COMBINATORIAL DIVISOR THEORY FOR GRAPHS

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To my parents,

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for their unwavering support.

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SUMMARY

Chip-firing is a deceptively simple game played on the vertices of a graph, which was independently discovered in probability theory, poset theory, graph theory, and statistical physics. In recent years, chip-firing has been employed in the development of a theory of divisors on graphs analogous to the classical theory for Riemann surfaces. In particular, Baker and Norin were able to use this set up to prove a combinatorial Riemann-Roch formula, whose classical counterpart is one of the cornerstones of modern algebraic geometry. It is now understood that the relationship between divisor theory for graphs and algebraic curves goes beyond pure analogy, and the primary operation for making this connection precise is tropicalization, a certain type of degeneration which allows us to treat graphs as "combinatorial shadows" of curves. The development of this tropical relationship between graphs and algebraic curves has allowed for beautiful applications of chip-firing to both algebraic geometry and number theory.

In this thesis we continue the combinatorial development of divisor theory for graphs. In Chapter 1 we give an overview of the history of chip-firing and its connections to algebraic geometry. In Chapter 2 we describe a reinterpretation of chip-firing in the language of partial graph orientations and apply this setup to give a new proof of the Riemann-Roch formula. We introduce and investigate transfinite chip-firing, and chip-firing with respect to open covers in Chapters 3 and 4 respectively. Chapter 5 represents joint work with Arash Asadi, where we investigate Riemann-Roch theory for directed graphs and arithmetical graphs, the latter of which are a special class of balanced vertex weighted graphs arising naturally in arithmetic geometry.

CHAPTER I

INTRODUCTION

Chip-firing is a simple and elegant graph theoretic process with connections to various areas of mathematics, and the sciences at-large. For describing chip-firing, we begin with a finite set of *chips* on the vertices of a graph. The fundamental operation is *firing*, whereby a vertex sends a chip to each of its neighbors and loses its degree number of chips in the process, so that the total number of chips in the graph is conserved. We remark that our use of the word chips is intended to connote a collection indistinguishable poker chips sitting at the vertices of the graph, as opposed to computer chips (although rather interestingly, the latter interpretation is not without merit, e.g. [17]). If one encodes a chip configuration by a vector $\vec{x}$ then the operation of firing the ith vertex can described in a linear algbraic fashion by subtracting the ith column of the Laplacian matrix from $\vec{x}$.

The history of chip-firing is quite complicated due to its independent discovery by several different communities. The term chip-firing seems to have been introduced by Björner, Lovász, and Shor [15] in their seminal paper where they develop ideas introduced by Spencer [72], and Anderson, Lovász, Shor, Spencer, Tardos, and Winograd [5]. The phrase chip-firing appeared in print for the first time in Tardos [73], but he refers to Björner, Lovász, and Shor so it seems that the latter paper simply took longer to publish.

In statistical physics, chip-firing was independently introduced around the same time by Bak, Tang, and Weisenfeld [7] for the square two-dimensional lattice as an example of a phenomenon which they called *self organized criticality*. When investigating chip-firing dynamics on large grid graphs, they encounters a phenomenon which they called an *avalanche* where large cascades of chip-firings occur in short succession, and it was their hope that chip-firing could be utilized for describing real world events such as forest fires and earthquakes. Their model is referred to as the *Abelian Sandpile Model*, often abbreviated as the ASM, and was rigorously developed by Dhar [26] shortly afterwards. The use of the term sandpile comes from their description of chips as grains of sand (suggestive of their large scale perspective), and the adjective abelian is used to emphasize the important property that the final outcome of chip-firing is independent of any choices made. In the ASM it is often assumed that there is a sink vertex, e.g. the contracted boundary of the square grid, which does not fire, so as to ensure that the chip-firing process terminates. This abelian property of chip-firing was also observed by Björner, Lovász, and Shor who noted that this made chip-firing into an example of an abstract rewriting system with the Church Rosser property, i.e., the confluence property. They also noted chip-firing was an example of a greedoid, or more precisely, an antimatroid. Cori, Rossin, and Salvy [25], and independently Postnikov and Shapiro [66] later realized that one could naturally associate a binomial ideal to chip-firing, and in this context the abelian property can be reinterpreted as saying that the binomials associated to the firings of the vertices form a graded reverse lexicographic Gröbner basis. We note that Gröbner bases are a well-known example of an abstract rewriting system with the confluence property. This commutative algebraic investigation of binomial ideals associated to chip-firing has gained much attention in recent years [64, 55, 60, 54, 54, 6]

By repeatedly adding chips and firing until it is no longer possible (stabilizing), we obtain a Markov chain, and Dhar prove that the recurrent states for this chain provide a collection of distinguished representatives for the cokernel of the reduced Laplacian. This cokernel is a finite abelian group which he referred to as the *sandpile group*. Chip-firing allows for a combinatorial presentation of this group, where the group law is defined by adding two recurrent states and stabilizing. It follows from basic linear algebra that the cokernel has order equal to the determinant of the reduced Laplacian and by Kirchoff's matrix-tree theorem, this is precisely the number of spanning trees of the graph. Dhar showed that by a beautiful process which he called *burning*, an explicit bijection could be obtained between the recurrent configurations of the ASM and the spanning trees of a graph. Variations on this burning process have been developed, some of which allow for bijections which preserve important tree statistics such as external activity [24] or tree inversion number [65].

Last summer the author attended a chip-firing workshop at the American Institute of Mathematics. On the last day, Jim Propp gave a short speech where he explained the fact, which surprisingly few audience member were aware of, that chip-firing is originally due to Engel [30] from the 1970's who called it the "probabilistic abacus" and treated it as a pedagogical tool for teaching 4th grade students about Markov chains! Engel is reportedly attempting to publish a text book about this perspective, but unfortunately has yet to receive an offer from any publishing company.

Also in the 1970's, while investigating Hasse diagrams for posets, Mosesian introduced the concept of a *sink reversal*, also called a *pushing down* for acyclic graph orientations. The idea is that given any sink t in an acyclic orientation, one can reverse the orientation of all of the incoming edges to produce another acyclic orientation. It is not hard to see that the indegree sequences of the two acyclic orientations are related by the firing of t. The notion of a sink reversal and the connection to chip-firing were addressed in Björner, Lovász, and Shor although they seemed to be unaware of Mosesian's previous work. These authors identified the indegree sequences of the acyclic orientations as the minimal recurrent states in the sinkless chip-firing model. Gioan [35] recently generalized sink reversals to arbitrary orientations by introducing cut reversals and cycle reversals. In Chapter 2 of this thesis the author systematically further generalizes Gioan's theory to the setting of partial graph orientations.

In arithmetic geometry, Raynaud found a description of the component group of the special fiber of the Neron model of a curve in terms of the special fiber of a regular semistable model. His result says that the component group is canonically isomorphic to the cokernel of the Laplacian of the dual graph of the special fiber of the model for the curve. Motivated by this result, Lorenzini proved several theorems about the structure of the cokernel of the Laplacian of a graph [50, 51, 48]. The work of Raynaud and Lorenzini may be viewed as the first important step in the development of an exciting and very active area of research relating chip-firing to algebraic geometry and number theory. It is worth noting that because Lorenzini was unaware of chip-firing, his approach to studying the cokernel of the Laplacian of a graph employed mostly linear algebraic techniques to get a handle on the Smith normal form of the Laplacian, which encodes this group.

The next major step in this direction came from a somewhat different angle. Bacher, De la Harpe, and Nagnibeda [62] developed, in a combinatorial way, but still without the aid of chip-firing, a detailed theory of cut and flow lattices, and the Jacobian of a graph. Their paper extends earlier results of Biggs and was written with the motivation to develop a theory of divisors on graphs analogous to the classical theory for Riemann surfaces. Lorenzini never explicitly referred to the cokernel of the Laplacian as the Jacobian of a graph, so their paper seems to be the first appearance of this phrase in the literature. Bacher, De la Harpe and Nagnibeda were unaware of Lorenzini's previous work and treated their results as being analogous to classical results for Riemann surfaces. In particular, they did not suggest the possibility that graphs might be viewed rigorously as "combinatorial shadows" of Riemann surfaces or more generally, algebraic curves.

Approaching the topic from the perspective of Arakelov theory and Berkovich analytic curves, Baker began to investigate the theory of divisors graphs. In particular, Baker was interested in the question of whether there existed a graphical version of the celebrated Riemann-Roch formula.

The Riemann-Roch theorem is a statement about the dimension, called the rank, of a linear space of meromorphic functions on a Riemann surface with prescribed lower bounds for zeroes and poles. Baker did not initially know how to define such a quantity in the setting of graphs where one is bereft of geometry. Cleverly, he observed that without having a working definition of rank, one could still conjecture the following special case of the Riemann-Roch theorem. Suppose you have an integral (not necessarily positive) configuration of chips on a graph. If the number of chips is at least $|E(G)| - |V(G)| + 1$, the genus of G, then there exists a sequence of chip-firing moves which brings every vertex out of debt. Baker had an REU student Dragos Ilas perform some computations, which indeed supported Matt's conjecture. Ilas presented these results in a talk at Georgia Tech where Sergey Norin was present, who then proved this special case of Riemann-Roch over the next couple of days. Baker and Norin began working together, and shortly after, the Riemann-Roch formula for graphs was established [9].

The fundamental combinatorial tool which Baker and Norin employed is a distinguished type of chip-configurations called a q-reduced divisor. This is a configuration which is nonnegative away from q, such that the firing any subset of vertices not including q causes some vertex to be sent into debt. These configurations are known elsewhere as G-parking functions, although reduced divisors are technically different in that they keep track of the number of chips at q. G-parking functions were first defined by Postnikov, and by taking G to be the complete graph, one recovers the classical parking function whose name is derived from a certain combinatorial problem about car parking. Parking functions are quite popular within Stanley's school of combinatorics, and it is the author's understanding that they were originally introduced by Pyke [67] and further studied by French combinatorialists before Pak introduced them to Stanley. Perhaps the most famous application of parking functions is due to Haiman [38] who utilized them in his algebraic geometric proof of positivity for MacDonald polynomials.

Baker and Norine observed that q-reduced divisors were in bijection with the recurrent states of the abelian sandpile model by a simple duality, which appears very similar to Riemann-Roch duality, and remains somewhat mysterious. Perhaps the most intriguing insight was given by Manjunath and Sturmfels [55], who observed that this duality could be interpreted as a manifestation of Alexander duality for monomial ideals. In Chapter 4 of this thesis, we describe a family of chip-firing models induced by simplicial complexes on the vertex set of a graph which provide a fine interpolation between these two models.

The Riemann-Roch formula was soon extended to the continuous setting of metric graphs independently by Gathmann and Kerber [33], and Mikhalkin and Zharkov [58]. The former authors' approach was to prove the statement by a taking a continuous limit of Baker and Norin's result. Both sets of authors were motivated to provide a "tropical" version of Baker and Norin's result. Tropical geometry is a certain piece-wise linear version of algebraic geometry, obtained by degenerating varieties to polyhedral complexes. The tropicalization of a variety has real dimension equal to that of the original variety, hence the tropicalization of a curve is a one dimensional object. For the purposes of divisor theory, the embedding of a tropical curve is unimportant, so without loss of generality, we may view this object as a metric graph. Tropical curves have unbounded rays (tentacles), which are also unimportant from the perspective of divisor theory, and so we may disregard these parts of tropical curves, and assume that the metric graphs in question are compact.

Baker [58] then proved a "specialization lemma" which states that when passing from a curve to its dual graph, the rank of a divisor cannot drop. This inequality is significant in that it allows certain questions about ranks of divisors on algebraic curves to be reduced to questions about ranks of divisors on graphs. Baker's result has had some very nice recent applications in geometry and number theory, such as

Cools, Draisma, Payne, and Robeva's tropical proof of the Brill-Noether theorem [22] and Katz and Zureick-Brown's contribution to the theory of effective bounds for the number of rational points on curves [42].

In this thesis, we continue the combinatorial investigation of divisors on both discrete and metric graphs. The outline of the paper is as follows. In the Chapter 2 we describe a complete reinterpretation of the linear equivalence of divisors on graphs via a generalization of Gioan's cycle-cocycle reversal system for partial graph orientations. We show that the Baker-Norine rank of a partially orientable divisor is one less than the minimum number of directed paths which need to be reversed in the generalized cocycle reversal system to produce an acyclic partial orientation. We apply this perspective in giving new proofs of Baker and Norine's Riemann-Roch theorem for graphs as well as Luo's topological characterization of rank determining sets [52]. We then describe a fundamental connection between divisor theory for graphs and the max-flow min-cut theorem from combinatorial optimization. We conclude with an overview of the ways in which these results extend to metric graphs.

In Chapter 3 we introduce and investigate transfinite chip-firing on metric graphs. Luo presented a metric version of Dhar's burning algorithm for the investigation of divisor theory on metric graphs [52]. We give a new proof of the finite termination of Luo's iterated Dhar algorithm, and then investigate Baker and Luo's question of whether the greedy reduction algorithm terminates in finite time. We provide a strongly negative answer to this question. We first show that the Euclidean algorithm can be modeled by the reduction of a certain degree 12 divisor on a metric graph of genus 7. By running this example on two incommensurable number, we obtain an example of greedy reduction which does not terminate. We remark that any infinite greedy reduction has a well defined limit, and so we may pass to the limit and begin the algorithm again. This allows us to investigate the greedy reduction of divisors on metric graphs using the language of ordinal numbers, and we show that the set of all running times for the greedy algorithm is precisely the set of ordinal numbers strictly less than ω^ω.

In Chapter 4 we introduce the notion of chip-firing with respect to an open cover. We begin with discrete graphs using the discrete topology, where our allowed firings are determined by an abstract simplicial complex on the vertices. It is shown that each divisor stabilizes uniquely, and is linearly equivalent to a unique recurrent configuration. These discrete models are equivalent to ones independently introduced by the statistical physicist Paoletti [63], and we generalize this set up to directed graphs by allowing vertex weighted abstract simplicial complexes. A generalization of the Cori-Le Borgne bijection [24] between the chip-firing recurrent states and the spanning trees of an undirected graph is presented which is applicable for any simplicial complex. We conclude with a discussion of the case of metric graphs where finite sets are replaces by open covers. We explain that the basic results extend, and thus each open cover of a metric graph induces a canonical presentation of the Jacobian. We explain how any two to one cover of the metric graph by stars serves as a continuous analogue of the abelian sandpile model, in particular, we obtain a continuous version of a duality due by Baker and Norine which is remarkably similar to Riemann-Roch duality.

In Chapter 5 we describe work with Arash Asadi extending Riemann-Roch theory to directed graphs. By the lattice reduction algorithm of Wilmes, this setup allows for a combinatorial interpretation of Amini and Manjunath's Riemann-Roch theory for lattices [3]. We generalize Dhar's burning algorithm for this setting, which is dual to an algorithm introduced by Speer, and use this to give a method for determining whether or not a given directed graph has the Riemann-Roch formula. We then apply this algorithm to the study of arithmetical graphs, which are certain balanced vertex weighted graphs introduced by Lorenzini. In particular we give a very satisfying solution to a question posed by Lorenzini, who asked for a combinatorial proof of the fact that if there are at least g_0 chips present in an arithmetical graph, there necessarily exists a way of bringing all of the vertices out of debt by chip-firing moves. Lorenzini's original proof of this result was algebraic geometric in nature. We conclude by presenting some examples of arithmetical graphs with and without the Riemann-Roch property.

CHAPTER II

RIEMANN-ROCH THEORY FOR GRAPH ORIENTATIONS

We develop a new framework for investigating linear equivalence of divisors on graphs using a generalization of Gioan's cocycle reversal system for partial orientations. An oriented version of Dhar's burning algorithm is introduced and employed in the study of acyclicity for partial orientations. We then show that the Baker-Norine rank of a partially orientable divisor is one less than the minimum number of directed paths which need to be reversed in the generalized cocycle reversal system to produce an acyclic partial orientation. These results are applied in providing new proofs of the Riemann-Roch theorem for graphs as well as Luo's topological characterization of rank-determining sets. We demonstrate that max-flow min-cut is equivalent to the Euler characteristic description of orientable divisors, and extend this characterization to the setting of partial orientations. Efficient algorithms for computing break divisors and constructing partial orientations are presented.

2.1 Introduction

Baker and Norine [9] introduced a combinatorial Riemann-Roch theorem for graphs analogous to the classical statement for Riemann surfaces. Their result employed chip-firing, a deceptively simple game on graphs with connections to various areas of mathematics. Given a graph G, we define a *configuration of chips* D on G as a function from the vertices to the integers. A vertex v *fires* by sending a chip to each of its neighbors, losing its degree number of chips in the process. If we take D to be a vector, firing the vertex v_i precisely corresponds to subtracting the ith column of the Laplacian matrix from D. In this way we may view chip-firing as a combinatorial language for describing the translates of the lattice generated by the columns of the Laplacian matrix.

Reinterpreting chip configurations as *divisors*, we say that two divisors are *linearly equivalent* if one can be obtained from the other by a sequence of chip-firing moves, and a divisor is *effective* if each vertex has a nonnegative number of chips. Baker and Norine define the *rank* of a divisor, denoted $r(D)$, to be one less than the minimum number of chips which need to be removed so that D is no longer equivalent to an effective divisor. Taking the *canonical divisor* K to have entries $K(v) = \deg(v) - 2$ and defining the *genus* of G to be $g = |E(G)| - |V(G)| + 1$, they prove the Riemann-Roch formula:

$$r(D) - r(K - D) = \deg(D) - g + 1.$$

Baker and Norine's proof depends in a crucial way on the theory of *q-reduced divisors*, known elsewhere as a *G-parking functions* or *superstable configurations*. A divisor D is said to be *q-reduced* if (i) $D(v) \geq 0$ for all $v \neq q$, and (ii) for any non-empty subset $A \subset V(G) \setminus \{q\}$, firing the set A causes some vertex in A to go into debt. They show that every divisor D is linearly equivalent to a unique q-reduced divisor D', and $r(D) \geq 0$ if and only if D' is effective. We note that q-reduced divisors are dual, in a precise sense, to the *recurrent configurations* (also known as *q-critical configurations*), which play a prominent role in the abelian sandpile model [9, Lemma 5.6]

There is a second story, which runs parallel to that of chip-firing, describing certain constrained reorientations of graphs, first introduced by Mosesian [61] in the context of Hasse diagrams for posets. Given an acyclic orientation of a graph $\mathcal{O}$ and a sink vertex q, we can perform a *sink reversal*, reorienting all of the edges incident to q.

This operation is directly connected to the theory of chip-firing: we can associate to $\mathcal{O}$ a divisor $D_{\mathcal{O}}$ with entries $D_{\mathcal{O}}(v) = \text{indeg}_{\mathcal{O}}(v) - 1$, and performing a sink reversal at v_i we obtain the orientation $\mathcal{O}'$ with associated divisor $D_{\mathcal{O}'}$ given by the firing of v_i. Mosesian observed that, provided an acyclic orientation $\mathcal{O}$ and a vertex q, there exists an acyclic orientation $\mathcal{O}'$ having q as the unique source, which is obtained from $\mathcal{O}$ by sink reversals. The divisors associated to these q-rooted acyclic orientations are the maximal noneffective q-reduced divisors. This connection between acyclic orientations and chip-firing dates back at least to Björner, Lovász, and Shor's original paper on the topic [15].

Gioan [35] generalized this setup to arbitrary (not necessarily acyclic) orientations by introducing the *cocycle reversal*, wherein all of the edges in a consistently oriented cut can be reversed, and a dual *cycle reversal*, in which the edges in a consistently oriented cycle can be reversed. Using these two operations, he defined the *cycle-cocycle reversal system* as the collection of full orientations modulo cycle and cocycle reversals, and proved that the number of equivalence classes in this system is equal to the number of spanning trees of the underlying graph. He also showed that each orientation is equivalent in the cocycle reversal system to a unique *q-connected orientation*. These are the orientations in which every vertex is reachable from q by a directed path. Bernardi [11] combined these results, presenting an activity-preserving bijection between the *minimal q-connected orientations* and spanning trees of a graph, where minimal refers to a standardized choice of the orientation's cyclic part. Recently An, Baker, Kuperberg, and Shokrieh [4] showed that the divisors associated to the q-connected orientations are precisely the break divisors of Mikhalkin and Zharkov [58] offset by a chip at q. They then applied this observation to give a "volume proof" of Kirchoff's matrix-tree theorem via a polyhedral decomposition of Pic^g.

A limitation of the orientation-based perspective is that the divisor associated to an orientation will always have degree $g - 1$. In this work, we introduce a generalization of the cycle-cocycle reversal system for investigating *partial orientations*, thus allowing for a discussion of divisors with degrees less than $g - 1$. The *generalized cycle-cocycle reversal system* is defined by the introduction of *edge pivots*, whereby an edge (u, v) oriented towards v is unoriented and an unoriented edge (w, v) is oriented towards v (see Figure 1). Note that edge pivots, as with cycle reversals, leave the divisor associated to a partial orientation unchanged. We demonstrate that this additional operation is dynamic enough to allow for a characterization of linear equivalence, that is, we prove that two partial orientations are equivalent in the generalized cycle-cocycle reversal system if and only if the associated divisors are linearly equivalent. Moreover, we use edge pivots and cocycle reversals to prove, via an explicit construction, that a divisor with degree at most $g - 1$ is linearly equivalent to a divisor associated to a partial orientation, unless D has negative rank, in which case we obtain a certificate in the form of an acyclic partial orientation.

Dhar's burning algorithm is one of the key tools in the study of chip-firing. Originally discovered in the context of the abelian sandpile model, Dhar's algorithm provides a linear-time test for determining whether a given configuration is q-reduced. There are variants of Dhar's algorithm which produce bijections between q-reduced divisors and spanning trees, some of which respect important tree statistics such as external activity [24] or tree inversion number [65]. In the work of Baker and Norine, this algorithm was implicitly employed in the proof of their core lemma RR1, which states that if a divisor has negative rank then it is dominated by a divisor of degree $g - 1$ divisor which also has negative rank. We present an "oriented" version of Dhar's algorithm whose iterated application provides a method for determining whether a partial orientation is equivalent in the generalized cocycle reversal system to an acyclic partial orientation or a sourceless partial orientation. We introduced q-connected partial orientations and use them to prove that the Baker-Norine rank of

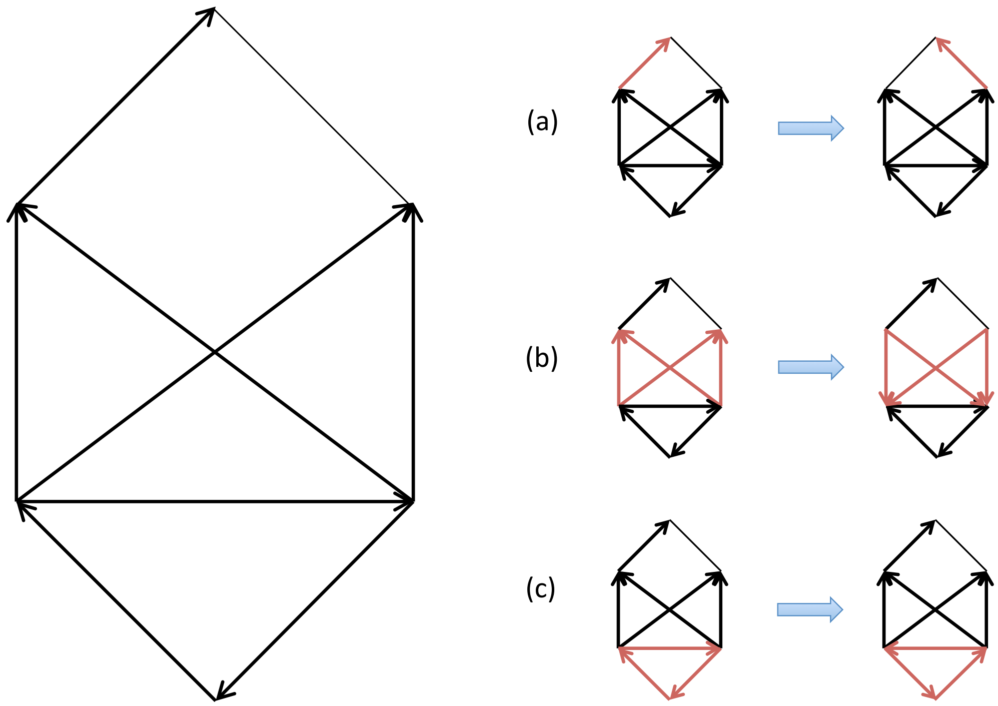

Figure 1: A partial orientation with (a) an edge pivot, (b) a cocycle reversal, and (c) a cycle reversal.

a divisor associated to a partial orientation is one less than the minimum number of directed paths which need to be reversed in the generalized cocycle reversal system to produce an acyclic orientation. We then apply these results in providing a new proof of the Riemann-Roch theorem for graphs. For this, we employ a variant of Baker and Norine's formal reduction involving strengthened versions of RR1 and RR2.

The Riemann-Roch theorem was extended to metric graphs and tropical curves by Gathmann and Kerber [33], and Mikhalkin and Zharkov [58]. We are currently writing an extension of the results from this chapter to the setting of metric graphs, and in the final section 2.8 we present a preliminary description of this work.

Luo [52] investigated the notion of a *rank-determining set* of a metric graph, a collection A of points such that the rank of any divisor can be computed by removing chips only from points in A. As a second application of the path-reversal description of ranks, we provide a new proof of Luo's topological characterization of rank-determining sets as those which intersect every special open set. Our proof involves a reduction to the case of full orientations and hence does not require any techniques involving partial orientations of metric graphs.

We discuss a close relationship between network flows and divisor theory. A polynomial-time method for computing break divisors is provided, combining the observation (originally due to Felsner [31]) that max-flow min-cut can be used to construct orientations, with An, Baker, Kuperberg, and Shokrieh's reinterpretation of break divisors as the q-connected orientations offset by a chip at q. We demonstrate that the max-flow min-cut theorem is logically equivalent to the Euler characteristic description of orientable divisors [4], and provide an extension of this result for partial orientations. We conclude with an efficient algorithm for constructing a partial orientation whose associated divisor is linearly equivalent to a given divisor, which integrates max-flow min-cut and the oriented Dhar's algorithm.

The perspective given by partial orientations is more "matroidal" than the divisor theory of Baker and Norine. In future work, we plan to extend the ideas from this chapter to *partial reorientations of oriented matroids*.

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2.2 Notation and Terminology

Graphs: We take G to be a finite loopless undirected connected multigraph with vertex set $V(G)$ and edge set $E(G)$. For $X, Y \subset V(G)$, we write (X, Y) for the set of edges with one end in X and the other in Y. Therefore (X, X^c) is the cut defined by X. Given $v \in V(G)$, we write $\mathrm{outdeg_X}(v)$ for the number of edges incident to v leaving the set X. We take Δ to be the *Laplacian* matrix $\Delta = D - A$, where D is diagonal with $(i,i)th$ entry $\deg(v_i)$, the negative of the number of edges incident to v_i, and A is the adjacency matrix with $(i,j)th$ entry equal to the number of edges between v_i and v_j. A *divisor*, or a *configuration of chips*, is a function $D : V(G) \to \mathbb{Z}$ and denote the set of divisors on G, by $\mathrm{Div}(G)$. We define $\mathrm{Pic}^d(G)$ to be the set of divisors of degree d modulo linear equivalence. If a vertex v_i *fires*, it send a chip to each of its neighbors, losing its degree number of chips in the process, and we obtain the new divisor $D - \Delta e_i$. We define the firing of a set of vertices to be the firing of each vertex in that set. We say that two divisors D and D' are *linearly equivalent*, written $D \sim D'$, if there exists a sequence of firings bringing D to D', i.e., $D - D'$ is in the $\mathbb{Z}$-span of Δ. A vertex v is in debt if $D(v) < 0$, and D is *effective* if no vertex is in debt. The *rank* of a divisor is the quantity $r(D) = \min_{E \geq 0} \deg(E) - 1$ such that there exists no $E' \geq 0$ with $D - E \sim E'$. The genus of a graph $g = |E(G)| - |V(G)| + 1$, also known as the cyclomatic number of G, is the dimension of the cycle space of G. The *canonical divisor* K is the divisor with with values $K(v_i) = \deg(v_i) - 2$. A divisor D is said to be *q-reduced* for some $q \in V(G)$ if (i) $D(v) \geq 0$ for all $v \neq q$, and (ii) for any set $A \subset V(G) \setminus \{q\}$, firing A causes some vertex to be sent into debt. We take the set of *non-special divisors* to be $\mathcal{N} = \{\nu : \deg(D) = g - 1, r(\nu) = -1\}$. Let $D_1, D_2 \geq \vec{0}$ with disjoint supports such that $D_1 - D_2 = D$. We write $\mathrm{deg^+(D)}$ and $\mathrm{deg^-(D)}$ for $\mathrm{deg(D_1)}$ and $\mathrm{deg(D_2)}$ respectively.

Orientations: An *orientation* of an edge $e = (u,v) \in E(G)$ is a pairing (e,v). In this case we say that e has tail u and head v, and is oriented away from u and oriented towards v. We draw an oriented edge, i.e., directed edge as an arrow pointing from u to v. A *partial orientation* $\mathcal{O}$ of a graph is an orientation of a subset of the edges. A partial orientation is said to be *full*, or simply an orientation, if each edge in the graph is oriented. A directed path is a sequence of oriented edges such that the head of each oriented edge is tail of its successor, and the heads of the edges are all distinct. For a partial orientation $\mathcal{O}$ and a set $X \subset V(G)$, we write $\bar{X}_{\mathcal{O}}$ for the set of vertices reachable from X by a directed path in $\mathcal{O}$, or simply $\bar{X}$ when $\mathcal{O}$ is clear from the context. The indegree of a vertex v in $\mathcal{O}$, written $\text{indeg}_{\mathcal{O}}(v)$ or simply $\text{indeg}(v)$, is the number of edges oriented towards v in $\mathcal{O}$. We associate to each partial orientation a divisor $D_{\mathcal{O}}$ with $D_{\mathcal{O}}(v) = \text{indeg}(v) - 1$. We note that the importance of the -1 here is not expected to be immediately clear upon introduction. We say that a divisor is partially orientable, resp. orientable, if it is the divisor associated to some partial, resp. full, orientation. We say a vertex is a *source* in a partial orientation if it has no incoming edges. We say that a partial orientation is *acyclic* if it contains no directed cycles and *sourceless* if each vertex has an incoming edge. We note that a partial orientation is sourceless if and only if the associated divisor is effective. Given a partially orientable divisor D we denote by $\mathcal{O}_D$ any partial orientation with associated divisor D.

An *edge pivot* at a vertex v is an operation on a partial orientation $\mathcal{O}$ whereby an edge oriented towards v is unoriented and an unoriented edge incident to v is oriented towards v. A *cocycle* or *cut* in the graph, which we use interchangeably, is the set of edges connecting a set of vertices A and its complement A^c. We say that a cut is *saturated* if each edge in the cut is oriented. A cut is *consistently oriented* in $\mathcal{O}$ if the cut is saturated and each edge in the cut is oriented in the same direction. We may also refer to this cut as being *saturated towards* A if the cut is consistently oriented towards A. We similarly define a consistently oriented cycle in $\mathcal{O}$. A *cut reversal*, resp. *cycle reversal*, in $\mathcal{O}$ is performed by reversing all of the edges in a consistently oriented cut, resp. cycle. The *cycle*, resp. *cocycle*, resp. *cycle-cocycle reversal systems* describe the collection of full orientations of a graph modulo cycle, resp. cocycle, resp. cycle and cocycle reversals. The *generalized cycle*, resp. *cocycle*, resp. *cycle-cocycle* *reversal systems* are the previous systems extended to partial orientations by the inclusion of edge pivots. If two partial orientations $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle-cocycle reversal system, we simply say that they are *equivalent* and write $\mathcal{O} \sim \mathcal{O}'$. A partial orientation is said to be *q-connected* if there exists a directed path from q to every other vertex. We refer to those edges in a partial orientation $\mathcal{O}$ which belong to a directed cycle as the *cyclic part* of $\mathcal{O}$.

For a non-empty $S \subset V(G)$, we take $G[S]$ to be the induced subgraph on S and let $D|_S$ be the divisor D restricted to S. We define $\chi(S)$ to be the Euler characteristic of $G[S]$, i.e., $|S| - |E(G[S])|$. Given a divisor D and a non-empty subset $S \subset V(G)$, we define $\chi(S, D) = \deg(D|_S) + \chi(S)$, $\chi(G, D) = \min_{S \subset V(G)} \chi(S, D)$, $\bar{\chi}(S, D) = |E(G)| - |E(G[S^c])| - |S| - \deg(D|_S)$, and $\bar{\chi}(G, D) = \min_{S \subset V(G)} \bar{\chi}(S, D)$

2.3 Generalized Cycle, Cocycle, and Cycle-Cocyle Reversal Systems

The following two statements generalize results of Gioan [35, Proposition 4.10 and Corollary 4.13] to the setting of partial orientations. That is, if we remove the words "partial" and "generalized" from the following two statements, we obtain results of Gioan.

Lemma 2.3.1. *Two partial orientations $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle reversal system if and only if $D_{\mathcal{O}} = D_{\mathcal{O}'}$.*

Proof. Clearly, if $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle reversal system then $D_{\mathcal{O}} = D_{\mathcal{O}'}$. We now demonstrate the converse.

Suppose there exists some vertex v incident to an edge e which is oriented towards v in $\mathcal{O}$ and is unoriented in $\mathcal{O}'$. Because $D_{\mathcal{O}} = D_{\mathcal{O}'}$, there exists another edge e' which is oriented towards v in $\mathcal{O}'$ such that e' is not also oriented towards v in $\mathcal{O}$. We can perform an edge pivot so that e' becomes unoriented and e is now oriented towards v in both $\mathcal{O}$ and $\mathcal{O}'$. By induction on the number of of edges with different orientations in $\mathcal{O}$ and $\mathcal{O}'$, we can assume that no such edge exists, and claim that the orientations differ by cycle reversals.

Let e be some edge oriented towards v in $\mathcal{O}$ and away from v in $\mathcal{O}'$. Again, because $D_{\mathcal{O}} = D_{\mathcal{O}'}$ there exists another edge e' which is oriented away from v in $\mathcal{O}$ and towards v in $\mathcal{O}'$. We may perform a directed walk along edges in $\mathcal{O}$ which are oriented oppositely in $\mathcal{O}'$. Eventually this walk must reach a vertex which had already been visited. This gives a cycle which is consistently oriented in $\mathcal{O}$ and $\mathcal{O}'$ with opposite orientations. We can reverse the orientation of this cycle in $\mathcal{O}$ and again induct on the number of edges with different orientation in $\mathcal{O}$ and $\mathcal{O}'$, thus proving the claim. $\square$

Definition 2.3.2. *Given a directed path P from u to v in G, and an unoriented edge e incident to v, we may perform successive edge pivots along P causing the initial edge of the path to become unoriented. We call this sequence of edge pivots a Jacob's ladder cascade (see Figure 2).*

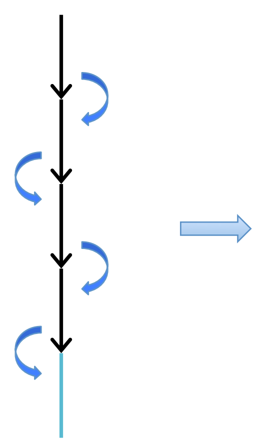


Figure 2: A Jacob's ladder cascade.

Theorem 2.3.3. *Two partial orientations $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle-cocycle reversal system if and only if $D_{\mathcal{O}}$ is linearly equivalent to $D_{\mathcal{O}'}$.*

Proof. Clearly, if $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle-cocycle reversal system then $D_{\mathcal{O}} \sim D_{\mathcal{O}'}$. We now demonstrate the converse.

By the previous lemma, it suffices to show that in the generalized cocycle reversal system there exists $\mathcal{O}'' \sim \mathcal{O}$ such that $D_{\mathcal{O}''} = D_{\mathcal{O}'}$. Without loss of generality, we may assume that $D_{\mathcal{O}'} = D_{\mathcal{O}} - \Delta f$, where $f \geq 0$ and there exists some $v \in V(G)$ such that $f(v) = 0$. This is because the kernel of the Laplacian of an undirected connected graph is generated by the all 1's vector. Let a and b be the minimum and maximum positive values of f respectively. We take $A = \{v \in V(G) : f(v) \geq a\} = \operatorname{supp}(f)$ and $B = \{v \in V(G) : f(v) = b\}$.

We first claim that we may perform edge pivots so that the boundary of A does not contain any edges oriented away from A. Suppose this is not true, let $\mathcal{O}$ be an orientation equivalent by edge pivots which minimizes the number of edges oriented towards A^c, and let e be an edge oriented away from A with head $v \in A^c$. Let X denote the set of vertices reachable from v by a directed path in A^c. If any vertex in X is incident to an unoriented edge, we can perform a Jacob's ladder cascade so that e is unoriented and the number of edges oriented away from A has decreased, thus contradicting the minimality of $\mathcal{O}$. The induced subgraph $G[X]$ is fully oriented and (X, X^c) is saturated. Moreover, the edges in $(X, X^c) \cap G[A^c]$ are all oriented towards X and by assumption $(X, X^c) \cap (A, A^c)$ has at least one edge e oriented towards X. This contradicts the fact that X has at least $|(X, X^c) \cap (A, A^c)|$ more chips in $D_{\mathcal{O}'}$ than $D_{\mathcal{O}}$ which are fired from the set A. We remark that the previous statement can be written compactly as $\bar{\chi}(X, D_{\mathcal{O}'}) < 0$, which is impossible as $D_{\mathcal{O}'}$ is partially orientable. See Theorem 2.7 for a proof of the converse.

