Measuring Closeness to Singularities of Parallel Manipulators with Application to the Design of Redundant Actuation

A Thesis
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy in Mechanical Engineering

George W. Woodruff School of Mechanical Engineering
Georgia Institute of Technology
March 2004

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Measuring Closeness to Singularities of Parallel Manipulators with Application to the Design of Redundant Actuation

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Date Approved: 1 April 2004
ACKNOWLEDGEMENTS

A wise man once said, “No man stands alone.” I am not different in this endeavor. This work would not have been possible without the support of numerous people. First, I would like to acknowledge my wife, Mara, for her loving support through this entire “process” who without I would have certainly lost my sanity. I also would like to thank all of the members of the Robotic Mechanisms Laboratory, past and present, who have helped me overcome my mental blocks by their numerous intellectual boxing matches. I would also be remiss if I didn’t mention both the financial and scholastic support from my advisor, Dr. Imme Ebert-Uphoff as well as her advice to a periodic skeptic. I would also like to thank Dr. Harvey Lipkin for his patience in explaining screw theory to a sometimes very stubborn student. Lastly, I would like to thank my reading committee, Dr. Shreyes Melkote, Dr. Frank Dellaert, and Dr. Michael Stanisic for their comments on my dissertation and help along the way.
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LIST OF SYMBOLS

\[0_{n \times m}\] n by m zero matrix
\[1_{n \times m}\] n by m identity matrix
\(A, A_i\) Leg Jacobian matrices and submatrices
\(B\) Platform Jacobian matrix or constraint equations for dynamics
\(A, B, C, D\) Points
\(C\) Dynamic damping matrix
\(c\) Constraint equations
\(c_i\) Intermediate variables
\(D\) Weighting matrix for screws in axis coordinates
\(E\) Weighting matrix for screws in ray coordinates
\(E\) Young’s Modulus
\(\mathcal{F}_i\) Coordinate frames
\(F_{ee}\) Force at the end effector
\(f\) Components of the force vector
\(f\) General multivariable function
\(f\) Force vector
\(f_{for}\) Forward kinematic function
\(f_{inv}\) Inverse kinematic function
\(F\) Optimization cost function
\(G\) Grasp map
\(g\) Vector of gravitational (potential force) terms
\(h\) Constraint equations
\(I\) Area moment of inertia (depending on context)
\(I_n, I_n\) Planar or spatial mass moment of inertia
\(i\) Index
\(J\) Jacobian matrix where \(J = A^{-1}B\)
\(J_{serial}\) Serial robot Jacobian matrix
$K$ General stiffness matrix
$K_d$ Derivative gain matrix
$K_p$ Proportional gain matrix
$\mathcal{K}$ Diagonal stiffness matrix
$k$ Stiffness
$L$ Length
$L$ Langrangian
$l_i$ Lengths of links
$M$ Inertia (mass) matrix
$M_{ee}$ Inertia (mass) matrix of the end effector
$M_{ee}$ Moment at the end effector
$\mathcal{M}$ Amount of unconstrained motion
$\mathcal{M}$ Combination of inertia matrices
$m$ Moment vector
$m$ Mass
$\mathcal{N}()$ Nullspace of a matrix
$n$ Counting number
$O$ Origin of coordinate frame
$P$ General point in $\mathbb{R}^n$
$P$ Prismatic Joint
$P$ Power
$\mathcal{P}$ Vector of generalized coordinates
$Q$ Vector of generalized joint coordinates
$q$ Vector of minimum number of generalized input coordinates
$R$ Weighting matrix
$R$ Revolute joint
$R_v$ Raleigh quotient
$r$ Radius measurement
$r_\perp$ Perpendicular distance to screw axis

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<table>
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<th>Description</th>
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<tbody>
<tr>
<td>$S$</td>
<td>General positive definite weighting matrix</td>
</tr>
<tr>
<td>$S$</td>
<td>General disk or annuli</td>
</tr>
<tr>
<td>$s$</td>
<td>Screw axis</td>
</tr>
<tr>
<td>$T$</td>
<td>General weighting matrix</td>
</tr>
<tr>
<td>$T$</td>
<td>Kinetic Energy</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$\bar{t}$</td>
<td>Translational distance</td>
</tr>
<tr>
<td>$V$</td>
<td>Potential energy</td>
</tr>
<tr>
<td>$v$</td>
<td>Vector of translational velocity of the end effector</td>
</tr>
<tr>
<td>$\mathbf{v}$</td>
<td>Vector of nonlinear dynamic terms</td>
</tr>
<tr>
<td>$\mathbf{W}_\theta, \mathbf{W}_s$</td>
<td>Weighting matrices for the input and output of a manipulator</td>
</tr>
<tr>
<td>$W$</td>
<td>Work</td>
</tr>
<tr>
<td>$\mathcal{W}$</td>
<td>Workspace</td>
</tr>
<tr>
<td>$w$</td>
<td>Workspace penalty function</td>
</tr>
<tr>
<td>$\mathbf{X}$</td>
<td>Vector of end effector coordinates</td>
</tr>
<tr>
<td>$\mathbf{x}$</td>
<td>Vector of optimized kinematic parameters</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>Cartesian coordinates</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>Intermediate angular coordinates</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Small increment of change</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Clearance region</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Clearance region</td>
</tr>
<tr>
<td>$\eta_i$</td>
<td>Intermediate angular coordinates</td>
</tr>
<tr>
<td>$\phi, \varphi$</td>
<td>End effector angle</td>
</tr>
<tr>
<td>$\mathbf{\theta}$</td>
<td>Vector of joint coordinates</td>
</tr>
<tr>
<td>$\theta_i$</td>
<td>Individual joint coordinates</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Condition number of a matrix</td>
</tr>
<tr>
<td>$\mathbf{\lambda}$</td>
<td>Vector of Lagrange multipliers</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Eigenvalue of a matrix</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Lagrange multiplier</td>
</tr>
</tbody>
</table>
\( \nu \)  
Ratio between a length of vectors

\( \rho \)  
Pitch of a screw

\( \sigma_{\text{min}}, \sigma_{\text{max}} \)  
Minimum and maximum singular values

\( \tau \)  
Vector of input torques

\( \Upsilon \)  
Flipper matrix

\( \omega \)  
Vector of angular velocity of the end effector

\( \xi^t \)  
Twist (screw) vector in axis coordinates

\( \xi^t_{\text{trans}} \)  
Pure translation twist (screw) vector in axis coordinates

\( \xi_{\text{load}} \)  
Load wrench on the end effector

\( \xi^w \)  
Wrench (screw) vector in ray coordinates

\( \cdot \)  
Dot product

\( \oplus \)  
Minkowski sum
SUMMARY

At a platform singularity, a parallel manipulator loses constraint. Adding redundant actuation in an existing leg or new leg can eliminate these types of singularities. However, redundant manipulators have been designed with little attention to frame invariant techniques.

In this dissertation, physically meaningful measures for closeness to singularities in non-redundant manipulators are developed. Two such frameworks are constructed. The first framework is a constrained optimization problem that unifies seemingly unrelated existing measures and facilitates development of new measures. The second is a clearance propagation technique based on workspace generation.

These closeness measures are expanded to include redundancy and thus can be used as objective functions for designing redundant actuation. The constrained optimization framework is applied to a planar three degree of freedom redundant parallel manipulator to show feasibility of the technique.
CHAPTER 1

INTRODUCTION

This dissertation improves upon a particular class of robots commonly known as parallel manipulators. Specifically, two new frameworks to better understand and measure closeness to platform singularities are presented. Several researchers have investigated, but not adequately completed, this task for manipulators with both translational and rotational degrees of freedom. This dissertation develops new methods that overcome the traditional shortcomings and compares these methods.

1.1 Parallel Manipulators

Serial manipulators have open kinematic chains whereas Parallel Manipulators (PMs) contain closed kinematic chains. In other words, the end effector is connected to the ground by one kinematic chain in serial manipulators and more than one kinematic chain in PMs. Examples of PMs are shown in Figure 1.

Because of their parallel nature, the actuators of PMs can be placed at stationary joints which allows for a lower inertia end effector and higher speed than its serial counterparts. Additionally, because of the parallel nature of the legs, PMs are claimed to have larger stiffnesses and accuracy [117].

With the advantages given above, one might wonder why PMs have not received more attention than serial manipulators. First, PMs typically have a small workspace. Second, PMs are typically more complex than their serial counterparts. Finally, PMs suffer from a class of singularities that is unique to them: platform\(^1\) singularities.

\(^{1}\)Platform singularities are also called uncertainty, static, wrench, or forward kinematic singularities.
These singularities cause a loss of constraint in one or more directions and occur within the workspace of the manipulator. As will be shown in this dissertation, singularities can cause large unconstrained motion, a need for large actuator torques/forces, and a double solution to the inverse kinematics.

1.2 Research Purpose

The purpose of this dissertation is to eliminate the platform singularities so that more effective parallel manipulators may be created. The best way to visualize how this can work is to examine a simple example.

Figure 2 shows another simple parallel manipulator known as the 2RR robot\(^2\) (a 5-bar mechanism). In this mechanism, \(\theta_1\) and \(\theta_2\) are actuated, \(\alpha_1\) and \(\alpha_2\) are passive, and the \((x, y)\) position is the output. On the left side is shown a nonsingular

\(^2\)This designation follows the scheme outlined by Merlet [80]. The number in the front represents the number of identical legs. The series of letters represents the joint types (R for revolute, P for prismatic, S for spherical, etc.). The underline denotes which joint in the mechanism is actuated, where all other joints are passive.
configuration, on the right, a singular configuration. Because the secondary links are two force members, it is easily seen that forces perpendicular to the secondary links cannot be resisted. This condition defines a platform singularity.

There are several ways to eliminate this type of singularity. One possible way is to actuate one of the passive joints, $\alpha_1$ or $\alpha_2$, in addition to $\theta_1$ and $\theta_2$. However,
doing so will add inertia to the moving links and thus takes away one of the benefits of a PM. Another option is to add a redundant leg to the manipulator as shown in Figure 3. This choice ensures the inertia of the end effector remains small, but further increases the complexity of the mechanism and may reduce the range of motion.

Some of the questions that a robot designer is concerned with are the following:

- What is the best actuation scheme: adding a redundant leg or actuating a passive joint?
- If choosing a redundant leg, where is the most appropriate place to add the leg? And what dimensions should be used?
- What measures can be used to determine which is better?

This dissertation does not answer these questions specifically, but provides tools that can be used to answer them for any parallel manipulator. The ultimate goal of this research is to provide a framework to determine the “best” design for redundant actuation. Part of this process is to develop scale and frame invariant measures for the closeness to singularities. Whereas in the past, researchers have often focused on a somewhat “blind” application of linear algebra analysis of the Jacobian, this dissertation focuses on the physical symptoms of singularities and how to eliminate these specific problems.

1.3 Organization of Dissertation

Since much research has gone into the study of singularities, there is a wealth of literature on the topic. Chapter 2 outlines the relevant literature. The chapter is divided into several parts that look at singularities from a historical perspective that will show how singularities been treated in the past and why the symptoms of singularities have been overlooked. The chapter also reviews how other researchers have approached the problem and where many of them have gone astray.
Since much of the research is erroneous, this dissertation examines singularities from the beginning. Chapter 3 uses two example manipulators to demonstrate singularities and their effects. The kinematics of the manipulators as well as the differential kinematics (i.e. the Jacobian relationship) are derived. From the Jacobian relationship, singularities are defined and discussed. The relationship between the differential kinematics to statics and topology are also briefly reviewed.

Chapter 4 takes the results of Chapter 3 and explains the symptoms of singularities from a physical standpoint. The chapter then explains how redundancy can effectively remove platform singularities and, more importantly, how redundancy solves the symptoms of platform singularities.

Before singularities can be eliminated, there must be measures that determine how effective singularity elimination is, or in other words, the closeness of singularities must be determined. Chapter 5 outlines the current measures and where they fall short. This chapter provides the groundwork for the development of the general framework to consolidate the measures in Chapters 6 and 7.

Chapter 6 introduces a new way to look at how to measure closeness to singularities. The chapter overviews the process from a physical standpoint. The closeness to singularities is formulated as a constrained optimization problem in which both existing and new measures can be incorporated. In this new framework, there is a choice of a suitable metric that has frame and unit invariance. Several suitable metrics are introduced and examined to show the power of the methodology.

Chapter 7 completes the analysis introduced in Chapter 6. Like the differential kinematics and statics duality, there are two domains in which this problem can be solved: the velocity domain and the force domain. An exhaustive list of possible weighting matrices and the resulting measures are summarized for both domains. The chapter concludes with the scope of what can and cannot be included in the measures. Chapter 8 gives detailed examples of the measures applied to a 3RRR
manipulator and shows how some of the singularities can be missed if the measures are not used properly.

Chapter 9 changes gears somewhat and looks at the measures differently by examining the amount of motion that is allowed at the end effector due to clearances in the joints. A novel method to determine the amount of motion at the end effector is developed which is based on the concepts of workspace generation. This procedure is also applied to a 3RRR manipulator in order to show its ease of use.

Chapter 10 expands the frameworks of Chapters 7 and 9 to redundancy. Because of the way the measures were developed in Chapters 7 and 9, they can easily be expanded to handle redundancy. The measures are applied to a redundant manipulator to show how effective a redundant leg is at eliminating singularities. The next logical step is to use the measures to optimize a 4RRR manipulator and is done so in Chapter 11.

The dissertation concludes in Chapter 12 with the contributions of this research and also gives avenues for future study.
CHAPTER 2

LITERATURE REVIEW

Parallel manipulators have been widely studied. The first six degree of freedom PM was Gough’s universal tyre [sic] testing machine [35] followed by Stewart’s platform [27]. The parallel structure was chosen for its strength, rigidity, accuracy as well as its six degrees of freedom. However, the added benefits did not come without a cost. These mechanisms suffer from singularities within their workspaces. The singularities effectively partition the workspace into smaller usable areas, and typically need to be avoided. Because of these singularities, the usefulness of PMs are limited to small workspace situations. The study and elimination of these singularities allows for the wider use of parallel manipulators.

2.1 Singularities

Chapter 3 and 4 derive the kinematics and singularities for two simple manipulators. While these chapters derive analytical expressions for singularities, the enumeration of the singular configurations is in general very difficult to do. As an example, an entire master’s thesis is dedicated to finding the singularities of a Gough-Stewart platform in an efficient numerical fashion [77].

The physical interpretation of singularities is also important to the analysis of singularities in Chapter 4 and the measures discussed in Chapter 5. Initial research in PMs showed certain configurations that are singular [41, 27]. Merlet [78, 82] (and similarly Hao and McCarthy [37] and Notash and Podhorodeski [92]) simplify locating all the singularities of PMs by establishing a connection between the singularities and Grassman Geometry. Merlet exhibits the singularities by showing that the Jacobian
matrix becomes singular when the space spanned by the transpose of the Jacobian matrix loses rank. By placing the vectors of the transpose of the Jacobian into a projective space, one can look at the rank of the lines spanned by this space. When this space is less than maximal, the manipulator is in a singular configuration. These singularities are represented in “real” space by observing when a certain geometry of the manipulator is present.

2.2 Measures of Closeness to Singularities

Because singular positions are dangerous and control of the manipulator at these positions is very difficult, if not impossible in most instances, several measures have been introduced to help solve the problem of how close a manipulator is to a singular position. This research is highly relevant to the measures in Chapters 5 and as a historical context for the new framework outlined in Chapters 6 and 7.

Most of these methods involve the conditioning number of the matrix. Yoshikawa [123, 124] defines a manipulability measure by determining the condition number of a matrix. Gosselin [32] apply this method to a PM. His method involves finding the stiffness matrix by multiplying, $JJ^T$ by a constant, $k$. He compares the smallest to the largest eigenvalues of the stiffness matrix (i.e. the condition number) as a measure of how close one is to a singular position. The graphs of this measure are called conditioning maps. By doing so, one can determine how close one is to a singular position, and better design a manipulator. The method appears to have merit, but only applies if there are no mixed dimensions. As brought up by Lipkin and Duffy [65] and later further expanded by Duffy [22] and Doty et al. [20], these methods are not unit or frame invariant (i.e. the values change with unit or coordinate changes) if the output or input vectors are of mixed dimensions.

In response to these problems, different manipulability measures that solve the invariance issue have been suggested. Park and Kim [96] solve the problem by adding
weighting matrices. By doing this, the condition number can become frame and unit invariant. However, choosing the weighting matrices is somewhat arbitrary, and is dependent upon how one defines the metric in the joint and task spaces. Typically, the metric is chosen depending upon the application in question. One then can integrate the measure over the joint space manifold to create a global manipulability measure that can be utilized in optimization algorithms. Out of this research comes another method to classify the singularities of parallel manipulators [97]. The approach follows closely to that presented in Gosselin [33] and allows for classification of redundant manipulators rather easily.

Lee et al. [62] also recognize the frame invariance problem and look at it differently. Since the determinant of the Jacobian matrix does not suffer from dimensional problems, they define an index that is the ratio of the determinant of the Jacobian to the maximum of the determinant of the Jacobian matrix for one particular robot. By doing so, the ratio will be between 0 (at a singularity) and 1. This ratio is frame and unit invariant. Because it is a ratio, it is also scale invariant (i.e. it does not change with changes in the size of the manipulator). The exact meaning of the determinant can be visualized for this example as the volume of the octahedron formed by the legs of the manipulator. Therefore, the ratio gives a measure for how close one is to a singularity. However, how it applies to other manipulators is not fully known. Further work in this area (for example see [127]) gives a similar index for a redundant manipulator.

2.3 Self-motion

As discussed in Chapter 4, one of the symptoms with platform singularities is degenerated accuracy. One of the ways to show the severity of a singularity is the concept of self-motion. This concept is a stepping stone for the new procedure presented in Chapter 9.
In an architecture singularity\(^1\) a manipulator can collapse since large regions of the workspace are singular. However, there are other cases where the mechanism’s platforms are not similar in shape, but can still have large unconstrained motion. Husty and Zsombor-Murray [43] show that a special type of Gough-Stewart platform could be created which had one degree of freedom even with all the actuators locked. In this mechanism, the attachment points on the moving platform (end effector) are constrained to move on spherical paths. This type of motion was first studied in the context of general constrained motion by Borel and Bricard\(^2\) in the early 1900’s. In Borel-Bricard motion, the moving platform rotates around a fixed axis, but the translational motion rate changes with its location on the axis. An interesting side note on this mechanism is that the moving platform and base platforms do not need to lie in a plane. Additionally, an arbitrary number of legs could be added to the mechanism (an eighteen legged example was presented), and the end effector could still be move while all the actuators were locked. As long as the attachment points satisfy a specified series of equations, then motion occurs.

Karger and Husty [50] call this phenomenon (large unconstrained motion of the end effector with all the actuators locked) self motion.\(^3\) They outline a method to find all the self motions of the Gough-Stewart platform. The method involves writing the kinematics of the manipulator in terms of the Study parameters (see [10] or [110] for description of Study Parameters). They write out the constraint equations (along with the Study Quadric) for the mechanism in terms of these parameters. In

\(^1\)Ma and Angeles [71] discuss Architecture Singularities which occur in a large section of the workspace, and sometime contain the whole workspace.

\(^2\)These results are contained in the references [9] and [11], but the author has not read these himself.

\(^3\)This designation is in direct contrast to the meaning of self motion for serial manipulators. For serial manipulators, self-motion is defined as motion of the manipulator without moving the end effector. Typically, this designation is used in conjunction with redundantly actuated serial manipulators. Self-motion in relation to parallel manipulators (and how it is used in this dissertation) refers to uncontrolled motion of the end effector while all the input actuators are locked. See [6] for further explanation and discussion of the serial manipulator usage.
this way, they establish an algebraic relationship between the end effector and the base. Basically, they come up with eight equations with eight unknowns (the Study parameters). When the system becomes degenerate, self motion can occur. They establish conditions where this can happen. However, the equations used to establish this relationship can turn out to be rather complex (703 terms in one instance). Additionally, the equations are only good for the case of a specific type of Gough-Stewart platform. Additional papers show the case for other types of mechanisms (see [42] and [49]) but use the same method for characterizing the motion.

