MINIMAL SURFACES IN THE THREE-SPHERE WITH SPECIAL SPHERICAL SYMMETRY

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MINIMAL SURFACES IN THE THREE-SPHERE WITH SPECIAL SPHERICAL SYMMETRY

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To my mother, Yvonne Patricia Barrett

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TABLE OF CONTENTS

Acknowledgements	iv
List of Figures	vi
Summary	vii
Chapter 1 Introduction	1
Chapter 2 Preliminaries	2
2.1 Mean curvature of hypersurfaces	3
2.2 Mean curvature formulae	4
2.3 Rotations of $\mathbb{R}^n$	8
2.4 Stereographic projection	9
2.5 Orthogonality	11
Chapter 3 Spherical symmetry	14
3.1 Spherical reflection	14
3.2 Steiner circles and circles of Apollonius	17
3.3 Special spherical symmetry	21
Chapter 4 Classification	26
4.1 Parametrization of symmetric surfaces	26
4.2 Minimal surface equation	29
4.3 Solutions	31
References	39

LIST OF FIGURES

Figure 1	Orthogonal Circles.	12
Figure 2	Circles of Apolonnius, Steiner circles, and both sets together.	18
Figure 3	The Stereographic projection of the Clifford Torus and its meridian curve.	25
Figure 4	A surface that corresponds to a solution to (12) for $0 < c < 1/2$.	33

SUMMARY

We introduce the notion of special spherical symmetry and classify the complete regular minimal surfaces in $\mathbb{S}^3$ having this symmetry. We also show that the Clifford torus is the unique embedded minimal torus in $\mathbb{S}^3$ possessing special spherical symmetry.

CHAPTER 1
INTRODUCTION

We study minimal surfaces in $\mathbb{S}^3$ that are either 1) invariant under spherical reflection with respect to each sphere in a family of spheres that are each orthogonal to $\mathbb{S}^3$ and whose centers comprise a line $\ell \subset \mathbb{R}^4 \setminus \mathbb{S}^3$ or 2) invariant under planar reflection with respect to each hyperplane in a family of hyperplanes whose intersection contains a great circle in $\mathbb{S}^3$. We briefly review mean curvature of hypersurfaces, rotations in $\mathbb{R}^n$, stereographic projection, and orthogonality of surfaces, and then we introduce reflection about spheres and special spherical symmetry. Finally we show that if M is a complete regular minimal surface in $\mathbb{S}^3$ that possesses special spherical symmetry, then M is either a great sphere or a rotation of the Clifford torus.

CHAPTER 2

PRELIMINARIES

$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) | \, x_i \in \mathbb{R}, i = 1, \ldots, n\}$ will denote n-dimensional cartesian space equipped with the Euclidean metric

$$|x - y| = |(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)| = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

and the inner product

$$\langle x, y \rangle = \langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle = \sum_{i=1}^{n} x_i y_i.$$

A subset $M \subseteq \mathbb{R}^n$ $(n \geq 2)$ is a *regular hypersurface* if for each $p \in M$ there is a neighborhood $V \subseteq \mathbb{R}^n$ of p and a continuously differentiable mapping $X : U \subset \mathbb{R}^{n-1} \to V \cap M$ of an open set U onto $V \cap M$ such that
(i) X is a homeomorphism (i.e. X has a continuous inverse), and
(ii) the derivative of X at p, $dX(p) = dX_p : \mathbb{R}^{n-1} \to \mathbb{R}^n$, is injective.
The pair (X, U) is called a *local coordinate system* or a *parametrization* of M at p. The hypersurfaces we consider will be regular, and when $n = 3$, hypersurfaces will be referred to as $surfaces$.

Remark: (i) allows for a meaningful notion of differentiability of smooth functions defined on M. That is, (i) can be used to show that coordinates on M can be changed in a differentiable manner and thus derivatives computed on M are independent of the choice of coordinates used [D]. (ii) will allow us to talk about the tangent plane to M at each $p \in M$.

Two examples of hypersurfaces in $\mathbb{R}^n$ are the *(n-1)-sphere* or *hypersphere* centered at a point c of radius $r > 0$, $\{x \in \mathbb{R}^n : |x - c| = r\}$, and the *hyperplane* containing p with normal ν, $\{x \in \mathbb{R}^n : \langle \nu, x - p \rangle = 0\}$.

Hypersurfaces in $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ are defined analogously to those of $\mathbb{R}^n$. An example of a hypersurface in $\mathbb{S}^n$ is the *hypersphere* which is defined to be the intersection of a hyperplane in $\mathbb{R}^{n+1}$ with $\mathbb{S}^n$. Another example is the *great sphere*, which is a hypersphere whose associated hyperplane contains the origin.

For two points p, q of a hypersurface M, $d(p, q)$ is defined to be the infimum of the lengths of all piecewise differentiable curves in M joining p and q. M is *complete* if d makes M into a complete metric space. Complete hypersurfaces are connected and non-extendable, and there is always a path of least length joining any two points in a complete hypersurface [D].

2.1 MEAN CURVATURE OF HYPERSURFACES

Observe that if $M \subseteq \mathbb{R}^n$ is a hypersurface, and (X, U) is a parametrization of M at p, $dX_p : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is a linear map. And if $e_j \in \mathbb{R}^{n-1}$, $j = 1, ..., n-1$, is the vector with 1 in its *jth* coordinate and 0 in the other coordinates, then

$$dX_p(e_j) = \frac{\partial X(p)}{\partial u_j} = X_{u_j}(p).$$

We define

$$T_pM = \text{span}\left(\left\{X_{u_1}(p), X_{u_2}(p), \ldots, X_{u_{n-1}}(p)\right\}\right)$$

to be the *tangent plane* of M at p.

$\mathbb{R}^n \subset \mathbb{R}^{n+1}$ is a hypersurface and we may parameterize $\mathbb{R}^n$ globally by inclusion $i : (x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_n, 0)$. For any p, $i_{x_j}(p) = e_j$, so $T_p\mathbb{R}^n$ =span$(\{e_1, \ldots, e_n\}) =$

$\mathbb{R}^n$. $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is also a hypersurface and it can be shown that $T_p\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \langle x, p\rangle = 1\}$.

Let $N = \mathbb{R}^{n+1}$ or $\mathbb{S}^{n+1}$, $M \subseteq N$ a hypersurface and $p \in M$. We have that $T_pM \subset T_pN$ and $\dim(T_pM) = n < n+1 = \dim(T_pN)$. Therefore there are precisely two unit length, normal vectors in $T_pN \cap (T_pM)^\perp$. Locally fixing a smooth normal field $\nu : U \cap M \to \mathbb{S}^n$, where $|\nu| = 1$ and $U \subseteq N$ is a neighborhood of p, we define the *mean curvature* of M at p to be

$$H(p) = -\frac{1}{n}\operatorname{tr}(d\nu_p), \tag{1}$$

where $d\nu_p : T_pM \to T_p\mathbb{S}^n$ is the derivative of ν at p and $\operatorname{tr}(A)$ denotes the trace of a linear map A. M is said to be a *minimal hypersurface* or simply *minimal* if its mean curvature vanishes at every point in M.

Remark: Suppose $\alpha_j : (-\varepsilon_j, \varepsilon_j) \subset \mathbb{R} \to M$ ($\varepsilon_j > 0$) $j = 1, \ldots, n$ are smooth unit-speed curves satisfying $\alpha_j(0) = p$ and $\langle \alpha_j'(0), \alpha_k'(0)\rangle = \delta_{jk}$. It turns out that

$$H(p) = \frac{1}{n}\sum_{j=1}^{n} k_j$$

where $k_j = -\langle \alpha_j''(0), \nu\rangle$. That is, $H(p)$ is the average of the respective curvatures, as measured in M, of n unit-speed curves that meet orthogonally at p.

2.2 MEAN CURVATURE FORMULAE

Let $M \subseteq \mathbb{R}^3$ be a surface, $p \in M$, and (X, U) be a parametrization of M at p. Let (u, v) denote the coordinates on U and ν the normal field defined on a neighborhood of p. Below, all derivatives are evaluated at p.

We have that $\langle \nu, \nu\rangle = 1$, so $\langle \nu, \nu_u\rangle = \langle \nu, \nu_v\rangle = 0$, and thus $\nu_u, \nu_v \in T_pM = \operatorname{span}(\{X_u, X_v\})$.

Hence there are constants a, b, c, d such that

$$\nu_u = aX_u + bX_v \quad \nu_v = cX_u + dX_v \tag{2}$$

Taking inner products of X_u and X_v with (both sides of) both equations in (2), we find that $d\nu_p$ has the matrix representation

$$d\nu_p \doteq \frac{1}{EG - F^2} \begin{pmatrix} fF - eG & gF - fG \\ eF - fE & fF - gE \end{pmatrix}$$

in the basis $\{X_u, X_v\}$, where

$$E = |X_u|^2, \quad F = \langle X_u, X_v \rangle, \quad G = |X_v|^2,$$

and

$$e = \langle X_{uu}, \nu \rangle, \quad f = \langle X_{uv}, \nu \rangle, \quad g = \langle X_{vv}, \nu \rangle.$$

Thus,

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}. \tag{3}$$

Inverse stereographic projection (see section 2.4 for the definition and properties of stereographic projection) is given by

$$\sigma : \mathbb{R}^3 \to \mathbb{S}^3; (u_1, u_2, u_3) \mapsto \frac{(2u_1, 2u_2, 2u_3, u_1^2 + u_2^2 + u_3^2 - 1)}{u_1^2 + u_2^2 + u_3^2 + 1}.$$

We will write

$$\sigma(u) = \frac{(2u, |u|^2 - 1)}{|u|^2 + 1}, \quad u = (u_1, u_2, u_3).$$

Observe that $\sigma(M) \subseteq \mathbb{S}^3$ is a surface with parametrization $(\sigma \circ X, U)$ at $\sigma(p) \in \sigma(M)$.