We now assume that none of the edges in (A, A^c) are oriented towards A^c. If it were possible to perform edge pivots so that (B, B^c) was saturated towards B, we could reverse this cut and induct on $\deg(f)$. Therefore we assume that this is not the case, and take $\mathcal{O}$ be an orientation which is equivalent by edge pivots and minimizes the number of edges oriented away from B. It follows by the previous claim that $\bar{B}$ is contained in A. Moreover, the boundary of $\bar{B}$ is saturated towards $\bar{B}$, otherwise we could perform a Jacob's ladder cascade decreasing the number of edges in (B, B^c) oriented towards B^c, therefore we can reverse the cut $(\bar{B}, \bar{B}^c)$ and induct on $\deg(f)$. □

Corollary 2.3.4. *Let $\mathcal{O}$ and $\mathcal{O}'$ be partial orientations with $\mathcal{O}'$ acyclic. Then $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle-cocycle reversal system if and only if they are equivalent in the generalized cocycle reversal system.*

Proof. It is clear that if $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cocycle reversal system then they are equivalent in the generalized cycle-cocycle reversal system. For the converse, suppose that $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cocycle reversal system. By the proof of Theorem 2.3.3, $\mathcal{O}$ is equivalent in the generalized cocycle reversal system to some partial orientation $\mathcal{O}''$ such that $D_{\mathcal{O}''} = D_{\mathcal{O}'}$. Then by the proof of Lemma 2.3.1, $\mathcal{O}''$ is equivalent to $\mathcal{O}'$ in the generalized cycle reversal system using only edge pivots as $\mathcal{O}'$ is acyclic. □

In the following sections, we will be interested in the question of when a partially orientable divisor $D_{\mathcal{O}}$ is linearly equivalent to a partially orientable divisor $D_{\mathcal{O}'}$ where $\mathcal{O}'$ is acyclic. By Corollary 2.3.4, it is sufficient to restrict our attention to the generalized cocycle reversal system.

2.4 Oriented Dhar's Algorithm

Let D be a divisor such that $D(v) \geq 0$ for all $v \neq q$. *A priori* we would need to check firing every subset of $V(G) \setminus \{q\}$ to determine whether D is q-reduced, but Dhar's algorithm [26] guarantees that we only need to check a maximal chain of sets. In Dhar's algorithm, we begin by firing $S = V(G) \setminus \{q\}$. At each each step, if firing S causes some vertex v to be sent into debt, we remove v from S and continue. Dhar showed that the algorithm terminates at the empty set if and only if D is q-reduced (recurrent in his setting). If the algorithm terminates early, we obtain a set which can be fired without causing any vertex to be sent into debt, thus bringing the divisor closer to being reduced. We now extend this idea to the generalized cocycle reversal system.

Algorithm 2.4.1. *Oriented Dhar's Algorithm*

Input: A partial orientation $\mathcal{O}$ containing a directed cycle and a source.
Output: A partial orientation $\mathcal{O}'$ with $D_{\mathcal{O}'} = D_{\mathcal{O}}$ which is either acyclic or certifies that no such acyclic partial orientation exists.

Initialize by taking X to be the set of sources in $\mathcal{O}$. At the beginning of each step, look at the cut (X, X^c) and perform any available edge pivots at vertices on the boundary of X^c which bring oriented edges into the cut directed towards X^c. Afterwards, for each v on the boundary of X^c with no incoming edge contained in $G[X^c]$, add v to X. If no such vertex exists, output $\mathcal{O}'$.

Correctness: At each step, there are no edges oriented towards X. To prove this, we first observe that X satisfies this condition at the beginning of the algorithm, and note that the vertices added to X at each step do not cause any such edge to be introduced because any vertex added does not have an incoming edge in $G[X^c]$. It follows that X will never contain a vertex from a directed cycle: when a vertex v from a cycle hits the boundary of X^c, either the cycle is broken or v stays in X^c. Moreover, the algorithm will never construct directed cycles. Thus, if the algorithm terminates at $X = V(G)$, we obtain $\mathcal{O}'$ which is acyclic. If the algorithm terminates with $X \neq V(G)$, then $\mathcal{O}'$ has a cut saturated towards X^c and $G[X^c]$ is sourceless. It follows by Lemma 2.3.1 that an acyclic partial orientation $\mathcal{O}'$ with $D_{\mathcal{O}'} = D_{\mathcal{O}}$ is obtainable from $\mathcal{O}$ by edge pivots. Any other orientation $\mathcal{O}''$ obtained from $\mathcal{O}'$

by edge pivots will still have $G[X^c]$ sourceless and therefore contain a cycle: If we perform a directed walk backwards in $G[X^c]$, this walk will eventually cycle back on itself.

Corollary 2.4.2. *A partially orientable divisor D is the divisor associated to an acyclic partial orientation $\mathcal{O}$ if and only if there exists no set $A \subset V(G)$ which is out of debt and can fire without sending a vertex into debt, i.e., it is reduced with respect to the set of sources in $\mathcal{O}$.*

Proof. Suppose that D is partially orientable. Run the oriented Dhar algorithm 2.4.1 on D. We have that D is not the divisor associated to an acyclic orientation if and only if the oriented Dhar's algorithm produces a set $X^c \subset V(G)$ such that each vertex $v \in X^c$ has at least $\mathrm{outdeg}_{X^c}(v) + 1$ edges oriented inward. It follows that $D(v) \geq \mathrm{outdeg}_{X^c}(v)$, hence firing the set X^c does not cause any vertex to be sent into debt.

□

Algorithm 2.4.3. *Unfurling Algorithm*

Input: A partial orientation $\mathcal{O}$ containing a directed cycle and a source.
Output: A partial orientation $\mathcal{O}'$ equivalent to $\mathcal{O}$ in the generalized cocyle reversal system which is either acyclic or sourceless.

At the kth step, run the oriented Dhar's algorithm. Stop if $X = V(G)$, otherwise reverse the consistently oriented cut given by Dhar and reset X (see Figure 3).

Correctness: This follows directly from the correctness of the oriented Dhar's algorithm.

Termination: The collection of partially orientable divisors linearly equivalent to $D_{\mathcal{O}}$ is finite, hence the collection of firings which defines them is as well, modulo the all 1's vector which generates the kernel of the Laplacian Δ. Let $\mathcal{O}_k$ be the orientation obtained after the k-th step of the unfurling algorithm, that is, after the k-th cut reversal, and let f_k be such that $D_{\mathcal{O}_k} = D_{\mathcal{O}} - \Delta f_k$. We prove that if the algorithm were to persist, it would require vertices a and b such that $f_k(a) - f_k(b)$ diverges with k. Let A_k be the set of sources in $\mathcal{O}_k$ and B_k be the set of vertices belonging to the directed cycles in $\mathcal{O}_k$. Observe that the sets A_k and B_k are both weakly decreasing with k: vertices never become sources, and cycles are never created. Therefore, given any $a \in A_k$ and $b \in B_k$ for all k, the value $f_{k+1}(b) - f_{k+1}(a) = f_k(b) - f_k(a) + 1$, which diverges with k.

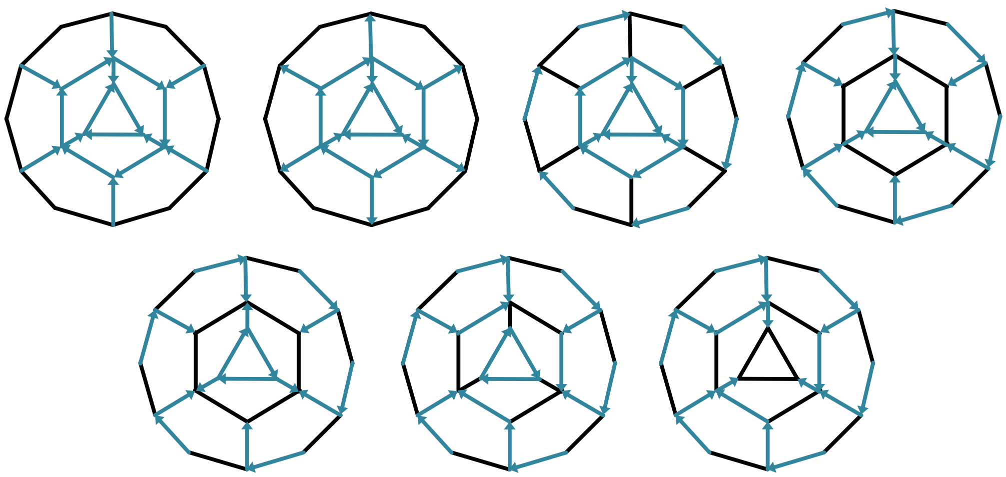


Figure 3: The unfurling algorithm applied to the partial orientation on the top left, terminating with the acyclic partial orientation on the bottom right.

Baker and Norine described the following game of solitaire [9, Section 1.5]. Suppose you are given a configuration of chips, can you perform chip-firing moves to bring every vertex out of debt? There is a natural version of this game for partial orientations: given a partial orientation, can you find an equivalent partial orientation which is sourceless? Interestingly, there exists a dual game in this setting, which does not make much sense in the context of chip-firing: given a partial orientation, can you find an equivalent partial orientation which is acyclic? Our unfurling algorithm gives a winning strategy for at least one of the two games. We now show that winning strategies in these games are mutually exclusive.

Theorem 2.4.4. *A sourceless partial orientation $\mathcal{O}$ is not equivalent to an acyclic partial orientation $\mathcal{O}'$.*

Proof. First we observe that $\mathcal{O}$ necessarily contains a directed cycle. Indeed, if we perform a directed walk backwards starting at an arbitrary vertex, this walk must eventually reach a vertex which has already been visited, demonstrating the existence of a directed cycle in $\mathcal{O}$. Suppose for the sake of contradiction that $D_{\mathcal{O}} \sim D_{\mathcal{O}'}$ with $\mathcal{O}'$ an acyclic partial orientation and that $D_{\mathcal{O}} - \Delta f = D_{\mathcal{O}'}$ with $f \geq 0$ and $f(v) = 0$ for all $v \in S \subset V(G)$ with S non-empty. It follows that even if it were possible to perform edge pivots in $\mathcal{O}'$ at each vertex on the boundary of S to saturate the cut (S, S^c) towards S, we will still have $G[S]$ sourceless, implying the existence of a directed cycle in S, a contradiction. $\square$

Before describing our algorithm for constructing partial orientations, we first introduce a modified version of the unfurling algorithm.

Algorithm 2.4.5. *Modified Unfurling Algorithm*

Input: A partial orientation $\mathcal{O}$ and a set of sources S.

Output: A partial orientation $\mathcal{O}'$ equivalent to $\mathcal{O}$ in the generalized cocyle reversal system which either has an edge oriented toward some vertex in S or is acyclic and certifies that no such orientation exists.

Initialize with $S := X_0$. We proceed as in the unfurling algorithm, but with the following changes. At the kth step, after performing all available edge pivots, if the edges in (X_k, X_k^c) are consistently oriented towards X_k^c, reverse this cut and reset X_k. Otherwise, take some v on the boundary of X_k^c incident to an unoriented edge in (X_k, X_k^c), and set $X_{k+1} := X_k \cup \{v\}$.

Correctness: If the algorithm terminates with $X = V(G)$, then the orientation $\mathcal{O}'$ produced is acyclic by an argument similar to the one given for the correctness of the oriented Dhar's algorithm. We next prove that this acyclic orientation certifies there is no partial orientation equivalent to $\mathcal{O}$ in the generalized cocycle reversal system with an edge oriented towards some $v \in S$. Without loss of generality, we assume that no cut is reversed in $\mathcal{O}$ prior to the termination of the algorithm. Towards a contradiction, assume there exists some $\mathcal{O}''$ equivalent to $\mathcal{O}$ by edge pivots and a set Y with $S \subset Y$ such that the edges in (Y, Y^c) are consistently oriented towards Y^c. Let k be the largest integer such that $X_k \subset Y$. It follows that there is some vertex $v \in X_k \cap Y$ with $v \notin X_{k+1}$. This vertex was added to X_k because it was incident to an unoriented edge in (X_k, X_k^c). In $\mathcal{O}''$, the vertex v must also be incident to an unoriented edge contained in (Y, Y^c), a contradiction.

Algorithm 2.4.6. *Construction of partial orientations*

Input: A divisor D with $\deg(D) \leq g - 1$.

Output: A divisor $D' \sim D$ and a partial orientation $\mathcal{O}$ such that $D' = D_{\mathcal{O}}$ or $D' \leq D_{\mathcal{O}}$ with $\mathcal{O}$ acyclic certifying that $D \nsim D''$ partially orientable.

We work with partial orientation-divisor pairs $(\mathcal{O}_i, D_i)$ such that at each step, $D_{\mathcal{O}_i} + D_i \sim D$. Initialize with $(\mathcal{O}_0, D_0) = (\mathcal{O}', D - D_{\mathcal{O}'})$, where $\mathcal{O}'$ is an arbitrary partial orientation. At the ith step, let R_i be the negative support of D_i, S_i be the positive support in D_i, and T_i be the set of vertices incident to an unoriented edge in $\mathcal{O}_i$. While $D_i \neq \vec{0}$, we are in one of the two following cases:

Case 1: The set S_i is non-empty and $\mathcal{O}_i$ is not a full orientation.

Take $s \in S_i$. If $\bar{s} \cap T_i \neq \emptyset$, perform a Jacob's ladder cascade to free up an unoriented edge e incident to s. Orient e towards s, set $D_{i+1} = D_i - (s)$, and updated $\mathcal{O}_{i+1}$. Eventually no such paths exists, and either $\mathcal{O}_i$ is a full orientation or the cut $(\bar{s}, \bar{s}^c)$ is saturated towards $\bar{s}$. In the latter case, reverse the cut, update $\bar{s}$, and continue. By induction on the size of $|V(G) \setminus \bar{s}|$, this process will eventually terminate.

Case 2: The sets S_i and R_i are both non-empty, and $\mathcal{O}_i$ is a full orientation.

Let $s \in S$. If $\bar{s} \cap R_i \neq \emptyset$, reverse a path from s to $r \in R_i$, set $D_{i+1} = D_i - (s) + (r)$, and update $\mathcal{O}_{i+1}$. Otherwise, the cut $(\bar{s}, \bar{s}^c)$ is saturated towards $\bar{s}$. Reverse the cut, update $\bar{s}$, and continue.

Case 3: The set S_i is empty and the set R_i is non-empty.

Apply the modified unfurling algorithm 2.4.5 to $\mathcal{O}_i$ with $S := R_i$ to find an equivalent orientation $\mathcal{O}$ in generalized cocycle reversal system which is either acyclic or has an edge oriented towards some $r \in R_i$. In the latter case we may unorient an edge pointing towards r, set $D_{i+1} := D_i + (r)$, update $\mathcal{O}_{i+1}$, and continue. Thus we may take $\mathcal{O}$ to be acyclic and observe that $D_{\mathcal{O}} \geq D_{\mathcal{O}} + D_i \sim D$.

Corollary 2.4.7 (An-Baker-Kuperberg-Shokrieh, Theorem 4.7 [4])**.**

Every divisor D of degree $g-1$ is linearly equivalent to an orientable divisor.

Proof. Suppose that D is not linearly equivalent to an orientable divisor. It follows from Algorithm 2.4.6 that D is linearly equivalent to $D' \leq D_{\mathcal{O}}$, where $\mathcal{O}$ is an acyclic partial orientation. But then $g-1 = \deg(D') \leq \deg(D_{\mathcal{O}}) \leq g-1$, therefore $D' = D_{\mathcal{O}}$, a contradiction. $\square$

Corollary 2.4.8. *An effective divisor of degree at most $g-1$ is linearly equivalent to a divisor associated to a sourceless partial orientation.*

Proof. When applying Algorithm 2.4.6 to an effective divisor beginning with the empty orientation, we will remain in Case 1. First observe that each vertex will eventually receive an edge oriented inwards. When performing cut reversals in order to obtain a directed path from s to an undirected edge, we have that s is the only vertex which may have all its incoming edges removed. Immediately after performing a Jacob's ladder cascade, we orient an edge towards s, and we update $\mathcal{O}_{i+1}$. $\square$

Theorem 2.4.9. *A divisor D with degree at most $g-1$ is linearly equivalent to a partially orientable divisor $D_{\mathcal{O}}$ if and only if $r(D+\vec{1}) \geq 0$.*

Proof. If D is linearly equivalent to a partially orientable divisor $D_{\mathcal{O}}$, then $D_{\mathcal{O}} + \vec{1}$ is effective. Conversely, suppose that $D \sim D' \geq -\vec{1}$. If we apply Algorithm 2.4.6 starting with the empty orientation, we will always be in Case 1 and the algorithm will necessarily succeed in producing a partial orientation $\mathcal{O}$ with $D_{\mathcal{O}} \sim D' \sim D$. $\square$

In section 2.7 we will describe a second method for constructing partial orientations which applies max-flow min-cut.

2.5 Directed Path Reversals and the Riemann-Roch Formula

In this section we investigate directed path reversals and their relationship to Riemann-Roch theory. The Baker-Norine rank of a divisor associated to a partial orientation is shown to be one less than the minimum number of directed paths which need to be reversed in the generalized cocycle reversal system to produce an acyclic orientation. To prove this characterization, we introduce q-connected partial orientations, which generalize the q-connected orientations of Gioan [35]. We then apply this characterization of rank, together with results from section 2.4, to give a new proof of the Riemann-Roch theorem for graphs. Baker and Norine's original argument proceeds by a formal reduction to statements which they call RR1 and RR2, we instead employ a variant of this reduction introducing strengthened versions of RR1 and RR2. We note that while Strong RR2 is an immediate consequence of Riemann-Roch, Strong RR1 is not, and appears to be new to the literature.

Every full orientation is equivalent in the cocycle reversal system to a q-connected orientation [4, Theorem 4.7] and [35, Proposition 4.7]. A simple proof proceeds as follows: suppose that $\mathcal{O}$ is a full orientation which is not q-connected, then $\bar{q} \neq V(G)$ and $(\bar{q}, \bar{q}^c)$ is saturated towards $\bar{q}$. We can reverse this cut and induct on $|\bar{q}^c|$. In Theorem 2.5.5, we prove a generalization of this statement for divisors of degree less than $g - 1$.

Lemma 2.5.1 (RR1). *If* $r(D) = -1$ *then* $D \leq D'$ *with* $\deg(D') = g - 1$ *and* $r(D') = -1$.

Proof. We claim that if $r(D) = -1$ then there exists $D' \sim D$ such that $D' \leq D_{\mathcal{O}}$ where $\mathcal{O}$ is a full acyclic orientation. By Theorem 2.4.4 and Corollary 2.4.8 this is sufficient to establish the Lemma.

We first argue that if $r(D) = -1$ then $\deg(D) \leq g-1$. Suppose that $\deg(D) \geq g$, and let $D' = D - E$ with $\deg(D') = g - 1$ and $E \geq \vec{0}$. Let $\mathcal{O}$ be an orientation with $D_{\mathcal{O}} \sim D'$ as guaranteed by Corollary 2.4.7. Without loss of generality, we take $\mathcal{O}$ to be q-connected with $q \in \text{supp}(E)$. It follows that $D \sim D_{\mathcal{O}} + E \geq 0$ and $r(D) \geq 0$.

Given D with $r(D) = -1$ we can apply Algorithm 2.4.6 followed by Algorithm 2.4.3 to obtain $D' \sim D$ such that $D' \leq D_{\mathcal{O}}$ where $\mathcal{O}$ is an acyclic partial orientation. It is a classical fact, whose proof we now give, that any acyclic partial orientation can be greedily extended to a full acyclic orientation. Let $e = (u, v)$ be some unoriented edge in $\mathcal{O}$ and suppose that both orientations of e cause a directed cycle to appear.

This implies that there exist directed paths in $\mathcal{O}$ from u to v and v to u, hence a directed cycle was already present in $\mathcal{O}$, a contradiction. $\square$

Theorem 2.4.4 and Corollary 2.4.8 were combined in the previous argument to show that an effective divisor is not linearly equivalent to a divisor associated to an acyclic partial orientation. In the case of full orientations, this argument is "dual" to the one which has been given in the past [23, Proposition 6].

Corollary 2.5.2.

$$r(D) = \min_{\nu \in \mathcal{N}} \deg^+(D - \nu) - 1$$

Proof. Let $E_1, E_2 \geq \vec{0}$ be effective divisors such that $D - E_1 + E_2 = \nu$ with $\mathrm{supp}(E_1) \cap \mathrm{supp}(E_2) = \emptyset$ and $\nu \in \mathcal{N}$ achieving the minimum value of $\deg^+(D-\nu) - 1$. We have that $\deg^+(D - \nu) = \deg(E_1)$ and $D - E_1 = \nu - E_2$ so $r(D - E_1) = -1$, which implies $r(D) \leq \deg^+(D - \nu) - 1$.

Take $E_1 \geq \vec{0}$ with $r(D) = \deg(E_1) - 1$ and $r(D - E_1) = -1$. By RR1 there exists some effective divisor E_2 such that $D - E_1 + E_2 = \nu$ for some $\nu \in \mathcal{N}$. We observe that $\mathrm{supp}(E_1) \cap \mathrm{supp}(E_2) = \emptyset$, thus $r(D) \geq \deg^+(D - \nu) - 1$. $\square$

Lemma 2.5.3 (Strong RR1)**.**

If $\deg(D) \leq g - 1$ *then there exists a divisor* $D \leq D'$ *with* $\deg(D') = g - 1$ *and* $r(D) = r(D')$.

Proof. Let $E_1, E_2 \geq 0$ be such that $E_1 - E_2 = D - \nu$ which achieves the minimum value of $\deg^+(D - \nu)$ for $\nu \in \mathcal{N}$. Now take $0 \leq E \leq E_2$ such that $\deg(E) = \deg(g - 1 - D)$. We claim that $r(D+E) = r(D)$. Clearly $r(D+E) \geq r(D)$, and the reverse inequality follows by Corollary 2.5.2 as $r(D+E) = \min_{\nu \in \mathcal{N}} \deg^+(D+E-\nu) \leq \min_{\nu \in \mathcal{N}} \deg^+(D-\nu) = r(D)$.

$\square$

Lemma 2.5.4. *A partial orientation $\mathcal{O}$ which is either sourceless or has q as its unique source is equivalent in the generalized cocycle reversal system to a q-connected partial orientation $\mathcal{O}'$.*

Proof. Suppose there exists a potential edge pivot at a vertex on the boundary of $\bar{q}^c$ which would bring an oriented edge from $G[\bar{q}^c]$ into the cut pointing towards $\bar{q}^c$. Performing this edge pivot would enlarge $\bar{q}$, therefore by induction on $|\bar{q}^c|$, we assume that no such edge pivot is available. Because $\mathcal{O}$ is sourceless, we conclude that the cut $(\bar{q}, \bar{q}^c)$ is saturated towards $\bar{q}$. We can then reverse this cut and again induct on $|\bar{q}^c|$.

□

Theorem 2.5.5. *A divisor D with $\deg(D) \leq g-1$ is linearly equivalent to a divisor associated to a q-connected partial orientation if and only if $r(D+(q)) \geq 0$.*

Proof. The necessity of the condition is clear. Sufficiency follows by the proof of Corollary 2.4.8 and Lemma 2.5.4. □

We remark that the q-rooted spanning trees are precisely the q-connected partial orientations associated to the divisor $-(q)$. Additionally, the q-connected partial orientations associated to $\vec{0}$ are the divisors obtained from q-rooted spanning trees by orienting an new edge towards q, i.e., they are the directed spanning unicycles.

Any two q-connected full orientations which are equivalent in the cycle-cocycle reversal system are equivalent in the cycle reversal system, i.e., they have the same associated divisors. This theorem does not extend to the setting of partial orientations, as the example in Figure 4 shows.

Lemma 2.5.6. *A divisor D with $D(q) = -1$ is q-reduced if and only if D is the divisor associated to a q-connected acyclic partial orientation $\mathcal{O}$.*

Proof. It is follows by Corollary 2.4.2 that if $\mathcal{O}$ is a q-connected acyclic partial orientation, then $D_{\mathcal{O}}$ is q-reduced. Conversely, supposing that D is q-reduced, it again follows

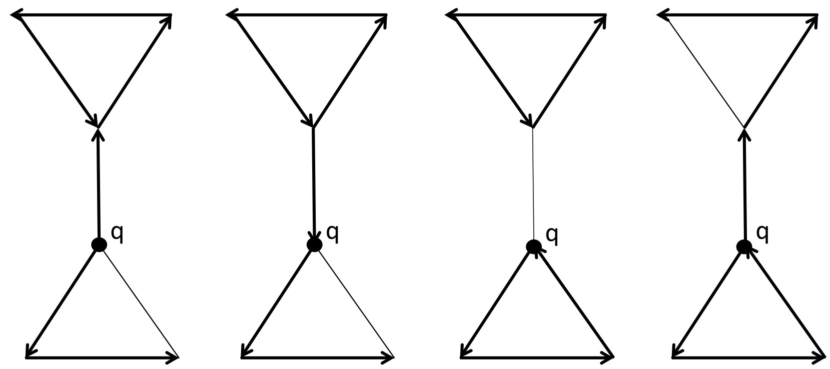


Figure 4: A sequence of equivalent partial orientations. The left and right partial orientations are both q-connected, but have different associated divisors. The partial orientation on the right is a directed spanning unicycle.

by Corollary 2.4.2 that there exists some acyclic partial orientation $\mathcal{O}_D$. Following the argument of Lemma 2.5.4, we can perform edge pivots to make $\mathcal{O}_D$ q-connected without performing any cut reversals, otherwise, D would not be q-reduced. $\square$

Theorem 2.5.7. *The Baker-Norine rank of a divisor $D_{\mathcal{O}}$ associated to a partial orientation $\mathcal{O}$ is one less than the minimum number of directed paths which need to be reversed in the generalized cocycle reversal system to produce an acyclic orientation.*

Proof. Let $\mathcal{N}_k = \{D \in \mathrm{Div}(G) : \deg(D) = k, r(D) = -1\}$ so that $\mathcal{N}_{g-1} = \mathcal{N}$. We claim that

$$r(D) = \min_{\nu \in \mathcal{N}_{\deg(\mathrm{D})}} \deg^+(D - \nu) - 1.$$

The formula follows by the same argument as Corollary 2.5.2, which we omit. Let $f_D = D - \nu$ for $\nu \in \mathcal{N}_{\deg(\mathrm{D})}$ which achieves the minimum value of $\deg^+(\mathrm{D} - \nu) - 1$. By assumption $\deg(f_D) = 0$ and we can write

$$f_D = \sum_{i=0}^{r(D)} (q_i) - (p_i).$$

By Corollary 2.5.5 there exists a partial orientation $\mathcal{O}$ which is q_0-connected and $D_{\mathcal{O}} \sim D$. We can reverse a path from q_0 to p_0 to obtain $\mathcal{O}'$ with $D_{\mathcal{O}'} = D_{\mathcal{O}} + (q_0) - (p_0)$. Proceeding in this way, we arrive an orientation $\mathcal{O}''$ with $D_{\mathcal{O}''} \sim D + f_D$. Therefore,

$r(D_{\mathcal{O}''}) = -1$ and by the proof of RR1, $\mathcal{O}''$ is equivalent in the generalized cocycle revesal system to an acyclic orientation. The reverse inequality follows similarly. See Figure 5 for an illustrating example. ☐

Corollary 2.5.8. *The Baker-Norine rank of a divisor $D_{\mathcal{O}}$ associated to a partial orientation $\mathcal{O}$ is the maximum number k such that the reversal of any k directed paths in the generalized cocycle reversal system does not produce an acyclic partial orientation.*

Proof. This is a tautological consequence of Theorem 2.5.7. ☐

Theorem 2.5.7 and Corollary 2.5.8 hold in the generalized cycle-cocycle reversal system as well, which follows by Corollary 2.3.4 and Theorem 2.5.5

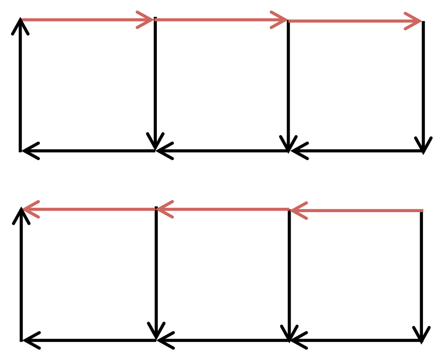


Figure 5: A directed path whose reversal produces an acyclic orientation. By Theorem 2.5.7 it follows that the divisor associated to the top orientation has rank 0.

Corollary 2.5.9 (Strong RR2)**.** *If* deg*(D) = g – 1 then r(D) = r(K-D).*

Proof. If D is equivalent to an orientable divisor D' then $K-D$ is equivalent to $K-D'$, and these two divisors are coming from opposite orientations. It is clear by the path-reversal interpretation of rank for orientable divisors that $r(D') = r(K - D')$. ☐

Theorem 2.5.10 (Baker-Norine [9])**.** *For every divisor D on G,*

$$r(D) - r(K - D) = \deg(D) - g + 1.$$

Proof. Either D or $K-D$ has degree at most $g-1$, therefore without loss of generality, we take D to be a divisor with $\deg(D) \le g-1$. By Strong RR1, there exits $E \ge 0$ such that $D+E = D'$ with $r(D') = r(D)$, and by Strong RR2 we know that $r(D') = r(K-D')$. To prove the theorem, it suffices to show that

$$r(K-D) - r(K-D') = \deg(K-D) - g + 1 = \deg(E).$$

We know that

$$r(K-D) - r(K-D') \le \deg(K-D) - g + 1 = \deg(E),$$

and for the sake of contradiction, we suppose that

$$r(K-D) - r(K-D') < \deg(E).$$

Let $K - D'' = K - D - E'$ with $E' \ge 0$, $\deg(E) = \deg(E')$, and

$$r(K-D) - r(K-D'') = \deg(K-D) - g + 1 = \deg(E').$$

We have $D \le D + E' = D''$, but

$$r(D'') = r(K-D'') < r(K-D') = r(D') = r(D),$$

a contradiction, thus proving the theorem. □

For a comparison with other proofs of the Riemann-Roch formula for graphs which appear in the literature, we refer the reader to [2, 3, 9, 23, 55, 75].

2.6 Luo's Theorem on Rank-Determining Sets

For an introduction to the theory of linear equivalence of divisors on metric graphs, see section 3.2. Luo [52] defined a set of points A to be *rank-determining* for a metric graph Γ if when computing the rank of any divisor on Γ, we only need to subtract chips from points in A. A *special open set* $\mathcal{U}$ is a nonempty, connected, open subset of Γ such that every connected component X of $\Gamma \setminus \mathcal{U}$ has a boundary point p with

$\mathrm{outdeg}_X(p) \geq 2$. Luo introduced a metric version of Dhar's burning algorithm and applied this technique to obtain the following beautiful result, which we reprove using the language of acyclic orientations and directed path reversals.

In what follows, we work with the canonical minimal model of Γ, whose vertex set is taken to be the collection of points with a number of tangent directions different from two.

Before presenting the proof, we first note a motivating special case: given an acyclic orientation $\mathcal{O}$ of a metric graph and an edge e in which the orientation changes direction, we can "push" the change of direction to one of the two incident vertices without creating a directed cycle. This follows by the same argument which was used in our proof of RR1 for showing that any acyclic partial orientation may be extended greedily to a acyclic full orientation. By the reduction at the beginning of Theorem 2.6.1, this observation may be converted into a proof that the vertices of Γ are rank-determining, which is [52, Theorem 1.5]. See Figure 6.

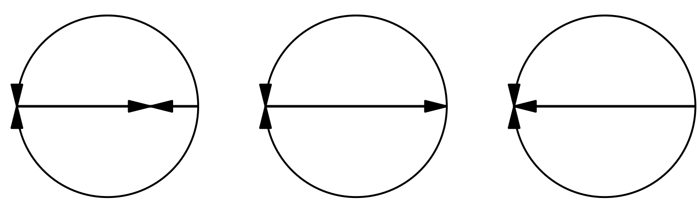


Figure 6: A full orientation of a metric graph and two other orientations obtained by "pushing" the change of orientation along the middle edge to the right and left. The push to the right causes directed cycles to appear while the push to the left does not.

Theorem 2.6.1 (Luo [52], Theorem 3.16)**.** *A finite subset $A \subset \Gamma$ is rank-determining if and only if it intersects every special open set $\mathcal{U}$ in Γ.*

Proof. We first give a reduction to the study of negative rank divisors. Suppose that E is effective, $r(D-E) = -1$, $\deg(E) = r(D)+1$, and $q \in \mathrm{supp}(E)$. The set A is rank-determining if and only if there exists some $a \in A$ such that $r(D-(E-(q)+(a)) = -1$.

This follows by induction on $\deg(E|_{\Gamma\setminus A})$. Therefore, we can reduce to the case of showing that for every D with $r(D) = -1$ and every point $q \in \Gamma$, there exists some $a \in A$ with $r(D + (q) - (a)) = -1$.

By the (metric version) of RR1, if a divisor has degree at least g, it has nonnegative rank. Therefore, we need only study divisors of degree at most $g - 1$. Let D be a divisor such that $\deg(D) \leq g - 1$, $r(D) = -1$, and $\nu \in \mathcal{N}$ be such that $D \leq \nu$. If for every $q \in \Gamma$, there exists some $a \in A$ such that $\nu + (q) - (a) \in \mathcal{N}$, then the same holds for D. Conversely, we know that if A is rank determining for all divisors with degree at most $g-1$ then it is certainly rank determining for divisors of degree $g-1$. Therefore, A is rank determining if and only if for every $\nu \in \mathcal{N}$ and every $q \in \Gamma$, there exists some $a \in A$ such that $\nu + (q) - (a) \in \mathcal{N}$

Suppose that A is not rank-determining. By the previous reductions, we may assume that there exists an acyclic orientation $\mathcal{O}$ and a point $q \in \Gamma$ such that $D_{\mathcal{O}} + (q) - (a)$ has nonnegative rank for each $a \in A$. Taking $\mathcal{O}$ to be q-connected [4, Theorem 4.4], this says that whenever a path from q to A is reversed, it causes a directed cycle to appear in the graph. Equivalently, there exist at least two paths from q to each point of A. Let $\mathcal{U}$ be the set of points which are reachable from q by a unique directed path. We claim that $\mathcal{U}$ is a special open set not intersecting A.

Nonempty: The point $q \in \mathcal{U}$, otherwise there would be a path from q to itself, implying the existence of a directed cycle.

Connected: Every point in $\mathcal{U}$ lies on a path P from q. Moreover, $P \subset \mathcal{U}$, hence by transitivity, ignoring orientation, $\mathcal{U}$ is connected.

Open: We prove that the complement is closed. Suppose we have a sequence S of points in $\mathcal{U}^c$ converging to some point p. There exists some convergent subsequence

S' of S which is contained in an edge e incident to p. If we go far enough along in S' we may assume that all of the points in the sequence are contained in a consistently oriented segment of e and that all of the points on this segment have degree 2 in Γ. If this segment is oriented towards p, it is clear that p is contained in $\mathcal{U}^c$. On the other hand, if the edge is oriented away from p, the points in our sequence must be twice reachable through p, and so p is in $\mathcal{U}^c$.

Special: If $\mathcal{U}$ is not special then there exists some connected component X of $\mathcal{U}^c$ with $\text{outdeg}_X(p) = 1$ for all boundary points p of X. Not all of these points can be sinks in $\mathcal{O}$, otherwise there would be no way of reaching the interior of X from q and this would contradict q-connectivity. Let p be a boundary point which is not a sink in $\mathcal{O}$. Because $p \in \mathcal{U}^c$, p is twice reachable from q, as are all of the points in a small neighborhood of p, but this contradicts the assumption that p is a boundary point of $\mathcal{U}$.

Conversely, if we are given a special open set $\mathcal{U}$ not intersecting A, we may construct an acyclic orientation $\mathcal{O}$ for which A is not rank-determining. Let $q \in \mathcal{U}$ and take a q-connected acyclic orientation of $\mathcal{U}$. It follows because $\mathcal{U}$ is connected and open that $\mathcal{O}$ will have sinks at each of the boundary points of $\mathcal{U}$.

For any connected component X of $\mathcal{U}^c$ and boundary point $p \in X$ with $\text{outdeg}_X(p) \geq 2$, we can construct a p-connected acyclic orientation for X. Proceeding in this way for each component X, we obtain a full acyclic orientation $\mathcal{O}$. There exist two paths from q to a for each $a \in A$, hence the reversal of any path from q to a will cause a directed cycle to appear in Γ. This implies that A is not rank-determining for $D_{\mathcal{O}} + (q)$.

$\square$

2.7 Max-Flow Min-Cut and Divisor Theory

In this section we investigate the intimate relationship between network flows, a topic of fundamental importance in combinatorial optimization, and the theory of divisors on graphs. We recall that a *network* N is a directed graph $\vec{G}$ together with a source vertex $s \in V(\vec{G})$, a sink vertex $t \in V(\vec{G})$, and a capacity function $c : E(\vec{G}) \to \mathbb{R}_{\geq 0}$. A *flow* f on N is a function $f : E(\vec{G}) \to \mathbb{R}_{\geq 0}$ such that $f(e) \leq c(e)$ for all $e \in E(\vec{G})$ and

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$$

for all $v \neq s, t$, where $E^+(v)$ and $E^-(v)$ are the set of edges pointing towards and away from v, respectively. Let $X \subset V(\vec{G})$ such that $s \in X$. A simple calculation shows that

$$\sum_{v \in X} (\sum_{e \in E^-(v)} f(e) - \sum_{e \in E^+(v)} f(e)) =$$
$$\sum_{e \in \langle X, X^c \rangle} f(e) - \sum_{e \in \langle X^c, X \rangle} f(e)),$$

where $\langle X, X^c \rangle$ and $\langle X^c, X \rangle$ are the set of edges in the cut (X, X^c) directed towards X^c and X respectively. This sum is independent of the choice of X, in particular it is equal to

$$\sum_{e \in E^-(s)} f(e) - \sum_{e \in E^+(s)} f(e) = \sum_{e \in E^+(t)} f(e) - \sum_{e \in E^-(t)} f(e),$$

which we call the the flow value from s to t.