Another method that combines the classification and the motion of the end-effector is developed by Wohlhart [121]. In that method, one leg of a manipulator is chosen to have a certain amount of translational backlash (clearance). This clearance creates a certain amount of end effector motion. Several different poses and configurations with different “degrees of shakiness” are investigated.

A similar method is introduced by Karger [48]. Although the method is for a serial manipulator, it is included in this discussion because of its relation to the other methods and the procedure introduced in Chapter 9. Unlike the previous example, he does not examine a specific leg, but investigates the Jacobian matrix itself. The singularities of the Jacobian occur when the determinant of the Jacobian matrix is zero. By expanding the determinant of the Jacobian in a Taylor series, he is able to determine the number of partial derivatives that go to zero. He calculates the partial derivative terms by using screw cross products (basically a Lie Bracket). One then can classify the singularity by how many of the partial derivatives go to zero. Also, by using a Taylor series approximation, the behavior of the singular surface can be investigated. He utilizes the derivatives to approximate up to the second order how the singularity surface locally appears. Several examples show the nature of the singularity around the singularity pose. Additionally, one can derive an equation for the determinant of the Jacobian matrix in its entirety (not just up to the second
order). This method is performed with a general six parameter serial manipulator. For the general case, the determinant has 736 terms. Using this equation, he is able to determine how many joints need to be actuated to get out of a singular configuration. Given that the Jacobian matrix for a parallel manipulator is typically more complex, the method does not lend itself well for a dual analysis on PMs. However, it shows that the singular surface can be studied to gain further insight into the manipulator.

There is yet another method in Tchoń and Muszyński [115] to classify the singular positions that is similar to the method outlined by Karger. This method uses the singularity theory of maps (see [3]) to transform the kinematics of the manipulator into what is called a normal form. Through this transformation, the singularity manifold can be described by a simple algebraic equation. However, the method outlined in Tchoń and Muszyński details a procedure for solving the inverse kinematic problem, i.e. a leg-singularity which is a characteristic of serial manipulators, and not a platform singularity which is contained in PMs. This procedure produces a form utilized in control algorithms that allows the inverse kinematic problem be solved and the manipulator be controlled. It is unclear how this method can be applied to the platform singularity problem and additionally if it would give any benefit to the study of singularities of PMs.

2.4 Measures of Amount of Unconstrained Motion

While self-motion is a binary measure, there exists other ways using the clearances and tolerances to determine the amount of unconstrained motion away from singularities. These tools also form the background for the techniques in Chapter 9.

There are basically four schools of thought in the literature on how to determine the amount of unconstrained end effector motion. Many of these methods were not originally meant for use in the context of parallel manipulator singularities, but rather just general mechanism tolerance analysis. However, they can be extended to use with
parallel manipulators.

1. **Jacobian Matrix Approximation** This technique uses the Jacobian matrix as a linear approximation between actuator clearances and end effector motion [98, 111]. This method does not work for configurations close to singularities since the Jacobian matrix becomes singular. The method also does not take into account the role of the clearances in the passive joints which may be significant. Another significant drawback is in the theory from which the approximation is derived. These theories start with the Jacobian matrix relationship:

\[ \dot{\theta} = J\delta_t \]  

(1)

Taking the finite difference approximation, and assuming that one can “cancel out” the time element gives:

\[ \Delta \theta = J \Delta X \]  

(2)

and then small variations in the input are transferred to the output. While Equation 2 is defined for small \( \Delta X \), the \( X \) is not physically defined. Since the twist contains quasi-coordinates for general 6DOF motion [24], they do not have a clear physical meaning when a finite displacement is taken.

2. **Numerical Approximation** Another method is to model each clearance and tolerance as a homogeneous transformation with certain bounds that are concatenated along the links [119]. Then, a statistical analysis is completed to determine how much the end effector moves. Another technique is to use finite element analysis to determine the amount of deflection [21]. These techniques give a bound on the amount of motion, but not the actual amount of motion at the end effector. Acquiring the models and bounds also requires much calculation and effort to achieve. Additionally, the method does not give an intuitive feel for what is happening.
3. *Equivalent Mechanisms* A method that has been used quite extensively for planar mechanisms is to model the clearances as if it were another link inside the joint [18, 64, 104]. Usually, stochastic methods are employed to the different link lengths. The advantage to this procedure is that it allows both clearances and tolerances to be modeled together.

4. *Static Force Analysis* Another set of techniques use the principle of virtual work. Forces are placed on the end effector, and the corresponding deflection with regards to the constraints are calculated. Variants of this method include those by Innocenti [44] and Parenti-Castelli and Venanzi [95]. However, this method is only good at poses away from singularities, i.e. the mechanism with all the actuators locked is a structure.

The above methods either do not show how much end effector motion can occur at a singularity, or take a stochastic approach to the problem. Chapter 9 proposes a new simple technique to determine the amount of end effector unconstrained motion that is based on techniques that do not rely on the linear Jacobian relationship or statistical techniques.

### 2.5 Redundancy

The goal of this dissertation is to create a mathematical framework for evaluating redundancy in parallel manipulators. This dissertation proposes two new frameworks (Chapters 6 and 7 and Chapter 9) and extends these procedures in Chapter 10. However, this type study has been attempted previously with differing ranges of success.

In Dasgupta and Mruthyunjaya [19], it is shown that adding a redundant actuator or leg decreases the dimension of the singularity manifold. This is in contrast to serial manipulators, where the number of singularities remain the same after adding
redundancy. To distinguish between the two types, redundancy is called force and kinematic redundancy for the parallel and serial cases, respectively. Given this fact, some research has gone into determining where redundancy can be placed to get the most help. In Notash [91], actuators are added to existing links to determine how many would be necessary to totally eliminate the platform singularities. Similarly, it is seen in Matone and Roth [76] that the placement of actuators will affect the singularity manifold. The drawback is that extra motors placed on the legs increase the weight of the links as well as the complexity. Alternately, an additional leg could be added to decrease or eliminate the singularity surface (see [93, 54]). However, adding legs can cause more leg interference, which for PMs is already an issue. The benefit to adding another leg is enhanced robustness to actuator failure. If one actuator fails, the mechanism could still function in a decreased workspace and manner. With the addition of extra actuators on existing or new legs comes the problem of controlling them for which there are many techniques (see the references outlined in Appendix E).

There are successful applications of redundant parallel mechanisms. Merlet [79] outlines the key concepts to be considered in designing and using a redundant parallel manipulator. Reboulet and Durand-Leguay [102] design a redundant spherical parallel manipulator for endoscopic surgery. Kurtz and Hayward [61] also design a spherical manipulator taking into account several different criteria. Kock and Schumacher [54] design a redundant pick and place positional parallel manipulator for high-speed applications. All these methods use condition numbers and isotropy for the design. Since the output of the mechanism and the input of the mechanism are of consistent units, the condition number (as well as isotropy) is well defined and is an appropriate measure. Marquuet et al. [73] develop a redundant planar manipulator based on certain substructures but do not give explanations on the measures utilized for the design. Kim et al. [53] add a redundant actuator to their Eclipse mechanism...
to eliminate a singularity in the middle of the workspace. However, this additional actuator was added after the machine was designed.

On a similar, but albeit different, approach, O’Brien [94] uses redundant braking at singular configurations. Instead of adding an additional actuator, a brake is added in one of the kinematic chains and is actuated when the manipulator becomes close to a singular configuration. This braking gives the mechanism more stability at and near singular configurations. However, a problem arises when trying to escape from a singular configuration.

One of the few works on design of redundant actuation that does not use condition number was by Chan and Ebert-Uphoff [15]. In this work, the nullspace of the Jacobian is utilized to determine the nature of the singularity. They denote the nullspace as the end effector deficiency twist. As its name implies, this twist shows the direction of unconstrained motion of the end effector. By plotting this twist screw over the workspace, a general notion for the nature of the deficiency can be ascertained. Additionally, they investigate how actuation (and more generally leg architectures) could be implemented to eliminate singularities. By use of the reciprocal product, the effectiveness of different actuation schemes is determined and compared.

2.6 Relation to Grasping, Fixturing, and Wire Driven Robots

Redundancy as discussed in this dissertation is analogous to other realms of mechanical engineering. Three of these analogous areas appear in grasping, fixturing, and wire driven robots [26]. Since in all of these realms the constraints are unidirectional, a “redundant” actuator or leg must be supplied to add the necessary constraint to ensure all the unidirectional constraints are satisfied. However, most of the work in these areas focuses on the larger problem of dealing with the unidirectional problem, and not the redundancy problem.
Rimon and Burdick [106, 107] develop second order properties to determine when a grasp was in a closure position. Most of the work in fixturing has to do with the optimal placement of the locating fingers (fixels). Wang [120] propose a method using the determinant of the Jacobian like matrix (i.e. the D optimality condition on the contact information matrix). Cai et al. [14] use variational methods for the robust design of the fixels. Asada and By [4] develop reconfigurable fixtures for a variety of parts.

However, all of these methods deal with determination of a “good” grasp or fixture design at a particular configuration. Therefore, the applicability of these methods to the problem in this dissertation is suspect. One would need to average the procedure over the entire workspace of the manipulator. By doing so, some of the methods become the aforementioned procedures. The rest do not have a clear abstraction to the general case.

Several wire driven robots have been designed, but the field is not as mature as that of grasping or parallel manipulators. Kawamura et al. [51] design a redundant wire driven parallel manipulator called FALCON7, but only provide a small heuristic list on why the particular configuration was chosen. A similar 3 degree of freedom system is designed and called the FALCON4 [17], but again the exact placement of the wires is not discussed. Maeda et al. [72] also design a redundant wire driven parallel manipulator. The manipulator design is optimized for maximum workspace using heuristic rules [114]. Ming and Higuchi [84, 85] develop a planar redundant wire driven manipulator using unknown design criteria. Albus et al. [1] develop a wire driven robot based on a Gough-Stewart platform, but it is not redundantly actuated.

2.7 Literature Summary

In summary, much research has been done on singularities of PMs.
• Their locations have been rigorously studied.

• Much insight has been gained through the use of screw theory and geometry to determine where they are located.

• The singularities have been classified into many different types. The types used in this dissertation include only platform and leg singularities.

• Several metrics have been introduced to evaluate how close one is to a singularity.

• Some researchers have shown that it is possible to have large unconstrained motion at the end effector, but the techniques currently done only show where such a motion exists.

• Further research has led to eliminating singularities by adding actuators either in an existing or added leg.

• Some redundant manipulators have been designed and fabricated using techniques that are applicable to dimensionally consistent problems. A few mixed dimension redundant manipulators have been built mainly using heuristic techniques.

• Related areas include grasping, fixturing, and wire driven robots. However, these realms typically deal with other issues than the design for redundancy.

However, there is no method so far that uses a frame invariant technique to design and analyze a redundant parallel manipulator. With this impetus, this dissertation provides two methodologies to measure closeness to singularities and applies them to the design of a redundantly actuated parallel manipulator.
CHAPTER 3

THE BASICS OF PARALLEL MANIPULATORS

This chapter is dedicated to the calculation and the understanding of singularities for two sample manipulators that are used throughout the dissertation.

3.1  Notational Note

In this document, singularities are defined as in Gosselin and Angeles [33]. A brief introduction to them is provided below, but the reader is advised to go to the reference for more information.

3.2  Kinematics of Parallel Manipulators

The kinematics of a robotic manipulator relates the input angles/lengths, designated by $\theta$, with the output position and orientation of the end effector, designated by $X$. In general, there is a non-linear relationship between the inputs and the outputs:

$$f(\theta, X) = 0$$  \hspace{1cm} (3)

The forward kinematics have the outputs in terms of the inputs, i.e.

$$X = f_{for}(\theta)$$  \hspace{1cm} (4)

The forward kinematics are typically used for simulation, and for parallel robots is, in general, very difficult to do. Similarly, the inverse kinematics have the inputs in terms of the output, i.e.

$$\theta = f_{inv}(X)$$  \hspace{1cm} (5)

The inverse kinematics are typically used for the control of the robot, and are much easier to perform. One word of comment needs to be made here as the difficulty
of the forward and inverse kinematics are reverse of their serial counterparts. For serial robots, the forward kinematics are typically easy, while the inverse kinematics is typically very difficult.

The kinematic relationships for PMs typically are determined using vector loop closure equations. These equations establish the connection between the end effector coordinates to the input angles by systematically creating a vector closure equation for each loop in the mechanism. The passive joint angles are then eliminated yielding a function between the inputs and outputs.

One should note that the forward and inverse kinematics for a parallel manipulator are in general not unique (i.e. there are several solutions). Serial manipulators have unique forward kinematics, but typically have multiple solutions to their inverse kinematics. Since parallel manipulators have serial legs with passive joints, the inverse kinematics have multiple solutions due to their serial legs. Also, since parallel manipulators have passive joints, their forward kinematics have multiple solutions as well.

### 3.2.1 Two Examples of Parallel Manipulator Kinematics

Since two manipulators are utilized in this dissertation quite regularly, the inverse kinematics for these mechanisms are derived here in detail. The forward kinematics are not derived since, in general, it is very difficult to do. The manipulators that are discussed are the 2RR\(^1\) (a 5 bar mechanism) and the 3RRR manipulator. Similar derivations of these kinematics are found in Gosselin and Wang [34], Tsai [116], and Merlet [80].

- **2RR** This manipulator is shown in Figure 4. Let \(\theta_1\) and \(\theta_2\) be the inputs (i.e. \(^1\)Again, this designation follows the scheme outlined by Merlet [80].
Figure 4: 2RR Manipulator with notation

they are actuated). The vector loop through the left leg is

\[
x = l_1 \cos \theta_1 + l_3 \cos \alpha_1 \\
y = l_1 \sin \theta_1 + l_3 \sin \alpha_1
\]  

(6)

and the right leg is:

\[
x = l_2 \cos \theta_2 + l_4 \cos \alpha_2 + l_0 \\
y = l_2 \sin \theta_2 + l_4 \sin \alpha_2
\]  

(7)

The kinematics are computed by isolating the intermediate angles and eliminating them. First, isolating the angles gives:

\[
l_3 \cos \alpha_1 = \underbrace{x - l_1 \cos \theta_1}_{c_1} \\
l_3 \sin \alpha_1 = \underbrace{y - l_1 \sin \theta_1}_{c_2}
\]  

(8)

\[
l_4 \cos \alpha_2 = \underbrace{x - l_2 \cos \theta_2 - l_0}_{c_3} \\
l_4 \sin \alpha_2 = \underbrace{y - l_2 \sin \theta_2}_{c_4}
\]  

(10)
Squaring and adding the equations yields:

\[ l_3^2 = c_1^2 + c_2^2 \]
\[ = x^2 - 2x l_1 \cos \theta_1 + l_1^2 \cos^2 \theta_1 + y^2 - 2y l_1 \sin \theta_1 + l_1^2 \sin^2 \theta_1 \]
\[ = x^2 + y^2 + l_1^2 - 2l_1 (x \cos \theta_1 + y \sin \theta_1) \quad (12) \]
\[ l_4^2 = c_3^2 + c_4^2 \]
\[ = x^2 - 2x l_2 \cos \theta_2 - 2x l_0 + 2l_0 l_2 \cos \theta_2 + l_2^2 \cos^2 \theta_2 + l_0^2 + y^2 \]
\[ - 2l_2 y \sin \theta_2 + l_2^2 \sin^2 \theta_2 \]
\[ = x^2 + y^2 + l_2^2 + l_0^2 - 2l_2 (x \cos \theta_2 + y \sin \theta_2) - 2x l_0 + 2l_0 l_2 \cos \theta_2 \quad (13) \]

One should note that Equations 12 and 13 resemble the law of cosines. In actuality, using trigonometry is an easier geometric way to perform the inverse kinematics for this particular manipulator. However, the vector loop closure method is more general and thus is utilized here.

These two equations contain the unknowns \( \theta_1 \) and \( \theta_2 \) while the rest of the variables are known. Although not done so here, these equations can be solved for \( \theta_1 \) and \( \theta_2 \) by further algebraic manipulations or by a non-linear equation solver. Once the input angles \( \theta_1 \) and \( \theta_2 \) are found, they can be used to find the intermediate angles \( \alpha_1 \) and \( \alpha_2 \) if desired.

- **3RRR** The 3RRR robot is shown in Figure 5. Following the same procedure, the vector loops generate the equations:

\[ x = x_{0i} + l_{1i} \cos \theta_i + l_{2i} \cos \eta_i + l_{3i} \cos \alpha_i \quad i = 1, 2, 3 \quad (14) \]
\[ y = y_{0i} + l_{1i} \sin \theta_i + l_{2i} \sin \eta_i + l_{3i} \sin \alpha_i \quad i = 1, 2, 3 \quad (15) \]

where \((x_{0i}, y_{0i})\) is the location of the \(i\)th motor mount position. Due to the geometry of the manipulator, the \(\alpha\) values are a given function of end effector
Figure 5: 3RRR Manipulator with Notation

angle, φ. For this manipulator,

\[
\begin{align*}
\alpha_1 &= \phi \\
\alpha_2 &= \phi + \frac{2\pi}{3} \\
\alpha_3 &= \phi - \frac{2\pi}{3}
\end{align*}
\] (16)

The problem now becomes eliminating the \(\eta_i\) values from Equations 14 and 15. Isolating the values on the left hand side, the equations become very similar to those of the 2RR robot. This result should not be a surprise as the leg structure of the two robots are very similar. Performing the same operations gives:

\[
\begin{align*}
-l_{2i} \cos \eta_i &= x_{0i} + l_{1i} \cos \theta_i + l_{3i} \cos \alpha_i - x \quad i = 1, 2, 3 \\
-l_{2i} \sin \eta_i &= y_{0i} + l_{1i} \sin \theta_i + l_{3i} \sin \alpha_i - y \quad i = 1, 2, 3
\end{align*}
\] (17)
Squaring the two equations and adding them together yields:

\[
\begin{align*}
    l_2^2 &= x_{0i}^2 + 2x_{0i}l_3 \cos \alpha_i - 2xx_{0i} + l_1^2 \cos^2 \theta_i + 2l_1l_3 \cos \theta_i \cos \alpha_i - 2xl_1 \cos \theta_i \\
    &\quad + l_3^2 \cos^2 \alpha_i - 2xl_3 \cos \alpha_i + x^2 + y_{0i}^2 + 2y_{0i}l_3 \sin \alpha_i - 2yy_{0i} + l_3^2 \sin^2 \theta_i \\
    &\quad + 2l_1l_3 \sin \theta_i \sin \alpha_i - 2yl_1 \sin \theta_i + l_3^2 \sin^2 \alpha_i - 2yl_3 \sin \alpha_i + y^2 \\
    &= x_{0i}^2 + y_{0i}^2 + x^2 + y^2 + l_1^2 + l_3^2 - 2xx_{0i} - 2yy_{0i} + 2l_1l_3 \cos(\theta_i + \alpha_i) \\
    &\quad + 2l_3(x_{0i} \cos \alpha_i + y_{0i} \sin \alpha_i) - 2xl_1(\cos \theta_i + \sin \theta_i) \\
    &\quad - 2l_3(x \cos \alpha_i - y \sin \alpha_i) \\
    &\quad i = 1, 2, 3
\end{align*}
\]

(19)

These equations are solved for \( \theta_i \) using further algebraic simplifications or by a non-linear equation solver. Again, using trigonometry would be an easier way to perform the inverse kinematics for this manipulator. However, this method does not generalize like the vector loop equation method.

One should note that in these examples, care needs to be taken so that all the inverse kinematic solutions are obtained. Each leg can take on an “elbow up” and an “elbow down” configuration. The two solutions are mirror images around the line \( \overline{AC} \) for the 2RR manipulator. In Figure 4, the left leg is in the elbow up configuration while the right side and all legs of Figure 5 are shown in the elbow down configuration. In this dissertation, these are denoted by a plus or minus superscript. Therefore, the solution to the inverse kinematics for the 2RR manipulator to the inverse kinematics is not unique, and in this case, has four solutions: \((\theta_1^+, \theta_2^+)\), \((\theta_1^-, \theta_2^+)\), \((\theta_1^+, \theta_2^-)\), and \((\theta_1^-, \theta_2^-)\). For the 3RRR manipulator, there are 8 solutions.

