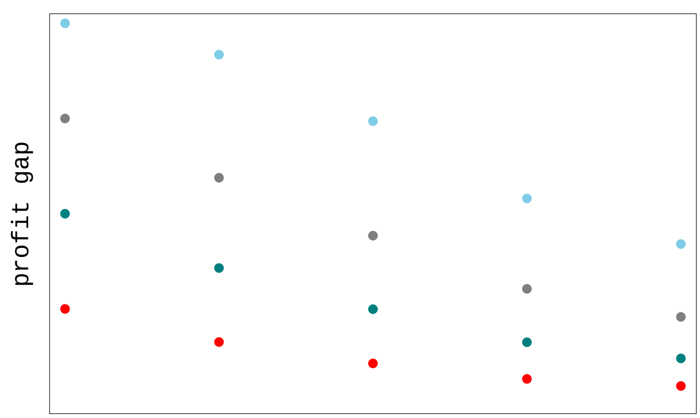


Figure 4: Gap Between Optimal Supply Chain Profit and Supply Chain Profit under Powerful Supplier as Coefficient of Variation of Demand Increases: Case 2 – Both Supplier and Supply Chain Critical Ratio Changing

For two random variables X and Y with distribution functions $F(\cdot)$ and $G(\cdot)$, X is said to be larger than Y in *dispersive order* if $F^{-1}(\beta) - F^{-1}(\alpha) \geq G^{-1}(\beta) - G^{-1}(\alpha)$ whenever $0 < \alpha \leq \beta < 1$ (denoted as $X \geq_{disp} Y$) (Shaked and Shanthikumar [67]). Dispersive order requires the difference between two quantiles of X_i to be smaller than the difference between the corresponding quantiles of X_j; therefore dispersive order compares the variability of the two distributions. Assuming there is dispersive order between the demand distributions, the following theorem shows that the gap is always larger under the more dispersive demand.

Theorem 3.2.1 *The gap between optimal supply chain profit and the supply chain profit when the supplier sets the pooled inventory level as is optimal for herself is greater under the more dispersive demand.*

Proof Assume $D_1 \leq_{disp} D_2$. Let $Q_i^{'} = F_i^{-1}(\frac{p-c}{p+h}) = F_i^{-1}(\alpha)$ and $Q_i^{''} = F_i^{-1}(\frac{p+p_M-c}{p+p_M+h}) = F_i^{-1}(\beta)$ where $i \in \{1,2\}$. We can write supply chain profit for demand D_i as a function of the total inventory in the system as follows:

$$\pi_i(Q_i) = (\ p + p_M + h\ )\mu_i - (\ c + h\ )Q_i - (\ p + p_M + h\ ) \int_{Q_i}^{\infty} (1 - F_i(x))\, dx$$

Our aim is to show that $\pi_1(Q_1^{'}) - \pi_1(Q_1^{''}) \geq \pi_2(Q_2^{'}) - \pi_2(Q_2^{''})$ where for $i \in \{1,2\}$

$$\pi_i(Q_i^{'}) - \pi_i(Q_i^{''}) = (p + p_M + h) \int_{F_i^{-1}(\alpha)}^{F_i^{-1}(\beta)} (-1 + F_i(x))\, dx\ +\ (h + c)(F_i^{-1}(\alpha) - F_i^{-1}(\beta))$$

We need to show that the following holds

$$\begin{aligned}(p + p_M + h) &\left(\int_{F_2^{-1}(\alpha)}^{F_2^{-1}(\beta)} (-1 + F_2(x))\, dx - \int_{F_1^{-1}(\alpha)}^{F_1^{-1}(\beta)} (-1 + F_1(x))\, dx \right) \\ &\geq (h + c) \left(F_2^{-1}(\beta) - F_2^{-1}(\beta) - F_1^{-1}(\beta) + F_1^{-1}(\alpha)\right)\end{aligned} \tag{9}$$

Because $p \geq c$ (otherwise the supplier would not be in business), $p + p_M + h \geq c + h$ and the following is equivalent to expression 9.

$$\begin{aligned}&(p + p_M - c) \left(\int_{F_2^{-1}(\alpha)}^{F_2^{-1}(\beta)} (-1 + F_2(x))\, dx - \int_{F_1^{-1}(\alpha)}^{F_1^{-1}(\beta)} (-1 + F_1(x))\, dx \right) \\ &+ (h + c) \left(F_2^{-1}(\beta) - F_2^{-1}(\beta) - F_1^{-1}(\beta) + F_1^{-1}(\alpha)\right) \\ &+ (h + c) \left(\int_{F_2^{-1}(\alpha)}^{F_2^{-1}(\beta)} (F_2(x))\, dx - \int_{F_1^{-1}(\alpha)}^{F_1^{-1}(\beta)} (F_1(x))\, dx \right) \\ &\geq\ (h + c) \left(F_2^{-1}(\beta) - F_2^{-1}(\beta) - F_1^{-1}(\beta) + F_1^{-1}(\alpha)\right)\end{aligned} \tag{10}$$

where the second term on the left hand side cancels out with the term on the right hand side. Dispersive order implies $F_2(F_2^{-1}(\alpha)+\delta) \leq F_1(F_1^{-1}(\alpha)+\delta)$ for $\delta \in \left[0, F_1^{-1}(\beta) - F_1^{-1}(\alpha)\right]$. Therefore the following inequality holds

$$\int_{F_1^{-1}(\alpha)}^{F_1^{-1}(\beta)} F_1(x)\,dx \geq \int_{F_2^{-1}(\alpha)}^{F_2^{-1}(\alpha)+F_1^{-1}(\beta)-F_1^{-1}(\alpha)} F_2(x)\,dx \tag{11}$$

In addition since $\beta = \frac{p+p_M-c}{p+p_M+h}$ the following inequality holds

$$\begin{aligned}(p+p_M-c)\left(F_2^{-1}(\beta)-F_2^{-1}(\beta)-F_1^{-1}(\beta)+F_1^{-1}(\alpha)\right) \\ \geq \ (\,p+p_M+h) \int_{F_2^{-1}(\alpha)+F_1^{-1}(\beta)-F_1^{-1}(\alpha)}^{F_2^{-1}(\beta)} F_2(x)\,dx\end{aligned} \tag{12}$$

Inequalities 11 and 12 together imply (10) which completes the proof. □

The next result directly follows from Theorem 3.2.1 since for two random variables Y and Z, $Y \leq_{disp} Z$ implies $Var(Y) \leq Var(Z)$.

Corollary 3.2.1 *Assume we can order the total demands in two different markets in the dispersive sense. Then the gap between optimal supply chain profit and the supply chain profit when the supplier sets the inventory level to maximize her expected profit is greater under the more variable demand.*

3.2.2 Supplier Profit as Demand Variance Changes

We go back to the example in Section 3.2.1 and analyze the gap between supplier profit at supply-chain -optimal and supplier-optimal inventory levels. Again we consider two different scenarios depending on whether $\frac{p-c}{p+h}$ is kept constant and obtain Figures 3.2.2 and 3.2.2. We observe that for all of the service levels, the supplier profit is higher when demand coefficient of variation (thus demand variance since means are kept constant) is lower. This is intuitive since the supplier bears the inventory risk in this supply chain. We also look at the gap between the supplier's optimum profit level and her expected profit when the inventory is set at the supply-chain-optimal level. Both of the figures reveal that the gap is larger when demand coefficient of variation is higher. If the other supply chain members force the supplier to carry the supply-chain-optimal level of inventory, she loses more under more variable demand. However these observations are not always true as we demonstrate with the following example. For a special case where the demand follows a normal distribution, we show in Proposition 3.2.1 that the observations hold.

Example 3.2.1 *Assume the supplier operates in two different markets where the demand at market 1 follows a Weibull*$(1, \frac{1}{2})$ *distribution and the demand at market 2 follows a Weibull*$(\frac{1}{3}, \frac{1}{3})$ *distribution. The critical ratio for the supply chain is 0.9. The mean demand at both markets is 2 and the demand variance at market 1 is 20 whereas it is 76 at market 2. When the inventory level is set optimally for the supply chain, the supplier's expected profit in market 1 is* -10.944 *and her expected profit at market 2 is* -9.2958 *so the supplier profit is higher in the more variable market. However as one would expect, if the supplier carries the supplier-optimal level of inventory, her expected profit at market 1 is* 0.024 *and her maximum profit at market 2 is* 0.0017 *– the supplier makes more money in the less variable market. The profit gap is* 10.968 *and* 9.297 *respectively which means the supplier loses more by carrying the supply-chain-optimal inventory level if the demand is less variable.*

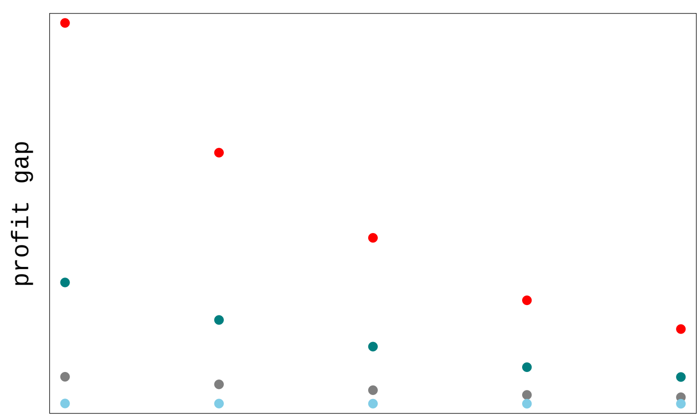


Figure 5: Gap Between Optimal Supplier Profit and Supplier Profit when Supply Chain Profit is Optimized as Coefficient of Variation of Demand Increases: Case 1 – Supplier Critical Ratio Fixed, Supply Chain Critical Ratio Changing

Proposition 3.2.1 *Let the demand in two non-overlapping markets be normally distributed with mean* μ_i *and standard deviation* σ_i *where* $i \in \{1, 2\}$*. Assume* $\mu_1 = \mu_2 = \mu$ *and* $\sigma_1 \leq \sigma_2$*.*

a. Under the supply-chain-optimal inventory level, the supplier's profit is higher in the market with $N(\mu, \sigma_1)$ *demand.*

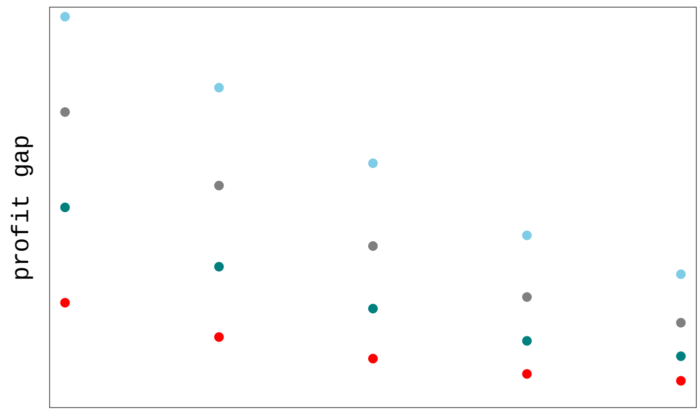


Figure 6: Gap Between Optimal Supplier Profit and Supplier Profit when Supply Chain Profit is Optimized as Coefficient of Variation of Demand Increases: Case 2 – Both Supplier and Supply Chain Critical Ratio Changing

b. The gap between supplier's optimal profit and her profit when the inventory level is set at the supply chain optimal level is larger in the market with $N(\mu, \sigma_2)$ demand.

c. The gap between the optimal supply chain profit and the total supply chain profit when the supplier sets the inventory level to maximize her expected profit is larger in the market with $N(\mu, \sigma_2)$ demand.

Proof For this proof we let u in the subscript denote the unit Normal distribution.

Part a. Using the unit Normal distribution we can write the expected profit in market i as follows (see Appendix A for a detailed derivation)

$$\pi_i = (p+h)\mu - (c+h)(\sigma_i k + \mu) - (p+h)\sigma_i G_u(k)$$

where

$$G_u(k) = \int_k^\infty (u_0 - k) f_u(u_0)\, du_0$$

and the inventory level is set at $\sigma_i k + \mu$ where $k = F_u^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)$. The difference between the expected profits of the two markets is

$$\begin{aligned}\pi_1 - \pi_2 &= -(c+h)k(\sigma_1 - \sigma_2) - (p+h)G_u(k)(\sigma_1 - \sigma_2) \\ &= (c+h)k(\sigma_2 - \sigma_1) + (p+h)G_u(k)(\sigma_2 - \sigma_1) \geq 0\end{aligned}$$

which proves part a.

Part b. Let $k' = F_u^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)$ and $k'' = F_u^{-1}\left(\frac{p-c}{p+h}\right)$. Let $\pi_i^{'}$ and $\pi_i^{''}$ be the expected supplier profit at market i given the inventory level is set at $\sigma_i k'+\mu$ and $\sigma_i k''+\mu$ respectively. Then the difference between the expected profit levels can be written as follows for $i \in \{1, 2\}$.

$$\begin{aligned}
\pi_i^{''} - \pi_i^{'} &= -(c+h)\sigma_i(k''-k') - (p+h)\sigma_i\left(G_u(k'') - G_u(k')\right) \\
&= \sigma_i\left((c+h)(k'-k'') + (p+h)\left(G_u(k') - G_u(k'')\right)\right)
\end{aligned}$$

and this difference is greater than or equal to zero by the definition of $\pi_i^{'}$ and $\pi_i^{''}$. Since $\sigma_1 \le \sigma_2$, $\pi_1^{''} - \pi_1^{'} \le \pi_2^{''} - \pi_2^{'}$ which concludes the proof.

Part c. Proof similar to part b. □

Proposition 3.2.1 indicates a conflict of interest between the supplier and the supply chain. As demand gets more variable, the supplier has less incentive to carry the supply-chain-optimal level of inventory. At the same time, the supply chain is losing increasingly more money due to this behavior. This proposition points out that, at least for normally distributed demand, designing the right incentives to induce the supplier not just to pool inventory but also carry the optimal level of stock becomes increasingly more important as demand variance increases.

3.3 Concluding Remarks

Two common intuitions, both in industry and academia, regarding inventory pooling are that inventory pooling is always beneficial (either reduces costs or increases profits) and that inventory pooling is of greater value when demand is more variable. The academic reasoning behind these intuitions is the fact that traditionally papers analyzing inventory pooling assume normally distributed demand (for example [26], [27]) where both of the intuitions are always correct.

In a multi-echelon supply chain however Anupindi and Bassok [2] were the first to observe that members of one of the echelons or the whole supply chain may lose profits due pooling. In the type of supply chain we consider, it is the retailers who may lose profits if the supplier unilaterally pools inventory. We analyze the observation by Anupindi and

Bassok further and find that even a service contract based on the probability of stock-out does not necessarily guarantee profits for the retailers. A service contract based on fill rate can guarantee profits for the retailers but is not frequently used since fill rate is hard to measure. However even when a service contract is able to guarantee profits such a contract may still be undesirable for a subset of the retailers. In Section 3.1.4 we identify two regions in the service level matrix where it is optimal for the supplier to satisfy the service level constraint of the retailer requiring higher service as an equality. Then without carrying any additional inventory, the supplier can still satisfy the demand from the second retailer requiring a significantly low service level. The retailer requiring the higher service is creating extra sales for the supplier and is in a way underwriting the service of his competitor. However according to the terms of the contract he does not make any additional money and thus may prefer to preclude pooling. We are aware of a similar concern at Home Depot where they believe whenever the Home Depot shipments are less-than-truckload, one of their suppliers fills up the extra space with shipments to a smaller Home Depot competitor. Home Depot was concerned about this arrangement since they suspected the supplier offered a transportation discount to their competitor. At the time of our conversation, Home Depot was trying to preclude this practice. We believe Home Depot may have acted differently had they been receiving benefits from this pooling practice. A similar concern has been voiced by the U.S. Defense Contract Audit Agency. The agency was concerned that the Department of Defense may be subsidizing the inventory costs of the commercial customers of its suppliers. Commercial demand is generally more variable and the suppliers benefit by pooling this demand with the less variable demand from the Department of Defense [1].

We find that, in general, service contracts provide inadequate coordination incentives for pooling since they do not necessarily distribute the total additional value in a manner consistent with the contributions of the supply chain members. Instead we propose a profit allocation mechanism that ensures expected profits of all parties involved in the contract remain at before-pooling levels. The rest of the dissertation deals with the analysis of this profit allocation mechanism.

As for the second intuition regarding inventory pooling, Gerchak and He [31] show by example that benefits of pooling may decrease in increasing demand variance. Similarly we look at the gap between the profits when pooled inventory levels are determined optimally and not. We find that even though loss may be greater under less variable demand, under mild conditions on the demand distributions we are able to show that the gap gets larger in demand variance. Therefore in markets characterized by high demand variance, utilizing suboptimal coordination mechanisms such as service contracts is costlier for the supply chain.

CHAPTER IV

MODELLING METHODOLOGY: COOPERATIVE GAME THEORY

4.1 Cooperative Game Theory and Solution Concepts

We analyze the inventory pooling problem among the retailers and the supplier as a cooperative game, which allows for the possibility of coalitions among players. Coalitions are possible because players are assumed to negotiate effectively with each other and this assumption differentiates cooperative game theory from non-cooperative theory [53]. Non-cooperative game theory is concerned with the specific actions and best responses players take while cooperative game theory focuses on the value created by coalitions of players [14].

Let $N = 1, 2, \ldots, n$ be the set of players and J, $J \subseteq N$, be a coalition of the players. The largest possible coalition, N, is called the grand coalition. What defines a cooperative game (under the transferable utility assumption) is called a *characteristic function* which we denote by v. We let $v(J)$ be the value assigned to coalition J by the characteristic function. For each coalition J, $v(J)$ is the total worth players in J can create by themselves without any help from players in $N \backslash J$. By definition, $v(\emptyset) = 0$.

A solution concept for cooperative games assigns a portion of the total worth of the coalition to each of the players and we call the assignments allocations. An allocation $\phi(v)$ is a vector, where each $\phi_i(v)$ is the payoff to player i for the game characterized by the characteristic function v (we suppress the dependence on v whenever it is clear from the context which characteristic function the allocation depends on). Here we briefly review some of the many allocation rules proposed for cooperative games (for a more detailed discussion we refer the reader to a game theory book such as Myerson [53]).

- **Fractional rule:** This rule allocates a fixed proportion of the grand coalition value to members of the coalition. The fractions can be fixed exogenously or may be derived based on some characteristics of the players. For example in allocating inventory costs, the fractions can be derived as the proportion of demand each player generates (see Gerchak and Gupta [30] for a discussion of this rule). Granot and Sosic [36], Cachon and Netessine [14], and Anupindi et al. [3] also discuss this rule as a possible allocation mechanism for cooperative games within the context of supply chain management.

- **Louderback's allocation rule:** This mechanism allocates the grand coalition's worth such that the allocation to each player is a weighted average of that player's incremental and individual values. The Louderback's allocation to player i is [50]

$$\phi_i = v(N) - v(N\backslash\{i\}) + \alpha(v(\{i\}) - v(N) - v(N\backslash\{i\}))$$
$$\text{where } \alpha = \frac{v(N) - \sum_{k\in N}(v(N) - v(N\backslash\{k\}))}{\sum_{k\in N} v(\{k\}) - \sum_{k\in N}(v(N) - v(N\backslash\{k\}))}$$

  This allocation rule allocates the complete worth of the grand coalition to the players due to the way α is defined. This allocation rule has been considered as a cost allocation rule in the cost accounting literature [8] and is discussed by Robinson [63] and Hartman and Dror [41] as a possible cost allocation rule in the context of inventory centralization.

- **Shapley value:** Shapley [68] derives the expected payoff to player i under characteristic function v, $\phi_i(v)$, as the unique solution to the following axioms. For the second axiom, a *carrier* of v is any set $U \subseteq N$ with $v(S) = V(U \cap S)$, $\forall S \subseteq N$.

  - **Symmetry** For all permutations $\Pi(N)$ of N, $\phi_{\pi i}(\pi v) = \phi_i(v)$ for each permutation π in $\Pi(N)$.
  - **Efficiency** For each carrier U of v $\sum_U \phi_i(v) = v(U)$.
  - **Law of aggregation** $\phi_i(v + w) = \phi_i(v) + \phi_i(w)$.

  The Shapley value as stated in Expression 13 may be interpreted as the expected marginal contribution of player i to a coalition.

$$\phi_i(v) = \sum_{J\subseteq N-i} \frac{\mid J \mid!(\mid N \mid - \mid J \mid -1)!}{\mid N \mid!}(v(J \cup \{i\}) - v(J)) \tag{13}$$

Robinson [63], Hartman and Dror [41], and Granot and Sosic [36] discuss Shapley value as an allocation rule in the supply chain management context.

There are other allocation rules such as nucleolus, stable sets, and kernel that are frequently used in other disciplines. However we do not discuss them in this chapter since they are yet to be used in the supply chain context.