One may view a flow as a fluid flow from s to t through a system of one-way pipes where the capacity of a given edge represents the maximum rate at which water can travel through the pipe. The flow across any given cut separating s from t is restricted by the sum of the capacities of the edges crossing a cut (X, X^c) towards t, which we denote $c(X)$. The "max-flow min-cut" theorem, abbreviated as MFMC, states that equality is obtained, that is, the greatest flow from s to t is equal to the minimum capacity of a cut separating s from t. This theorem was first proven by Ford and Fulkerson [32] in their investigation of the max flow problem posed by Harris and

Ross [?] in the classified RAND document concerning military railroad transportation in the Soviet Union, which was declasified in 1999 per Alexander Schrijver's request. The result was independently discovered by Elias, Feinstein, and Shannon [29], and Kotzig [44] the following year. We refer the reader to Schrijver [69], for an interesting account of the problem's history.

There are two standard methods of proving MFMC, the first is to demonstrate that a flow of maximum value can be obtained greedily by so-called augmenting paths which leads to the classical Fork-Fulkerson algorithm, and the second is to rephrase the max flow problem as a linear program and establish MFMC via linear programming duality. We remark that it has recently been shown that this theorem may be also be viewed as a manifestation of directed Poincaré duality [34].

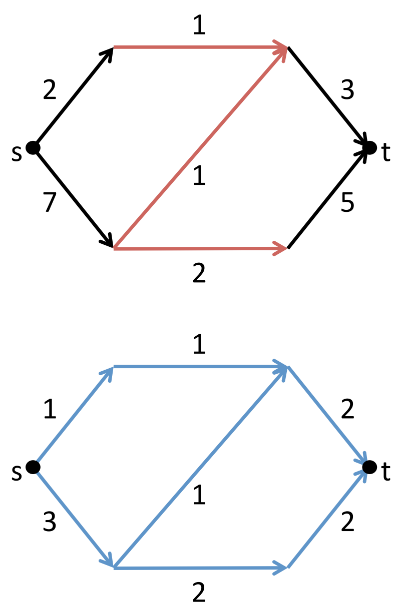


Figure 7: Top: A network with source s, sink t, capacities listed next to edges, and a minimum cut of size 4 colored red. Bottom: A maximum flow on this network with flow value 4. Note that the flow along each edge in the minimum cut is equal to the capacity of that edge.

Momentarily switching gears, we mention the following theorem which characterizes the collection of orientable divisors on a graph in terms of Euler characteristics. This result has been rediscovered multiple times, but appears to originate with S. L.

Hakimi[39]. It might be natural to view his theorem historically as an extension to arbitrary graphs of Landau's characterization of score vectors for tournaments [45], i.e., divisors associated to orientations of the complete graph, although it seems that Hakimi was unaware of Landau's result which was presented in a paper on animal behavior a decade earlier.

Recall we define the Euler characteristic of $G[S]$ to be $\chi(S) = |S| - |E(G[S])|$. Given a divisor D and a non-empty subset $S \subset V(G)$, we define

$$\chi(S, D) = \deg(D|_S) + \chi(S)$$

$$\chi(G, D) = \min_{S \subset V(G)} \chi(S, D)$$

$$\bar{\chi}(S, D) = |E(G)| - |E(G[S^c])| - |S| - \deg(D|_S)$$

$$\bar{\chi}(G, D) = \min_{S \subset V(G)} \bar{\chi}(S, D).$$

Theorem 2.7.1 (Hakimi[39], Felsner[31], An-Baker-Kuperberg-Shokrieh[4])**.** *A divisor D of degree $g-1$ is orientable if and only if $\chi(G, D) \geq 0$.*

There is a "dual" formulation of this theorem which is better suited for our approach.

Lemma 2.7.2. *Let D be a divisor of degree $g-1$. We have that $\chi(G, D) \geq 0$ if and only if $\bar{\chi}(G, D) \geq 0$.*

Proof. Informally, $\chi(S, D) \geq 0$ says that the total number of chips in S should be at least as large as the contribution of the edges in $G[S]$, and $\bar{\chi}(S, D) \geq 0$ says that the total number of chips in S should not exceed contribution of the edges in $G[S]$ and the cut (S, S^c). Because $\deg(D) = g - 1$, we have that $\chi(S, D) \geq 0$ if and only if $\bar{\chi}(S^c, D) \geq 0$. The lemma follows by taking the minimum of these values over all $S \subset V(G)$. □

Before providing a proof of Theorem 2.7.1, we remark that the result has a similar flavor to MFMC; it states that a certain obviously necessary condition is also sufficient. The following proof originally due to Felsner (and rediscovered independently by the author) reduces the problem to an application of MFMC.

Proof. 2.7.1

Let D be a divisor of degree $g-1$ satisfying $\chi(G,D) \geq 0$. By Lemma 2.7.2 it follows that $\bar{\chi}(G,D) \geq 0$. We now demonstrate by explicit construction that this condition is sufficient to guarantee the existence of an orientation $\mathcal{O}_D$. Let $\mathcal{O}$ be an arbitrary orientation and take $\tilde{D} = D - D_{\mathcal{O}}$. Denote the negative and positive support of $\tilde{D}$ as S and T respectively. Add two auxiliary vertices s and t with directed edges from s to each vertex $s' \in \text{supp}(S)$ with capacity $\tilde{D}(s')$ and from each vertex $t' \in \text{supp}(T)$ to t with capacity $-\tilde{D}(t')$. Assign each edge in $\mathcal{O}$ capacity 1, and take N be the corresponding network.

We claim that there is a flow from s to t with flow value $\deg^+(\tilde{D}) = \deg^-(\tilde{D})$. Any $s-t$ cut in N is determined by a set $X \subset \{V(G) \cup \{s\}\}$. By MFMC, to show that that such a flow exists, we need to that show the minimum capacity of a cut is at least $\deg^+(\tilde{D})$. For each set $X \subset V(G)$ let $X \cap T = T_1$, $T \setminus T_1 = T_2$, $X \cap S = S_1$, and $S \setminus S_1 = S_2$. The capacity of the cut, $c(X)$ is equal to $\deg^-(\tilde{D}|_{S_2}) + \deg^+(\tilde{D}|_{T_1}) + \bar{\chi}(X \setminus \{s\}, D_{\mathcal{O}})$. We claim that $\bar{\chi}(X \setminus \{s\}, D_{\mathcal{O}}) \geq \deg^-(\tilde{D}|_{S_1}) - \deg^+(\tilde{D}|_{T_1})$. Supposing the claim, we have that $c(X) \geq \deg^-(\tilde{D}|_{S_2}) + \deg^+(\tilde{D}|_{T_1}) + \deg^-(\tilde{D}|_{S_1}) - \deg^+(\tilde{D}|_{T_1})) = \deg^-(\tilde{D}|_{S_2}) + \deg^-(\tilde{D}|_{S_1}) = \deg^-(\tilde{D}|_S)$ as desired.

To prove that $\bar{\chi}(X \setminus \{s\}, D_{\mathcal{O}}) \geq \deg^-(\tilde{D}|_{S_1}) - \deg^+(\tilde{D}|_{T_1})$ we note that $\bar{\chi}(X \setminus \{s\}, D_{\mathcal{O}}) = \bar{\chi}(X \setminus \{s\}, D) + \deg^-(\tilde{D}|_{S_1}) - \deg^+(\tilde{D}|_{T_1})$ and $\bar{\chi}(X \setminus \{s\}, D) \geq 0$ by assumption, and the claim follows. Now let f be an $s-t$ flow in N with flow value $\deg^+(\tilde{D}|_S)$. To complete the proof we simply reverse the direction of each edge in $\mathcal{O}$ in the support of f to obtain a reorientation of N which when restricted to G gives a desired orientation $\mathcal{O}_D$. See Figure 8 for an illustrating example.

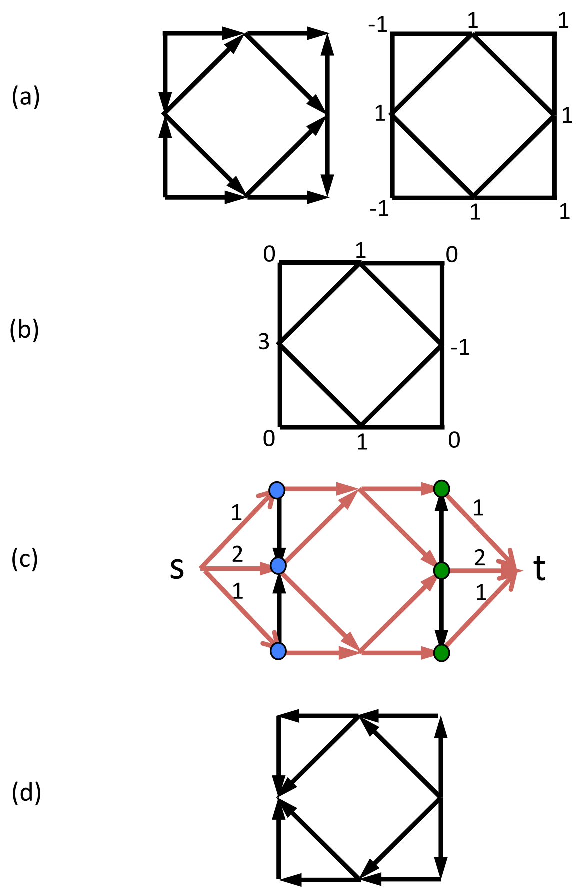


Figure 8: (a) Left: An orientation $\mathcal{O}$ of a graph G Right: The divisor $D_{\mathcal{O}}$ on G, (b) A divisor D on G, (c) The network N, with auxiliary vertices s and t. The set S is colored blue, the set T is colored green, and the additional edges are labeled by their capacities. The remaining edges have capacity 1. The edges colored red are the support of a maximal flow f. (d) An orientation $\mathcal{O}_D$ obtained by reversing the flow f on N and then restricting to G.

$\square$

We now demonstrate the converse implication. To the best of the author's knowledge, this argument has not appeared previously in the literature.

Theorem 2.7.3. *The max-flow min-cut theorem is equivalent to Theorem 2.7.1.*

Proof. The previous argument shows that max-flow min-cut implies the Euler characteristic description of orientable divisors Theorem 2.7.1. We now demonstrate that

Theorem 2.7.1 can be applied in proving MFMC. Let N be some network with integer valued capacities which we can view as an orientation on a multigraph G where the number of parallel edges is given by the capacities. Suppose that the minimum capacity of a cut between s and t is of size k. Let $\tilde{D} = k(t) - k(s)$. We claim that $D = D_N - \tilde{D}$ is orientable. By Theorem 2.7.1 and Lemma 2.7.2, it suffices to prove that $\bar{\chi}(G, D) \geq 0$. Let $X \subset V(G)$ with $s, t \notin X$. We have that $\bar{\chi}(X, D) = \bar{\chi}(X, D_N) \geq 0$. Now take $X \subset V(G)$ with $s \in X$ and $t \notin X$, and let $c(X)$ be the capacity associated to this cut. By definition, $\bar{\chi}(X, D) + k = \bar{\chi}(X, D_N) \geq c(X)$, therefore $\bar{\chi}(X, D) = c(X) - k \geq 0$. Finally, we have that $\bar{\chi}(X^c, D) = \bar{\chi}(X^c, D_N) + k \geq 0$, and the claim follows.

We next claim that the symmetric difference of orientations $\mathcal{O}_D$ and N is a flow in N with flow value k. Perform a directed walk on the symmetric difference of $\mathcal{O}_D$ and N in N starting at s. This walk either terminates at t or it loops back on itself. In the former case, we can reverse the path and in the latter case we can reverse the associated cycle. In both instances the claim follows by induction.

It is a classical fact that integer MFMC implies rational MFMC by scaling, and rational MFMC implies real MFMC by taking limits. □

We remark that if $\mathcal{O}'$ is an integer network, i.e. a full orientation with distinguished vertices s and t, and we wish to find a flow from s to t of value k, we can take $D = k(s) - k(t) + D_{\mathcal{O}'}$. Applying Algorithm 2.4.6, we will always be in Case 2, and we recover the Ford-Fulkerson algorithm. The algorithm produces an orientation $\mathcal{O}$ such that the symmetric difference of $\mathcal{O}$ and $\mathcal{O}'$ is a flow of value k from s to t.

In our proof of Theorem 2.7.1, it was crucial that we start with an arbitrary orientation $\mathcal{O}$ and find an appropriate flow whose reversal gave a desired orientation $\mathcal{O}_D$. Implicit in this approach is the following result. This statement holds for metric graphs as well.

Theorem 2.7.4. *The set* $\operatorname{Pic}^{g-1}(G)$ *is canonically isomorphic as a* $\operatorname{Pic}^0(G)$*-torsor to the collection of equivalence classes in the cycle-cocycle reversal system acted on by path reversals.*

Proof. Let S denote the collection equivalence classes of full orientations in the cycle-cocycle reversal system. By Corollary 2.4.7 and Theorem 2.3.3, we can canonically identify the sets S and $\operatorname{Pic}^{g-1}(G)$. Let $p, q \in V(G)$, $[\mathcal{O}] \in S$, and $\mathcal{O}_q$ be a q-connected orientation in $[\mathcal{O}]$. The divisor $(q) - (p)$ maps $[\mathcal{O}] \in S$ to $[\mathcal{O}'] \in S$ where $[\mathcal{O}'_p]$ is obtained from $\mathcal{O}_q$ by reversing the path from q to p. Because $D_{\mathcal{O}'_p} = D_{\mathcal{O}_q} + (q) - (p)$, this self-map of S is compatible with the action of $(q)-(p)$ on $\operatorname{Pic}^{g-1}(G)$. By linearity, this extends to an action of $\operatorname{Div}^0(G)$ on S. Moreover, this action respects linear equivalence, and hence defines an action of $\operatorname{Pic}^0(G)$ on S. □

We recall that a *break divisor* is a divisor of degree g such that for all $p \in \Gamma$ there is an injective mapping of chips at p to tangent directions at p, such that if we cut the graph at the specified tangent directions, we obtain a connected contractable space, i.e., a spanning tree. These divisors were first introduced in the work of Mikhalkin and Zharkov [58], and the following theorem states that they are precisely the divisors associated to q-connected orientations offset by a chip at q. Following [4], we call the divisors associated to q-connected orientations, *q-orientable.*

Theorem 2.7.5 (An-Baker-Kuperberg-Shokrieh [4])**.** *A divisor D of degree g is a break divisor if and only if $D - (q)$ is q-orientable for any point $q \in \Gamma$.*

Let $\to_q$ denote the map which adds a chip at q to a divisor. An important property of break divisors is that they provide distinguished representatives for the divisor classes of degree g. Indeed, by Theorem 2.7.5, the image of the map $\to_q$ applied to $\{q-$orientable divisors$\}$ is independent of the choice of q. We offer the following short proof of this result which does not make use of Theorem 2.7.5. If we compose $\to_q$ with the inverse of $\to_p$ and apply this map to $\{q-$orientable divisors$\}$, we obtain the set $\{q\text{−orientable divisors}\}+(q)-(p)$. These are the divisors associated to orientations obtained from the q-connected orientations by reversing a path from q to p. It is easy to verify that these are precisely the p-connected orientations.

We now describe a simple MFMC based algorithm to obtain the unique break divisor linearly equivalent to a given divisor of degree g.

Algorithm 2.7.6. *Efficient method for calculating break divisors*

Input: A divisor D of degree g.

Output: A break divisor $D' \sim D$.

Take $q \in V(G)$, and let D' be a divisor of degree $g-1$ with $D' = D-(q)$. Take $\mathcal{O}$ an arbitrary orientation and construct an auxiliary network for D' as in the proof of Theorem 2.7.1. We can perform any preferred MFMC algorithm to find a maximal flow in this network. After reversing this flow, either we obtain an orientation $\mathcal{O}'$ with $D_{\mathcal{O}'} \sim D-(q)$, or we obtain a directed cut which can be reversed. In this way we proceed alternating between flow reversals and cut reversals until we obtain an orientation $\mathcal{O}$ with $D_{\mathcal{O}} \sim D-(q)$. Executing further cut reversals if necessarily, we can achieve a q-connected orientation $\mathcal{O}_q$. By Theorem 2.7.5, $D_{\mathcal{O}_q}+(q)$ is a break divisor linearly equivalent to D.

Algorithm 2.7.7. *A second construction of partial orientations*

Take D with $\deg(D) \leq g-1$, and let $D' = D+E$ with $E \geq 0$ and $\deg(D') = g-1$. First, obtain $\mathcal{O}$ with $D_{\mathcal{O}} \sim D'$ by reversing flows obtained via some MFMC algorithm, and reversing cuts. Then perform the modified unfurling algorithm to obtain an orientation with some edge pointed towards a vertex in the support of E. We unorient this edge, subtract a chip from E and repeat. Eventually we either obtain a partial orientation $\mathcal{O}'$ with $D_{\mathcal{O}'} \sim D$ or $\mathcal{O}'$ acyclic and $D_{\mathcal{O}'} \geq D'$ with $D' \sim D$.

We conclude with an extension of Theorem 2.7.1 to the setting of partially orientable divisors.

Lemma 2.7.8. *Submodularity of $\bar{\chi}$ and χ:*

$$\bar{\chi}(S_1 \cup S_2, D) + \bar{\chi}(S_1 \cap S_2, D) \leq \bar{\chi}(S_1, D) + \bar{\chi}(S_2, D)$$

and

$$\chi(S_1 \cup S_2, D) + \chi(S_1 \cap S_2, D) \leq \chi(S_1, D) + \chi(S_2, D).$$

Proof.

$$\deg(D|_{S_1 \cup S_2}) + \deg(D|_{S_1 \cap S_2}) = \deg(D|_{S_1}) + \deg(D|_{S_2}).$$

$$|S_1 \cup S_2| + |S_1 \cap S_2| = |S_1| + |S_2|.$$

$$|E(G[S_1 \cup S_2])| + |E(G[S_1 \cap S_2])| = |E(G[S_1])| + |E(G[S_2])| + |(S_1 \setminus S_2, S_2 \setminus S_1)|.$$

$$\Rightarrow |E(G[S_1 \cup S_2])| + |E(G[S_1 \cap S_2])| \geq |E(G[S_1])| + |E(G[S_2])|.$$

The Lemma follows by the above relations. $\square$

Theorem 2.7.9. *A divisor D is partially orientable if and only if $D(v) \geq -1$ for all $v \in V(G)$ and $\bar{\chi}(G, D) \geq 0$.*

Proof. 2.7 The necessity of this condition is clear. We prove sufficiency by induction on $g - 1 - \deg(D)$. By Lemma 2.7.8 and the fact that $\bar{\chi}(G, D) \geq 0$, we have that $\bar{\chi}(S_1, D) = \bar{\chi}(S_2, D) = 0$ implies $\bar{\chi}(S_1 \cup S_2, D) = \bar{\chi}(S_1 \cap S_2, D) = 0$.

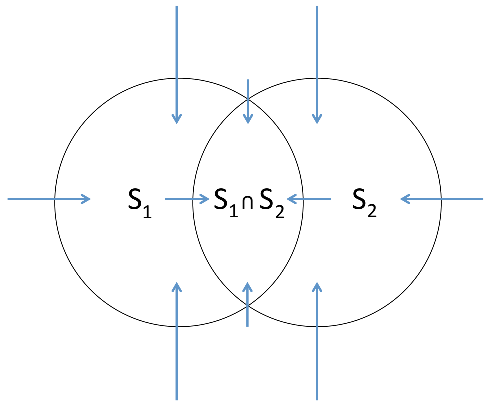


Figure 9: A picture proof that for partially orientable divisors, $\bar{\chi}(S_1, D) = \bar{\chi}(S_2, D) = 0$ implies $\bar{\chi}(S_1 \cup S_2, D) = \bar{\chi}(S_1 \cap S_2, D) = 0$. This figure does not immediately apply in the proof of Theorem 2.7 because we cannot presuppose the divisors in question are partially orientable, although it can be converted into a proof if we contract $G[(S_1 \cup S_2)^c]$ and reduce to the case of full orientations.

We claim that if D is a divisor with $\deg(D) < g - 1$ which satisfies the conditions of the theorem, then there exists some vertex $v \in V(G)$ such that $D + (v)$ also satisfies the conditions of the theorem. Suppose not, then for every vertex v we may associate a set $S_v \subset V(G)$ such that $\bar{\chi}(S_v, D) = 0$ since $\bar{\chi}(S, D + (v)) = \bar{\chi}(S, D) - 1$ for all S containing v . Taking the union of S_v over all vertices, the previous observation gives that $\bar{\chi}(V(G), D) = 0$, which says that $\deg(D) = g - 1$, a contradiction.

Alternately, we could have made following argument. Assume that S_v is minimal in the sense that $\bar{\chi}(S, D) > 0$ for all $S \subsetneq S_v$. It follows that $S_u \cap S_v = \emptyset$ for all $u, v \in V(G)$. By connectivity, there exists some edge $e \in (S_u, S_v)$, but this implies that $\bar{\chi}(S_u \cup S_v, D) < 0$, a contradiction.

To complete the proof of the Theorem, we add a chip to some $v \in V(G)$, so that $D + (v)$ satisfies $\bar{\chi}(G, D + (v)) \geq 0$. It follows by induction on $g - 1 - \deg(D)$ that $D + (v)$ is partially orientable. Given some partial orientation $\mathcal{O}_{D+(v)}$ we may unorient an edge directed towards v to obtain a partial orientation $\mathcal{O}_D$. $\square$

We now provide an algorithmic proof using edge pivots.

Proof. Starting with the empty orientation, apply Algorithm 2.4.6 to a divisor D satisfying the conditions of the Theorem 2.7. While building our partial orientation $\mathcal{O}$, we will only be in Case 1 since $D(v) \geq -1$ for all $v \in V(G)$. Moreover, by utilizing Jacob's ladder cascades in Case 1, we will never need to perform a cocycle reversal unless we obtain a set X such that $\bar{\chi}(X, D) < 0$, thus proving the sufficiency of the conditions from the theorem. $\square$

We remark that the previous argument provides a potentially new proof of Theorem 2.7.1 and thus, by Theorem 2.7.3, a potentially new proof of MFMC.

2.8 Partial Orientations of Metric Graphs

In this section we discuss partial orientations of metric graphs and outline the ways in which this setting differs from the discrete one which we have investigated in previous sections.

Before giving a description of a partial orientation of a metric graph, we must first give a suitable working definition of the tangent space of Γ. A tangent direction t associated to a point $p \in \Gamma$ is an equivalence class of paths emanating from p, where two paths are said the be the equivalent if they share some initial segment. An orientation $\mathcal{O}$ of Γ is an assignment of values 1 (an inward orientation) and 0 (an outward orientation) to the points in the tangent space of Γ with the following property. For any tangent direction t at a point p, there is some small initial segment in the direction of t for which all of the values at the corresponding tangent directions agree. We define a partial orientation $\mathcal{O}$ of Γ to be obtained from a full orientation $\mathcal{O}'$ by omitting a finite number of incoming tangent directions in $\mathcal{O}'$. We call such tangent directions *missing*. Note that unlike the case of discrete graphs, any partial orientation extends **uniquely** to a full orientation, and we denote this orientation $c(\mathcal{O})$ for the *closure of* $\mathcal{O}$.

For completeness sake, we describe a second equivalent definition of a partial orientation of a metric graph. Let G be a model for Γ and G' be another model obtained from G by adding in the point $p \in \Gamma$. Given a partial orientation $\mathcal{O}$ of G, we say that the partial orientation $\mathcal{O}'$ of G' is a *refinement* of $\mathcal{O}$ if one of the following holds. If p is interior to an edge $e \in G$ which is oriented in $\mathcal{O}$, then the two new edges in G' which replace e are both oriented as e is in G. Otherwise, p is interior to an edge e which is unoriented in $\mathcal{O}$ and we require that exactly one of the two new edges is oriented towards p, while the other edge is unoriented. We say that a partial orientation is a *refinement* of another partial orientation if it is obtained by a sequence of refinements as above. We say that two partial orientations $\mathcal{O}_1$ and $\mathcal{O}_2$ of models G_1 and G_2 respectively are Γ- *equivalent* if they have a common refinement. Given a set of equivalent partial orientations such that each point of Γ belongs to some model underlying a partial orientation in our set, we define the direct limit of this set of partial orientations under refinement to be a partial orientation of Γ.

We define a partial orientation to be *sourceless* if every point has an inwardly oriented tangent direction and acyclic if it contains no directed cycle. Note that for any (not necessarily induced) path in Γ, each interior point has a naturally associated pair of tangent directions. We say that a path γ in $\mathcal{O}$ is *consistently oriented* if every point in γ has oppositely oriented associated tangent directions in γ. To give the right notion of an edge pivot for a partial orientation of a metric graph, we take the continuous limit of a Jacob's ladder cascade. Given a consistently oriented half open path, whose frontier point p is missing the associated tangent direction t, we can reverse the orientation of this path, assign t value 0, and remove the associated tangent direction at the other boundary point. As in the discrete case, this operation does not effect $D_{\mathcal{O}}$. We call such an operation a *half open path reversal.* When the path is degenerate and consists of a single point we refer to this operation as a *tangent pivot.* The generalized cycle, cocycle, and cycle-cocycle are defined as in the previous sections.

It is very tempting to try and combine the notions of a half open path reversals and a cycle reversals, although at the time of writing we are not aware of a completely natural method for doing this. This would allow for the cycle-cocycle reversal system to be subsumed by the cocycle reversal system. One reason that this is a particularly attractive choice is that, as in the discrete, the ranks of partial orientations in the two systems agree.

Although we were not able to write down the proofs of our main results by the time of submission of this thesis, we have checked them carefully. We describe informally how the results of the previous sections extend to this setting. Lemma 2.3.1 and Theorem 2.3.3 both extend, as do the oriented Dhar's algorithm and the unfurling algorithm after appropriate modifications. The one thing which needs to be checked carefully in the unfurling algorithm is that the process terminates in finite time, but this follows by an argument which is essentially the same as the one given in our proof Luo's Theorem 3.2.1.

In the discrete case, for determining whether a divisor is linearly equivalent to a partially orientable divisor we needed to introduce a modified version of the unfurling algorithm. Perhaps one of the most interesting differences between the metric case and discrete case is that this modified unfurling algorithm for partial orientations of metric graphs is unnecessary because of the following fact.

Theorem 2.8.1. *Every divisor of degree at most $g-1$ on a metric graph Γ is linearly equivalent to a partially orientable divisor.*

Most of the work on section 2.7 extends due to the fact that the lengths of the edges in a metric graph do not correspond to capacities. Therefore, if we wish to use max-flow min-cut for constructing an orientation of a metric graph, it suffices to work with a model of that metric graph and forget the underlying edge lengths. We conclude this section by extending our Euler characteristic description of partially orientable divisors.

Theorem 2.8.2. *A divisor D is partially orientable if and only if $D(p) \geq -1$ for all $p \in \Gamma$ and $\bar{\chi}(S, D) \geq 0$ for every closed subset $S \subset \Gamma$.*

Proof. The necessity of the conditions of the theorem are obvious. Rather than give a self contained proof of sufficiency, we choose to provide a reduction to the case of finite graphs. Note that the property described in the statement of the theorem is preserved under homeomorphisms of Γ. Therefore, we may perturb the edge lengths of Γ by some small amounts so that they become commensurable as do the positions the chips in Γ. We may then scale Γ so that the lengths become integral and reduce to the case of discrete graphs. $\square$

The advantage of the previous proof is that it provides a method of finite verification. *A priori* the conditions of the theorem might not be checkable.

CHAPTER III

TRANSFINITE CHIP-FIRING

We demonstrate that the greedy algorithm for reduction of divisors on metric graphs need not terminate by modeling the Euclidean algorithm in this context. We observe that any infinite reduction has a well defined limit, allowing us to treat the greedy reduction algorithm as a transfinite algorithm and to analyze its running time via ordinal numbers. Matching lower and upper bounds on worst case running time of $O(\omega^n)$ are provided.

3.1 Introduction

Chip-firing on graphs has been studied in various contexts for over 20 years. The theory has found new applications in the recent work of Baker and Norine [9], who showed that by studying chip-firing, one may extend the work of Bacher, de la Harpe, and Nagnibeda [62] on the theory of linear equivalence of divisors on graphs. In particular, they were able to demonstrate the existence of a Riemann-Roch theorem for graphs analogous to the classical statement for curves. Gathmann and Kerber [33], and independently Mikhalkin and Zharkov [58], proved a Riemann-Roch theorem for tropical curves. The approach of Gathmann and Kerber was to establish the tropical Riemann-Roch theorem as a limit of Baker and Norine's result for graphs under subdivision of edges. Hladký, Král, and Norine [58] then showed that this theorem may be proven in an elementary way by studying the combinatorics of chip-firing on abstract tropical curves, i.e., metric graphs. Several papers have pursued this approach further along with other consequences for the theory[58] of linear equivalence of divisors on tropical curves [1] [22] [37] [52].

The central combinatorial objects in this study, for both graphs and tropical curves, are the so-called q-reduced divisors (known elsewhere in the literature as superstable configurations or G-parking functions). A q-reduced divisor is a special representative from the class of divisors linearly equivalent to a given divisor. There is an algorithmic method for obtaining the unique q-reduced divisor consisting of two parts. In this chapter, we investigate the second, more subtle part of this process known as reduction. We offer a new short proof of Luo's result that Dhar's reduction algorithm terminates after a finite number of iterations. We then investigate the greedy reduction algorithm, which in the graphical case is known to succeed. We show that the Euclidean algorithm may be modeled by the greedy reduction of divisors on metric graphs. By evaluating this algorithm on two incommensurable numbers, we obtain a run of the greedy reduction algorithm which does not terminate.

After observing that any infinite reduction has a well-defined limit, we analyze the running time of the greedy algorithm via ordinal numbers. We demonstrate matching upper and lower bounds on worst case running time of $O(\omega^n)$. The lower bound is obtained by gluing n copies of the Euclidean algorithm example together and ordering the firings lexicographically. The upper bound of $\omega^{deg(D)}$ is provided by an inductive argument.

3.2 Metric Chip-Firing and Reduced Divisors

A metric graph Γ is a metric space which can be obtained from an edge weighted graph G by viewing each edge with weight $w_{i,j}$ as being isometric to an interval of length $w_{i,j}$. Each point interior to an edge has a neighborhood homeomorphic to an open interval and each vertex has a small neighborhood homeomorphic to a star. The *degree* of a point $p \in \Gamma$ is the number of tangent directions at p. A vertex is called a *combinatorial vertex* if it has degree other than 2.

This chapter concerns certain combinatorial aspects of chip-firing on metric graphs, so we will take a rather concrete working definition of chip-firing. For completeness sake, we begin with a slightly more abstract definition. Fix a metric graph Γ and a parameterization of the edges of Γ. Let f be a piecewise affine function with integer slopes on Γ. We define the Laplacian operator Q applied to f at a point to be the sum of the slopes of the function as we approach p along each of the tangent directions at p. We note that $Q(f)(p) = 0$ if f is differentiable at p. We define a *divisor* D on Γ to be a formal sum of points from Γ with integer coefficients, all but a finite number of which are zero. We say that D has $D(p)$ *chips* at p. Given some divisor D on Γ, we define the chip-firing operation f applied to D to be $D - Q(f)$. We say that two divisors are linearly equivalent if they differ by some chip-firing move. A divisor E is said to be *effective* if it has a nonnegative number of chips at each point.

We now give the definition of chip-firing on metric graphs which will be used for the remainder of the chapter. Let X and Y be two disjoint open connected subsets of Γ such that the $\Gamma \setminus (X \cup Y) = Z$ is isometric to a disjoint collection of closed intervals of length ϵ. Note that the set Z defines a minimal cut in Γ. Now, we define the divisor $Q(f)$ as the divisor which is negative one at the end points of these intervals on the boundary of X and positive one at the endpoints on the boundary of Y. One may intuitively understand this divisor as pushing a chip along each edge in this cut a fixed distance ϵ. We take this to be the basic type of chip-firing move and call ϵ the *length* of the firing. Note that the chip-firing divisor is of the form $Q(f)$ where f is the piecewise affine function with integer slopes which is 0 on X, ϵ on Y, and has slope 1 on the each open interval in Z. We write $\epsilon(f)$ for the length of the firing f. As is noted in [8], any piecewise affine function with integer slopes can be expressed as a finite sum of the functions just described, so we will not sacrifice any generality by restricting our definition of chip-firing to be basic chip-firing moves.

A *q-reduced divisor* is a divisor which is nonnegative at each point other than $q \in \Gamma$, such that any firing $Q(f)$ which pushes chips toward q causes some point to go into debt. It is proven in [58] that given any divisor D on a metric graph Γ, there exists a unique q-reduced divisor ν which can be reached from D by a sequence of chip-firing moves. Moreover, there exists an effective divisor E equivalent to D if and only if ν is effective. An algorithmic way of obtaining such a divisor was described by Luo [52]. His method is to first bring every point other than q out of debt by some sequence of chip-firing moves. Once we have obtained such a configuration, we may perform firings which push chips back toward q without causing any vertex to go into debt. We call this second part of the process *reduction*. Luo's method for reducing a divisor is to use a generalization of Dhar's burning algorithm originally investigated in the study of the sandpile model.

Dhar's burning algorithm may be described in the following informal way: Let D be a divisor which is nonnegative at every point of Γ other than q. Place $D(p)$ firefighters at each point p other that q. Light a fire at q and let the fire spread through Γ along the edges. Every time the fire reaches a firefighter, it stops. If the fire approaches a point from more directions than there are firefighters present, these firefighters are overpowered and the fire continues to spread through the Γ. It is not hard to check that a divisor is q-reduced if and only if the fire burns through the entirety of Γ.

Let D be nonnegative at all points other than q. We say that a firing f is *legal* if $D - Q(f)$ is also nonnegative at all points other than q. A firing f is a *maximal legal firing* for D if the legal firing f is taken to have maximum associated length, i.e., the firing describes a push of chips along a cut so that at least one of the chips hits a combinatorial vertex and therefore cannot be pushed any further without choosing a different cut. Every time a fire is prevented from burning through Γ, the collection of chips, i.e., firefighters which stop the fire define a maximal legal firing towards q which does not cause any point to go into debt. Luo showed that if we take a divisor which is non-negative away from q and reduce according to the maximal legal firings obtained from this algorithm, we will obtain a reduced divisor after a finite number of steps. We offer the following new short proof that this process terminates.

Theorem 3.2.1. *(Luo) Let Γ be a metric graph and D be a divisor on Γ such that $D(p) \geq 0$ for all $p \in \Gamma$ with $p \neq q$. The reduction of D with respect to q by maximal legal firings obtained from Dhar's algorithm terminates after a finite number of iterations.*

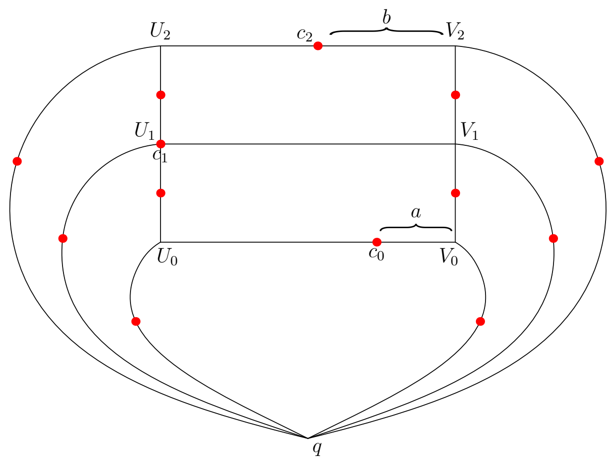
Proof. Let $\#E(\Gamma)$ be the number of edges in Γ and without loss generality take D to be a divisor which has 0 total chips. We proceed by double induction on $\#E(\Gamma)$ and $-D(q)$, i.e., induction on $k = \#E(\Gamma) - D(q)$. We will restrict our attention to the set S of edges adjacent to q. First observe that for any edge e in S which has a chip, the closest chip p in e to q will never leave e, although chips in e further away from q may. This is because when the fire burns from q, it will always reach p. Also, if every edge in S contains a chip, on the next iteration of the algorithm, the fire will burn from q and be stopped by precisely these edges. Hence, these chips will be fired towards q, which will receive a chip and we may induct on k. Therefore, we may assume that there exists an edge in S which will never receive a chip. We can contract this edge and again induct on k. □

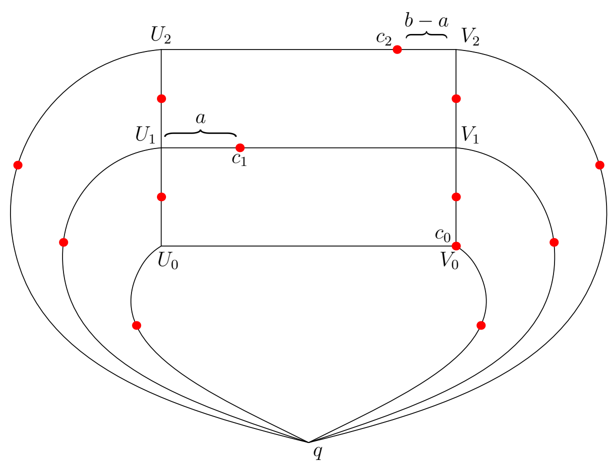
3.3 Infinite Greedy Reduction

We can always reduce by performing any legal firings we wish. If this process terminates, we know by uniqueness that we have reached the q-reduced divisor equivalent to D. In the case of discrete graphs, it is clear that this process will terminate. We begin by answering Matthew Baker and Ye Luo's question of whether the greedy reduction algorithm for metric graphs also terminates after a finite number of iterations. The answer, as we will see, is a resounding no. To this effect, we will provide an example which demonstrates that we can model the Euclidean algorithm in this context and therefore, by taking our input to be a pair of incommensurable numbers, obtain a run of the greedy reduction algorithm which does not terminate in finite time.

We define a *greedy reduction* of a divisor $D = D_0$ to be a sequence of divisors D_i such that $D_i = D_{i-1} - Q(f_{i-1})$, where $Q(f_{i-1})$ is a maximal legal firing for D_{i-1}.