$$T_{\sigma(p)}\sigma(M) = \operatorname{span}\left(\{(\sigma \circ X)_u, (\sigma \circ X)_v\}\right),$$

and since $|\sigma \circ X| = 1$, $\langle \sigma \circ X, (\sigma \circ X)_u \rangle = \langle \sigma \circ X, (\sigma \circ X)_u \rangle = 0$. Thus, in some neighborhood V of $\sigma(p)$ there exists a smooth unit normal field $\eta : V \cap \sigma(M) \to \mathbb{S}^3$ such that

$$\left[T_{\sigma(p)}\sigma(M)\right]^{\perp} = \operatorname{span}\left(\{\sigma \circ X, \eta\}\right),$$

and

$$\left[T_{\sigma(p)}\sigma(M)\right]^{\perp} \cap T_{\sigma(p)}\mathbb{S}^3 = \operatorname{span}\left(\{\eta\}\right).$$

A calculation similar to the one performed above to derive (3) shows

$$H_S = \frac{|(\sigma \circ X)_u|^2 \langle (\sigma \circ X)_{vv}, \eta \rangle - 2\langle (\sigma \circ X)_u, (\sigma \circ X)_v \rangle \langle (\sigma \circ X)_{uv}, \eta \rangle + |(\sigma \circ X)_v|^2 \langle (\sigma \circ X)_{uu}, \eta \rangle}{2(|(\sigma \circ X)_u|^2 |(\sigma \circ X)_v|^2 - |\langle (\sigma \circ X)_u, (\sigma \circ X)_v \rangle|^2)}, \tag{4}$$

where H_S is the mean curvature of $\sigma(M)$ at $\sigma(p)$. We have that

$$\begin{aligned}
|(\sigma \circ X)_u|^2 &= \frac{4|X_u|^2}{(1+|X|^2)^2} = \frac{4E}{(1+|X|^2)^2}, \\
\langle (\sigma \circ X)_u, (\sigma \circ X)_v \rangle &= \frac{4\langle X_u, X_v \rangle}{(1+|X|^2)^2} = \frac{4F}{(1+|X|^2)^2}, \\
|(\sigma \circ X)_v|^2 &= \frac{4|X_v|^2}{(1+|X|^2)^2} = \frac{4G}{(1+|X|^2)^2}, \\
(\sigma \circ X)_{uu} &= \frac{4}{(1+|X|^2)^2}\left(\frac{1+|X|^2}{2} X_{uu} - (|X_u|^2 + \langle X, X_{uu} \rangle) X, |X_u|^2 + \langle X, X_{uu} \rangle\right) \\
&\quad - \frac{4\langle X, X_u \rangle}{1+|X|^2} (\sigma \circ X)_u,
\end{aligned}$$

$$\begin{aligned}(\sigma \circ X)_{vv} &= \frac{4}{(1+|X|^2)^2}(\frac{1+|X|^2}{2}X_{vv} - (|X_v|^2 + \langle X, X_{vv}\rangle)X, |X_v|^2 + \langle X, X_{vv}\rangle) \\ &\quad -\frac{4\langle X, X_v\rangle}{1+|X|^2}(\sigma \circ X)_v,\end{aligned}$$

and

$$\begin{aligned}(\sigma \circ X)_{uv} &= \frac{4}{(1+|X|^2)^2}(\frac{1+|X|^2}{2}X_{uv} + \langle X, X_v\rangle X_u - \langle X, X_u\rangle X_v - (\langle X_u, X_v\rangle + \\ &\quad \langle X, X_{uv}\rangle)X, \langle X_u, X_v\rangle + \langle X, X_{uv}\rangle) - \frac{4\langle X, X_v\rangle}{1+|X|^2}(\sigma \circ X)_u.\end{aligned}$$

Substituting these calculations into (4) and using the fact that $|\eta| = 1$ and $\langle \eta, \sigma \circ X\rangle = \langle \eta, (\sigma \circ X)_u\rangle = \langle \eta, (\sigma \circ X)_v\rangle = 0$, we find [1]

$$H_S = \frac{1+|X|^2}{2}H + \langle X, \nu\rangle. \tag{5}$$

Remark: Expressions for the inner products involving η in (4) are relatively simple if the inner products are computed in the ordered (orthonormal) basis for $\mathbb{R}^4$

$$\beta = \left\{\frac{X_u}{\sqrt{E}}, \frac{\sqrt{E}}{\sqrt{EG-F^2}}\left(X_v - \frac{F}{E}X_u\right), \nu, e_4\right\}.$$

In β, η has the representation

$$\begin{aligned}\eta &\doteq \Bigg(-\frac{\langle X, X_u\rangle\langle X, \nu\rangle}{\sqrt{E}}, \frac{\langle X, X_u\rangle\langle X, \nu\rangle F - \langle X, X_v\rangle\langle X, \nu\rangle E}{\sqrt{E}\sqrt{EG-F^2}}, \\ &\quad 1 + \frac{\langle X, X_u\rangle^2}{E} - \frac{1+|X|^2}{2} + \frac{(\langle X, X_u\rangle\langle X, \nu\rangle F - \langle X, X_v\rangle\langle X, \nu\rangle E)^2}{E(EG-F^2)}, \langle X, \nu\rangle\Bigg).\end{aligned}$$

[1]The author would like to thank Sungho Park for introducing the author to equation (5).

2.3 ROTATIONS OF $\mathbb{R}^n$

$f : \mathbb{R}^n \to \mathbb{R}^n$ is a *rigid motion* of $\mathbb{R}^n$ if for all $x, y \in \mathbb{R}^n$, $|f(x) - f(y)| = |x - y|$.

Proposition 1 Let f be a rigid motion of $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n; x \mapsto f(x) - f(0)$. Then for all $x, y \in \mathbb{R}^n$ and $a \in \mathbb{R}$,

1. $|T(x)| = |x|$
2. $|T(x) - T(y)| = |x - y|$
3. $\langle T(x), T(y)\rangle = \langle x, y\rangle$, and
4. $T(x + ay) = T(x) + a\,T(y)$.

Proof: Parts 1 and 2 follow directly from definition of a rigid motion. 3. $\langle T(x), T(y)\rangle = \frac{1}{2}\left(|T(x) - T(y)|^2 - |T(x)|^2 - |T(y)|^2\right) = \frac{1}{2}\left(|x - y|^2 - |x|^2 - |y|^2\right) = \langle x, y\rangle$.

$$\begin{aligned}
4.\ |T(x+ay) - T(x) - aT(y)|^2 &= |(T(x+ay) - T(x)) - aT(y)|^2 \\
&= |T(x+ay) - T(x)|^2 + a^2|T(y)|^2 - \\
&\quad 2a\langle T(x+ay) - T(x), T(y)\rangle \\
&= |(x+ay) - x)|^2 + a^2|y|^2 - \\
&\quad 2a\left(\langle T(x+ay), T(y)\rangle - \langle T(x), T(y)\rangle\right) \\
&= 2a^2|y|^2 - 2a(\langle x+ay, y\rangle - \langle x, y\rangle) \\
&= 0. \quad \square
\end{aligned}$$

Hence if T is a rigid motion that fixes the origin, then T is a linear map and $T^tT = TT^t = id_{\mathbb{R}^n}$. Thus, $\det(T) = \pm 1$. T is said to be a *rotation* of $\mathbb{R}^n$ if $\det(T) = +1$.

$$R_\theta^{x_j x_k} = \begin{pmatrix} I_{j-1} & & & & \\ & \cos\theta & & -\sin\theta & \\ & & I_{k-j-1} & & \\ & \sin\theta & & \cos\theta & \\ & & & & I_{n-k} \end{pmatrix}$$

is the *elementary rotation* of the $x_j\, x_k$ coordinate plane by an angle θ, where I_m is the $m \times m$ identity matrix and zeros fill the empty spaces. $R_\theta^{x_j x_k}$ is a rotation of $\mathbb{R}^n$ and we refer the reader to the appendix of $[M]$ for a proof of the following proposition.

Proposition 2

1. If $u, v \in \mathbb{R}^n$, there exists a finite composition of elementary matrices R such that $Ru = v$, and

2. every rotation of $\mathbb{R}^n$ is a finite composition of elementary rotations.

2.4 STEREOGRAPHIC PROJECTION

$$\pi : \mathbb{S}^n \setminus \{e_{n+1}\} \to \mathbb{R}^n; (x_1, \ldots, x_{n+1}) \mapsto \frac{(x_1, \ldots, x_n)}{1 - x_{n+1}}$$

is defined to be *stereographic projection.* We will sometimes write $x = (x_1, \ldots, x_{n+1}) = (\underline{x}, x_{n+1})$, where $\underline{x} = (x_1, \ldots, x_n)$. With this convention

$$\pi(x) = \frac{\underline{x}}{1 - x_{n+1}}.$$

Geometrically, $\pi(x) = \mathbb{R}^n \bigcap \ell(x)$, where $\ell(x) = \{x + t(e_{n+1} - x) \in \mathbb{R}^{n+1} : t \in \mathbb{R}\}$. π is a useful tool when doing geometry in $\mathbb{S}^n$ for the following reasons

Proposition 3

1. π is bijective,

2. π is *conformal* (or angle preserving), and

3. π maps hyperspheres in $\mathbb{S}^n$ to $n-$spheres and hyperplanes in $\mathbb{R}^n$.

Proof: 1. $\pi^{-1}(u) = (2u, |u|^2 - 1)/(|u|^2 + 1)$, $u \in \mathbb{R}^n$.