4.2 Stability Properties of Allocation Rules

An allocation rule should satisfy certain properties such that under the allocation rule, the players have incentive to participate in the coalition and no incentive to break away from it. We call these the stability properties of the allocation rules. The first such property we discuss is *individual rationality.*

Definition 4.2.1 *An allocation rule is individually rational if*

$$\phi_i(v) \geq v(\{i\}), \quad \forall i \in N \tag{14}$$

Equation 14 means that when individual rationality is satisfied for the allocation rule, each player's allocation is at least as much as he or she would get without the coalition. Individual rationality corresponds to the participation constraints of the players. When this condition is not satisfied for one or more of the players, they do not have any incentive to participate in the coalition.

The second major stability condition is the *core* which is defined in 4.2.2.

Definition 4.2.2 *An allocation ϕ is said to be in the core of v if and only if*

$$\begin{aligned} \textstyle\sum_{i \epsilon N} \phi_i &= v(N) \\ \textstyle\sum_{i \epsilon J} \phi_i &\geq v(J), \quad \forall J \subseteq N \end{aligned}$$

The first condition in the definition of the core guarantees that the allocation rule distributes the total worth of the grand coalition to the players completely. This condition by itself is known as the *efficiency* property. In terms of profit allocation, non-efficient rules are clearly suboptimal since we would not expect money to be left on the table in an optimal rule. The second condition in the definition of the core means that for any coalition, the total worth of the members under the grand coalition is at least as much as their worth would be if they were to break away from the grand coalition and form a smaller coalition themselves.

A core solution is desirable because it is stable; but the core of a cooperative game may be empty or may not be unique. In addition, even when the core exists, an allocation in the core may have other undesirable characteristics. For example, it may be extreme and/or sensitive to system parameters [53, page 429]. In general, it is hard to determine whether the core of a coalitional game exists or not. Even when it does, the more important question is whether a proposed allocation scheme is actually in the core. We further discuss the applicability of the core as a stability measure in Section 6.3.4.

Another measure, *coalitional monotonicity*, is mentioned by Granot and Sosic [36] and we define it below.

Definition 4.2.3 *An allocation rule is coalitionally monotonic if* $\forall\, i \in N, \forall\, v, w$

$$\left.\begin{array}{rcl} v(S) & \geq & w(S),\ \forall\, i \in S \\ v(S) & = & w(S),\ \forall\, i \notin S \end{array}\right\} \Rightarrow \phi_i(v) \geq \phi_i(w)\ .$$

If the worths of the coalitions containing player i increase while the worths of the coalitions not containing i remain the same, the allocation to player i will not decrease when the underlying allocation rule satisfies coalitional monotonicity. This is an important and desirable property for a profit allocation rule because otherwise players have incentive to reduce their profit contributions to the coalitions in order to increase their allocations [36, 70].

Finally Hartman and Dror [41] introduce the concept of *justifiability.* They originally describe justifiability in the context of cost allocation rules and below we slightly modify their definition and characterize it for profit allocation rules. We first define the benefit game with characteristic function b such that for coalition S

$$b(S) = v(S) - \sum_{j \in S} v(\{i\})$$

By this definition $b(S)$ corresponds to the increase in profit due to forming coalition S.

Definition 4.2.4 *An allocation rule is justifiable if for each player* $i \in N$

$$\phi_i(b) = \phi_i(v) - v(\{i\})$$

Therefore if the same justifiable allocation rule is used to allocate both the profit and the benefit for the same player, the difference is equal to the profit that player can create on his or her own.

4.3 Derivation of the Characteristic Function

In the inventory centralization context, coalitions are formed when a subset of players agree to pool inventory. We denote such a coalition by J and use the subscript notation to represent the elements of set J; that is if $J = \{1, 2, S\}$, $v(J) = v_{12S}$ denotes the expected profit of a coalition consisting of retailers 1 and 2 and the supplier, denoted by S.

For our problem, $v(J)$ is the total expected profit of coalition J and to derive $v(J)$ we note the following key observations:

- Since there is only one supplier, the retailers who do not agree to pool inventory still obtain their stock from the same supplier.
- Again since there is only one supplier, the retailers cannot unilaterally decide to pool inventory without the supplier agreeing to pooling.

In defining $v(J)$ if the supplier is in J, we have to take into account the supplier's profit due to the non-participating retailers as well; otherwise we would be double counting the supplier both as part of the coalition and not. Therefore for each coalition J, $v(J)$ consists of two parts: the total expected profit of the retailers and the supplier in the coalition and the total profit the supplier earns due to the retailers who are not in the coalition. By definition, $v(\emptyset) = 0$. We formalize this characterization in Definition 4.3.1

Definition 4.3.1 *Let S denote the set of coalitions which include the supplier.*
For $J \subset S$ and $\mid J \mid \geq 2$,

$$v(J) = \max_{x_J} \{(p + p_M)S_J - hH_J - cx_J\} + \sum_{k \in N \setminus J} \max_{x_k} \{pS_k - hH_k - cx_k\} \tag{15}$$

For $J \subset S$ and $\mid J \mid < 2$,

$$v(J) = p_M S_k + \sum_{k \in N} \max_{x_k} \{pS_k - hH_k - cx_k\} \tag{16}$$

Otherwise,

$$v(J) = \sum_{k}^{|J|} p_M S_k \tag{17}$$

where

$$x_k = F_k^{-1}\left(\max\left\{\frac{p-c}{p+h}, \rho_k\right\}\right) .$$

The definition of the characteristic function reflects the fact that the worth of a coalition exceeds the sum of the profits of the players in the coalition only if the players include the supplier and at least two retailers. Equation 15 denotes the characteristic function for this case. The first part of the equation denotes the expected profit of the pooling coalition and the second part denotes the expected profit of the supplier due to the retailers who do not participate in the pooling coalition. Equation 16 denotes the characteristic function when the coalition consists of a single retailer and the supplier. In this case, there is no inventory pooling and therefore the supplier's profit equals to the sum of her profits due to each of the retailers. The retailer gets the markup times expected sales. Finally, Equation 17 denotes the characteristic function of a coalition consisting only of retailers. Following from the second observation above, the retailers cannot pool inventory unless the supplier is in the coalition. Therefore the total expected profit in this case is the sum of their individual profits.

A characteristic function is called *superadditive* if and only if for every J and K, $v(J \cup K) \geq v(J) + v(K)$ when $J \cap K = \emptyset$ [53]. We establish that our characteristic function is superadditive with Proposition 4.3.1.

Proposition 4.3.1 *The characteristic function for the inventory holding game as defined in 4.3.1 is superadditive.*

Proof We consider two cases:

Case a. Either J and K consist only of retailers or without loss of generality J consists of the supplier and K consists of a single retailer. We have $v(J \cup K) = v(J) + v(K)$ in both cases since coalitions create additional value only when they consist of at least two retailers *and* the supplier.

Case b. Either J or K contains the supplier and the total number of retailers in $J \cup K$ is at least two. By definition, $v(J \cup K)$ corresponds to expected profit when the inventory level is set optimally. If the inventory level in $J \cup K$ equals the sum of the inventory levels in J and K, the coalition will achieve at least $v(J) + v(K)$ and the result follows. □

The economic implication of superadditivity is that adding one more player to the coalition increases the total worth of the coalition by at least the stand-alone worth of that player.

4.4 Choice of Shapley Value as the Solution Concept

Of the solution concepts we discussed in Section 4.1, we are going to use the Shapley value as the solution concept for our inventory centralization game. This choice depends on both the desirability of the Shapley value axioms for our problem and also the extent to which Shapley value satisfies the stability properties discussed in Section 4.2.

The axioms on which Shapley value is based are meaningful and practical in terms of our problem. We would, for example, expect players of equal power to receive the same allocation and the first axiom ensures that the Shapley value allocation only depends on the contribution of the player to the coalitions. The second axiom makes sure that the Shapley value allocation mechanism allots the total worth of the coalition to the players and a player who is not in the carrier receives zero allocation. Again, in our context we would expect any reasonable allocation mechanism to exhaustively distribute the total profit of the system to the players and to assign zero value to a player who does not increase the value of a coalition. Finally, if the players play two different games with value functions v and w, then the total Shapley value allocation to player i is the same as if the players were to play a game with value function $v + w$. This axiom shows that Shapley value allocations are not dependent on the time of bargaining between the players.

Individual rationality is the bare-minimum stability property that must hold for an allocation rule to be applicable as a contract in the inventory centralization context. The

Shapley value is individually rational if the underlying characteristic function is superadditive. As we show with Proposition 4.3.1, superadditivity holds for our characteristic function. Therefore the Shapley value allocations are individually rational and the players have incentive to participate in the pooling coalitions.

The second stability condition is membership in the core. Without knowing the properties of the characteristic function, it is not possible to tell whether the Shapley value is in the core of the game. We know, for example, that the Shapley value is always in the core when the characteristic function is convex for a benefit game and concave for a cost game [69]. However for our inventory centralization game we cannot prove convexity in general. In Chapters 5 and 6, we discuss the special conditions under which the Shapley value is always in the core of the inventory game and some shortcomings of the core as a stability measure. We would like to note here however that according to Myerson [53] the core is *"derived to avert coalitional objections"* and he calls the core an *unobjectionable* solution concept. He compares the core to a Nash equilibrium and argues that unobjectionable solution concepts may require some knowledge about the strategic behavior of the players. However our aim in this dissertation is not to model the strategic behavior of the players and check whether they reach an equilibrium; rather we aim to design a contract such that each player will receive an allocation consistent with his contribution to the coalition. Myerson categorizes the Shapley value as an *equitable* solution concept meaning the allocation rule distributes the total worth of the coalition to ensure allocations are consistent with each player's contribution. He also argues that equitable solutions are *"arbitration guidelines taking into account power structures represented by the characteristic function"*. Hence the Shapley value is not necessarily an equilibrium but it is an arbitration rule taking into account "equity" and "fairness"; therefore is a suitable profit allocation mechanism to be built in a contract even when it is not in the core.

The other two conditions we defined in Section 4.2 are coalitional monotonicity and justifiability. Shubik [70] and Young [76] show independently that the Shapley value satisfies coalitional monotonicity. In addition Young shows that no core allocation rule satisfies coalitional monotonicity for five or more players. If one would have to choose between a core allocation rule and one that satisfies coalitional monotonicity, we believe that the allocation rule that satisfies coalitional monotonicity would be a better choice to enforce as a contract in the inventory centralization context. Since Shapley value allocations are individually rational, there is incentive for all the players to sign the pooling contract. Once the contract is signed, we may perceive the contract as the means of establishing stability and the fact that the allocation rule is not in the core becomes irrelevant in terms of the stability of the coalition. On the other hand, an allocation rule that does not satisfy coalitional monotonicity may induce a player to increase his profit allocation by reducing his profit contribution which may lead to falsifying profit information and gaming among the players. These types of behavior are almost impossible to control through contracts and may require costly monitoring of individual profits.

Justifiability was introduced by Hartman and Dror [41] since they believe that a cost allocation mechanism must be consistent with a benefit allocation mechanism from an accounting point of view. They prove that Shapley value is a justifiable allocation rule for a characteristic function based on expected costs and it is straightforward to extend their proof to a profit allocation setting. In the profit allocation case, the allocations based on expected total profit and the allocations based on expected increase in profit due pooling differ only by the individual profit the player would have obtained without the coalition. This is consistent with the interpretation of Shapley value as the marginal contribution of that player to the coalition. Justifiability is a desirable property of Shapley value since it establishes the fact that each player's allocation is his expected profit without the coalition plus his contribution to the profit increase in the coalition.

4.5 Concluding Remarks

In an interesting recent survey on game theory as a tool in supply chain analysis, Cachon and Netessine emphasize that cooperative game theory has not received much attention in the supply chain literature in spite of its potential usefulness [14]. In the same chapter, Cachon and Netessine also indicate that the Shapley value has not yet been employed in supply chain research in spite of its desirable characteristics such as uniqueness. Robinson [63] and Hartman and Dror [41] consider Shapley value as a cost-allocation scheme but do not analyze the operational implications of using it. Granot and Sosic [36] appear to have been the first to mention Shapley value as a profit-allocation mechanism that may induce supply-chain-optimal inventory decisions but, as far as we know, this idea has not been followed up.

Shubik [70] is the first to propose using Shapley value as an incentive mechanism in the management of decentralized systems. He argues that the axioms underlying Shapley value coincide with the properties the incentive mechanism requires for the joint cost allocation problem he is analyzing. The axioms of the Shapley value are meaningful and desirable for our inventory centralization problem, too. In addition, we find that the Shapley value allocations satisfy individual rationality, coalitional monotonicity, and justifiability. With such desirable properties, we believe Shapley value is a strong incentive mechanism to induce the players in the decentralized supply chain to take the system-optimal decisions. We offer this research as one initial step in understanding the uses of Shapley value as a value-sharing mechanism to affect the operational decisions of supply chain partners. We analyze the operational characteristics of the Shapley value allocation mechanism in the next two chapters – first for a stylized supply chain of two retailers, then for a general $N - retailer$ supply chain.

CHAPTER V

SHAPLEY VALUE PROFIT ALLOCATIONS: CASE OF 2-RETAILER SUPPLY CHAINS

5.1 Introduction

Consider two retailers selling a single product procured form a single, common supplier. The retailers face uncertain demand and do not carry inventory. When they observe demand, they place an order at the supplier and receive shipments without significant delay. Ownership passes from the supplier to a retailer after the retailer places the order and pays for the product and so the supplier bears all the inventory risk. Sales are lost in case of a stock-out at the supplier. To service the retailers, the supplier may either keep inventory reserved for each of her customers or else pool inventory to share among all of her customers.

Inventory-pooling is known to reduce costs and so increases profits for the supply chain party that owns the inventory, in this case, the supplier [26]. However, the retailers may object to inventory-pooling because of two concerns. First is the concern of how inventory will be allocated among the retailers when there are shortages and how this rationing mechanism will affect their profits. With reserved inventory, the retailer can control his risk of stock-out by specifying minimum-inventory levels to be held by the supplier. But if the retailers draw on a common, pooled inventory, which of the competing retailers has priority when requesting the last of the inventory? Any inventory-pooling contract will need to address this issue either directly (by specifying a stock-rationing mechanism) or indirectly (by specifying reservation profits to the parties such that their profits are at least as much as their before-pooling profits).

The second concern is how much information should be shared in the supply chain to facilitate inventory-pooling. In the case of reserved inventories, each company shares demand information only with the supplier. However, in the case of inventory pooling, a company can, by observing his own service level, infer something about the demand faced by the competitor with whom he is sharing inventory.

In Chapter 2, we considered supply chain members with varying degrees of power and showed that the supply-chain-optimal inventory level cannot be attained under powerful retailers who preclude pooling or a powerful supplier who pools inventory to maximize her profits. In addition, we concluded that a service contract based on the frequently used service measure, probability of no stock-out, does not induce supply-chain-optimal inventory levels in the system.

Instead of a service contract, in this chapter, we propose a *value-sharing method* based on Shapley value from cooperative game theory and derive closed-form expressions of the Shapley values. We find that the Shapley value induces coordination and the allocations under this mechanism satisfy individual rationality conditions for all players and belong to the core of the game. Though stable, an allocation based on Shapley value may induce envy among some players. In particular, we find that the allocation mechanism may be interpreted as "unfair" by some players. We show that the mechanism favors retailers in the sense that retailer allocations may exceed their contribution to total supply chain profit at the expense of the supplier.

Under the proposed contract, the retailers prefer to form pooling coalitions with retailers with either very high or very low service requirements. Up to a threshold service level a retailer prefers to be the one requesting the higher service level because it ensures him the greater share of total profits. Beyond the threshold level a coalition partner with very high service requirements forces the supplier to overstock, increasing sales for both of the retailers. We also show that when the supplier has the power to maximize her profits by manipulating the service levels she provides for the retailers, the retailer with lower demand variance has a better chance of increasing his profits. The Shapley value scheme rewards the retailer introducing less risk into the supply chain and one can reasonably argue that this is "fair".

In this chapter, we explore the Shapley value allocations and their properties in Section 5.2. In Section 5.3, we discuss the possible instabilities that may be caused by the Shapley value allocation scheme. Finally, in Section 5.4, we analyze the question "With whom to form a coalition" from the (different) perspectives of a retailer and the supplier given the service level constraints of each of the retailers. We conclude with a discussion of our findings and future research directions.

5.2 Shapley Value Allocations for Two-Retailer Games

For two retailers and one supplier, the value of the coalition increases only when all three players agree to inventory pooling. Therefore, the value of a 2-player coalition is the sum of the individual expected profits of the players before pooling. This simplifies the calculation of the Shapley value for player i ($i \in \{1, 2, S\}$) to

$$\phi_i(v) = \frac{2}{3}\, v_i + \frac{1}{3}\left(v_{12S} - \sum_{j \in \{1,2,S\}, j \neq i} v_j \right) \tag{18}$$

where v_{12S} is the value of the coalition when all three players agree to pooling and v_1,v_2, and v_S are the individual expected profits of the players before pooling. Equation 18 tells us that in the Shapley value allocation, for each player i, the weight of his contribution to the coalition is half the weight of his before-coalition payoff. The Shapley value formalizes the allocation rule for the total profit to the three players. However to fully characterize the value-sharing mechanism we also need to define a rule for calculating the individual expected profits of the players before pooling. Before pooling, the supply chain has the structure described in Section 3.1.1. Therefore v_1, v_2, v_S are calculated with respect to the stock levels set at $F_i^{-1}\left(\max\left(\underline{\rho}_i, \frac{p-c}{p+h}\right)\right)$ for each retailer.

Writing Expression 18 in a different way, we obtain the equivalent expression

$$\phi_i(v) = v_i + \frac{1}{3}\left(v_{12S} - v_1 - v_2 - v_S\right) \tag{19}$$

which shows that for two retailers, the three players share the extra revenue due to pooling equally. Each player's expected payoff is his expected payoff before pooling plus one third of the increase in total expected system profit due to pooling.

We next establish some stability properties of the Shapley value allocations.

Proposition 5.2.1 *The core of the inventory holding game among the supplier and the two retailers is non-empty.*

Proof We use a theorem by Bondareva (1963) and Shapley (1967) that says that the core of the coalitional game is non-empty if

$$v(N) \quad \geq \quad \textstyle\sum_{J \subseteq N} \lambda(J) v(J) \tag{20}$$

where $\lambda(J)$'s are balanced maps such that $\lambda : 2^N \rightarrow [0, 1]$ and $\sum_{j:i \in J} \lambda(J) = 1,\ \forall i \in N$. In our case, using the subscript notation for $v(S)$ we can write the right hand side of Equation 20 as follows:

$$\begin{aligned}
&\sum_{J \subseteq N} \lambda(J) v(J) \\
&\quad = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_S v_S + \lambda_{12} v_{12} + \lambda_{1S} v_{1S} + \lambda_{2S} v_{2S} + \lambda_{12S} v_{12S} \\
&\quad = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_S v_S + \lambda_{12}(v_1 + v_2) + \lambda_{1S}(v_1 + v_S) + \lambda_{2S}(v_2 + v_S) + \lambda_{12S} v_{12S} \\
&\quad = (1 - \lambda_{12S})(v_1 + v_2 + v_S) + \lambda_{12S} v_{12S}
\end{aligned}$$

Since $v_{12S} \geq v_1 + v_2 + v_S$, $v(N) = v_{12S} \geq (1 - \lambda_{12S})(v_1 + v_2 + v_S) + \lambda_{12S} v_{12S}$ □

Theorem 5.2.1 *The Shapley value allocation scheme induces coordination of the supply chain.*

Proof Using Expression 19, one can see that ϕ_i for $i = 1, 2, S$ is maximized when $v(N) = v_{12S}$ is maximized, which happens when the pooled-inventory level for the 2-retailer coalition is set at the supply chain optimum level. □

Even though the Shapley value profit allocation mechanism coordinates the supply chain, it will not be implementable if the players do not have incentive to participate. Recall from Chapter 4 that in the cooperative game theory context the participation constraints for the players correspond to the individual rationality of the allocations. In Section 4.4, we state that the Shapley value allocations are individually rational for our inventory problem. For completeness, we state this result as Proposition 5.2.2.

Proposition 5.2.2 *The Shapley value allocations for the inventory holding game are individually rational for all of the players.*

Proposition 5.2.3 *The Shapley value allocations are in the core of the inventory holding game.*

Proof By using Expression 19 we can easily verify that the allocations add up to v_{12S}, the value of the grand coalition. In addition, by Theorem 6.2.2 and Proposition 5.2.2, the second condition on the definition of core is satisfied. □

Thus when the Shapley value is used as the profit allocation scheme in a 2-retailer supply chain, the retailers and the supplier have incentive to form pooling coalitions. In addition, the resulting coalition is stable (in the core) and the total joint profit is the maximum the supply chain can attain.