Another consideration is when the inverse kinematic relationships break down. At these poses (i.e. the end effector position and orientation) there are an infinite number of solutions to the inverse kinematics. For the 2RR robot this only happens in special circumstances. For example, when \( l_1 = l_3 \) and \((x, y)\) is at \((0, 0)\), \( \theta_1 \) can take on any value. These are related to configurations where there are leg singularities which is discussed in the next section.
3.3 Differential Kinematics of Parallel Manipulators: The Jacobian Relationship

The differential kinematics relate the input velocities to the output velocities. These are found by differentiating the kinematics. These equations are, by definition, a linear relationship between the input and output. However, the linear portion depends on the kinematic parameters as well as the pose of the manipulator.

This linear relationship is useful because it finds other singularities that are not captured in the kinematics. At these singularities, the mechanism “gains” or “loses” a degree of freedom depending on the singularity.

Again, let $\theta$ denote the vector of actuated input angles and/or actuated link lengths for a Parallel Manipulator (PM). $\dot{\theta}$ represents the actuated joint velocities and $v, \omega$ represents the linear and angular velocity of the end-effector, respectively. Differentiating Equation 3 one can obtain the well known relationship [33]:

$$A \dot{\theta} = B \begin{bmatrix} v \\ \omega \end{bmatrix},$$

(20)

where $A, B$ are called Jacobian matrices. Since the vector on the right is of mixed dimensions, one can more appropriately designate it as a twist in axis coordinates,$^2$

$$s' = \begin{bmatrix} v \\ \omega \end{bmatrix}.$$  

Assuming that matrix $A$ is invertible, we get

$$\dot{\theta} = (A^{-1}B) \ s' = J \ s',$$

(21)

where $J$ is the combined Jacobian matrix.$^3$

---

$^2$Note that on purpose the notation $\dot{X} = \begin{bmatrix} v \\ \omega \end{bmatrix}$ has been avoided since $\begin{bmatrix} v \\ \omega \end{bmatrix}$ cannot be integrated and thus $X$ would be ill defined. This distinction is very important for the development of screw theory and will be explained in more detail in Section 5.3.

$^3$Note that this relationship is the inverse of what is normally done for serial manipulators where $J = B^{-1}A$. 

25
3.3.1 Two Examples of Parallel Manipulator Jacobians

To further understand the differential kinematics, the Jacobian matrix for the two sample manipulators (the 2RR and 3RRR) are derived.

- 2RR: The Jacobian relationship is found from differentiating the vector loop equations in Equations 6 and 7:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = 
\begin{bmatrix}
-l_1 \sin \theta_1 & -l_3 \sin \alpha_1 \\
l_1 \cos \theta_1 & l_3 \cos \alpha_1
\end{bmatrix}
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\alpha}_1
\end{pmatrix} = \mathbf{A}_1
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\alpha}_1
\end{pmatrix}
\] (22)

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = 
\begin{bmatrix}
-l_2 \sin \theta_2 & -l_4 \sin \alpha_2 \\
l_2 \cos \theta_2 & l_4 \cos \alpha_2
\end{bmatrix}
\begin{pmatrix}
\dot{\theta}_2 \\
\dot{\alpha}_2
\end{pmatrix} = \mathbf{A}_2
\begin{pmatrix}
\dot{\theta}_2 \\
\dot{\alpha}_2
\end{pmatrix}
\] (23)

Inverting the matrices using Cramer’s rule gives,

\[
det \mathbf{A}_1
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\alpha}_1
\end{pmatrix} = 
\begin{bmatrix}
l_3 \cos \alpha_1 & l_3 \sin \alpha_1 \\
-l_1 \cos \theta_1 & -l_1 \sin \theta_1
\end{bmatrix}
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix}
\] (24)

\[
det \mathbf{A}_2
\begin{pmatrix}
\dot{\theta}_2 \\
\dot{\alpha}_2
\end{pmatrix} = 
\begin{bmatrix}
l_4 \cos \alpha_2 & l_4 \sin \alpha_2 \\
-l_2 \cos \theta_2 & -l_2 \sin \theta_2
\end{bmatrix}
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix}
\] (25)

where

\[
det \mathbf{A}_1 = l_1 l_3 (\cos \theta_1 \sin \alpha_1 - \sin \theta_1 \cos \alpha_1)
\]

\[
= l_1 l_3 \sin(\alpha_1 - \theta_1)
\] (26)

\[
det \mathbf{A}_2 = l_2 l_4 (\cos \theta_2 \sin \alpha_2 - \sin \theta_2 \cos \alpha_2)
\]

\[
= l_2 l_4 \sin(\alpha_2 - \theta_2)
\] (27)

Note that the determinants are placed on the left side of the equation in order to obtain the two Jacobian type matrices as well as to avoid dividing by zero.

Taking the first row from each equation yields:

\[
\begin{bmatrix}
det \mathbf{A}_1 & 0 \\
0 & \det \mathbf{A}_2
\end{bmatrix}
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\alpha}_2
\end{pmatrix} = 
\begin{bmatrix}
l_3 \cos \alpha_1 & l_3 \sin \alpha_1 \\
l_4 \cos \alpha_2 & l_4 \sin \alpha_2
\end{bmatrix}
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix}
\] (28)
The only potential problem with this equation is that the values of the intermediate angles $\alpha_1$ and $\alpha_2$ are still in the definition of the $A$ and $B$ matrices. However, these values are easily determined from the inverse kinematics which we assume we know from the inverse kinematic solution given previously. This step is not completed so that the location of the singularities would be easier to see geometrically and is discussed in the next section.

- **3RRR** Differentiating the vector loop equations in Equations 14 and 15 gives

$$
\begin{align*}
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} &=
\begin{bmatrix}
-l_1 \sin \theta_i & -l_2 \sin \eta_i & -l_3 \sin \alpha_i \\
l_1 \cos \theta_i & l_2 \cos \eta_i & l_3 \cos \alpha_i
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_i \\
\dot{\eta}_i \\
\dot{\alpha}_i
\end{bmatrix}
\text{ for } i = 1, 2, 3
\end{align*}
$$

(29)

We need to note that since $\alpha_i$ differs from $\phi$ by a constant value, then

$$\dot{\alpha}_i = \dot{\phi}$$

(30)

Substituting this result and rearranging the equations gives:

$$
\begin{align*}
\begin{bmatrix}
1 & 0 & l_3 \sin \alpha_i \\
0 & 1 & -l_3 \cos \alpha_i
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\phi}
\end{bmatrix} &=
\begin{bmatrix}
-l_1 \sin \theta_i & -l_2 \sin \eta_i \\
l_1 \cos \theta_i & l_2 \cos \eta_i
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_i \\
\dot{\eta}_i
\end{bmatrix}
\end{align*}
$$

(31)

Using Cramer’s rule, the matrix on the right is inverted and yields a equation of the form:

$$
\det A_i \begin{bmatrix}
\dot{\theta}_i \\
\dot{\eta}_i
\end{bmatrix} =
\begin{bmatrix}
l_2 \cos \eta_i & l_2 \sin \eta_i \\
l_1 \cos \theta_i & l_1 \sin \theta_i
\end{bmatrix}
\begin{bmatrix}
1 & 0 & l_3 \sin \alpha_i \\
0 & 1 & -l_3 \cos \alpha_i
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\phi}
\end{bmatrix}
$$

(32)
where

\[
\det \mathbf{A}_i = l_1 l_2 (\sin \eta_i \cos \theta_i - \sin \theta_i \cos \eta_i) = l_1 l_2 \sin(\eta_i - \theta_i)
\]  (33)

Taking the first row of all the columns and applying the trigonometric addition identities gives:

\[
\begin{bmatrix}
\det \mathbf{A}_1 & 0 & 0 \\
0 & \det \mathbf{A}_2 & 0 \\
0 & 0 & \det \mathbf{A}_3
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix}
= \begin{bmatrix}
l_2 \cos \eta_1 & l_2 \sin \eta_1 & l_2 l_3 \sin(\alpha_1 - \eta_1) \\
l_2 \cos \eta_2 & l_2 \sin \eta_2 & l_2 l_3 \sin(\alpha_2 - \eta_2) \\
l_2 \cos \eta_3 & l_2 \sin \eta_3 & l_2 l_3 \sin(\alpha_3 - \eta_3)
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\phi}
\end{bmatrix}
\]  (34)

### 3.4 Definition of Leg and Platform Singularities in Parallel Manipulators

Singular configurations occur if either matrix \( \mathbf{A} \) or \( \mathbf{B} \) is singular \(^4\). If \( \mathbf{A} \) is singular, a **leg singularity** is encountered and the end-effector is over-constrained, i.e. it instantaneously loses at least one degree-of-freedom. For example, there exists a non-zero input velocity, \( \dot{\theta} \), which results in a zero output, \( \mathbf{S}^T \mathbf{f}_0 = \mathbf{0} \). In other words:

\[
\mathbf{A} \dot{\theta} = \mathbf{0}.
\]  (35)

This type of singularity is where the branches of the **inverse kinematics** meet. Typically, it is found at the boundary of the workspace or when a leg folds upon itself. This type of singularity is due to the serial nature of the legs and is discussed extensively in literature.

If \( \mathbf{B} \) is singular, a **platform singularity**\(^5\) is encountered and the end-effector is under-constrained, i.e. the end-effector can move instantaneously even if all actuators

\(^4\)Zlatanov et al. [129] augments these two types to include another singularity called a constraint singularity that occurs in reduced degree of freedom manipulators. This type of singularity is not discussed here due to clarity.

\(^5\)Note these are also called uncertainty, wrench, static, or forward kinematic singularities.
are locked. At these configurations, the manipulator gains an uncontrollable degree of freedom at the end effector. For example, there exists a non-zero output velocity, $\dot{s}^t$, corresponding to a zero input velocity, $\dot{\theta}$, in other words:

$$0 = B\dot{s}^t \quad (36)$$

This pose is where the different branches of the forward kinematics meet. This second type of singularity happens within the workspace.

A third type of singularity happens when both the $A$ and $B$ matrices are singular. This situation was first inadvertently classified as dependent on a certain special design of the manipulator, but was later proven incorrect by Mohammadi et al. [87]. By way of examples, they demonstrate that the third type of singularity does not occur only in “special” mechanism. Rather, it can occur in “regular” mechanisms.

One last thing to note is in defining the differential kinematics to have two Jacobian like matrices (the $A$ and $B$ matrices), neither matrix is unique. This does not cause a problem unless care is not exercised and one inadvertently divides by zero. The number and location of singularities do not change (e.g. the determinant will be zero at the same place), but the magnitude of the determinant does change depending on what side of the equations the constants are placed.

### 3.4.1 Two Examples of Parallel Manipulator Singularities

To further understand how platform singularities act in practice, the 2RR and 3RRR manipulators are again examined from the point of view of singularities.

- **2RR**

  - Leg Singularities: For the 5 bar mechanism, the matrix $A$ is singular if either the terms on the diagonals in Equation 28 are zero. In other words:

    $$\det A_1 = l_1 l_3 \sin(\alpha_1 - \theta_1) = 0 \quad (37)$$
    $$\det A_2 = l_2 l_4 \sin(\alpha_2 - \theta_2) = 0 \quad (38)$$
These equations are singular when:

\[ \alpha_i - \theta_i = n\pi, \quad i = 1, 2 \quad n = 0, \pm 1, \pm 2, \ldots \] (39)

This only happens when either leg is fully extended or folded back upon itself.

- Platform singularities: Platform singularities occur when the \( B \) matrix is singular. This happens when its determinant is zero:

\[
\det B = (l_3 \cos \alpha_1)(l_4 \sin \alpha_2) - (l_3 \sin \alpha_1)(l_4 \cos \alpha_2) \\
= l_3l_4(\cos \alpha_1 \sin \alpha_2 - \sin \alpha_1 \cos \alpha_2) \\
= l_3l_4(\sin \alpha_2 \cos \alpha_1 - \cos \alpha_2 \sin \alpha_1) \\
= l_3l_4 \sin(\alpha_2 - \alpha_1) \] (40)

This equation is zero if

\[ \alpha_1 = \alpha_2 + n\pi \quad n = 0, \pm 1, \pm 2, \ldots \] (41)

This condition occurs when the secondary links align. These conditions are denoted as platform singularities and result in a gain in the instantaneous degree of freedom of the end effector.

- 3R R R R R

- Leg Singularities: For the leg singularities, the \( A \) matrix in Equation 34 goes to zero. This occurs when any of the diagonal elements becomes zero or, in other words, when the determinant of the \( A_i \) matrices becomes zero:

\[
\det A_i = l_1l_2 \sin(\eta_i - \theta_i) = 0 \quad i = 1, 2, 3 \] (42)

Just like in the previous case, this happens when:

\[ \eta_i - \theta_i = n\pi \quad n = 0, \pm 1, \pm 1, \ldots \quad i = 1, 2, 3 \] (43)
Figure 6: 3RRR Manipulator in the two types of platform singularities

and happens when the legs are fully extended or folded back upon themselves.

– Platform singularities: Returning to Equation 34, we need to find when the $B$ matrix becomes singular. That is:

$$
\det B = l_2 \begin{vmatrix} \cos \eta_1 & \sin \eta_1 & l_3 \sin(\alpha_1 - \eta_1) \\ \cos \eta_2 & \sin \eta_2 & l_3 \sin(\alpha_2 - \eta_2) \\ \cos \eta_3 & \sin \eta_3 & l_3 \sin(\alpha_3 - \eta_3) \end{vmatrix} = 0 \quad (44)
$$

Instead of computing the determinant of this matrix, it is better to examine the matrix geometrically. This matrix does not depend upon the input angles or on the first link length. The rows of the matrix are the $x$ and $y$ component of a unit vector along the distal link. The third column is the “moment” of that unit vector around the end effector point. Geometrically, singularities occur when these three lines become dependent upon each other. This can only happen when all three lines intersect at a point or at infinity (i.e. they are all parallel). Figure 6 shows these two cases.
3.5 Statics Duality of the Jacobian Relationship

Before continuing, one word needs to be made with regards to the dual relationship between the kinematic and static equations.\textsuperscript{6} The principle of virtual work is utilized to derive the relationship between the input and output wrenches [80] and results in:

\[
\begin{bmatrix}
\mathbf{f} \\
\mathbf{m} \\
\$^w
\end{bmatrix} = J^T \mathbf{\tau}
\]  

(45)

where \( \mathbf{\tau} \) is a vector of input torques/forces and \( \$^w \) is a vector of output force and moment. This quantity is a wrench screw in ray coordinates, and thus is denoted by the dollar sign. From this equation, it is seen that not only does the Jacobian relate the input and output velocities, but it relates the static input to static output forces and moments. Therefore, there is another interpretation of singularities.

Platform singularities occur when there exists a force/moment in one or more directions that cannot be resisted by the inputs. Referring back at Figure 6, the distal links are two force members and their line of action occurs along their links. When all the forces intersect at one point (the right side of the figure), the manipulator cannot resist a moment around that point. When they all are parallel (the left side of the figure), there exists a force perpendicular to the lines of action which cannot be resisted by the manipulator.

3.6 Platform Singularities from a Topological Viewpoint

From another viewpoint, singularities occur when the multiple solutions to the forward or inverse kinematics coincide. An easy way to see this effect is to use the 2RR manipulator shown in Figure 4 as an example. The specific values for the kinematic parameters are \( l_0 = 2m, \ l_1 = 1.5m, \ l_2 = 1.3m, \) and \( l_4 = l_5 = 1m. \)

\textsuperscript{6}This is sometimes denoted as the kinestatic relationship.
Figure 7: Two inverse kinematic solutions for $\theta_1$ for the 2RR Manipulator

As was already discussed in Section 3.2.1, both the forward and inverse kinematics yield multiple answers. Figure 7 shows the different inverse kinematic solutions for $\theta_1$ over the entire workspace. The leg singularities occur when the forward kinematic solutions meet. In the figure, the two values for $\theta_1$ value are shown over the workspace. When the leg is fully extended or folded back upon itself, the top surface and bottom surface intersect.\(^7\) Recall, this is the same solution that was derived previously from the Jacobian relationship.

Conversely, Figure 8 shows the multiple forward kinematic solutions for the 2RR manipulator. Platform singularities occur when the forward kinematic solutions meet. For this figure, the singular configurations meet within the workspace. Further analysis (not included here) shows that the points very close to each other are indeed identical to the platform singularities from the Jacobian analysis, thus verifying the previous statement.

Figures 7 and 8 exhibit the difficulty in determining when the inverse and forward kinematic solutions coalesce. For higher degree of freedom manipulators, graphical methods are very difficult, if not impossible to construct. However, if the supporting

\(^7\)Although not shown in the figure, finer resolution would show two solutions do coalesce.
Figure 8: X-coordinate of the two forward kinematic Solutions for the 2RR Manipulator

legs are of a specific type, graphical methods could be used to show some of the trends.

Another possible way around such difficulties is to understand the topology underlying in surfaces. This possibility was brought up by Burdick in his dissertation [12]. In this dissertation, the underlying topology of an n DOF **serial** robot with only revolute joints was studied. For this particular case, the underlying topology in the joint space is an n-torus. Because of this special structure, numerous results from topology are extended to this particular robot. However, for parallel robots, this special topology does not exist. Even for a four bar mechanism (i.e. a single DOF manipulator) the kinematic parameters change the underlying topology of the mechanism. In fact, this area is still an active area of research for topologists [83]. Consequently, this aspect of singularities is discussed later in the dissertation only in low DOF manipulators and where applicable.
CHAPTER 4

PARALLEL MANIPULATOR SINGULARITY
ANALYSIS

While the previous chapter looked at the definition of singularities and their physical interpretations, this chapter discusses further how singularities affect the operation of parallel manipulators and why they are typically avoided. These results are then be utilized to develop measures of the closeness to singularities.

There are three basic reasons why singularities become an issue in real life situations. They follow directly from the interpretations of singularities and are: reduced accuracy, large internal forces, and loss of knowledge of solution tree.

4.1 Scope

This dissertation is only concerned with platform singularities. Leg singularities are easier to detect and move outside of the desired workspace by changing leg lengths. However, since platform singularities happen within the workspace, they effectively partition the effective workspace into smaller usable portions. It is this effect that this dissertation is trying to overcome. Therefore, when referring to singularities, platform singularities are what is inferred.

4.2 Performance Problems Caused By Singularities

4.2.1 Degenerated Accuracy

The Jacobian analysis relates the input and output velocities. At a singularity the end effector can have an instantaneous velocity at the output for zero input velocity.
If the differentials inputs and outputs are taken, the Jacobian relations becomes:

\[ \dot{\theta} = J^t \]  
\[ \frac{\Delta \theta}{\Delta t} \approx J \frac{\Delta X}{\Delta t} \]  

“canceling” out the \( \Delta t \) term gives:

\[ \Delta \theta \approx J \Delta X \]  

Therefore, not only does the Jacobian relate the input and output velocities and forces, it also approximates the mechanical advantage between the input and the output. When the manipulator is at a singular pose, the end effector only has an instantaneous amount of motion, in other words, no motion at all. However, all manipulators have some amount of clearances (and similarly some compliance) and allow some finite motion at the end effector. This motion is denoted as the *unconstrained end effector motion*. The issue now becomes how to find out how much unconstrained end effector motion is allowed.

### 4.2.2 Large Internal Forces

Perhaps the best way to observe how platform singularities affect the internal forces is through the use of an example. A MATLAB simulation is created that shows how the static forces required by the input forces (actuators) change when a manipulator goes through a singularity.

For the 3RRR robot shown in Figure 1a, a static force analysis is performed to show the forces required by actuators. Using \( l_a = l_b = 1.1m \) and \( l_c = 0.18m \) with motor positions at \((0,0),(3m,0)\) and \((1.5m,2.6m)\), the manipulator position is held constant while it is rotated around 360°. Arbitrary end effector forces and moment of \( F_x = 6N, F_y = 7N \) and \( M = 8Nm \) are applied to the end effector, and a graph of the required static actuator torques is given in Figure 9a and b.

---

1Care should be exercised with this equation. Since the output contains angular velocities, a quasi-velocity, it has no real integral.
Figure 9: Static equilibrium input forces for the 3RRR manipulator.