2. For any smooth curve $\alpha : (a, b) \subset \mathbb{R} \to \mathbb{S}^n \setminus \{e_{n+1}\}; t \mapsto \alpha(t)$ we have

$$|(\pi \circ \alpha)'(t)| = |d\pi_{\alpha(t)}(\alpha'(t))| = \lambda(t)|\alpha'(t)|, \quad \lambda(t) = \frac{1}{1 - \alpha_{n+1}(t)} > 0.$$

So if two smooth curves $x, y \subset \mathbb{S}^n \setminus \{e_{n+1}\}$ satisfy $x(0) = y(0) = p$, and thus $\lambda(0) = 1/(1 - x_{n+1}(0)) = 1/(1 - y_{n+1}(0)) > 0$, then

$$\begin{aligned}
\frac{\langle(\pi \circ x)'(0), (\pi \circ y)'(0)\rangle}{|(\pi \circ x)'(0)||(\pi \circ y)'(0)|} &= \frac{\frac{1}{2}\left(|d\pi_p(x'(0)) - |d\pi_p(y'(0))|^2 - |d\pi_p(x'(0))|^2 - |d\pi_p(y'(0))|^2\right)}{\lambda(0)^2|x'(0)||y'(0)|} \\
&= \frac{\frac{1}{2}\left(|d\pi_p(x'(0) - y'(0))|^2 - \lambda(0)^2|x'(0))|^2 - \lambda(0)^2|y'(0))|^2\right)}{\lambda(0)^2|x'(0)||y'(0)|} \\
&= \frac{\frac{1}{2}\left(|(x'(0) - y'(0))|^2 - |x'(0))|^2 - |y'(0))|^2\right)}{|x'(0)||y'(0)|} \\
&= \frac{\langle x'(0), y'(0)\rangle}{|x'(0)||y'(0)|}.
\end{aligned}$$

3. If $P = \{x \in \mathbb{R}^{n+1} : \langle x, u\rangle = \alpha\}$

$$\begin{aligned}
\pi(P \cap \mathbb{S}^n) &= \{x \in \mathbb{R}^n : \langle x, \underline{u}\rangle = \alpha\}, \quad \text{if} \quad e_{n+1} \in P \\
&= \left\{x \in \mathbb{R}^n : \left|x - \frac{\underline{u}}{\alpha - u_{n+1}}\right| = \frac{\sqrt{|u|^2 - \alpha^2}}{|u_{n+1} - \alpha|}\right\}, \quad \text{otherwise.}
\end{aligned}$$

If $S = \{x \in \mathbb{R}^n : |x - c| = r\}$,

$$\pi^{-1}(S) = \{x \in \mathbb{S}^n : \langle x, (-2c, r^2 - |c|^2 + 1)\rangle = r^2 - |c|^2 - 1\},$$

and if $Q = \{x \in \mathbb{R}^n : \langle x, v\rangle = \beta\}$,

$$\pi^{-1}(Q) \cup \{e_{n+1}\} = \{x \in \mathbb{S}^n : \langle x, (v, \beta)\rangle = \beta\}. \quad \square$$

Remark: Observe that π is defined on $U = \{x \in \mathbb{R}^{n+1} : \langle x, e_{n+1}\rangle \neq 1\}$ and its extension to U is surjective.

2.4 ORTHOGONALITY

Two hyperplanes in $\mathbb{R}^n$ are *orthogonal* if their normal vectors are orthogonal. Two hypersurfaces in $M, N \subseteq \mathbb{R}^n$ are *orthogonal* if

1. $M \cap N \neq \emptyset$, and
2. for each $p \in M \cap N$, T_pM and T_pN are orthogonal hyperplanes.

Proposition 4

1. A hyperplane P is orthogonal to a hypersphere S if and only if P contains the center of S.
2. Two hyperspheres are orthogonal if and only if the square of the distance between their centers is equal to the sum of the squares of their radii.

Proof: 1. Let $S = \{x \in \mathbb{R}^n : |x - c| = r\}$ and $P = \{x \in \mathbb{R}^n : \langle x - x_0, u\rangle = 0\}$. If (X, U) is a parametrization of S at p, we have $|X - c|^2 = r^2$ and thus $\langle X - c, X_{u_j}\rangle = 0$ for $j = 1, \ldots, n-1$. Hence, if $p \in S \cap P$, we can take u as the normal of $T_pP = P$ and $p - c$ as the normal to T_pS. Since $\langle x - p, u\rangle = 0$ for all $x \in P$, we have that $\langle c - p, u\rangle = 0$ if and only

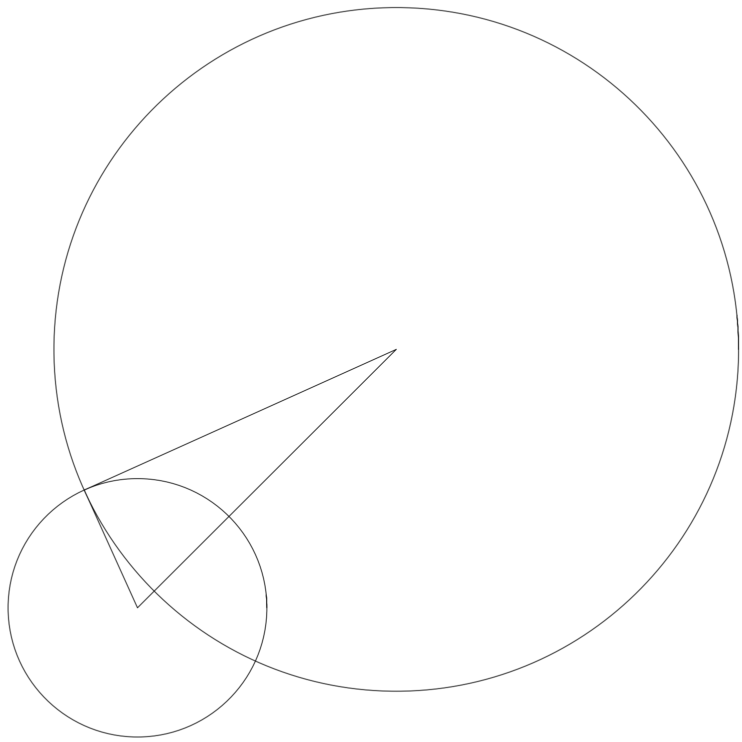

Figure 1: Orthogonal circles.

if $c \in P$ if and only if S and P are orthogonal.

2. Let $S_1 = \{x \in \mathbb{R}^n : |x - c_1| = r_1\}$ and $S_2 = \{x \in \mathbb{R}^n : |x - c_2| = r_2\}$. Suppose that S_1 and S_2 are orthogonal. As in part 1, we have that for $p \in S_1 \cap S_2$, $v_1 = p - c_1$ $v_2 = p - c_2$ can be taken as the normals to T_pS_1 and T_pS_2, respectively, and orthogonality requires $\langle v_1, v_2 \rangle = 0$. Hence

$$|c_1 - c_2|^2 = |v_1 - v_2|^2 = |p - c_1|^2 + |p - c_2|^2 = r_1^2 + r_2^2.$$

Conversely, suppose that $|c_1 - c_2|^2 = r_1^2 + r_2^2$. Observe that S_1 and S_2 intersect since $|c_1 - c_2|^2 < (r_1 + r_2)^2$. If $p \in S_1 \cap S_2$, we have that $v_1 = p - c_1$ $v_2 = p - c_2$ can be taken as the normals to T_pS_1 and T_pS_2, respectively. Since $|c_1 - c_2|^2 = r_1^2 + r_2^2$, $|v_1 - v_2|^2 = |v_1|^2 + |v_2|^2$

and consequently $\langle v_1, v_2 \rangle = 0$. Thus S_1 and S_2 are orthogonal. $\square$

CHAPTER 3

SPHERICAL SYMMETRY

Below we introduce spherical reflection and special spherical symmetry. We also establish two results that will help us identify the stereographic projections of sets in $\mathbb{S}^3$ possessing special spherical symmetry; this will in turn simplify the analysis needed for the proof of our main theorem. Theorem 10 asserts a subset $A \subset S^3$ has special spherical symmetry if and only its stereographic projection has an analogous symmetry; Theorem 8 asserts that it is possible to study the intersections of $\pi(A)$ with a specific family of planes to determine if $\pi(A)$ has this "analogous" symmetry.

3.1 SPHERICAL REFLECTION

Let $P = \{x \in \mathbb{R}^n : \langle x, u \rangle = \alpha, |u| = 1\}$ and $a \in P$. We define the mapping

$$\psi_P : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto x - 2\langle x - a, u \rangle u$$

to be *planar reflection* about P. ψ_P does not depend on the choice of $a \in P$, $\psi_P^2 = id_{\mathbb{R}^n}$, ψ_P is conformal, and ψ_P is a rigid motion. Any set $A \subseteq \mathbb{R}^n$ satisfying $\psi_P(A) = A$ is said to be *symmetric* with respect to P or *invariant* under reflection about P.

By proposition 3, $S = \pi^{-1}(P) \cup \{e_{n+1}\}$ is a hypersphere in $\mathbb{S}^n$. Now consider the map $\Psi_S : \mathbb{S}^n \to \mathbb{S}^n : x \mapsto \pi^{-1} \circ \psi_P \circ \pi(x)$. It is straightforward to show that if $0 \notin P$

$$\Psi_S(x) = \frac{v}{\alpha} + \frac{1}{\alpha^2} \frac{x - v/\alpha}{|x - v/\alpha|^2}, \tag{6}$$

where $v = (u, \alpha)$; and if $0 \in P$, $\Psi_S(x) = x - 2\langle x, v \rangle v$ where $v = (u, 0)$. Hence, if S is a great sphere, Ψ_S is just the restriction of a planar reflection to $\mathbb{S}^n$.

Ψ_S is a mapping that is completely determined by $S = \pi^{-1}(P) \cup \{e_{n+1}\}$, which is a hypersphere in $\mathbb{S}^n$. However, Ψ_S does not depend on whether or not S contains e_{n+1}. Therefore, we consider any hypersphere $S \subseteq \mathbb{S}^n$ (which does not necessarily contain e_{n+1}) and use its associated mapping Ψ_S to determine a reflection $\psi_Q : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto \pi \circ \Psi_S \circ \pi^{-1}(x)$. It can be verified that if $Q = \pi(S \cap \mathbb{S}^n)$ is a hyperplane, ψ_Q is planar reflection about Q; and if Q is an n-sphere with center a and radius ρ

$$\psi_Q(x) = a + \rho^2 \frac{x-a}{|x-a|^2}, \quad x \in \mathbb{R}^n \setminus \{a\}.$$

ψ_Q is called *spherical reflection* about Q, and any set A such that $\psi_Q(A) = A$ is said to be *spherically symmetric* with respect to Q or *invariant* under spherical reflection about Q.

Remark: Now it is evident that the mapping defined in (6) is spherical reflection about the $(n+1)$-sphere $S_0 \subseteq \mathbb{R}^{n+1}$ with radius $1/\alpha$ and center v/α. By proposition 4, S_0 is orthogonal to $\mathbb{S}^n$. Furthermore, for any hypersphere $S \subseteq \mathbb{S}^n$ that is not a great sphere there is a unique $(n+1)$-sphere $S_0 \in \mathbb{R}^{n+1}$ such that $S = S_0 \cap \mathbb{S}^n$ and S_0 is orthogonal to $\mathbb{S}^n$. S_0 is the *horizon sphere* corresponding to S.