5.3 Induced Instabilities

That the profit allocations under Shapley value allocation scheme are individually rational may not be adequate to prevent what we call *induced instabilities.* These kinds of instabilities may arise if one or more of the players believe there is asymmetric, unfair profit allocation to some other player(s). In cooperative game theory, it is assumed that players would not be willing to deviate from coalitions if individual rationality constraints are satisfied and the allocations are in the core. However, players may hesitate to form coalitions if they believe their competitor benefits more than he should from the coalition. They may require further adjustments to the coalition contract, for example in the form of side payments.

In the remainder of this paper we use the BP and AP notation in the superscript to differentiate the values each variable (such as inventory level, expected sales) takes before pooling and after pooling respectively.

5.3.1 Shapley Value Allocations Favor Retailers

Retailer profit is the product of sales by the mark-up per item and so we define *effective sales* at retailer i as the Shapley value allocation to retailer i divided by the unit mark-up, and

$$\text{E[effective sales at retailer } i] = \frac{\phi_i}{p_M}$$

Comparing total expected effective sales by total expected actual sales after pooling, we can determine whether the retailers get more than their actual contribution to total after-pooling profit, in which case the supplier gets less than her contribution. More specifically, we are interested in knowing when the following inequality occurs:

$$\text{E[total effective sales]} = \frac{\phi_1 + \phi_2}{p_M} > \text{E[total sales after pooling]} \tag{21}$$

Theorem 5.3.1 *Total retailer allocations are greater than actual retailer contribution to after-pooling profit if and only if the expected change in supplier profit exceeds the expected change in average retailer profit.*

In other words, when the change in expected profit for the supplier after pooling is greater than the average change for the retailers, the supplier is forced to give up a portion of her extra profits to the retailers, the size of which is determined by the Shapley value calculations.

Proof In terms of S_1^{AP} and S_2^{AP} Expression 21 is:

$$\frac{\phi_1 + \phi_2}{p_M} \quad \geq \quad \frac{S_1^{AP} + S_2^{AP}}{2} \tag{22}$$

An equivalent expression to (22) is:

$$\begin{aligned} \frac{S_1^{BP} + S_2^{BP}}{3}\left(1 - \frac{2p}{p_M}\right) + \frac{2c}{3p_M}\left[x_1^{BP} + x_2^{BP} - x_1^{AP} - x_2^{AP}\right] \\ + \tfrac{2h}{3p_M}\left(H_1^{BP} + H_2^{BP} - H_1^{AP} - H_2^{AP}\right) \quad \geq \frac{S_1^{AP} + S_2^{AP}}{3}\left[1 - \frac{2p}{p_M}\right] \end{aligned} \tag{23}$$

Change in expected supplier profit exceeding average change in total expected retailer profit is represented as

$$\Delta\text{E[supplier profit]} \geq \frac{\Delta\text{E[total retailer profit]}}{2} \tag{24}$$

Using the definition of E[profit], we can rewrite inequality 24 as follows

$$\begin{aligned} 2p(S_1^{AP} + S_2^{AP} - S_1^{BP} + S_2^{BP}) - 2c(x_1^{AP} + x_2^{AP} - x_1^{BP} - x_2^{BP}) \\ -2h(H_1^{AP} + H_2^{AP} - H_1^{BP} - H_2^{BP}) \quad \geq p_M(S_1^{AP} + S_2^{AP} - S_1^{BP} + S_2^{BP}) \end{aligned} \tag{25}$$

Algebraic manipulation reveals that inequality 25 is equivalent to Expression 22, which proves the claim. $\square$

Even when Expression 21 holds, it is possible that only one of the retailers benefits from the extra allocation:

Example 5.3.1 *Consider two retailers with iid* $U(0,1)$ *demand. Service level is set at* 0.9 *by retailer 1 and at 0.65 by retailer 2. Let* $p = 4$*,* $p_M = 4$*,* $c = 2$*, and* $h = 0.1$*. The ex-post profit allocations are:* $\phi_1 = 2.367776$ *and* $\phi_2 = 1.911776$*. E[total sales after pooling]* $= 0.980813$ *and E[total effective sales] is* $(2.367776 + 1.911776)/2 = 1.012138$*. Comparing the two,* $1.012138 > 0.980813$ *implies that the retailers' total allocation is greater than their total expected profit. In addition, the effective sales for retailer 2 is* $1.911776/4 = 0.477944$*. However,* $0.980813 - 0.477944 > 0.5$*, which implies his effective sales is less than his expected sales (because expected sales at retailer 1 cannot exceed 0.5). Therefore retailer 2's allocation under Shapley value scheme is less than his expected sales revenue after pooling.*

In this example both retailer 2 and the supplier get allocations less than their individual contributions to total after pooling profit, while retailer 1 gets a higher allocation. In this example, this is a fair allocation because retailer 1 requests a higher service level before pooling. Retailer 2, by forming a pooling coalition with retailer 1, gains access to a larger stock but has to to give up some of his profits to retailer 1.

Define the following notation for ease of presentation. Let Λ be the change in the supplier's expected cost and Δ_i be the change in expected sales at retailer i due to pooling.

$$\begin{aligned}
\Lambda &= c\left(x_1^{BP} + x_2^{BP} - x_1^{AP} - x_2^{AP}\right) + h\left(H_1^{BP} + H_2^{BP} - H_1^{AP} - H_2^{AP}\right) \\
\Delta_i &= S_i^{AP} - S_i^{BP}, \quad i \in \{1,2\}
\end{aligned}$$

Proposition 5.3.1 *Given E[total effective sales]* $\geq$ *E[total sales after pooling], if the change in expected sales at retailer* i *is greater than or equal to the change in expected sales at retailer* j *then E[effective sales at retailer* j*]* $\geq$ *E[sales at retailer* j *after pooling].*

Proof In the proof of Theorem 5.3.1 we have established the equivalency of $\frac{\phi_1+\phi_2}{p_M} \geq S_1^{AP} + S_2^{AP}$ to Expression 25. Now rewriting Expression 25 using the Λ and Δ_i notation, we obtain

$$\frac{\phi_1+\phi_2}{p_M} \geq S_1^{AP}+S_2^{AP} \Leftrightarrow 2\Lambda \geq (p_M-2p)(\Delta_1+\Delta_2)$$

Similarly, we can write the following equivalent conditions.

$$\begin{aligned} \frac{\phi_1}{p_M} &\geq S_1^{AP} \Leftrightarrow \Lambda \geq (2p_M-p)\Delta_1-(p_M+p)\Delta_2 \\ \frac{\phi_2}{p_M} &\geq S_2^{AP} \Leftrightarrow \Lambda \geq (2p_M-p)\Delta_2-(p_M+p)\Delta_1 \end{aligned}$$

Without loss of generality, assume $\Delta_1 \geq \Delta_2$. The proposition states $2\Lambda \geq (p_M - 2p)(\Delta_1+\Delta_2)$. This inequality along with $\Delta_1 \geq \Delta_2$ implies $\Lambda \geq (2p_M-p)\Delta_2-(p_M+p)\Delta_1$ which proves the result. □

Proposition 5.3.1 says that the expected change in retailer i's sales after pooling is greater than the change in retailer j's sales ensures that retailer j's final profit allocation will correspond to an effective sales level higher than his expected sales. However the same condition is not adequate to ensure the same for retailer i. This result is counterintuitive because we would normally expect retailer i would be ensured a greater portion of the extra profit due to pooling since he is making the more positive impact on expected sales.

5.4 With Whom to Form a Coalition?

In the previous section, we have shown that even though the Shapley value allocation scheme ensures profit allocations higher than before-pooling profit levels for all players, some players may get more favorable allocations. Therefore it is important for all players to know with whom it is most advantageous to form pooling coalitions. In this section, we analyze this question from the points of view of the retailers and the supplier separately. We take required service level and the demand distribution as the defining characteristics of the retailers. Cost and revenue parameters are still assumed to be identical for both of the retailers.

5.4.1 The Retailer's Perspective

The question we seek to answer is: "Given a fixed service level for retailer i, at what service level for retailer j would retailer i form a coalition with retailer j?" Throughout this section we make use of the following rule in the contract: before-pooling profit levels, v_i, v_j, and v_S

are calculated with respect to the stock levels set at $F_i^{-1}\left(\max\left(\underline{\rho}_i, \frac{p-c}{p+h}\right)\right)$ for each retailer. Therefore, our region of interest is $\rho_j \in (\frac{p-c}{p+h}, 1)$ because in the region $(0, \frac{p-c}{p+h}]$ the stock level is set at $F_j^{-1}(\frac{p-c}{p+h})$ regardless of the service level requirement. When the stock level for retailer j is fixed at $F_j^{-1}(\frac{p-c}{p+h})$, the service level requirement of retailer j does not have an impact on the ex-post profit allocation to retailer i.

Theorem 5.4.1 *The Shapley value profit allocation to retailer i is a unimodal function of service level ρ_j of retailer j. In addition, $\rho_j^* = \frac{p+p_M-c}{p+p_M+h}$ is the global minimizer.*

Proof Let π_c^{AP} be the expected supply chain profit after pooling and π_S^{BP} be expected supplier profit before pooling. Rewriting Expression 18, the Shapley value allocation to retailer i is

$$\phi_i = \tfrac{2}{3}S_i^{BP} + \tfrac{1}{3}(\pi_c^{AP} - \pi_S^{BP} - p_M S_j^{BP})$$

By definition, only the last two terms of the above equation depend on ρ_j. Let $x_j(\rho_j)$ be the before-pooling stocking level for retailer j as a function of the service level. Then, $x_j(\rho_j) = F_j^{-1}(\rho_j)$. Let $\Omega = \frac{\partial F_j^{-1}(\rho_j)}{\partial \rho_j}$. Then,

$$\frac{\partial \phi_1(\rho_j)}{\partial \rho_j} = -\tfrac{\Omega}{3}(-c - h\rho_j + (p + p_M)(1 - \rho_j))$$

Due to the assumptions we made on $F(\cdot)$, Ω is always positive. When $\rho_j < \frac{p+p_M-c}{p+p_M+h}$, then $\frac{\partial \phi_i(\rho_j)}{\partial \rho_j}$ is negative which means the function is decreasing and when $\rho_j > \frac{p+p_M-c}{p+p_M+h}$, the derivative is positive, which means the function is increasing. Therefore, the function is unimodal and $\rho_j^* = \frac{p+p_M-c}{p+p_M+h}$, is the global minimizer. □

In all examples we studied, $\phi_i(\rho_j)$ has always been a convex function. However, we could not prove this in general because $\frac{\partial^2 \phi_1(\rho_j)}{\partial \rho_j^2}$ is a function of $\frac{\partial^2 F_2^{-1}(\rho_j)}{\partial \rho_j^2}$, which is difficult to sign. However, proving unimodality is sufficient for our purposes because the interesting point in this proposition is that the ex-post profit allocation to a retailer decreases if he forms a coalition with a retailer with service level in the range $(\frac{p-c}{p+h}, \frac{p+p_M-c}{p+p_M+h})$.

The next natural question is whether there is a threshold service level ρ_j in the region $(\frac{p+p_M-c}{p+p_M+h}, 1)$ beyond which $\phi_i(\rho_j)$ is greater than $\phi_i(\frac{p-c}{p+h})$. The answer is "not necessarily".

Proposition 5.4.1 *When the demand distribution for retailer j has infinite support, then the ex-post profit allocation for retailer i goes to infinity as ρ_j goes to 1.*

Proof The Shapley value allocation to retailer i as a function of the service level of retailer j is

$$\begin{aligned}
\phi_i(\rho_j) &= \frac{2}{3}S_i^{BP} + \frac{1}{3}(\pi_c^{AP} - \pi_S^{BP} - p_M S_j^{BP}) \\
&= \frac{2}{3}S_i^{BP} + \frac{1}{3}\left(\pi_c^{AP} - \left(p(S_i^{BP} + S_j^{BP}) - h(H_i^{BP} + H_j^{BP}) - c(x_i + x_j)\right) - p_M S_j^{BP}\right) \\
&= K - \frac{1}{3}\left[(p + p_M - c)F_j^{-1}(\rho_j) - (p + p_M + h)\int_0^{F_j^{-1}(\rho_j)} F_j(x)dx\right]
\end{aligned}$$

where the term K represents the part of the $\phi_i(\rho_j)$ expression that does not depend on ρ_j and K is a function of ρ_i, p, p_M, h, and c. We can find the limit of the term in the parenthesis when $F_j(\cdot)$ has infinite support as follows:

$$\begin{aligned}
&\lim_{\rho_j \to 1}\left[(p + p_M - c)F_j^{-1}(\rho_j) - (p + p_M + h)\int_0^{F_j^{-1}(\rho_j)} F_j(x)dx\right] \\
&= \lim_{\rho_j \to 1}\left[(p + p_M + h)\int_0^{F_j^{-1}(\rho_j)} (1 - F_j(x))\,dx - (h + c)F_j^{-1}(\rho_j)\right] \\
&= (p + p_M + h)E[x] - (h + c)\lim_{\rho_j \to 1} F_j^{-1}(\rho_j) \\
&= -\infty
\end{aligned}$$

This implies $\lim_{\rho_j \to 1} \phi_i(\rho_j) = \infty$. □

Thus when $F_j(\cdot)$ has infinite support there is a range of ρ_j beyond $\frac{p+p_M-c}{p+p_M+h}$, where $\phi_i(\rho_j)$ is greater than $\phi_i(\frac{p-c}{p+h})$, and retailer i always prefers to form a pooling coalition with a retailer requiring high service level. However, when $F_j(\cdot)$ has finite support, whether such a region exits or not depends on the system parameters as we demonstrate with the following example.

Example 5.4.1 *Let the demand function for retailer 2 be U(0,1). The demand function for retailer 1 arbitrary but independent from that of retailer 2. Let $\rho_1 = 0.96, p = 5, c = 2, h = 0.1, p_M = 5.5$.*

In Figure 7: Case 1, the highest value $\phi_1(\rho_2)$ attains beyond $\frac{p+p_M-c}{p+p_M+h}$ is still lower than $\phi_1(\frac{p-c}{p+h})$. However, if we change p_M to 2, Figure 7: Case 2 shows that higher profit allocations are possible for retailer 1 beyond $\frac{p+p_M-c}{p+p_M+h}$.

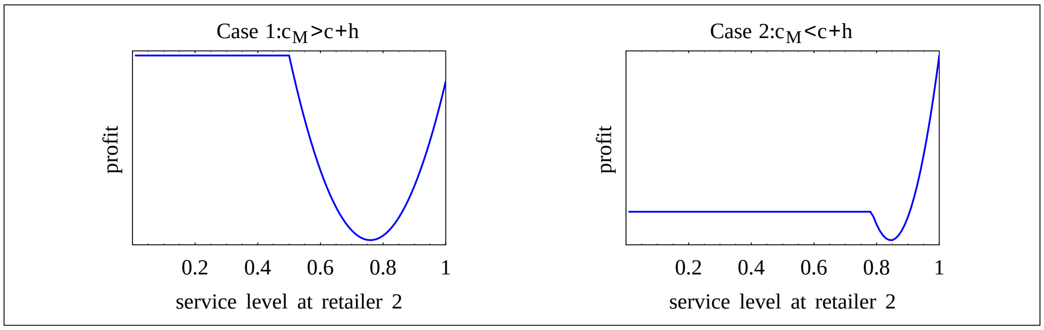


Figure 7: Profit allocations to retailer 1 as retailer 2's service changes (graphs not to scale)

If the demand distribution of retailer 2 is U(0,1), $\lim_{\rho_2 \to 1} \phi_1(\rho_2) > \phi_1(\frac{p-c}{p+h})$ when $p_M < p + h$. This condition does not depend on the value of c. We can interpret this result if we consider p_M to be the potential profit to the whole supply chain from the sale of a single item and $p + h$ to be the potential loss to the supplier when an item does not sell. When the potential loss to the supplier is large, she will tend to under-stock and this hurts the retailers. However, when the service level requirement of one or both of the retailers is very high, the supplier will have to stock enough to cover the requirement even if it is suboptimal for herself. Therefore, when the overage cost is very high, it is better for a retailer to form a coalition with a retailer with high service level requirement since this would force the supplier to stock more.

5.4.2 The Supplier's Perspective

In section 3.1.1 we set the contract such that the before-pooling profits are calculated to maximize the before-pooling supplier profit as long as the service level constraints set by the retailers are satisfied. This means that the before-pooling inventory levels are calculated as $F_i^{-1}(\max(\underline{\rho_i}, \frac{p-c}{p+h}))$ for each retailer i. Although this maximizes the supplier profit before pooling and guarantees at least ρ_i level of service for each retailer, this calculation may not maximize the supplier's after-pooling profit according to the Shapley value allocation scheme. The next theorem shows that the Shapley value allocation to the supplier is a unimodal function of the service level requirements of the retailers. Figure **??** is an example of how supplier profit changes as the service level requirement of one of the retailers changes.

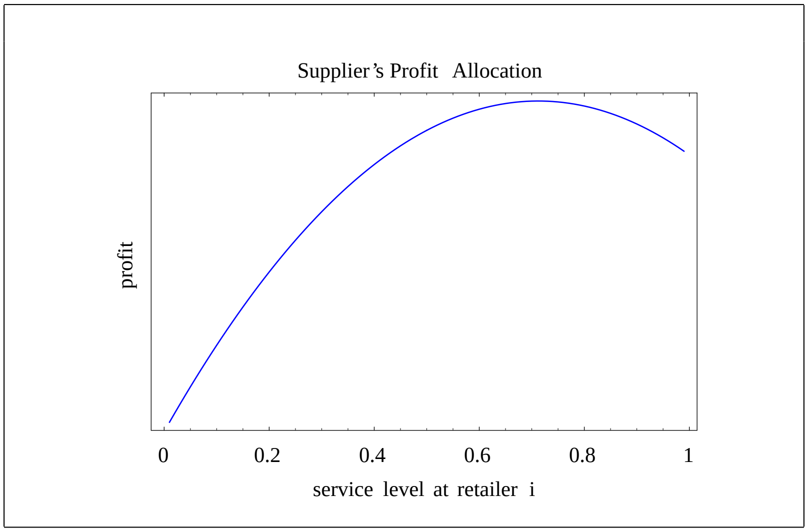


Figure 8: Supplier profit allocation as a function of service level

Theorem 5.4.2 *The Shapley value allocation to the supplier is unimodal in the service level requirements of the retailers and the global maximum occurs at*

i. $(\rho_1, \rho_2) = \left(\max\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right), \max\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)\right)$ *if* $\frac{2(p-c)-p_M}{2(p+h)-p_M} < 1$,

ii. $(\rho_1, \rho_2) = (0, 0)$ *otherwise.*

Proof Let $\Omega_i = \frac{\partial F_i^{-1}(\rho_i)}{\partial \rho_i}$. Then

$$\frac{\partial \phi_S(\rho_i,\rho_j)}{\partial \rho_i} = \frac{\Omega_i}{3}(2(p-c) - p_M - (2(p+h) - p_M)\rho_i)$$

Since $F_i(\cdot)$ is a cumulative distribution function $\Omega_i > 0$ for $i \in \{1, 2\}$. It is sufficient to consider the following three cases.

- Case 1: $2(p-c) - p_M \geq 0$ and $2(p+h) - p_M \geq 0$

  In this case both $\frac{\partial \phi_S(\rho_1,\rho_2)}{\partial \rho_1}$ and $\frac{\partial \phi_S(\rho_1,\rho_2)}{\partial \rho_2}$ are positive over the interval $\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$ and negative over the interval $\left(\frac{2(p-c)-p_M}{2(p+h)-p_M}, \infty\right)$. Therefore both $\phi_S(\rho_1)$ and $\phi_S(\rho_2)$ are increasing over the interval $\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$ and decreasing over the interval $\left(\frac{2(p-c)-p_M}{2(p+h)-p_M}, \infty\right)$, which shows $\phi_S()$ is unimodal in both ρ_1 and ρ_2. For this region, the global maximum is at $(\rho_1, \rho_2) = \left(\frac{2(p-c)-p_M}{2(p+h)-p_M}, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$.

- Case 2: $2(p-c) - p_M < 0$ and $2(p+h) - p_M \geq 0$

  In this region, for $\rho_1 \geq 0$ $\frac{\partial \phi_S(\rho_1,\rho_2)}{\partial \rho_1}$ is negative meaning $\phi_S(\rho_1)$ is decreasing. The same argument is true for $\frac{\partial \phi_S(\rho_1,\rho_2)}{\partial \rho_2}$ and $\phi_S(\rho_2)$. Therefore in this region $(\rho_1, \rho_2) = (0, 0)$ is the global maximum.