These plots show the required forces approach infinity at singularities (approximately 150° and 322°). Near singularities, the magnitude of the input forces are very large. A interpretation of this result is that the force transmission (i.e. the mechanical advantage) is very low at the singularities. Therefore, any inertial forces at the end effector would require very large forces at the actuators. Also, the forces change in direction as the manipulator goes through the platform singularity.

In reality, the manipulator is not static, nor can it provide infinite forces. The question then becomes, how does the force deficiency manifest itself at singularities? To answer such a question, the dynamics of the manipulator needs to be taken into account. Returning to the 3RRR robot, it is shown how this can be manifested. The robot dynamics are derived and included in Appendix A. This dynamic model is placed into a simulation (included in Appendix B) with independent PD joint control. To be more real, torque limits are placed on the actuators along with 100:1 ratio gear heads.

The manipulator starts at $x = 1.5m, y = 1.3m, \phi = 0^\circ$ and rotates to $\phi = 5^\circ$. After settling, the manipulator is again rotated through a singularity (at approximately
Figure 10: Position results from the dynamic analysis of the 3RRR manipulator

Figure 11: Angular output from the dynamic analysis of the 3RRR manipulator

\( \phi = -7^\circ \) to \( \phi = -30^\circ \). The prescribed trajectory moves the manipulator through a singularity in order to examine the internal forces in the legs. Figures 10 and 11 show the pose of the manipulator as it goes through the specified trajectory. Figure 12 shows the magnitude of the forces that would result in the joints in the first leg.\(^2\) The actuators also reach their maximum value along this trajectory. In fact, the motors in this example are back-driven. Back-driving gears in many cases cannot be done

\(^2\)The first leg is only shown as the other legs exhibit similar results due to the symmetry of the mechanism.
**Figure 12:** Forces in the joints for leg 1 from the dynamic analysis of the 3RRR manipulator

without serious damage to the teeth because of the forces internal to the gear heads. Also, the forces typically change sign through the singularity. Even if the motors and links were designed to withstand the forces generated as the manipulator goes through a singularity, fatigue life of the bearings and links is still an issue.

### 4.2.3 Solution Tree

The solutions for the forward kinematics coincide at singularities. In practice, the two different solutions cause control problems. If the manipulator moves close to a singularity and slips into the other solution due to overshoot, the manipulator mechanically is not where the control believes it is. The manipulator will be at a different end effector position than predicted and could lead the manipulator into poses where it was not intended to operate.

Using the dynamic analysis tools from the previous section, one can see how this problem manifests itself. Figure 11 shows how the angle of the end effector approaches the second forward kinematic solution. The angular value approaches a positive angular value before jumping back into the desired solution at \(-30^\circ\). If the desired trajectory is changed slightly (not shown here), the manipulator jumps to the
second unintended solution.\footnote{One might be tempted to believe that to go from one part of the solution tree to another would require traversing through a singularity. However, as is shown by Innocenti \cite{45}, one can get from one solution to the other without going through a singularity.}

4.3 Solution Methods to Singularity Problems

The preceding sections show that singularities are not only a mathematical problem, but also cause issues with accuracy and control. Therefore, the focus is now how to deal with singularities. There are two different ways to approach the problem: avoid them or eliminate them.

Avoiding the singularities drastically reduces the workspace of the manipulator, because not only should the singular poses be avoided, the poses \emph{near} singular ones should be avoided as well. As is shown in the force and solution tree analysis, the manipulator loses stiffness as it approaches singularities. Therefore, the effective natural frequency goes to zero at the singularities. Thus, as the manipulator approaches singularities, the amount of vibration increases and could force the manipulator into a singularity. Therefore, it is desirable to eliminate singularities to increase the usable workspace of the manipulator.

4.3.1 Relation Between Redundant Actuation and Singularity Elimination

Singularities can be eliminated by using redundant actuation, but this result is not trivial. For serial robots, adding actuation does not decrease the number of the singularities. However, as was shown by Dasgupta and Mruthyunjaya \cite{19}\footnote{In this article, the analysis concludes that getting from one assembly mode to another necessitate passing through a singularity. This assumption is proven incorrect by Innocenti \cite{45}. However, the remainder of the analysis is believed to be sound.} adding actuation to a parallel robot can decrease the degree of the singularity manifold.

Perhaps the easiest way to see this phenomenon is to look at the Jacobian relationships for both serial and parallel robots. As is shown in Equation 21, the
combined Jacobian matrix for parallel manipulators is:

\[
\dot{\theta} = J \$^t.
\]

(49)

For a non-redundant manipulator, the \( J \) matrix is \( n \times n \) where \( n \) is the degree of freedom of the manipulator. The manipulator is singular when the nullspace of the Jacobian matrix no longer is the null set. Since in general the Jacobian matrix is a function of the pose of the manipulator, singularities are dependent upon the pose of the manipulator. If the manipulator is redundant, the Jacobian matrix effectively adds a row to become \( m \times n \) where \( m > n \).\(^5\) A singularity is encountered when the nullspace is no longer the null set. So, if thought is put into where the extra actuator is placed, the manipulator’s nullspace is effectively reduced.

The combined Jacobian matrix for a serial robot is:

\[
\$^t = J_{\text{serial}} \dot{\theta}
\]

(50)

For the non-redundant case, the Jacobian matrix is again \( n \times n \) where \( n \) is the degree of freedom of the system. Just like the case with parallel robots, leg singularities occur when the nullspace of \( J_{\text{serial}} \) is no longer trivial. Adding *kinematic* redundancy adds a column to the matrix.\(^6\) Therefore, adding a column will not decrease the dimension of the nullspace.

### 4.3.2 Benefits and Drawbacks of Redundant Actuation

Knowing that singularities can be eliminated, let’s revisit the problems that are encountered as one goes through singularities. They are: degenerated accuracy, large internal forces, and loss of solution tree. How each is affected is examined below to show how redundant actuation helps.

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\(^5\)The reason the redundancy adds a row can be seen by a simple analysis of the size of the matrices. The output vector \( \$^t \) always remains at size \( n \) where the \( \dot{\theta} \) vector will add components to be size \( m \). Therefore, the Jacobian has to be \( m \times n \).

\(^6\)Again, this can be seen with a simple analysis of the sizes of all the matrices.
1. **Degenerated Accuracy** How redundant actuation helps the accuracy of the manipulator is difficult to observe. Recall, the linear approximation of the Jacobian equation:

$$\Delta \theta = J \Delta X$$  \hspace{1cm} (51)

If the Jacobian matrix is no longer singular due to redundancy, then the nullspace of the matrix is trivial. However, the linear relationship only applies at non-singular configurations. Therefore, it is very difficult to compare the resulting non-singular (redundant) pose with the singular (non-redundant) poses. Chapter 9 is dedicated to quantifying this error.

2. **Large Internal Forces** Obviously, if another actuator is added to the mechanism via another leg or in an existing leg, the mechanical advantage is drastically improved. The drawback of the improved force characteristics is the over-actuation of the manipulator. $J^T$ becomes an $m \times n$ matrix and there is not a unique solution to the inverse problem of how to determine the input forces for a given output wrench. In fact, if care is not exercised, redundant actuation can cause high internal forces - the very effect it is trying to eliminate. However, grasping and cooperating robots have redundant actuation and appropriate control methods. Therefore, this dissertation does not cover these methods, but leaves it to the references (contained in Appendix E) if more information is desired by the reader.

3. **Loss of Forward Solution Tree** Adding redundant actuation allows the number of solutions to the forward kinematics to be decreased by the degree of redundancy. In most cases, the solution to the forward kinematic analysis is unique.\(^7\) However, care must now be exercised to ensure that the inputs are a

\(^7\)It is not proven here, but at the remaining singular configurations, there still are multiple solutions to the forward kinematics.
consistent set, i.e. the joint positions actually have a solution.

4. **Workspace and Interference** On the practical side, two areas that are affected by adding an extra leg is the workspace and interference. Adding extra constraints to the manipulator necessarily forces the workspace to be smaller. Also, adding a redundant leg may also has an increased chance of interference with other legs. These effects need to be constantly weighed against the benefits of redundant actuation.

### 4.4 Conclusions

This chapter shows that singularities cause three separate problems when trying to use parallel manipulators: degenerated accuracy, large internal forces, and changes in solution tree. Therefore, it is desirable to eliminate these effects. It is shown that adding redundant actuators can actually decrease the dimension of the singularity manifold, unlike for serial robots. The next step is to determine how to efficiently choose kinematic parameters of the actuation. Before doing this, measures of effectiveness of redundant actuation need to be created and is the focus of the next chapter.
CHAPTER 5

DIFFERENTIAL KINEMATIC MEASURES OF SINGULARITIES

The goal of this research is to introduce measures of effectiveness for singularity elimination. A large sub-goal involves developing measures of how close a pose is to a singularity. More precisely, this chapter examines how this problem has been attempted in the past. The explanations in this chapter will be brief since the next chapter focuses on a new generalized theory that combines these measures into one framework. Also, the loss of solution tree is no longer discussed because it is still an active area of topologists.

5.1 Criteria for Closeness to Singularity Measures

Before beginning, criteria need to be established by which the measures put forth in this chapter should be gauged against. The criteria are frame invariance, scale invariance, and unit invariance. These three are defined as follows:

- **Frame Invariant** A quantity that does not change due to a change of the coordinate system’s location or orientation is said to be frame invariant.

  **Example** The distance between two points is frame invariant because it is defined as the difference between two vectors. Figure 13 graphically depicts this property. The distance is the magnitude of the vector $\overrightarrow{AB}$. However, this vector can be expressed in either frame as a difference between two vectors. That is in frame 1, $\mathcal{F}_1$:

  \[
  \overrightarrow{AB} = \overrightarrow{O_1B} - \overrightarrow{O_1A}
  \]  
  (52)
Figure 13: Distance between points $A$ and $B$ using two different coordinate systems or in frame 2, $\mathcal{F}_2$:

$$\overrightarrow{AB} = \overrightarrow{O_2B} - \overrightarrow{O_2A}$$  \hspace{1cm} (53)

Obviously, this vector is the same no matter what coordinate system it is expressed in, i.e. it is frame invariant. However, the vectors that go from the origin to the points $A$ and $B$ ($\overrightarrow{O_1A}$ and $\overrightarrow{O_1B}$) depend on the frame they are expressed from.

- **Unit Invariant** A quantity that does not change due to a change in the units of the coordinate frame (e.g. meters or inches) is said to be *unit invariant.*

**Example** Using the example in Figure 13, the distance between the points stays the same if different units are used. In other words, if the distance between the points is $1m$, then if centimeters are used, it becomes $100cm$. Even though the number before the units change, the units change accordingly. This problem may seem absurdly simple to many readers, however, it is this property that many measures do not obtain. Some of the measures ignore the units and try

---

1Some researchers classify unit invariance as a sub-class of frame invariance. These concepts are kept separate in this thesis to stress the point that units must be considered in all situations.
to define ratios that make no sense if the units are carried through. In other words, a unitless measure is not unit invariant if changing the measuring system changes the measure.

However, this statement does not mean that all ratios are bad. For example, the measure, $\nu$:

\[
\nu = \frac{\|\overrightarrow{AB}\|}{\|\overrightarrow{CD}_{\text{char}}\|}
\]  

(54)

where $\overrightarrow{CD}_{\text{char}}$ is some characteristic length is unit invariant. No matter what units are used, the measure is the same and still unitless.

- **Scale Invariant** A quantity that does not change due to a change of the size (or scaling) of the manipulator is said to be **scale invariant**.

**Example** Many robots are based on the same kinematic structure. The kinematics are the same no matter how large the mechanism becomes. For a measure to be scale invariant, the size of the robot does not matter, and the value of the measure remains the same.

In order to be meaningful, a measure should have frame and unit invariance. In other words, the measure should be the same no matter what coordinate system or measuring system is chosen. Scale invariance in the context of this dissertation is not necessary.

### 5.2 Measures Based on Linear Algebra

Mathematically, singularities are defined where the Jacobian relationship degenerates. Therefore, the most obvious method to evaluate singularities is to use techniques from linear algebra.
5.2.1 Condition Number

As was first suggested by Yoshikawa [124] in the context of serial robots, the condition number is a local measure of how close one is to a singularity. The condition number is defined as:

$$ \kappa = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} $$  \hspace{1cm} (55)

where $\sigma_{\text{min}}$ and $\sigma_{\text{max}}$ are the minimum and maximum singular values of the Jacobian matrix, $\mathbf{J}$, described earlier. The singular values are determined using a Singular Value Decomposition (SVD) routine.\(^2\)

5.2.1.1 Problems

As was pointed out in the literature review, this measure is not frame or unit invariant if the entries of the Jacobian matrix are not uniform [65, 22, 20]. That is, all the elements of the Jacobian matrix must have the same units.

Because of the wealth of literature and discussion on the topic, an example exhibiting the problems with the condition number is explained. The purpose of the example is two-fold. The first is to confirm by example that the condition is not unit invariant.\(^3\) Even though one may think that the units will “cancel out” as shown in Equation 55, this example shows the contrary. The problem occurs before one can even apply Equation 55, as the units on the $\sigma_{\text{min}}$ and $\sigma_{\text{max}}$ are not even defined and thus cannot be used. The second purpose is to refute a theory of some researchers that the condition number still shows the same general trend even if it is not well defined. This example will show that this is not the case. The trend of the condition number is vastly different depending on the frame that is used.

The example is conducted on the 2RR parallel manipulator (the five bar

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\(^2\)For a short description of the SVD and an example routine to calculate the SVD, see Press et al. [101]

\(^3\)A natural follow up to the problem is frame invariance. It is not proven here, but a simple example can be devised to show this problem as well.
mechanism) shown in Figure 4 and discussed in length in Chapter 3 and 4. In the previous chapter, the output is chosen to be the \((x, y)\) position of the joint \(C\). However, one has the freedom to choose the output of the mechanism differently. Two cases are considered in this section. The first case has an output such that the Jacobian matrix has consistent units (i.e. the output is the \((x, y)\) position of the joint \(C\)). In the second case, the output causes the Jacobian matrix to have mixed units (i.e. the \(x\) position of joint \(C\) and the angle of the secondary link, \(l_2\)).

**Case 1** Input: \(\theta_1\) and \(\theta_2\). Output: \((x, y)\) position of point \(C\).

In this particular case, Chapter 3 shows:

\[
\begin{bmatrix}
\det A_1 & 0 \\
0 & \det A_2
\end{bmatrix}
\begin{Bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{Bmatrix}
= \begin{bmatrix}
l_3 \cos \alpha_1 & l_3 \sin \alpha_1 \\
l_4 \cos \alpha_2 & l_4 \sin \alpha_2
\end{bmatrix}
\begin{Bmatrix}
\dot{x} \\
\dot{y}
\end{Bmatrix}
\]

where

\[
\det A_1 = l_1 l_3 (\cos \theta_1 \sin \alpha_1 - \sin \theta_1 \cos \alpha_1) \tag{57}
\]

\[
\det A_2 = l_2 l_4 (\cos \theta_2 \sin \alpha_2 - \sin \theta_2 \cos \alpha_2) \tag{58}
\]

Assuming \(A\) is of full rank (i.e. not in a leg singularity), the \(A\) matrix is inverted to find the combined Jacobian matrix:

\[
\begin{Bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{Bmatrix} = -A^{-1}B \begin{Bmatrix}
\dot{x} \\
\dot{y}
\end{Bmatrix} = -\begin{bmatrix}
\frac{1}{\det A_1} l_3 \cos \alpha_1 & \frac{1}{\det A_1} l_3 \sin \alpha_1 \\
\frac{1}{\det A_2} l_4 \cos \alpha_2 & \frac{1}{\det A_2} l_4 \sin \alpha_2
\end{bmatrix}
\begin{Bmatrix}
\dot{x} \\
\dot{y}
\end{Bmatrix}
\]

\[
= J_1 \begin{Bmatrix}
\dot{x} \\
\dot{y}
\end{Bmatrix} \tag{59}
\]

The condition number, \(\kappa\), is calculated by performing a SVD on the Jacobian matrix, \(J\), for all points in the workspace. Figure 14a shows the condition number
Case 2 Input: $\theta_1$ and $\theta_2$. Output: $x$ position of point C, and the angle $\alpha_1$.

Reverting back to Equations 23 in the kinematics section, one can solve the equations for $\dot{x}$ and $\dot{\alpha}_1$. After some algebra, the equations become:

$$
\begin{bmatrix}
-l_1 \cos \theta_1 & l_2 \cos \theta_2 - l_2 \cot \alpha_2 \sin \theta_2 \\
-l_1 \sin \theta_1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix}
=
\begin{bmatrix}
\cot \alpha_2 & l_3 \cos \alpha_1 \\
1 & l_3 \sin \alpha_1
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{\alpha}_1
\end{bmatrix}
$$

(60)
Figure 15: Case 2: Condition number for meters and millimeters for a mixed unit Jacobian

Inverting the $A$ matrix yields:

$$\begin{align*}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 
\end{bmatrix} &= -A^{-1}B \begin{bmatrix}
\dot{x} \\
\dot{\alpha}
\end{bmatrix} = J_2 \begin{bmatrix}
\dot{x} \\
\dot{\alpha}
\end{bmatrix}
\end{align*}$$

(61)

Ignoring the dimensions for a moment, and performing a similar analysis on $J_2$ as above, similar graphs are created for the condition number, $\kappa$, over the workspace. Figure 15a and 15b show the results for the condition number. Before continuing, some comments need to be made about Figure 15b. Figure 15b has a very low condition number over all of the workspace, and thus shows very little contours. The unlabeled solid lines are where the condition number is zero. Also, near the point (2, 0), the condition number has a spike that is approximately 0.013.

First observation of the plots shows Figure 15a has a different trend from Figures 14a and 14b. However, this is to be expected. The output chosen is different, and thus its condition number plot should be different. However, what may not be expected is how different Figure 15b is from 15a. The lack of paths on Figure 15b is due to
the fact that the majority of the workspace is very ill conditioned. An analysis (not presented here) shows that the maximum condition number is less than 0.014.

Even further, the trend in the condition number does not even remain the same. For example the location of the maximum condition number moves as one changes units. However, the singularity curve remains the same on both plots (i.e. the zero contour). The minimum singular value is zero (i.e. the Jacobian matrix is degenerate) at these configurations, and does not change with change in units. This fact is exploited later on when using screw theory to come up with measures for singularity elimination.

The lesson to be learned from this short example is that the condition number should be used with extreme caution when evaluating singularities of a manipulator. If the manipulator has a Jacobian with mixed units, then one needs to be concerned with what is actually being measured. The problem discussed above not only exists for the condition number, but for a variety of other metrics as well: the maximum singular value, the minimum singular value, manipulability ellipsoids, etc. For this reason, the commonly used condition number and all its variants are **NOT** utilized in this study.

**5.2.1.2 Potential Solutions**

The problem with the condition number does not lie with the SVD or the linear algebra methods. These are sound linear algebra devices. The problem lies in treating the rows of the Jacobian matrix as homogeneous vectors, not as mixed dimensional vectors (i.e. some of the components have length units while others do not). There are remedies for the condition number scale and frame variance. Some of them are:

- **Point Velocities** A possible method to get around the problems with the condition number is to change the output to be the velocities of points only as shown in Gosselin [31] (i.e. the output is linear velocities and has no angular
velocity components). If the input velocities are uniform, this causes the Jacobian matrix to be uniform as well. Therefore, the condition number is well defined and can be utilized after the Jacobian matrix is redrived. However, there is a choice on what points to choose.

- **Division** Zanganeh and Angeles [125] look at multiple condition numbers. They divide the Jacobian matrix into subparts of consistent units where the condition number becomes well defined. There exists a condition number for the rotation as well as one for the translation. However, with this method two measures must be evaluated and compared.

- **Characteristic Length, Weighting Matrices** Another solution to the frame variance problem is the use a weighting matrix. This weighting matrix creates a characteristic length between the linear velocity and the angular velocity measures.