Theorem 3.3.1. *There exists a metric graph Γ, a divisor D on Γ such that $D(p) \geq 0$ for all $p \in \Gamma$ with $p \neq q$, and a greedy reduction of D with respect to q which does not terminate after a finite number of steps.*





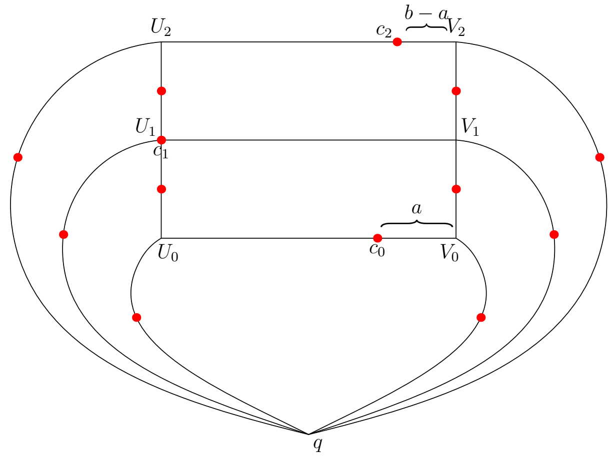


Figure 1. A divisor on a metric graph with infinite greedy reduction via firings which model the Euclidean algorithm.

Proof. We now present an example which demonstrates that the greedy reduction algorithm may not terminate after a finite number of iterations. We refer to Figure 1 which illustrates a certain divisor D on a metric graph Γ. We will take all of the edge lengths to be sufficiently large. What is meant by sufficiently large will become clear after we have completed the proof. We take D to have chips (with labels for the clarity's sake) c_0, c_1, and c_2 with c_1 at u_1, c_0 at distance a from v_0 on (u_0, v_0) and c_2 at distance b from v_2 on (u_2, v_2) with $a < b$. We take D to have a chip at the midpoint of every other edge in Γ. It is not important that chips be at the midpoints, only that they be sufficiently far from both endpoints. The idea of the example is to show that given D, we can perform the subtraction of a from b without changing the rest of the divisor much. We may then perform the Euclidean algorithm on inputs a and b. By taking a and b so that $\frac{a}{b} \notin \mathbb{Q}$ it follows (after verifying the convergence of a certain series) that we can obtain a run of the greedy reduction algorithm which does not terminate. We now describe the pair of firings which allows us to subtract a from b.

Firing 1: We would like to perform a maximal legal chip-firing move towards

q which will push chips c_0, c_1, and c_2 length a toward v_0, v_1, and v_2 so that c_0 hits v_0. We can achieve this by taking a firing corresponding to the cut $(X, Y) = (\{u_0, u_1, u_2\}, \{q, v_0, v_1, v_2\})$. Given this cut, we can push c_0, c_1, and c_2 as described and extend this to a maximal legal firing towards q by pushing the chips interior to the edges $(u_0, q), (u_1, q)$ and (u_2, q) distance a towards q.

Firing 2: Now, we would like to perform a maximal legal chip-firing move towards q which will push c_0 and c_1 distance a back towards u_0 and u_1 respectively so that c_1 reaches u_1. As in Firing 1, we can achieve this by the taking the cut corresponding to the partition $(X, Y) = (\{v_0, v_1\}, \{v_2, u_0, u_1, u_2, q\})$ and pushing chips in each of the other edges of this cut length a towards $\{v_2, u_0, u_1, u_2, q\}$.

By ignoring the position of all of the chips other than c_0, c_1, and c_2, we observe that we have returned to the original divisor with b replaced by $b - a$, so we have subtracted a from b. We can now perform the Euclidean algorithm by subtracting a, n times from b so that $0 \leq b - na < a$. By the symmetry of the construction, we may now reverse the roles of c_0 and c_2 and repeat. The one subtlety here is that we need to be sure that none of the other chips in the metric graph eventually reach either of the endpoints of the edge they are contained in, otherwise we might not be able to perform the firings described above. This is why we take the chips to be at the midpoints of sufficiently long edges. If a and b are such that $\frac{a}{b} \notin \mathbb{Q}$, this process will never terminate, but the series of length of the firings will converge, and we can take the lengths of the edges to be the twice this series of the lengths corresponding to the firings performed. It remains to prove that the corresponding series of lengths converges. To this end, we will assign some notation to the quantities appearing in the Euclidean algorithm. Given two numbers a_i and b_i with $0 < a_i < b_i$, we define $b_{i+1} = a_i$ and $a_{i+1} = b_i - n_i a_i$ with $n_i \in \mathbb{N}$ and $0 \leq b_i - n_i a_i < b_i$. Letting $l_i = n_i a_i$, it needs to be shown that $\sum_{i \geq 0} l_i$ converges. We claim that taking $a = a_0$ and $b = b_0$, $\sum_{i \geq 0} l_i \leq 4b$. This follows from the simple observations that $l_{i+1} \leq l_i$ and

$l_{i+2} < \frac{1}{2}l_i$, which allow us to conclude that $\sum_{i\geq 0} l_i$ is bounded geometrically and the claim follows.

$\square$

3.4 Running Time Analysis via Ordinal Numbers

We now prove than any reduction of a divisor which does not terminate has a well-defined limit. This will allow us to interpret the greedy algorithm as a transfinite algorithm and to analyze its running time in the language of ordinal numbers. We first prove that for any infinite reduction, the sum of the lengths of the firings must converge.

Lemma 3.4.1. *Let Γ be a metric graph, D a divisor on Γ such that $D(p) \geq 0$ for all $p \neq q$, and let f_i be an infinite sequence of maximal legal firings reducing D. Then the series $\sum_i f_i$ converges and the greedy reduction of D has a well defined limit.*

Proof. Let $l(p) = \sum_{i=0}^{\infty} f_i(p)$ for $p \in \Gamma$. We now take v and v' to be combinatorial vertices. If $l(v)$ is finite then $l(v')$ is finite – if this were not the case, it would mean that v' sent an infinite number of chips towards v which were never able leave the set of edges incident to v and so we would have an infinite number of chips clustered around v, a clear contradiction. Take some v adjacent to q. Clearly $l(v)$ is finite, otherwise v will send an infinite number of chips to q. By the connectedness of the metric graph, it follows that $l(v)$ is finite for each combinatorial vertex, hence $l(v)$ is finite for each v and it follows that $\sum_i f_i$ converges. We now show that $\sum_{i=0}^{\infty} \epsilon(f_i)$ converges. Because Γ is compact and the sum of f_i is convergent, we see that the limit of $\sum_{i=0}^{\infty} \int_\Gamma f_i$ is well defined. Moreover, $\int_\Gamma f_i \geq \epsilon(f_i)m(\Gamma)$, where $m(\Gamma)$ is the sum of the lengths of the edges of Γ, as f is a nonnegative piecewise affine function with slopes ± 1, therefore $\sum_{i=0}^{\infty} \epsilon(f_i)$ converges.

Label the chips in D arbitrarily. For each passage from D_i to D_{i+1}, a given chip c either stays fixed or travels $\epsilon(f_i)$. The series of these lengths which c travels must converge because it is an increasing sequence bounded above by $\sum_{i=0}^{\infty} \epsilon(f_i)$. Therefore as we follow the path which this chip traces out, we see that it must have a well defined limit. Hence the limit of the greedy reduction has a well defined limit. $\square$

It is now natural, given an infinite greedy reduction, to pass to the limit and begin the process again. We will analyze the running time of the greedy reduction algorithm in terms of ordinal numbers. For an introduction to ordinal numbers, we refer the reader to [41]. We remark that in what follows, we will not use any advanced properties of ordinal numbers, rather they serve as a bookkeeping tool for rigorously investigating the question of how long it takes for the greedy reduction of a divisor to terminate. It has been proven [10][58] that any convergent sum of basic chip-firing moves is itself a finite sum of basic chip-firing operations, so we can be confident that in passing to the limit of a chip-firing process, we never leave the class of divisors linearly equivalent to the one we started with. In what follows, we would like to emphasize that ω^n is not $n\omega$, that is, even informally we should not think of ω^n as n copies of ω concatenated, rather we should consider this quantity as a nest of ω's with depth n.

Theorem 3.4.2. *For every $n \in \mathbb{N}$, there exists a metric graph Γ and a divisor D on Γ with $D(p) \geq 0$ for all $p \in \Gamma$ with $p \neq q$ such that the greedy reduction of D with respect to q takes time at least ω^n.*

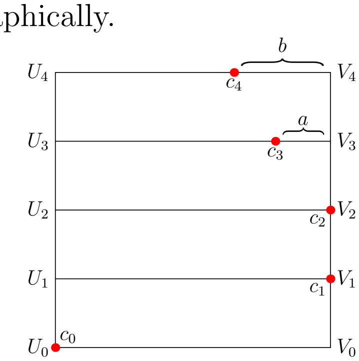
Proof. This is again a proof by construction. The idea is that by "gluing together" n copies of the previous Euclidean example we can obtain running time ω^n. See Figure 2 for an illustration of a piece of the example which will allow us to obtain running time ω^2. Missing from the figure are the edges (u_i, q) and (v_i, q) for all i. Once again, we imagine that all of the edges are sufficiently long and that there are chips at each of the midpoints of the edges not drawn. The idea is that we can run the example described previously for the bottom 3 rows (letting (u_0, v_0, c_0) and (u_1, v_1, c_1) switch roles). In the limit, c_0, c_1, and c_2 move to u_0, v_1 and v_2 respectively, after which we may use the top two rows to "recharge" c_1 and c_2. This recharging is achieved by the following two firings, which are illustrated in Figure 2:

Firing 1: We would like to push c_0 and c_3 distance a towards v_0 and v_3 respectively so that c_3 hits v_3. We can achieve this by taking a maximal legal firing corresponding to the cut $(X, Y) = (\{u_0, u_3\}, \{q, u_1, u_2, u_4, v_0, v_1, v_2, v_3, v_4\})$.

Firing 2: We would like to push c_0, c_1 and c_3 distance a towards v_0, v_1 and v_3 respectively so that c_0 hits v_0. We can achieve this by taking a maximal legal firing corresponding to the cut

$$(X, Y) = (\{v_0, v_1, v_3\}, \{q, u_0, u_1, u_2, u_3, u_4, v_2, v_4\}).$$

The figure shows how we can use c_3 and c_0 to recharge c_1. We can then perform a similar pair of firings using c_4 and c_0 to recharge c_2. We then iterate the process. The one subtle point is that we again need convergence of the double series of lengths coming from the firings. In order to do this, we should perform one step of the Euclidean algorithm with c_0, c_3 and c_4 after taking a limit of the bottom three rows and before recharging c_1 and c_2. A simple calculation shows that this minor adjustment ensures convergence. We leave the extension to n copies of the Euclidean example as an exercise for the reader. In order to ensure convergence of the associated n nested series of lengths, we should order the firings on the product of the n copies of the Euclidean example lexicographically.



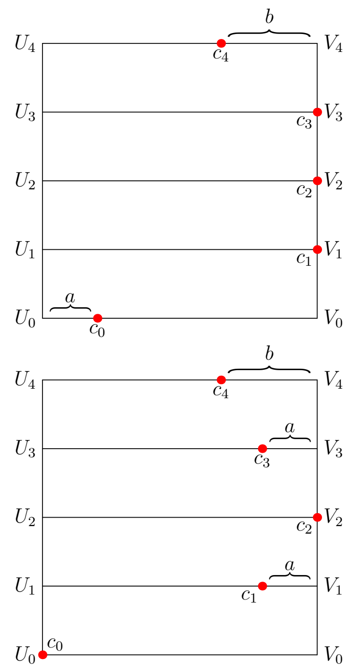


Figure 2. Two copies of the Euclidean example glued together and a recharging move achieved by two firings.

□

We now show that in some sense, the previous example is worst possible. The previous example shows that for any ordinal number $\alpha < \omega^\omega$, there exists a divisor D with a greedy reduction which takes more than α steps. While it is more or less obvious that we cannot have a greedy reduction with an uncountable number of iterations, ω^ω is still a countable ordinal and so *a priori* there might exist a divisor with a greedy reduction which takes ω^ω steps. The following result shows that this cannot occur.

Theorem 3.4.3. *Let Γ be a metric graph and D be a divisor on Γ such that $D(p) \geq 0$ for all $p \in \Gamma$ with $p \neq q$. Any greedy reduction of D with respect to q takes at most $\omega^{deg(D)}$ steps.*

Proof. Insight into this claim can be derived from inspection of the Euclidean example. As was noted previously, when we pass to the limit of this reduction process,

c_0, c_1, and c_2 approach the combinatorial vertices u_0, v_1, and v_2 respectively. We claim that more generally at step ω^{n-1} of any greedy reduction, there must be at least n chips present at the combinatorial vertices. Eventually, all of the chips will be at combinatorial vertices. During the next firing, some chip will traverse an edge from one combinatorial vertex to another. Thus the length of the firing will be at least the minimum the edge lengths. This cannot happen an infinite number of times otherwise the sum of the lengths will diverge contradicting Lemma 1. This will in turn give an upper bound on the running time of the greedy reduction algorithm of $\omega^{deg(D)}$.

We will proceed by induction. Because we are performing maximal legal firings, we always have a chip at a combinatorial vertex, e.g., one of the chips which arrived at a combinatorial vertex after the previous firing. We take this to be the base case of the claim. For the inductive step, assume that there are at least n chips at the combinatorial vertices at time ω^{n-1}. For each step $k\omega^{n-1}$, we can associate a set of chips S_k which are present at the combinatorial vertices at this time. Let A be some set of chips which is equal to S_k for infinitely many k. At time ω^n, the set of chips A will lie at combinatorial vertices. Moreover, if there exists some set of chips $B \neq A$ which is equal to S_k for infinitely many k, then the union $A \cup B$ will be present at combinatorial vertices at time ω^{n+1}, and we will have proved the claim, therefore we may assume that there exists a unique A equal to S_k for infinitely many k. At time $k\omega^n + 1$ some chip c_k must reach a combinatorial vertex. Observe that the chip $c_k \in S_k$ for only finitely many k, otherwise the nested series of lengths will diverge. Therefore there exist some chip $c = c_k$ for infinitely many k such that $c \notin S_k$. We conclude that $A \cup c$ are living at combinatorial vertices in the limit at time ω^n thus completing the proof.

$\square$

We note that although we are working with ordinal numbers, the previous argument employed only finite induction, not transfinite induction. We conclude with a question first posed to the author by Sergey Norin. The previous bound is a function of the degree of the divisor, but not the metric graph Γ. It would be nice to have a bound on the running time of an arbitrary greedy reduction on Γ in terms of the structure of Γ. For example, does there exist a bound of the form $\omega^{f(g)}$ where f is a polynomial in g, the genus of Γ? Can we take f to be linear?

There is a famous example showing non termination of Ford-Fulkerson for the case of real edge capacities. This example and the Euclidean example exist for the following dual reasons. Real edge capacities can be viewed as a limit of multigraphs where ratios of the multiplicities of edges converge to irrational quantities. For metric graphs, divisors with chips at irrational locations can be obtained as a limit of divisors on discrete graphs under subdivisions of the edges of the so that the ratios of distances between points converge to irrational numbers.

In future work, we hope to make this duality more precise. In particular, it would be nice to relate Luo's metric Dhar's algorithm to the Edmond-Karp variant on Ford-Fulkerson, both of which necessarily terminate in finite time. It may be interesting to analyze the running time of Ford-Fulkerson for real edge capacities via ordinal numbers and see if we obtain the same set of running times as with the greedy reduction algorithm.

CHAPTER IV

CHIP-FIRING VIA OPEN COVERS

Given a graph G, an sink vertex v_0, and an abstract simplicial complex σ on the nonsink vertices of G, we define a *hereditary chip-firing models* by requiring that only those vertices which form a face of σ may fire simultaneously, and only if they do not cause any vertex to be sent into debt. These models give a fine interpolation between the abelian sandpile model, where σ is a disjoint collection of points, and the cluster firing model, i.e., the unconstrained chip-firing model, where σ is the full simplex. The hereditary chip-firing models retain some very desirable properties, e.g. stabilization is independent of firings chosen and each chip-firing equivalence class contains a unique recurrent configuration. These models are equivalent to the ones independently discovered by Paoletti [?]. In this chapter we give self contained proofs of these results and explain how this framework generalizes to directed graphs using weighted abstract simplicial complexes. We present an explicit bijection between the recurrent configurations of a hereditary chip-firing model σ on an undirected graph G and the spanning trees of G, which generalizes the Cori-Le Borgne algorithm [24], and conclude with a description of how these results extend to metric graphs, where abstract simplicial complexes are replaced by open covers of Γ.

Acknowledgements: Thanks to Lionel Levine for bringing the work of Guglielmo Paoletti to my attention, and to Guglielmo Paoletti for his encouragement.

4.1 Introduction

In the ableian sandpile model (ASM), vertices are restricted to fire individually. This is in contrast to the cluster firing model (CFM) where vertices are allowed to fire simultaneously. A chip-firing model is a collection σ of subsets of the vertex set, those subsets which are allowed to fire simultaneously if no vertex is sent into debt. If every vertex appears somewhere in σ, and the family σ has the hereditary property, i.e. $A \in \sigma$ and $B \subset A$ implies $B \in \sigma$, we say that σ is a *hereditary chip-firing model*. This is equivalent to the statement, σ forms an abstract simplicial complex on the nonsink vertices of G. From this perspective, the sandpile model is the coarsest hereditary chip-firing model, described by taking σ to be the collection of all singleton sets from $V(G)\setminus\{v_0\}$, and the cluster model is the finest hereditary chip-firing model, described by taking σ to be the power set of $V(G) \setminus \{v_0\}$, i.e., the full simplex.

Some of the fundamental properties of the ableian sandpile model and the cluster firing model extend to arbitrary hereditary chip-firing models: the stabilization of a configuration is independent of the firings chosen and each chip-firing equivalence class contains a unique recurrent configuration. It is well known that the number of chip-firing equivalence classes is the same as the number of spanning trees of a graph. It follows that the number of recurrent configurations in a hereditary chip-firing model is the same as the number of spanning trees.

For the case of ASM and CFM, several bijections between recurrent configurations and spanning trees exist in the literature, e.g. [26] [24] [12] [20]. There is a simple relationship between the recurrent configurations in ASM and CFM which allows a bijection in one model to be "dualized" to produce a bijection in the other model. The recurrent configurations in CFM go by several names: G-parking functions, v_0-reduced divisors, superstable configurations. It is the aim of this chapter to present an explicit bijection between the recurrent configurations in an arbitrary hereditary chip-firing model and the spanning trees of a graph. Our bijection is a modification of the Cori-Le Borgne algorithm [24].

If we order the elements of σ by inclusion, we have a set of maximal elements $A_1, \dots A_k$ which in turn, by the hereditary property, determine σ. We note that these maximal elements of σ need not be disjoint, i.e. hereditary chip-firing models are not determined by partitions of the vertex set, instead we should think of them as being described by covers of $V(G) \setminus \{v_0\}$. Moreover, these covers need not be minimal, instead we ask that the elements of the cover be incomparable. This allows us to naturally identify hereditary chip-firing models with maximal antichains in the Boolean lattice. Calculating the number of such maximal antichains is an extremely challenging problem [28], but this quantity, called the Dedekind number, is known to be doubly exponential in n.

4.2 Notation and Terminology

We take G to be a connected undirected loopless multigraph with vertices labeled $v_0, v_1, \ldots, v_n$. Given $X, Y \subset V(G)$, we let $(X,Y) = \{e \in E(G) : e = (x,y), x \in X, y \in Y\}$, and let X^c denote $V(G) \setminus X$. To describe chip-firing, we begin with a graph G and a configuration D of *chips* on G. Formally, a configuration of chips is a function $D : V(G) \to \mathbb{Z}$. For the purposes of this chapter we will usually restrict our attention to D such that $D(v_i) \geq 0$ for all $i \neq 0$ and $D(v_0) = -\sum_{i=1}^{n} D(v_i)$ so that the sum of the values of D, called the *the degree of* D, is 0. If a vertex v in a configuration of D is seen to have $D(v) < 0$, we say that this vertex is *in debt*. The basic operation is *firing* whereby a vertex v sends a chip along each of its edges to its neighbors and loses $\deg(v)$ chips in the process so that the total number of chips is conserved. We designate v_0 to be the *sink vertex* and say that it cannot fire. This ensures that we cannot continue firing vertices indefinitely. The adjacency matrix A of a graph is an $(n+1) \times (n+1)$ matrix with entries $A_{i,j} = \#$ of edges between v_i and v_j. Taking D to be the diagonal matrix with $D_{i,i} =$ degree of v_i, the Laplacian Q of a graph is defined as the difference $D - A$.

For $S \subset V(G)$, we take χ_S to be the characteristic vector of S. As an abuse of notation we denote $\chi_{\{v_i\}}$ by χ_i. From a linear algebraic perspective, viewing a configuration D as a vector, if a vertex v_i fires then D is replaced by $D - Q\chi_i$, and more generally if a set S fires we obtain $D - Q\chi_S$. We say that two configurations D and D' are equivalent if there exists some sequence of firings which brings D to D' (possibly including firings by v_0 and passing through intermediate configurations which are negative at vertices other than v_0). Two configurations are seen to be equivalent if their difference is in the integral span of the columns of the Laplacian. We call a collection of configurations which are equivalent, a chip-firing equivalence class.

The ASM (ableian sandpile model) is defined by placing the additional restriction that vertices may only fire one at a time, whereas in the CFM (cluster firing model), vertices are allowed to fire simultaneously. We fix a collection σ of subsets of $V(G) \setminus \{v_0\}$, those sets which are allowed to fire simultaneously if no vertex is sent into debt, and call this collection a *chip-firing model*. If each vertex v_i with $i \neq 0$ appears somewhere in σ, we say that σ covers G. If σ covers G and σ is hereditary, i.e. for every $A \in \sigma$ and $B \subset A$, we have that $B \in \sigma$, we say that σ is a *hereditary chip-firing model*.

Let σ be a hereditary chip-firing model on a graph G. If a configuration of chips D has no set of vertices $M \in \sigma$ which can fire without some $v \in M$ being sent into debt, we say that D is *stable*. The process of firing sets from σ until a configuration becomes stable is called *stabilization*. We say that a set $M \in \sigma$, is *ready in* D if this set can fire without sending any vertex into debt, and call a vertex v *active* in a configuration D if there exists some $M \subset V(G) \setminus \{v_0\}$ with $v \in M$ which is ready. Suppose $v \in V(G)$ is active in a configuration D. There may very well be several different ready sets which contain v, and these different ready sets might cause v to lose different numbers of chips if they were to fire. Therefore, we let $N_{min}(v, D)$ denote the minimum amount that an active vertex v can lose by firing a ready set in D which contains v.

Lemma 1 states that the stabilization of a configuration in a hereditary chip-firing model is well defined, so we denote the stable configuration obtained from D by stabilization as D°. A configuration D is said to be *reachable* from another configuration D' if there exists a way of adding chips to D' and then firing ready sets to reach D. Because of our convention that the degree of D be zero, we are actually adding configurations of the form $\chi_i - \chi_0$, i.e. subtracting from v_0 exactly as many chips as we add to other vertices. A configuration D is *globally reachable* if it is reachable from every other configuration. Finally, we call D *recurrent*, if it is both stable and globally reachable. The original motivation for this terminology comes from the observation that if we continue adding chips and stabilizing, the configurations we will see infinitely many times are the recurrent ones. The recurrent configurations in CFM (G-parking functions) are precisely the stable configurations, so there is no need for a discussion of global reachability. We say that a configuration D is *critical* if it is stable and $(D - Q\chi_o)^{\circ} = D$. As with the ASM, we will show that a configuration in a hereditary chip-firing model is recurrent if and only if it is critical. This statement is trivially true for the CFM.

4.3 Preliminary Results for Discrete Graphs

In this section we present the basic results of hereditary chip-firing models. Hereditary chip-firing models as well as the results of this section were discovered independently of the author by Paoletti [**?**], and Caracciolo, Paoletti and Sportiello [18]. They observe that stabilization in a chip-firing model σ is independent of firings if and only if σ is closed under subtraction, i.e. for all $A, B \in \sigma$, we have $A \setminus B \in \sigma$. They then restrict to the case where for each $v \in V(G)$, $\{v\} \in \sigma$. It is easy to see that a family of subsets of $[n]$ is closed under subtraction and contains all singletons if and only if it is hereditary and covers $[n]$, i.e., is an abstract simplicial complex on $[n]$.

Lemma 4.3.1. *Given a fixed hereditary chip-firing model σ on a graph G, and a* *chip-firing configuration D on G, the stabilization of D is independent of the firings chosen.*

Proof. First, we observe that if $M, N \subset V(G) \setminus \{v_0\}$, M is ready and N fires first, then $M \setminus N$ is ready. This is because if we fire N and then fire $M \setminus N$, a vertex $v \in M$ loses at most as many chips as if M had fired alone. More generally, if M is ready and a multiset N fires, i.e. we fire vertices in N a number of times corresponding their multiplicity in N, then $M \setminus N$ is ready. Let $M_1, \ldots, M_s \in \sigma$ and $N_1, \ldots, N_t \in \sigma$ correspond to sequences of sets which are fired in two different stabilizations of D. Let $X_{M_q} = \sum_{i=1}^{q} \chi_{M_i}$ and $X_{B_r} = \sum_{i=1}^{r} \chi_{B_r}$. Suppose that $D - QX_{M_s}$ and $D - QX_{N_t}$ are not equal, i.e. the two stabilizations of D are different. We note that this can occur if and only if $X_{M_s} \neq X_{N_t}$, as v_0 does not fire and the kernel of the Laplacian is generated by the all one's vector. Without loss of generality, there exists some l maximum such that $X_{M_l} \leq X_{N_t}$ and $X_{M_{l+1}} \not\leq X_{N_t}$. By construction M_{l+1} is ready for $D - QX_{M_l}$. Now let $\chi_P = X_{N_t} - X_{M_l}$ be the characteristic vector corresponding to the multi set P . By the first observation, $M_{l+1} \setminus P$ is nonempty and ready for $D - QX_{M_l} - Q\chi_P = D - QX_{N_t}$, but this contradicts the fact that $D - QX_{N_t}$ is stable. $\square$

Theorem 4.3.2. *Given a fixed hereditary chip-firing model σ on a graph G, there exists a unique recurrent configuration ν in each chip-firing equivalence class.*

Proof. We begin by observing that every chip-firing equivalence class contains at least one recurrent configuration. In a stable configuration, each vertex v has at most $\deg(v) - 1$ chips. Therefore, if we can show that each equivalence class contains a configuration with more than $\deg(v)$ chips at each vertex v, it would follow that this configuration is globally reachable and hence its stabilization is recurrent. The technique which we now apply also appears in [9]. Partition the vertices according to their distance from v_0. Let d be the maximum distance of a vertex from v_0. Begin by firing all of the vertices of distance at most $d-1$ from v_0. This has the effect of sending money to the vertices of distance d. Repeat until each such vertex v has at least $\deg(v)$ chips. Now fire all of the vertices of distance at most $d-2$ from v_0 until the vertices of distance $d-1$ have at least their degree number of chips. Working backwards in this way towards v_0, we obtain the desired configuration.

We now show that there is at most one recurrent configuration in each equivalence class. This proof is identical to the argument presented in [40]. First, we would like to show that there exists a configuration ϵ with $\epsilon(v_i) > 0$ for all $i \neq 0$, such that when we add ϵ to a recurrent configuration ν and stabilize, we obtain ν. Let D be a configuration such that $D(v_i) \geq \deg(v_i)$ for all $i \neq 0$. We will take $\epsilon = D - D^\circ$. Because ν is recurrent, it is globally reachable, hence there exists some configuration ζ such that $(D+\zeta)^\circ = \nu$. We are interested in computing $\gamma^\circ = (D+\zeta+\epsilon)^\circ$. Because stabilization is independent of firings chosen, we can stabilize γ by first stabilizing $D+\zeta$, i.e. $\gamma^\circ = ((D+\zeta)^\circ + \epsilon)^\circ = (\nu+\epsilon)^\circ$. On the other hand, this is also equal to $(D^\circ + \zeta + \epsilon)^\circ = (D^\circ + \zeta + D - D^\circ)^\circ = (\zeta + D)^\circ = \nu$.

Assume that there are two different equivalent recurrent configurations ν and ν' such that $\nu \sim \nu'$. By definition, there exists some $f \in \mathbb{Z}^{n+1}$ such that $\nu - \nu' = Qf$, moreover we can take f to be such that $f(v_0) = 0$ because the all ones vector is in the kernel of Q. Let $f^+, f^- \in \mathbb{Z}^{n+1}$ be such that $f^+ \geq \vec{0}, f^- \leq \vec{0}$, and $f^+ + f^- = f$. Therefore, there is some configuration D such that $D = \nu - Qf^+ = \nu' - Q(-f^-)$. Note that because ν and ν' are stable, it follows that D may have vertices which are in debt. For any $k \in \mathbb{N}$, $\nu + k\epsilon$ and $\nu' + k\epsilon$ will stabilize to ν and ν' respectively, as was shown above. On the other hand, if we take k to be sufficiently large, we can perform firings defined by f^+ and $-f^-$ (by individual vertices for example) to $\nu + k\epsilon$ and $\nu' + k\epsilon$ respectively to obtain the configuration $D + k\epsilon$. But now we arrive at the contradiction that $D + k\epsilon$ should stabilize to both ν and ν'.

$\square$

We remark that Lemma 4.3.1 and Theorem 4.3.2 both extend to the setting of strongly connected directed graphs (or at least those with a spanning tree rooted a v_0), where they generalize in a curious way by the admittance of weighted abstract simplicial complexes. An abstract simplicial complex can be encoded by the incidence vectors of its faces which are in turn described as the downward closure of the incidence vectors of the facets. We can generalize this idea naturally by taking a *weight abstract simplicial complex* to be the downward closure of a finite collection of positive integral vectors. In the case of undirected graphs, one can easily show that this extra level of generality provides nothing new, the reason being that undirected graphs are special cases of Eulerian directed graphs, those whose digraphs whose Laplacian has a left kernel generated by the all one's vector. See Chapter 5 for a discussion of left kernels of directed Laplacians. In general, strongly connected directed graph are described equivalently as those directed graphs whose Laplacian which has a left kernel generated by positive vector, which is unique up to scaling. Let $\vec{G}$ be a strongly connected directed graph, and $\vec{R}$ the primitive (shortest integral) vector in the left kernel. It turns out that we lose no generality by restricting the incidence vectors for our weighted simplicial complex to be dominated by $\vec{R}$, thus for undirected graphs, and more generally Eulerian directed graphs, standard abstract simplicial complexes suffice.

The following remark requires that the reader have some working knowledge of commutative algebra. As was briefly mentioned in the introduction, the study of binomial ideals associated to chip-firing is a very active topic of research in combinatorial commutative algebra. The two previously studied ideals are those associated to the ASM and CFM. The former is called the *sandpile ideal*, and is contained in the latter, called the *Laplacian lattice ideal*. Given a hereditary chip-firing model we can naturally associate a binomial ideal with one generator coming from each allowed firing move, and we refer to these ideals as *hereditary chip-firing ideals*. Lemma 4.3.1

can be then be interpreted as saying that these binomials form a grevlex Gröbner basis for the ideal which they generate. This is simply because firing translates in this setting as polynomial division, and one characterization of a Gröbner basis is that division with respect to the given term order yields remainders which are independent of any choices made. As with the sandpile ideal, each hereditary chip-firing ideal is contained in the Laplacian lattice ideal, which is obtained by saturating with respect to the product of the variables. These are all zero dimensional ideals, so their associated variety is a finite collection of points. Moreover, each associated variety is set theoretically the same, they differ only by the "thickness" of the zero at the origin. One can show that the multiplicity of the origin is given by the number of stable configurations in the associated hereditary chip-firing ideal which are not recurrent. In future work we hope to further investigate hereditary chip-firing ideals and their minimal free resolutions.

Lemma 4.3.3. *Given a fixed hereditary chip-firing model on a graph G, a chip-firing configuration ν on G is recurrent if and only if it is critical.*

Proof. Suppose first that ν is recurrent, but not critical, that is $(\nu - Q\chi_0)^\circ = D \neq \nu$. Let ϵ be as in Theorem 4.3.2, then $(\nu + k\epsilon - Q\chi_0)^\circ = ((\nu + k\epsilon)^\circ - Q\chi_0)^\circ = (\nu - Q\chi_0)^\circ = D$. Because $\epsilon(v_i) > 0$ for all $i \neq 0$, we can take k sufficiently large so that $(\nu + k\epsilon - Q\chi_0)(v_i) > \deg(v_i)$ for all $i \neq 0$ and it follows that D is recurrent, a contradiction. Conversely, suppose that D is not recurrent, but that D is critical, then $(D - kQ\chi_0)^\circ = D$ for all $k \in \mathbb{N}$. If we take k to be sufficiently large, then we can perform firings as in the beginning of Theorem 1 to spread the chips around in the graph and reach a configuration which has at least degree number of chips at each vertex. It follows that D is globally reachable, hence recurrent, a contradiction.

□

Lemma 4.3.4. *The number of chip-firing equivalence classes on a graph G is the same as the number of spanning trees of G.*

Proof. Let $\bar{Q}$ denote the matrix obtained from Q by deleting the row and column corresponding to v_0. This matrix, called the reduced Laplacian of a graph, is known to have full rank as G is connected, and by the matrix-tree theorem $\det(\bar{Q})$ is equal to the number of spanning trees of G [43]. By ignoring the values of v_0 in our configuration, we see that the number of different chip-firing equivalence classes is the number of cosets for the image of $\bar{Q}$ and this index is given by $\det(\bar{Q})$. □

4.4 Spanning Tree Bijection

This algorithm is a modification of the Cori-Le Borgne algorithm [24] as presented in [10]. Their algorithm can be viewed as a variant of Dhar's burning algorithm [53]. We will call Dhar's burning algorithm as a subroutine, so we first begin by describing this method, and do so in the context of the cluster firing model where the author believes it is more naturally understood. One might argue that the brilliance of Dhar's algorithm is that its discovery occurred in the context of the ableian sandpile model, where its application is less obvious.

Given a recurrent configuration ν for the sandpile model $K^+ - \nu = \bar{\nu}$ is a recurrent configuration in the cluster firing model, where $K^+(v) = deg(v) - 1$ for all $V(G) \setminus \{v_0\}$. The interested reader can prove this fact using Lemma 3 or look to [9] for an alternate proof. This allows a bijection for one model to be "dualized" to produce a bijection for the other model. The bijection presented here is the first bijection which the author is aware of that applies directly to both models without exploiting this duality.

As was mentioned in the introduction, the recurrent configurations in the cluster firing model are precisely the stable configurations, therefore to check that a configuration ν is recurrent, we need only check that there exists no set $A \subset V(G) \setminus \{v_0\}$ which can fire without sending a vertex into debt. *A priori* we would need to check an exponential number of sets to be sure that ν was reduced, but Dhar's observation is that it's sufficient to check only n such sets. Begin by firing $A_1 = V(G) \setminus \{v_0\}$. By assumption, there exists at least one vertex v which is sent into debt. Remove v from A_1 and continue firing sets in ν and removing vertices sent into debt until reaching the empty set.

Here is why this works: suppose that $B \in V(G) \setminus \{v_0\}$ is ready in ν, but that we have a collection $A_1, \ldots, A_n$ of sets which were obtained from a run of Dhar's algorithm. There exists i maximum such that $B \subset A_i$. It follows that $A_{i-1} = A_i \setminus v$, with $v \in B$, where v was sent into debt by A_i, but if we fire $A_i \setminus B$, v may only gain chips, and v is supposedly able to fire in B without being sent into debt. Firing $A_i \setminus B$ and then B is the same as firing A_i, contradicting the fact that v was sent into debt by A_i.

Dhar's burning algorithm earns its name from the following alternate description: Place $D(v)$ firefighters at each vertex and start a fire at v_0. The fire spreads through the graph along the edges, but is prevented from passing through vertices by the firefighters located there. When the number of edges burned incident to a vertex is greater than the number of firefighters present, the firefighters are overpowered and the fire burns through the vertex. A configuration is stable in the cluster firing model if and only if the fire burns through the entire graph. Dhar noticed that by burning in a systematic way, this algorithm produces a bijection between the recurrent configurations and the spanning trees.

In the Cori-Le Borgne algorithm, the edges are burned in an order which produces an "activity preserving" bijection. To describe the Cori-Le Borgne algorithm, we begin with an arbitrary ordering of the edges $e_1, e_2, \ldots, e_m \in E(G)$. The setup is the same as with Dhar, except that we burn one edge at a time, always taking the edge with the smallest label connecting the burnt vertices to the non burnt vertices. When an edge burned causes the firefighters at a vertex to be overpowered and the vertex to be burnt, we mark this edge. It is clear that if the fire burns through the graph, these marked edges form a spanning tree . Conversely, if we start with a tree and begin burning the edges of our graph one at a time, the edges of the tree tell us when we should burn a vertex, hence how many firefighters (chips) a vertex should have. This shows that the algorithm produces a bijection between the recurrent configurations and spanning trees.

Before describing our algorithm, we introduce a third characterization of recurrent configurations. This definition is the the one which will be used in our bijection.

Lemma 4.4.1. *A configuration ν is critical if and only if any maximal sequence of firings by active vertices brings $\nu - Q\chi_0$ back to ν.*

Proof. Here we are allowing active vertices to fire even though this may cause them to go into debt. If a configuration ν is critical, it is clear that we can continue firing active vertices in the ready sets and eventually return to ν. Conversely, suppose that there exists some firing of individual active vertices which brings $\nu - Q\chi_0$ back to ν, but that ν is not critical. If this is the case, there must be some vertex $v \in V(G)\backslash\{v_0\}$ which was never fired in the stabilization of $\nu - Q\chi_0$. We might take v to be the first such vertex, but observe that this situation may only occur if a vertex of the same type has already been fired causing v to become active, a contradiction. □

The definition just described can be viewed as a quasi-local characterization of the recurrent states. It is local in the sense that vertices fire individually rather than as collections, but it is nonlocal in that whether a vertex is allowed to fire or not is based on nonlocal data. Recall $N_{min}(v, D)$ is the minimum amount that v can lose by firing a ready set in D which contains v. We now explain our bijection between recurrent configurations in a fixed hereditary chip-firing model σ on a graph G and the spanning trees of G. First we explain the map from critical configurations to spanning trees. Let X be the vertices which were fired at the ith step of the process. Let Y be the collection of maximal ready sets. The primary observation is that for each ready set S in Y, there exists a vertex v which, before X fired, would have been sent into debt if S fired. This means that $D(v) < N_{min}(v, D - Q\chi_X) < D - Q\chi_X$. As we burn edges from X across the cut to Y, eventually the number of edges burnt plus $D(v)$ is equal to $N_{min}(v, D - Q\chi_X)$. At this point we mark the last edge as part of the spanning tree, fire v, "unburn" the burnt edges and continue. We remark that although it is aesthetically displeasing to "unburn" the burnt edges and start anew with each iteration, it is necessary for the algorithm to work. In order to compute the value $N_{min}(v, D - Q\chi_X)$ more quickly than by simply checking all subsets of X complement, we can run the Dhar algorithm on each maximal element of σ contained in X complement.