We should also note that a horizon sphere is completely determined by its center: given any $c \in \mathbb{R}^{n+1}$ with $|c| > 1$, orthogonality requires that the horizon sphere centered at c has radius $\sqrt{|c|^2 - 1}$. The center of a horizon sphere is called a *cone point.*

Like planar reflection, spherical reflection has many nice properties. We mention just a few in the following proposition.

Proposition 5 Let $\psi : \mathbb{R}^n \setminus \{a\} \to \mathbb{R}^n \setminus \{a\}; x \mapsto a + \rho^2(x-a)/|x-a|^2$.

1. ψ is bijective.

2. The restriction of ψ to any line or hyperplane passing through a is bijective.

3. ψ maps n-spheres and hyperplanes to n-spheres and hyperplanes.

4. $\psi(S) = S$, $S = \{x \in \mathbb{R}^n : |x - a| = \rho\}$, and $\psi(x) = x$ if and only if $x \in S$.

5. ψ is conformal.

6. $\psi(S') = S'$ for any n-sphere or hyperplane S' that is orthogonal to S.

7. For $x \notin S$, $\psi(x) = (\ell \cap S_0) \setminus \{x\}$, where $\ell = \{a + t(x - a) \in \mathbb{R}^n : t \in \mathbb{R}\}$ and

$$S_0 = \left\{ y \in \mathbb{R}^n : \left| y - \left(a + \frac{\rho^2 + |x-a|^2}{2|x-a|} \frac{x-a}{|x-a|} \right) \right| = \sqrt{\left(\frac{\rho^2 + |x-a|^2}{2|x-a|} \right)^2 - \rho^2} \right\}.$$

Proof: 1.

$$\begin{aligned} \psi^2(x) &= \psi(\psi(x)) \\ &= a + \rho^2 \frac{\psi(x) - a}{|\psi(x) - a|^2} \\ &= a + \rho^2 \frac{(a + \rho^2(x-a)/|x-a|^2) - a}{|(a + \rho^2(x-a)/|x-a|^2) - a|^2} \\ &= a + (x - a) \\ &= x, \end{aligned}$$

so $\psi = \psi^{-1}$.

2. Suppose $a \in P = \{x \in \mathbb{R}^n : \langle x, n \rangle = \alpha, |n| = 1\}$. If $x \in P$, $\langle \psi(x), n \rangle = \langle a + \rho^2(x - a)/|x-a|^2, n \rangle = \langle a, n \rangle = \alpha$, so $\psi(P) \subset P$. Again we have that if $x \in P$, then $y = \psi(x) \in P$; but then $x = \psi^{-1}(y) = \psi(y) \in \psi(P)$, and thus $\psi(P) = P$. $\psi(a + tv) = a + \rho^2 v/t$, and $\psi^2(a + tv) = a + tv$, so $\psi(\ell) = \ell$ for $\ell = \{a + tv \in \mathbb{R}^n : |v| = 1, t \in \mathbb{R}\}$.

3. If $P = \{x \in \mathbb{R}^n : \langle x, n \rangle = \alpha, |n| = 1\}$, then

$$\begin{aligned}\psi(P) &= P, \quad \text{if} \quad \langle a, n \rangle = \alpha \\ &= \left\{ x \in \mathbb{R}^n : \left| x - \left(a + \frac{\rho^2/2}{\alpha - \langle a, n \rangle} n \right) \right| = \frac{\rho^2/2}{|\langle a, n \rangle - \alpha|} \right\}, \quad \text{otherwise;}\end{aligned}$$

and if $Q = \{x \in \mathbb{R}^n : |x - x_0| = r\}$,

$$\begin{aligned}\psi(Q) &= \{x \in \mathbb{R}^n : \langle x_0 - a, x - a \rangle = \rho^2/2\}, \quad \text{if} \quad |a - x_0| = r \\ &= \left\{ x \in \mathbb{R}^n : \left| x - \left(a + \rho^2 \frac{x_0 - a}{|x_0 - a|^2 - r^2} \right) \right| = \frac{\rho^2 r}{||x_0 - a|^2 - r^2|} \right\}, \quad \text{otherwise.}\end{aligned}$$

4. If $\rho = |x - a|$, $\psi(x) = x$. $\psi(x) = a + \rho^2 (x - a)/|x - a|^2 = x \Leftrightarrow |x - a| = \rho \Leftrightarrow x \in S$.

5. For any smooth curve $\alpha : (a, b) \subset \mathbb{R} \to \mathbb{R}^n \setminus \{a\} : t \mapsto \alpha(t)$, we have that

$$|(\psi \circ \alpha)'(t)| = \frac{\rho^2}{|\alpha(t) - a|^2} |\alpha'(t)|.$$

We can now repeat the argument given in the proof of proposition 3.

6. The result follows from part 3 of this proposition and proposition 4.

7. $x \in \ell \cap S_0$ and ℓ contains the center of S_0, so $\ell \cap S_0 = \{x, y\}$ for some $y \in \mathbb{R}^n$. Moreover, $\psi(\ell) = \ell$ and $\psi(S_0) = S_0$, by parts 2 and 6 of this proposition, respectively. Since $\psi = \psi^{-1}$, we have that $\psi(\ell \cap S_0) = \psi(\ell) \cap \psi(S_0) = \ell \cap S_0$. Then $\psi(\{x, y\}) = \{\psi(x), \psi(y)\} = \{x, y\}$, and by hypothesis $x \notin S$, so part 4 requires that $\psi(x) \neq x$. Hence, $\psi(x) = y$. □

3.2 STEINER CIRCLES AND CIRCLES OF APOLLONIUS

The set of circles in $\mathbb{R}^2$ passing through two points $a, b \in \mathbb{R}^2$ is the set of *Steiner Circles* corresponding to a and b.

Given two points $p, q \in \mathbb{R}^2$, the family of circles

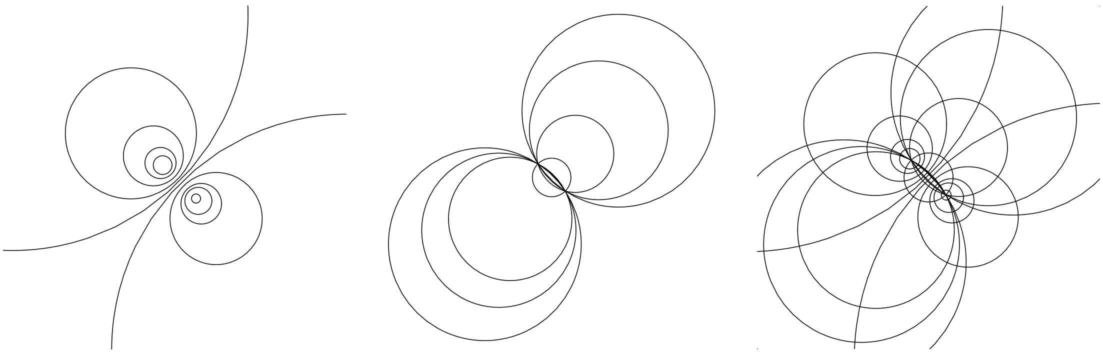

Figure 2: Circles of Apolonnius, Steiner circles, and both sets together.

$$\left\{x \in \mathbb{R}^2 : \frac{|x-p|}{|x-q|} = t\right\}_{t\in(0,1)\cup(1,\infty)}$$

are the *circles of Apollonius* corresponding to p and q.

Lemma 6 Each circle of Apollonius corresponding to a and b is orthogonal to every Steiner circle corresponding to a and b.

Proof: Let $\rho = |a-b|/2 > 0$ and without any loss of generality we may assume that $a = -\rho e_1$ and $b = \rho e_1$.

$$S = \left\{x \in \mathbb{R}^2 : \frac{|x+\rho e_1|}{|x-\rho e_1|} = t\right\} = \left\{x \in \mathbb{R}^2 : \left|x - \frac{\rho(1+t^2)}{t^2-1} e_1\right| = \frac{2\rho t}{|t^2-1|}\right\}$$

and

$$T_{\pm} = \{x \in \mathbb{R}^2 : |x \pm \sqrt{r^2-\rho^2} e_2| = r\}.$$

Any circle of Apollonius corresponding to $\pm\rho e_1$ can be represented by S for some $t \in (0,1) \cup (1,\infty)$ and any Steiner circle corresponding to $\pm\rho e_1$ is given by T_+ or T_- for some $r \geq \rho$. The result follows from proposition 4, since

$$\left| \frac{\rho(1+t^2)}{t^2-1} e_1 \pm \sqrt{r^2-\rho^2} e_2 \right|^2 = \frac{4\rho^2 t^2}{(t^2-1)^2} + r^2. \quad \square$$

Proposition 7 $A \subset \mathbb{R}^2$ is invariant with respect to each reflection about each circle in the set of Steiner circles corresponding to a and b, if and only if A is a union of circles of Apollonius corresponding to a and b and possibly the line orthogonal to the Steiner circles corresponding to a and b.