For example, Bicchi et al. [8], define a Rayleigh quotient as the ratio of the input velocity vector, $\dot{\theta}$, and output velocity vector, $\dot{s}^t$:

$$R_v \equiv \frac{\|s^t\|}{\|\dot{\theta}\|} \quad (62)$$

where the norms are the weighted 2-norms:

$$\|\dot{\theta}\| = \sqrt{\dot{\theta}^T W_\theta \dot{\theta}}$$

$$\|s^t\| = \sqrt{s^T W_s s^t}$$

An issue arises on how one appropriately chooses the elements of the weighting matrices. This choice depends upon the application at hand, and thus may be difficult to perform.\(^5\) Chapters 6 and 7 will address the issue of weighting

\(^5\)One may be tempted to use the characteristic length as a free design parameter. However, as was shown in Stocco [112], doing so will negatively affect the results.
matrices in a more general framework, namely as a constrained optimization problem. The results will be valid and consistent no matter what units or coordinate frame is used.

5.2.2 Determinant of the Jacobian

Another possible measure for how close one is to a singularity is the determinant of the Jacobian matrix. As is shown in Murray et al. [88], the determinant of the Jacobian matrix is frame and unit invariant. The determinant is, in differential geometry terms, a volume form of the columns of the Jacobian matrix, i.e. the determinant is a multiplication of all the terms involved. Thus, changing frames does not affect its value. Likewise, since it is a volume, the measure is also unit invariant if the units are kept in the derivation. Scale invariance is not guaranteed with the determinant of the Jacobian since the determinant is a function of the kinematic parameters.

However, there is an issue that need to be addressed when looking at the determinant. The geometric meaning of the determinant is not clear. In other words, what is a “good” value for a determinant? For specific manipulators, a geometric meaning for the determinant can be developed. For example, Lee et al. [63], Lee et al. [62], Zhang and Duffy [126] and Zhang et al. [127] find relationships between the determinant and a volume on the manipulator for several specific, Gough-Stewart type mechanisms. However, in general, finding a physical meaning for the determinant is very difficult.

5.3 Measures Based on Screw Theory

Screws were introduced in 1900 by Sir Robert Stawell Ball [5]. This analysis decomposes all instantaneous rigid body motion into a rotation around an axis and

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6More abstractly, Ebert-Uphoff et al. [25] showed how the volume of a characteristic tetrahedron was related to whether or not a singular configuration was encountered.

7For a more concise and palatable introduction to screw theory, see the article by Lipkin and Duffy [66].
a translation along the axis. A “dual” to this concept is that of all forces acting on a rigid body can be decomposed as a force along an axis and a moment around that axis. Appendix C overviews some of the more elementary theory and calculations of screws that are important to this dissertation.

For example, Pottman et al. [100], develop techniques to determine how close one is to a singularity by use of the power product. The power product is defined as the normal dot product between a twist in axis coordinates, $\$^t$:

$$\$^t = \begin{bmatrix} \mathbf{v} \\ \mathbf{\omega} \end{bmatrix}$$  \hspace{1cm} (63)

and a wrench in ray coordinates $\$^w$:

$$\$^w = \begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix}$$  \hspace{1cm} (64)

as

$$P = \$^t \cdot \$^w = \mathbf{v} \cdot \mathbf{f} + \mathbf{\omega} \cdot \mathbf{m}$$  \hspace{1cm} (65)

where ‘ $\cdot$ ’ is the standard vector dot product. What results is the instantaneous power between the force and the twist. Pottman et al. find the twist that minimizes the square of the power\footnote{The square of the power is used to keep two different power calculations from canceling out much like the square of the error in linear regression.} with all the rows of the Jacobian (i.e. the wrenches). This power is used as a measure to see how close one is to a singularity. In other words, they find the twist that goes against the constraints the least, and then calculate the power via the reciprocal product. This procedure is the basis for the general technique that is introduced in Chapter 6 and completed in Chapter 7.

### 5.4 Measures Based on Dynamics

As was shown in Kozak et al. [57] and Kozak [56], the non-holonomic dynamic equations of motion can be linearized around a quasi-static point. A linearized
natural frequency (and damping ratio) of a PM can be derived using these equations of motion with constrained coordinates.\textsuperscript{9} Although the linearized natural frequency is not a differential kinematic measure, it is included here in the discussion because of its relationship to later chapters. The linearized natural frequency is really a dynamic measure.

What is interesting to note is that the lowest linearized natural frequency goes to zero at a singularity. The natural frequency in the single degree of freedom linear system is defined as:

$$\omega_n = \sqrt{\frac{k}{m}}$$  \hspace{1cm} (66)

It seems reasonable that the lowest linearized natural frequency goes to zero at a platform singularity since the stiffness goes to zero at that point [59]. Therefore, this linearized natural frequency (at a particular pose) can be used as a measure to determine how close one is to a singularity. Because of its formulation, the linearized natural frequency is frame and unit invariant. The dynamic equations take care of the units by use of both the mass and stiffness matrices.

\textsuperscript{9}Appendix A derives the dynamic equations of motion for the 3R\textsubscript{RR} manipulator as well as how these equations can be linearized.
CHAPTER 6

NEW PERSPECTIVE ON CLOSENESS TO SINGULARITIES

Most of the existing measures outlined in Chapter 5 have *one or more* of the following limitations:

1. No clear physical meaning,

2. Not applicable to any arbitrary parallel manipulator (in particular uniform actuator types are often implicitly assumed), and/or

3. Result in two or more values rather than a single value for any considered configuration.\(^1\)

This chapter determines closeness to singularities by formulating the question in terms of a constrained optimization problem. The constrained optimization problem results in a corresponding generalized eigenvalue problem. The resulting eigenvalue has physical meaning and is utilized as a measure of the performance near singularities, and thus is a measure of closeness to singularities.

This optimization approach is not novel to the application of singularity analysis, but has been utilized previously in other incomplete forms and applications. Pottmann et al. [100] use the constrained optimization problem to determine the linear complex that is closest to a singularity. Wolf and Shoham [122] use the methodology to describe the instantaneous behavior near singularities. Joh and Lipkin [47] use

\(^1\)Since these measures generally result in a different value for each configuration, a related question is how those values should best be combined throughout the workspace to obtain a single number for manipulator design, etc. This question will be addressed in Chapter 11
the constrained optimization problem to determine the internal forces in a redundant grasp. Lipkin and Patterson [67, 68] use constrained optimization to generalize the concept of center-of-compliance.

This chapter expands the previous work by incorporating several other seemingly non-related measures into the constrained optimization framework. It is utilized to exhibit the connections between the different measures so that a unified approach to singularity analysis can be devised. Also, as a side benefit, new measures for parallel manipulators (e.g. natural frequency) can easily be put into this framework.

To keep a global picture of the material, this chapter takes a first look at the measures from a physical standpoint and quickly derives some measures so the overall framework is not lost in the details. The next chapter completes the framework for all known possible weighting matrices and thus is quite lengthy.

6.1 Requirements for Measures of Closeness to Singularities

Translating the limitations given above into desired properties, a measure, $M(X)$, at a particular configuration, $X$, (i.e. position and orientation)

\[ M : \{\text{configuration space}\} \rightarrow \mathbb{R}, \]

that evaluates closeness to singularity should have the following three properties:

**Property 1**: $M(X) = 0$ if and only if $X$ is a singular configuration.

**Property 2**: If $X$ is non-singular, $M(X) > 0$.

**Property 3**: $M(X)$ has a clear physical meaning.

Existing measures generally address Properties 1 and 2, but not Property 3. In other words, it is clear what it means if $M(X) = 0$, but what does $M(X) = 10$ mean? Examples are given in the following subsections.
6.2 Motivation

If one approaches the problem of creating a measure from a physical standpoint, one first needs to define the physical quantity, \( M(X) \), one cares about. As is shown in Chapter 4, a singular configuration has many different effects, including loss of constraint, increased end-effector error, loss of stiffness and degenerated actuator torque transmission. The effect that one is most concerned about should form the basis of the singularity measure to be used for a particular application.

For example, if one cares primarily about the loss of constraint that occurs in at least one direction at a singular configuration, an appropriate measure that also applies to non-singular configurations is the amount of constraint provided by the mechanism in the least constrained direction. In this example the physical quantity of interest is the amount of constraint in a particular direction but minimized over all instantaneous directions in order to identify the direction that is least constrained.

More generally, denoting by \( F(X, \S^t) \) the value of the physical quantity of interest at configuration \( X \) for twist direction \( \S^t \), a meaningful measure is obtained by minimizing \( F \) over all “unit” motions, \( \S^t \), as follows:

\[
M(X) = \left\{ \begin{array}{l}
\min_{\S^t} \quad F(X, \S^t) \\
\text{subject to} \quad \| \S^t \|^2 = c
\end{array} \right.
\]  

(67)

where \( \| \cdot \|_t \) represents some type of norm for twists, and \( c \) is a constant. The twist is constrained to have a certain norm because otherwise the trivial solution, \( \S^t = 0 \), would always be the minimum.

It remains to select an objective function, \( F \), and appropriate normalization of \( \S^t \) for minimization problem (67). The former is discussed in Section 6.6, the latter in section 6.3. Sections 6.4 and 6.5 review a few concepts from optimization theory that are relevant to the subsequent sections.
6.3 Some Norms for Twists

The general minimization problem presented in Equation (67) requires normalization of a twist, which raises many issues on how this vector should be normalized [20]. Since there exists no natural norm for this space [88], the normalization requires a choice. The three norms considered in this chapter are defined below. This is by no means an exhaustive list, but represents some of the most common choices. The following chapter provides a complete list of norms.

1. The Euclidean Norm treats the twist as a homogeneous vector:

   \[ \| \mathbf{S}^t \|_{\text{Eucl}} = \sqrt{\mathbf{S}^t \mathbf{S}^t} = \sqrt{\mathbf{v} \cdot \mathbf{v} + \mathbf{\omega} \cdot \mathbf{\omega}} \]

   \[ = \sqrt{\mathbf{S}^T \mathbf{1}_{(6 \times 6)} \mathbf{S}^t}, \quad (68) \]

   where \( \mathbf{1}_{(6 \times 6)} \) is a \((6 \times 6)\) identity matrix.

   Comment: The Euclidean norm ignores the mixed dimensions of a twist. Thus, in actuality, it is ill-defined as is apparent when evaluating its units, e.g. \( \sqrt{\frac{m^2}{s^2} + \frac{rad^2}{s^2}} \).

2. The Invariant Norm takes the magnitude of only the frame-invariant portion of the screw:

   \[ \| \mathbf{S}^t \|_{\text{Inv}} = \sqrt{\mathbf{\omega} \cdot \mathbf{\omega}} \]

   \[ = \sqrt{\mathbf{S}^T \mathbf{D} \mathbf{S}^t}, \quad (69) \]

   where

   \[ \mathbf{D} = \begin{bmatrix} \mathbf{0}_{(3 \times 3)} & \mathbf{0}_{(3 \times 3)} \\ \mathbf{0}_{(3 \times 3)} & \mathbf{1}_{(3 \times 3)} \end{bmatrix}. \quad (70) \]

   Since a norm must vanish if and only if its input is identical to zero, the invariant norm must be supplemented for the case of pure translation. One potential solution is the following supplemental definition:
If \( \omega = 0 \), then \[ \|\$^t\|_{\text{Inv}} = \sqrt{v \cdot v} \] (71)

Comment: The Invariant Norm deals well with the mixed dimensions for \( \omega \neq 0 \), but it is discontinuous in the neighborhood of \( \omega = 0 \).

3. **The Kinetic Energy Norm** assigns the square root of double a body’s kinetic energy corresponding to twist \( \$^t \):

\[
\|\$^t\|_{\text{K.E.}} = \sqrt{mv^T v + \omega^T \mathbf{I}_n \omega} = \sqrt{\$^{t^T} M \$^t},
\] (72)

where \( m \) is the mass of the body, \( \mathbf{I}_n \) is its \( (3 \times 3) \) inertia matrix, and

\[
M = \begin{bmatrix}
m \mathbf{1}_{(3 \times 3)} & \mathbf{0}_{(3 \times 3)} \\
\mathbf{0}_{(3 \times 3)} & \mathbf{I}_n
\end{bmatrix}
\] (73)

is its generalized \( (6 \times 6) \) inertia matrix about its center of mass.

Comment: The Kinetic Energy Norm deals well with mixed dimensions, but introduces a non-kinematic quantity, namely the inertia matrix.

### 6.4 Specific Form of Minimization Problem

As is shown in Section 6.6, all of the objective functions investigated here can be expressed in the form:

\[
F(\mathbf{X}, \$^t) = \$^{t^T} R(\mathbf{X}) \$^t,
\] (74)

where \( R(\mathbf{X}) \) is a matrix that depends on configuration \( \mathbf{X} \).

Using any of the three norms of the proceeding subsection, the constraint that a twist be of constant magnitude can be formulated as

\[
\|\$^t\|_{\text{Eucl/Inv/K.E.}}^2 = \$^{t^T} T \$^t = c,
\] (75)
where \( \mathbf{T} \) equals \( \mathbf{1}_{(6 \times 6)} \), \( \mathbf{D} \) or \( \mathbf{M} \), respectively.

For simplicity, the dependence on configuration \( \mathbf{X} \) is no longer pointed out explicitly in the remainder of this chapter wherever it is clear from the context. In summary, the following constrained minimization problem is proposed to measure closeness to singularities of configuration \( \mathbf{X} \):

\[
M(\mathbf{X}) = \min_{\mathbf{s}^t} F(\mathbf{s}^t) = \mathbf{s}^{t^T} \mathbf{R} \mathbf{s}^t, \quad \text{subject to} \quad h(\mathbf{s}^t) = \mathbf{s}^{t^T} \mathbf{T} \mathbf{s}^t - c = 0, \tag{76}
\]

where \( \mathbf{R} \) and \( \mathbf{T} \) are \( n \times n \) symmetric positive semi-definite matrices, and \( c \) is some positive constant.\(^2\) The constant \( c \) has different units depending upon what norm is being utilized. Also, due to the positive semi-definiteness of matrix \( \mathbf{R} \), function \( F \) only takes on non-negative values, \( F(\mathbf{s}^t) \geq 0 \).

### 6.5 Corresponding Eigenvalue Problem

To solve the constrained optimization problem (76), it is transformed into an unconstrained optimization problem.\(^3\) The transformation is achieved by forming the Lagrangian, \( L(\mathbf{s}^t, \lambda) = F(\mathbf{s}^t) - \lambda h(\mathbf{s}^t) \) and solving the resulting unconstrained optimization problem,

\[
\min_{\mathbf{s}^t, \lambda} L(\mathbf{s}^t, \lambda), \tag{77}
\]

where for this particular problem, the Lagrangian becomes

\[
L(\mathbf{s}^t, \lambda) = \mathbf{s}^{t^T} \mathbf{R} \mathbf{s}^t - \lambda \left( \mathbf{s}^{t^T} \mathbf{T} \mathbf{s}^t - c \right). \tag{78}
\]

Using elementary calculus techniques, the minimization of the Lagrangian is now performed by taking the partial derivative of the Lagrangian with respect to \( \lambda \) and \( \mathbf{s}^t \) and setting both equations equal to zero. By doing so, all extrema of the problem are identified which, since \( F \) is bounded from below by 0, must include the absolute

---

\(^2\) The positive semi-definite and symmetric restrictions will be discussed further in Chapter 7.

\(^3\) For details, see Reklaitis et al. [103] or any other standard optimization textbook.
minimum of $F$. Differentiating the Lagrangian with respect to $\lambda$ results in

$$\frac{\partial L}{\partial \lambda} = \left( s_i^T T s_i^t - c \right) = 0,$$

which is the constraint from Equation (76). Differentiating the Lagrangian with respect to $s_i^t$ and using the fact that $R$ and $T$ are symmetric yields:

$$\frac{\partial L}{\partial s_i^t} = 2 R s_i^t - 2 \lambda T s_i^t = 0$$

$$(R - \lambda T) s_i^t = 0$$

(80)

For a nontrivial solution to exist, the matrix expression in the parentheses must be singular. In other words:

$$\det (R - \lambda T) = 0,$$

(81)

which is called the corresponding general eigenvalue problem. From this, the eigenvalues, $\lambda_i$, (i.e. the stationary points in a minimization sense) and the associated eigenvectors, $s_i^t$, are computed. The eigenvectors are then scaled to satisfy the constraint, $s_i^T T s_i^t - c = 0$, and the resulting scaled vectors are substituted into the original objective function, $F$, from Equation (76) to yield its minimum value.

Furthermore, the objective function’s minimum is the smallest eigenvalue, $\lambda_{min}$, of Equation (81) multiplied by the constant, $c$. This is proven by first going back to Equation (80) which rewritten yields

$$R s_i^t = \lambda T s_i^t$$

(82)

Substituting this relationship into objective function $F$ and using the constraint $h$ yields:

$$F(s_i^t) = s_i^T R s_i^t = \lambda \underbrace{s_i^T T s_i^t}_{c} = c \lambda,$$

(83)

Since $c$ is a positive constant, $F$ is minimized for the smallest eigenvalue, $\lambda_{min}$, thus

$$\min_{s_i^t} F(s_i^t) = c \lambda_{min}.$$

(84)

Another result that can be inferred from this analysis is all eigenvalues are non-negative, $\lambda_i \geq 0$, since the objective function is non-negative.
6.6 Minimization-Based Measures - EXAMPLES

The remaining item to complete is to determine the objective function and the appropriate measures.

6.6.1 Power-Inspired Measure

Pottmann et al. [100] propose the following measure for closeness to singularities of Gough-Stewart parallel manipulators:

$$\min_{\$^t} \quad F(\$^t) = \sum_{i=1}^{k} (J_i^T \$^t)^2,$$

subject to $$\|\$^t\|_{\text{INV}}^2 = 1$$

(85)

where $J_i$ are the rows of Jacobian matrix $J$ and $k$ is the number of its rows. The motivation for this measure, using the Gough-Stewart platform as example, comes from the dual meaning of the Jacobian matrix. Namely the rows $J_i$ of $J$ are also the columns of $J^T$ and can be interpreted as the wrenches applied by the legs for unit actuator torques. Each term $(J_i^T \$^t)^2$ of $F$ can thus be interpreted as the square of the power of the $i$th leg on end-effector twist, $\$^t$. In a singular configuration there exists a twist $\$^t$ for which none of the leg forces can do any work and thus the minimum of $F$ goes to zero. Away from singular configurations, the minimization identifies the least constrained twist - with the restriction that its angular part be of norm one - and uses the power done by the constraining leg forces on that twist as a measure. Organizing Equation (85) in the form of Equation (76), this measure becomes:

$$M(X) = \begin{cases} 
\min_{\$^t} \quad F(\$^t) = \$^{\text{TR}} \bar{J}^T \bar{J} \$^t \\
\text{subject to} \quad h(\$^t) = \$^{\text{TR}} \bar{D} \$^t - c = 0.
\end{cases}$$

(86)

\(^4\)One word of caution should be noted. If the actuators are prismatic, the rows have units consistent with a wrench. However, if revolute actuators are used, then the rows of $J$ will have other units, but still can be loosely interpreted as a wrench.
where $\mathbf{D}$ comes from Equation (69).

This approach has two weaknesses. The first one comes from the normalization procedure, namely the fact that the angular part of the twist is constrained to have unit norm. Thus singular configurations with a purely translational deficiency are not detected by the measure. Pottmann et al. propose to supplement this approach with a second optimization for which the twist is normalized to have unit linear part. However, this results in two different values for any configuration and it is unclear how these two values are to be combined to yield a single value for closeness to singularity.

The second weakness comes from the objective function itself which is only well defined if all actuators are of the same type, i.e. all revolute or all linear.\footnote{Since Pottmann et al. only discuss the Gough-Stewart mechanism in their article there is no problem in their work, but it limits the applicability of the proposed measure to other parallel manipulators.} This becomes very obvious when rewriting $F(\mathbf{s}^t)$ as follows:

$$
F(\mathbf{s}^t) = (\mathbf{Js}^t)^T(\mathbf{Js}^t) = \dot{\mathbf{\theta}}^T \dot{\mathbf{\theta}} = \|\dot{\mathbf{\theta}}\|^2, 
$$

(87)

i.e. $F(\mathbf{s}^t)$ is the Euclidean norm of the vector of actuator velocities, which is ill-defined if the units of $\dot{\mathbf{\theta}}$ are not all identical. Thus Properties 1 and 3 are both violated for manipulators with mixed dimensions. The measure presented in the following subsection seeks to remedy some of these problems.