Input:
$G = (V, E)$, a graph with a fixed ordering on E,
$v_0 \in V(G)$,
σ, a hereditary chip-firing model on $V(G)\backslash\{v_0\}$
$\nu = \sum_v a_v(v)$, a v_0-critical divisor of degree d.
Output:
T_ν, a spanning tree of G.

Initialization:
$X = \{v_0\}$ ("burnt" vertices),
$R = \emptyset$ ("burnt" edges),
$T = \emptyset$ ("marked" edges).

while $X \neq V(G)$ **do**
  $f = \min\{e = \{s, t\} \in E(G) \,|\, e \notin R,\, s \in X, t \notin X\}$,
  let $v \in V(G)\backslash X$ be the vertex incident to f,
  if $a_v = N_{min}(v, \nu - Q\chi_X) - |\{e \text{ incident to } v \,|\, e \in R\}|$ **then**
    $X \leftarrow X \cup \{v\}$,
    $T \leftarrow T \cup \{f\}$,
    $R \leftarrow \emptyset$
  end
  else $R \leftarrow R \cup \{f\}$
end
Output $T_\nu = T$.

Algorithm 1: Reduced divisor to spanning tree.

We now describe our algorithm γ for taking a tree T and producing a recurrent configuration, ν_T. This process has two parts. First we use T to construct a total order on the vertices. The idea is to mimic the Cori– Le Borgne algorithm for taking a tree and producing a critical configuration ν. The problem is that, in this more general setting, we are no longer able to determine ν because we do not know what the ready sets are at each step. Still, we are able to obtain a total order on the vertices which corresponds to the order in which the vertices should be fired, and we can then use this total order to reconstruct ν by running the algorithm a second time, this time "backwards".

Input:
$G = (V, E)$, a graph with a fixed ordering on E,
$v_0 \in V(G)$,
σ, a hereditary chip-firing model on $V(G)\backslash\{v_0\}$,
T, a spanning tree of G.
Output:
$w_0 < w_1 < \cdots < w_n$, a total oder on $V(G)$

Initialization:
$i = 0$,
$w_0 = v_0$ ("burnt" vertices),
$R = \emptyset$ ("burnt" edges).

while $i \neq n$ **do**
  $f = \min\{e = \{s,t\} \in E(G) \,|\, e \notin R,\ s \in X, t \notin X\}$,
  if $f \in T$ **then**
    let $v \in V(G)\backslash X$ be the vertex incident to f,
    $w_i := v$,
    $R \leftarrow \emptyset$,
    $i \leftarrow i + 1$.
  end
  else $R \leftarrow R \cup \{f\}$
end
Output: $w_0 < w_1 < \cdots < w_n$.

Algorithm 2: Spanning tree to reduced divisor: part 1

Now, let σ and γ be the maps from critical configuration to spanning trees and spanning trees to critical configurations respectively as described in the algorithms above.

Input:
$G = (V, E)$, a graph with a fixed ordering on E,
$v_0 \in V(G)$,
σ, a hereditary chip-firing model on $V(G) \setminus \{v_0\}$,
T, a spanning tree of G,
$w_0 < w_1 < \cdots < w_n$, a total order of $V(G)$ obtained in part 1.
Output:
$\nu_T = \sum_v a_v(v)$, a v_0-critical divisor.

Initialization:
$X = V(G) \setminus \{w_n\}$ ("burnt" vertices),
$R = \emptyset$ ("burnt" edges),
$i = n$.

while $X \neq v_0$ **do**
  $f = \min\{e = \{s, t\} \in E(G) \mid e \notin R,\ s \in X, t \notin X\}$,
  if $f \in T$ **then**
    $w_i \in V(G) \backslash X$ is the vertex incident to f,
    $a_{w_i} := N_{min}(w_i, \nu - Q\chi_X) - |\{e \text{ incident to } v \mid e \in R\}|$,
    $X \leftarrow X \setminus \{w_i\}$,
    $R \leftarrow \emptyset$,
    $i \leftarrow i - 1$.
  end
  else $R \leftarrow R \cup \{f\}$.
end
Output $\nu_T = \sum_v a_v(v)$.

Algorithm 3: Spanning tree to reduced divisor: part 2

Theorem 4.4.2. *The operations, σ and γ are inverse to each other and induce a bijection between the recurrent configurations of a hereditary chip-firing model σ on a graph G and the spanning trees of G.*

Proof. First we claim that $\gamma \circ \sigma$ is the identity map on the recurrent configurations. Let D be recurrent and $\sigma(D) = T$ a spanning tree. Observe that the total order produced on the vertices of G during the run of γ on T is the same as the order in which the vertices are processed during σ run on D. Given this total order on the vertices, the algorithm γ is designed so as to produce the configuration D such that $\sigma(D) = T$. It follows that σ is injective, and by Lemma 3, σ is an injective map between two sets with the same cardinality. It follows that σ is a bijection with explicit inverse γ. $\square$

4.5 Chip-firing via Open Covers of Metric Graphs

In this section we briefly discuss continuous analogues of the previously investigated model for metric graphs. Let Γ be a compact metric graph, and $\mathcal{U}$ an open cover of Γ with maximal sets $U_1, \ldots U_n$, and $q \in \Gamma$. Given a divisor D which is effective away from q. We call a firing function f *allowable* if $f \geq 0$, $f(q) = 0$, and there exists some U_i such that the support of f is contained in U_i. We would like to talk about stabilization of D with respect to $\mathcal{U}$ as the repeated application of firing until it is no longer possible, but the immediate problem is that such a process might never terminate. Thus, to give an appropriate notion of stabilization, we allow for transfinite firing processes, which will terminate in time less than ω^ω. Given this notion of stabilization we remark that we have the following natural metric versions of Lemma 4.3.1.

Lemma 4.5.1. *Given a metric graph Γ, a point $q \in \Gamma$, $\mathcal{U}$ an open cover of Γ with maximal sets $U_1, \ldots U_n$, and a divisor D which is effective away from q, the $(\mathcal{U}, q)$-stabilization of D is independent of any firing choices.*

We could like to also define *recurrent configurations* with respect to $\mathcal{U}$, but this is problematic for the following reason. If we define a Markov chain by adding chips and stabilization, we will eventually only see certain configurations, but because this Markov chain is infinite, we see any fixed recurrent configuration again with probability 0. Instead we work with the following equivalent notion. We say that a divisor D which is effective away from q is $(\mathcal{U}, q)$-critical if when we fire away from q some arbitrarily small distance ϵ and then $(\mathcal{U}, q)$-stabilize, we return to D. We now describe a metric version of Theorem 4.3.2.

Theorem 4.5.2. *Given a metric graph Γ, a point $q \in \Gamma$, $\mathcal{U}$ an open cover of Γ with maximal sets $U_1, \ldots U_n$, and a divisor D on Γ, D is linearly equivalent to a unique $(\mathcal{U}, q)$-critical divisor.*

Putting these two results together, we have the following corollary.

Corollary 4.5.3. *Given a metric graph Γ, a point $q \in \Gamma$, and $\mathcal{U}$ an open cover of Γ with maximal sets $U_1, \ldots U_n$ induce a canonical presentation of the Jacobian.*

Proof. By the previous Theorem, any elements $[D_1], [D_2] \in Pic^0(\Gamma)$ are linearly equivalent contain unique $(\mathcal{U}, q)$-critical divisors D_1 and D_2. We can add these two divisors and $(\mathcal{U}, q)$-stabilize to obtain the unique $(\mathcal{U}, q)$-critical configuration in $[D_1]+[D_2]$. □

If we take a collection of points $S \subset \Gamma$, which contains all of the points which have a number of tangent direction other than 2, this set induces a canonical cover $\mathcal{U}_S$ of Γ by stars, which is two to one away from S. We find that this model serves as a metric version of the abelian sandpile model. In particular, we obtain a duality between the $(\mathcal{U}_S, q)$-critical configurations and the q-reduced divisors.

Theorem 4.5.4. *Let $S \subset \Gamma$ be a set of points containing all of the points from Γ which have a number of tangent direction different than 2, $\mathcal{U}_S$ be the canonical cover of Γ by stars. and $K_S^+ = \sum s \in S(\deg(s) - 1)(s)$. There exists a canonical pairing of* *the $(\mathcal{U}_S, q)$-critical configurations and the q-reduced divisors so that the sum of each pair $(\mathcal{U}_S, q)$-stabilizes to $K_S^+ = \sum_{s \in S}(\deg(s) - 1)(s)$.*

If Γ has all edge lengths one, take S to be the minimal set satisfying the desired property, and we restrict this pairing to divisors supported on S, we retain a duality of Baker and Norine [9]. Every open cover has a refinement of a 2 to 1 cover by stars. Therefore, if there were method of making sense of a limit of $(\mathcal{U}_S, q)$-criticial divisors under refinement, then these star shaped covers would allows for a computation of the limit. We allow ourselves this one moment in the thesis to be completely speculative and suggest that this duality, viewed through the appropriate lens, ought to be interpretable as a combinatorial version of Serre duality.

CHAPTER V

RIEMANN-ROCH THEORY FOR DIRECTED GRAPHS AND ARITHMETICAL GRAPHS

In this chapter we investigate Riemann-Roch theory for directed graphs and arithmetical graphs. The Riemann-Roch criteria of Amini and Manjunath is generalized to all integer lattices orthogonal to some positive vector. Using generalized notions of v_0-reduced divisors and Dhar's algorithm, we investigate two chip-firing games coming from the rows and columns of the Laplacian of a strongly connected directed graph. We discuss how the "column" chip-firing game is related to directed $\vec{G}$-parking functions and the "row" chip-firing game is related to the directed sandpile model. Wilmes' lattice reduction algorithm shows that the "row" chip-firing game gives a graph theoretic model for the work of Amini and Manjunath. We conclude with a discussion of arithmetical graphs, which after a simple transformation may be viewed as a special class of directed graphs which will always have the Riemann-Roch property for the column chip-firing game. We answer a question of Lorenzini who asked for a combinatorial proof of the fact that if there are g_0 chips present in an arithmetical graph, then there exists a sequence of chip-firing moves which brings all of the vertices out of debt. Examples of arithmetical graphs are provided which demonstrate that either, both, or neither of the two Riemann-Roch conditions may be satisfied for the row chip-firing game. This chapter represents joint work with Arash Asadi.

5.1 Introduction

This project with Arash Asadi was the first one which the author pursued as a graduate student. It began when Matt Baker suggested that we answer a question posed by Lorenzini, who asked for a combinatiorial proof of the fact that if there are at least g_0 chips present in an arithmetical graph, then there necessarily exists a way of bringing all of the vertices out of debt by chip-firing. He also asked for a chip-firing proof that if g_0 agrees with g_{max}, the geometric genus of the associated lattice, then we have a natural canonical divisor. His original proofs of theses results relied on specialization arguments from curves to graphs. Eventually we found graph theoretic proofs of these statements, which required the introduction of generalized notions of reduced divisors and Dhar's algorithm for arithmetical graphs. After solving this problem, we generalized, per Farbod Shokrieh's suggestion, Amini and Manjunath's work on Riemann-Roch theory for full-dimensional lattices orthogonal to the all 1's vector to full-dimensional lattices orthogonal to an arbitrary positive integer vector, and used these results to investigate Riemann-Roch theory for arithmetical graphs. We remark that this extension requires little more than carefully checking that their arguments extend. Omid Amini visited Georgia Tech, and when we got to chat together about our work, his first reaction was that by scaling the Laplacian lattice coming from an arithmetical graph, one obtains chip-firing on a special class of directed graphs. By my work with Arash, we were able to fact check that this type of scaling is legitimate. We began to read about chip-firing on directed graphs and were immediately disappointed to notice that a dual version of our Dhar's algorithm had been discovered by the mathematical physicist Speer in 1994, who called it the script algorithm, as were a dual notion of reduced divisors. In section, illustrate this duality, and note that this has recently been extended to Gabrielov's M-matrices by Guzman and Klivans. Both our row and column chip-firing games for strongly connected directed graphs were both generalized in the much overlooked unpublished work of Gabrielov. Independently, and at the same time as us, Perkinson, Perlman, and Wilmes discovered directed reduced divisors and the dual script algorithm. It seems then, looking back on this work, that the real contribution is our chip-firing analysis of arithmetical graphs, in particular our combinatorial proofs of Lorenzini's theorems.

In this chapter we investigate Riemann-Roch theory for two dual chip-firing games coming from the Laplacian of a strongly connected directed graph, i.e., a directed graph for which there exists a direct path from any vertex to any other vertex. These digraphs can be algebraically characterized as those digraphs for which the left kernel of the Laplacian is 1-generated by a single positive (integer) vector, and it is the primitive vector in this left kernel of the Laplacian which determines the dynamics for both chip-firing games. The unconstrained *row chip-firing game* and *column chip-firing game* are defined similarly to the unconstrained chip-firing game of Baker and Norin, but they are determined by the row and column spans of the directed Laplacian. We recall that the Laplacian of an undirected graph can be obtained as the Laplacian of a directed Laplacian by viewing each undirected edge as a pair of directed edges. In this way, the row and column and chip-firing games may be viewed as dual extensions of the undirected chip-firing game. The row chip-firing game is both the more intuitive and important of the two games, so we begin by describing it first. Given a directed graph and a (not necessarily positive) chip configuration on the vertices, a vertex v fires by sending a chip along each of it's outgoing edges, and losing this many chips in the process, so that the number of chips is conserved. This game was investigated first in chip-firing by Lovasz, and In the column chip-firing game, the vertex v firing still loses its degree number of chips, but now the vertices with edges pointed towards v gain a chip. It appears at first that the total number of chips is not conserved, but for strongly connected directed graphs, the primitive vector in the left kernel gives a list of currencies for the vertices which makes the game conservative. We show how the directed G-parking functions are the appropriate generalization of the reduced divisors for this column chip-firing game when determining whether we have a winning strategy in the Baker-Norin game. By Amini's scaling argument, we may scale the associated lattice by the left kernel to reduce the study of Riemann-Roch theory for the column-chip-firing to the row chip-firing game on Eulerian directed-graphs.

We then present our work on arithmetical graphs. We prove Lorenzini's theorems using reduced divisors for the row chip-firing game and our dual script algorithm. We then present examples of arithmetical graphs with and without the Riemann-Roch property. Our main example of arithmetical graphs with the Riemann-Roch property are a class we call Euclidean stars, whose proof involves several techniques developed in the chapter. Amini and Manjunath [3] showed that by viewing the chip-firing game of Baker and Norine geometrically as a walk through the lattice spanned by its Laplacian, a pair of necessary and sufficient Riemann-Roch conditions, equivalent to those of Baker and Norine, could be generalized to all sub-lattices of the lattice $\Lambda_{\vec{\mathbf{1}}}$. They refer to these conditions as *uniformity* and *reflection invariance.*

5.1.1 Basic Notations and Definitions

For any two vectors $x, y \in \mathbb{R}^{n+1}$, let $x \cdot y$ denote the inner product of x and y. For any $x = (x_0, \dots, x_n)^T \in \mathbb{R}^{n+1}$, define $x^+ = (x_0^+, \dots, x_n^+)^T \in \mathbb{R}_+^{n+1}$ and $x^- = (x_0^-, \dots, x_n^-)^T \in \mathbb{R}_-^{n+1}$ to be the *positive part* and *negative part* of x respectively where $x = x^+ + x^-$ and $x_i^+ x_i^- = 0$, for all $0 \leq i \leq n$. Define $\deg_R(x) = R \cdot D$ and call it the *degree* of x. We denote $\deg_R(x^+)$ by $\deg_R^+(x)$ and we call it the *degree plus* of x.

Assume $\vec{\mathbf{0}}$ and $\vec{\mathbf{1}}$ are the vectors in $\mathbb{R}^{n+1}$ all of whose coordinates are 0 or 1, respectively. For any $x = (x_0, \dots, x_n)^T \in \mathbb{R}^{n+1}$, we say $x \geq \vec{\mathbf{0}}$ ($x > \vec{\mathbf{0}}$) if and only if for all $0 \leq i \leq n$, $x_i \geq 0$ ($x_i > 0$). We define a *partial order* in $\mathbb{R}^{n+1}$ as follows: for any $x, y \in \mathbb{R}^{n+1}$, we say $x \geq y$ ($x > y$) if and only if $x - y \geq \vec{\mathbf{0}}$ ($x - y > \vec{\mathbf{0}}$). For any vector $x \in \mathbb{R}^{n+1}$, define $C^+(x) = \{y \in \mathbb{R}^{n+1} : y \geq x\}$ and $C^-(x) = \{y \in \mathbb{R}^{n+1} : x \geq y\}$. We denote the standard basis for $\mathbb{R}^{n+1}$ by $\{e_0, \dots, e_n\}$. Suppose that $R \in \mathbb{N}^{n+1}$ is a vector, and define $H_R = \{x \in \mathbb{R}^{n+1} : R \cdot x = 0\}$. Let $\Lambda_R = H_R \cap \mathbb{Z}^{n+1}$ be the integer lattice in the hyperplane H_R where $R \in \mathbb{N}^{n+1}$. Let $\|\cdot\|$ denote the ℓ^2-norm, i.e., $\|x\| = \sqrt{x \cdot x}$, for all $x \in \mathbb{R}^{n+1}$.

Let G be graph and let $\{v_0, \ldots, v_n\}$ be an ordering of vertices of G. Let $Div(G)$ be the free Abelian group on the set of vertices of G. By analogy with the Riemann surface case as noted also in [9], we refer to elements of $Div(G)$ as *divisors* on G. In the case that the graph G is implied by context, we simply refer to elements of $Div(G)$ as divisors. Because there is a fixed ordering on vertices of G, we think of an element $\alpha \in Div(G)$, which is a formal integer linear combinations of vertices of G, as a vector $D = (d_0, \ldots, d_n) \in \mathbb{Z}^{n+1}$ where d_i is the coefficient of v_i in α for all $0 \leq i \leq n$. We denote to the ith coordinate of D by $D(v_i)$, for all $0 \leq i \leq n$. We refer to both vectors in $\mathbb{Z}^{n+1}$ and elements of $Div(G)$ as divisors.

5.2 *Riemann-Roch Theory for Sub-lattices of* Λ_R

5.2.1 Main Theorems

Throughout this section, R will denote a vector in $\mathbb{N}^{n+1}$.

Definition 5.2.1. *Let $\Lambda \subseteq \Lambda_R$ be a sub-lattice of rank n. Define*

$$\Sigma(\Lambda) = \{D \in \mathbb{Z}^{n+1} : D \ngeq p \text{ for all } p \in \Lambda\},$$

$$\Sigma_{\mathbb{R}}(\Lambda) = \{x \in \mathbb{R}^{n+1} : x \ngeq p \text{ for all } p \in \Lambda\}.$$

Note that the set $\Sigma(\Lambda)$ defined in Definition 5.2.1 is the negative of the *Sigma region* set defined by Amini and Manjunath [3]. We denote by $\overline{\Sigma}_{\mathbb{R}}(\Lambda)$ the topological closure of the set $\Sigma_{\mathbb{R}}$ in $\mathbb{R}^{n+1}$. Let $B(x, r) = \{y \in \mathbb{R}^{n+1} : \|y - x\| \leq r\}$ denote the ball of radius r with center at x. For any set $S \subset \mathbb{R}^{n+1}$, let $int(S)$ denote the *relative interior* of S.

Define $H_R^+ = \{x \in \mathbb{R}^{n+1} : x \cdot R \geq 0\}$. For any vector $p \in H_R^+$, define $\Delta_R(p) = H_R \cap C^-(p)$ to be the n-dimensional simplex in the hyperplane H_R. For the definitions of simplex and facet and their properties, we refer the reader to [56, 68]. For simplicity we denote $\Delta_R(R)$ by Δ_R.

It is easy to see that for any $p \in H_R^+$ there exists a unique $\lambda \geq 0$ and $p' \in H_R$ such that $p = p' + \lambda R$. Define the *projection* function $\pi : H_R^+ \to H_R$ as follows: for any $p \in H_R^+$, define $\pi(p) = p'$. It is also easy to see that $\lambda = (p \cdot R)/\|R\|^2$. We refer to $\pi(p)$ as the *projection* of the point p into the hyperplane H_R along the vector R.

Definition 5.2.2. *For any two points $p, q \in H_R$, define the Δ_R-distance function between p and q as follows:*

$$d_{\Delta_R}(p, q) = \inf\{\lambda \geq 0 : q \in p + \lambda \Delta_R\}.$$

The Δ_R-distance function defined above is a *gauge function* (which is often used in the study of convex bodies). For more on gauge functions and their properties, see [70].

For any point $p \in \Lambda$ define $d_{\Delta_R}(p, \Lambda) = \min\{\lambda \geq 0 :$ there exists $q \in \Lambda$ such that $q \in p + \lambda \Delta_R\}$.

Definition 5.2.3. *Define*

$$\begin{aligned}
Ext(\Sigma(\Lambda)) &= \{\nu \in \Sigma(\Lambda) : \deg_R(\nu) \geq \deg_R(p),\ \textit{for all } p \in N(\nu) \cap \Sigma(\Lambda)\}, \\
Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda)) &= \{\nu \in \overline{\Sigma}_{\mathbb{R}}(\Lambda) : \exists\, \delta > 0,\ \textit{such that } \deg_R(\nu) \geq \deg_R(p), \\
&\qquad \textit{for all } p \in B(\nu, \delta) \cap \overline{\Sigma}_{\mathbb{R}}(\Lambda)\}, \\
Crit(\Lambda) &= \{\nu \in H_R : \exists\, \delta > 0 \textit{ such that } d_{\Delta_R}(\nu, \Lambda) \geq d_{\Delta_R}(p, \Lambda), \\
&\qquad \textit{for all } p \in B(\nu, \delta) \cap H_R\}.
\end{aligned}$$

where $N(\nu)$ consists of all points $D \in \mathbb{Z}^{n+1}$ such that $\|D - \nu\|_{\vec{\mathbf{1}}} \leq 1$. We call $Ext(\Sigma(\Lambda))$, $Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$ the set of extreme points or extreme divisors of $\Sigma(\Lambda)$ and $\overline{\Sigma}_{\mathbb{R}}(\Lambda)$ respectively. The set of critical points of Λ is denoted $Crit(\Lambda)$.

Definition 5.2.4. *Let Λ be a sub-lattice of Λ_R of rank n, and $Ext(\Sigma(\Lambda))$ be the set of extreme points of $\Sigma(\Lambda)$. Define*

$$\begin{aligned} g_{\min} &= \min\{\deg_R(\nu) : \nu \in Ext(\Sigma(\Lambda))\} + 1, \\ g_{\max} &= \max\{\deg_R(\nu) : \nu \in Ext(\Sigma(\Lambda))\} + 1. \end{aligned}$$

We say the lattice Λ is uniform if $g_{\min} = g_{\max}$.

Definition 5.2.5. *Let Λ be a sub-lattice of Λ_R of rank n. We say Λ is reflection invariant if $-Crit(\Lambda)$ is a translate of $Crit(\Lambda)$, i.e., if there exists $v \in \mathbb{R}^{n+1}$ such that $-Crit(\Lambda) = Crit(\Lambda) + v$.*

Definition 5.2.6. *Let Λ be a sub-lattice of dimension n of Λ_R. We say a divisor $K \in \mathbb{Z}^{n+1}$ is a canonical divisor of Λ, or equivalently Λ has a canonical divisor K, if for all divisors $D \in \mathbb{Z}^{n+1}$,*

$$\deg_R(D) - 3g_{\max} + 2g_{\min} + 1 \leq r(D) - r(K - D) \leq \deg_R(D) - g_{\min} + 1.$$

Theorem 5.2.7. *Let Λ be a reflection invariant sub-lattice of Λ_R of rank n. Then Λ has a canonical divisor, i.e. there exists a divisor K such that for all $D \in \mathbb{Z}^{n+1}$,*

$$\deg_R(D) - 3g_{\max} + 2g_{\min} + 1 \leq r(D) - r(K - D) \leq \deg_R(D) - g_{\min} + 1.$$

Definition 5.2.8. *Let Λ be a uniform sub-lattice of dimension n of Λ_R. We say Λ has the Riemann-Roch property if there exists a divisor K with degree $2g - 2$, where $g = g_{\min} = g_{\max}$, such that for all divisor $D \in \mathbb{Z}^{n+1}$:*

$$r(D) - r(K - D) = \deg(D) - g + 1.$$

Theorem 5.2.9. *Let Λ be a uniform sub-lattice of dimension n of Λ_R. Then Λ is reflection invariant if and only if Λ has the Riemann-Roch property.*

Definition 5.2.10. *We say a sub-lattice Λ of Λ_R has the Riemann-Roch formula if there exists a an integer $m \in \mathbb{Z}$ and a divisor K of degree $2m-2$ such that for all $D \in \mathbb{Z}^{n+1}$:*

$$r(D) - r(K-D) = \deg_R(D) - m + 1.$$

Theorem 5.2.11. *Let Λ be a sub-lattice of dimension n of Λ_R. Then Λ has a Riemann-Roch formula if and only if Λ is uniform and reflection invariant, in particular Λ has the Riemann-Roch property.*

Let $R = (r_0, \ldots, r_n) \in \mathbb{N}^{n+1}$ and $\mathcal{R} = diag(r_0, \ldots, r_n)$ be a matrix mapping Λ_R to $\Lambda_{\vec{1}}$. To be more precise, for any $p \in \Lambda_R$ the image of p is $\mathcal{R}p$. For any set $S \subseteq \mathbb{R}^{n+1}$, let $\mathcal{R}S$ denote the set $\{\mathcal{R}p : p \in S\}$. It is easy to see that if $\Lambda \subseteq \Lambda_R$ is a sub-lattice of dimension n then $\mathcal{R}\Lambda$ is a sub-lattice of $\Lambda_{\vec{1}}$ of dimension n. The proceeding theorem follows immediately from Theorem 5.2.11, Corollary 5.2.30 and Lemma 5.2.31 appearing in Appendix A 5.2.2.

Theorem 5.2.12. *Let Λ be a uniform sub-lattice of dimension n of Λ_R. Then Λ has the Riemann-Roch property if and only if $\mathcal{R}\Lambda \subseteq \Lambda_{\vec{1}}$ has the Riemann-Roch property.*

5.2.2 Amini and Manjunath's Riemann-Roch theory for lattices

Many of the proofs and statements presented in this section are similar to those which appeared in Amini and Manjunath [3]. Essentially, what is being demonstrated is that if one replaces each statement about lattices orthogonal to the all one's vector with the same statement for lattices orthogonal to some fixed positive vector, the proofs will go through without much extra effort. This in itself is not a very strong observation, but it is necessary for proving Theorems 5.2.7, 5.2.9, 5.2.11 and 5.2.12, which are used several times in the preceding sections so, for the sake of completeness, we have decided to provide all of the necessary lemmas and theorems with proofs.

Recall Definitions 5.2.1, 5.2.2 and 5.2.3.

Lemma 5.2.13. *If $\Lambda \subseteq \Lambda_R$ is a sub-lattice of rank n, then*

$$\overline{\Sigma}_{\mathbb{R}}(\Lambda) = \{x \in \mathbb{R}^{n+1} : x \not> p, \text{ for all } p \in \Lambda\}.$$

Proof. Suppose $x \in \mathbb{R}^{n+1}$ such that $x > p$ for some $p \in \Lambda$. Thus there exists $\delta > 0$ such that for all $y \in B(x, \delta)$, $y > p$. Thus $x \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. Now, suppose $x \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. Then there exists $\delta > 0$ and $p \in \Lambda$ such that $x - \frac{\delta}{2}\vec{\mathbf{1}} \geq p$. Hence $x > p$, and this completes the proof of the lemma. □

Lemma 5.2.14. *If $D \in \mathbb{Z}^{n+1}$ then $D \in \Sigma(\Lambda)$ if and only if $D + \vec{\mathbf{1}} \in \overline{\Sigma}_{\mathbb{R}}(\Lambda)$.*

Proof. If $D \notin \Sigma(\Lambda)$ then there exists $p \in \Lambda$ such that $D \geq p$. Hence $D + \vec{\mathbf{1}} > p$ and by Lemma 5.2.13 $D + \vec{\mathbf{1}} \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. If $D + \vec{\mathbf{1}} \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$ then Lemma 5.2.13 implies that $D + \vec{\mathbf{1}} > p$ for some $p \in \Lambda$. Since $D, p \in \mathbb{Z}^{n+1}$, it follows that $D \geq p$ and this implies that $D \notin \Sigma(\Lambda)$. □

Suppose $R = (r_0, \ldots, r_n) \in \mathbb{R}_+^{n+1}$ and $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$. Define $\|x\|_R = \sum_{i=0}^n r_i |x_i|$. It is easy to see that $\|\cdot\|_R$ is a norm on $\mathbb{R}^n$. For any two points $x, y \in \mathbb{R}^{n+1}$, we define $dist_R(x, y) = \|x - y\|_R$. One can consider $\|\cdot\|_R$ as a *weighted taxi-cab* distance. For any set $S \subseteq \mathbb{R}^{n+1}$ and $p \in \mathbb{R}^{n+1}$, we define $dist_R(p, S) = \inf\{dist_R(p, x) : x \in S\}$. Observe that $r(D) = -1$ if D is not equivalent to any effective divisor and $-1 \leq r(D) \leq \deg_R(D)$.

Lemma 5.2.15. *If $D \in \mathbb{Z}^{n+1}$ is a divisor then*

(i) $r(D) = -1$ *if and only if* $D \in \Sigma(\Lambda)$.

(ii) $r(D) = dist_R(D, \Sigma(\Lambda)) - 1 = \min\{dist_R(D, p) : p \in \Sigma(\Lambda)\} - 1.$

Proof. (i) For $D \in \mathbb{Z}^{n+1}$, $r(D) = -1$ if and only if for all $p \in \Lambda$, $D - p \ngeq \vec{\mathbf{0}}$ if and only if $D \in \Sigma(\Lambda)$.

(ii) Since $\Sigma(\Lambda)$ is a closed set, $\inf\{dist_R(D,p) : p \in \Sigma(\Lambda)\} = \min\{dist_R(D,p) : p \in \Sigma(\Lambda)\}$.

$$\begin{aligned}
r(D) &= \min\{\deg(E) : |D-E| = \emptyset, E \geq \vec{\mathbf{0}}\} - 1 \\
&= \min\{\deg(E) : r(D-E) = -1, E \geq \vec{\mathbf{0}}\} - 1 \\
&= \min\{\deg(E) : D-E \in \Sigma(\Lambda), E \geq \vec{\mathbf{0}}\} - 1 \\
&= \min\{\deg_R(D-p) : D-p \geq \vec{\mathbf{0}}, p \in \Sigma(\Lambda)\} - 1 \\
&= dist_R(D, \Sigma(\Lambda)) - 1.
\end{aligned}$$

Note that the last equality follows from the fact that if $p \in \Sigma(\Lambda)$ and $(D-p)_i < 0$ for some $0 \leq i \leq n$ then $dist_R(D, p-e_i) \leq dist_R(D,p)$ and $p - e_i \in \Sigma(\Lambda)$.

□

Lemma 5.2.16. *If $p = (p_0, \ldots, p_n) \in H_R^+$ and $p = \pi(p) + \lambda R$, then*

(i) $\Delta_R(p) = \pi(p) + \lambda\Delta_R$.

(ii) $F_i = \Delta_R(p) \cap \{x \in \mathbb{R}^n : x_i = p_i\}$ *for all* $0 \leq i \leq n$, *define all of the facets of the simplex* $\Delta_R(p)$.

It is easy to see that Δ_R is the simplex in H_R with vertices $b^0, \ldots, b^n \in H_R$, whose coordinates are:

$$b_j^i = \begin{cases} -\sum_{k \neq i} \frac{r_k^2}{r_i} & \text{if } i = j \\ r_i & \text{otherwise} \end{cases}$$

for all $0 \leq j \leq n$. The following remark can be considered as a generalization of Lemma 4.7 in [3], and its proof easily follows from Definition 5.2.2.

Remark 5.2.17. *Given any two vectors $p, q \in H_R$,*

$$d_{\Delta_R}(p,q) = \max_{0 \leq i \leq n}\{\frac{q_i - p_i}{r_i}\}.$$

Proof. By Definition 5.2.2,

$$d_{\Delta_R}(p,q) = \inf\{\lambda \geq 0 : q \in p + \lambda\Delta_R\} = \inf\{\lambda \geq 0 : q \in p + C^-(\lambda R)\}$$

$$= \inf\{\lambda \geq 0 : q \leq p + \lambda R\} = \max_{0 \leq i \leq n}\{\frac{q_i - p_i}{r_i}\}.$$

□

Lemma 5.2.18. *If $p, q \in H_R^+$, then $p \leq q$ if and only if $\Delta_R(p) \subseteq \Delta_R(q)$. In particular, $p < q$ if and only if $\Delta_R(p) \subsetneq int(\Delta_R(q))$.*

Proof. It is easy to see that $p \leq q$ if and only if $C^-(p) \subseteq C^-(q)$. Now the second part of Lemma 5.2.16 implies that $C^-(p) \subseteq C^-(q)$ if and only if $(C^-(p) \cap H_R) \subseteq (C^-(q) \cap H_R)$. □

Recall Definition 5.2.3. An easy application of Lemma 5.2.13 is that if $p \in Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$, then $p \notin \Lambda$. The following theorem characterizes the set of extreme points of $\overline{\Sigma}_{\mathbb{R}}(\Lambda)$.

Theorem 5.2.19. *If $p \in \overline{\Sigma}_{\mathbb{R}}(\Lambda) \setminus \Lambda$ then $p \in Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$ if and only if each facet of the simplex $\Delta_R(p)$ contains a point of Λ in its interior.*

Proof. Assume that $p = (p_0, \ldots, p_n) \in \overline{\Sigma}_{\mathbb{R}}(\Lambda) \setminus \Lambda$. Let F_i, $0 \leq i \leq n$ be the facets of $\Delta_R(p)$. Let $0 \leq i \leq n$ be such that $int(F_i)$ contains no point of Λ. By Lemma 5.2.16 (ii), there exists an $\epsilon > 0$ such that $\Delta_R(p + \epsilon e_i)$ does not contain any points of Λ in its interior. Hence Lemma 5.2.18 and Lemma 5.2.13 imply that $p + \epsilon e_i \in \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. Since $\deg_R(p) < \deg_R(p + \epsilon e_i)$, the point p is not an extreme point.

Conversely, assume that $p \in \overline{\Sigma}_{\mathbb{R}}(\Lambda) \setminus \Lambda$ is such that the interior of each facet F of $\Delta_R(p)$ contains a point of Λ. We claim that for any $v = (v_0, \ldots, v_n) \in \mathbb{R}^{n+1}$, either $\deg_R(p + \epsilon v) \leq \deg_R(p)$ for all $\epsilon \geq 0$, or there exists $\lambda > 0$ such that for all $0 < \epsilon \leq \lambda$, $p + \epsilon v \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. If $v \leq \vec{\mathbf{0}}$, then for all $\epsilon \geq 0$, $\deg_R(p + \epsilon v) \leq \deg_R(p)$. Now, without loss of generality assume that $v_0 > 0$ and $v_1 \leq 0$. Suppose $x \in int(F)$ where

$F = \Delta_R(D) \cap \{y \in \mathbb{R}^n : (y - D) \cdot e_0 = 0\}$. Since $x \in int(F)$, we can pick $\lambda > 0$ small enough such that for all $0 < \epsilon \leq \lambda$, $x \in int(\Delta_R(p + \epsilon v))$. Thus Lemma 5.2.18 and Lemma 5.2.13 imply that $x \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$ for all $0 < \epsilon \leq \lambda$. This completes the proof of the claim. It is easy to see that the proof of the theorem follows from the claim. $\square$

Corollary 5.2.20. *The set $Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$ is a subset of $\mathbb{Z}^{n+1}$.*

Proof. Let $p \in Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$. Theorem 5.2.19 shows that the interior of every facet F of $\Delta_R(p)$ contains a point of Λ. Since $\Lambda \subseteq \mathbb{Z}^{n+1}$, the second part of Lemma 5.2.16 implies that $p \in \mathbb{Z}^{n+1}$. $\square$

Theorem 5.2.21. *A divisor $\nu \in Ext(\Sigma(\Lambda))$ if and only if $\nu + \vec{\mathbf{1}} \in Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$.*

Proof. Corollary 5.2.20 implies that $Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda)) \subseteq \mathbb{Z}^{n+1}$. The theorem immediately follows from Lemma 5.2.14. $\square$

The set of critical points of Λ ($Crit(\Lambda)$ in Definition 5.2.3) is the set of local maxima of the function $d_{\Delta_R}(\cdot, \Lambda)$. The following theorem characterizes critical points of Λ in terms of extreme points of $\overline{\Sigma}_{\mathbb{R}}(\Lambda)$.

Theorem 5.2.22. *For $p \in H_R$, let $\lambda = d_{\Delta_R}(p, \Lambda)$ and $p' = p + \lambda R$. Then $p' \in Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$ if and only if $p \in Crit(\Lambda)$.*

Proof. If $p' \in Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$ then by Theorem 5.2.19 each facet of the simplex $\Delta_R(p + \lambda R) = p + \lambda \Delta_R$ contains a point of Λ in its interior. This shows that $p \in Crit(\Lambda)$.