Proof: We will assume that $a = -\rho e_1$ and $b = \rho e_1$ where $\rho = |b-a|/2 > 0$. Let $\mathcal{R}$ be the set of Steiner circles corresponding to $\pm\rho e_1$ and suppose $\psi_S(A) = A$ for each circle $S \in \mathcal{R}$. Then,

$$\begin{aligned} A &= \bigcup_{S \in \mathcal{R}} A \\ &= \bigcup_{S \in \mathcal{R}} \psi_S(A) \\ &= \bigcup_{S \in \mathcal{R}} \left(\bigcup_{x \in A} \{\psi_S(x)\} \right) \\ &= \bigcup_{x \in A} \left(\bigcup_{S \in \mathcal{R}} \{\psi_S(x)\} \right). \end{aligned}$$

If $\langle x, e_1 \rangle = 0$, $\bigcup_{x \in A} \{\psi_S(x)\} = \{w \in \mathbb{R}^2 : \langle w, e_1 \rangle = 0\}$, which is the line orthogonal to all the

Steiner circles; otherwise,

$$\begin{aligned}\bigcup_{x\in A}\{\psi_S(x)\} &= \left\{w\in\mathbb{R}^2 : \left|w-\frac{|x|^2+\rho^2}{2\langle x,e_1\rangle}e_1\right| = \sqrt{\left(\frac{|x|^2+\rho^2}{2\langle x,e_1\rangle}\right)^2-\rho^2}\right\}\\ &= \left\{w\in\mathbb{R}^2 : \frac{|w+\rho e_1|}{|w-\rho e_1|} = \frac{|x+\rho e_1|}{|x-\rho e_1|}\right\},\end{aligned}$$

which is circle of Apollonius. Conversely, if A is a union of circles of Apollonius corresponding to $\pm\rho e_1$, lemma 6 has that each circle in A is orthogonal to every circle in $\mathcal{R}$. The result now follows from proposition 5. □

Theorem 8 Let $\gamma\subset\mathbb{R}^3$ be a circle. The centers of the family of spheres $\mathcal{R}$ that contain γ comprise a line ℓ. A subset $M\subseteq\mathbb{R}^3\setminus\{\ell\}$ is symmetric with respect to each sphere in $\mathcal{R}$ if and only if $M\cap P$ is a union of circles of Apollonius corresponding to $\gamma\cap P$ for each plane P containing ℓ.

Proof: γ is contained in a plane with a normal direction v. ℓ passes through the center of γ in the direction of v.

Assume that $M\subseteq\mathbb{R}^3\setminus\{\ell\}$ is symmetric with respect to each sphere in $\mathcal{R}$, let $S\in\mathcal{R}$ and P be a plane containing ℓ. Since $\psi_S=\psi_S^{-1}$ we have $\psi_S(M\cap P)=\psi_S(M)\cap\psi_S(P)$. By proposition 7, $\psi_S(P)=P$. Hence, $\psi_S(M\cap P)=M\cap P$. The restriction of ψ_S to P is reflection about a Steiner circle corresponding to $\gamma\cap P$, and since $S\in\mathcal{R}$ was arbitrary, proposition 7 has that $M\cap P$ is a union of circles of Apollonius corresponding to $\gamma\cap P$.

Now suppose that $M\cap P$ is a union of circles of Apollonius corresponding to the points $\gamma\cap P$ for each plane P passing through ℓ. Proposition 7 asserts that $\psi_S(M\cap P)=M\cap P$ for each $S\in\mathcal{R}$. It's clear that

$$\mathbb{R}^3=\bigcup_{P\supset\ell}P$$

where the union is taken over all planes $P \subset \mathbb{R}^3$ that contain ℓ, and since $\psi_S = \psi_S^{-1}$ for all $S \in \mathcal{R}$ we have

$$
\begin{aligned}
\psi_S(M) &= \psi_S\left(M \cap \mathbb{R}^3\right) \\
&= \psi_S\left(M \cap \left(\bigcup_{P \supset \ell} P\right)\right) \\
&= \psi_S\left(\bigcup_{P \supset \ell} (M \cap P)\right) \\
&= \bigcup_{P \supset \ell} \psi_S(M \cap P) \\
&= \bigcup_{P \supset \ell} (M \cap P) \\
&= M. \qquad \square
\end{aligned}
$$

3.3 SPECIAL SPHERICAL SYMMETRY

$A \subseteq \mathbb{S}^3$ has *special spherical symmetry* if 1) there is a *line of spherical symmetry* $\ell \subset \mathbb{R}^4 \setminus \mathbb{S}^3$ such that for each $x \in \ell$, $\psi_S(A) = A$, where S is the horizon sphere centered at x or 2) $\psi_S(A) = A$ for each hyperplane in a family of hyperplanes whose intersection contains a *great circle* in $\mathbb{S}^3$, which is the intersection of two distinct great spheres in $\mathbb{S}^3$.

Lemma 9 Suppose that $A \subseteq \mathbb{S}^3$ has special spherical symmetry and that A has a line of spherical symmetry ℓ. Then all the horizon spheres associated with A intersect in a two-sphere W. Moreover, $W \cap \mathbb{S}^3$ is a circle.

Proof: We may assume that there exist $p, v \in \mathbb{R}^4$ such that $|p| > 1$, $|v| = 1$ $\langle p, v\rangle = 0$, and $\ell = \{p + tv : t \in \mathbb{R}\}$. Observe that the horizon sphere S_t with center $p + tv$ has radius

$\sqrt{|p|^2+t^2-1}$ (this follows from the fact that S_t and $\mathbb{S}^3$ are orthogonal).

A sufficient and necessary condition for two spheres, with radii r_1 and r_2 and whose centers are separated by a distance d, to intersect nontrivially is that $r_1+r_2>d$. Since $d=|(p+tv)-(p+sv)|=|s-t|$ and $r_1+r_2=\sqrt{|p|^2+t^2-1}+\sqrt{|p|^2+s^2-1}>|s|+|t|\geq |s-t|=d$, S_t and S_s intersect nontrivially. Hence we may suppose that there is $x\in S_s\cap S_t$ and comparing the equations

$$|x-(p+tv)|^2=|p|^2+t^2-1 \quad \text{and} \quad |x-(p+sv)|^2=|p|^2+s^2-1$$

we find $(s-t)\langle x,v\rangle=0$. Since $s\neq t$, we must have $\langle x,v\rangle=0$. This also gives that

$$|p|^2+t^2-1=|x-(p+tv)|^2=|x-p|^2+t^2$$

or that $|x-p|=\sqrt{|p|^2-1}$. Hence,

$$S_t\cap S_s\subseteq\{x\in\mathbb{R}^4|\langle x,v\rangle=0\}\cap S_0$$

for all $s,t\in\mathbb{R}$. And if $y\in\{x\in\mathbb{R}^4|\langle x,v\rangle=0\}\cap S_0$,

$$|y-(p+tv)|^2=|y-p|^2+t^2-2t\langle y-p,v\rangle=|p|^2+t^2-1\subset S_t$$

for all $t\in\mathbb{R}$.

$$W=\{x\in\mathbb{R}^4|\langle x,v\rangle=0\}\cap S_0=\bigcap_{t\in\mathbb{R}}S_t$$

which is two-sphere and $W\cap\mathbb{S}^3=\{x\in\mathbb{S}^3|\langle x,v\rangle=0,\langle x,p\rangle=1\}$ which is a circle. $\square$

Theorem 10 $A\subseteq\mathbb{S}^3$ has special spherical symmetry if and only if there is either a circle or a line $\gamma\in\mathbb{R}^3$ such that $\psi_S\circ\pi(A)=\pi(A)$ for each sphere or plane S containing γ.

Proof: Suppose that $A \subset \mathbb{S}^3$ has special spherical symmetry. For each horizon sphere or symmetry hyperplane Q, we define $\psi_S = \pi \circ \Psi_Q \circ \pi^{-1}$ where $S = \pi(Q \cap \mathbb{S}^3)$ and Ψ_Q is spherical or planar reflection about Q. Hence $\psi_S \circ \pi(A) = \pi \circ \Psi_Q(A) = \pi(A)$. If A has a line of spherical symmetry ℓ, Lemma 9 asserts that all the horizon spheres of A intersect in a two-sphere and the intersection of the two-sphere with $\mathbb{S}^3$ is a circle W. Thus $\pi(A)$ has either spherical or reflectional symmetry with respect to each sphere or plane passing through $\pi(W) = \gamma$, which is either a circle or a line in $\mathbb{R}^3$. If A does not have a line of spherical symmetry ℓ, then the intersection of all the symmetry hyperplanes of A intersects $\mathbb{S}^3$ in a great circle and so we may repeat the argument above.

Conversely suppose that $\pi(A)$ is symmetric with respect to each sphere S passing through a circle γ. We have that $\Psi_{S'}(A) = \pi^{-1} \circ \psi_S \circ \pi(A) = A$, where S' is a symmetry sphere or hyperplane for A determined by $\pi^{-1}(S)$ and ψ_S is spherical reflection with respect to S. If γ has radius ρ, is centered a point a, and is contained in a plane with normal u, for each sphere S passing through a γ there is a unique $t \in \mathbb{R}$ such that S has center $a + tu$ and radius $\sqrt{\rho^2 + t^2}$. We write $S_t = \{x \in \mathbb{R}^3 : |x - (a + tu)| = \sqrt{\rho^2 + t^2}\}$. If $a = 0$ and $\rho = 1$, we have that the line of centers $\{tu \in \mathbb{R}^3 : t \in \mathbb{R}\}$ contains the origin, so $\pi^{-1}(\gamma)$ is a great circle. Hence, A has special spherical symmetry. If $a \neq 0$,

$$\pi^{-1}(S_t) \quad = \quad \{x \in \mathbb{R}^4 : \langle x, (-2(a+tu), \rho^2 - |a|^2 - 2t\langle a, v\rangle + 1)\rangle = \rho^2 - |a|^2 - 2t\langle a, v\rangle - 1\}$$

which is a non-great hypersphere for $t \neq t^* = (\rho^2 - |a|^2 - 1)/2\langle a, v\rangle$. The cone points associated with $\pi^{-1}(S_t)$ are

$$\alpha(t) = \frac{(-2(a+tu), \rho^2 - |a|^2 - 2t\langle a, v\rangle + 1)}{\rho^2 - |a|^2 - 2t\langle a, v\rangle - 1}, \quad t \in \mathbb{R} \setminus \{t^*\}.$$

We have that

$$|\alpha(t)| = \sqrt{\frac{4(\rho^2 + t^2)}{(\rho^2 - |a|^2 - 2t\langle a, u\rangle - 1)^2} + 1} > 1$$

and

$$\begin{aligned} \alpha'(t) &= -\frac{2}{(\rho^2 - |a|^2 - 2t\langle a, v\rangle - 1)^2}((\rho^2 - |a|^2 - 2t\langle a, v\rangle - 1)v + \langle a, v\rangle a, -2\langle a, v\rangle) \\ &= f'(t)w \end{aligned}$$

where

$$f(t) = -\frac{1}{\langle a, v\rangle}\frac{1}{\rho^2 - |a|^2 - 2t\langle a, v\rangle - 1}$$

is a real-valued, monotone function on $\mathbb{R} \setminus \{t^*\}$ and $w = ((\rho^2 - |a|^2 - 2t\langle a, v\rangle - 1)v + \langle a, v\rangle a, -2\langle a, v\rangle)$ is a constant vector. Consequently, $\ell = \alpha(\mathbb{R} \setminus \{t^*\}) \subset \mathbb{R}^4 \setminus \mathbb{S}^3$ is a line for which A is invariant under reflection about each horizon sphere centered on ℓ. We conclude that A has special spherical symmetry.