- Case 3: $2(p-c) - p_M < 0$ and $2(p+h) - p_M < 0$

  In this region $\frac{2(p-c)-p_M}{2(p+h)-p_M} \geq 1$ and beyond the meaningful service level region $[0, 1)$. For $\rho_1 \in [0, 1)$ $\frac{\partial \phi_S(\rho_1,\rho_2)}{\partial \rho_1}$ is negative so $\phi_S(\rho_1)$ is decreasing. The same argument is true for $\frac{\partial \phi_S(\rho_1,\rho_2)}{\partial \rho_2}$ and $\phi_S(\rho_2)$. Therefore in this region $(\rho_1, \rho_2) = (0, 0)$ is the global maximum. □

Theorem 5.4.2 states that the supplier has incentive to relax the terms of the contract. The current contract calculates before-pooling profits using $x_i = F_i^{-1}(\max(\rho_i, \frac{p-c}{p+h}))$ for each retailer. However, the supplier prefers a contract that calculates before-pooling profits based on $x_i = F_i^{-1}(\rho_i)$. Then she is allowed to maximize her after-pooling profits by setting the contracted service level at $\left(\frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$ if at least one of the retailers requires a service level that is smaller than $\left(\frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$. Otherwise, the supplier does not have room for manipulation since the contract still guarantees that the after-pooling profit allocations are at least as much as the before-pooling profits (as set through the service level constraints).

5.4.3 Conflict Between Retailers and Supplier

In the previous two sections, we looked at how service level requirements can be used to optimize profits by both the retailers and the supplier. However, we did not analyze the effects of these decisions on the other parties in the coalition. The total supply chain profit does not increase when the supplier maximizes her profits by varying the contracted service level. Therefore the Shapley value allocation to one or both of the retailers must be reduced. We would like to know "what happens to the profits of the retailers when the supplier maximizes her profit?".

We define the *base case* as the case where the stocking levels are determined by $F^{-1}(\frac{p-c}{p+h})$. From Theorem 5.4.2 we know that supplier profit is maximized at either $(\rho_1, \rho_2) = (0, 0)$ or $(\rho_1, \rho_2) = \left(\max\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right), \max\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)\right)$. Clearly $(\rho_1, \rho_2) = (0, 0)$ is not implementable. Therefore the supplier wants to set $(\rho_1, \rho_2) = \left(\frac{2(p-c)-p_M}{2(p+h)-p_M}, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$ and this requires $p-c \geq p_M/2$. The supplier's per-unit profit $(p-c)$ needs to be at least as much as half the retailers' total per-unit profit (p_M) for the supplier to be able to maximize her after-pooling profits. We can interpret this condition as a measure of the relative power of the supplier. If the supplier is making a high per-unit margin on each item she sells, she has the ability to manipulate the contracted service levels whenever the retailer requirements allow it.

For two random variables X and Y with distribution functions $F(\cdot)$ and $G(\cdot)$, X is said to be larger than Y in *dispersive order* if $F^{-1}(\beta) - F^{-1}(\alpha) \geq G^{-1}(\beta) - G^{-1}(\alpha)$ whenever $0 < \alpha \leq \beta < 1$ (denoted as $X \geq_{disp} Y$) (Shaked and Shanthikumar [67]). Dispersive order requires the difference between two quantiles of X_i to be smaller than the difference between the corresponding quantiles of X_j; therefore dispersive order compares the variability of the two distributions. Assuming there is dispersive order between the demand distributions, the following theorem identifies which one of the retailers (if either) will be better off when compared to the base case.

Theorem 5.4.3 *Assume $D_i \geq_{disp} D_j$. When the supplier maximizes her own after-pooling profit allocation by changing (ρ_i, ρ_j), either the after-pooling profit allocations to both of the retailers are reduced or the profit allocation to the one with smaller demand in dispersive order is increased while the profit allocation to the other is reduced.*

Proof Let ϕ_i' and ϕ_j' denote the profit allocations to retailer i and j after the supplier maximizes her profit allocation. Define the following notation:

$$\begin{aligned}
\alpha &= \frac{2(p-c) - p_M}{2(p+h) - p_M} \\
\beta &= \frac{p-c}{p+h} \\
\nu &= F_i^{-1}(\alpha) \\
\eta &= F_i^{-1}(\beta) \\
\varepsilon &= F_j^{-1}(\alpha) \\
\gamma &= F_j^{-1}(\beta)
\end{aligned}$$

That the profit allocation to retailer i after the supplier maximizes her profit allocation is greater than or equal to retailer i's allocation under the base case, that is $\phi_i' \geq \phi_i$, is equivalent to

$$3p_M\left(\nu-\eta+\int_\nu^\eta F_i(x)\,dx\right)$$
$$\geq(-p-p_M+c)(\gamma-\varepsilon+\eta-\nu)+(p_M+p+h)\left[\int_\varepsilon^\gamma F_j(x)dx+\int_\nu^\eta F_i(x)\,dx\right]$$

and similarly $\phi_j' \geq \phi_j$ is equivalent to

$$3p_M\left(\varepsilon-\gamma+\int_\varepsilon^\gamma F_j(x)\,dx\right)$$
$$\geq(-p-p_M+c)(\gamma-\varepsilon+\eta-\nu)+(p_M+p+h)\left[\int_\varepsilon^\gamma F_j(x)\,dx+\int_\nu^\eta F_i(x)\,dx\right]$$

The total after-pooling profit of the supply chain does not increase when the supplier maximizes her own after-pooling profit allocation. Then both of the inequalities cannot hold at the same time. Either neither of the equalities will hold or only one of them will hold. Therefore we need to compare $\nu-\eta+\int_\nu^\eta F_i(x)dx = \int_\nu^\eta(-1+F_i(x))\,dx$ and $\varepsilon-\gamma+\int_\varepsilon^\gamma F_i(x)dx=\int_\varepsilon^\gamma(-1+F_j(x))\,dx$ to find which retailer's profit allocation increases, if any. Since $D_i \geq_{disp} D_j$, we have $\eta-\nu\geq\gamma-\varepsilon$.

Since $D_i \geq_{disp} D_j$, we have

$$F_i^{-1}(1-y)-F_j^{-1}(1-y)\geq F_i^{-1}(1-x)-F_j^{-1}(1-x) \tag{26}$$

for $y \leq x$ and $y,x\in[1-\beta,\ 1-\alpha]$. Expression 26 implies that $1-F_i(\nu+\delta)\geq 1-F_j(\varepsilon+\delta)$ for $\delta\in[0,\gamma-\varepsilon]$ and that $\eta-\nu\geq\gamma-\varepsilon$. Then $\int_\nu^\eta(-1+F_i(x))\ dx \leq \int_\varepsilon^\gamma(-1+F_j(x))\ dx$, which concludes the proof. □

The next result directly follows from Theorem 5.4.3 since for two random variables Y and Z, $Y \leq_{disp} Z$ implies $Var(Y)\leq Var(Z)$.

Corollary 5.4.1 *If the demand of one retailer is greater than the demand of the other retailer in dispersive order and the supplier maximizes her own after-pooling profit allocation, either the profit allocation to the retailer with the smaller demand variance will increase or the profit allocations to both of the retailers will decrease.*

This result is intuitive in terms of the supply chain because when the supplier maximizes her profits, if the Shapley value allocation to one of the retailers will increase then it will be the one with smaller demand variance. This result is not surprising because the retailer with the smaller demand variance brings less risk into the pooling coalition and we would expect that retailer to receive a higher allocation.

The next theorem states that convolutions of random variables with logconcave densities can be ordered in the dispersive sense. This result implies that assuming dispersive order between the demand variables is not very restrictive.

Theorem 5.4.4 (Shaked and Shanthikumar [67], Theorem 2.B.3, p 71) *The random variable X satisfies $X \leq_{disp} X + Y$ for any random variable Y independent of X if and only if X has a logconcave density.*

Normal and gamma (with $k \geq 1$) distributions are frequently invoked models of demand distributions and they have logconcave densities. Therefore, by Theorem 5.4.4 normal and gamma demands with different shape parameters can be ordered in the dispersive sense and thus satisfy the condition on Theorem 5.4.3.

Another interesting property of the dispersive order is $X \leq_{disp} Y$ if and only if $X + c \leq_{disp} Y$ for any real number c. This means that the dispersive order between two random variables is preserved even if there is a shift in the mean(s). This property has an interesting implication on our results: the retailer whose profits decrease due to the supplier maximizing her profits cannot reverse the situation (become the retailer whose profits increase) even if his mean demand increases and thus creates more sales. However he can reverse the situation by changing the shape of his demand distribution by reducing the demand variance, because the allocation mechanism favors the retailer with lower risk.

5.5 Concluding Remarks

As we discussed in the previous chapter, several authors suggested that Shapley value has potential in supply chain analysis [14] and as an incentive mechanism to control decentralized systems [70]. However, this solution concept has not yet been utilized in the supply chain management context as a means of aligning the interests of supply chain partners. To our knowledge, ours is the first research attempt in understanding the uses of Shapley value as an incentive scheme to affect operational decisions in a decentralized supply chain.

Our model shares some limitations with most work in this area. For example, like others [4, 65, 74], we are limited by analytic tractability mostly to 2-retailers. Similarly, to derive more particular results we have to make some simplifying assumptions about the demand distributions experienced by the retailers. In the next chapter, we discuss the results we were able to extend to arbitrary numbers.

In this chapter, in addition to proposing a supply-chain coordinating mechanism, we analyzed the behaviors of the supply chain members under the proposed mechanism. It is important to compare various mechanisms for coordinating the supply chain by studying the strategic behavior that they might induce. For example, how will supply chain players answer such questions as with whom to form a coalition or whether one can game the system?

We are assuming a long-term relationship among the supply chain partners because we model the pooling problem as an *allocation game in expectation* (AGE) [3]. Another approach is a *snapshot allocation game* (SAG), which is used by Anupindi et al. [4]. In SAG, the value of the game is calculated based on each realization of random demand. While allocations in the core of SAG are renegotiation proof, allocations for AGE implicitly assume the players will not break from the contract based on individual realizations of demand [3].

The Shapley value allocations for the 2-retailer supply chain correspond to equal sharing of extra revenue due to pooling. Cachon and Lariviere [12] analyze revenue-sharing contracts and identify their limitations. They conclude that revenue sharing is not prevalent in practice partly because of high administrative costs and difficulties in monitoring revenues of retailers. Similar shortcomings apply to our value-sharing mechanism as well. We are proposing a contract where the three players first pool their profits and then the total is redistributed to them according to the Shapley values. We can think of this as a taxing mechanism where some players pay their taxes (return some of their profit) and some players get refunds (receive payments). This framework would work best if the supply chain members are in a long-term relationship, which is also the implicit assumption underlying AGE. All members are better off pooling inventory and sharing it based on Shapley value; however the mechanism will not work if there is doubt some player will break away from the coalition after getting a refund and will not be there to pay his tax when it is his turn. Cachon [17] discusses industry experience from the Duke University Medical Center (DUMC) where DUMC outsourced its inventory management to Baxter International (now Allegiance Healthcare) under the contract that any savings would be shared with respect to a pre-negotiated percentage. This example provides evidence that revenue sharing is used as a contract in practice under which supply chain members delegate control (in our case control of inventory management) to one member of the supply chain to whom the contract provides the incentive to make the supply-chain-optimal decisions.

As Cachon and Lariviere [12] emphasize, to share value, it must be possible to monitor revenues of the retailers. The Shapley-value mechanism, in addition, requires visibility of both the stocking level of the supplier and her costs. Our proposed value-sharing mechanism also raises the issue of information guessing at the retailers: Can players infer information about their coalition partners that might allow them to gain advantages?

CHAPTER VI

SHAPLEY VALUE PROFIT ALLOCATIONS: CASE OF N-RETAILER SUPPLY CHAINS

6.1 Introduction

In Chapter 5 we limited our analysis to 2-retailer supply chains. In this chapter, we extend some of our results to arbitrary numbers of retailers. However this generalization comes at the expense of an additional assumption – in addition to independence of retailers we further assume they are identical. It is not difficult to calculate the Shapley value allocations for non-identical retailers; however we need closed form expressions for some of the other results of this chapter and we are able to obtain closed form expressions only when the retailers are identical.

We are mainly interested in three types of questions:

- For two retailers, the Shapley value allocation rule is individually rational, induces coordination, and is in the core. Can we extend these results to supply chains with arbitrary numbers of retailers?

- The main motivation for pooling is to smooth demand variance across retailers. How would the Shapley value allocations behave if demand variance changes? Would the retailers and the supplier be affected in the same way?

- In the supply chain we consider, there is a single supplier who enjoys a higher bargaining power. As a response, the retailers may want to collude and act as a single retailer to increase their bargaining power and hence their allocations. Would this type of strategic bargaining always benefit the colluding retailers?

We find that the Shapley value allocations, even though individually rational and coordination inducing, are not necessarily in the core for $N > 2$. Counter to our intuition, increasing demand variance may increase the profit allocation to either the supplier or the retailers. The direction of change depends on the cost and revenue parameters, particularly on the markup the retailers charge to the end customer. Finally, we show that retailer colluding will not increase the profit allocations of the retailers if they do not have sufficient power in the supply chain which we measure by the level of markup they are able to charge to the end customer.

6.2 N-Retailer Games and Shapley Value Allocations

One of the criticisms directed at the Shapley value is the number of calculations involved in obtaining the allocations for all players. The number of calculations grow exponentially as new players join the game because the Shapley value allocations require the calculation of the value for each $i-retailer$ coalition as i goes from 1 to N. However, because we assume all our retailers are identical, we can show that the Shapley value calculations in our game simplify and grow linearly in the number of players.

Proposition 6.2.1 *For the $N-retailer$ inventory holding game among N identical retailers and a single supplier, the Shapley value allocations are as follows:*

$$\phi_i = \frac{N^2+N+2}{2N(N+1)}\, v_1 + \sum_{j=2}^{N} \frac{j}{N(N+1)}(v_{1....j,S} - v_{1....j-1,S}) \tag{27}$$

$$\phi_S = \frac{2}{N+1}\, v_S + \sum_{j=2}^{N} \frac{1}{N+1}(v_{1....j,S} - v_{1....j}) \tag{28}$$

Derivation

Recall the definition of the Shapley value

$$\phi_i = \sum_{J\subseteq N-i} \frac{\mid J \mid!(\mid N \mid - \mid J \mid -1)!}{\mid N \mid!}(v(J\cup\{i\}) - v(J))$$

We first derive the Shapley value for the retailers and then that for the supplier. For the derivation of the Shapley value for the retailer, recall that since coalitions can only increase value if the supplier is part of the coalition, the value of an $i-retailer$ game without the supplier is equal to the sum of the values of the retailers before pooling.

To derive ϕ_i, we first obtain the coefficient of v_1 and then the coefficients of $v_{1\ldots j,S} - v_{1\ldots j-1,S}$ for $j \in \{2,\ldots,N\}$. The difference $v(J \cup \{i\}) - v(J))$ equals v_1 under three different conditions. The first one is if $J = \emptyset$. The first term in Expression 29 corresponds to the coefficient of v_1 under this condition. The second condition is if $\mid J \mid= 1$ and either a retailer or the supplier is in J. Then the coefficient is $\frac{1!(N-1)!}{((N+1)!}$ multiplied by the number of different times such a coalition J can be formed. Once the retailer i is fixed there are $\binom{N-1}{1}$ different ways we can choose a retailer to be in J and only one way we can choose the supplier. Therefore the number of different ways such a coalition can be formed is $\left(\binom{N-1}{1}+1\right)$ as in the second term of Expression 29. Finally $v(J \cup \{i\}) - v(J)) = v_1$ if J consists of j retailers. Again after fixing i, there are $\binom{N-1}{j}$ different ways to form such a coalition J and hence the coefficient of v_1 under this condition corresponds to the last term in Expression 29. By considering all three conditions, we obtain the coefficient of v_1 as

$$\begin{aligned}
&= \frac{0!(N)!}{(N+1)!} + \frac{1!(N-1)!}{((N+1)!}\left(\binom{N-1}{1}+1\right) + \sum_{j=2}^{N}\frac{j!(N-j)!}{(N+1)!}\binom{N-1}{j} \qquad (29)\\
&= \frac{1}{N+1} + \frac{1}{N+1} + \frac{N-2}{N(N+1)} + \frac{N-3}{N(N+1)} + \cdots + \frac{1}{N(N+1)}\\
&= \frac{\sum_1^{N-2} x}{N(N+1)} + \frac{2}{N+1}\\
&= \frac{N^2+N+2}{2N(N+1)}
\end{aligned}$$

Coefficient of $v_{1\ldots j,S} - v_{1\ldots j-1,S}$

$$\begin{aligned}
&= \frac{j!(N-j)!}{N!}\binom{N-1}{j-1} \qquad (30)\\
&= \frac{j}{N(N+1)}
\end{aligned}$$

The first term in Expression 30 is the number of different ways a coalition consisting of $j-1$ retailers and the supplier can be formed. Since the retailers are identical, to obtain the coefficient of $v_{1\ldots j,S} - v_{1\ldots j-1,S}$ we multiply this by the number of different ways we can choose $j-1$ retailers among $N-1$ retailers.

Derivation of ϕ_S:

Again we consider the coefficients of v_S and $v_{1\ldots j,S} - v_{1\ldots j}$ for $j \in \{2,\ldots,N\}$ separately. There are two conditions under which $v(J \cup \{i\}) - v(J)$ equals v_S. The first is when $J = \emptyset$

and the corresponding coefficient is the first term of Expression 31. The second is when J consists of a single retailer, which can happen in N different ways since there are N retailers. The corresponding coefficient is the second term in Expression 31. Finally for each coalition J such that J consists of j retailers, we multiply $\frac{j!(N-j)!}{(N+1)!}$ by the number of different ways we can choose j retailers among N retailers. By summing these terms over feasible values of j we obtain the last term in Expression 31.

$$\phi_S = \frac{0!(N)!}{(N+1)!} v_S + \frac{1!(N-1)!}{(N+1)!} N \, v_S + \sum_{j=2}^{N} \frac{j!(N-j)!}{(N+1)!} \binom{N}{j} (v_{1\ldots.j,S} - v_{1\ldots.j}) \quad (31)$$

Simplification yields Expression 28. □

By definition, the coefficients of v_1 and $v_{1\ldots j,S} - v_{1\ldots j-1,S}$ in Expression 27 and the coefficients of v_S and $v_{1\ldots j,S} - v_{1\ldots j}$ in Expression 28 add up to 1. In addition, due to the superadditivity of the characteristic function, which we show in Proposition 4.3.1, the difference terms, $v_{1\ldots j,S} - v_{1\ldots j-1,S}$ and $v_{1\ldots j,S} - v_{1\ldots j}$ never attain negative values for $j \in \{2 \ldots N\}$. Based on these two facts, we make Observation 6.2.1.

Observation 6.2.1 *The Shapley value allocations to both the retailers and the supplier are convex combinations of the value of that player before pooling and the contribution of the player to coalitions of different sizes.*

In addition, the coefficients of v_1 and v_S are decreasing functions of N in ϕ_i and ϕ_S respectively. The coefficient of v_S in ϕ_S decreases faster (and in the limit goes to zero) than the coefficient of v_i in ϕ_i. On the other hand $lim_{N\to\infty} \frac{N^2+N+2}{2N(N+1)} = \frac{1}{2}$ which means the retailer's before-pooling expected profit makes up at least 50% of his Shapley value allocation after pooling. In terms of the size of the after-pooling allocation, it is more important for the retailers to start with a higher before-pooling value.

Using Observation 6.2.1 we rewrite ϕ_i and ϕ_S as

$$\phi_i = v_1 + \sum_{j=2}^{N-1} \frac{j}{N(N-1)} (v_{1\ldots.j,S} - v_{1\ldots.j-1,S} - v_1) \quad (32)$$

$$\phi_S = v_S + \sum_{j=2}^{N-1} \frac{1}{N} (v_{1\ldots.j,S} - v_{1\ldots.j} - v_S) \quad (33)$$

We next establish some stability properties of the Shapley value allocation rule for the $N - retailer$ game.