Conversely, assume that $p \in Crit(L)$and $p' \notin Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$. As the proof of Theorem 5.2.19 shows, there exist $0 \leq i \leq n$ and $\delta > 0$ such that for all $0 < \epsilon \leq \delta$, $p'_\epsilon = p' + \epsilon e_i \in \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. For each $0 < \epsilon \leq \delta$, let $p_\epsilon = \pi(p'_\epsilon)$ to be the projection of p'_ϵ along R into H_R. Lemma 5.2.24 implies that $d_{\Delta_R}(p_\epsilon, \Lambda) = \left(\frac{p'_\epsilon \cdot R}{\|R\|^2}\right)$. Since $p'_\epsilon \cdot R > p' \cdot R$, we conclude that $d_{\Delta_R}(p_\epsilon, \Lambda) > d_{\Delta_R}(p, \Lambda)$, a contradiction. $\square$

Corollary 5.2.23. *Let $\varphi : Ext(\Sigma(\Lambda)) \to Crit(\Lambda)$ be as follows: For any $\nu \in Ext(\Sigma(\Lambda))$, $\varphi(\nu) = \pi(\nu + \vec{\mathbf{1}})$. Then φ is a bijection.*

Proof. This follows from Theorems 5.2.22 and 5.2.21. □

Lemma 5.2.24. *Let* $p \in H_R$, $\lambda = d_{\Delta_R}(p, \Lambda)$, *and* $\lambda' = \max\{t \geq 0 : p + tR \in \overline{\Sigma}_{\mathbb{R}}(\Lambda)\}$. *Then* $\lambda = \lambda'$.

Proof. First note that since $p \in \overline{\Sigma}_{\mathbb{R}}(\Lambda)$ and $\overline{\Sigma}_{\mathbb{R}}(\Lambda)$ is a closed set, $\max\{t \geq 0 : p+tR \in \overline{\Sigma}_{\mathbb{R}}(\Lambda)\}$ is well-defined. The first part of Lemma 5.2.16 implies that $p + t\Delta_R = \Delta_R(p + tR)$. Now, for all $0 \leq t \leq \lambda$, by applying Lemma 5.2.13 and Lemma 5.2.18, we conclude that $p + tR \in \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. So $\lambda' \geq \lambda$. Conversely, suppose $t \geq 0$ is such that $\Lambda \cap (p + t\Delta_R) \neq \emptyset$. Lemma 5.2.13 and Lemma 5.2.18 imply that $p + tR \in \overline{\Sigma}_{\mathbb{R}}(\Lambda)$ if and only if $\Lambda \cap int(p + t\Delta_R) = \emptyset$. This shows that $\lambda' \leq \lambda$, completing the proof of the lemma. □

Lemma 5.2.25. *There exists a constant* C *depending only on the lattice* Λ *and the vector* R *such that for any point* $p \in \Sigma(\Lambda)$, *we have:*

(i) $\deg_R(p) \leq C$,

(ii) there exists some $\nu \in Ext(\Lambda)$ *such that* $p \leq \nu$.

Proof. (i): First, we claim that there exists c such that for all $p \in H_R, d_{\Delta_R}(p, \Lambda) \leq c$. We start by noting that there exists a constant K depending only on R such that $d_{\Delta_R}(p, q) \leq K \cdot \|p - q\|$. This follows immediately by taking the constant K to be the largest radius of a sphere in H_R with center at the origin contained in Δ_R.

Let $\{l_0, ..., l_{n-1}\}$ be a set of generators of Λ, and let P be the parallelotope generated by $l_0, ...l_{n-1}$. Because the Δ_R-distance function is invariant under translation by lattice points, it is sufficient to prove the claim for all $p \in P$. By letting c be K times the maximum ℓ^2-distance from a point in P to the vertices of P (diameter of P by ℓ^2-norm), the claim is proved.

To prove the first part, it is enough to show that for all $p \in H_R^+ \cap \Sigma(\Lambda)$, $\deg_R(p) \leq C$. Let $p' = \pi(p)$, $\lambda \geq 0$ be such that $p = p' + \lambda R$. Lemma 5.2.18 implies that $p \in \Sigma(\Lambda)$

if and only if $\Delta_R(p)$ contains no points of Λ. Lemma 5.2.24 and Theorem 5.2.21 imply that $\lambda \leq dist_{\Delta_R}(p, \Lambda)$, so $\lambda \leq c$. Therefore, $\deg_R(p) = \lambda\|R\|^2 \leq c\|R\|^2$. This shows that $C \leq c\|R\|^2$, which completes the proof of the first part.

(ii): Let $p \in \Sigma(\Lambda)$. The first part shows that the degrees of points in $Ext(\Lambda)$ are bounded above by C. Therefore $C^+(p) \cap \Sigma(\Lambda)$ is a finite set. This immediately shows that there exists $\nu \in Ext(\Lambda)$ such that $p \leq \nu$. To be more precise, one can find an extreme point $\nu \in Ext(\Lambda)$ greedily by starting at point p and walking in positive directions as much as possible while remaining in Σ. $\square$

Lemma 5.2.26. *For any divisor $D \in \mathbb{Z}^{n+1}$, $r(D) = \min\{\deg_R^+(D-\nu) : \nu \in Ext(\Lambda)\} - 1$.*

Proof. First we show that $\min\{\deg_R^+(D-\nu) : \nu \in Ext(\Lambda)\} \leq r(D)+1$. Let $E \geq \vec{\mathbf{0}}$ with $\deg_R(E) = r(D)+1$ be such that $D - E \in \Sigma(\Lambda)$, where the existence of E guaranteed by Lemma 5.2.15. By Lemma 5.2.25, there exists $\nu \in \Sigma(\Lambda)$ such that $\nu \geq D - E$. Let $E' = \nu - (D-E)$. We claim that $E' \cdot E = 0$. Suppose not and assume there exists $0 \leq i \leq n$ such that $E_i, E'_i \geq 1$. Note that $D - (E - e_i) \in \Sigma(\Lambda)$ as $\nu \geq D - (E - e_i)$, but $\deg_R(E - e_i) < \deg_R(E) = r(D)+1$, a contradiction. This gives that $\deg_R^+(D-\nu) = \deg_R^+(E-E') = \deg(E) = r(D)+1$.

For proving the reverse inequality, let $\nu \in Ext(\Lambda)$ be such that $\deg^+(D-\nu)$ is minimum. Because $\nu \geq \nu + (D-\nu)^- = D - (D-\nu)^+$, it follows that $D - (D-\nu)^+ \in \Sigma(\Lambda)$. Hence Lemma 5.2.15 implies that $r(D) \leq \min\{\deg_R^+(D-\nu) : \nu \in Ext(\Lambda)\} - 1$, which completes the proof. $\square$

Lemma 5.2.27. *Suppose $\phi : \mathcal{A} \to \mathcal{A}'$ is a bijection between sets, and $f : \mathcal{A} \to \mathbb{Z}$ and $f' : \mathcal{A}' \to \mathbb{Z}$ are functions whose values are bounded from below. If there exist constants $c_1, c_2 \in \mathbb{Z}$ such that for all $a \in \mathcal{A}$,*

$$c_1 \leq f(a) - f'(\phi(a)) \leq c_2,$$

then

$$c_1 \leq \min_{a\in\mathcal{A}} f(a) - \min_{a'\in\mathcal{A}'} f'(a') \leq c_2.$$

Proof. Since f and f' are integer valued functions whose values are bounded from below, there exists $x \in \mathcal{A}$ and $y \in \mathcal{A}'$ such that $f(x) = \min_{a\in\mathcal{A}} f(a)$ and $f'(y) = \min_{a'\in\mathcal{A}'} f'(a')$. The choice of x and y implies that $f(x) - f'(y) \leq f(\phi^{-1}(y)) - f'(y) \leq c_2$, and $f(x) - f'(y) \geq f(x) - f'(\phi(x)) \geq c_1$. Hence $c_1 \leq f(x) - f'(y) \leq c_2$, as desired. $\square$

Recall Definitions 5.2.4, 5.2.5 and 5.2.6. Here we are going to present the proof of Theorem 5.2.7.

Proof of Theorem 5.2.7. First we construct the canonical divisor K and then we show it has the desired property. Since Λ is reflection invariant, there exists a vector $v \in \mathbb{R}^{n+1}$ such that $-Crit(\Lambda) = Crit(\Lambda) + v$. Therefore there exists a bijection function η from $Crit(\Lambda)$ to itself such that $\eta(c)+c = v$. Let $\varphi : Ext(\Sigma(\Lambda)) \to Crit(\Lambda)$ be the bijection described in Corollary 5.2.23. Define the bijection ϕ from $Ext(\Sigma(\Lambda))$ to itself so that for all $\nu \in Ext(\Sigma(\Lambda))$, $\phi(\nu) = \varphi^{-1}\eta\varphi(\nu)$. Since for all $\nu \in Ext(\Sigma(\Lambda))$, $\deg_R(\nu + \phi(\nu)) \leq 2g_{\max}$, there exists $\nu_0 \in Ext(\Sigma(\Lambda))$ such that $\deg_R(\nu_0 + \phi(\nu_0))$ is as large as possible. Let the canonical divisor K be $\nu_0 + \phi(\nu_0)$.

For any $\nu \in Ext(\Sigma(\Lambda))$, let $c = \varphi(\nu)$; then we have:

$$\phi(\nu) + \nu = \phi(\varphi^{-1}(c)) + \varphi^{-1}(c) = \varphi^{-1}\eta(c) + \varphi^{-1}(c) = \lambda R + v - 2 \times \vec{\mathbf{1}},$$

where $\lambda \in \mathbb{R}$ is a constant depends on ν (or equivalently c). Hence, the choice of K implies that for any $\nu \in Ext(\Sigma(\Lambda))$, there exists $E_\nu \in \mathbb{R}_+^{n+1}$ such that $\phi(\nu)+\nu+E_\nu =$

K. Therefore, for all divisor $D \in \mathbb{Z}^{n+1}$ and $\nu \in Ext(\Sigma(\Lambda))$ we have:

$$\begin{aligned}
\deg_R^+(D-\nu) - \deg_R^+(K-D-\phi(\nu)) &= \deg_R^+(D-\nu) - \deg_R^+(\phi(\nu)+\nu+E_\nu-D-\phi(\nu)) \\
&= \deg_R^+(D-\nu) - \deg_R^+(\nu+E_\nu-D) \\
&\leq \deg_R^+(D-\nu) - \deg_R^+(\nu-D) \\
&= \deg_R(D) - \deg_R(\nu) \\
&\leq \deg_R(D) - g_{\min} + 1.
\end{aligned}$$

Note that for all $\nu \in Ext(\Sigma(\Lambda))$, $E_\nu = K - (\nu + \phi(\nu)) \leq 2g_{\max} - 2g_{\min}$. Hence,

$$\begin{aligned}
\deg_R^+(D-\nu) - \deg_R^+(K-D-\phi(\nu)) &= \deg_R^+(D-\nu) - \deg_R^+(\phi(\nu)+\nu+E_\nu-D-\phi(\nu)) \\
&= \deg_R^+(D-\nu) - \deg_R^+(\nu+E_\nu-D) \\
&\geq \deg_R^+(D-\nu) - \deg_R^+(\nu-D) - 2(g_{\max}-g_{\min}) \\
&= \deg_R(D) - \deg_R(\nu) - 2g_{\max} + 2g_{\min} \\
&\geq \deg_R(D) - 3g_{\max} + 2g_{\min} + 1.
\end{aligned}$$

Therefore for all $D \in \mathbb{Z}^{n+1}$ and all $\nu \in Ext(\Sigma(\Lambda))$,

$$\deg_R(D) - 3g_{\max} + 2g_{\min} + 1 \leq \deg_R^+(D-\nu) - \deg_R^+(K-D-\varphi(\nu)) \leq \deg_R(D) - g_{\min} + 1.$$

For a fixed $D \in \mathbb{Z}^{n+1}$, $\deg_R(D) - 3g_{\max} + 2g_{\min} + 1$ and $\deg_R(D) - g_{\min} + 1$ are constant integers, $\deg_R^+(D-\nu)$ and $\deg_R^+(K-D-\varphi(\nu))$ are integer valued functions bounded from below by zero, and φ is a bijection from $Ext(\Sigma(\Lambda))$ to itself, hence Lemma 5.2.27 implies that

$$\begin{aligned}
&\deg_R(D) - 3g_{\max} + 2g_{\min} + 1 \\
&\leq \min_{\nu \in Ext(\Sigma(\Lambda))} \deg_R^+(D-\nu) - \min_{\nu \in Ext(\Sigma(\Lambda))} \deg_R^+(K-D-\nu) \\
&\leq \deg_R(D) - g_{\min} + 1.
\end{aligned}$$

The assertion of the theorem now follows from Lemma 5.2.26. $\square$

Recall Definitions 5.2.8 and 5.2.10, the following are the proof of Theorems 5.2.9 and 5.2.11, respectively.

Proof of Theorem 5.2.9. Assume Λ is reflection invariant and let K be the canonical divisor obtained in the proof of Theorem 5.2.7. By applying Theorem 5.2.7, it is enough to show that $\deg(K) = 2g - 2$. The construction of K shows that $K = \nu + \phi(\nu)$, where ϕ is the bijection obtained in proof of Theorem 5.2.7. Since Λ is uniform, $g_{\min} = g_{\max} = g$. Hence $\deg_R(\nu) = \deg_R(\phi(\nu)) = g - 1$ and this implies that $\deg_R(K) = 2g - 2$.

Now, assume that Λ has the Riemann property. Assume ν is an extreme divisor of $\Sigma(\Lambda)$, so the first part of Lemma 5.2.15 implies that $r(\nu) = -1$. Since Λ is uniform $\deg_R(\nu) = g - 1$ and this shows that $r(K - \nu) = r(\nu) = -1$. By Lemma 5.2.15, $K - \nu \in \Sigma(\Lambda)$, and is hence an extreme divisor of $\Sigma(\Lambda)$. Hence the function ψ defined as $\psi(-\nu) = K - \nu$, for all $\nu \in Ext(\Lambda)$ is a bijection from $Ext(\Lambda)$ to itself. If φ is the function defined in Corollary 5.2.23, the function $\varphi o \psi o \varphi^{-1}$ is a bijection from $Crit(\Lambda)$ to itself. It is easy to see that for any $p \in Crit(\Lambda)$, $\varphi(\psi(\varphi^{-1}(p))) = -p + \pi(K) + 2\pi(\vec{\mathbf{1}})$, and by picking $v = -\pi(K) - 2\pi(\vec{\mathbf{1}})$, we have $-Crit(\Lambda) = Crit(\Lambda) + v$. □

Proof of Theorem 5.2.11. If Λ is uniform and reflection invariant, then Theorem 5.2.9 implies that Λ has Riemann-Roch property and therefore Λ has the Riemann-Roch formula with $m = g_{\max}$.

For proving the other direction it is enough by Theorem 5.2.9 to show that Λ is uniform and $m = g_{\max}$. First, we show that $m = g_{\max}$. Let D be a divisor with $\deg_R(D) \geq m$. The Riemann-Roch formula implies that $r(D) - r(K - D) \geq 1$ and since $r(K - D) \geq -1$, we have $r(D) \geq 0$. It follows that $g_{\max} \leq m$.

We know that for any divisor $D \in \mathbb{Z}^{n+1}$, if the degree of D is more that $g_{\max} - 1$ then the divisor is effective, so $\deg_R(D) - r(D) \leq g_{\max}$. On the other hand, if

$\deg_R(D) > 2m-2$, then $\deg_R(K-D) < 0$, therefore $r(K-D) = -1$. The Riemann-Roch formula implies that $\deg(D) - r(D) = m$. Therefore, $m \leq g_{\max}$. This shows that $m = g_{\max}$.

To prove uniformity, let $\nu \in Ext(\Sigma(\Lambda))$ and $\deg_R(\nu) < g_{\max} - 1$. Since $\deg_R(K) = 2g_{\max} - 2$, $\deg_R(K-\nu) \geq g_{\max}$, so $K - \nu \notin \Sigma(\Lambda)$, and by Lemma 5.2.15 is equivalent to an effective divisor. The Riemann-Roch formula implies that $r(K-\nu) = g_{\max} - \deg(\nu) - 2$, so there exists an effective divisor E of degree $g_{\max} - \deg(\nu) - 1 > 0$ such that $|K - \nu - E| = \emptyset$. We claim that $\nu + E$ is not equivalent to an effective divisor. The Riemann-Roch formula implies that $r(\nu+E) - r(K-\nu-E) = \deg_R(\nu+E) - g_{\max} + 1 = 0$ and therefore $r(\nu+E) = -1$. By Lemma 5.2.15, $\nu + E \in \Sigma(\Lambda)$, contradicting the fact that $\nu \in Ext(\Sigma(\Lambda))$. $\square$

The following lemmas and corollaries preparing the ground for proving Theorem 5.2.12

Lemma 5.2.28. *Let Λ be a sub-lattice of dimension n of Λ_R. Then $\mathcal{R}\Sigma(\Lambda) = \Sigma(\mathcal{R}\Lambda)$.*

The proof of above lemma follows easily from Definition 5.2.1 and the fact that $\mathcal{R}$ is an invertible matrix with positive diagonal entries.

Lemma 5.2.29. *Let Λ be a sub-lattice of dimension n of Λ_R. Then $\mathcal{R}Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda)) = Ext(\overline{\Sigma}_{\mathbb{R}}(\mathcal{R}\Lambda))$.*

Proof. Let $\nu \in Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$ so that there exists some $\delta > 0$ such that for all $p \in B(\nu, \delta) \cap \overline{\Sigma}_{\mathbb{R}}(\Lambda)$, $\deg_R(\nu) \geq \deg_R(p)$. Let $\delta' = \delta$. It is easy to see that if $q \in B(\mathcal{R}\nu, \delta')$, we have $\mathcal{R}^{-1}q \in B(\nu, \delta)$. Hence $\deg_R(\mathcal{R}^{-1}q) \leq \deg_R(\nu)$ and therefore $\deg_{\vec{\mathbf{1}}}(q) \leq \deg_{\vec{\mathbf{1}}}(\mathcal{R}\nu)$. Here we have used the fact that for any $D \in \mathbb{Z}^{n+1}$, $\deg_R(D) = \deg_{\vec{\mathbf{1}}}(\mathcal{R}D)$ and Lemma 5.2.28. This proves that $\mathcal{R}Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda)) \subseteq Ext(\overline{\Sigma}_{\mathbb{R}}(\mathcal{R}\Lambda))$. The other direction is proved similarly. $\square$

The following corollary immediately follows from Lemma 5.2.29 and Theorem 5.2.21.

Corollary 5.2.30. *Let Λ be a sub-lattice of dimension n of Λ_R. Then Λ is uniform if and only if $\mathcal{R}\Lambda \subseteq \Lambda_{\vec{\mathbf{1}}}$ is uniform.*

Lemma 5.2.31. *Let Λ be a uniform sub-lattice of dimension n of Λ_R. Then Λ is reflection invariant if and only if $\mathcal{R}\Lambda \subseteq \Lambda_{\vec{\mathbf{1}}}$ is reflection invariant.*

Proof. First suppose Λ is reflection invariant. Then there exists a vector $v \in \mathbb{R}^{n+1}$ such that $-Crit(\Lambda) = Crit(\Lambda) + v$. By applying Lemma 5.2.29 and Theorem 5.2.22, let $\mathcal{R}\nu - \vec{\mathbf{1}} - \deg_{\vec{\mathbf{1}}}(R\nu - \vec{\mathbf{1}})\vec{\mathbf{1}}$ be an arbitrary point of $Crit(\mathcal{R}\Lambda)$ where ν is an arbitrary point of $Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$. Now, by applying Theorem 5.2.22,

$$\nu - \vec{\mathbf{1}} - \deg_R(\nu - \vec{\mathbf{1}})R \in Crit(\Lambda).$$

Since Λ is reflection invariant, there exists $\nu' \in Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$ such that

$$-\nu + \vec{\mathbf{1}} + \deg_R(\nu - \vec{\mathbf{1}})R = \nu' - \vec{\mathbf{1}} - \deg_R(\nu' - \vec{\mathbf{1}})R + v,$$

therefore

$$-\mathcal{R}\nu + \mathcal{R}\vec{\mathbf{1}} + \deg_R(\nu - \vec{\mathbf{1}})\mathcal{R}R = \mathcal{R}\nu' - \mathcal{R}\vec{\mathbf{1}} - \deg_R(\nu' - \vec{\mathbf{1}})\mathcal{R}R + \mathcal{R}v.$$

Since Λ is uniform $\deg_R(\nu - \vec{\mathbf{1}})$ is a constant independent from the choice of $\nu \in Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$. Hence, $\mathcal{R}\nu - \mathcal{R}\nu' = u$ where u is constant vector in $\mathbb{R}^{n+1}$ which does not depend on ν or ν'. Since $\mathcal{R}\Lambda$ is uniform, $\deg_{\vec{\mathbf{1}}}(\mathcal{R}\nu - \vec{\mathbf{1}})$ is a constant independent from the choice of $\nu \in Ext(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$. This shows that

$$\mathcal{R}\nu - \mathcal{R}\nu' = u + 2\deg_{\vec{\mathbf{1}}}(\mathcal{R}\nu - \vec{\mathbf{1}}) + 2 \times \vec{\mathbf{1}}.$$

Hence $\mathcal{R}\Lambda$ is reflection invariant. The other direction is proved similarly. $\square$

Recall the definition of the canonical vector (Definition 5.2.6) and the argument in the proof of Lemma 5.2.7 in constructing a canonical vector for a reflection invariant sublattice of Λ_R. So we can consider the following corollary as a consequence of Theorem 5.2.21, Lemma 5.2.29, and Lemma 5.2.31.

Corollary 5.2.32. *Let Λ be a reflection invariant sub-lattice of dimension n of Λ_R. If K is a canonical vector of $\mathcal{R}\Lambda$ then $\mathcal{R}^{-1}(K + 2 \times \vec{\mathbf{1}}) - 2 \times \vec{\mathbf{1}}$ is a canonical vector of Λ.*

5.2.3 Wilmes' Lattice Reduction Algorithm

John Wilmes' Senior Thesis from Reed College includes the following result: Given a full dimensional sub-lattice of $\mathbb{Z}^n$, we can find a basis for this lattice coming from the rows of the reduced Laplacian of a directed graph. We first show how to use this result to prove that any full dimensional sub-lattice L of the root lattice has a basis coming from the rows of a strongly connected directed graph. It follows from this observation that the work of this chapter provides a combinatorial framework for Amini and Manjunath [3].

Take B to be a basis for L. Because this lattice is codimension one, we can choose a vector v in B such that the remaining vectors span the root lattice over $\mathbb{Q}$. Let v' be some integral vector lying in the root lattice which is positive in the first entry and negative in all other entries. The vector v' is in the $\mathbb{Q}$-span of $B \setminus v$ therefore there exists some positive integer k such that kv' is in the span of $B \setminus v$ over $\mathbb{Z}$. Taking k to be large enough, we can be sure that $v + kv'$ has the same sign pattern as kv'. Now we can take the vectors in $B \setminus v$ and apply Wilmes' reduction algorithm. We claim that the basis obtained along with $v + kv'$ is coming from the rows of the Laplacian of a strongly connected digraph. We will first describe Wilmes' algorithm informally as it appears in [64, 75], the claim will follow by construction.

Delete the first entry of each vector in $B \setminus v$ and arrange the resulting vectors in a square matrix M. We describe a set of elementary row operations on M which turns M into a reduced directed Laplacian matrix. We note that the operations being performed are also being performed on the first entry of the basis, but because the row sums are one, we can easily recover these values at the end of the algorithm. The defining qualities of such a matrix are that $(i)M_{i,i} \geq 0, (ii)M_{i,j} \leq 0$ for $i \neq j$ and (iii) the sum of the entries in each row, i.e. the degree of each row is nonnegative.

First observe that not all of the degrees of the rows are zero, otherwise they would linearly dependent. By performing the Euclidean algorithm on the degrees of the rows, which only involves adding integer multiples of rows to each other, we can take the degrees of all but the first row to be zero. Moreover, we can take the degree of the first row to be positive by negating this row if necessary.

Next we restrict attention to the remaining $n-1$ rows and apply the Euclidean algorithm to the entries in the second column, and by possibly permuting these rows, the entries below the second entry are zero. Now we restrict attention to the bottom $n-2$ rows and again apply Euclidean algorithm to the entries in the third column. Continuing this way, we may make M so that that all of the entries below the supra diagonal entries, i.e. those entries directly below the diagonal, are zero. Moreover, by negating rows when necessary, we can assume that these supra diagonal entries are negative. The matrix M now satisfies (iii) and this condition will be maintained for the remainder of the algorithm.

The last row now satisfies the conditions (i) and (ii). We now perform a bootstrapping procedure: assuming that the last k rows satisfy (i) and (ii), we can make the $n-k$th row also satisfy (i) and (ii) by adding the appropriate multiples of these bottom k rows. A corollary of this construction is that the directed graph we obtain whose Laplacian is obtain from M and $v+kv'$ has a special form. It is a path from v_n to v_0 (the supra diagonal entries are nonzero) with edges added from v_0 (v+kv' has no zero entries) to all other vertices and potentially additional edges (v_i, v_j) where $j < i$ (no nonzero entries below the supra diagonal).

5.3 Chip-Firing Games on Directed Graphs

5.3.1 Row Chip-Firing Game, The Sandpile Model, and Riemann-Roch Theory

Let $\vec{G}$ be a directed graph with vertex set $\{v_0, ..., v_n\}$ and adjacency matrix $\vec{A}$ whose entry $\vec{A}_{i,j}$ for $0 \leq i, j \leq n$ is the number of edges directed from v_i to v_j. Let $\vec{\mathcal{D}} = diag(\deg^+(v_0), \ldots, \deg^+(v_n))$ where $\deg^+(v)$ denotes the number edges leaving vertex $v \in V(\vec{G})$. We call the matrix $\vec{Q} = \vec{\mathcal{D}} - \vec{A}$ the *Laplacian matrix* of the directed graph $\vec{G}$. We define $\Lambda_{\vec{G}}$ to be the lattice spanned by the rows of $\vec{Q}$.

In this section we study the following row chip-firing game on vertices of $\vec{G}$. Begin with $D \in \mathbb{Z}^{n+1}$, which we call a configuration or a divisor, whose ith entry $D(v_i)$ is the number of chips at vertex v_i. In each *move* of the game a vertex either *fires* or *borrows*. We say a vertex *fires* if it sends a chip along each of its outgoing edges to its neighbors and *borrows* if it receives a chip along each of its incoming edges from its neighbors. We say that a vertex is in *debt* if the number of chips at that vertex is negative. The objective of the game is to bring every vertex out of debt by some sequence of moves. Note that the game is "commutative" in the sense that the order of firings and borrowings does not effect the final configuration. For $f \in \mathbb{Z}^{n+1}$, we may interpret the divisor $D' = D - \vec{Q}^T f$ as the divisor obtained from D by a sequence of moves in which the vertex v_i fires $f(v_i)$ times if $f(v_i) \geq 0$ and it borrows $f(v_i)$ times if $f(v_i) \leq 0$. We refer to f as a *firing strategy*. Note that both firing strategies and divisors are vectors in $\mathbb{Z}^{n+1}$. We say a configuration is a *winning configuration* if all of the vertices are out of debt. We call a sequence of moves which achieves a winning configuration a *winning strategy*. The question of whether a winning strategy exists is equivalent to the question of whether there exists a firing strategy $f \in \mathbb{Z}^{n+1}$ and an effective divisor $E \in \mathbb{Z}^{n+1}_{\geq 0}$ such that $E = D - \vec{Q}^T f$, i.e., $D - E \in \Lambda_{\vec{G}}$, $|D| \neq \emptyset$ or $r(D) \geq 0$. In what follows we will restrict our attention to strongly connected directed graphs. The main motivation for this consideration is given in the following lemma which, interpreted combinatorially, characterizes strongly connected digraphs in terms of which firings leave a divisor unaffected.

Lemma 5.3.1. *A directed graph $\vec{G}$ is strongly connected if and only if there exists a vector $R \in \mathbb{N}^{n+1}$, unique up to multiplication by a real constant, such that $\vec{Q}^T R = 0$.*

Proof. Let $\vec{G}$ be strongly connected. For the sake of contradiction suppose there exists $R \ngeq 0$ such that $\vec{Q}^T R = 0$. Let V^+ be the set of vertices of $\vec{G}$ such that $R(v) > 0$ for all $v \in V^+$. Let $D = \vec{Q}^T R$. Since the net amount of chips leaving V^+ is positive, there must exist some $v \in V^+$ such that $D(v) < 0$, a contradiction. Now assume there exist two linearly independent firing strategies R_1 and R_2. It is easy to see that there exists a linear combination of R_1 and R_2, say R, such that $R \ngeq 0$. This proves the uniqueness. Note that we can take R to be an integral vector.

Conversely, suppose $\vec{G}$ is not strongly connected. Let $V_1, \ldots, V_t$ be the decomposition of vertices of $\vec{G}$ into maximal strongly connected components. Without loss of generality, let V_1 be a set of vertices such that there exists no edges from u to v where $u \in V_i$, $2 \leq i \leq t$ and $v \in V_1$. As above there exists $v \in V_1$ such that $\vec{Q}^T R(v) < 0$, a contradiction. □

5.3.1.1 Reduced Divisors

Let $f, f' \in \mathbb{Z}^{n+1}$ be firing strategies. We define an equivalence relation $\sim$ on $\mathbb{Z}^{n+1}$ by declaring $f \sim f'$ if $\vec{Q}^T(f - f') = \vec{\mathbf{0}}$. For any set $S \subseteq V(\vec{G})$, the *characteristic vector of* S, denoted by χ_S, is the vector $\sum_{v_i \in S} e_i$. We say a vector $f \in \mathbb{Z}^{n+1}$ is a *natural* firing strategy if $f \leq R$, and $f \nleq \vec{\mathbf{0}}$. We say a nonzero vector $f \in \mathbb{Z}^{n+1}$ is a *valid* firing strategy with respect to v_0 if $f(v_0) = 0$, and $\vec{\mathbf{0}} \leq f \leq R$. The following lemma is an immediate consequence of Lemma 5.3.1.

Lemma 5.3.2. *Let $f \in \mathbb{Z}^{n+1}$ be a nonzero firing strategy then there exists a unique $f' \in \mathbb{Z}^{n+1}$ such that $f \sim f'$ and f' is a natural firing strategy.*

Definition 5.3.3. *Let $\vec{G}$ be a directed graph. We call a divisor D v_0-reduced if the following two conditions hold:*

(i) for all $v \in V(\vec{G}) \setminus \{v_0\}, D(v) \geq 0$,

(ii) for every valid firing f with respect to v_0, there exists a vertex $v \in V(\vec{G}) \setminus \{v_0\}$ such that $(D - \vec{Q}^T f)(v) < 0$.

The proceeding remark immediately follows from Definition 5.3.3.

Remark 5.3.4. *If $D' \sim D$ is a v_0-reduced divisor then for all $k \in \mathbb{Z}$, $D' + k\chi_{\{v_0\}}$ is a v_0-reduced divisor and $D' + k\chi_{\{v_0\}} \sim D + k\chi_{\{v_0\}}$.*

Lemma 5.3.5. *Let D be a v_0-reduced divisor and let f be a firing strategy such that $f(v_0) \leq 0$ and $f(v) > 0$ for some vertex $v \in V(\vec{G}) \setminus \{v_0\}$. Then there exists $v \in V(\vec{G}) \setminus \{v_0\}$ such that $(D - \vec{Q}^T f)(v) < 0$.*

Proof. Lemma 5.3.2 implies that there exists a natural firing strategy $f' \sim f$ with $f'(v_0) \leq f(v_0) = 0$. Suppose f^+ and f^- are the positive and negative part of f'. It is easy to see that f^+ is a valid firing strategy with respect to v_0. Hence there exists a vertex $v \in V(\vec{G}) \setminus \{v_0\}$ such that $(D - \vec{Q}^T f^+)(v) < 0$. Therefore,

$$(D - \vec{Q}^T f)(v) = (D - \vec{Q}^T f')(v) = (D - \vec{Q}^T f^+ - \vec{Q}^T f^-)(v) \leq (D - \vec{Q}^T f^+)(v) < 0.$$

□

Lemma 5.3.6. *Let $\vec{G}$ be a directed graph and let D be a divisor. Then there exists a divisor $D' \sim D$ such that D' is v_0-reduced.*

Proof. The proof that we present here is similar to the proof given by Baker and Norine [9](§3.1). The process of obtaining a v_0-reduced divisor $D' \sim D$ has two steps: first we bring every $v \in V(\vec{G}) \setminus \{v_0\}$ out of debt, so that it satisfies the first condition of Definition 5.3.3, and then we "reduce" the divisor with respect to v_0, in order to satisfy the second condition of Definition 5.3.3. For performing the first step, define $d(v)$, for all $v \in V(\vec{G}) \setminus \{v_0\}$, to be the length of the shortest directed path from v_0 to v. Let $d = \max_{v \in V(\vec{G}) \setminus \{v_0\}} d(v)$. For all $1 \leq i \leq d$, define $A_i = \{v \in V(\vec{G}) : d(v) = i\}$. Now we bring the A_i's out of debt consecutively, starting at A_d. We recursively define sequences of integers b_i and divisors D_i as follows. Let $b_d = \max\left(\{-D(v) : v \in A_d, D(v) \leq 0\} \cup \{0\}\right)$. Define $D_d = D - \vec{Q}^T f_d$ where f_d is the all zero vector except $f_d(v_j) = b_d$ if $v_j \notin A_d$. It is easy to see that $D_d(v_j) \geq 0$ for all $v_j \in A_d$. Now suppose $1 \leq i \leq d-1$, and define $b_i = \max\left(\{-D(v) : v \in A_i, D_{i+1}(v) \leq 0\} \cup \{0\}\right)$. Define $D_i = D_{i+1} - \vec{Q}^T f_i$ where f_i is the all zero vector except $f_i(v_j) = b_i$ if $v_j \notin \bigcup_{k=i}^{d} A_k$. It is easy to see that $D_i(v_j) \geq 0$ for all $v_j \in A_i$ and $D_i(v_j) = D_{i+1}(v_j)$ for all $v_j \in \bigcup_{k=i+1}^{d} A_k$. Since d is a finite number and the b_i's are bounded, the above procedure terminates. It is easy to verify that $D_1 \sim D$ is a divisor such that no vertex other than v_0 is in debt. This completes the description of the first step.

Now, we are going to explain the second step. Let $D' = D_1$ be the divisor obtained from the first step. While there exists a valid firing strategy f with respect to v_0 such that $(D' - \vec{Q}^T f)(v) \geq 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$, replace D' by $D' - \vec{Q}^T f$. If we show that the procedure terminates, it is obvious that D' is a v_0-reduced divisor. Since $f(v_0) = 0$ for any valid firing strategy with respect to v_0, the vertex v_0 must stop receiving money at some point. At this point, none of its neighbors fires, so they must eventually stop receiving money. By iterating this argument we see that, since v_0 is reachable from every vertex, each vertex must stop receiving money at some point. Hence, the above procedure terminates at a v_0-reduced divisor. $\square$

Corollary 5.3.7. *Let D be a divisor satisfying the property (i) in Definition 5.3.3. Then there exists a sequence of valid firings $f_1, \ldots, f_k$ with respect to v_0 such that $D' = D - \vec{Q}^T(\sum_{i=1}^{k} f_i)$ is v_0-reduced.*

Lemma 5.3.8. *For any divisor D, there exist exactly $R(v_0) = r_0$ distinct v_0-reduced divisors equivalent to D.*

Proof. First, we show that there exist at most r_0 distinct reduced divisors equivalent to D. Suppose not, so by the pigeonhole principle, there exist two distinct reduced divisors, $D' = D - \vec{Q}^T f'$ and $D'' = D - \vec{Q}^T f''$ with $f'(v_0) \equiv f''(v_0) \pmod{r_0}$. Pick $k \in \mathbb{Z}$ so that $(f' - f'' - kR)(v_0) = 0$ and let $f^* = f' - f'' - kR$. By our assumption $D' \neq D''$ and so $\vec{Q}^T(f' - f'') \neq 0$. Hence by Lemma 5.3.1, either f^* or $-f^*$ satisfies the assumptions of Lemma 5.3.5. Without loss of generality, suppose f^* satisfies the assumption of Lemma 5.3.5. But $D' = D'' - \vec{Q}^T f^*$ is a v_0-reduced divisor, contradicting Definition 5.3.3(i).

Now, we show that there exist at least r_0 distinct reduced divisors equivalent to D. Lemma 5.3.6 implies that there exists at least one v_0-reduced divisor equivalent to D, so if $r_0 = 1$ we are done. Therefore for the rest of the proof we will assume that $r_0 > 1$. Take a v_0-reduced divisor $D' \sim D$ and observe that $D'' = D' - \vec{Q}^T(\chi_{\{v_0\}})$ satisfies the condition (i) of Definition 5.3.3. Hence Corollary 5.3.7 implies that D'' can be reduced without firing v_0 to achieve a new reduced divisor from D'. We can acquire r_0 v_0-reduced divisors equivalent to D by repeated application of this method. We claim that all of the v_0-reduced divisors obtained are distinct. Suppose that there exist $0 \leq i < j < r_0$ and firing strategies f' and f'' such that $f'(v_0) = i$, $f''(v_0) = j$, and $D^* = D' - \vec{Q}^T f' = D' - \vec{Q}^T f''$ is v_0-reduced. This implies that $\vec{Q}^T(f'' - f') = \vec{\mathbf{0}}$ but $0 < (f'' - f')(v_0) < r_0$, contradicting the statement of Lemma 5.3.1. $\square$

Corollary 5.3.9. *Let $\vec{G}$ be a directed graph and let D be a divisor. There exist r_0 v_0-reduced divisors $D_i = D - \vec{Q}^T f_i$ where $f_i(v_0) = i$ for all $0 \leq i \leq r_0 - 1$.*

Lemma 5.3.10. *Let $\vec{G}$ be a directed graph and let D be a divisor. Then*

(i) D is equivalent to an effective divisor if and only if there exists a v_0-reduced divisor $D' \sim D$ such that D' is effective;

(ii) *Suppose D is not equivalent to an effective divisor. Then D is an extreme divisor if and only if for any $v \in V(\vec{G})$, there exists a v-reduced divisor $D' \sim D$ such that $D'(v) = -1$.*

Proof. (i): One direction is obvious. So assume D is equivalent to an effective divisor, call it D''. If D'' is v_0-reduced then we are done. Otherwise, Corollary 5.3.7 implies that there exists a valid firing strategy f with respect to v_0 such that $D'' - \vec{Q}^T f$ is v_0-reduced. Since D'' is effective and f is valid with respect to v_0, $D'' - \vec{Q}^T f$ is effective.