Suppose that $\pi(A)$ is symmetric with respect to each plane P containing a line γ. Again we have that $\Psi_{S'}(A) = \pi^{-1} \circ \psi_P \circ \pi(A) = A$, where S' is a symmetry sphere or hyperplane for A determined by $\pi^{-1}(P) \cup \{e_4\}$ and ψ_P is planar reflection about P. If γ contains the origin, $\pi^{-1}(\gamma)$ is a great circle and thus A has special spherical symmetry. Otherwise $\pi^{-1}(\gamma) \cup \{e_4\}$ is circle passing through e_4 and a calculation very similar to the above shows that A has special spherical symmetry. $\square$

In view of the equivalence established in theorem 10, we will say that a set $A \subseteq \mathbb{R}^3$ has special spherical symmetry if $\psi_S(A) = A$ for each sphere S containing a circle in $\mathbb{R}^3$ or $\psi_P(A) = A$ for each plane P containing a line in $\mathbb{R}^3$. With this convention, Theorem 12 also shows that rotationally symmetric surfaces (or surfaces of revolution) and their inverse stereographic projections have special spherical symmetry.

Example 11 Since spheres are rotationally symmetric, spheres (in $\mathbb{R}^3$ and $\mathbb{S}^3$) have special spherical symmetry.

Example 12 The *Clifford torus* is defined to be $\mathcal{C} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 = 1/2 = x_3^2 + x_4^2\}$. It can be shown that $\mathcal{C}$ is a minimal surface in $\mathbb{S}^3$ and that $\pi(\mathcal{C})$ is the surface of revolution obtained by rotating the circle $\Gamma = \{(x, 0, z) \in \mathbb{R}^3 : (x - \sqrt{2})^2 + z^2 = 1\}$ around the z-axis. Theorem 12 implies that $\mathcal{C}$ possesses special spherical symmetry. In fact,

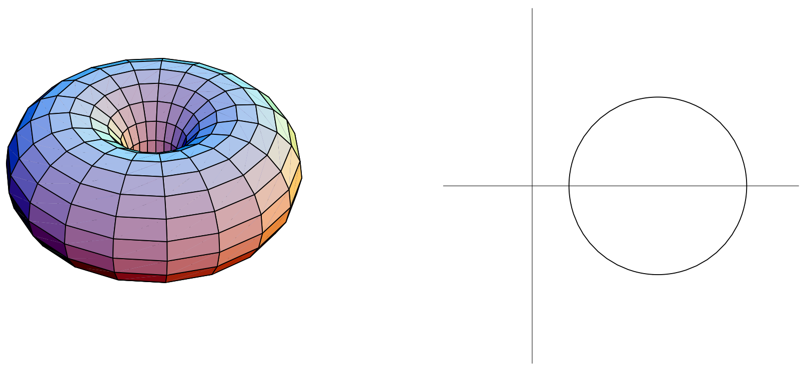

Figure 3: The stereographic projection of the Clifford torus and its meridian curve.

McCuan and Speitz have shown that every rotation of $\mathcal{C}$ has special spherical symmetry [M].

CHAPTER 4

CLASSIFICATION

In this section we establish the main result of this paper

Theorem 13 Let M be a complete regular minimal surface in $\mathbb{S}^3$ with special spherical symmetry. Then M is either a great sphere or a rotation of the Clifford torus.

An immediate consequence of Theorem 13 is a special case of Lawson's conjecture, which states that the Clifford torus is the unique complete embedded minimal torus in $\mathbb{S}^3$ [Y].

Corollary 14 If $M \subseteq \mathbb{S}^3$ is a complete embedded minimal torus with special spherical symmetry, M is necessarily a rotation of the Clifford torus.

Proof: If M is an embedded minimal torus, M is regular and has genus 1, so Theorem 13 applies. $\square$

This section is outlined as follows. First, we obtain a useful parametrization for the stereographic projections of complete regular surfaces possessing special spherical symmetry. Using this parametrization and formula (5), we then compute the mean curvature of these surfaces and obtain a minimal surface equation. Theorem 13 is a result of the classification of the solutions of the minimal surface equation.

4.1 PARAMETRIZATION OF SYMMETRIC SURFACES IN $\mathbb{S}^3$

The following lemma is a major step towards the proof of Theorem 13. The success of our strategy depends crucially on our choice of parametrization.

Lemma 15 Let $M \subset \mathbb{S}^3$ be a complete regular surface. M possesses spherical symmetry if and only if there is a rotation R of $\mathbb{R}^4$ such that $\pi \circ R(M)$ is a plane or at each point $\pi \circ R(M)$ admits a local parametrization of the form

$$X(\theta,\phi) = ([\sqrt{r(\theta)^2+\rho^2} + r(\theta)\cos\phi]\cos\theta, [\sqrt{r(\theta)^2+\rho^2} + r(\theta)\cos\phi]\sin\theta, r(\theta)\sin\phi + h) \quad (7)$$

$(\theta,\phi) \in (\theta_0,\theta_1) \times [0,2\pi)$, where $\rho,\theta_0,\theta_1,h \in \mathbb{R}$ and $r : (\theta_0,\theta_1) \longrightarrow [0,\infty]$ is a smooth function of θ.

Proof: ($\Rightarrow$) Suppose M has special spherical symmetry and that M is invariant under reflection about each hyperplane in a family of hyperplanes whose intersection contains a great circle $\xi \subset \mathbb{S}^3$. Then there exist $n_1, n_2 \in \mathbb{S}^3$ such that $\xi = \{x \in \mathbb{S}^3 : \langle x, n_1\rangle = \langle x, n_2\rangle = 0\}$; and for any rotation R of $\mathbb{R}^4$ satisfying $Rn_1 = e_3$ and $Rn_2 = e_4$, we have that $\pi \circ R(\xi)$ is the unit circle in the xy plane. It follows that $\psi_S \circ \pi \circ R(M) = \pi \circ R(M)$ for each sphere S containing the unit circle.

Now suppose that $\pi \circ R(M)$ is not a plane and let p be a point in $\pi \circ R(M)$ that is not on the z-axis in $\mathbb{R}^3$. By Corollary 10, the intersection of the half plane Π_0 containing p and the z-axis is a union of circles of Apolonnius (corresponding to $\pm e_1$). Hence p lies on a circle of Apolonnius in Π_0. Since M is regular, there is some $\epsilon > 0$ such that the intersection of the halfplane Π_θ, obtained by rotating Π_0 about the z-axis by an angle θ, and $\pi \circ R(M)$ is non-trivial and thus contains a circle of Apolonnius for $|\theta| < \epsilon$. If we denote the radius of these circles by $r(\theta)$, (7) is a local parametrization of M at p with $h = 0$ and $\rho = 1$. For if we fix $\theta = \theta_0 \in (-\epsilon, \epsilon)$ we have that

$$\left| X(\theta_0, \phi) - \sqrt{r(\theta_0)^2 + 1}(\cos\theta_0, \sin\theta_0, 0) \right| = r(\theta_0) \quad \phi \in [0, 2\pi),$$

and so $X(\theta_0, \cdot)$ is a circle of Apolonnius in the halfplane Π_{θ_0} with radius $r(\theta_0)$. The smoothness of r follows from the regularity of M.

Now suppose that p is on the z-axis. The special spherical symmetry of M requires that $\pi \circ R(M)$ contains the z-axis and that $\pi \circ R(M)$ is unbounded. For each small $\delta > 0$, the completeness of M assures that there is $x \in \pi \circ R(M)$ (that is not on the z-axis) such that $|x - p| \leq d(x, p) < \delta$. From the above arguments, (7) is a parametrization of M at x and thus x lies on a circle of Apolonnius α corresponding to $(\pm e_1)$. The closest point (in the sense of $|\cdot|$) on α to the z-axis is $X(0, \pi)$ with

$$|X(0, \pi) - (0, 0, 0)| = \sqrt{r(0)^2 + 1} - r(0) \leq |x - p| < \delta,$$

and thus

$$r(0) > \frac{1}{2}\left(\delta^{-1} - \delta\right).$$

Hence, we can always find a sequence of circles in $\pi \circ R(M)$ that converge to the z-axis with corresponding radii that converge to ∞. Since $\pi \circ R(M)$ is regular we can assume that this convergence happens in a (single) neighborhood of the p. Hence, $\pi \circ R(M)$ can be parametrized at p by (7) ($h = 0$ and $\rho = 1$), where there is a $\theta^* \in (\theta_0, \theta_1)$ such that $\lim_{\theta \to \theta^*} X(\theta, \cdot)$ is the z-axis.

Suppose M has special spherical symmetry and that M is invariant under reflection about each horizon sphere centered on a line $\ell \subset \mathbb{R}^4 \setminus \mathbb{S}^3$. By Proposition 2, there is a rotation R_0 of $\mathbb{R}^4$ such that $R_0(\ell)$ intersects the x_4 axis in $\mathbb{R}^4$. Then $\pi \circ R_0(\ell)$ is a line that contains the origin in $\mathbb{R}^3$. Again by Proposition 2, there is a rotation P of $\mathbb{R}^3$ such that

$P \circ \pi \circ R_0(\ell)$ is the z-axis. Trivially extending P to a rotation of $\mathbb{R}^4$ via $Pe_4 = e_4$ we have $P \circ \pi \circ R_0(\ell) = \pi \circ PR_0(\ell)$ which by design is the z-axis. Hence there is a rotation $R = PR_0$ of $\mathbb{R}^4$ such that $\pi \circ R(\ell)$ is the z-axis. Now we can repeat the argument given in the previous case where the the unit circle is now replaced with the circle γ of radius ρ that is centered at a point $(0, 0, h)$ on the z-axis and all the symmetry spheres of $\pi \circ R(\ell)$ contain γ.