Proposition 6.2.2 *The Shapley value allocation mechanism induces the supplier to make the optimal inventory decision for the pooling-coalition.*

Proof We use the definition of ϕ_S as given in Expression 33. For a given N, that is for a given coalition size, all the terms except $v_{1\ldots N,S}$ are constants. Therefore the supplier wants to set the inventory level so as to maximize $v_{1\ldots N,S}$ in order to maximize her expected profit allocation. The $v_{1\ldots N,S}$ term is the total expected profit of the coalition consisting of N retailers and the supplier and is maximized at a stock level of $F_c^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)$ where $F_c^{-1}(\cdot)$ denotes the convolution of the demand functions of the N retailers. From Chapter 2 $F_c^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)$ is the supply chain coordinating inventory level and this concludes the proof. $\square$

In Chapter 2 we note that in a decentralized supply chain the supplier, as the bearer of inventory risk, maximizes her profit by understocking. By Proposition 6.2.2, however, we show that under the Shapley value allocation rule, the supplier carries the supply-chain-optimal level of inventory. How does the Shapley value profit allocation mechanism induce the supplier to carry the optimal level of inventory? Comparing Expressions 32 and 33, we find that for $\mid j \mid= 2$, $(v_{1\ldots.j,S} - v_{1\ldots.j-1,S} - v_1)$ equals $(v_{1\ldots.j,S} - v_{1\ldots.j} - v_S)$ and the coefficients of these two terms are the same as well. Therefore for two retailers, the retailers and the supplier share the additional value due to pooling equally. For all $j \in \{3\ldots N\}$, $(v_{1\ldots.j,S} - v_{1\ldots.j} - v_S)$ is greater than $(v_{1\ldots.j,S} - v_{1\ldots.j-1,S} - v_1)$ and the coefficient of $(v_{1\ldots.j,S} - v_{1\ldots.j} - v_S)$ is also greater than that of $(v_{1\ldots.j,S} - v_{1\ldots.j-1,S} - v_1)$. This means that the supplier gets a larger portion of the extra profit due to pooling. We argue this is a fair allocation because the value of coalitions increase only when the supplier is part of the coalition. In other words, in our model the retailers cannot cooperate to pool inventory at a central facility other than at the supplier site. This relative power of the supplier is reflected in the Shapley value allocation to the supplier. We formalize this observation as follows:

Observation 6.2.2 *Under the Shapley value profit allocation mechanism the supplier has incentive to carry the supply-chain-optimal level of inventory since she reaps more of the additional profit due to pooling.*

Even though the supplier reaps more of the additional profit due to inventory pooling, the retailers still have incentive to participate in pooling under the proposed allocation mechanism as we show with the next result.

Proposition 6.2.3 *The Shapley value allocations for any retailer $i \in \{1 \ldots N-1\}$ and the supplier are individually rational.*

Proof Follows from the superadditivity of the characteristic function. □

6.3 Individual Profits vs. Coalition Size

One would expect both the retailers' and the supplier's profit allocations to benefit from larger coalitions since larger coalitions pool inventory risk over a larger number of retailers. However as the following example demonstrates this is not always true and the retailers' profit allocations may actually be larger under smaller coalitions.

Example 6.3.1 *Consider a supply chain with three retailers. Let demand, D_i, at each retailer i be independent, identically distributed such that for $i \in \{1, 2, 3\}$*

$$D_i = \begin{cases} 0 & w.p. \quad \frac{1}{2} \\ 1 & w.p. \quad \frac{1}{2} \end{cases}$$

Let $p_M = p = 3.5$, $c = 2.5$, and $h = 0.5$. In this case, when there is no pooling, the optimal stock the supplier carries for each retailer is $x^* = 0$ which corresponds to $v_i^* = v_S^* = 0$. When two of the retailers decide to pool inventory, the optimal pooled-stock level is $x_2^* = 1$ (let the subscript denote the number of retailers in the pooling coalition). The Shapley value profit allocation to each retailer is $\phi_i^2 = \frac{7}{8}$. The supplier's profit allocation is also $\frac{7}{8}$. When all three of the retailers agree to pool inventory, the optimal stock level is $x_3^* = 2$ and the Shapley value allocation to each retailer is $\phi_i^3 = \frac{55}{64}$. In this case, the Shapley value profit allocation of the supplier is $\frac{111}{64}$.

In Example 6.3.1, a coalition consisting of two retailers and the supplier is the optimum coalition size for the retailers. However, the supplier prefers the larger coalition consisting of three retailers. We formalize our findings regarding the change in Shapley value profit allocations as the coalition size changes in the following two theorems.

The first theorem concerns the behavior of Shapley value profit allocations to the retailers as the number of retailers in the coalition increases from 2 to N, as the supply chain size is kept fixed. We are interested in the behavior of the retailer's Shapley value as an increasing number of the retailers (who are already part of the supply chain) agree to pooling. We find that, depending on the demand function, the retailer's Shapley value is maximized at either the smallest-size coalitions with 2 retailers or the largest N-retailer coalitions.

Before we state the theorem, we introduce new notation. Recall from Equation 15 that $v_{1\ldots i,S}$ consists of two terms. Let $v_i^c = \max_x \{(p+p_M)S - hH - cx\}$ and let $(N-i)v_{Si} = \sum_{k \ni \{1\ldots i\}} \max_{x_k} \{pS_k - hH_k - cx_k\}$ where $v_{Si} = \max_{x_k} \{pS_k - hH_k - cx_k\}$. With the new notation, v_i^c represents the part of $v_{1\ldots i,S}$ that is due to the pooling coalition and v_{Si} is the profit the supplier makes due to a single retailer without pooling. We use ϕ_i^j to denote the Shapley value allocation to retailer i when the number of retailers in the coalition is $|j|$.

Theorem 6.3.1 *For given N, suppose $\frac{v_i^c}{i+1}$ is bounded for all i. Then*
a)if v_i's are concave and $\phi_i^3 - \phi_i^2 \leq 0$ then ϕ_i is monotone decreasing in i and convergent.
b)if v_i's are convex and $\phi_i^3 - \phi_i^2 \geq 0$ then ϕ_i is monotone increasing in i and convergent.

Proof We provide the proof for part a only since the proof for part b follows in the same manner.

Assume $\phi_i^{k+1} - \phi_i^k \leq 0$, where

$$\begin{aligned}\phi_i^{k+1} - \phi_i^k &= \frac{i(i+1)(v_{i+1}^c - v_i^c - v_{Si} - v_1)}{i(i+1)(i+2)} \\ &- \frac{2(2((n-2)v_{Si} + v_2^c - 2v_1 - v_S) + \cdots + i(v_i^c - v_{i-1}^c - v_{Si} - v_1))}{i(i+1)(i+2)}\end{aligned}$$

Let $Y = 2((n-2)v_{Si} + v_2^c - 2v_1 - v_S) + 3(v_3^c - v_2^c - v_{Si} - v_1) + \cdots + i(v_i^c - v_{i-1}^c - v_{Si} - v_1)$. Then $\phi_i^{k+1} - \phi_i^k \leq 0$ implies

$$2Y \geq i(i+1)(v_{i+1}^c - v_i^c - v_{Si} - v_1) \tag{34}$$

Now assume $\phi_i^{k+2} - \phi_i^{k+1} > 0$, where

$$\begin{aligned}
\phi_i^{k+2} - \phi_i^{k+1} &= \frac{(i+1)(i+2)(v_{i+2}^c - v_{i+1}^c - v_{Si} - v_1)}{(i+1)(i+2)(i+3)} \\
&- \frac{2(2((n-2)v_{Si} + v_2^c - 2v_1 - v_S) + \cdots + (i+1)(v_{i+1}^c - v_i^c - v_{Si} - v_1))}{(i+1)(i+2)(i+3)}
\end{aligned}$$

Again using the Y notation, $\phi_i^{k+2} - \phi_i^{k+1} > 0$ implies

$$(i+1)(i+2)(v_{i+2}^c - v_{i+1}^c - v_{Si} - v_1) > 2(Y + (i+1)(v_{i+1}^c - v_i^c - v_{Si} - v_1) \tag{35}$$

Due to Expression 34, adding $2Y$ to the left hand side and $i(i+1)(v_{i+1}^c - v_i^c - v_{Si} - v_1)$ to the right hand side of Expression 35 does not change the direction of the inequality . We get

$$\begin{aligned}
(i+2)v_{i+2}^c - 2(i+1)v_{i+1}^c + iv_i^c &> 2(v_{i+1}^c - v_i^c) \\
(i+2)(v_{i+2}^c - v_{i+1}^c) - i(v_{i+1}^c - v_i^c) &> 2(v_{i+1}^c - v_i^c) \\
v_{i+2}^c - v_{i+1}^c &> v_{i+1}^c - v_i^c \\
\frac{v_{i+2}^c + v_i^c}{2} &> v_{i+1}^c
\end{aligned}$$

which is a contradiction to the concavity assumption. Now given $\phi_i^3 - \phi_i^2 \leq 0$, we can show that ϕ_i is monotone decreasing. To show convergence, we also need to show that ϕ_i is bounded. We rewrite ϕ_i^k as

$$\phi_i^k = \frac{v_1}{2} - \frac{v_1 + 2v_S}{k(k+1)} - \frac{\sum_{j=2}^{k-1} v_j^c}{k(k+1)} + \frac{v_k^c}{k+1}$$

Define $\widehat{\phi_i^k}$:

$$\begin{aligned}
\widehat{\phi_i^k} &= \frac{v_1}{2} - \frac{v_1 + 2v_S}{k(k+1)} - \frac{\sum_{j=2}^{k-1} v_1}{k(k+1)} + \frac{v_k^c}{k+1} \\
&= \frac{v_1}{2} - \frac{v_1 + 2v_S}{k(k+1)} - \frac{(k-2)v_1}{k(k+1)} + \frac{v_k^c}{k+1} \\
\lim_{i\to\infty} \widehat{\phi_i^k} &= \frac{v_1}{2} + \lim_{i\to\infty} \frac{v_k^c}{k+1}
\end{aligned}$$

Since we assumed $\frac{v_i^c}{i+1}$ is bounded, the last line implies $\widehat{\phi_i^k}$ is bounded. Clearly, $\widehat{\phi_i^k} > \phi_i^k$. Therefore, ϕ_i^k is also bounded. Then by monotone convergence, ϕ_i is convergent. $\square$

Theorem 6.3.2 *The Shapley value profit allocation to the supplier is a non-decreasing function of the number of retailers in the coalition.*

Proof Recall that $v_S = Nv_{Si}$. The change in the supplier's profit allocation when a third retailer joins a $2-retailer$ coalition is:

$$\phi_S^3 - \phi_S^2 = N \cdot v_{Si} + \frac{1}{4}v_3^c + \frac{1}{4}v_4^c - \frac{5}{4}(v_{Si} + v_1) - \left[N \cdot v_{Si} + \frac{1}{3}v_3^c - \frac{2}{3}(v_{Si} + v_1)\right] \quad (36)$$

$$= \frac{1}{4}v_4^c - \frac{1}{12}v_3^c - \frac{7}{12}(v_{Si} + v_1) \quad (37)$$

From Expression 37, $\phi_S^3 - \phi_S^2 \leq 0$ is equivalent to $3v_4^c \leq v_3^c + 7(v_{Si} + v_1)$. This last expression implies $3v_3^c + 3(v_{Si} + v_1) \leq v_3^c + 7(v_{Si} + v_1)$ (because $3v_3^c + 3(v_{Si} + v_1)$ is a lower bound for v_4^c) which is a contradiction since $2v_3^c \geq 4(v_{Si} + v_1)$.
Assume $\phi_S^N - \phi_S^{N-1} \geq 0$. In addition, $\phi_S^{N+1} - \phi_S^N \geq 0$ is equivalent to

$$Nv_{N+1} \geq v_N + v_{N-1} + \ldots + v_3 + \frac{(N+1)^2 - (N+1) + 2}{2}(v_{Si} + v_1) \quad (38)$$

$$= v_N + v_{N-1} + \ldots + v_3 + 2N(v_{Si} + v_1) \quad (39)$$

Since we assumed $\phi_S^N - \phi_S^{N-1} \geq 0$, we know that

$$(N-1)Nv_N \geq v_{N-1} + v_{N-2} + \ldots + v_3 + \frac{N^2 - N + 2}{2}(v_{Si} + v_1) \quad (40)$$

Adding $N(v_{Si} + v_1) + v_N$ to both sides of Inequality 40, we obtain

$$N(v_{Si} + v_1) + Nv_N \geq v_N + v_{N-1} + \ldots + v_3 + \frac{(N+1)^2 - (N+1) + 2}{2}(v_{Si} + v_1) \quad (41)$$

Observe that the right hand side of Inequality 41 is the same as the right hand side of Inequality 39. We also know that

$$Nv_{N+1} \geq N(v_{Si} + v_1) + Nv_N \quad (42)$$

Together with (42), (41) implies Inequality 39 which concludes the proof. □

6.3.1 Is the Shapley Value Profit allocation Mechanism in the Core?

In Section 4.2, we have defined the *core* as one of the stability criteria of the allocation mechanisms for cooperative games. In Theorem 6.3.2, we show that the supplier's allocation is always maximized under the grand coalition. In addition, if the conditions in Theorem 6.3.1 Part (a) are satisfied then the retailers also prefer the grand coalition and the Shapley value allocations are in the core. However, Theorem 6.3.1 also shows that under some conditions the retailer allocations are maximized when there are only two retailers in the coalition. This, by itself, does not imply that the Shapley value allocations are not in the core when the retailers prefer smaller coalitions. Nevertheless we can find examples, like Example 6.3.2 below, where the Shapley value allocations are not in the core.

Example 6.3.2 *Consider a 3-retailer supply chain. In terms of v_1 and v_{Si}, we get that $v_2^c = 2(v_{Si} + v_1)$. Let $v_3^c = 2.5(v_{Si} + v_1)$ and $v_4^c = 3.6(v_{Si} + v_1)$. When only two of the retailers participate in inventory pooling, the corresponding Shapley value allocations are:*

$$\begin{aligned}
\phi_R^3 &= v_1 + \frac{1}{3}(2.5 - 2)(v_1 + v_{Si}) = \frac{14}{12}v_1 + \frac{2}{12}v_{Si} \\
\phi_S^3 &= v_S + \frac{1}{3}(2.5 - 2)(v_1 + v_{Si}) = 3v_{Si} + \frac{1}{6}(v_1 + v_{Si})
\end{aligned}$$

The total allocation to the supplier and two participating retailers is:

$$2\phi_R^3 + \phi_S^3 = 2\left(\frac{14}{12}v_1 + \frac{2}{12}v_{Si}\right) + \frac{38}{12}v_{Si} + \frac{2}{12}v_1 = \frac{30}{12}v_1 + \frac{42}{12}v_{Si} \tag{43}$$

When all three of the retailers participate in inventory pooling, the corresponding allocations are:

$$\begin{aligned}
\phi_R^4 &= v_1 + \frac{2}{12}\left(2.5(v_1 + v_{Si}) - 2v_{Si} - v_1\right) + \frac{3}{12}\left(3.6(v_1 + v_{Si}) - 2.5(v_1 + v_{Si}) - v_{Si}\right) \\
&= \frac{13.3}{12}v_1 + \frac{1.3}{12}v_{Si} \\
\phi_S^4 &= v_S + \frac{1}{4}(2.5 - 2)(v_1 + v_{Si}) + \frac{1}{4}(3.6 - 3)(v_1 + v_{Si}) = 3v_{Si} + \frac{1.1}{4}(v_1 + v_{Si})
\end{aligned}$$

The total allocation to the supplier and any two of the three retailers is:

$$2\phi_R^4 + \phi_S^4 = 2\left(\frac{13.3}{12}v_1 + \frac{1.3}{12}v_{Si}\right) + \frac{39.3}{12}v_{Si} + \frac{2}{12}v_1 = \frac{29.9}{12}v_1 + \frac{41.9}{12}v_{Si} \tag{44}$$

By comparison of Expressions 43 and 44, we find that the total profit is higher if one of the retailers is out of the coalition. Therefore the grand coalition is not optimal, which means the Shapley value allocation is not in the core for this example.

We summarize our findings regarding the membership of the Shapley value allocations in the core with the following theorem.

Theorem 6.3.3 *The Shapley value allocation rule is*

a. in the core when the Shapley value allocation to the retailers is non-decreasing in the number of retailers in the coalition. One special case is when the expected profit function is convex in the number of retailers.

b. may not be in the core when the Shapley value allocation to the retailers is decreasing in the number of retailers in the coalition. One special case is when the expected function is concave in the number of retailers.

Proof Theorems 6.3.1 and 6.3.2 together with Example 6.3.2 imply this result. □

6.3.2 Shapley Value Allocations and Normally Distributed Demand

In addition to establishing the fact that the Shapley value allocations may not always be in the core, Theorem 6.3.3 also provides a condition through which we can check whether the Shapley value allocations are actually in the core – we need to check the convexity of the profit function in the number of retailers. By Theorem 6.3.1 we establish that the expected profit function is convex in the number of retailers in the coalition when demand is normally distributed. Therefore the Shapley value allocations are in the core when demand is normally distributed.

Lemma 6.3.1 *When retailer demand is normally distributed, the expected profit of the coalition is convex in the number of retailers.*

Proof Let demand at each retailer be normally distributed with mean μ and standard deviation σ. Then the total demand for a pooling coalition with N retailers will be distributed normally with mean $N\mu$ and standard deviation $\sqrt{N}\sigma$. For normally distributed demand, we know the optimal stock level to be mean demand plus k times the standard deviation, where k is the safety factor whose size depends on the inventory management policy followed (reserved versus pooled) as well as the system's cost and revenue parameters. Let

k_1 be the safety factor under the reserved inventory management policy and determine k_1 such that Equation 45 is satisfied.

$$p_{u<}(k_1) = \frac{p-c}{p+h} \tag{45}$$

where $p_{u<}(k) = P(\text{a unit normal variable is less than } k)$

Similarly let k_2 be the safety factor under the pooled inventory management policy and determine k_2 such that Equation 46 is satisfied.

$$p_{u<}(k_2) = \frac{p+p_M-c}{p+p_M+h} \tag{46}$$

Finally define $G_u(k)$, a special function of the unit normal distribution which is used frequently in inventory calculations.

$$\begin{aligned} G_u(k) &= \int_k^\infty (u_0 - k)\ f_u(u_0)\ du_0 && (47) \\ &= f_u(k) - kp_{u\geq}(k) && (48) \end{aligned}$$

Silver et al. [71, page 407] derive the objective function of the newsvendor problem as a function of μ, σ, k, and $G_u(k)$ when the demand follows a normal distribution and the objective is to minimize expected cost. We do the same for the case the objective is profit maximization instead of cost minimization. The expression for expected excess stock, H, as a function of these variables is (see Appendix A for a detailed derivation):

$$H = k\sigma + \sigma G_u(k) \tag{49}$$

The expression for expected sales, S, is:

$$S = \mu - \sigma G_u(k) \tag{50}$$

Using Expressions 49 and 50 we obtain π^N, the expected total profit of a coalition with N retailers (where $N \geq 2$).

$$\begin{aligned} \pi^N &= N(p+p_M-c)\mu - \sqrt{N}\sigma\left[(p+p_M+h)G_u(k_2) + (c+h)k_2\right] \\ &= N(p+p_M-c)\mu - \sqrt{N}\sigma(p+p_M+h)f_u(k_2) \end{aligned}$$

The difference between the expected profit of a coalition with N retailers and that of a coalition with $N-1$ retailers is:

$$\pi^N - \pi^{N-1} = (p + p_M - c)\mu - (\sqrt{N} - \sqrt{N-1})\sigma(p + p_M + h) f_u(k_2) \tag{51}$$

Since $\sqrt{N} - \sqrt{N-1}$ is a decreasing function of N, Expression 51 is an increasing function of N which means π^N is convex in N. □

Theorem 6.3.4 *When demand is normally distributed, allocations due to Shapley value are always in the core.*

Proof Result follows from Theorems 6.3.3 and 6.3.1 . □

6.3.3 A Core Allocation Rule

A core allocation may not always exist for every cooperative game. For our inventory game, however, we show that there exists a core allocation even when the Shapley value is not in the core. Therefore we conclude that the core of the inventory pooling game is non-empty.

Theorem 6.3.5 *The core of the inventory pooling game among the supplier and the retailers is non-empty.*

Proof We prove the result by constructing an allocation that is always in the core of the game. Consider an allocation scheme where the retailer allocations are equal to their expected profit before pooling and the supplier gets the remaining profit; that is expected after pooling profit of the coalition minus the retailers' allocations. Then the supplier's and the retailers' allocations under the grand coalition are:

$$\begin{aligned} \phi_i^N &= v_1, \; \forall\, i \in \{1 \ldots N\} \\ \phi_S^N &= v_N - N v_1 \end{aligned}$$

Condition 1 of the core definition is satisfied since

$$\sum_{i=1}^{N} \phi_i + \phi_S = v_N$$

Consider a smaller allocation with $N-j$ retailers. Using the same allocation rule, the allocations under this smaller coalition are

$$\begin{aligned}
\phi_i^{N-j} &= v_1,\ \forall\, i \in \{1 \ldots N-j\} \\
\phi_S^{N-j} &= v_{N-j} - (N-j)v_1
\end{aligned}$$

where $j \in \{1 \ldots N-2\}$. The total worth of the $N-j$ retailers under this smaller coalition is the same as the total worth of these retailers under the grand coalition. The total worth of the $N-j$ retailers and the supplier under the grand coalition is $v_N - Nv_1 + (N-j)v_1 = v_N - jv_1$. We know that $v_N - jv_1$ is at least as much as v_{N-j} since the expected profit of a supply chain with all N retailers agreeing to pooling is greater than or equal to the expected profit of a supply chain where only $N-j$ of the retailers agree to pooling and the remaining j dictate a reserved inventory policy. Therefore the total worth of the coalition consisting of $N-j$ retailers and the supplier is less than or equal to the worth of the same coalition under the grand coalition. By a similar argument we find that the supplier's allocation under the grand coalition is at least as much as her allocation under any smaller coalition. Therefore the second condition in the definition of the core is also satisfied and the proposed allocation mechanism is in the core. $\square$

We check the remaining two desired properties, individual rationality and ability to induce coordination, in order to better compare the Shapley value allocation rule with the mechanism proposed in Theorem 6.3.5. The second condition in the definition of the core corresponds to the individual rationality conditions for each of the players when the subsets considered are the singletons. Therefore the fact that the allocation mechanism is in the core directly implies that the allocations are individually rational, which we formalize in Corollary 6.3.1.