(ii): First assume that D is an extreme divisor. The assertion of part (i) implies that for all $v \in V(D)$, if $D' \sim D$ is a v-reduced divisor, $D'(v) \leq -1$. Suppose there exists $v \in V(\vec{G})$ such that for all v-reduced divisor $D' \sim D$ we have that $D'(v) < -1$. Then by Remark 5.3.4, for all v-reduced divisors $D' \sim D$, $D' + \chi\{v\}$ is not effective and it is v-reduced. So by part (i), $D + \chi_{\{v\}}$ is not effective, a contradiction.

For proving the other direction, it is enough to show that for all $v \in V(\vec{G})$, $D + \chi_{\{v\}}$ is equivalent to an effective divisor. So let v be a vertex and let $D' \sim D$ be the v-reduced divisor such that $D'(v) = -1$. Then $D' + \chi_{\{v\}}$ is effective and so $D + \chi_{\{v\}}$ is also. $\square$

5.3.1.2 Dhar's Algorithm

Dhar [26], while studying the sand pile model, found a simple algorithm for checking whether a given divisor in an undirected graph G is v_0-reduced or not. We discuss the directed sandpile model in the next section. Here we generalize his algorithm so that it applies to an arbitrary strongly connected directed graph $\vec{G}$. The authors found this generalization independently from Speer [71].

The input of the algorithm is a divisor D satisfying the condition (i) of Definition 5.3.3. The output of the algorithm is a finite sequence f_i of firing strategies which is decreasing with respect to the $\leq$ relation. The description of the algorithm is as follows.

We construct a sequence of firing strategies f_i's recursively. Set $f_0 = R$, the primitive vector in the left kernel of the Laplacian. For $t \geq 0$, if there exists some $v \in V(\vec{G}) \setminus \{v_0\}$ such that

$$(D - \vec{Q}^T f_t)(v) \leq -1, \tag{1}$$

pick one such vertex v and set $f_{t+1} = f_t - \chi_{\{v\}}$. If for all $v \in V(\vec{G}) \setminus \{v_0\}, (D - \vec{Q}^T f_t)(v) \geq 0$ and $f_t(v_0) > 0$, set $f_{t+1} = f_t - \chi_{\{v_0\}}$. Otherwise the algorithm terminates and the output of the algorithm is the decreasing sequence of f_i's.

We call the above algorithm the *generalized Dhar's algorithm.*

Theorem 5.3.11. *Let D be a divisor satisfying condition (i) in Definition 5.3.3. Then*

(i) the divisor D is v_0-reduced if and only if the generalized Dhar's Algorithm terminates at $f_{\vec{\mathbf{1}} \cdot R} = \vec{\mathbf{0}}$.

(ii) if D is a v_0-reduced divisor then for each $0 \leq t \leq \vec{\mathbf{1}} \cdot R - 1$ such that $f_{t+1} = f_t - \chi_{\{v_0\}}$, $D - \vec{Q}^T f_t$ is a v_0-reduced divisor.

Proof. (i): Clearly if D is reduced then the algorithm terminates at $f_{\vec{\mathbf{1}} \cdot R} = 0$.

So assume that the algorithm terminates on the divisor D. Take a valid firing f with respect to v_0 and pick t as large as possible such that $f_t \geq f$. The choice of t implies that $f_{t+1} = f_t - \chi_{\{v\}}$ for some vertex $v \in V(\vec{G}) \setminus \{v_0\}$ since $f(v_0) = 0$. Therefore $f_t = f + f'$ where $f' \geq 0$ and $f'(v) = 0$. Hence $(D - \vec{Q}^T f)(v) = (D - \vec{Q}^T f_t - \vec{Q}^T f')(v) \leq (D - \vec{Q}^T f_t)(v) < 0$ so the divisor D satisfies the second condition of Definition 5.3.3. Hence D is v_0-reduced.

(ii): For the sake of contradiction, let t be such that $f_{t+1} = f_t - \chi_{\{v_0\}}$ and $D - \vec{Q}^T f_t$ is not a v_0-reduced divisor. There exists a valid firing strategy f with respect to v_0 such that $((D - \vec{Q}^T f_t) - \vec{Q}^T f)(v) \geq 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$. Let $f' = f_t + f$, then we have two cases. Assume there exists $v_i \in V(\vec{G}) \setminus \{v_0\}$ such that $f'(v_i) > r_i$

then $f'' = f' - R$ is a firing strategy which satisfies the conditions of Lemma 5.3.5, contradicting the fact that for all $v \in V(\vec{G}) \setminus \{v_0\}$, $(D - \vec{Q}^T f')(v) > 0$. Therefore, we can choose s as large as possible such that $f_s \geq f'$. The choice of s implies that there exists $v \in V(\vec{G})$ such that $f_s(v) = f'(v)$ and $f_{s+1} = f_s - \chi_{\{v\}}$. If $v = v_0$, since $t > s$, $f_{s+1} \geq f_t$ but $f_{s+1}(v_0) < f_t$, a contradiction. Hence $v \in V(\vec{G}) \setminus \{v_0\}$ and $(D - \vec{Q}^T f_s)(v) < 0$. But $(D - \vec{Q}^T f')(v) \leq (D - \vec{Q}^T f_s)(v) < 0$ and this contradicts the choice of f and f_t. $\square$

The following two paragraphs are not central to this section, and require a working knowledge of commutative algebra. The generalized Dhar's algorithm was independently discovered by Perkinson, Perlman, and Wilmes [64] in their investigation of directed Laplacian lattice ideals. Building on an work of Cori, Rossin, and Salvy [25], and independently Postnikov and Shapiro [66], they observed that the binomial coming from the firings in the generalized Dhar's algorithm are a grevlex Gröbner basis for the directed Laplacian lattice ideal which they generate.

The author and Madhusudan Manjunath have recently answered, in the full dimensional case, a question posed by Miller and Sturmfels [59], who asked for an explicit deformation of a lattice ideal. By Wilmes' lattice reduction algorithm, it suffices to study directed Laplacian lattice ideals. The Gröbner basis coming from the generalized Dhar's algorithm has the property that it respects perturbations of the lattice coming from perturbations of the graph. We then use this observation to deterministically perturb the graph so that the associated Gröbner basis has full support, implying that the ideal it generates is generic.

We conclude this section with the following definition which will appear in each of the subsequent sections.

Definition 5.3.12. *Let $\vec{G}$ be a directed graph with the Riemann-Roch property. Then $\vec{G}$ has the natural Riemann-Roch property if its canonical divisor K has ith entry $\deg^+(v_i) - 2$ for $0 \leq i \leq n$.*

5.3.1.3 The Sandpile Model

The sandpile model for a directed graph is a constrained version of the "row" chip-firing game. We define a divisor D to be a v_0*-sandpile configuration* if D satisfies the condition (i) from Definition 5.3.3. The vertex v_0 does not participate in this game and a vertex $v \in V(\vec{G}) \setminus \{v_0\}$ may only fire if it has at least as many chips as its out-degree (so that v does not go in debt), and it never borrows. Moreover, we say that two configurations are the same if they agree at all vertices other than v_0. This model has been studied in [40, 46, 71]. The goal of this section is to show a connection between the sandpile model and the Riemann-Roch property for the row chip-firing game on a strongly connected directed graph. To do this we will first show a connection between this model and v_0-reduced divisors. We begin with some necessary definitions.

We now restrict our attention to the sandpile model. We call a v_0-sandpile configuration v_0*-stable* if no vertex $v \in V(\vec{G}) \setminus \{v_0\}$ can fire. We note that while some authors require v_0 to be a global sink (in order to guarantee that a divisor will eventually stabilize), we simply insist that v_0 never fires. We say that a v_0-sandpile configuration D' *stabilizes* to D, a v_0-stable configuration, if D is v_0-sandpile achievable from D'. To see that any v_0-sandpile configuration will eventually stabilize to a v_0-stable configuration, one may follow an argument similar to the one from Lemma 5.3.6. We note that, as the language suggests, D is unique, i.e., stabilization is independent of the choice of firings, and a simple proof by induction on k, the length of the sequence of firings, gives this fact. A v_0-stable configuration D is said to be v_0*-reachable* from another v_0-sandpile configuration D' if there exists an effective divisor E such that $D' + E$ stabilizes to D. A v_0-stable configuration is v_0*-recurrent* if it is v_0-reachable from any other v_0-sandpile configuration.

Lemma 5.3.13. *A divisor D is v_0-recurrent if and only if there exists a divisor D' such that $D'(v) \geq \deg^+(v)$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ and D' stabilizes to D.*

Proof. We begin with the easier of the two directions. Assume that D is v_0-recurrent and let D'' be some divisor such that $D''(v) \geq \deg^+(v)$. By definition, D is v_0 reachable from D'', therefore there exists some effective divisor E such that $D'' + E = D'$ stabilizes to D. This gives the existence of D' from the statement of the lemma.

Conversely, given some v_0-sandpile configuration D' such that $D'(v) \geq \deg^+(v)$ for all $v \in V(\vec{G}) \setminus \{v_0\}$, which stabilizes to D, we will show that D is v_0-recurrent. Take some D'', a v_0-sandpile configuration. We will show that D is v_0-reachable from D''. First let D'' stabilize to the configuration D'''. Now $D''' \leq D'$ so that D is v_0-reachable from D'''. Let $D' - D''' = E \geq 0$. We claim that $D'' + E$ stabilizes to D. By the observation made above, that stabilization is independent of a choice of firings, it is sufficient to show that there exists a sequence of firings which brings $D'' + E$ to D. Because $D'' + E \geq D''$ we can perform the sequence of firings which brought D'' to D'''. This sequence of firings brings $D'' + E$ to $D''' + E = D'$ and this now stabilizes to D.

□

The following definition is for the unconstrained row chip-firing game introduced in the previous section. We say that a divisor D is v_0-*negatively achievable* from D' if there exists a sequence of borrowings by individual vertices such that at each step the vertex which borrows has a negative number of chips prior to borrowing.

Lemma 5.3.14. *A divisor ν is v_0-reduced if and only if there exists a divisor D with $D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ such that ν is v_0-negatively achievable from D.*

Proof. We will first show that if ν, a v_0-sandpile divisor, is v_0-negatively achievable from D with $D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ then ν is v_0-reduced. We now introduce some notation, which will be useful for this proof. Let $S : v_{a_1}, \ldots, v_{a_k}$ be the sequence of vertices which borrow and let $f_S \leq 0$ be the corresponding firing so that $D - Q^T f_S = \nu$. Let $f_{S,j}$ be the firing strategy defined as $f_{S,j}(v) = |\{i :$

$v_{a_i} = v, i \leq j\}|$ for $1 \leq j \leq k$, with $f_{S,0} = \vec{0}$. Assume that ν is not v_0-reduced and let $f \neq \vec{0}$ be a natural firing such that $\nu - Q^T f = \nu'$ is a v_0-sandpile divisor. If $f + f_S \not\leq 0$ then there exists a maximal connected subset A of $V(\vec{G}) \setminus \{v_0\}$ such that $(f + f_S)(v) > 0$ for all $v \in A$, but the set A loses a net positive amount of money via the firing $(f + f_S)$ contradicting the fact that $D - Q^T(f + f_S) = \nu'$ is a v_0 sandpile configuration and $D(v) < 0$ for all $v \in A$. Because $f + f_S \leq 0$ we may take j maximum so that $f_{S,j} \geq f + f_S$ but $f_{S,j+1} \not\geq f + f_S$. This shows that $0 \leq \nu'(v_{a_{j+1}}) = (D - Q^T(f + f_S))(v_{a_{j+1}}) \leq (D - Q^T f_{S,j})(v_{a_{j+1}}) < 0$, a contradiction.

We now show that for any v_0-reduced divisor ν there exists some D with $D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ such that ν is v_0-negatively achievable from D. Take ν and greedily fire vertices in $v \in V(\vec{G}) \setminus \{v_0\}$ with a nonnegative number of chips until you obtain D with $D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$. To see that this process will eventually terminate, adapt the argument given in Lemma 5.3.6 for why greedy reduction of a divisor terminates. We claim that D is the desired divisor. If we now, as above, greedily borrow by vertices in $v \in V(\vec{G}) \setminus \{v_0\}$ which are in debt, we will stop at a v_0-reduced divisor ν'. To see that this process eventually terminates, again mimic the argument from Lemma 5.3.6. The fact that ν' is v_0-reduced was proven above. The divisor ν' is clearly equivalent to ν, and v_0 did not participate in the above process, hence the divisor obtained is equal to ν. □

The authors, independently from Speer [71], discovered the following theorem.

Theorem 5.3.15. *A v_0-sandpile configuration D is v_0-recurrent if and only if the divisor ν is a v_0-reduced divisor, where $\nu(v_i) = \deg^+(v_i) - 1 - D(v_i)$ for all $0 \leq i \leq n$.*

Proof. Let K be the divisor such that $K(v_i) = deg^+(v_i) - 2$. We first note that the map $\phi(D) = K + \vec{1} - D$ is a bijection between divisors D such that $D(v) \geq \deg^+(v)$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ and divisors D such that $D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$. The theorem then follows by observing that ν is v_0-negatively achievable from D with

$D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ if and only if $\phi(\nu)$ is v_0-sandpile achievable from $\phi(D)$ with $(\phi(D))_i \geq \deg^+(v_i)$ for all $v \in V(\vec{G}) \setminus \{v_0\}$. $\square$

We note that using the notion of equivalence given by the unconstrained row chip-firing game, the previous theorem shows that there are exactly r_0 v_0-recurrent divisors in each equivalence class. This is different from the case of undirected graphs or directed graphs with v_0 a global sink, where the recurrent state in each equivalence class is unique.

We define a divisor D to be minimally v_0-recurrent if, ignoring the value of $D(v_0)$, it is minimal with respect to dominance among all v_0-recurrent divisors. Using this definition we have a new way of describing the natural Riemann-Roch property in terms of the sandpile model for strongly connected directed graphs.

Theorem 5.3.16. *A directed graph, $\vec{G}$ has the natural Riemann-Roch property if and only if for each minimal v_0-recurrent divisor D there exists $D' = D + ke_0$, $k \in \mathbb{Z}$, $E_i \in \mathbb{Z}_{\geq 0}$ for $0 \leq i \leq n$ such that $E_i(v_i) = 0$ and $E_i(v_j) > 0$ for $j \neq i$ and $D' \sim E_i$ and each D' is of fixed degree $g - 1 \in \mathbb{N}$.*

Proof. Clearly D is minimally v_0 recurrent if and only if, by Theorem 5.3.15, we may fix D' as in the statement of the theorem such that $\nu = K - D' + \vec{\mathbf{1}}$ is extreme v_0-reduced. Hence, $\vec{G}$ has the natural Riemann Roch property if and only if $\nu' = D' - \vec{\mathbf{1}} \in Ext(\Sigma(\Lambda)$ and is fixed degree $g - 1$, which occurs precisely when $D' \in Ext(\Sigma_{\mathbb{R}}(\Lambda))$ and is of fixed degree $g - 1$. By Lemma 5.2.19, the Theorem follows. $\square$

5.3.2 Column Chip-Firing Game, $\vec{G}$-Parking Functions, and Riemann-Roch Theory

In this section we present a chip-firing game which comes from the columns of the Laplacian matrix.

Definition 5.3.17. *We call a divisor D a directed $\vec{G}$-parking function (or simply $\vec{G}$-parking) with respect to v_0 if the following two conditions hold:*

(i) for all $v \in V(\vec{G}) \setminus \{v_0\}, D(v) \geq 0$,

(ii) for every set $A \subseteq V(\vec{G}) \setminus \{v_0\}$, there exists some $v \in A$ such that $|\{(v, u) \in E(\vec{G}) : u \notin A\}| \geq D(v)$.

We introduce the following "column" chip-firing game wherein if a vertex v *fires*, it loses $\deg^+(v)$ chips and sends a chip along each incoming edge $(u, v) \in E(\vec{G})$ (*borrowing* is defined as the inverse of firing). Note that the total number of chips is not preserved by firing in contrast to the previous "row" chip-firing game. It is not hard to see that if all vertices in a set A fire once then a vertex $v \in A$ will lose as many chips as it has edges leaving A, i.e., $|\{(v, u) : u \notin A\}|$, while a vertex $u \notin A$ will gain as many chips as it has edges entering to it from A, i.e., $|\{(v, u) : v \in A\}|$. One may view this game as a walk through the lattice spanned by the columns of the Laplacian of $\vec{G}$ and it follows immediately that if D is a divisor then $(D - \vec{Q}\chi_A)(v) = D(v) - |\{(v, u) : u \notin A\}|$ if $v \in A$ and $(D - \vec{Q}\chi_A)(u) = D(u) + |\{(v, u) : v \in A\}|$ if $u \notin A$. Because $\vec{Q}\vec{\mathbf{1}} = \vec{\mathbf{0}}$, we have that for any firing strategy f, there exists some firing strategy f' such that $\vec{Q}(f - f') = \vec{\mathbf{0}}$ and $f' \leq \chi_A$ for some $A \subseteq V(\vec{G}) \setminus \{v_0\}$. It is also worth mentioning that if $R = (r_0, \ldots, r_n) \in \mathbb{N}^{n+1}$ is the vector guaranteed by Lemma 5.3.1 such that $R^T\vec{Q} = \vec{\mathbf{0}}^T$, then $\deg_R(\vec{Q}f) = 0$ for all $f \in \mathbb{Z}^{n+1}$, i.e., the total number of chips is preserved in the "column" chip-firing game with respect to $\deg_R(\cdot)$. One may interpret this fact combinatorially by assigning to each vertex v_i its own "chip currency" worth r_i of a "universal chip currency" making the game conservative. Similar notions of "currencies" and "exchange rates" are employed when discussing chip-firing on arithmetical graphs in Section 5.4.

The definition of a $\vec{G}$-parking function is the "column" chip-firing analogue of a v_0-reduced divisors from the "row" chip-firing game. More specifically, if we change $\vec{Q}^T$ to $\vec{Q}$ in definition of v_0-reduced divisor (Definition 5.3.3), then we get the definition of $\vec{G}$-parking function with respect to v_0 (Definition 5.3.17). Hence, Dhar's algorithm introduced in [9, 26] applies in verifying whether D is $\vec{G}$-parking function with respect to v_0. Note that for undirected graphs, the notion of a v_0-reduced divisor and a G-parking function agree as the Laplacian is symmetric, i.e., the "row" and "column" chip-firing games are identical. It is a well known fact, and has several combinatorial proofs, that the $\vec{G}$-parking functions are in bijection with set of rooted directed spanning trees [20].

An *Eulerian* directed graph $\vec{H}$ is a directed graph such that $\deg^+(v) = \deg^-(v)$ for each $v \in V(\vec{H})$. The name is derived from the fact that they are exactly those directed graphs which possess a directed Eulerian circuit.

Theorem 5.3.18. *Let $\vec{G}$ be a strongly connected directed graph with Laplacian $\vec{Q}$ and let $\vec{G}'$ be the Eulerian directed graph with Laplacian $\vec{Q}^T\mathcal{R}$ where $\mathcal{R} = diag(r_0, \ldots, r_n)$ where $\vec{1}^T\mathcal{R}\vec{Q} = 0$. The directed graph $\vec{G}$ has the Riemann-Roch property for the column chip-firing game if and only if the directed graph $\vec{G}'$ has the Riemann-Roch property for the row chip-firing game.*

Proof. Let $\Lambda'_{\vec{G}} = \{\vec{Q}f : f \in \mathbb{Z}^{n+1}\}$ be the lattice spanned by the columns of $\vec{Q}$. It follows by Theorem 5.2.12 that $\Lambda'_{\vec{G}}$ has the Riemann-Roch property if and only if $\mathcal{R}\Lambda'_{\vec{G}}$ does. This is the lattice spanned by the rows of $\vec{Q}^T\mathcal{R}$ completing the proof. □

We note that the column chip-firing game for an Eulerian digraph is the same game as the row chip-firing game played on the same directed graph with as of the orientations of all of the arrows reversed. This explains why we are passing to the transpose of the Laplacian in the proof.

Amini and Manjunath [3] have some results related to Eulerian directed graphs (which they call regular digraphs). By the previous theorem, all of these results extend to the column chip-firing game on strongly connected directed graphs. We also remark that for testing whether a divisor is v_0-reduced, the burning algorithm of Dhar may be applied (burning along incoming edges) and this algorithm can be used to obtain several of the results of Amini and Manjunath related to Eulerian directed graphs.

5.4 Arithmetical Graphs

5.4.1 A Combinatorial Proof of Lorenzini's Theorem

Let G be a connected undirected multigraph, choose an ordering $\{v_0, \ldots, v_n\}$ of vertices of G, and let A be the corresponding *adjacency matrix* of G. Let $R = (r_0, \ldots, r_n)^T \in \mathbb{N}^{n+1}$ be such that $gcd(r_0, r_1 \ldots, r_n) = 1$ and let $\delta_0, \ldots, \delta_n \in \mathbb{N}$ be such that $(\mathcal{D} - A)R = \vec{\mathbf{0}}$, where $\mathcal{D} = diag(\delta_0, \ldots, \delta_n)$. We say (G, R) is an *arithmetical graph* with Laplacian $Q = \mathcal{D} - A$ and corresponding multiplicity vector R, where for all $0 \leq i \leq n$ the value r_i is the *multiplicity* of the vertex v_i. Note that an undirected graph G can be considered as an arithmetical graph $(G, \vec{\mathbf{1}})$.

Consider the following chip-firing game played on the vertices of an arithmetical graph (G, R). Suppose we have a "universal chip currency" and each vertex v_i has its own "v_i-chip currency" such that each v_i-chip is worth r_i of the "universal chip currency". If a vertex v_i *fires*, it loses δ_i of its own v_i-chips and sends $m_{i,j}$ v_j-chips to each v_j adjacent to v_i, where $m_{i,j}$ is the number of edges between v_i and v_j. We define *borrowing* to be the inverse of firing. Let $\Lambda_{(G,R)}$ be the lattice spanned by the columns of Q. It is easy to see that moves in this chip-firing game correspond to translations of some divisor D by a lattice point $l \in \Lambda_{(G,R)}$. This observation allows us to make use of definitions and theorems from Section 2 when discussing the chip-firing game.

Let (G, R) be an arithmetical graph and $\mathcal{R} = diag(r_0, \ldots, r_n)$. Let $\vec{G}_R$ be the directed graph obtained from (G, R) by replacing each undirected edge (v_i, v_j) with r_j edges directed from v_i to v_j and r_i edges directed from v_j to v_i. The chip-firing game for (G, R) corresponds to the row chip-firing game for $\vec{G}_R$ by converting each vertex's currency to the universal chip currency. Omid Amini observed that if we define $\vec{Q}_R$ to be the Laplacian of $\vec{G}_R$, $\vec{Q}_R^T = \mathcal{R}Q$. It then follows by Theorem 5.2.12 that the chip-firing game on (G, R) will have the Riemann-Roch property if and only if the row chip-firing game on $\vec{G}_R$ has the Riemann-Roch property. The row chip-firing game on $\vec{G}_R$ is strictly "finer" than the chip-firing game on (G, R) in the sense that a vertex, v_i need not have a multiple of r_i universal chips, although so the role of Theorem 5.2.12 here is to say that this difference does not effect whether the Riemann-Roch property holds.

In our discussion of the chip-firing game for arithmetical graphs we will borrow several definitions and methods from the row chip-firing game whose interpretation will be clear from the context in which they are used. In particular the definition of a v_0-reduced divisor and the generalized Dhar's algorithm will be frequently employed.

Theorem 5.4.1. *Let (G, R) be an arithmetical graph with Laplacian Q and let $\vec{G}_R$ be the associated directed graph. Then $\vec{G}_R$ has the Riemman-Roch property for the column chip-firing game.*

Proof. By Theorem 5.3.18 it is equivalent to ask the question for the row chip-firing game on the directed graph $\vec{H}$ whose Laplacian is $\mathcal{R}\vec{Q}'$ where $\vec{Q}'$ is the Laplacian for $\vec{G}_R$. But $\vec{Q}'$ is simply $\vec{Q}\mathcal{R}$ and so $\vec{H}$ has Laplacian $\mathcal{R}\vec{Q}\mathcal{R}$ which as one can easily check is the Laplacian of the undirected graph obtained from G by replacing each edge (v_i, v_j) with $r_i r_j$ edges. By Baker and Norine, this graph has the Riemman-Roch property and this completes the proof.

$\square$

Let $\mathcal{N} = \{D \in Ext(\Sigma(\Lambda_{(G,R)})) : \deg_R(D) = g_{\max} - 1\}$. For each $0 \leq i \leq n$, let $N(v_i)$ denote the family of vertices which are adjacent to v_i, counting their multiplicities. We call $|N(v_i)|$ the *degree* of the vertex v_i and we denote it by $\deg(v_i)$. Recall the definition of g_0, the number such that $2g_0 - 2 = \sum_{i=0}^{n} r_i(\delta_i - 2)$. It is not hard to verify, and is noted in [50], that g_0 is an integer. It is also easy to see that by firing all of the vertices of the G, we get $\sum_{i=0}^{n} r_i\delta_i = \sum_{i=0}^{n} r_i \deg(v_i)$. Therefore $2g_0 - 2 = \sum_{i=0}^{n} r_i(\deg(v_i) - 2)$.

The following Theorem 5.4.2 and Theorem 5.4.5 are due to Lorenzini [49]. His approach in proving these theorems is purely algebraic and employs the classical Riemann-Roch Theorem for curves. As mentioned in [49], he was interested in combinatorial proofs of these facts, which we now present.

Theorem 5.4.2. *Let (G, R) be an arithmetical graph. Then $g_{\max} \leq g_0$.*

Proof. The following proof is an averaging argument employing the generalized Dhar's algorithms and gives a bound twice as good as the naive bound. If one looks closely at the proof, it becomes apparent that arithmetical graphs are precisely those "directed graphs" for which such an averaging argument is successful. Let $D \in \mathcal{N}$. Choose a v_0-reduced divisor $D' \sim D$ such that $D'(v_0)$ is as large as possible. For proving the theorem, it is enough to show that $\deg_R(D') \leq g_0 - 1$. Apply the generalized Dhar's algorithm to D'. For all $0 \leq i \leq n$ and $1 \leq k \leq r_i$, define $\mathcal{F}_{i,k}$ to be the firing strategy obtained from the generalized Dhar's algorithm such that $\mathcal{F}_{i,k}(v_i) = k$, and the successor of $\mathcal{F}_{i,k}$ is the firing strategy $\mathcal{F}_{i,k} - \chi_{\{v_i\}}$. For each $v_i \in V(\vec{G}) \setminus v_0$ we obtain r_i inequalities as follows:
for each k where $1 \leq k \leq r_i$, we have:

$$D'(v_i) \leq k\delta_i - \left(\sum_{v_j \in N(v_i)} \mathcal{F}_{i,k}(v_j) \right) - 1, \tag{2}$$

which follows from the fact that $(D' - Q\mathcal{F}_{i,k})(v_i) < 0$ by choice of $\mathcal{F}_{i,k}$.
For the vertex v_0, we know that for all $1 \leq k \leq r_0$,

$$k\delta_0 - \sum_{v_j \in N(v_0)} \mathcal{F}_{0,k}(v_j) \geq 0,$$

by the choice of D' and the second assertion of Lemma 5.3.11. Because $D' \in \mathcal{N}$, by (ii) of Lemma 5.3.10 we have that $D'(v_0) < 0$. Hence, for all $1 \leq k \leq r_0$,

$$D'(v_0) \leq k\delta_0 - \left(\sum_{v_j \in N(v_0)} \mathcal{F}_{0,k}(v_j) \right) - 1. \tag{3}$$

Note that $\sum_{i=0}^{n} \sum_{k=1}^{r_i} D'(v_i) = D' \cdot R = \deg_R(D')$.

Now, taking the sum over all inequalities in (2) and (3), we have:

$$\sum_{i=0}^{n}\sum_{k=1}^{r_i} D'(v_i) \le \sum_{i=0}^{n} r_i((r_i+1)\delta_i - 2)/2 - \sum_{i=0}^{n}\sum_{k=1}^{r_i}\sum_{v_j\in N(v_i)} \mathcal{F}_{i,k}(v_j). \qquad (4)$$

We will now restrict our attention to $\sum_{i=0}^{n}\sum_{k=1}^{r_i}\sum_{v_j\in N(v_i)} \mathcal{F}_{i,k}(v_j)$. By reordering the sums, we have

$$\sum_{i=0}^{n}\sum_{k=1}^{r_i}\sum_{v_j\in N(v_i)} \mathcal{F}_{i,k}(v_j) = \sum_{i<j,\ v_iv_j\in E(G)} \left(\sum_{k=1}^{r_i} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i)\right).$$

We claim that if $v_iv_j \in E(G)$ then $\sum_{k=1}^{r_i} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i) = r_ir_j$. We prove the claim by induction on $r_i + r_j$. If $r_i + r_j = 2$, then the claim holds trivially, since $r_i = r_j = 1$. Now suppose $r_i + r_j = m \ge 3$. Without loss of generality, assume $\mathcal{F}_{i,r_i}$ is generated before $\mathcal{F}_{j,r_j}$ in the run of the generalized Dhar's algorithm on D'. Hence

$$\sum_{k=1}^{r_i} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i) = r_j + \sum_{k=1}^{r_i-1} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i) = r_j + (r_i-1)r_j = r_ir_j.$$

The equality $\sum_{k=1}^{r_i-1} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i) = (r_i-1)r_j$ follows from the induction hypothesis. This completes the proof of the claim. So

$$\sum_{i<j,\ v_iv_j\in E(G)} \left(\sum_{k=1}^{r_i} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i)\right) = \sum_{i<j,\ v_iv_j\in E(G)} r_ir_j = \frac{1}{2}\left(\sum_{i=0}^{n} r_i \sum_{v_j\in N(v_i)} r_j\right).$$

Since $QR = 0$, for all $0 \le i \le n$, $\sum_{v_j\in N(v_i)} r_j = r_i\delta_i$. Hence

$$\sum_{i=0}^{n}\sum_{k=1}^{r_i}\sum_{v_j\in N(v_i)} \mathcal{F}_{i,k}(v_j) = \frac{1}{2}\left(\sum_{i=0}^{n} r_i^2\delta_i\right). \qquad (5)$$

Now by substituting (5) into inequality (4), we have:

$$\deg_R(D') \le \sum_{i=0}^{n}(r_i((r_i+1)\delta_i - 2)/2 - \frac{1}{2}\left(\sum_{i=0}^{n} r_i^2\delta_i\right) = \sum_{i=0}^{n} r_i(\delta_i - 2)/2 = g_0 - 1.$$

□

It follows from the above theorem shows that if, in a configuration of the game identified by $D \in Div((G,R))$, $\deg_R(D) \ge g_0$, then D has a winning configuration.

Corollary 5.4.3. *We have that $g_{\max} = g_0$ if and only if all inequalities in (2) and (3) obtained in a run of the generalized Dhar's algorithm on a v_0-reduced divisor $D \in \mathcal{N}$ are tight, i.e., if f_i is the sequence of firing strategies obtained from the run of the generalized Dhar's algorithm on a v_0-reduced divisor $D \in \mathcal{N}$, for all $0 \leq t \leq \vec{\mathbf{1}} \cdot R - 1$, if $f_{t+1} = f_t - \chi_{\{v\}}$ then $(D - Q(f_t))(v) = -1$.*

It is clear, and demonstrated below, that if $D \in \mathcal{N}$ and $\deg(D) = g_{max} - 1$, then for each $v \in V(G)$ and $D' \sim D$ such that D' is v-reduced, we have $D'(v) = -1$. The following theorem shows that the converse is also true.

Theorem 5.4.4. *Let $D \in \mathcal{N}$. Then deg(D)=$g_{max} - 1$ if and only if for each $D' \sim D$ such that D' is a v-reduced divisor, $D'(v) = -1$.*

Proof. Suppose $D \in \mathcal{N}$ with $\deg(D) = g_{max} - 1$. Take $v \in V(\vec{G})$. By applying (ii) of Lemma 5.3.10 we may pick $D' \sim D$ to be a v-reduced divisor such that $D'(v) = -1$. Corollary 5.4.3 implies that all the inequalities are tight, so for all v-reduced divisor $D'' \sim D$, $D''(v) = -1$.

Conversely, assume that $D \in \mathcal{N}$ is v_0-reduced and suppose that for each $D' \sim D$ which is an extreme v-reduced divisor, $D'(v) = -1$. We wish to show that deg(D)=$g_{max} - 1$. Apply the generalized Dhar's algorithm to D, and define $\mathcal{F}_{i,k}$ to be the firing strategy obtained from the generalized Dhar's algorithm such that $\mathcal{F}_{i,k}(v_i) = k$ and the successor of $\mathcal{F}_{i,k}$ is the firing strategy $\mathcal{F}_{i,k} - \chi_{\{v_i\}}$.

$$D(v_i) \leq k\delta_i - \left(\sum_{v_j \in N(v_i)} \mathcal{F}_{i,k}(v_j) \right) - 1, \tag{6}$$

which follows from the fact that $(D - Q\mathcal{F}_{i,k})(v_i) < 0$ by choice of $\mathcal{F}_{i,k}$. By the previous corollary, to show that $\deg(D) = g_{max} - 1$, it is enough to show that each of the inequalities from (6) holds with equality.

For the vertex v_0, we know that for all $1 \leq k \leq r_0$,

$$k\delta_0 - \sum_{v_j \in N(v_0)} \mathcal{F}_{0,k}(v_j) \geq 0,$$

this follows from the choice of D and the second assertion of Lemma 5.3.11. Because D is extreme, by (ii) of Lemma 5.3.10 we have that $D(v_0) < 0$. Hence for all $1 \leq k \leq r_0$,

$$D(v_0) \leq k\delta_0 - \left(\sum_{v_j \in N(v_0)} \mathcal{F}_{0,k}(v_j)\right) - 1. \tag{7}$$

By assumption, all of the inequalities for v_0 above hold with equality. So take $v_i \in V(\vec{G})\backslash v_0$ and $1 \leq k \leq r_i$. For finishing the proof, we will show that $(D - Q(\mathcal{F}_{i,k}))(v_i) = -1$. Let the firing strategy f be such that $D - Qf$ is v_i-reduced and $f(v_i) = k$, where the existence of f is guaranteed by Corollary 5.3.9. Assume $f' \sim f$ is a natural firing strategy. Let f_t's be the sequence of firing strategies obtained from a run of the generalized Dhar's algorithm on D. Take j as large as possible such that $f_j \geq f'$. Let $v \in V(\vec{G})$ be such that $f_{j+1} = f_j - \chi_{\{v\}}$ and let the firing strategy f'' be such that $f' = f_j - f''$ where $f'' \geq \vec{\mathbf{0}}$ and $f''(v) = 0$. We claim that $v = v_i$. If $v \notin \{v_0, v_i\}$ then $(D - Qf')(v) = (D' - Q(f_j - f''))(v) \leq (D - Q(f_j))(v) < 0$, contradicting the fact that $D - Qf'$ is a v_i-reduced. If $v = v_0$, then $(D - Qf')(v_0) = (D - Q(f_j - f''))(v_0) \leq (D - Q(f_j))(v_0) = -1$ since $D - Qf_j$ is a v_0-reduced divisor by the second part of Theorem 5.3.11. But this again contradicts the fact that $D - Qf'$ is a v_i-reduced divisor. Hence $v = v_i$ and this finishes the proof of the claim. Therefore $f_j = \mathcal{F}_{i,k}$ and we have:

$$-1 = (D - Qf')(v_i) = (D - Q(f_j - f''))(v_i)$$

$$= (D - Q(\mathcal{F}_{i,k} - f''))(v_i) \leq (D - Q(\mathcal{F}_{v_i,k}))(v_i) \leq -1.$$

Hence $(D - Q(\mathcal{F}_{i,k}))(v_i) = -1$ as desired. $\square$

We note that a more general version of the previous theorem can be stated for strongly connected directed graphs and might have been included in the section on Dhar's algorithm, but because we do not have statement like Corollary 5.4.3 for all strongly connected directed graphs, the statement of this more general theorem would have been awkwardly phrased.

Theorem 5.4.5. *Let $K = (\delta_0 - 2, ..., \delta_n - 2)$ be a vector in $\mathbb{Z}^{n+1}$. If $g_{\max} = g_0$ then $D \in \mathcal{N}$ if and only if $K - D \in \mathcal{N}$.*

Proof. Without loss of generality, we may assume D is a v_0-reduced divisor. Apply the generalized Dhar's algorithm to D and let f_i be the output sequence. Let $\mathcal{F}_{i,k}$ be the firing strategies defined in the proof of Theorem 5.4.2.

Define the divisor D' such that for all $0 \le i \le n$,

$$D'(v_i) = k\delta_i - \left(\sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k})(v_j) \right) - 1.$$

We claim that D' is well-defined. For proving the claim, it is enough to show that for all $0 \le i \le n$, the value of $D'(v_i)$ does not depend upon k. We will show $D' = K - D$.

Since $g_{\max} = g_0$, Corollary 5.4.3 implies that for all $0 \le i \le n$, $\sum_{v_j \in N(v_i)} \mathcal{F}_{i,r_i+1-k}(v_j) = (r_i + 1 - k)\delta_i - D(v_i) - 1$. For all $0 \le i \le n$, we have:

$$\begin{aligned}\sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k})(v_j) &= \left(\sum_{v_j \in N(v_i)} r_j \right) - ((r_i + 1 - k)\delta_i - D(v_i) - 1) \\ &= -\delta_i + k\delta_i + D(v_i) + 1.\end{aligned}$$

Therefore,

$$\begin{aligned}D'(v_i) &= k\delta_i - \left(\sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k})(v_j) \right) - 1 \\ &= k\delta_i - (-\delta_i + k\delta_i + D(v_i) + 1) - 1 = \delta_i - 2 - D(v_i).\end{aligned}$$

Since $\deg_R(K - D) = g_0 - 1$, for finishing the proof we only need to show that $K - D$ is not equivalent to an effective divisor.