($\Leftarrow$) If $\pi \circ R(M)$ is a plane, then example 11 implies that M has special spherical symmetry. If $\pi \circ R(M)$ is not a plane and can be parametrized by (10) at each point, for some rotation R of $\mathbb{R}^4$, then this implication follows directly from Theorem 8 and Theorem 10. $\square$

4.2 MINIMAL SURFACE EQUATION

Suppose that $M \subset \mathbb{S}^3$ is a complete regular minimal surface with special spherical symmetry. Formula (5) implies that

$$H_S = \frac{1 + |X|^2}{2} H + \langle X, N \rangle \equiv 0,$$

where X is a parametrization for $\pi(M)$ with unit normal field N and H is the mean curvature of $\pi(M)$. If H vanishes identically, then $\langle X, N \rangle = 0$. In this case, $\pi(M)$ is a plane containing the origin and hence M is a great sphere. If H does not vanish identically, Lemma 15 asserts that there is a rotation R of $\mathbb{R}^4$ such that $\pi \circ R(M)$ admits the parametrization given in (7). Therefore we may assume without any loss of generality that $\pi(M)$ can be parametrized by (7).

We have

$$\begin{aligned} N &= \frac{X_\theta \times X_\phi}{|X_\theta \times X_\phi|} \\ &= \frac{\sqrt{r^2 + \rho_0^2} \cos \phi \, u_1 - r' u_2 + \sqrt{r^2 + \rho_0^2} \sin \phi \, e_3}{\sqrt{r'^2 + r^2 + \rho_0^2}} \end{aligned} \qquad (8)$$

where $u_1 = (\cos\theta, \sin\theta, 0)$, $u_2 = (-\sin\theta, \cos\theta, 0)$, and using (3)

$$\begin{aligned} H &= \frac{a_0 + a_1\cos\phi + a_2(\cos\phi)^2}{2r(r\cos\phi + \sqrt{r^2+\rho_0^2})^2(r^2+\rho_0^2+r'^2)^{3/2}} \\ a_0 &= \sqrt{r^2+\rho_0^2}\left[(r^2+\rho_0^2)rr'' - ((r^2+\rho_0^2)^2 + (2r^2+\rho_0^2)r'^2)\right] \\ a_1 &= r\left[(r^2+\rho_0^2)(rr'' - 3(r^2+\rho_0^2)) - (3\rho_0^2+4r^2)r'^2\right] \\ a_2 &= -2r^2\sqrt{r^2+\rho_0^2}(r^2+\rho_0^2+r'^2). \end{aligned} \tag{9}$$

Moreover,

$$\begin{aligned} 0 &= (H_S)_\phi\,|_{\phi=0} \\ &= \left(\langle X, X_\phi\rangle H + \frac{1+|X|^2}{2}H_\phi + \langle X, N_\phi\rangle + \langle X_\phi, N\rangle\right)|_{\phi=0} \\ &= h\left(rH|_{\phi=0} + \sqrt{\frac{r^2+\rho^2}{r^2+\rho^2+r'^2}}\right) \end{aligned}$$

where $r' = dr/d\theta$. If $h \neq 0$, then

$$rH|_{\phi=0} + \sqrt{\frac{r^2+\rho^2}{r^2+\rho^2+r'^2}} = 0. \tag{10}$$

The general solution to (10) is

$$r(\theta) = \sqrt{a^2 - (a^2+\rho^2)\left(\sin(\theta-\theta_0)\right)^2},$$

with $a \neq 0$. Since

$$\left|X(\theta,\phi) - \left(\sqrt{a^2+\rho^2}\cos\theta_0, \sqrt{a^2+\rho^2}\sin\theta_0, h\right)\right| = |a|,$$

M is necessarily a great sphere. Therefore we have proved

Proposition 16 If M is a complete regular minimal surface whose stereographic projection M is parametrized by (7), $h \neq 0$ implies that M is a great sphere.

This leads us to the following surprising result

Corollary 17 If M is a complete regular minimal surface whose stereographic projection M is parametrized by (7), $h = 0$ implies that $\rho = 1$.

Proof: If $h = 0$, the xy-plane is a plane of symmetry for $P = \pi(M)$, where is M is not a great sphere. $\pi \circ R^{zw}_{\pi/2} \circ \pi^{-1}$ maps the xy-plane to $\mathbb{S}^2$. Thus $\mathbb{S}^2$ is a symmetry sphere for $\pi \circ R^{zw}_{\pi/2} \circ \pi^{-1}(P) = P'$. Since M was not a great sphere, $\pi^{-1}(P')$ is not a great sphere. On the other hand P' possesses special spherical symmetry (where all the symmetry spheres are centered along the z-axis) so P' can be parametrized by (7) and again we have that $h = 0$. Hence the xy-plane is a plane of symmetry for P'. It follows that the symmetry spheres of P' all meet in the unit circle on the xy plane and thus $\rho = 1$. $\square$

From (8) and (9), we find that when $h = 0$ (and thus $\rho = 1$)

$$H_S = \frac{\sqrt{r^2+1}\,[r(1+r^2)r'' + r^4 - 1 - r'^2]}{2r(r'^2 + r^2 + 1)^{3/2}} = 0,$$

or

$$r(1+r^2)r'' + r^4 - 1 - r'^2 = 0. \tag{11}$$

4.3 SOLUTIONS

$r(\theta) = 1$ is the lone positive constant solution to (11). In this case, (7) becomes

$$X(\theta, \phi) = \left([\sqrt{2} + \cos\phi] \cos\theta, [\sqrt{2} + \cos\phi] \sin\theta, \sin\phi \right), \quad (\theta, \phi) \in [0, 2\pi)^2.$$

Thus $X([0, 2\pi)^2)$ is the surface of revolution with generating curve $x^2 + (\sqrt{2} - z)^2 = 1$ which implies that M is the Clifford torus (see example 12).

More generally, (11) has the first integral

$$c = \frac{r}{\sqrt{1 + r^2}\sqrt{1 + r^2 + r'^2}}$$

where c is a constant of integration. Hence, each solution to (11) is *periodic* and satisfies

$$r'^2 = \frac{r^2 - c^2(1 + r^2)^2}{c^2(1 + r^2)}. \tag{12}$$

We must have $0 < c \leq 1/2$ for r'^2 to exist and be nonnegative. $c = 1/2$ corresponds to the Clifford torus, so now we will only consider solutions for $0 < c < 1/2$.

In order to complete the proof of Theorem 13, we need to show that there are no solutions to (12) for $0 < c < 1/2$ that correspond to complete regular minimal surfaces. This can be done through analyzing the periods of the solutions. For a fixed $c \in (0, 1/2)$, the only way for the surface that corresponds to the solution to (12) to be regular is that there is a natural number n such that $nT(c) = 2\pi$, where $T(c)$ is the period of the solution (i.e. $r(\theta) = r(\theta + T(c))$ for all θ). Otherwise, the surface will be self-intersecting and thus non-regular (see figure 4 for an example of such a surface). We now proceed to show that this is precisely the case.

A solution to (12) for a fixed $0 < c < 1/2$ assumes minimum and maximum values r_0

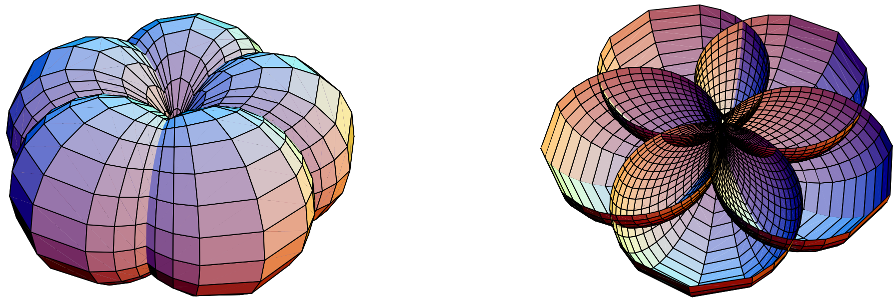

Figure 4: A surface that corresponds to a solution to (12) for $0 < c < 1/2$.

and r_1, respectively, where

$$r_0 = \frac{1-\sqrt{1-4c^2}}{2c} \quad \text{and} \quad r_1 = \frac{1+\sqrt{1-4c^2}}{2c}$$

(since (12) is translation invariant, we may assume $r(0) = r_0$). It follows that r is monotone increasing on some interval $[0, \theta_{max}]$, where $r(\theta_{max}) = r_1$, and r is then monotone decreasing on $[\theta_{max}, T(c)]$ where $r(T(c)) = r(T(c)+0) = r(0) = r_0$. We have that

$$\theta_{max} = \int_{r_0}^{r_1} \frac{c\sqrt{1+r^2}}{\sqrt{r^2 - c^2(1+r^2)^2}} dr,$$

and

$$T(c) - \theta_{max} = -\int_{r_1}^{r_0} \frac{c\sqrt{1+r^2}}{\sqrt{r^2 - c^2(1+r^2)^2}} dr.$$

Thus,

$$T(c) = \int_{r_0}^{r_1} \frac{2c\sqrt{1+r^2}}{\sqrt{r^2 - c^2(1+r^2)^2}} dr. \tag{13}$$

The following proposition describes the periods of the solutions to (12). From our comments above, it is easily seen that the first assertion of the proposition completes the proof of Theorem 13.