### 6.6.2 Stiffness-Inspired Measure

The first problem to be addressed is that the units of the objective function do not match if the actuators are not all of the same type. This problem can be fixed, as mentioned by several research groups in a different context [32], by introducing a diagonal stiffness matrix, $\mathbf{K}$, that represents actuator stiffness. Using the stiffness matrix in the end-effector space, $\mathbf{K}$, results as

$$
\mathbf{K} = \mathbf{J}^T \mathbf{K} \mathbf{J},
$$

(88)
see [80]. This motivates the use of the following modified minimization problem:

\[
M(X) = \begin{cases}
\min_{\$^t} & F(\$^t) = \$^t \begin{pmatrix} J^T \mathcal{K} J \end{pmatrix} \$^t \\
\text{subject to} & h(\$^t) = \$^t D\$^t - c = 0,
\end{cases}
\] (89)

where \( F \) now represents the potential energy stored in the actuator springs, based on a finite displacement represented by \( \$^t \). Thus \( F \) has a clear physical meaning for any parallel manipulator and is thus well defined.

However, since the invariant norm is used, its related problem that is seen in the Power-Inspired Measure still applies. Namely, a second measure would have to be used to detect singularities of pure translation, i.e. Property 1 is still violated.

### 6.6.3 Minimal Eigenvalue of \((J^T J)\) and of \((J^T \mathcal{K} J)\)

It turns out that two of the more traditional measures, namely the minimal eigenvalue of \((J^T J)\) and that of \((J^T \mathcal{K} J)\) can also be formulated as the result of a minimization problem of the form discussed above.

In fact, using the Euclidean norm instead of the invariant norm in the Power-Inspired Measure (86) leads to the following minimization problem:

\[
M(X) = \begin{cases}
\min_{\$^t} & F(\$^t) = \$^t J^T J \$^t \\
\text{subject to} & \|\$^t\|^2_{\text{Eucl}} = c.
\end{cases}
\] (90)

The corresponding eigenvalue problem, namely Equation (81) in Subsection 6.5, with \( R = J^T J, T = 1_{(6 \times 6)} \), becomes:

\[
\det (J^T J - \lambda 1_{(6 \times 6)}) = 0.
\] (91)

Thus the solution of the minimization problem is in fact the minimal eigenvalue of \((J^T J)\) times \( c \).

---

6This quantity is not exactly the potential energy since the twist is linear and angular velocity, but can be interpreted as potential energy for a small time. It would be more appropriate to replace \( \$^t \) with \( \$^t dt \). However, the \( \$^t \) is done for notational consistency. See Chapter 7 for further discussion.
Likewise, using the Euclidean norm instead of the invariant norm in the stiffness measure (89) leads to a very similar minimization problem:

\[
M(X) = \left\{ \min_{s^t} \begin{array}{c} F(s^t) = s^T J^T K J s^t \\ \text{subject to} \quad \|s^t\|_{\text{Eucl}}^2 = c. \end{array} \right. \tag{92}
\]

Using the same procedure as above, the solution to this minimization problem turns out to be the minimal eigenvalue of \((J^T K J)\) times \(c\).

There are three interesting observations to be made:

1. Two common measures, namely the minimal eigenvalue of \((J^T J)\) and of \((J^T K J)\) can be described in the minimization framework discussed here.

2. The fact that both of these measures are based on the Euclidean norm, whose physical units are ill-defined for a twist, confirms the concerns raised by other groups about this measure.

3. While the benefit of this measure is arguable in the neighborhood of a singularity, it at least guarantees to pick up all singularities and thus **is actually more reliable** than the Power-Inspired and Stiffness-based measures discussed in the preceding section.

### 6.6.4 Natural Frequency Measure

As a last measure let us use the objective function from the stiffness measure (Equation 89) and combine it with a constraint based on the kinetic energy norm:

\[
M(X) = \left\{ \min_{s^t} \begin{array}{c} F(s^t) = s^T K s^t \\ \text{subject to} \quad \|s^t\|_{\text{K.E.}}^2 = c. \end{array} \right. \tag{93}
\]

Choosing \(R = K = J^T K J, T = M\), leads to the corresponding eigenvalue problem,

\[
\det (K - \lambda M) = 0, \tag{94}
\]

66
where $M$ is for now assumed to represent the mass of only the moving platform. Since the expression in Equation 94 is the same as those derived from the linearized dynamics [59], the eigenvalue is the same as the square of the natural frequency, $\lambda = \omega_n^2$. As a result, the solution to minimization (93) is $c\lambda_{\text{min}} = c\omega_n^2$, i.e. a constant multiple of the square of the lowest linearized natural frequency at this pose. Therefore, the measure represents the lowest linearized natural frequency of the manipulator in a specific configuration.

The benefit of this measure is that it uses both a physically meaningful objective function and a physically meaningful constraint. As a consequence, the solution also has a very clear physical meaning and has been shown to be important in the design of a parallel manipulator [99]. The drawback of this approach is that in addition to stiffness, another non-kinematic quantity has been introduced, namely the inertia of the manipulator.

### 6.6.5 Summary of Results

Table 1 overviews the results of the measures. The first measure (i.e. the eigenvalue of $J^T J$) is more reliable for detecting singularities than the second and third, but suffers from the frame invariance problem. As one moves down the table in order to fix this problem, the measures add non-kinematic parameters to the objective function and the norm. The fourth measure solves all the problems, but uses two non-kinematic quantities. The power of the optimization framework is that it allows an easier comparison of the measures and how they are related.

### 6.7 Limitation of the Natural Frequency Measure

The natural frequency measure, which is relatively new [59], has great potential and is discussed in more detail below.
**Table 1: Summary of Results of Constrained Optimization Framework**

<table>
<thead>
<tr>
<th>#</th>
<th>Measure</th>
<th>Objective Function</th>
<th>Norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Minimal Eigenvalue</td>
<td>$J^T J$, $J^T K J$</td>
<td>Euclidean</td>
</tr>
<tr>
<td>2</td>
<td>Power</td>
<td>$J^T J$</td>
<td>Invariant</td>
</tr>
<tr>
<td>3</td>
<td>Stiffness</td>
<td>$J^T K J$</td>
<td>Invariant</td>
</tr>
<tr>
<td>4</td>
<td>Natural Frequency</td>
<td>$J^T K J$</td>
<td>Kinetic Energy</td>
</tr>
</tbody>
</table>

### 6.7.1 Choosing Level of Modeling Detail

The stiffness matrix, $K = J^T K J$, on which the natural frequency measure is based, only presents a very simple version of the stiffness of the whole manipulator, since it only models the stiffness of the actuators.

To address this problem, one can substitute $K$ in Equation 93 by a more detailed stiffness matrix that also includes the stiffness of the links, etc. as long as it remains symmetric and positive semi-definite. The same approach can be taken to refine the inertia matrix, $M$, of the manipulator to also include masses of the leg links, etc. However, there are two limitations to these approaches: (1) It is very time consuming to derive detailed models of a manipulator. At an early design stage that may not even be possible. (2) While the mass matrix can always be made symmetric positive definite, that is not the case for the stiffness matrix of the whole manipulator [23]. Symmetry and positive semi-definiteness, however, was assumed in the derivation of the corresponding eigenvalue problem in Subsection 6.5.

For those two reasons, it seems plausible to only use a very simple model of $M$ and $K$ to develop a first-order measure that describes the manipulator performance reasonably well. More on this topic is described at the end of Chapter 7.
6.7.2 Acceptable Threshold Value

Recall that the goal is to determine how close a parallel manipulator is to a singularity. The natural frequency measure is a physically meaningful measure of how much the performance degenerates close to a singularity. Due to the way the problem has been formulated, the natural frequency measure goes to zero at a singularity, and represents the natural frequency squared multiplied by the constant $c$ everywhere else. However, what the threshold value for $\omega_{min}^2$ should be remains an open question. In fact, the threshold value is application dependent and deserves further investigation.

6.7.3 Relation to End Effector Motion

Another drawback to the natural frequency is that it does not relate to the amount of unconstrained motion at the end effector. While the natural frequency will show how the robot will react in a dynamic sense, it will not show the other effects that singularities may exhibit, namely the amount of unconstrained motion. For this type of analysis, a different technique has been developed based on workspace generation techniques and is discussed in Chapter 9.
CHAPTER 7

GENERAL FRAMEWORK FOR
DIFFERENTIAL KINEMATIC MEASURES

Chapter 6 introduces the concept of using constrained optimization as a general framework in which many of different measures in Chapter 5 are placed. This chapter now completes the analysis by providing an exhaustive list of weighting matrices that can be utilized. In order to perform this complete analysis, the formulation needs to change to be more mathematically based. However, the physical interpretation presented in Chapter 6 is still valid and beneficial to the understanding of closeness to singularities.

In broad mathematical terms, the optimization technique exploited in Chapter 6 is a generalization of a matrix norm. If all units are consistent, the norm of the Jacobian matrix, \( J \), is defined as [39]:

\[
\|J\| = \max_{\|\mathbf{s}^i\|=1} \|\dot{\mathbf{\theta}}\| \quad \text{where} \quad \dot{\mathbf{\theta}} = JS^i
\]

Intuitively, this measures the size of the input velocities, \( \dot{\mathbf{\theta}} \), for a “unit” output velocity, \( S^i \). In the case of finding closeness to singularities, the Jacobian does not have consistent units. Therefore, weighting matrices must be used to make sure the units remain consistent.

This chapter further generalizes the framework from Chapter 6. Specifically, the weighting matrix in the objective function changes from a matrix that depends upon the Jacobian to be just a constant matrix. This change is performed to be able to perform the optimization in both the velocity domain (i.e. with twists) and the force domain (i.e. with wrenches)
7.1 Mathematical Preliminaries

As is seen in Chapter 6, the general framework is based on non-linear constrained optimization, and thus a more detailed overview of the method is presented below. For completeness, the derivation is repeated since there is a slight change in the formulation.

7.1.1 Optimization in the Velocity Domain

It is assumed that measure is the result of the constrained optimization problem where the cost function, \( F \), is a quadratic equation with both linear, \( h_1 \) and quadratic constraints, \( h_2 \). This is formulated mathematically as:

\[
\begin{align*}
\text{min or max} & \quad F(\hat{\theta}) = \dot{\theta}^T S \dot{\theta} \\
\text{subject to} & \quad h_1 = \dot{\theta} - J S^t = 0 \\
& \quad h_2 = S^t T S^t - 1 = 0
\end{align*}
\]

(96)

where \( S \) and \( T \) are \( n \times n \) symmetric matrices.\(^1\) The objective function, \( F \), is some scalar function that will be used as the measure of closeness to singularities. Note that this formulation has changed slightly from Chapter 6 to be more closely related to the matrix norm formulation given in the introduction. Specifically, now the input velocities, \( \dot{\theta} \), are explicitly included in the cost function where they were implicitly done in Chapter 6. This change is also the reason the notation has changed from \( R \) to \( S \) in the objective function.

Because there is both the input and output included, the first constraint, \( h_1 \), is the Jacobian relationship. The second constraint, \( h_2 \), is a limit on the size of the twist. Therefore, the optimization problem looks at the size of the manipulator inputs, \( \dot{\theta} \), for a “unit” output, \( S^t \), while satisfying the Jacobian relationship. Since both the objective function and the second constraint are quadratic, they can be

\(^1\)The symmetric restriction will be discussed further at the end of this chapter.
thought of as the square of a "pseudo-norm" and thus approximate the matrix norm. The optimization is to be performed at a particular pose, and therefore, $J$ is constant. Also, the units on the second constraint change depending on the units on $T$, and need to be consistent.

The solution to Equation (96) is found with a straightforward application of Lagrange multipliers. More specifically, Lagrange multipliers transform the constrained optimization problem into an unconstrained optimization problem by forming the Lagrangian, $\mathcal{L} = F - \lambda_v^T h_1 - \mu_v h_2$. Since there are now two constraints, there are two Lagrange multipliers, a $n \times 1$ vector, $\lambda_v$, and a constant $\mu_v$. The subscript $v$ is to distinguish the multipliers for the velocity domain.

For this particular problem, the Lagrangian becomes:

$$\mathcal{L}(\dot{\theta}, \dot{s}^t, \lambda_v, \mu_v) = \dot{\theta}^T S \dot{\theta} - \lambda_v^T \left( \dot{\theta} - J \dot{s}^t \right) - \mu_v \left( s^{tT} T s^t - 1 \right)$$

The optimum for this equation is found by minimizing the Lagrangian with respect to all variables. The minimization of the Lagrangian is performed by taking the partial derivative of the Lagrangian with respect to $\dot{\theta}$, $\dot{s}^t$, $\lambda_v$, and $\mu_v$ and setting all equations equal to zero. Differentiating the Lagrangian gives:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 2S \dot{\theta} - \lambda_v = 0$$

(98)

$$\frac{\partial \mathcal{L}}{\partial \dot{s}^t} = J^T \lambda_v - 2\mu_v T s^t = 0$$

(99)

$$\frac{\partial \mathcal{L}}{\partial \lambda_v} = \dot{\theta} - J \dot{s}^t = 0$$

(100)

$$\frac{\partial \mathcal{L}}{\partial \mu_v} = s^{tT} T s^t - 1 = 0$$

(101)

Equations 100 and 101 return the original constraints while Equations 98 and 99 give further requirements on the optimum. Rearranging Equation 98 gives:

$$\lambda_v = 2S \dot{\theta}$$

(102)

\footnote{For a more detailed explanation of the techniques of optimization, refer to Reklaitis et al. [103] or any other standard textbook on optimization.}
Substituting this result into Equation 99 gives:

\[ 2J^T S \hat{\theta} - 2\mu_v T S^t = 0 \]  

(103)

Using the constraint from Equation 100 (i.e. the Jacobian relationship) yields:

\[ 2J^T S J S^t - 2\mu_v T S^t = 0 \]  

(104)

which after some algebra gives:

\[ (J^T S J - \mu_v T) S^t = 0 \]  

(105)

For a nontrivial solution to exist, the matrix expression in the parenthesis must be singular. In other words:

\[ \text{det} (J^T S J - \mu_v T) = 0 \]  

(106)

This result is the generalized eigenvalue problem. From this, the eigenvalues, \( \mu_{v_i} \), (i.e. the stationary points in a minimization sense) and the associated eigenvectors, \( S_i^t \), are computed.\(^3\) These eigenvectors must also satisfy the constraint in Equation 101, i.e. \( S_i^T T S_i^t - 1 = 0 \). Once the eigenvectors have been computed to satisfy \( h_2 \), they can be back substituted into \( h_1 \) to find \( \hat{\theta} \) which plugged into the original function, \( F \), determines the maximum or minimum value. This maximum or minimum value is used as a measure of the closeness to singularities.

To show that this problem is the same as the one discussed in Chapter 6, the constraint, \( h_1 \) can be substituted into the original function to produce the following reduced problem:

\[
\min \quad F(S^t) = S^t J^T S J S^t \\
\text{subject to} \quad h_2(S^t) = S^T T S^t - 1 = 0
\]  

(107)

\(^3\)As an interesting side note, the Lagrange multipliers, \( \lambda_v \), and the eigenvectors, \( S^t \), have interesting properties. The Lagrange multipliers create a screw known as the Lagrangian screw [47, 46] which are important for the pseudoinverse solution to redundant grasping or underconstrained serial robots. Under appropriate choice of weighting matrices \( T \) and \( R \), the eigenvectors also are known as eigentwists [67, 68], and are useful for decomposing the stiffness matrices in an invariant way.
where in Chapter 6 $R = J^T SJ$. Since there is only one constraint, the Lagrangian becomes:

$$\mathcal{L} = s^T J^T SJ s^t - \mu_v \left( s^T T s^t - 1 \right)$$

(108)

Differentiating the Lagrangian gives:

$$\frac{\partial \mathcal{L}}{\partial s^t} = 2 J^T SJ s^t - 2 \mu_v T s^t = 0$$

(109)

$$\frac{\partial \mathcal{L}}{\partial \mu_v} = s^T T s^t - 1 = 0$$

(110)

Equation 109 gives the generalized eigenvalue problem:

$$(J^T SJ - \mu_v T)s^t = 0$$

(111)

which is consistent with the result derived in Equation 105 and Chapter 6.

Due to the way the problem is formulated, the smallest (largest) eigenvalue will be the minimum (maximum) value of the cost function. This is seen with the following proof. The eigenvalue problem can be written as:

$$J^T SJ s^t - \mu_v T s^t = 0$$

$$J^T SJ s^t = \mu_v T s^t$$

(112)

We plug these results back into the calculation for the cost function which gives:

$$F = s^T J^T SJ s^t$$

(113)

$$= s^T \mu_v T s^t$$

(114)

$$= \mu_v \frac{s^T T s^t}{1}$$

(115)

Since the constraint forces the second part to be unitary for any eigenvector:

$$F_{\text{min}} = \mu_{v_{\text{min}}} \quad \text{or} \quad F_{\text{max}} = \mu_{v_{\text{max}}}$$

(116)

This shows that the smallest eigenvalue minimizes the cost function. It also shows that if $S$ is positive definite, all the eigenvalues must be real and positive.
7.1.2 Optimization in the Force Domain

Due to the dualities between twists and wrenches in the static analysis, a similar problem can be formulated with forces and moments instead of velocities. Formulating the problem with the Jacobian relationship as a constraint allows the dual problem to be investigated. This formulation completes the analysis and identifies the singularities that are missed in the velocity domain.

Suppose now that the cost function deals with forces of the input with the wrench on the output to be constrained to be unitary. Therefore in the general framework of optimization, the problem is stated as:

\[
\begin{align*}
\text{min or max} & \quad F(\tau) = \tau^T S \tau \\
\text{subject to} & \quad h_1 = \$^w - J^T \tau = 0 \\
& \quad h_2 = \$^{wT} T \$^w - 1 = 0
\end{align*}
\]

Creating the Lagrangian gives:

\[
\mathcal{L}(\tau, \$, \lambda_f, \mu_f) = \tau^T S \tau - \lambda_f^T (\$^w - J^T \tau) - \mu_f (\$^{wT} T \$^w - 1)
\]

Differentiating the Lagrangian yields:

\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial \tau} &= 2S\tau - J\lambda_f = 0 \\
\frac{\partial \mathcal{L}}{\partial \$^w} &= \lambda_f - 2\mu_f T \$^w = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_f} &= \$^w - J^T \tau = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu_f} &= \$^{wT} T \$^w - 1 = 0
\end{align*}
\]

Again, Equations 121 and 122 return the original constraints. Rearranging Equation 119 gives:

\[
\tau = \frac{1}{2} S^{-1} J \lambda_f
\]

Substituting this into Equation 121 gives:

\[
\$^w = \frac{1}{2} J^T S^{-1} J \lambda_f
\]
Solving for $\boldsymbol{\lambda}_f$ in Equation 120, and plugging it into the previous equation gives:

$$\$^w = \frac{1}{2} \boldsymbol{J}^T \boldsymbol{S}^{-1} \boldsymbol{J} (2\mu_f \boldsymbol{T}\$^w)$$

(125)

$$= \mu_f \boldsymbol{J}^T \boldsymbol{S}^{-1} \boldsymbol{J} \boldsymbol{T}\$^w$$

(126)

which, as expected, turns into the eigenvalue problem:

$$(\mathbf{1}_{6 \times 6} - \mu_f \boldsymbol{J}^T \boldsymbol{S}^{-1} \boldsymbol{J} \boldsymbol{T}) \$^w = \mathbf{0}$$

(127)

Like the result of the velocity domain optimization, the force domain optimization results in the general eigenvalue problem. If a little algebra is done on Equation 127, it is easier compared to the result in Equation 105. More specifically,

$$\begin{align*}
\text{Velocity Domain} & \quad \frac{\boldsymbol{J}^T \boldsymbol{S} \boldsymbol{J} - \mu_v \boldsymbol{T}}{(\boldsymbol{J}^T \boldsymbol{S}^{-1} \boldsymbol{J})^{-1} - \mu_f \boldsymbol{T}} \$^t = \mathbf{0} \\
\text{Force Domain} & \quad \frac{\boldsymbol{J}^T \boldsymbol{S}^{-1} \boldsymbol{J}}{(\boldsymbol{J}^T \boldsymbol{S}^{-1} \boldsymbol{J})^{-1} - \mu_f \boldsymbol{T}} \$^w = \mathbf{0}
\end{align*}$$

(105)  (128)

Note that for implementation, the inverse of $\boldsymbol{J}^T \boldsymbol{S}^{-1} \boldsymbol{J}$ would not be computed, but is done here to show that both problems yield very similar results. The two different domains differ by the inverses. This subtle difference makes a large difference in the outcome of the different measures.