Corollary 6.3.1 *The allocations due to the allocation scheme proposed in Theorem 6.3.5 are individually rational.*

Proposition 6.3.1 *The allocation mechanism proposed in Theorem 6.3.5 coordinates the supply chain.*

Proof The supplier's allocation under the proposed mechanism is $v_N - Nv_1$. In order to maximize her profit allocation, the supplier maximizes v_N (the other terms are fixed) and v_N is maximized when the inventory level in the supply chain equals to the optimal inventory level for the centralized supply chain. □

When compared to the Shapley value allocation mechanism, the proposed allocation mechanism looks more favorable since it is in the core in addition to inducing coordination and being individually rational. However even though the theoretical stability conditions are satisfied, we argue that the proposed mechanism will be harder to implement than the Shapley value. The proposed mechanism reflects the relative power of the supplier; however at the same time gives zero incentive to the retailers to participate in the coalition. One may argue that since the retailers are indifferent between participating or not, they may as well agree to participate in pooling. On the other hand, the retailers also know that pooling increases total supply chain profit and the benefit is solely awarded to the supplier. Thus in a realistic negotiation setting, the retailers are more likely to preclude pooling than to agree to it.

Even when it is not in the core, the appeal of the Shapley value allocation mechanism is that the Shapley value is the unique expected payoff to the players which corresponds to their marginal contribution to coalitions of different sizes. When a player is able to calculate the size of her marginal contribution to the coalition under complete information (when all the demand and revenue information is known to all the players as is the case here), it will be extremely hard to make her participate in a coalition unless she is allocated her contribution or more.

6.3.4 Shortcomings of the *Core* as a Stability Measure

Even though the *core* is considered the predominant measure of stability in cooperative game theory since no player has incentive to break away from a core allocation, the core has shortcomings as a stability measure. In light of these shortcomings, we discuss the Shapley value as a practical profit allocation scheme even when the corresponding allocations are not in the core.

The first problem with the core is that it may be empty for a given cooperative game. However as we have shown in Section 6.3.3 the core is not empty for our inventory problem. Another problem is that the core may not be a unique allocation when it is nonempty and there is no easy way to determine whether the core of a game is unique or not. The obvious problem with multiple allocations which are all in the core is that it is not possible to predict one of them as the likely outcome of the cooperative game. When the core is not a singleton, it is most likely that different players will be better off under different core allocations. This would make it difficult to employ one of the allocations as a contract to induce supply chain optimality.

Even when the core exists and is unique, it may be extremely sensitive to system parameters, which may render it non-implementable. We demonstrate this with the following example from Myerson [53, page 429].

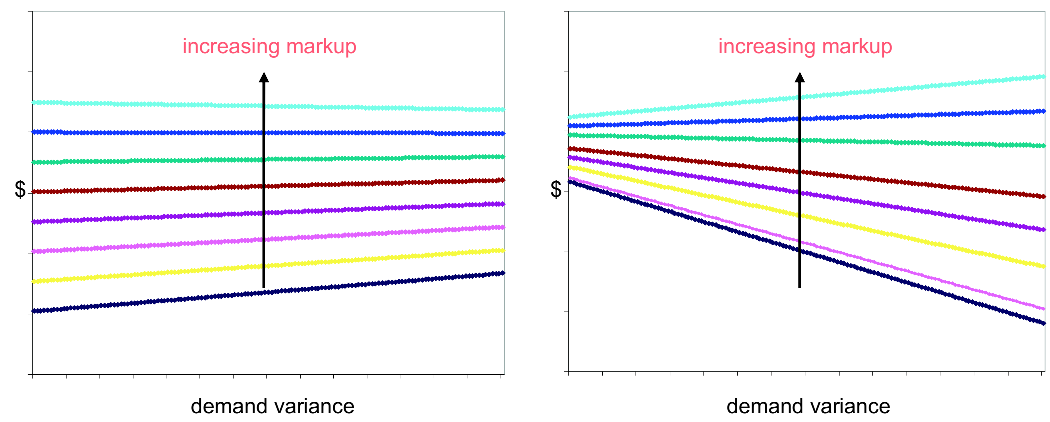
Example 6.3.3 *Consider a game with* $2,000,001$ *players where* $1,000,000$ *of the players can supply a left glove and* $1,000,001$ *of the players can supply a right glove. The worth of a coalition corresponds to the number of matched pairs in the coalition. The unique core allocation assigns* 1 *to the players who supply the left glove and* 0 *to the players who supply the right glove.*

The intuitive reasoning behind the core allocation in Example 6.3.3 which assigns an allocation of 0 to the players who supply the right gloves is that the right gloves are in excess supply. However, if we added just two left-glove suppliers to the game then the right-glove suppliers would get an allocation of 1 and the left-glove suppliers would get an allocation of 0. This property of being *sensitive* to game parameters and the tendency to assign a zero allocation to the player with the excess resource is inherent in the core since the characterization of the core derives from the duality idea in linear programming (the interested reader can refer to Myerson [53] and the references therein for a detailed discussion). Another example is the core allocations proposed by Anupindi et al. [4]. The allocations based on the values of the dual variables exhibit the same property – the players with the scarce resource divide the total worth of the coalition among themselves and the players with the excess resource get an allocation of zero. Even though we agree that the allocation mechanism should recognize and reward the players with the scarce resource, we believe assigning an allocation of 0 to the players with the excess resource will not be acceptable in practice.

For the game discussed in Example 6.3.3, the Shapley value allocation to each of the right-glove suppliers is 0.500443 and the Shapley value allocation to each of the left-glove suppliers is 0.499557. The Shapley value allocation scheme recognizes the slightly higher bargaining power of the right-glove suppliers because they have the scarce resource. In addition, the Shapley value also recognizes the fact that without the left-glove suppliers there will not be any coalitions and therefore the right-glove suppliers will not be able to make any money either. As a result, the Shapley value allocation to each of the left-glove suppliers is not only positive but also only slightly lower than the allocation to the right glove suppliers to reflect the fact that the left gloves are in excess only by one glove given the $2,000,001$ gloves. Theoretically, in this example, allocations due to the Shapley value would be considered unstable since they are not in the core. In practice, however, the unique core allocation may be more unstable since the left-glove suppliers are indifferent between participating or not.

6.4 Effect of Demand Variance on Allocations

One of the main reasons behind inventory pooling is to mitigate against negative effects of demand variance. Therefore we analyze how changes in demand variance affect the Shapley value allocations to both the supplier and the retailers. We are not able to prove results analytically for general distributions; however we discuss some illustrative examples and present our findings for a special case – where the demand at the retailers is normally distributed. Figure 9 shows that both the retailer's and the supplier's profit allocations may either increase or decrease in demand variance. We obtain Figure 9 based on an example where demand is normally distributed with mean 10 and standard deviation varying on the range $[1 \dots 10]$. We consider a supply chain with ten retailers and $p = 4$, $c = 1.5$, $h = 0.1$. The markup, p_M, is varied on the range $[3 \dots 13]$. Numerical experiments yield that the



(a) Retailer's profit allocation (b) Supplier's profit allocation

Figure 9: Change in retailer and supplier profit allocations as demand variance increases at different levels of markup

relative size of p_M with respect to the values of the other cost and revenue parameters affects the direction of change. Figure 9 shows that when markup increases, profit allocations to both the retailers and the supplier increase; however as demand variance increases, the direction of change in the profit allocations shifts as we increase the markup. As the markup increases, the retailer allocations first increase then decrease with demand variance whereas the supplier allocation first decreases then increases. We are able to derive a sufficient condition on p_M under which the supplier's Shapley value allocation is decreasing in standard deviation.

Proposition 6.4.1 *When demand at each of the retailers is normally distributed with mean μ and standard deviation σ, the Shapley value allocation to the supplier is a decreasing function of σ if $p + h > p_M$.*

Proof We need to represent the Shapley value allocation to the supplier in terms of σ.

$$\begin{aligned}
\phi_S &= v_S + \frac{1}{N+1}\left(v_2^c - 2v_1 + (N-2)v_{Si} + Nx\right) + \cdots + \frac{1}{N+1}(v_N^c - Nv_1 - Nx) \\
&= v_S + \frac{1}{N+1}(v_2^c + \cdots + v_N^c) - \frac{1}{N+1}(2+\cdots N)v_1 - \frac{1}{N+1}(2+\cdots N)v_{Si} \\
&= v_S + \frac{1}{N+1}(v_2^c + \cdots + v_N^c) - \frac{1}{N+1}\left(\frac{(N+1)N}{2} - 1\right)(v_1 + v_{Si}) \\
&= Nx + \frac{1}{N+1}(v_2^c + \cdots + v_N^c) - \left(\frac{N}{2} - \frac{1}{N+1}\right)(v_1 + v_{Si}) \\
&= \frac{Nx}{2} + \frac{v_{Si}}{N+1} - \frac{Nv_1}{2} + \frac{v_1}{N+1} + \frac{1}{N+1}\left(\frac{N(N+1)}{2} - 1\right)(p + p_M - c)\mu \\
&\quad - \frac{1}{N+1}\left(\sqrt{N} + \cdots + \sqrt{2}\right)\sigma\left((p + p_M + h)G_u(k_2) + (c+h)k_2\right)
\end{aligned}$$

The last line is obtained by plugging in the expected profit function for normally distributed demand as derived in Expression 51. We can write v_{Si} and v_1 in terms of μ and σ in the same manner; however we need to use k_1 instead of k_2. By making use of the equality $G_u(k) = f_u(k) - kp_{u\geq}(k)$, we obtain the complete representation of ϕ_S in terms of μ and σ as

$$\begin{aligned}
\phi_S &= N\left((p-c)\mu - (p+h)\sigma f_u(k_1)\right) - \frac{1}{N+1}\left(\sqrt{N} + \cdots + \sqrt{2}\right)\sigma f_u(k_2)(p + p_M + h) \\
&\quad + \frac{N}{2}\left(\sigma f_u(k_1)(p + p_M + h) - p_M\sigma k_1\left(\frac{h+c}{p+h}\right)\right) \\
&\quad + \frac{1}{N+1}\left(-\sigma f_u(k_1)(p + p_M + h) + p_M\sigma k_1\left(\frac{h+c}{p+h}\right)\right)
\end{aligned}$$

Taking the derivative of ϕ_S with respect to σ we obtain

$$\begin{aligned}
\frac{\partial\phi_S}{\partial\sigma} &= -\frac{N(p+h)}{2}f_u(k_1) - \frac{1}{N+1}\left(\sqrt{N} + \ldots + \sqrt{2}\right)f_u(k_2)(p + p_M + h) \\
&\quad - \frac{N}{2}\left(-f_u(k_1)p_M + p_M k_1\left(\frac{c+h}{p+h}\right)\right) \\
&\quad + \frac{1}{N+1}\left(-f_u(k_1)(p + p_M + h) + p_M k_1\left(\frac{c+h}{p+h}\right)\right)
\end{aligned} \tag{52}$$

From the definition of $G_u(k_1)$ we know that

$$\frac{1}{N+1}\left(-f_u(k_1)(p + p_M + h) + p_M k_1\left(\frac{c+h}{p+h}\right)\right) \leq 0$$

In addition, when $(p+h) > p_M$

$$\frac{N(p+h)}{2}f_u(k_1) \geq \frac{N}{2}\left(-f_u(k_1)p_M + p_M k_1\left(\frac{c+h}{p+h}\right)\right)$$

Therefore $\frac{\partial \phi_S}{\partial \sigma} < 0$ which proves the result. $\square$

The proof of Proposition 6.4.1 shows that surprisingly the direction of change does not depend on the demand distribution's parameters (mean and variance). The derivative of the supplier's Shapley value with respect to standard deviation (Expression 52) contains neither μ nor σ. As long as the cost and revenue parameters are kept constant, the direction of change for the supplier's Shapley value allocation cannot be altered by changing the values of the demand mean and variance.

We derive another sufficient condition on the relative size of p_M with respect to the other parameters under which the retailer's Shapley value allocation is increasing in demand variance. For the rest of the results in this section we require that $\frac{p-c}{p+h} \geq 0.5$ which means that the supplier guarantees at least a service level of 0.5 to the retailers before pooling, which is usually the case in real life examples (for example [75] makes the same assumption).

Proposition 6.4.2 *When demand is normally distributed with mean μ and standard deviation σ, the Shapley value allocation to each of the retailers is an increasing function of σ if $p + h \geq N(N+1)p_M$.*

Proof Writing ϕ_i in terms of σ we obtain

$$\begin{aligned}
\phi_i &= v_1 + \frac{2}{N(N+1)}(v_2^c + (N-2)v_{Si} - 2v_1 - Nx) \\
&\quad + \ldots + \frac{N}{N(N+1)}(v_N^c - v_{N-1}^c - v_{Si} - v_1) \\
&= v_1 + \frac{Nv_N^c}{N(N+1)} - \frac{v_{N-1}^c}{N(N+1)} - \ldots - \frac{v_2^c}{N(N+1)} - \frac{v_{Si}+v_1}{N(N+1)} - \frac{v_{Si}+v_1}{2} \\
&= p_M\mu - p_M\sigma f_u(k_1) + p_M\sigma k_1 \frac{h+c}{p+h} \\
&\quad + \frac{N}{N(N+1)}\left(N(p+p_M-c)\mu - \sqrt{N}(p+p_M+h)\sigma f_u(k_2)\right) \\
&\quad - \cdots - \frac{1}{N(N+1)}\left((2)(p+p_M-c)\mu - \sqrt{2}p + p_M + h)\sigma f_u(k_2)\right) \\
&\quad - \frac{(p+p_M-c)\mu - \sigma f_u(k_1)(p+p_M+h) + p_M\sigma k_1\frac{h+c}{p+h}}{N(N+1)} \\
&\quad - \frac{(p+p_M-c)\mu - \sigma f_u(k_1)(p+p_M+h) + p_M\sigma k_1\frac{h+c}{p+h}}{2} \\
&= p_M\mu - p_M\sigma f_u(k_1) + p_M\sigma k_1\frac{h+c}{p+h} \\
&\quad + (p+p_M-c)\mu\left(\frac{N^2}{N(N+1)} - \left(\frac{N(N-1)}{2} - 1\right)\frac{1}{N(N+1)} - \frac{1}{N(N+1)} - \frac{1}{2}\right) \\
&\quad - \frac{1}{N(N+1)}\sigma f_u(k_2)\left(N\sqrt{N} - \sqrt{N-1} - \cdots - \sqrt{2}\right) \\
&\quad + \sigma f_u(k_1)(p+p_M+h)\left(\frac{1}{N(N+1)} + \frac{1}{2}\right) - p_M\sigma k_1\frac{h+c}{p+h}\left(\frac{1}{N(N+1)} + \frac{1}{2}\right) \\
&= p_M\mu - p_M\sigma G_u(k_1)\left(\frac{1}{2} - \frac{1}{N(N+1)}\right) - \frac{N\sqrt{N} - \cdots - \sqrt{2}}{N(N+1)}\sigma f_u(k_2)(p+p_M+h) \\
&\quad + \sigma f_u(k_1)(h+p)\left(\frac{1}{2} + \frac{1}{N(N+1)}\right)
\end{aligned}$$

The last step follows from the definition of $G_u(k)$. Taking the derivative of ϕ_i with respect to σ we obtain

$$\begin{aligned}
\frac{\partial\phi_i}{\partial\sigma} &= -p_M G_u(k_1)\left(\frac{1}{2} - \frac{1}{N(N+1)}\right) - \frac{N\sqrt{N} - \cdots - \sqrt{2}}{N(N+1)} f_u(k_2)(p+p_M+h) \\
&\quad + f_u(k_1)(h+p)\left(\frac{1}{2} + \frac{1}{N(N+1)}\right)
\end{aligned}$$

At optimality we know that $k_1 \le k_2$ which implies $f_u(k_1) \ge f_u(k_2)$. Therefore we have

$$\frac{1}{2}f_u(k_1)(p+p_M+h) \ge \frac{\sqrt{N}}{N+1}f_u(k_2)(p+p_M+h) \tag{53}$$

In addition when $p + h \geq N(N+1)p_M$ we have

$$\frac{(p+h)f_u(k_1)}{N(N+1)} \geq p_M f_u(k_1) \tag{54}$$

Expressions 53 and 54 imply that $\frac{\partial \phi_i}{\partial \sigma} > 0$ which proves the result. $\square$

The reason why the markup affects the change in retailer and supplier profit allocations in different directions lies in the fact that both the retailer and supplier Shapley value allocations consist of two parts – the part due to the respective player's expected profit before pooling and the part due to the player's contribution to coalitions of different sizes (recall Expressions 27 and 28). We find that depending on the level of markup either one of these parts become dominant in determining the player's final allocation. For the retailer, at low levels of markup the contribution to the coalitions becomes the dominant factor and for normally distributed demand, the value of forming pooling coalitions increases in increasing demand variance. Therefore as the demand variance increases, the retailer's Shapley value allocation also increases. On the contrary, for high levels of markup, the retailer's expected profit before pooling becomes the dominant factor and that term decreases as demand variance increases. As a result, for high levels of markup, the retailer's Shapley value allocation decreases as demand variance increases. The markup has the opposite effect on the supplier's profit allocation; that is for low levels of markup, the contribution to the coalitions becomes the dominant factor for the supplier and for high levels of markup vice versa.

By Theorem 6.4.1 below we also show that there exist two levels of markup, which we denote by p_M^1 and p_M^2, such that below p_M^1 the supplier's profit allocation decreases and the retailers' profit allocations increase with demand variance and beyond p_M^2 we observe the opposite behavior. We summarize this result in Table 2.

Theorem 6.4.1 *There exist two levels of markup which we denote by p_M^1 and p_M^2 such that (a) for markup levels less than or equal to p_M^1, the supplier's profit allocation is decreasing and the retailer's profit allocation is increasing in demand variance.*
(b) for markup levels greater than or equal to p_M^2, the retailer's profit allocation is decreasing

and the supplier's profit allocation is increasing in demand variance.

(c) p_M^1 *is not necessarily equal to* p_M^2.

Proof From Expression 51 we know that expected total profit of the coalition is a decreasing function of demand variance. Therefore when demand variance increases, profit allocations to at least one type of player will decrease. As a result, to prove parts (a) and (b), it suffices to show that the profit allocation to one type of player increases in the given parameter region.

(a) The existence of p_M^1 follows from Proposition 6.4.2.

(b) Rewriting Expression 52 by using the definition of $G_u(k)$ we obtain

$$\begin{aligned}\frac{\partial \phi_S}{\partial \sigma} &= -(p+h)f_u(k_1)\left(\frac{N}{2}+\frac{1}{N+1}\right)-\frac{\sqrt{N}+\cdots+\sqrt{2}}{N+1}f_u(k_2)(p+p_M+h) \\ &\quad + G_u(k_1)\left(\frac{N}{2}-\frac{1}{N+1}\right)p_M\end{aligned} \tag{55}$$

The first term in Equation 55 is a constant in terms of p_M and the last term is an increasing function of p_M. We take the derivative of the second term with respect to p_M to obtain

$$\begin{aligned}&\frac{\partial}{\partial p_M}\left[(p+p_M+h)f_u(k_2)\right] \\ &= f_u(k_2) \\ &\quad -(p+p_M+h)f_u(k_2)\Theta^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)\frac{\partial}{\partial p_M}\Theta^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)\frac{h+c}{(p+p_M+h)^2} \\ &= f_u(k_2)\left[1-\Theta^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)\frac{\partial}{\partial p_M}\Theta^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)\frac{h+c}{p+p_M+h}\right]\end{aligned}$$

where $\Theta(\cdot)$ is the cumulative distribution function of the unit normal distribution.