Assume to the contrary that D' is equivalent to some effective divisor E and let f be such that $D' - Qf = E$. Let $f' \sim f$ be a natural firing strategy guaranteed by Lemma 5.3.2. Define a "reverse sequence" of firing strategies $f'_i = R - f_{\vec{\mathbf{1}} \cdot R - i}$ for all $0 \le i \le \vec{\mathbf{1}} \cdot R$. Take t as large as possible such that $f'_t \ge f'$, so there exists $v_i \in V(\vec{G})$ such that $f'(v_i) = f'_t(v_i)$. By the definition of the reverse sequence, there exists $1 \le k \le r_i$ such that $f'_t = R - \mathcal{F}_{i,r_i+1-k} + \chi_{\{v_i\}}$. Therefore,

$$E(v_i) \le (D' - Qf'_t)(v_i)$$

$$= k\delta_i - \left(\sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k})(v_j)\right) - 1 - (r_i - (r_i + 1 - k) - 1)\,\delta_i$$

$$+ \left(\sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k} + \chi_{\{v_i\}})(v_j)\right)$$

$$= k\delta_i - (r_i - (r_i + 1 - k) - 1) - 1 = -1.$$

Note that $\sum_{v_j \in N(v_i)}(R - \mathcal{F}_{i,r_i+1-k} + \chi_{\{v_i\}})(v_j) = \sum_{v_j \in N(v_i)}(R - \mathcal{F}_{i,r_i+1-k})(v_j)$. This contradicts the choice of E. Hence $D' = K - D$ is not equivalent to an effective divisor. $\square$

Theorem 5.4.6. *Let (G, R) be an arithmetical graph. If $g_0 = g_{\min} = g_{\max}$, then (G, R) has the Riemann-Roch property. Moreover, the corresponding directed graph has the natural Riemann-Roch property.*

Proof. The first part of the theorem follows as an immediate consequence of Theorem 5.2.11 and Theorem 5.4.5. The second part of the theorem follows by Corollary 5.2.32, which in this context says that if $g_0 = g_{\min} = g_{\max}$, then the canonical divisor for the corresponding digraph $\vec{G_R}$ has ith entry $\deg^+(v_i) - 2$, i.e., $\vec{G_R}$ satisfies Definition 5.3.12 for the row chip-firing game. Moreover, we note that $(\delta_0 - 2, \ldots, \delta_n - 2) \sim (\deg(v_0) - 2, \ldots, \deg(v_n) - 2)$ as is easily observed by computing $Q\vec{1}$.

$\square$

5.4.2 Arithmetical Graphs with the Riemann-Roch Property

In this section we provide some examples of arithmetical graphs with the Riemann-Roch property applying several techniques developed in the previous sections. We begin with a very simple lemma.

Lemma 5.4.7. *Let (G, R) be an arithmetical graph. If $\Lambda_{(G,R)}$ has a unique class of extreme divisors, i.e. $Ext(\Sigma(\Lambda_{(G,R)})) = \{\nu + \ell : \ell \in \Lambda_{(G,R)}\}$, then $\Lambda_{(G,R)}$ has the Riemann-Roch property.*

Theorem 5.4.8. *Let (G, R) be an arithmetical graph. If $g_0 \leq 1$ then (G, R) has the Riemann-Roch property.*

Proof. Let v_0 be a vertex such that $r_0 \leq r_i$ for all $1 \leq i \leq n$. Let D be an extreme v_0-reduced divisor with $D(v_0) = -1$. By Theorem 5.4.2 $g_{\max} \leq g_0$, so $\deg(D) \leq g_{\max} - 1 \leq 0$. Now we have two cases:

(i) $D(v_i) = 0$ for all $1 \leq i \leq n$, part (ii) of Lemma 5.3.10 and the choice of r_0 implies that D is the unique extreme v_0-reduced divisor, and the assertion of the lemma holds by Corollary 5.4.7. Note that in this case $g_{\max} \neq g_0$ unless $g_0 = 0$ and $r_0 = 1$.

(ii) There exists $1 \leq i \leq n$ such that $D(v_i) > 0$. Since $\deg(D) \leq 0$, $r_i = r_0$ and v_i is the only vertex with $D(v_i) > 0$. This implies that the divisor D' with $D'(v_0) = -1$ and $D'(v_j) = 0$ for all $1 \leq j \leq n$ is not an extreme divisor. Hence, $g_0 = g_{\min} = g_{\max} = 1$, and assertion of the lemma follows by Theorem 5.4.6.

□

Using the definition of g_0, the following is an immediate consequence of Theorem 5.4.8.

Corollary 5.4.9. *Let (G, R) be an arithmetical graph with all δ_i's equal to two or all* $\deg(v_i)$*'s equal to two. Then (G, R) has the Riemann-Roch property.*

The former arithmetical graphs are those coming from the connection between Lie algebras or elliptical curves which have been classified [19] and the latter arithmetical graphs where the underlying graph is a cycle. The following two examples show that both cases described in the proof of Theorem 5.4.8 occur.

Example 1. *Let (G, R) be an arithmetical graph where G is the even cycle $v_0, \ldots, v_{2n-1}$ for $n \geq 2$, and for all $0 \leq i \leq n-1$, the multiplicities of the vertices v_{2i} and v_{2i+1}*

are 1 *and* 2*, respectively. Then* $g_{\min} = g_{\max} = g_0 = 1$*, and in particular* (G, R) *has the Riemann-Roch property.*

Proof. We claim that the set of extreme v_0-reduced divisors for (G, R) are the set of divisors $D_i = \chi_{\{v_{2i}\}} - \chi_{\{v_0\}}$ for all $1 \le i \le n-1$. Assume $1 \le i \le n-1$, and the vector f is a valid firing strategy with respect to v_0 such that $D_i - Qf \ge \vec{\mathbf{0}}$. Observe that if $f(v_{2i}) = 1$, then in order to $(D_i - Qf)(v_{2i}) \ge 0$ we must have $f(v_{2i-1}) + f(v_{2i-1}) \ge 3$. By symmetry, assume that $f(v_{2i-1}) \ge 2$. Since $(D_i - Qf)(v_{2i-1}) \ge 0$, we have $f(v_{2i-2}) = 1$. By repeating the argument, we conclude that $f(v_0) = 1$, a contradiction. This shows that D_i is v_0-reduced and since $r_0 = 1$, (i) of Lemma 5.3.10 implies that D_i is not equivalent to an effective divisor. For proving the fact that D_i is an extreme divisor, it is enough to show that $D_i + \chi_{\{v_j\}}$ is equivalent to an effective divisor, for all $0 \le j \le 2n-1$.

It is easy to see that $g_0 = 1$. If $0 \le j \le 2n-1$ is odd, then the divisor $D_i + \chi_{\{v_j\}}$ has degree $2 > g_0$, thus Theorem 5.4.2 implies that $D_i + \chi_{\{v_j\}}$ is effective. We claim that for all $0 \le j \le i \le n-1$, the divisor $D_i + \chi_{\{v_{2j}\}}$ is equivalent to an effective. We prove the claim by induction on j. If $j = 0$, then the assertion of the claim trivially holds. So, assume $j > 0$ and let $f = \chi_{\{v_{2j-1},\ldots,v_{2i+1}\}}$. A simple computation gives that $D_i + \chi_{\{v_{2j}\}} - Qf = D_{i+1} + \chi_{\{v_{2j-2}\}}$. The induction hypothesis implies that $D_{i+1} + \chi_{\{v_{2j-2}\}}$ is equivalent an effective divisor, so is $D_{i+1} + \chi_{\{v_{2j}\}}$. This shows that D_i's are extreme v_0-reduced divisors.

Now assume that D is an extreme v_0-reduced divisor. Part (ii) of Lemma 5.3.10 implies that $D(v_0) = -1$. If $D(v_{2i+1}) = 1$ for some $0 \le i \le n-1$, then D is not a v_0-reduced divisor. The above argument shows that if $D(v_{2i}) = 2$ or $D(v_{2i}) = D(v_{2j}) = 1$ for some $0 \le i \ne j \le n-1$, the divisor D is equivalent to an effective divisor. Obviously $D \ne -\chi_{\{v_0\}}$, and this completes the proof of the claim.

Since each extreme v_0-reduced divisor D_i, $1 \le i \le n-1$ has degree zero, $g_{\min} = g_{\max} = g_0$. Theorem 5.4.6 implies that (G, R) has the Riemann-Roch property. □

Example 2. *Let (G, R) be an arithmetical graph where G is a cycle $v_1, \dots, v_n$ for $n \geq 3$ and the multiplicity of vertex v_i is i for all $1 \leq i \leq n$. Then (G, R) has Riemann-Roch property.*

Proof. It is easy to see that $g_0 = 1$. Now assume D is an extreme v_1-reduced divisor. The part (ii) of Lemma 5.3.10 implies that $D(v_1) = -1$. If there exists $2 \leq i \leq n$ such that $D(v_i) \geq 1$, then degree of D is at least one. Thus, Theorem 5.4.2 implies that D is equivalent to an effective divisor. This shows that $D = -\chi_{\{v_1\}}$ is the unique extreme v_1-reduced divisor and the assertion of the lemma follows Corollary 5.4.7. □

The following example introduced in [49] has the Riemann-Roch property.

Example 3. *Let (G, R) be an arithmetical graph where G is a graph with vertex set $\{v_0, v_1\}$ such that v_0 is connected to v_1 with $r_0 r_1$ edges where r_0 and r_1 are the multiplicity of the vertex v_0 and v_1, respectively. Then (G, R) has the Riemann-Roch property.*

Proof. The proof follows from Corollary 5.4.7, since there exists a unique extreme v_0-reduced divisor, $D = -\chi_{\{v_0\}} + (r_0^2 - 1)\chi_{\{v_1\}}$. Hence $g_{\min} = g_{\max} = g_0$. □

Given any two integers $r_0 > r_1$, we can recursively construct a decreasing sequence r_i's where $r_{i+1} = \delta_i r_i - r_{i-1}$, $r_{i+1} < r_i$ and $\delta_i \in \mathbb{N}$ for all $i \geq 1$. We call such a sequence the *Euclidean sequence generated by r_0 and r_1*. Note that the Euclidean sequence generated by r_0 and r_1 is finite and it comes from a simple variation of Euclid's algorithm.

Let (G, R) be an arithmetical graph. We define a *Euclidean chain leaving v_0 generated by r_0 and r_1* to be an induced path $C = v_0, v_1, \dots, v_n$ of length $n + 1 \geq 2$ in G such that $\deg_G(v_n) = 1$ where the corresponding sequence of multiplicities, $r_0, r_1, \dots, r_n$ is the Euclidean sequence generated by r_0 and r_1. Note that $r_n =$

$gcd(r_i, r_{i+1})$ for all $0 \le i \le n-1$. If v_0, r_0 and r_1 are clear from the context, we may simply refer to the path as a *Euclidean chain*.

Lorenzini [50] uses a slight variation of the Euclidean chain for building arithmetical graphs. We use Euclidean chains to construct a family arithmetical graph with the Riemann-Roch property.

A *Euclidean star generated by* r_0 and r_1 is an arithmetical graph (G, R) with the *center* vertex v_0 with multiplicity r_0 and r_0 identical Euclidean chains leaving v_0 generated by r_0 and r_1. We call the vertex v_0 the *center vertex*. When r_0 and r_1 are clear from the context, we will simply say *Euclidean star*.

We will show that every Euclidean star generated by r_0 and r_1 with $gcd(r_0, r_1) = 1$, has the Riemann-Roch property.

Definition 5.4.10. *Let $r_0 > r_1$ be two positive integers with $gcd(r_0, r_1) = 1$. Assume $r_0, r_1, \ldots, r_m$ is the Euclidean sequence generated by r_0 and r_1. Given a nonnegative integer x, we say x has a good representation with respect to r_0 and r_1 if there exist $0 \le t_i \le \delta_i - 1$, for all $1 \le i \le m$ such that $x = \sum_{i=1}^{m} t_i r_i$, and there exist no $1 \le i < j \le m$ such that $t_i = \delta_i - 1$, $t_j = \delta_j - 1$ and for all $i < k < j$, $t_k = \delta_k - 2$.*

Lemma 5.4.11. *Let r_0 and r_1 be positive integers with $gcd(r_0, r_1) = 1$. Given a nonnegative integer x, x has a good representation with with respect to r_0 and r_1 if and only if $0 \le x \le r_0 - 1$. Moreover, if $0 \le x \le r_0 - 1$ such a representation is unique.*

Proof. Assume $r_0, r_1, \ldots, r_m$ is the Euclidean sequence generated by r_0 and r_1. We prove by induction on m. If $m = 1$, the assertion of the lemma is obvious. Now assume $m \ge 2$ and x is an arbitrary nonnegative integer. It is easy to see that $t_1 \le \lfloor \frac{x}{r_1} \rfloor$. If $t_1 < \lfloor \frac{x}{r_1} \rfloor$, then $x - t_1 r_1 \ge r_1$, so by the induction hypothesis $x - t_1 r_1$ does not have a good representation with respect to r_1 and r_2 because $gcd(r_1, r_2) = 1$ and the Euclidean sequence obtained from r_1 and r_2 is $r_1, r_2, \ldots, r_m$.

Hence, we may assume $t_1 = \lfloor \frac{x}{r_1} \rfloor$, so by induction hypothesis $x - t_1 r_1$ has a good representation with respect to r_1 and r_2. If $t_1 \leq \delta_1 - 2$, then the good representation of $x - t_1 r_1$ with respect to r_1 and r_2 extends to a good representation of x with respect to r_0 and r_1.

If $t_1 = \delta_1 - 1$, then $x - (\delta_1 - 1)r_1 = x - r_0 - r_2 + r_1 < r_1 - r_2$, therefore $x - t_1 r_1 + r_2 = \sum_{i=2}^{m} t_i r_i$ is a unique good representation with respect to r_1 and r_2. We claim $t_2 \geq 1$. If $t_2 = 0$ then $x - t_1 r_1 + r_2$ has a good representation with respect to r_2 and r_3, therefore by induction $x - t_1 r_1 + r_2 < r_2$, so $x - t_1 r_1 < 0$, a contradiction. Therefore $(t_2 - 1)r_2 + \sum_{i=3}^{m} t_i r_i$ is the unique good representation of $x - t_1 r_1$ with respect to r_1 and r_2. We claim that $t_1 r_1 + (t_2 - 1)r_2 + \sum_{i=3}^{m} t_i r_i$ is the unique good representation of x with respect to r_0 and r_1. Uniqueness has been established, so it remains to show that the representation is good. Assume the representation is not good. It follows that there exists $i \geq 3$ such that $t_i = \delta_i - 1$ and for all $2 < k < i$, $t_k = \delta_k - 2$, and $t_2 - 1 = \delta_2 - 2$. Therefore, $t_2 = \delta_2 - 1$, which implies $\sum_{i=2}^{m} t_i r_i$ is not a good representation of $x - t_1 r_1 + r_2$ with respect to r_0 and r_1, a contradiction.

Suppose there exists an integer $x \geq r_0$ such that x has a good representation with respect to r_0 and r_1, $x = \sum_{i=1}^{m} t_i r_i$. If $t_1 \leq \delta_1 - 2$ then $x - t_1 r_1 \geq x - (r_0 + r_2) + 2r_1 \geq r_1$. So by induction hypothesis $x - t_1 r_1$ does not have a good representation respect to r_1 and r_2, a contradiction. Hence $t_1 = \delta_1 - 1$ and $x - t_1 r_1 < r_1$. This implies that $x - t_1 r_1 \geq x - (r_0 + r_2) + r_1 \geq r_1 - r_2$. Let $x - t_1 r_1 = \sum_{i=2}^{m} t_i r_i$ be the good representation of $x - t_1 r_1$ with respect to r_1 and r_2. By induction hypothesis $x - t_1 r_1 + r_2 \geq r_1$ does not have a good representation with respect to r_1 and r_2. Either there exists $3 \leq j \leq m$ such that $t_j = \delta_j - 1$, $t_2 + 1 = \delta_2 - 1$ and $t_i = \delta_i - 2$ for all $2 < i < j$, or $t_2 + 1 = \delta_2$, both of which contradict the fact that $\sum_{i=1}^{m} t_i r_i$ is a good representation of x with respect to r_0 and r_1 because $t_1 = \delta_1 - 1$. □

Lemma 5.4.12. *Let (G, R) be a Euclidean star generated by r_0 and r_1 with center vertex v_0. Then the set of all v_0-reduced divisors are the set of divisors such that* *for any Euclidean chain $C = v_0, v_1, \ldots, v_m$ leaving v_0, $x = \sum_{i=1}^{m} D(v_i)r_i$ is a good representation with respect to r_0 and r_1.*

Proof. Let D be a v_0-reduced divisor and $C = v_0, v_1, \ldots, v_m$ be a Euclidean chain leaving v_0. It is clear that if $x = \sum_{i=1}^{m} D(v_i)r_i$ is not a good representation with respect to r_0 and r_1 then D is not a v_0-reduced divisor.

Conversely, let D be a divisor such that for every Euclidean chain $C = v_0, v_1, \ldots, v_m$ leaving v_0, $x = \sum_{i=1}^{m} D(v_i)r_i$ is a good representation with respect to r_0 and r_1, but D is not a v_0-reduced divisor. Let $f \geq \vec{\mathbf{0}}$ be a firing strategy such that $f(v_0) = 0$ and $D' = D - Qf$ is a v_0-reduced divisor. Note that the existence of f is guaranteed by Corollary 5.3.7. Let $C = v_0, v_1, \ldots, v_m$ be a Euclidean chain leaving v_0. Without loss of generality, we may assume $f' \neq \vec{\mathbf{0}}$ where f' is the projection of f into the first $m+1$ coordinates. If $f'(v_1) > 0$ then $\sum_{i=1}^{m} D'(v_i)r_i < 0$, therefore there exists $1 \leq i \leq m$ such that $D'(v_i) < 0$, a contradiction. Hence, $\sum_{i=1}^{m} D'(v_i)r_i = \sum_{i=1}^{m} D(v_i)r_i$. Since $f' \neq \vec{\mathbf{0}}$, by Lemma 5.3.1 and the uniquenes of the representation of $\sum_{i=1}^{m} D(v_i)r_i$ implied by Lemma 5.4.11, $\sum_{i=1}^{m} D'(v_i)r_i$ is not a good representation. Therefore D' is not v_0-reduced, a contradiction. $\square$

Definition 5.4.13. *Let (G, R) be a Euclidean star generated by r_0 and r_1 with center vertex v_0. We say a divisor S is a staircase divisor if there exists a labeling $C_0, \ldots, C_{r_0-1}$ of the Euclidean chains leaving v_0 where $P_i = v_0, v_{i,1}, \ldots, v_{i,m}$ is the induced path of C_i such that $\sum_{j=1}^{m} S(v_{i,j})r_j$ is the good representation of i, for all $0 \leq i \leq r_0 - 1$, and $S(v_0) = -1$.*

Lemma 5.4.14. *Let (G, R) be a Euclidean star generated by r_0 and r_1 with center vertex v_0. A divisor D is an extreme v_0-reduced divisor if and only if D is a staircase divisor.*

Proof. Let S be a staircase divisor and $C_0, \ldots, C_{r_0-1}$ be a labeling of the Euclidean chains leaving v_0 where $v_0, v_{i,1}, \ldots, v_{i,m}$ are the vertices of C_i. We claim that S is not equivalent to an effective divisor. For proving the claim, it is enough to show that all v_0-reduced divisors equivalent to S are staircase divisors. Let $1 \leq k \leq r_0$ and f_k be the firing strategy guaranteed by Corollary 5.3.9, such that $f_k(v_0) = k$ and $S_k = S - Qf_k$ is a v_0-reduced divisor. Note that since S is a v_0-reduced divisor, by Lemma 5.4.12, the divisor S is v_0-reduced. So, as an application of part (ii) of Theorem 5.3.11, we may assume $f_k \geq \vec{\mathbf{0}}$. It is clear from the proof of Lemma 5.4.12, $\sum_{j=1}^{m} S_k(v_{i,j})r_j$ is a good representation of $i + kr_1 \bmod r_0$ for all $0 \leq i \leq r_0 - 1$. Note that S_k is a staircase divisor and $s_k(v_0) = -1$. So (i) of Lemma 5.3.10 implies that S_k is not equivalent to an effective divisor.

We now prove that for any v_0-reduced divisor D not equivalent to an effective, there exists a staircase divisor S such that $D' \sim D$ with $D' \leq S$. Let $C_0, \ldots, C_{r_0-1}$ be a labeling of the Euclidean chains leaving v_0 where $v_0, v_{i,1}, \ldots, v_{i,m}$ are the vertices of C_i such that $\sum_{j=1}^{m} D(v_{i,j})r_j \leq \sum_{j=1}^{m} D(v_{i+1,j})r_j$ for all $0 \leq i \leq r_0 - 2$. Let S be the staircase divisor defined by the same labeling of the Euclidean chains leaving v_0. If for all $0 \leq i \leq r_0 - 1$, $\sum_{j=1}^{m} D(v_{i,j})r_j \leq i$ then $D \leq S$, so we may assume that there exists $0 \leq i \leq r_0 - 1$ such that $\sum_{j=1}^{m} D(v_{i,j})r_j > i$. Let k be such that $kr_1 \equiv r_0 - i - 1 \pmod{r_0}$. By Corollary 5.3.9, there exist firing strategies f_D and f_S such that $f_D(v_0) = f_S(v_0) = k$ and the divisors $D_k = D - Qf_D$ and $S_k = S - Qf_S$ are v_0-reduced. We claim that D_k is effective, in particular $D_k(v_0) = 0$. We have $f_D(v_{\ell,1}) = f_S(v_{\ell,1}) = \lfloor \frac{kr_1}{r_0} \rfloor$ for all $0 \leq \ell \leq i-1$ and $f_D(v_{\ell,1}) = f_S(v_{\ell,1}) = \lceil \frac{kr_1}{r_0} \rceil$ for all $i+1 \leq \ell \leq r_0 - 1$, but $f_D(v_{i,1}) = \lceil \frac{kr_1}{r_0} \rceil$ while $f_S(v_{i,1}) = \lfloor \frac{kr_1}{r_0} \rfloor$. This proves the claim and completes the proof of the lemma. □

Theorem 5.4.15. *Let (G, R) be a Euclidean star then (G, R) has the Riemann-Roch property.*

Proof. By Lemma 5.4.14, we know that the set of staircase divisors is the set of extreme v_0-reduced divisors, hence

$$g_{\min} - 1 = g_{\max} - 1 = (\sum_{i=0}^{r_0-1} i) - r_0 = r_0(r_0-3)/2.$$

Let $V(\vec{G}) = \{v_0, \ldots, v_n\}$. Using the formula

$$g_0 - 1 = \sum_{i=0}^{n} r_i(\deg(v_i) - 2)/2 = r_0(r_0-3)/2 = \begin{pmatrix} r_0 - 1 \\ 2 \end{pmatrix} - 1.$$

Now the assertion of the theorem follows from Theorem 5.4.6. □

5.4.3 Arithmetical Graphs without the Riemann-Roch Property

It follows from Theorem 5.2.11 that an arithmetical graph (G, R) fails to have the Riemann-Roch property if (G, R) is not uniform or is not reflection invariant. The following examples show that there exist arithmetical graphs which are uniform, but not reflection invariant, arithmetical graphs which are reflection invariant, but not uniform, and arithmetical graphs which are neither reflection invariant nor uniform.

Example 4. *Let (G, R) be an arithmetical graph, where G is the graph obtained by adding two edges connecting v_0 to v_3 to the 6-cycle $v_0, \ldots, v_5$, and the multiplicity of the vertex v_i is 1 if $i \in \{0, 2, 4\}$ and is 2 otherwise. Then (G, R) is neither uniform nor reflection invariant.*

Proof. Let $\nu_1 = -\chi_{\{v_0\}} + \chi_{\{v_2,v_3,v_4\}}$, $\nu_2 = -\chi_{\{v_0\}} + \chi_{\{v_2\}} + 2\chi_{\{v_4\}}$ and $\nu_3 = -\chi_{\{v_0\}} + 2\chi_{\{v_2\}} + \chi_{\{v_4\}}$. We claim that $\mathcal{E} = \{\nu_1, \nu_2, \nu_3\}$ is the set of extreme v_0-reduced divisors of (G, R). Note that $\deg_R(\nu_1) = 3$ and $\deg_R(\nu_2) = \deg_R(\nu_3) = 2$. For proving the claim, we start by showing that ν_1 is v_0-reduced. Let f be a valid firing strategy with respect to v_0 such that $(D_1 - Qf)(v_i) \geq 0$, for all $1 \leq i \leq 5$. If $f(v_2) = 1$, since $(D_1 - Qf)(v_2) \geq 0$, we have $f(v_1) + f(v_3) \geq 3$. If $f(v_1) = 2$, since $(D_1 - Qf)(v_1) \geq 0$ we must have $f(v_0) \geq 1$, a contradiction. So $f(v_3) = 2$ and this implies that in order to have $(D_1 - Qf)(v_3) \geq 0$ we must have $f(v_4) = 3$, a contradiction. This shows that $f(v_1) = 0$, and by symmetry $f(v_5) = f(v_4) = 0$, which shows that $f(v_3) = 0$. This shows that $f = \vec{\mathbf{0}}$, which contradicts the fact that f is valid strategy with respect to v_0. Hence, ν_1 is v_0-reduced, as desired. By applying a similar argument, we can see that ν_2 and ν_3 are v_0-reduced divisors. Note that since $r_0 = 1$, by Lemma 5.3.10(i), the v_0-reduced divisors ν_1, ν_2, ν_3 are not effective and they are pairwise inequivalent.

It is easy to compute that $\deg_R(\nu_1) = 3 = g_0 - 1$, so Theorem 5.4.2 implies that ν_1 is extreme. Hence, by symmetry, we only need to prove that ν_2 is extreme. For proving this fact it is enough to show that $D = \nu_2 + \chi_{\{v_i\}}$ is equivalent to an effective divisor for all $0 \leq i \leq 5$. If $i \notin \{0, 2, 4\}$, then degree of D is $4 = g_0$, so Theorem 5.4.2 implies that D is equivalent to an effective divisor. If $i = 0$, then D is trivially effective. If $i = 2$, then we have a firing strategy $f = \vec{\mathbf{1}} - \chi_{\{v_0\}}$ such that $D - Qf = 3\chi_{\{v_0\}} \geq \vec{\mathbf{0}}$. Also, if $i = 4$, then we have $f = \chi_{\{v_4, v_5\}}$ such that $D - Qf = \chi_{\{v_2, v_3\}} \geq \vec{\mathbf{0}}$. This completes the proof of the fact that ν_1, ν_2, ν_3 are extreme v_0-reduced divisors.

Suppose ν is an extreme v_0-reduced divisor. It is easy to see that $\nu(v_2) \leq 2$ (by symmetry $\nu(v_4) \leq 2$), since otherwise $\nu - Qf \geq 0$, where $f = \chi_{\{v_1, v_2\}}$. Note that $\nu(v_1) = \nu(v_5) = 0$ and $\nu(v_3) \leq 1$. It follows that $\mathcal{E}$ is the set of v_0-reduced divisors and this completes the proof of the claim. This demonstrates that (G, R) is not uniform.

Now, we are going to show that (G, R) is not reflection invariant. Let Λ be the lattice spanned by Laplacian of (G, R). By applying Lemma 5.3.6 and (ii) of Lemma 5.3.10, we conclude that $Ext(\Sigma(\Lambda)) = \{\nu + \ell : \ell \in \Lambda, \nu \in \mathcal{E}\}$. Corollary 5.2.23 implies $Crit(\Lambda) = \mathcal{P} + \Lambda$, where $\mathcal{P} = \{\pi(\nu + \vec{\mathbf{1}}) : \nu \in \mathcal{E}\}$. Let $p_i = \pi(\nu_i + \vec{\mathbf{1}}) = (\nu_i + \vec{\mathbf{1}}) - \left(\frac{(\nu_i + \vec{\mathbf{1}}) \cdot R}{R \cdot R}\right) R$. An easy computation shows that $p_1 = \frac{1}{5}(-4, -3, 6, 2, 6, -3), p_2 = \frac{1}{15}(-11, -7, 19, -7, 34, -7)$ and $p_3 = \frac{1}{15}(-11, -7, 34, -7, 19, -7)$. For seeking a contradiction, assume there exists $v \in \mathbb{R}^6$ such that $-Crit(\Lambda) = Crit(\Lambda) + v$. Either there exist $\ell, \ell', \ell'' \in \Lambda$ such that $-p_1 = p_1 + \ell + v$, $-p_2 = p_2 + \ell' + v$ and $-p_3 = p_3 + \ell'' + v$, in this case $2(p_i - p_j) \in \Lambda$ for all $1 \leq i \neq j \leq 3$. Or, there exist $\ell, \ell' \in \Lambda$ and $\{i, j, k\} = \{1, 2, 3\}$ such that $-p_i = p_j + \ell + v$, and $-p_k = p_k + \ell' + v$, in this case $-p_j = p_i + \ell + v$ and we must have $-2p_k + p_i + p_j \in \Lambda$. Note that $\Lambda \subseteq \mathbb{Z}^6$, so an easy computation shows that none of the above cases happen. This proves that (G, R) is not reflection invariant. $\square$

Example 5. *Let (G, R) be an arithmetical graph, where G is a graph obtained from K_4 where $V(K_4) = \{v_0, v_1, v_2, v_3\}$, by subdividing the edge v_2v_3 twice. The multiplicity of the vertices v_0 and v_1 are 2 and 4 respectively, and the multiplicity of the other vertices are 3. Then (G, R) is uniform, but not reflection invariant.*

Proof. Let $P = v_2v_4v_5v_3$ be the induced path connecting v_2 to v_3, i.e., the path obtained by subdividing the edge v_2v_3 in the graph K_4.

Let $\nu_1 = -\chi_{\{v_0\}} + \chi_{\{v_2,v_4\}}$, $\nu_2 = -\chi_{\{v_0\}} + 2\chi_{\{v_2\}}$ and $\nu_3 = -\chi_{\{v_0\}} + 2\chi_{\{v_3\}}$. We claim that $\mathcal{E} = \{\nu_1, \nu_2, \nu_3\}$ is the set of extreme v_0-reduced divisors of (G, R). By running the generalized Dhar's algorithm on each ν_i, $1 \leq i \leq 3$, it is not hard to see that $\nu_1 \sim -\chi_{\{v_0\}} + \chi_{\{v_3,v_5\}}$, $\nu_2 \sim -\chi_{\{v_0\}} + \chi_{\{v_3,v_4\}}$ and $\nu_3 \sim -\chi_{\{v_0\}} + \chi_{\{v_2,v_5\}}$.

We will leave the details of the fact that ν_i, $1 \leq i \leq 3$ is v_0-reduced to the reader. (It follows from Lemma 5.3.11, or case analysis similar to that one used in the proof of the Example 4.) It is easy to compute that $g_0 = 7$, and for all $\nu \in \mathcal{E}$ and $0 \leq i \leq 5$, $\deg_R(\nu + \chi_{\{v_i\}}) \geq 7$. Now, Theorem 5.4.2 implies that $\nu + \chi_{\{v_i\}}$ is equivalent to an effective divisor. This shows that ν_i, $1 \leq i \leq 3$ is extreme v_0-reduced.

To finish the proof of the claim, it is enough to show that if ν is extreme v_0-reduced divisor, then $\nu \in \mathcal{E}$. Note that $\nu(v_1) = 0$ since otherwise $\nu - Qf \geq 0$ where $f = \chi_{\{v_0\}} + 3\chi_{\{v_1\}} + 2\chi_{\{v_2,v_3,v_4,v_5\}}$. Also, note that if $\nu(v_2) \geq 1$ and $\nu(v_3) \geq 1$, then $\nu - Qf \geq \chi_{\{v_1\}}$ where $f = \chi_{\{v_0,\ldots,v_5\}}$. This shows that there exists $1 \leq i \leq 3$ such that $\nu = \nu_i$ or $\nu \sim \nu_i$.

The uniformity of (G, R) immediately follows from the fact that for all $\nu \in \mathcal{E}$, $\deg_R(\nu) = 4$.

For proving the fact that (G, R) is not reflection invariant, we apply a similar argument we used in the proof of Example 4. Let $\mathcal{P} = \{p_1, p_2, p_3\}$ be the same set as defined in Example 4. An easy computation shows that $p_1 = \frac{1}{3}(-2, -1, 4, -1, 4, -1), p_2 =$

$\frac{1}{3}(-2,-1,7,-1,1,-1)$ and $p_3 = \frac{1}{5}(-4,-3,1,7,1,-3)$. For seeking a contradiction, assume there exists $v \in \mathbb{R}^6$ such that $-Crit(\Lambda) = Crit(\Lambda) + v$. Either there exist $\ell, \ell', \ell'' \in \Lambda$ such that $-p_1 = p_1 + \ell + v$, $-p_2 = p_2 + \ell' + v$ and $-p_3 = p_3 + \ell'' + v$, in this case $2(p_i - p_j) \in \Lambda$ for all $1 \leq i \neq j \leq 3$. Otherwise there exist $\ell, \ell' \in \Lambda$ and $\{i,j,k\} = \{1,2,3\}$ such that $-p_i = p_j + \ell + v$, and $-p_k = p_k + \ell' + v$, in this case $-p_j = p_i + \ell + v$ and we must have $-2p_k + p_i + p_j \in \Lambda$. Note that $\Lambda \subseteq \mathbb{Z}^6$, so an easy computation shows that none of the above cases occur. This proves that (G,R) is not reflection invariant. $\square$

Example 6. *Suppose $R = (r_0, r_1, r_2) = (1,2,3)$. Let (G,R) be an arithmetical graph where G is a graph with vertex set $\{v_0, v_1, v_2\}$ such that the multiplicity of v_i is r_i and v_i is connected to v_j with $r_i r_j$ edges for all $0 \leq i \neq j \leq 2$. Then (G,R) is not uniform, but it is reflection invariant.*

Proof. We claim that $\nu_1 = -\chi_{\{v_0\}} + 3\chi_{\{v_1\}} + 2\chi_{\{v_2\}}$ and $\nu_2 = -\chi_{\{v_0\}} + \chi_{\{v_1\}} + 3\chi_{\{v_2\}}$ are the only extreme v_0-reduced divisors. Suppose ν is an extreme v_0-reduced divisor. Lemma 5.3.10 (ii) implies that $\nu(v_0) = -1$. It is not hard to see that $\nu(v_1) \leq 3$ and $\nu(v_2) \leq 3$, otherwise $\nu - Qf$ is effective where $f = \chi_{\{v_1,v_2\}}$ and $f = \chi_{\{v_1\}} + 2\chi_{\{v_2\}}$ respectively. Moreover, if $D = -\chi_{\{v_0\}} + 2\chi_{\{v_1\}} + 3\chi_{\{v_2\}}$, then $D - Qf$ is effective where $f' = 2\chi_{\{v_1\}} + 3\chi_{\{v_2\}}$. Therefore the only possible extreme divisors are ν_1 and ν_2. By running the generalized Dhar's algorithm on ν_1 and ν_2, and applying Lemma 5.3.11, one can check that ν_1 are ν_2 are v_0-reduced and therefore they are not equivalent to effective divisors. Note that the above computation shows that we already checked some of the different possible firing strategies in a run of the generalized Dhar's Algorithm on ν_1 and ν_2.

So, we claim that if an arithmetical graph (G,R) has only two v_0-reduced divisors then (G,R) is reflection invariant. Let Λ be the lattice spanned by the Laplacian of (G,R) and $\mathcal{E}$ be the set of extreme divisors of Λ. By applying Lemma 5.3.6 and (ii) of Lemma 5.3.10, we conclude that $Ext(\Sigma(\Lambda)) = \{\nu + \ell : \ell \in \Lambda, \nu \in \mathcal{E}\}$. Corollary 5.2.23

implies $Crit(\Lambda) = \mathcal{P} + \Lambda$ where $\mathcal{P} = \{\pi(\nu + \vec{\mathbf{1}}) : \nu \in \mathcal{E}\}$. Let ν_1 and ν_2 be the only extreme v_0-reduced divisors of (G, R) and $p_1 = \pi(\nu_1 + \vec{\mathbf{1}})$ and $p_2 = \pi(\nu_2 + \vec{\mathbf{1}})$. For proving the claim its enough to show that $-Crit(\Lambda) = Crit(\Lambda) + v$ where $v = -p_1 - p_2$. Assume $p \in Crit(\Lambda)$, therefore there exists $1 \leq i \leq 2$ and $\ell \in \Lambda$ such that $p = p_i + \ell$. Now, it is easy to see that $p_i + \ell + v = -p_j + \ell = -(p_j - \ell)$ where $j = -i + 3$ and $p_j - \ell \in Crit(\Lambda)$. This completes the proof of the claim.

So by a similar argument mentioned in proof of Example 5, (G, R) is reflection invariant. Since $\deg_R(\nu) = 11$ and $\deg_R(\nu') = 10$, we have $g_{\max} = 12$ and $g_{\min} = 11$. This shows that (G, R) is not uniform.

□

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VITA

Spencer Christopher Foster Backman grew up in Calais, Vermont on a small farm which has been in his family for seven generations. He attended Oberlin College where he studied Mathematics and Physics. During his Junior year at Oberlin, he participated in the Budapest Semesters in Mathematics program where his passion for combinatorics was solidified. After graduating from Oberlin with Honors in Mathematics, he participated in the Math in Moscow program. In 2008, he began his graduate career in the Algorithms Combinatorics and Optimization program at The Georgia Institute of Technology. He visited The University of California Berkeley Department of Mathematics with his advisor Matthew Baker, and spent three semesters there as a Visiting Student Researcher. Additionally, he traveled to Dharamsala, India in the 2012 and 2013 Summers, where he served as a mathematics instructor in the Emory Tibet Science Initiative, teaching mathematics to a group of 50 Tibetan monks.