It is important to note is that the $\boldsymbol{S}$ matrix was assumed to be invertible (and symmetric), but the Jacobian matrix does not need to be full rank. Like the case with the twists, the minimal (maximal) eigenvalues of Equation 127 are the minimum (maximum) to the objective function. This fact is seen by the following proof. The objective function is:

$$\boldsymbol{F} = \boldsymbol{\tau}^T \boldsymbol{S} \boldsymbol{\tau}$$

(129)

Plugging Equation 123 yields:

$$\begin{align*}
\boldsymbol{F} &= \left( \frac{1}{2} \boldsymbol{S}^{-1} \boldsymbol{J} \boldsymbol{\lambda}_f \right)^T \boldsymbol{S} \left( \frac{1}{2} \boldsymbol{S}^{-1} \boldsymbol{J} \boldsymbol{\lambda}_f \right) \\
&= \frac{1}{4} \boldsymbol{\lambda}_f^T \boldsymbol{J}^T \boldsymbol{S}^{-1} \boldsymbol{J} \boldsymbol{\lambda}_f \\
&= \frac{1}{4} \boldsymbol{\lambda}_f^T \boldsymbol{J}^T \boldsymbol{S}^{-1} \boldsymbol{J} \boldsymbol{\lambda}_f
\end{align*}$$

(129)  (130)  (131)  (132)
where due to the symmetry assumption, $S^{-T} = S^{-1}$. Plugging in the value of $\lambda_f$ from Equation 120 gives:

$$F = \frac{1}{4} (2\mu_f T S^w)^T J^T S^{-1} J (2\mu_f T S^w)$$  \hspace{1cm} (133)

$$= \mu_f^2 S^w T^T J^T S^{-1} J T S^w$$  \hspace{1cm} (134)

Using Equation 127 and 122 gives:

$$F = \mu_f^2 S^w T^T J^T S^{-1} J T S^w$$ \hspace{1cm} (135)

$$= \mu_f S^w T S^w$$ \hspace{1cm} (136)

$$= \mu_f$$ \hspace{1cm} (137)

which shows that the stationary points of the objective function are the eigenvalues. Therefore,

$$F_{\text{min}} = \mu_f \text{min} \quad \text{or} \quad F_{\text{max}} = \mu_f \text{max}$$ \hspace{1cm} (138)

Now that the mathematical preliminaries are formulated, this framework is related to singularities.

### 7.2 Selection of Weighting Matrices $S$ and $T$

The eigenvalues, $\mu_v$ and $\mu_f$, are the solution to the optimization problem. It only remains to pick appropriate $S$ and $T$ matrices to make the “pseudo-norms” of the input and output. There are several different matrices that can be chosen for the weighting matrices. The only requirements for the weighting matrices are to have $S$ and $T$ be symmetric, and $S$ be invertible for use in the force domain.

There a plethora of different types of weighting matrices that can be used. However, the weighting matrices should have some physical meaning behind them when they are used in the objective function or the constraint. Since the constraint deals with either a twist or a wrench, the screw norms should be included. Since the
inputs can be of any type, the constitutive properties should also be included. These include the mass, stiffness, and damping matrices. For completeness, the identity matrix is included as well.

The ones that will be discussed here are the identity matrix, \( \mathbf{1} \), the flipper matrix, \( \mathcal{Y} \), the screw norm matrices, \( \mathbf{D} \) and \( \mathbf{E} \), the stiffness matrices, \( \mathcal{K} \) and \( \mathcal{K} \), the compliance matrices, \( \mathcal{C} \) and \( \mathcal{C} \), and the mass matrices, \( \mathcal{M} \) and \( \mathcal{M} \). Each of these are defined as follows:

\[
\mathbf{1}_{6\times6} = \begin{bmatrix}
1 \\
& \\ & \\ & \\ & \\ & 1
\end{bmatrix} \quad \text{Identity Matrix} \tag{139}
\]

\[
\mathcal{Y} = \begin{bmatrix}
\mathbf{0}_{3\times3} & \mathbf{1}_{3\times3} \\
\mathbf{1}_{3\times3} & \mathbf{0}_{3\times3}
\end{bmatrix} \quad \text{Flipper Matrix} \tag{140}
\]

\[
\mathbf{D} = \begin{bmatrix}
\mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\
\mathbf{0}_{3\times3} & \mathbf{1}_{3\times3}
\end{bmatrix} \quad \text{Twist Norm (axis coordinates)} \tag{141}
\]

\[
\mathbf{E} = \begin{bmatrix}
\mathbf{1}_{3\times3} & \mathbf{0}_{3\times3} \\
\mathbf{0}_{3\times3} & \mathbf{0}_{3\times3}
\end{bmatrix} \quad \text{Wrench Norm (ray coordinates)} \tag{142}
\]

\[
\mathcal{K} = \begin{bmatrix}
k_1 \\
& \\ & \\ & \\ & \\ & k_6
\end{bmatrix} \quad \text{Stiffness matrix of the inputs} \tag{143}
\]

\[
\mathbf{K} = \mathbf{J}^T \mathcal{K} \mathbf{J} \quad \text{Stiffness matrix due to inputs at the end effector} \tag{144}
\]

\[
\mathcal{C} = \mathcal{K}^{-1} = \begin{bmatrix}
\frac{1}{k_1} \\
& \\ & \\ & \\ & \\ & \frac{1}{k_6}
\end{bmatrix} \quad \text{Compliance matrix of the inputs} \tag{145}
\]

\[
\mathbf{C} = \mathbf{K}^{-1} = \mathbf{J}^{-1} \mathcal{C} \mathbf{J}^{-T} \quad \text{Compliance matrix due to inputs at the end effector} \tag{146}
\]
\[ D = \begin{bmatrix} d_1 \\ \vdots \\ d_6 \end{bmatrix} \quad \text{Velocity dependent dissipation in the inputs} \quad (147) \]

\[ M = \begin{bmatrix} I_n_1 \\ \vdots \\ I_n_6 \end{bmatrix} \quad \text{Inertia of the inputs} \quad (148) \]

\[ M_{ee} = \begin{bmatrix} m_{ee}I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{n_{ee3 \times 3}} \end{bmatrix} \quad \text{Mass of the end effector} \quad (149) \]

In order to simplify the procedure, the stiffness, compliance, and mass matrices are assumed to be diagonal. For the stiffness or compliance matrix, this assumes that all of the stiffness comes from the proportional gain on the controller. Likewise for a diagonal damping matrix, it assumes all the damping comes from the actuators. A diagonal mass matrix assumes that there exists gearing on the motors and thus the coupling effects of the mass are negligible compared to the mass of the rotors. However, all of these matrices can be expanded to symmetric matrices which include other effects. Section 7.4 discusses how this would be done.

### 7.2.1 Velocity Domain Analysis

For this section, the problem given in Equation 96 is examined to determine what matrices can be utilized to yield meaningful results.

#### 7.2.1.1 Selection of S

Since the \( S \) is in the objective function, it needs to work with the inputs. In general, the inputs are not screws, and thus, the screw norms, \( D, E \), and \( \Upsilon \) make no sense. Also, the constitutive properties at the end effector, \( M_{ee} \) and \( K \) should also not be utilized. This section now investigates the rest of the measures.
1. \( S = 1_{6 \times 6} \)

The identity matrix takes as a measure the square of the two (Euclidean) norm of the input velocities. In other words,

\[
F = \hat{\theta}^T 1_{6 \times 6} \hat{\theta} = (\|\hat{\theta}\|_2)^2
\]

From this equation, it is easy to see that unless the inputs are of the same type, this formulation will not work. That is, the units on the inputs must be the same and thus all need to be measured in \textit{rad/s} or \textit{m/s}. This restriction is also seen in Chapter 6.

2. \( S = \mathcal{K} \)

For the stiffness matrix, the objective function becomes:

\[
F = \hat{\theta}^T \mathcal{K} \hat{\theta} = k_1 \dot{\theta}_1^2 + \cdots + k_6 \dot{\theta}_6^2
\]

This allows for different types of actuators, by pre-multiplying by the stiffness in each actuator. This quantity is potential energy like.\(^4\) It is not the true potential energy since it deals with the velocities instead of positions. However, in a more abstract sense, the Jacobian relationship can be thought of as a first order relationship between the kinematics and thus, one can substitute \( \Delta \theta \) for \( \dot{\theta} \) and then it can be thought of as potential energy over a fixed time \( \Delta t \). The velocity formulation was kept for notational consistency, and thus this quantity is denoted the \textbf{velocity potential energy}.

3. \( S = \mathcal{D} \)

Since the objective function deals with velocities, another possibility is the

\(^4\text{One could choose } S = \frac{1}{2} \mathcal{K} \text{ so that the quantity would be more potential energy like, but is not done here to be consistent and keep from having a constant in some formulations and not in others.} \)
amount of energy dissipated in the joints. In other words,

\[ F = \theta^T \mathcal{D} \dot{\theta} \]  \hspace{1cm} (154)

\[ = \text{Power dissipated in the joints} \]  \hspace{1cm} (155)

While this is correct mathematically, the calculation of the dissipative terms is very difficult to come by. Additionally, the majority of manipulators seek to minimize the dissipative forces. Even though it is correct, for this reason, it is not utilized.

4. \( S = \mathcal{M} \)

Using the mass of the inputs as a weighting matrix gives the kinetic energy of the input,\(^5\)

\[ F = \dot{\theta}^T \mathcal{M} \dot{\theta} \]  \hspace{1cm} (156)

\[ = I_{n_1} \dot{\theta}_1^2 + \cdots + I_{n_6} \dot{\theta}_6^2 \]  \hspace{1cm} (157)

For manipulators that have gearboxes, this will give a good indication of the inertia of the manipulator.

In summary, since the inputs are vectors typically without any special form, the screw weighting matrices, i.e. \( \mathbf{Y}, \mathbf{D}, \) and \( \mathbf{E}, \) can not be used. If the joints are all the same, the identity matrix can be used. However, for a general manipulator, a constitutive quantity, i.e. \( \mathcal{K}, \mathcal{D}, \) or \( \mathcal{M}, \) should be utilized.

7.2.1.2 Selection of \( T \)

For the \( T \) matrix, the quantities are screws and thus the screw norms are applicable. Also, since the \( h_2 \) constraint deals with the end effector coordinates, the constitutive properties in the end effector space, \( \mathbf{K} \) and \( \mathbf{M}_{ee} \), should be utilized.

\(^5\)Again, the true kinetic energy would be \( S = \frac{1}{2} \mathbf{M} \), but the \( \frac{1}{2} \) factor is left off for consistency.
1. $\mathbf{T} = 1_{6 \times 6}$

Since the constraint of Equation 96, $h_2 = \mathbf{S}^T \mathbf{T} \mathbf{S} - 1 = 0$, deals with the twist in axis coordinates, use of the identity matrix does not give meaningful results because it adds linear and angular velocity:

$$h_2 = \mathbf{S}^T 1_{6 \times 6} \mathbf{S} - 1 = 0$$  \hspace{1cm} (158)

$$\mathbf{v} \cdot \mathbf{v} + \omega \cdot \omega = 1$$  \hspace{1cm} (159)

$$???$$  \hspace{1cm} (160)

2. $\mathbf{T} = \mathbf{D}$ or $\mathbf{E}$

Since a twist is in axis coordinates only the $\mathbf{D}$ matrix is applicable (not $\mathbf{E}$) to normalize the twist via the invariant part. In other words,

$$h_2 = \mathbf{S}^T \mathbf{D} \mathbf{S} - 1 = 0$$  \hspace{1cm} (161)

$$\omega \cdot \omega = 1$$  \hspace{1cm} (162)

$$\|\omega\|^2 = 1$$  \hspace{1cm} (163)

which is the standard twist norm. This weighting matrix is not a true norm since it gives a zero result for any pure translation, no matter what the size. However, this weighting matrix is kept to show how it relates to other current measures.

3. $\mathbf{T} = \Upsilon$

The flipper matrix “norms” the twist via:

$$h_2 = \mathbf{S}^T \Upsilon \mathbf{S} - 1 = 0$$  \hspace{1cm} (164)

$$2\mathbf{v} \cdot \omega = 1$$  \hspace{1cm} (165)

$$2\rho \omega \cdot \omega = 1$$  \hspace{1cm} (166)

where $\rho$ is the pitch of the twist.\textsuperscript{6} Its meaning is not truly understood, but is

\textsuperscript{6}See Appendix C for details on calculation of the pitch and its associated meaning.
kept here for completeness. It should be noted that this too is not a true norm and can give strange results.

4. $T = K$

Utilizing the stiffness at the end effector, $K$, in constraint $h_2$ gives:

$$h_2 = t^T K s^t - 1 = 0$$  \hspace{1cm} (167)

Similar to $S = K$, the meaning of this constraint is the velocity potential energy is constant (i.e. the potential energy of the manipulator over fixed time, $\Delta t$, is constant).

5. $T = M_{ee}$

Using the mass of the end effector, $M_{ee}$, the constraint becomes the kinetic energy of the end effector and results in:

$$h_2 = t^T M_{ee} s^t - 1 = 0$$  \hspace{1cm} (168)

In summary, since the constraint deals with screws, the screw norms, $D$ and $Y$, can be used while the identity matrix has no meaning. The constitutive quantities $K$ and $M_{ee}$, can also be utilized.

7.2.1.3 Summary of Velocity Domain Measures

Table 2 summarizes the different measures that can be used. From the table it is easy to see that only twelve different measures remain that can be utilized to have invariant properties and some type of physical meaning. However, three measures give non-meaningful results. One measure, $S = 1$, $T = K$, will be discussed later. The other two, i.e. $S = K$, $T = K$ and $S = M$, $T = M_{ee}$, can be eliminated immediately since the eigenvalue problem gives only trivial results. For this choice of weighting matrices, the objective function and the constraint are basically the same and lead to the eigenvalues being zero. Each of the remaining ten measures are now discussed in detail.
Table 2: Summary of Velocity Domain Optimization Problem

1. $S = 1_{6 \times 6}$, $T = D$ : Power

The Pottmann et al. measure discussed in Chapter 6 can be transformed into the standard form given in Equation 96 with $S = 1_{6 \times 6}$ and $T = D$. The solution is the generalized eigenvalue problem:

$$\text{det} \left( J^T 1_{6 \times 6} J - \mu v D \right) = 0$$

(169)

and

$$P^2 = \mu v_{\text{min}} \quad \text{or} \quad P = \sqrt{\mu v_{\text{min}}}$$

(170)

Therefore, the square root of the eigenvalue gives the minimum power.\(^7\)

Again, one should note that the $D$ matrix normalizes the screw to have the invariant portion be one. This normalization causes problems when the invariant part (i.e. angular velocity) goes to zero (i.e. a translation). Therefore, the procedure may leave out the “truly minimal” eigenvalue. In fact, the procedure yields only 3 eigenvalues in the spatial case. In other words, the

\(^7\)An interesting side note is that the maximum eigenvalue will necessarily maximize the power, i.e. $P^2_{\text{max}} = \mu v_{\text{max}}$. The associated eigenvector, $\mathbf{s}_{\text{max}}$, will then be the direction that has the most constraint and would be useful for the design of robots to be rigid in a certain direction.
eigenvalues associated with pure translation are left out of the analysis. This is seen by multiplying out the eigenvalue problem:

\[
\det \left( J^T 1_{6 \times 6} J - \mu_w D \right) = 0
\]

\[
\det \left( J^T J - \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_w & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_w & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_w \\
\end{bmatrix} \right) = 0
\]

which results in 3 eigenvalues. For the planar case, this degenerates to 1 eigenvalue.

Pottman et al. [100] define a way to overcome this shortfall by augmenting the problem with another optimization problem that only looks at the translation. In other words, the rotational part of the screw is set to zero. Mathematically, this becomes:

\[
\min \quad F = \sum_{i=1}^{k} (J_i^T \mathbf{s}_{trans}^t)^2 = \sum_{i=1}^{k} \left( J_i^T \begin{bmatrix} v \\ 0 \end{bmatrix} \right)^2
\]

subject to \( h = \|\mathbf{s}_{trans}^t\| = \|\mathbf{v}\| = 1 \)

(172)

However, there are several problems with this formulation. First, it requires a second optimization procedure and thus a method to determine how to compare the two results. Secondly, this optimization can not be placed in the generalized framework. Lastly, and most importantly, the measure is not frame or unit invariant. There is a measure (i.e. the input torque) in the force domain that is very close to the translational power that fits into the framework and completes the eigenvalue problem.
2. $S = 1_{6 \times 6}, \ T = \gamma$: Power (Flipper Normed) Another possible way to norm a screw is to use the flipper matrix. Therefore, the problem becomes:

$$\det (J^T J - \mu_v \gamma) = 0$$  \hspace{1cm} (173)

Since the flipper matrix is of full rank, the measure identifies all the singularities. However, the physical meaning of the measure is unknown.

3. $S = 1_{6 \times 6}, \ T = K$: Power (Velocity Potential Energy Normed)

A logical choice for a constitutive property that may help is the stiffness. Using the stiffness in the constraint, the corresponding eigenvalue problem becomes:

$$\det (J^T 1_{6 \times 6} J - \mu_v K) = 0$$  \hspace{1cm} (174)

Now the constraint normalizes the twist such that its velocity potential energy is unitary. However, the objective function in this case needs to have consistent actuators for this to work. Furthermore, if the stiffness are all the same, the eigenvalue problem becomes:

$$\det (J^T 1_{6 \times 6} J - \mu_v k J^T 1_{6 \times 6} J) = 0$$  \hspace{1cm} (175)

which gives no meaningful results. Therefore, the measure only gives meaningful results if the inputs are all consistent, but not identical. Even in this case, the measure yields non-meaningful results. That is:

$$\det \left( J^T 1_{6 \times 6} J - \mu_v J^T \begin{bmatrix} k_1 \\ \vdots \\ k_6 \end{bmatrix} J \right) = 0$$  \hspace{1cm} (176)

$$\mu_v = \frac{1}{k_1} \cdots \frac{1}{k_6}$$

which the optimization gives only the trivial result of the smallest stiffness value. Although the units are consistent and invariant, it only leads to the trivial result.
in the best case. Therefore, this measure should not be used unless the stiffness matrix is augmented to contain stiffness that is not included in the actuated joints. Therefore, Table 2 shows an “NA” since it does not yield a meaningful measure.

4. $S = I_{6 \times 6}$, $T = M$: Power (Kinetic Energy Normed)

Another way around the problem with the translational singularities is to normalize the twist such that the kinetic energy of the end effector is constant. The resulting eigenvalue problem is:

$$\det (J^T I_{6 \times 6} J - \mu v M_{ee}) = 0 \quad (177)$$

Since the mass of the end effector, $M_{ee}$, is used, the problem yields nontrivial results. Additionally, since the inertia matrix is of full rank, there are six eigenvalues. The physical meaning of the measure is the same as power, except now it is for a certain kinetic energy input of the end effector. In other words, it estimates the size of the manipulator input, $\dot{\theta}$, for a certain constant kinetic energy of the end effector. If the size of inputs is small, then the manipulator has very poor mechanical advantage and it is close to an instantaneous degree of freedom of the end effector, i.e. a singularity.

5. $S = K$, $T = D$: Stiffness

Taking $S = K$ gives the velocity potential energy objective function. The minimum eigenvalue measures the stiffness in the least constrained direction. This formulation assumes that the actuated joints contribute the most significant portion of the stiffness in the manipulator. This stiffness may be from by the motor stiffness introduced by independent joint control.\(^8\)

\(^8\)There will be more on the stiffness and the assumptions in the discussion section.
The resulting eigenvalue problem is:

$$(J^T \kappa J - \mu_v D) \mathbf{S}^t = 0$$  \hspace{1cm} (178)$$

which is the same as the stiffness measure in Chapter 6. Again, note that due to the degeneracy of the $D$ matrix, the problem only yields 1 or 3 eigenvalues for the planar and spatial cases, respectively. Also, the measure does not identify translational singularities.