For $\frac{p+p_M-c}{p+p_M+h} \geq 0.5$ (we can always satisfy this condition by increasing p_M)

$$f_u(k_2)\left[1-\Theta^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)\frac{\partial}{\partial p_M}\Theta^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)\frac{h+c}{p+p_M+h}\right] \leq f_u(k_2)$$

Taking the limit of both sides as p_M goes to infinity we get

$$\begin{aligned}&\limsup_{p_M\to\infty} f_u(k_2)\left[1-\Theta^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)\frac{\partial}{\partial p_M}\Theta^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)\frac{h+c}{p+p_M+h}\right] \\ &\quad \leq \lim_{p_M\to\infty} f_u(k_2) = 0\end{aligned}$$

Table 2: Direction of change in retailer and supplier profit allocations as demand variance increases at different levels of markup

	$p_M \leq p_M^1$	$p_M^1 < p_M < p_M^2$	$p_M \geq p_M^2$
supplier's profit allocation	$\downarrow$	$\downarrow$	$\uparrow$
retailer's profit allocation	$\uparrow$	$\downarrow$	$\downarrow$

On the other hand

$$\frac{\partial}{\partial p_M}\left[-\ (p+h)f_u(k_1)\left(\frac{N}{2}+\frac{1}{N+1}\right)+\ G_u(k_1)\left(\frac{N}{2}-\frac{1}{N+1}\right)p_M\right]$$
$$= G_u(k_1)\left(\frac{N}{2}-\frac{1}{N+1}\right)$$

Because limit of the derivative of $(p+p_M+h)f_u(k_2)$ is bounded above by zero, there exists a finite p_M^0 such that the derivative of $(p+p_M+h)f_u(k_2)$ evaluated at p_M^0 is equal to $G_u(k_1)\left(\frac{N}{2}-\frac{1}{N+1}\right)$. This means there is a point $p_M^0+\epsilon$ at which $G_u(k_1)\left(\frac{N}{2}-\frac{1}{N+1}\right)p_M-(p+h)\ f_u(k_1)\left(\frac{N}{2}+\frac{1}{N+1}\right)$ and $\frac{\sqrt{N}+\cdots+\sqrt{2}}{N+1}(p+p_M+h)f_u(k_2)$ intersect. Therefore the Shapley value allocation to the supplier is an increasing function of demand variance beyond $p_M^2 = p_M+\epsilon$ which proves part (b).

(c) We prove by example. Let $\mu = 10$, $p = 4$, $c = 1.5$, $h = 0.1$, and $N = 10$. For this example $p_M^1 = 3.92$ and $p_M^2 = 9.87$. For any markup level between these numbers the profit allocation to both the supplier and the retailers decreases. $\square$

The effects of the other parameters on the direction of change can be summarized as below:

- For large N, the allocation to the supplier starts increasing at smaller p_M.
- For given p_M, there is a high enough h that causes the allocation to retailers to decrease as demand variance increases.
- For high h allocation to the supplier increases over a larger range of p_M. For high p, the suppliers allocation increases only at very high p_M.

6.5 Strategic Bargaining and Colluding Retailers

Up to this point, we assumed that in the inventory centralization game the order of events is such that the supplier announces a pooling contract in which she specifies the proportions with which total supply chain profit will be divided. Then the retailers announce whether they will participate in this pooling coalition. However once the supplier announces the contract, the retailers may first want to form coalitions among themselves and act as a single retailer with a demand function corresponding to the sum of their demands. We call this behavior colluding of the retailers. This coalition of retailers acting as one retailer, which we call a *pseudo* retailer and denote by the subscript p, accepts or rejects the supplier's offer. We assume that once two of the retailers collude to act as a single retailer in negotiating their profit allocation within the pooling coalition, if they decide not to enter the coalition, the supplier still treats them as a single retailer and determines the stock level accordingly. We still assume that each of the individual retailers faces independent identically distributed demand. The questions we are interested in are: Do the profit shares of the colluding retailers always increase? Does the profit share of the supplier always decrease? What happens to the profit shares of the non-participating retailers?

Due to the pooling effect, the demand from the pseudo retailer has a reduced variance and one would expect the total profit allocation to the colluding retailers to increase. We would also expect the collusion to adversely affect the profit allocations to the supplier and the non-participating retailer. However we find that these intuitions do not always hold.

To address these questions, we analyze a three-retailer supply chain where two of the retailers collude before they negotiate with the supplier. Let v_1 denote the expected before-pooling profit of the non-participating retailer and v_p denote the expected before-pooling profit of the single pseudo retailer. Recall that v_{S_i} denotes the profit of the supplier due to a single, no-pooling retailer. We use v_{S_p} to denote the supplier's profit due to the pseudo retailer. When both the non-prticipating retailer and the pseudo retailer agree to pool inventory, the resulting coalition will have two non-identical retailers. However without the retailer collusion, the coalition will have three identical retailers. We compare the profit allocation to the pseudo retailer under the two retailer coalition with the total profit allocation to two of the retailers under the three retailer coalition.

The pseudo retailer's profit allocation, ϕ_i^2 (as before we use the superscript to denote the number of retailers in the coalition), is

$$\phi_p^2 = v_p + \frac{1}{3}\left(v_3 - v_1 - v_p - v_{Si} - v_{Sp}\right)$$

The total profit allocation to two of the retailers if they do not collude is

$$2\phi_i^3 = 2v_1 + \frac{1}{3}\left(v_2^c - 2v_1 - 2v_{Si}\right) + \frac{1}{2}\left(v_3 - v_2^c - v_1 - v_{Si}\right)$$

The difference between the profit allocation to the pseudo retailer and the total profit allocation to the two non-colluding retailers is

$$\phi_p^2 - 2\phi_i^3 = \frac{1}{6}\left(6v_p - 12v_1 + v_2^c - v_3 + 5(v_1 + v_{Si}) - 2(v_p + v_{Sp})\right) \tag{56}$$

For the non-participating retailer, the profit allocation without the retailer collusion is

$$\phi_i^3 = v_1 + \frac{1}{6}\left(v_2^c - 2v_1 - 2v_{Si}\right) + \frac{1}{4}\left(v_3 - v_2^c - v_1 - v_{Si}\right)$$

The non-participating retailer's profit allocation in case of a retailer collusion is

$$\phi_i^2 = v_1 + \frac{1}{3}(v_3 - v_1 - v_p - v_{Si} - v_{Sp})$$

We obtain the difference between the two allocations to be

$$\phi_i^2 - \phi_i^3 = \frac{1}{12}\left(v_3 + v_2^c + 3(v_1 + v_{Si}) - 4(v_p + v_{Sp})\right) \tag{57}$$

Our first result concerns the comparison of the signs of Expressions 56 and 57.

Proposition 6.5.1 *When $v_p \geq 2v_1$, an increase in the profit allocation of the pseudo retailer due to retailer collusion implies an increase in the profit allocation of the non-participating retailer.*

Proof When $\phi_p^2 \geq 2\phi_i^3$, from Expression 56 we have

$$12v_1 + v_3 + 2(v_p + v_{Sp}) \leq 6v_p + v_2^c + 5(v_1 + v_{Si}) \tag{58}$$

Since $v_3 \geq v_p + v_{Sp} + v_1 + v_{Si}$, replacing v_3 on the left hand side of Expression 58 by $v_p + v_{Sp} + v_1 + v_{Si}$ and adding $v_3 - (v_p + v_{Sp})$ to the right hand side will not change the direction of the inequality. Simplifying after these changes we obtain

$$12v_1 + 4(v_p + v_{Sp}) \leq 6v_p + v_2^c + v_3 + 3(v_1 + v_{Si}) \tag{59}$$

When $v_p \geq 2v_1$, Expression 59 implies $4(v_p + v_{Sp}) \leq v_3 + 3(v_1 + v_{Si})$ which concludes the proof. $\square$

The condition in Proposition 6.5.1, $v_p \geq 2v_1$, means that the expected sales when the supplier sets the stock level with respect to the total demand of two retailers is at least as much as the expected sales when the supplier sets stock levels separately for each of the retailers. Intuition suggests that this condition is a trivial one and would always be true. However as we show in Example 3.1.1 expected sales may decrease as a result of pooling.

The implication of Proposition 6.5.1 is that if the profit allocation to the pseudo retailer increases *and* the expected sales increases as a result of pooling then the profit allocation to the supplier decreases (since total supply chain profit does not increase, allocation to at least one player must decrease). When $v_p \geq 2v_1$, retailer collusion does not hurt the non-participating retailer but hurts the supplier. We formalize this observation as follows.

Observation 6.5.1 *Proposition 6.5.1 implies that an increase in the profit allocation to the colluding retailers implies a decrease in the profit allocation to the supplier and an increase in the profit allocation to the non-participating retailer whenever expected sales increases due to inventory pooling.*

One case where $v_p \geq 2v_1$ is always satisfied is when demand at each of the retailers is normally distributed. We can see this by calculating expected sales in both cases by using Equation 50.

We are also able to derive some conditions under which the profit allocations to the pseudo retailer and the non-participating retailer increase when demand is normally distributed. Again for the next two results we assume that $\frac{p-c}{p+h} \geq 0$ as we did in the previous section.

Proposition 6.5.2 *When demand at each of the retailers is normally distributed with mean μ and standard deviation σ, a necessary condition for the profit allocation to the non-participating retailer to increase is*

$$\frac{k_2^2}{2} - \frac{k_1^2}{2} \geq 0.169$$

Proof Using Equation 51 we can write (57) as below

$$\phi_i^2 - \phi_i^3 = (p + p_M + h)\left[-(\sqrt{2}+\sqrt{3})f_u(k_2) + (4\sqrt{2}-3)f_u(k_1)\right] - \frac{(4\sqrt{2}-3)p_M(c+h)k_1}{p+h}$$

When $\phi_i^2 \geq \phi_i^3$

$$(p + p_M + h)\left[-(\sqrt{2}+\sqrt{3})f_u(k_2) + (4\sqrt{2}-3)f_u(k_1)\right] \geq \frac{(4\sqrt{2}-3)p_M(c+h)k_1}{p+h}$$

By definition $\frac{(4\sqrt{2}-3)p_M(c+h)k_1}{p+h} \geq 0$. Therefore one necessary condition for $\phi_i^{(2)} \geq \phi_i^{(3)}$ to hold is $-(\sqrt{2}+\sqrt{3})f_u(k_2) + (4\sqrt{2}-3)f_u(k_1) \geq 0$ which is equivalent to

$$\begin{aligned}
\frac{f_u(k_1)}{f_u(k_2)} &\geq \frac{\sqrt{2}+\sqrt{3}}{4\sqrt{2}-3} \\
\frac{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{k_1^2}{2}}}{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{k_2^2}{2}}} &\geq \frac{\sqrt{2}+\sqrt{3}}{4\sqrt{2}-3} \\
\frac{k_2^2}{2} - \frac{k_1^2}{2} &\geq 0.169
\end{aligned}$$

and this concludes the proof. $\square$

Similarly, we are able to derive a sufficient condition under which the profit allocation to the pseudo retailer always increases.

Proposition 6.5.3 *When demand at each of the retailers is normally distributed and $p_M > p+h$, a sufficient condition under which the pseudo retailer's profit allocation increases is*

$$\frac{k_2^2}{2} - \frac{k_1^2}{2} \geq 1.922$$

For normally distributed demand $\phi_p^2 - \phi_p^3 \geq 0$ is equivalent to

$$
\begin{aligned}
&(12-6\sqrt{2})G_u(k_1)p_M + (\sqrt{3}-\sqrt{2})(p+p_M+h)f_u(k_2) \\
&\quad \geq (5-2\sqrt{2})G_u(k_1)p_M + (5-2\sqrt{2})f_u(k_1)(p+h) \iff \\
&(7-4\sqrt{2})G_u(k_1)p_M + (\sqrt{3}-\sqrt{2})(p+p_M+h)f_u(k_2) \\
&\quad \geq (5-2\sqrt{2})f_u(k_1)(p+h) \iff \\
&(2-2\sqrt{2})G_u(k_1)p_M + (5-2\sqrt{2})f_u(k_1)p_M + (\sqrt{3}-\sqrt{2})(p+p_M+h)f_u(k_2) \\
&\quad \geq (5-2\sqrt{2})f_u(k_1)(p+h) - \frac{(5-2\sqrt{2}k_1(h+c)p_M}{p+h}
\end{aligned}
$$

Since $p_M > p+h$, $(5-2\sqrt{2})f_u(k_1)p_M \geq (5-2\sqrt{2})f_u(k_1)(p+h)$. An upper bound on $\frac{(5-2\sqrt{2})k_1(h+c)p_M}{p+h}$ is $(5-2\sqrt{2})f_u(k_1)p_M$. Therefore if $(\sqrt{3}-\sqrt{2})(p+p_M+h)f_u(k_2) \geq (5-2\sqrt{2})f_u(k_1)p_M$ then we are done. Equivalently we need

$$\frac{f_u(k_2)}{f_u(k_1)} \geq \frac{(5-2\sqrt{2})p_M}{(\sqrt{3}-\sqrt{2})(p+p_M+h)}$$

Taking the natural logarithm of both sides we obtain

$$\frac{k_2^2}{2} - \frac{k_1^2}{2} \geq 1.922 + \ln p_M - \ln(p+p_M+h)$$

Since $\ln p_M - \ln(p+p_M+h) \leq 0$, $\frac{k_2^2}{2} - \frac{k_1^2}{2} \geq 1.922$ is a sufficient condition for $\phi_p^{(2)} - \phi_p^{(3)} \geq 0$ to hold which concludes the proof. $\square$

The conditions in both Proposition 6.5.2 and 6.5.3 will be satisfied when k_2 and k_1 are sufficiently apart. One way in which the gap between these two critical ratios grows apart is increasing the retailer markup, p_M. One can interpret a high p_M as an indication of high retailer bargaining power. When retailers are powerful enough, colluding before they negotiate will increase their profit allocations. However, if the supplier is the powerful one, they cannot increase their allocations even through colluding.

6.6 Concluding Remarks

Our analysis of supply chains with an arbitrary number of retailers reveals the higher bargaining power of the supplier – a fact that was not apparent in the analysis of 2-retailer supply chains. The expected profit of a pooling coalition only increases if it contains the supplier and at least two retailers. Therefore, as long as the supplier has the freedom to pick any two retailers among N, she has a higher bargaining power reflected by the fact that she reaps more of the profit due to pooling. In addition, the supplier may increase her profit allocation at the expense of decreasing that of the retailers by including more retailers in the pooling coalition. If the retailers do not have the bare minimum bargaining power, which we measure by the markup they charge, they will not be able to increase their profit allocations even though they collude against the supplier and the supplier may end up benefiting from the collusion.

Our results *are* dependent on our single-supplier assumption; however they still provide important insights about the electronics supply chain which provided the motivation for this research. In supply chains characterized by strong suppliers (consider the electronics supply chain and a supplier like Intel) which enjoy some monopolistic powers, depending on the profit allocation rule utilized, the supplier may maximize her profits at the expense of the retailers by bringing in more retailers to pool inventory. However we recognize the importance of extending our analysis to arbitrary number of suppliers and research in this direction is under way.

Another insight we obtain from this chapter is regarding the relationship between demand variance and the retailer markup as they affect the participants' profit allocations. We know that both the retailer and supplier profit allocations increase with increasing markup. However when coupled with the effect of the demand variance on the allocations, the supplier may prefer a low markup - low demand variance environment to a high markup - high demand variance environment. On the other hand, the retailer may be better off charging a low markup under high demand variance than she would be if she were charging a higher markup at a lower demand variance environment. Assuming the markup is market driven and exogenous to the model (as we do in this chapter) but that the players have freedom to choose which market to operate in where the markets are characterized by their demand variance and markup levels, these insights help determine which player will choose which type of market. On the other hand, if the retailers have some control over the markup they charge, our results indicate the possibility of conflict of interest between the retailers and the supplier, especially at extreme values of demand variance.

CHAPTER VII

CONCLUSIONS

As also observed by Anupindi and Bassok [2], the common intuition regarding inventory pooling is that it is always beneficial since it reduces inventory costs. However this intuition is flawed in the sense that it is true only for the supply-chain party (or parties) holding inventory. As in Anupindi and Bassok, we also consider a multi-echelon supply chain where only one of the echelons carries inventory and find that the party whose revenues depend only on sales (in our case the retailers) may lose profits due to pooling. We further analyze the applicability of service contracts in this context as a means of ensuring reservation profits for the retailers. Surprisingly, in addition to sometimes failing to guarantee profits, service contracts may lead to some unintended consequences such as causing one retailer to subsidize the service of another, which make them undesirable from the point of view of the retailers.

The profit-sharing mechanism we propose is based on a cooperative game theoretic model of the inventory transactions among the retailers and the supplier. As we discussed in Chapter 4, cooperative game theory has not received much attention in the operations research literature. Shapley value has been used in the supply chain context before [63, 41] but only as a cost allocation rule not as a means of affecting the operational decisions of the supply chain partners. This dissertation, in addition to being an initial step in understanding the uses of Shapley value as a supply-chain-coordinating incentive mechanism, also investigates the type of behavior the proposed profit-allocation mechanism induces in the supply-chain partners.

7.1 Discussion of Assumptions and Model Limitations

In order to examine the strategic behavior induced by the proposed profit-allocation mechanism, we require closed-form expressions for the retailer and the supplier Shapley values.

For analytical tractability we are limited to either 2-retailer supply chains where the retailers are not necessarily identical or supply chains with an arbitrary number of identical retailers. We also assume independent demand across the retailers. These assumptions are common to most work in the inventory centralization literature. For example, Anupindi and Bassok [2], Rudi et al. [65] and Tagaras [74] also restrict analysis to two retailers. On the other hand, Cachon [17], Schwarz et al. [66], and the first part of McGavin et al. [51] among others assume N independent, identical retailers. One of the important results regarding the supply chain with N identical retailers is that the retailers do not always prefer the grand coalition. For the 2-retailer supply chain we find that the retailers prefer to form pooling coalitions with retailers requiring either very high or low service levels. In a supply chain with N non-identical retailers (and when the grand coalition is not optimal for the retailers), it would be interesting to analyze whether retailers with similar service requirements or retailers with sufficiently different service requirements group together to form coalitions.

Since the motivation of this research comes from the electronics industry where product life cycles are short, we assume a single period model. Under the single period model with independent demands across the retailers, complete sharing is always optimal. However in a multi-period model where sharing inventory today may result in a costly stockout tomorrow or in the case of competitive retailers, one of the hybrid policies discussed in Chapter 3, where the retailers carry some reserved inventory, may be optimal. Anupindi et al. [4] indeed point out that in one of the examples they discuss they would expect the e-tailers working with Ingram Entertainment to carry some reserved stock to hedge against stockouts at the supplier.

We assume that the wholesale price is exogenous to the model and is market-driven rather than being a contract parameter among the supplier and the retailers. Supporting this assumption, Flextronics does not try to keep suppliers in a price competition because they believe the market is competitive enough and determines the price. However this assumption becomes more questionable in the analysis of strategic retailer coalitions. For the retailers, the main motivation behind forming such coalitions is to increase their bargaining power against the supplier. Therefore it may be less reasonable to assume that a coalition of $N-1$ retailers will get the same wholesale price as a single retailer out of N retailers.

We use the *allocation game in expectation* approach in the analysis of the pooling problem. The analysis is based on the expected value of the game rather than on individual realizations of demand. Theoretically allocations for games in expectation are not negotiation proof. This approach implicitly assumes a long-term relationship among the players; if the players have the tendency to break away based on individual demand realizations, the allocations will not be stable. In addition, the Shapley value allocation mechanism is a form of revenue-sharing among the players, which we can view as a taxing mechanism. We know that the total supply chain performance is optimized when inventory is pooled; however some players may indeed be worse off. The value-sharing mechanism *taxes* the players whose profits increase due to pooling and gives refunds to players whose profits decrease. Again, this mechanism works best when the players are in a long-term relationship.

We assume complete demand-information sharing between the supplier and each of the retailers – the demand distribution of the retailer is transparent to the supplier. This is a common assumption in many game theoretic supply chain models; however it is worthwhile to consider the truth-inducing properties of the proposed allocation mechanism (see [13] for such an analysis in a different context). If the retailers were to pass on demand forecasts to the supplier, are there any incentives for the retailers to inflate the forecasts? Complete and truthful sharing of information regarding supplier and retailer revenues is also required for a revenue-sharing contract to work. Ensuring transparency of revenue information usually entails information technology investments and high monitoring costs, which may be one of the reasons why revenue-sharing contracts are not prevalent in the industry.

Finally, revenue-sharing among supply chain players belonging to different companies may raise some legal questions. First of all, the legal aspects of this problem are beyond the scope of this research. We offer the present analysis as a series of insights for supply chains where revenue-sharing may be implementable and as a benchmark of optimal performance for others where revenue-sharing is not possible due to reasons such as legal concerns. For example, the agreements between major studios and Blockbuster is one successful implementation of revenue-sharing contracts [12].