Proposition 18

1. $\pi < T(c) \leq \sqrt{2}\pi$, $0 < c < 1/2$, and

2. T is monotone increasing on $(0, 1/2)$, $\lim_{c \searrow 0} T(c) = \pi$ and $\lim_{c \nearrow 1/2} T(c) = \sqrt{2}\pi$

Proof: 1. $\sqrt{1+r^2} > r$, so

$$\begin{aligned} T(c) &= \int_{r_0}^{r_1} \frac{2c\sqrt{1+r^2}}{\sqrt{r^2 - c^2(1+r^2)^2}} dr \\ &> \int_{r_0}^{r_1} \frac{2cr}{\sqrt{r^2 - c^2(1+r^2)^2}} dr \\ &= \int_{r_0}^{r_1} \frac{2cr}{\sqrt{1/4c^2 - 1 - (cr^2 + c - 1/2c)^2}} dr \\ &= \sin^{-1}\left(\frac{cr^2 + c - 1/2c}{\sqrt{1/4c^2 - 1}}\right)\Big|_{r_0}^{r_1} \\ &= \pi. \end{aligned}$$

$c^2(1+r^2)^2 \leq r^2$ which implies $c(r + 1/r) \leq 1$, and thus

$$2\sqrt{\frac{1+r^2}{1+r/c+r^2}} = 2\sqrt{1 - \frac{1}{c(r+1/r)+1}} \leq 2\sqrt{1 - \frac{1}{2}} = 2\frac{1}{\sqrt{2}} = \sqrt{2}.$$

Hence,

$$\begin{aligned} T(c) &= \int_{r_0}^{r_1} \frac{2c\sqrt{1+r^2}}{\sqrt{r^2 - c^2(1+r^2)^2}} dr \\ &= \int_{r_0}^{r_1} \frac{2\sqrt{1+r^2}}{\sqrt{1+r/c+r^2}} \frac{dr}{\sqrt{-1+r/c-r^2}} \\ &\leq \sqrt{2}\int_{r_0}^{r_1} \frac{dr}{\sqrt{(r-r_0)(r_1-r)}} \\ &= \sqrt{2}\pi. \end{aligned}$$

2.

$$
\begin{aligned}
T(c) &= \int_{r_0}^{r_1} \frac{2c\sqrt{1+r^2}}{\sqrt{r^2-c^2(1+r^2)^2}} dr \\
&= \int_{r_0}^{r_1} \sqrt{1+\frac{1}{r^2}} \frac{2cr}{\sqrt{r^2-c^2(1+r^2)^2}} dr \\
&= \sqrt{1+\frac{1}{r^2}} \sin^{-1}\left(\frac{cr^2+c-1/2c}{\sqrt{1/4c^2-1}}\right)\Big|_{r_0}^{r_1} \\
&\quad - \int_{r_0}^{r_1} \left(\sqrt{1+\frac{1}{r^2}}\right)' \sin^{-1}\left(\frac{cr^2+c-1/2c}{\sqrt{1/4c^2-1}}\right) dr \\
&= \frac{\pi\sqrt{1+2c}}{2c} + \int_{r_0}^{r_1} \frac{\sin^{-1}\left((cr^2+c-1/2c)/\sqrt{1/4c^2-1}\right)}{r^2\sqrt{1+r^2}} dr, \qquad (14)
\end{aligned}
$$

and now it's clear that we may use Leibnitz' rule for differentiating integrals. Doing so, we find

$$
T'(c) = \frac{2}{1-4c^2} \int_{r_0}^{r_1} \frac{(1-2c^2)r^2-2c^2}{r^2\sqrt{1+r^2}\sqrt{r^2-c^2(1+r^2)^2}} dr.
$$

For each fixed $0 < c < 1/2$,

$$
r \mapsto \frac{r+\frac{\sqrt{2}c}{\sqrt{1-2c^2}}}{r^2\sqrt{1+r^2}\sqrt{cr^2+c+r}}
$$

is a positive, monotone decreasing function of r, and

$$
\inf_{r\in[r_0,r_1]} \left\{ \frac{r+\frac{\sqrt{2}c}{\sqrt{1-2c^2}}}{r^2\sqrt{1+r^2}\sqrt{cr^2+c+r}} \right\} = \frac{\sqrt{2}c^{5/2}(2\sqrt{1-2c^2}+\sqrt{2}(1-\sqrt{1-4c^2}))}{\sqrt{1-2c^2}(1+\sqrt{1-4c^2})^2} > 0.
$$

We also have for each $c \in (0, 1/2)$

$$
\int_{r_0}^{r_1} \frac{r-\frac{\sqrt{2}c}{\sqrt{1-2c^2}}}{\sqrt{r-c(1+r^2)}} dr = \pi \frac{\sqrt{1-2c^2}-2\sqrt{2}c^2}{2c^{3/2}\sqrt{1-2c^2}} > 0
$$

since $\sqrt{1-2c^2}-2\sqrt{2}c^2>0$ for $c\in(0,1/2)$. With these estimates

$$\begin{aligned}
T'(c) &= \frac{2}{1-4c^2}\int_{r_0}^{r_1}\frac{(1-2c^2)r^2-2c^2}{r^2\sqrt{1+r^2}\sqrt{r^2-c^2(1+r^2)^2}}dr \\
&= \frac{2(1-2c^2)}{1-4c^2}\int_{r_0}^{r_1}\frac{r+\frac{\sqrt{2}c}{\sqrt{1-2c^2}}}{r^2\sqrt{1+r^2}\sqrt{cr^2+c+r}}\frac{r-\frac{\sqrt{2}c}{\sqrt{1-2c^2}}}{\sqrt{r-c(1+r^2)}}dr \\
&\geq \frac{2(1-2c^2)}{1-4c^2}\frac{\sqrt{2}c^{5/2}(2\sqrt{1-2c^2}+\sqrt{2}(1-\sqrt{1-4c^2}))}{\sqrt{1-2c^2}(1+\sqrt{1-4c^2})^2}\int_{r_0}^{r_1}\frac{r-\frac{\sqrt{2}c}{\sqrt{1-2c^2}}}{\sqrt{r-c(1+r^2)}}dr \\
&> 0.
\end{aligned}$$

Now we proceed to show that $\lim_{c\searrow 0}T(c)=\pi$ by proving that

$$\lim_{c\searrow 0}\int_{r_0}^{1}\frac{2c\sqrt{1+r^2}}{\sqrt{r^2-c^2(1+r^2)^2}}dr=0 \quad\text{and}\quad \lim_{c\searrow 0}\int_{1}^{r_1}\frac{2c\sqrt{1+r^2}}{\sqrt{r^2-c^2(1+r^2)^2}}dr=\pi. \tag{15}$$

Recall that for $r\in[r_0,r_1]$, $2\sqrt{r^2+1}/\sqrt{r/c+r^2+1}\leq\sqrt{2}$, and for $r\in[r_0,1]$, $1/\sqrt{r_1-1}\geq 1/\sqrt{r_1-r}$. Thus,

$$\begin{aligned}
\int_{r_0}^{1}\frac{2c\sqrt{1+r^2}}{\sqrt{r^2-c^2(1+r^2)^2}}dr &= \int_{r_0}^{1}\frac{2\sqrt{1+r^2}}{\sqrt{1+r/c+r^2}}\frac{dr}{\sqrt{-1+r/c-r^2}} \\
&= \int_{r_0}^{1}\frac{2\sqrt{1+r^2}}{\sqrt{1+r/c+r^2}}\frac{1}{\sqrt{r_1-r}}\frac{dr}{\sqrt{r-r_0}} \\
&\leq \frac{\sqrt{2}}{\sqrt{r_1-1}}\int_{r_0}^{1}\frac{dr}{\sqrt{r-r_0}} \\
&= \frac{\sqrt{2}}{\sqrt{1/r_0-1}}2\sqrt{1-r_0} \\
&= 2\sqrt{2r_0} \\
&= 2\sqrt{\frac{1-\sqrt{1-4c^2}}{c}} \\
&= 2\sqrt{\frac{4c}{1+\sqrt{1-4c^2}}} \\
&\leq 4\sqrt{c},
\end{aligned}$$

which establishes the first limit in (15). As for the second limit in (15), we first observe that since $\sqrt{1+r^2}-r<1/2r$ for $r>0$,

$$\begin{aligned}
\left|\int_1^{r_1}\frac{2c\sqrt{r^2+1}}{\sqrt{r^2-c^2(r^2+1)^2}}dr-\int_1^{r_1}\frac{2cr}{\sqrt{r^2-c^2(r^2+1)^2}}dr\right| &= \int_1^{r_1}\frac{2c(\sqrt{r^2+1}-r)}{\sqrt{r^2-c^2(r^2+1)^2}}dr\\
&< \frac{1}{2}\int_1^{r_1}\frac{1}{r}\frac{2c}{\sqrt{r^2-c^2(r^2+1)^2}}dr\\
&= \frac{1}{2}\int_{r_0}^{1}\frac{2cr}{\sqrt{r^2-c^2(r^2+1)^2}}dr\\
&= \frac{1}{2}\left(\frac{\pi}{2}-\sin^{-1}(\sqrt{1-4c^2})\right)
\end{aligned}$$

which goes to 0 as $c\searrow 0$. Hence

$$\begin{aligned}
\lim_{c\searrow 0}\int_1^{r_1}\frac{2c\sqrt{r^2+1}}{\sqrt{r^2-c^2(r^2+1)^2}}dr &= \lim_{c\searrow 0}\int_1^{r_1}\frac{2cr}{\sqrt{r^2-c^2(r^2+1)^2}}dr\\
&= \lim_{c\searrow 0}\left(\frac{\pi}{2}+\sin^{-1}(\sqrt{1-4c^2})\right)\\
&= \pi.
\end{aligned}$$

Finally, we note that

$$\begin{aligned}
\left|\int_{r_0}^{r_1}\frac{\sin^{-1}\left((cr^2+c-1/2c)/\sqrt{1/4c^2-1}\right)}{r^2\sqrt{1+r^2}}dr\right| &\le \frac{\pi}{2}\int_{r_0}^{r_1}\frac{dr}{r^2\sqrt{1+r^2}}\\
&= \frac{\pi}{2}\left(\sqrt{1+\frac{1}{r_0^2}}-\sqrt{1+\frac{1}{r_1^2}}\right)
\end{aligned}$$

which implies that

$$\lim_{c\nearrow 1/2}\int_{r_0}^{r_1}\frac{\sin^{-1}\left((cr^2+c-1/2c)/\sqrt{1/4c^2-1}\right)}{r^2\sqrt{1+r^2}}dr=0,$$

since $\lim_{c\nearrow 1/2}r_1=\lim_{c\nearrow 1/2}r_0=1$. From (14) it is evident that $\lim_{c\nearrow 1/2}T(c)=\sqrt{2}\pi$. $\square$

Remark: Our classification applies to any minimal surface in $\mathbb{S}^3$ that can be rotated so that its stereographic projection admits the parametrization (7). We believe that this result applies in general to complete minimal surfaces possessing special spherical symmetry that can be immersed in $\mathbb{S}^3$.

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