7.2 Research Directions

One of the main limitations of the proposed allocation mechanism is that it is a form of revenue sharing. The drawbacks of implementing revenue-sharing contracts have been discussed extensively in the literature. On the other hand, revenue-sharing contracts are strong contracts in the sense that they coordinate a wider range of supply chain situations when compared to other more frequently invoked contracts. One remedy the researchers offer for this dilemma is to design equivalent contracts such that the percentage of revenue each player will get is not explicitly the contract parameter but rather another decision variable (or set of variables) such as wholesale price coupled with a side payment is. Cachon and Lariviere [12] prove the equivalence of buy-back and revenue-sharing contracts for a newsvendor-based model with a single retailer and a single supplier. However the equivalence fails to hold if the underlying model is a price-dependent newsvendor. Cheng et al. [20] design an options contract which achieves the same share of revenues as the supply-chain coordinating revenue sharing contract. However in both of these papers, there is only a single player at both of the echelons. Designing an equivalent contract where there are multiple players in one of the echelons is a more challenging task; however we conjecture that designing an options contract that is equivalent to our Shapley value allocation mechanism is possible and this is one area in which we would like to pursue further research.

Another limitation of the present work is that we assume there is no competition at either of the echelons. We would like to extend our results in two directions. First, we would like to relax the independent demands assumption and incorporate retailer competition. This can be done in several different ways. For example, we can assume total market demand is fixed and all the retailers compete for their market share. In this scenario, increasing sales at one location implies decreasing sales at all the other locations. In this case, the retailers may be more reluctant towards shared inventory management policies. Another approach would be to assume a predetermined percentage of the customers are willing to search for the product or wait for it to be delivered from another retailer. In this scenario, rather than letting the supplier pool inventory, the retailers may prefer to reserve their own inventories and whenever they have excess supply, sell at a higher price to another retailer with excess demand.

For the electronics industry that motivated this research, single or dual sourcing is the industry norm for most products. Therefore we believe extending our analysis to include two suppliers will significantly enhance the insights of our model. One question that is of particular interest to us is: Given two suppliers and the Shapley value allocation mechanism, will the retailers form two mutually exclusive coalitions where each coalition works with a single supplier or will the retailers continue to dual source?

Finally, as discussed before, one concern the retailers have against inventory pooling is the fear that their pooling-partners may infer strategic information regarding their demand distributions. In relation to information sharing, we would like to extend our research in two directions. First we would like to analyze the truth-inducing properties (or lack there of) of the proposed profit-allocation mechanism. For example, in Chapter 5 we show that the retailer and supplier profit allocations are maximized at different service-level requirement pairs. The natural question that arises is whether the allocation mechanism gives incentive to the retailers to falsify their service level requirements. The second area of research is the investigation of possible information guessing under the allocation mechanism. Under reserved inventory management, the retailers share their demand information only with the supplier. However when they pool inventory and allot the benefits using a form of revenue-sharing, can the retailers infer information about their coalition partners and even more importantly can they use this information to gain advantages? If they can, this gives them incentive to falsify their demand information and/or service level requirements. This would require remedies to the Shapley value allocation mechanism to ensure truth-inducing behavior.

APPENDIX A

DERIVATION OF SUPPLY CHAIN PROFIT WHEN DEMAND IS NORMALLY DISTRIBUTED

Here we give the details of the derivation of the expected profit function when demand is normally distributed. Silver et al. [71, page 407] derive the expected total cost expression when there is a stock-out cost.

Recall that the optimal stock level is $x^* = \mu + k^*\sigma$ when demand is normally distributed where the value of the safety factor, k^*, depends on who makes the inventory decision [71]. For the case of a powerful supplier, $k^* = k_1$ satisfies

$$p_{u<}(k_1) = \frac{p-c}{p+h}$$

For the case of a centralized supply chain, $k^* = k_2$ satisfies

$$p_{u<}(k_2) = \frac{p+p_M-c}{p+p_M+h}$$

We use k^* to denote the safety factor in the derivation; whether k_1 or k_2 should be used will be apparent from the context. We first derive the expression for expected left-over inventory, H. Let $u_0 = \frac{x^*-\mu}{\sigma}$.

$$\begin{aligned}
H &= \int_{-\infty}^{x^*} (x^*-y)f(y)dy \\
&= \int_{-\infty}^{\mu+k^*\sigma} (\mu+k^*\sigma-y)f(y)dy \\
&= \sigma \int_{-\infty}^{k^*} (k^*-u_0)\frac{1}{\sqrt{2\pi}}\, e^{-\frac{u_0^2}{2}}\, du_0 \\
&= -\sigma \int_{-\infty}^{k^*} (u_0-k^*)\frac{1}{\sqrt{2\pi}}\, e^{-\frac{u_0^2}{2}}\, du_0 \\
&= -\sigma \left(\int_{-\infty}^{\infty} (u_0-k^*)\frac{1}{\sqrt{2\pi}}\, e^{-\frac{u_0^2}{2}}\, du_0 - \int_{k^*}^{\infty} (u_0-k^*)\frac{1}{\sqrt{2\pi}}\, e^{-\frac{u_0^2}{2}}\, du_0 \right) \\
&= -\sigma(0-k^*-G_u(k^*)) \\
&= \sigma k^* + \sigma G_u(k^*)
\end{aligned}$$

where we invoke the definition of $G_u(k) = \int_k^\infty (u_0 - k) f_u(u_0)\, du_0$.

Similarly we derive the expression for expected sales, S.

$$
\begin{aligned}
S &= \int_{-\infty} \mu + k^*\sigma y f(y) dy + \int_{\mu+k^*\sigma} \infty(\mu + k^*\sigma) f(y) dy \\
&= \int_{-\infty}^{\infty} y f(y) dy - \int_{\mu+k^*\sigma}^{\infty} y f(y) dy + \int_{\mu+k^*\sigma}^{\infty} (\mu + k^*\sigma) f(y) dy \\
&= \mu + \int_{\mu+k^*\sigma} \infty(\mu + k^*\sigma - y) f(y) dy \\
&= \mu + \int_{k^*}^{\infty} (u_0\sigma + k^*\sigma) f_u(u_0) du_0 \\
&= \mu - \sigma G_u(k^*)
\end{aligned}
$$

Using the expressions for expected left-over inventory and expected sales, we obtain the expected profit function, π as

$$
\begin{aligned}
\pi &= (p + p_M)(\mu - \sigma G_u(k^*)) - h(\sigma k^* + \sigma G_u(k^*)) - c(\mu + k^*\sigma) \\
&= (p + p_M - c)\mu - \ \sigma f_u(k^*)(p + p_M + h)
\end{aligned}
$$

where the last step follows from the second definition of $G_u(k)$ as given below.

$$
\begin{aligned}
G_u(k) &= \int_k^\infty (u_0 - k) f_u(u_0)\, du_0 \\
&= f_u(k^*) - k^* p_{u\geq}(k^*)
\end{aligned}
$$

REFERENCES

[1] Anonymous, "Government cost probe may scuttle mrp ii," *IE News, Production and Inventory Control*, vol. 22, no. 1, pp. 1,7, 1987.

[2] Anupindi, R. and Bassok, Y., "Centralization of stocks: Retailers vs. manufacturer," *Management Science*, vol. 45, no. 2, pp. 178–191, 1999.

[3] Anupindi, R., Bassok, Y., and Zemel, E., *Study of Decentralized Distribution Systems: Part I – A General Framework*. Kellogg Graduate School of Management. Northwestern University, Evanston, IL, 1999.

[4] Anupindi, R., Bassok, Y., and Zemel, E., "General framework for the study of decentralized distribution systems," *Manufacturing and Service Operations Management*, vol. 3, no. 4, pp. 349–368, 2001.

[5] Baker, K. R., Magazine, M. J., and Nuttle, H. L. W., "The effect of commonality on safety stock in a simple inventory model," *Management Science*, vol. 32, no. 8, pp. 982–988, 1986.

[6] Barnes, E., Dai, J., Deng, S., Down, D., Goh, M., Lau, H. C., and Sharafali, M., *Electronics Manufacturing Service Industry*. The Logistics Institute–Asia Pacific, Georgia Tech and The National University of Singapore, Singapore, 2000.

[7] Benjaafar, S., Cooper, W., and Kim, J.-S., *On the Benefits of Inventory Pooling in Production-Inventory Systems*. University of Minnesota, Minneapolis, MN, 2002.

[8] Biddle, G. and Steinberg, R., "Allocations of joint and common costs," *Journal of Accounting Literature*, vol. 3, pp. 1–45, 1984.

[9] Cachon, G., "Supply chain coordination with contracts," in *To appear in Handbooks in Operations research and Management Science: Supply Chain Management* (Graves, S. and de Kok, T., eds.), North-Holland, 2002.

[10] Cachon, G. and Lariviere, M., "Capacity allocation using past sales when to turn-and-earn," *Management Science*, vol. 45, no. 5, pp. 685–703, 1999.

[11] Cachon, G. and Lariviere, M., "Capacity choice and allocation: Strategic behavior and supply chain performance," *Management Science*, vol. 45, no. 8, pp. 1091–1108, 1999.

[12] Cachon, G. and Lariviere, M., *Supply Chain Coordination with Revenue-Sharing Contracts: Strengths and Limitations*. The Wharton School. University of Pennsylvania, Philadelphia, PA, 2000.

[13] Cachon, G. and Lariviere, M., "Contracting to assure supply: How to share demand forecasts in a supply chain," *Management Science*, vol. 47, no. 5, pp. 629–646, 2001.

[14] Cachon, G. and Netessine, S., "Game theory in supply chain analysis," in *To appear in Supply Chain Analysis in the eBusiness Era.* (Simchi-Levi, D., Wu, S. D., and Shen, Z. J., eds.), Kluwer Academic Press, 2003.

[15] Cachon, G. and Zipkin, P., "Competitive and cooperative policies in a two-stage supply chain," *Management Science*, vol. 45, no. 7, pp. 936–953, 1999.

[16] Cachon, G. P., *Competitive and Cooperative Inventory Management in a Two-Echelon Supply Chain with Lost Sales.* Fuqua School of Business, Duke University, Durham, NC, 1999.

[17] Cachon, G. P., *Stock Wars: Inventory Competition in a Two-Echelon Supply Chain with Multiple Retailers.* Fuqua School of Business, Duke University, Durham, NC, 1999.

[18] Cachon, G. P., *The Allocation of Inventory Risk and Advanced Purchase Discounts in a Supply Chain.* The Wharton School, University of Pennsylvania, Philadeplphia, PA, 2002.

[19] Cachon, G. and Lariviere, M., "An equilibrium analysis of linear, proportional and uniform allocation of scarce capacity," *IIE Transactions*, vol. 31, pp. 835–849, 1999.

[20] Cheng, F., Ettl, M., Lin, G., Schwarz, M., and Yao, D., *Flexible Supply Contracts via Options.* IBM Research Report, Yorktown Heights, NY, 2002.

[21] Cherikh, M., "On the effect of centralization on expected profits in a multi-location newsboy problem," *Journal of the Operational Research Society*, vol. 51, pp. 755–761, 2000.

[22] Dana, J. and Spier, K., "Revenue sharing and vertical control in the video rental industry," *The Journal of Industrial Economics*, vol. 49, no. 3, pp. 223–245, 2001.

[23] Deshpande, V., Cohen, M., and Donohue, K., "A threshold inventory rationing policy for service-differentiated demand classes," *Management Science*, vol. 49, no. 6, pp. 683–703, 2003.

[24] Diks, E. and de Kok, A., "Controlling a divergent 2-echelon network with transshipments using the consistent appropriate rationing policy," *International Journal of Production Economics*, vol. 45, pp. 369–379, 1996.

[25] Dong, L. and Rudi, N., *Supply Chain Interaction under Transshipments: Exogenous vs. Endogenous Wholesale Prices.* Working Paper. Olin School of Business, Washington University, St. Louis, MO, 2002.

[26] Eppen, G., "Effects of centralization on expected costs in multi-location newsboy problem," *Management Science*, vol. 25, no. 5, pp. 498–501, 1979.

[27] Eppen, G. and Schrage, L., "Centralized ordering policies in a multi-warehouse system with lead time and random demand," *Multi-Level Production/Inventory Systems: Theory and Practice, L. Schwarz(ed), North Holland*, 1981.

[28] Erkoc, M. and Wu, D., *Managing High-Tech Capacity Expansion via Reservation Contracts.* Department of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA, 2002.

[29] Eynan, A., "The impact of demands' correlation on the effectiveness of component commonality," *International Journal of Production Research*, vol. 34, no. 6, pp. 1581–1602, 1996.

[30] Gerchak, Y. and Gupta, D., "On apportioning costs to customers in centralized continuous review systems," *Journal of Operations Management*, vol. 10, no. 4, pp. 546–551, 1991.

[31] Gerchak, Y. and He, Q.-M., *On the Relation Between the Benefits of Risk Pooling and the Variability of Demand*. Tel-Aviv University, Tel-Aviv, Israel, 2001.

[32] Gerchak, Y. and Henig, M., "Inventory model with component commonality," *Operations Research Letters*, vol. 5, no. 3, pp. 157–160, 1986.

[33] Gerchak, Y., Magazine, M. J., and Gamble, A. B., "Component commonality with service requirements," *Management Science*, vol. 34, no. 6, pp. 753–760, 1988.

[34] Gerchak, Y. and Mossman, D., "On the effect of demand randomness on inventories and costs," *Operations Research*, vol. 40, no. 4, pp. 804–807, 1992.

[35] Gerchak, Y. and Wang, Y., *Revenue-Sharing vs. Wholesale-Price Contracts in Assembly Systems with Random Demand*. Department of Industrial Engineering, Tel-Aviv University, Tel-Aviv, Israel, 2000.

[36] Granot, D. and Sosic, G., "A three-stage model for a decentralized distribution system of retailers," *Operations Research*, vol. 51, no. 5, pp. 771–784, 2002.

[37] Granot, D. and Sosic, G., *Formation of Alliances in Internet-Based Supply Exchanges*. Sauder School of Business, University of British Columbia, Vancouver, Canada, 2003.

[38] Ha, A., "Inventory rationing in a make-to-stock production system with several demand classes and lost sales," *Management Science*, vol. 43, no. 8, pp. 1093–1103, 1997.

[39] Ha, A., "Stock rationing policy for a make-to-stock production system with two priority classes and backordering," *Naval Research Logistics*, vol. 43, pp. 458–472, 1997.

[40] Ha, A., "Stock rationing in an $m|e_k|1$ make-to-stock queue," *Management Science*, vol. 46, no. 1, pp. 77–87, 2000.

[41] Hartman, B. and Dror, M., "Cost allocation in continuous-review inventory models," *Naval Research Logistics*, vol. 43, pp. 549–561, 1996.

[42] Hartman, B. and Dror, M., "Optimizing centralized inventory operations in a cooperative game theory setting," *IIE Transactions*, vol. 35, pp. 243–257, 2003.

[43] Hartman, B., Dror, M., and Shaked, M., "Cores of inventory centralization games," *Games and Economic Behavior*, vol. 31, pp. 26–49, 2000.

[44] Jackson, P. and Muckstadt, J., "Risk poolong in a two-period, two-echelon inventory stocking and allocation problem," *Naval Research Logistics*, vol. 36, pp. 1–26, 1989.

[45] Jeuland, A. and Shugan, S., "Managing channel profits," *Marketing Science*, vol. 2, no. 3, pp. 239–272, 1983.

[46] Jin, M. and Wu, D., *Procurement Auction with Supplier Coalitions: Validity Requirements and Mechanism Design.* Department of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA, 2003.

[47] Lariviere, M., "Supply chain contracting and coordination with stochastic demand," in *Quantitative Models of Supply Chain Management* (Tayur, S., Magazine, M., and Ganeshan, R., eds.), Kluwer Academic Publishers, 1999.

[48] Lee, H. and Whang, S., "Decentralized multi-echelon supply chains: Incentives and information," *Management Science*, vol. 45, no. 5, pp. 633–640, 1999.

[49] Lippman, S. and McCardle, K., "The competitive newsboy," *Operations Research*, vol. 45, pp. 54–65, 1997.

[50] Louderback, J., "Another approach to allocating joint costs: A comment," *Accounting Review*, vol. 51, no. 3, pp. 683–685, 1976.

[51] McGavin, E., Ward, J., and Schwarz, L., "Balancing retailer inventories," *Operations Research*, vol. 45, no. 6, pp. 820–830, 1997.

[52] Muller, A., Scarsini, M., and Shaked, M., "The newsvendor game has a nonempty core," *Games and Economic Behavior*, vol. 38, pp. 118–126, 2002.

[53] Myerson, R. B., *Game Theory Analysis of Conflict.* Harvard University Press, Cambridge, Massachusetts, 1991.

[54] Netessine, S. and Rudi, N., *Supply Chain Structures on the Internet: Marketing-Operations Coordination under Drop-shipping.* Simon Graduate School of Business, University of Rochester, Rochester, NY, 2001.

[55] Netessine, S. and Rudi, N., *Centralized and Competitive Inventory Models with Demand Substitution.* The Wharton School, University of Pennsylvania, Philadelphia, PA, 2002.

[56] Parlar, M., "Game theoretic analysis of the substitutable product inventory problem with random demands," *Naval Research Logistics*, vol. 35, pp. 397–409, 1988.

[57] Parlar, M. and Goyal, S., "Optimal ordering decisions for two substitutable products with stochastic demands," *Opsearch*, vol. 21, no. 1, pp. 1–15, 1984.

[58] Pasternack, B. A. and Drezner, Z., "Optimal inventory policies for substitutable commodities with stochastic demand," *Naval Research Logistics*, vol. 38, pp. 221–240, 1991.

[59] Plambeck, E. and Taylor, T., *Renegotiation of Supply Contracts.* Graduate School of Business, Stanford University, Palo Alto, CA, 2002.

[60] Plambeck, E. and Taylor, T., *Sell the Plant? The Impact of Contract Manufacturing on Innovation, Capacity and Profitability.* Graduate School of Business, Stanford University, Palo Alto, CA, 2003.

[61] Porteus, E. L., "Stochastic inventory theory," *Handbooks in OR&MS. eds: D. P. Heyman and M. J. Sobel*, vol. 2, pp. 605–652, 1990.

[62] Raghunathan, S., "Impact of demand correlation in the value of and incentives for information sharing in a supply chain," *European Journal of Operational Research*, vol. 146, pp. 634–649, 2003.

[63] Robinson, L., "A comment on gerchak and gupta's "on apportioning costs to customers in centralized continuous review systems"," *Journal of Operations Management*, vol. 11, pp. 99–102, 1993.

[64] Robinson, L., "Optimal and approximate policies in multiperiod, multilocation inventory models with transshipments," *Operations Research*, vol. 38, no. 2, pp. 278–295, 1990.

[65] Rudi, N., Kapur, S., and Pyke, D., "A two-location inventory model with transhipment and local decision making," *To appear in Management Science*, 2001.

[66] Schwarz, L., Deuermeyer, B., and Badinelli, R., "Fill-rate optimization in a one-warehouse, n-identical retailer distribution system," *Management Science*, vol. 31, no. 4, pp. 488–498, 1985.

[67] Shaked, M. and Shanthikumar, J. G., *Stochastic Orders and Their Applications.* Academic Press, San Diego, CA, 1994.

[68] Shapley, L. S., "A value for n-person games," *Contribution to the Theory of Games, Princeton University Press, Princeton, NJ*, vol. 2, pp. 307–317, 1953.

[69] Shapley, L. S., "Cores of convex games," *International Journal of Game Theory*, vol. 1, pp. 11–26, 1971.

[70] Shubik, M., "The assignment of joint costs," in *Management Control: New Directions in Basic Research* (Bonini, C., Jaedicke, R., and Wagner, H., eds.), pp. 205–226, 1964.

[71] Silver, E., Pyke, D., and Peterson, R., *Inventory Management and Production Planning and Scheduling.* John Wiley & Sons, New York, 1998.

[72] Spengler, J. J., "Vertical integration and antitrust policy," *Journal of Political Economy*, vol. 58, no. 4, pp. 347–352, 1950.

[73] Swaminathan, J. and Srinivasan, R., "Managing individual service constraints under stochastic demand," *Operations Research Letters*, vol. 24, pp. 115–125, 1999.

[74] Tagaras, G., "Effects of pooling on the optimization and service levels of two-location inventory systems," *IIE Transactions*, vol. 21, no. 3, pp. 250–257, 1989.

[75] Yang, H. and Schrage, L., *An Inventory Anomaly: Risk Pooling May Increase Inventory.* Graduate School of Business, University of Chicago, Chicago, IL, 2002.

[76] Young, H., "Monotonic solutions of cooperative games," *International Journal of Game Theory*, vol. 14, pp. 65–72, 1985.

[77] Zhao, H., Deshpande, V., and Ryan, J., *Inventory Sharing and Rationing in Decentralized Dealer Networks.* School of Industrial Engineering, Purdue University, West Lafayette, IN, 2002.

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