If the stiffness in each actuator is identical, then

$$\kappa = k \mathbf{1}_{6 \times 6}$$  \hspace{1cm} (179)$$

and the problem degenerates to:

$$\begin{align*}
\text{min} \quad & F(\mathbf{S}^t) = k \mathbf{S}^{tT} J^T J \mathbf{S}^t \\
\text{subject to} \quad & h(\mathbf{S}^t) = \mathbf{S}^{tT} D \mathbf{S}^t - 1 = 0
\end{align*}$$  \hspace{1cm} (180)$$

which only differs from the power measure by a constant. Therefore, the power measure and the stiffness measure are the same if the stiffness matrix is a scalar multiple of the identity matrix.

6. $S = \kappa$, $T = \Upsilon$: Stiffness (Flipper Normed)

One could also use the flipper matrix in the constraint resulting in the following eigenvalue problem:

$$(J^T \kappa J - \mu_v \Upsilon) \mathbf{S}^t = 0$$  \hspace{1cm} (181)$$

The flipper matrix will find all the singularities, but again, the resulting physical meaning is unknown.

7. $S = \kappa$, $T = M$: Natural Frequency

It is well known that the natural frequency is found from an eigenvalue problem [29]. This problem solves the equations of the form:

$$\det (\mathbf{K} - \omega_n^2 \mathbf{M}) = 0$$  \hspace{1cm} (182)$$
where $M$ is the mass matrix which includes rotational inertia. This arises from assuming a solution of the form, $e^{i\omega_n t}$ into the general undamped equations of motion of the manipulator (see Appendix A):

$$M\ddot{X} + KX = 0$$  \hspace{1cm} (183)

As introduced in Chapter 6, the natural frequency results from a similar optimization problem with $S = K$ and $T = M$. The associated eigenvalue problem becomes:

$$\det\left(\frac{J^TKJ}{K} - \mu_v M\right) = 0$$  \hspace{1cm} (184)

Examining this equation shows that this equation is the same as Equation 182, but now $\mu_v = \omega_n^2$.

What also should be noticed is that the optimization minimizes $F = S^T K S^t$ which is double the “potential energy” like quantity in the system subject to the constraint $S^T M S^t = 1$ which is double the kinetic energy being unitary.

Due to the normalization matrix, $M$, having full rank, the eigenvalue problem yields a full set of eigenvalues and eigenvectors. Therefore, the problems that arose in the previous two formulations have been taken care of by the mass matrix. The mass matrix effectively provides the characteristic length between rotational and translational quantities.

8. $S = M$, $T = D$: Kinetic Energy

Another way of measuring the effects of singularities is to use the amount of kinetic energy of the input for a unit twist change of output. Therefore, the resulting eigenvalue problem becomes:

$$\det\left(J^T M J - \mu_v D\right) = 0$$  \hspace{1cm} (185)

\footnote{Again, care should be noted about the meaning of the derivative and the use of quasi-coordinates.}
If all the motors are of the same, the eigenvalue problem degrades to:

$$\det \left( InJ^T1_{6\times6}J - \mu_e D \right) = 0$$

(186)

However, if all the motors are identical, the measure differs from the stiffness and power measures by a constant. Therefore, this measure should only be used in instances where the inputs are of mixed units. Furthermore, since the $D$ matrix is used in the constraint, this measure also does not find translational singularities.

9. $S = \mathcal{M}$, $T = \mathcal{Y}$: Kinetic Energy (Flipper Normed)

Likewise, the flipper matrix can be utilized to normalize the kinetic energy. The resulting eigenvalue problem becomes:

$$\det \left( J^T \mathcal{M} J - \mu_e \mathcal{Y} \right) = 0$$

(187)

Like the measures in the velocity domain, the physical meaning behind the measure is unknown.

10. $S = \mathcal{M}$, $T = \mathcal{K}$: Kinetic Energy (Velocity Potential Energy Normed)

This measure has units of the reciprocal of the natural frequency. In other words, the resulting eigenvalue problem becomes:

$$\det \left( J^T \mathcal{M} J - \mu_e \mathcal{K} \right) \mathcal{S}^t = 0$$

(188)

If interpreted as the reciprocal of the natural frequency, it only computes the kinetic energy on the input masses and thus is applicable only if the motors have gearheads so that the inertia of the manipulator is small compared to that of the motors. Also, the stiffness is only due to the control, and will, in general, not contain the stiffness or nonlinearities of the gearheads. Therefore, in general, it is better to use the prior formulation which gives a more accurate natural frequency. Another interpretation of this measure is the kinetic energy of the inputs due to a certain velocity potential energy.
7.2.2 Force Domain Analysis

To make the analysis complete, the same procedure must be followed for the force domain problem in Equation 117. However, with the analysis in the velocity domain already performed, the analysis in the force domain is easier.

7.2.2.1 Selection of $S$

Like the case for the velocity domain analysis, the screw norms ($D$, $E$, and $\mathcal{T}$) or the constitutive properties at the end effector ($M_{ee}$ and $K$) do not work at the input, and thus are eliminated from consideration.

1. $S = \mathbf{1}_{6 \times 6}$

   The identity matrix in the force domain problem results in the sum of the squares of input torques/forces.

   \[
   F = \tau^T \mathbf{1}_{6 \times 6} \tau
   \]

   \[
   = (\|\tau\|)^2
   \]  

   Like the case in the velocity domain, all the inputs must be of the same type for this to be applicable.

2. $S = C$

   Using the compliance of the input gives the potential energy of the input.\textsuperscript{10} That is:

   \[
   F = \tau^T C \tau
   \]

   \[
   = 2(\text{Potential Energy})
   \]

3. $S = M^{-1}$

   The masses of the inputs, $M^{-1}$, can be used. However, the meaning of this

\textsuperscript{10}For a detailed proof of this, see Appendix B of Joh’s dissertation [46].
quantity is unknown:

\[ F = \tau^T M^{-1} \tau \] (193)

However, the units will work out, it provides a meaningful result when combined with the appropriate constraint.

7.2.2.2 Selection of \( T \)

1. \( T = 1_{6 \times 6} \)

Since the constraint deals with wrenches, this matrix does not give meaningful results since, it adds the square of forces and the square of moments together.

2. \( T = D \) or \( E \)

Since wrenches are assumed to be in ray coordinates, the \( E \) matrix should be utilized. In other words,

\[ h_2 = \$w^T E \$w - 1 = 0 \] (194)
\[ \mathbf{f} \cdot \mathbf{f} = 1 \] (195)
\[ \| \mathbf{f} \|^2 = 1 \] (196)

which is the standard wrench norm.

3. \( T = \Upsilon \)

The flipper matrix norms the wrench via:

\[ h_2 = \$w^T \Upsilon \$w - 1 = 0 \] (197)
\[ 2 \mathbf{f} \cdot \mathbf{m} = 1 \] (198)
\[ 2 \rho \mathbf{f} \cdot \mathbf{f} = 1 \] (199)

where \( \rho \) is the pitch of the wrench. Again, the meaning of this is unclear, but it is kept here for completeness.
4. \( T = C \)

Using the compliance at the end effector, \( C \), the potential energy of the manipulator is unitary. That is:

\[
\begin{align*}
  h_2 &= \$^{wT}C\$^w - 1 = 0 \\
  2(\text{Potential Energy}) &= 1
\end{align*}
\]  

(200) \hspace{1cm} (201)

However, from the definition of the compliance matrix, this is only defined at poses that are not singular because the inverse of the Jacobian matrix must be taken.

5. \( T = M_{ee}^{-1} \)

Letting \( T = M_{ee}^{-1} \) results in a constraint without any physical meaning, but is included since it yields some important results. That is:

\[
\begin{align*}
  h_2 &= \$^{wT}M_{ee}^{-1}\$^w - 1 = 0
\end{align*}
\]  

(202)

7.2.2.3 Summary of Force Domain Measures

Table 3 summarizes the different measures that can be used in the force domain. From the table it is easy to see that again only ten different measures remain which have invariant properties and some type of physical meaning. Each of these are now discussed in detail.

1. \( S = 1_{6 \times 6}, T = E : Input Torque \)

The first item to examine is the equivalent to power in the force domain. Here choosing \( S = 1_{6 \times 6} \) finds the input force for a given output wrench. As is shown in previous chapters, the input forces at a singularity increase toward infinity in order to maintain static equilibrium. Therefore, instead of minimizing the objective function, the maximum of the objective function is used to determine how close a pose is to a singularity. For consistency with the velocity domain
<table>
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<th>$T$</th>
<th>$C$</th>
<th>$M^{-1}$</th>
</tr>
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<td>Wrench Norm</td>
<td>Force times moment</td>
<td>Potential Energy</td>
</tr>
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<td>Comment</td>
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<td>Not a true norm</td>
</tr>
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</table>

<table>
<thead>
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<th>$S$</th>
<th>$C$</th>
<th>$M^{-1}$</th>
</tr>
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<tr>
<td>Potential Energy</td>
<td>If all compliances are the same it is equal to input torque</td>
<td>Potential Energy</td>
</tr>
<tr>
<td>$M^{-1}$</td>
<td>No Meaning</td>
<td>If all inertias are the same it is equal to input torque</td>
</tr>
</tbody>
</table>

**Table 3:** Summary of Force Domain Optimization Problem

and to keep very large numbers from occurring, the reciprocal of the eigenvalue is used as a measure of closeness to a singularity. In other words:

$$\det (1_{6\times 6} - \mu I J^T J E) = 0$$  \hspace{1cm} (203)

where measure $= \frac{1}{\mu f_{max}}$

Therefore, this measure approximates the amount of internal forces or the mechanical advantage of the manipulator in a specific pose. Like the velocity domain, normalizing the output wrench with respect to the screw norm misses some of the pure couple singularities (i.e. translational singularities in the velocity domain).

This measure is also very close to the translational power measure introduced by Pottmann et al. [100] and briefly discussed in item 1 in Section 7.2.1.3. The translational power works on a subset of the Jacobian matrix, and thus gives slightly different results from the input torque measure. However, the input torque measure is the dual to the power measure and completes the analysis in a frame and unit invariant way.
2. $S = 1_{6 \times 6}, \ T = \gamma : \text{Input Torque (Flipper Normed)}$

Using the input torque as the of objective function and norming the output by the flipper matrix results in the following eigenvalue problem:

$$\text{det} \left( 1_{6 \times 6} - \mu_f J^T J \gamma \right) = 0 \quad (204)$$

This measure identifies all the singularities, but the physical interpretation is unknown.

3. $S = 1_{6 \times 6}, \ T = C : \text{Input Torque (Potential Energy Normed)}$

Another way to overcome the problem with the input torque measure is to normalize the wrench to have unitary potential energy. Therefore, the resulting eigenvalue problem becomes:

$$\text{det} \left( 1_{6 \times 6} - \mu_f J^T J C \right) = 0 \quad (205)$$

which for stiffnesses only in the input joints becomes:

$$\text{det} \left( 1_{6 \times 6} - \mu_f J^T J J^{-1} CJ^{-T} \right) = 0 \quad (206)$$
$$\text{det} \left( 1_{6 \times 6} - \mu_f J^T CJ^{-T} \right) = 0 \quad (207)$$

where the measure now still is the inverse of the maximum eigenvalue. However, the compliance matrix is only defined if the Jacobian matrix is non-singular. Unlike its dual in the velocity domain, this measure gives meaningful results.

4. $S = 1_{6 \times 6}, \ T = M_{ee}^{-1} : \text{Reciprocal of Power (Kinetic Energy Normed)}$

This measure is the same as the kinetic energy power measure that was given previously. This is seen by observing the resulting eigenvalue problem:

$$\text{det} \left( 1_{6 \times 6} - \mu_f J^T J M_{ee}^{-1} \right) = 0 \quad (208)$$

This eigenvalue problem gives the reciprocal of eigenvalues of the kinetic energy normed power measure which was:

$$\text{det} \left( J^T J - \mu_v M_{ee} \right) = 0 \quad (209)$$
This is seen with the following proof:

\[
\det \left( \frac{1}{\mu_v} J^T J - M_{ee} \right) = 0 \quad (210)
\]

\[
\det \left( \frac{1}{\mu_v} J^T J M_{ee}^{-1} - 1_{6\times6} \right) M_{ee} = 0 \quad (211)
\]

\[
\det \left( \frac{1}{\mu_v} J^T J M_{ee}^{-1} - 1_{6\times6} \right) \det (M_{ee}) = 0 \quad (212)
\]

\[
\det \left( \frac{1}{\mu_v} J^T J M_{ee}^{-1} - 1_{6\times6} \right) = 0 \quad (213)
\]

\[
\det \left( 1_{6\times6} - \frac{1}{\mu_v} J^T J M_{ee}^{-1} \right) = 0 \quad (214)
\]

which is the same as the previous measure where \( \mu_f = \frac{1}{\mu_v} \). While the eigenvalues are the same in either domain, the corresponding eigenvectors are different.

5. \( S = C, T = E \) : Potential Energy

If the objective function is the potential energy, and the wrench is normed via the standard wrench norm, then the resulting eigenvalue problem becomes:

\[
\det \left( 1_{6\times6} - \mu_f J^T C^{-1} J E \right) = 0 \quad (215)
\]

\[
\det \left( 1_{6\times6} - \mu_f K E \right) = 0 \quad (216)
\]

While this formulation is close to the stiffness measure in the velocity domain, it is not the same. Since \( E \) is not invertible, it can not be transformed from its current form to that of its dual, the stiffness measure.

6. \( S = C, T = \Upsilon \) : Potential Energy (Flipper Normed)

Likewise, the flipper matrix can be utilized to normalize the output wrench. The resulting eigenvalue problem becomes:

\[
\det \left( 1_{6\times6} - \mu_f J^T C^{-1} J \Upsilon \right) = 0 \quad (217)
\]

\[
\det \left( 1_{6\times6} - \mu_f K \Upsilon \right) = 0 \quad (218)
\]

This measure also can not be transformed from its current state, but is close to its dual, the stiffness (flipper normed) measure in the velocity domain.
7. $S = C$, $T = M_{ee}^{-1}$: Reciprocal of the Natural Frequency

This measure does not yield any new information, since the resulting eigenvalue problem is:

\[
\det \left( \mathbf{1}_{6 \times 6} - \mu_f \mathbf{J}^T \mathbf{C}^{-1} \mathbf{J} \mathbf{M}_{ee}^{-1} \right) = 0 \tag{219}
\]
\[
\det \left( \mathbf{1}_{6 \times 6} - \mu_f \mathbf{K} \mathbf{M}_{ee}^{-1} \right) = 0 \tag{220}
\]

Using the same analysis that was used in the reciprocal of power (kinetic energy normed), the eigenvalues are the same as the reciprocal of the natural frequency:

\[
\det \left( \mathbf{K} - \mu_v \mathbf{M}_{ee} \right) = 0 \tag{221}
\]

where $\mu_f = \frac{1}{\mu_v}$.

8. $S = \mathcal{M}^{-1}$, $T = \mathbf{E}$: Unknown

Using the inverse of the mass matrix for the objective function and the screw norm for the output gives:

\[
\det \left( \mathbf{1}_{6 \times 6} - \mu_f \mathbf{J}^T (\mathcal{M}^{-1})^{-1} \mathbf{J} \mathbf{E} \right) = 0 \tag{222}
\]
\[
\det \left( \mathbf{1}_{6 \times 6} - \mu_f \mathbf{J}^T \mathcal{M} \mathbf{J} \mathbf{E} \right) = 0 \tag{223}
\]

which can not be transformed to any of the velocity domain measures. Since the objective function does not have any meaning, it will not be used, but is left here for completeness.

9. $S = \mathcal{M}^{-1}$, $T = \mathbf{Y}$: Unknown

Likewise, normalizing the output wrench with the flipper matrix does not help. The resulting eigenvalue problem becomes:

\[
\det \left( \mathbf{1}_{6 \times 6} - \mu_f \mathbf{J}^T \mathcal{M} \mathbf{J} \mathbf{Y} \right) = 0 \tag{224}
\]

which also has no meaning.
10. \( S = M^{-1}, \, T = C \): Reciprocal of Kinetic Energy (Potential Energy Normed)

The resulting eigenvalue problem is:

\[
\det(1_{6 \times 6} - \mu_f J^T M J C) = 0 \tag{225}
\]

which is the reciprocal of the potential energy normed kinetic energy measure. However, in this case, the compliance matrix is not defined at singularities, and thus the velocity domain formulation should be utilized.

### 7.3 Discussion of Results

What should be evident now is that many of the existing measures of closeness to singularities are really different variations on the same problem. The measures are related to a constrained optimization problem which is transformed to a corresponding eigenvalue problem. The difference in the measures is how one defines the cost function that is being minimized and how the associated constraint screw is being normalized. The resulting eigenvalue can be used as a measure of how close one is to a singularity. \( S \) and \( T \) are some general normalization matrices.

From the previous analysis, several comments can be made regarding the choice of a measure:

1. In general, a constitutive property should be utilized in the objective function to make the units uniform and give a physical meaning.

2. If all of the inputs are identical (mass and stiffness) then all the objective functions reduce to the power or input torque measures.

3. The use of the constitutive quantities for the constraint give reciprocal results across the velocity and force domains.

4. For the constraint, \( h_2 \), the screw norms (\( D \) and \( E \)) miss some of the singularities since the weighting matrices are singular.
5. The use of the screw norms in the velocity (force) domain result in the portion of the problem that is missing in either the force (velocity) domain. For example, the singularities missed in the power measure will be identified in the input torque measure. However, now there are two measures that need to be examined. Using constitutive properties eliminates this issue by giving one physically meaningful quantity.

6. The weighting matrices could be combined in any linear combination to create even more measures.

7. The determinant measures are not given under this framework.

The major issue with the entire analysis is that non-kinematic parameters are introduced to solve what many consider a kinematic problem. That is, singularities are a deficiency of the Jacobian. To measure the problem, the stiffness measure brings in the stiffness of the manipulator, and the natural frequency brings in the stiffness and the mass of the manipulator as well.

There are two responses to this concern. The first is there exists no invariant metric on the group of rigid body displacements [88]. Therefore, any metric necessitates a choice on how to weight the translational and rotational portions together. The stiffness and mass matrices are a very natural choice for this weighting. Secondly, as is pointed out in Chapter 4, singularities are not truly kinematic. They affect the stiffness and dynamic properties of the robot. Therefore, the stiffness and natural frequency are good ways to measure these effects.

### 7.4 A Word on Scope

During the analysis, the mass and stiffness matrices are considered to be diagonal and symmetric. The symmetry allows the derivative of the Lagrangian to be taken such that the eigenvalue problem was encountered. If these matrices are not symmetric, a
different problem would arise. This is best shown by example.

For now, assume that neither the mass or stiffness matrix is symmetric. Using the general constrained optimization problem presented in Section 7.1 gives.

$$\min \quad F(t) = t^T S t$$

subject to \( h(t) = t^T T t - 1 = 0 \) \hspace{1cm} (226)

The analysis with the Lagrangian would be the same:

$$\mathcal{L}(t, \lambda) = t^T S t - \lambda \left( t^T T t - 1 \right)$$ \hspace{1cm} (227)

But now, differentiating the Lagrangian would result in two terms:

$$\frac{\partial \mathcal{L}}{\partial t} = S t + \left( t^T S \right)^T - \lambda \left( T t + \left( t^T T \right)^T \right)$$

$$\hspace{1cm} = \left[ (S + S^T) - \lambda (T + T^T) \right] t$$ \hspace{1cm} (228)

However, any matrix can be decomposed into its symmetric and anti-symmetric parts [39]. In this dissertation, it is defined:

$$S = S_s + S_a \quad T = T_s + T_a$$ \hspace{1cm} (229)

where the subscripts \( s \) and \( a \) represent the symmetric and anti-symmetric parts, respectively. Substituting this equation into the previous equation gives:

$$\frac{\partial \mathcal{L}}{\partial t} = \left[ (S_s + S_a + (S_s + S_a)^T) - \lambda \left( T_s + T_a + (T_s + T_a)^T \right) \right] t$$

$$= \left[ (S_s + S_a + S_s - S_a) - \lambda (T_s + T_a + T_s - T_a) \right] t$$

$$= \left[ 2S_s - 2\lambda T_s \right] t$$

$$= 2 (S_s - \lambda T_s) t = 0$$ \hspace{1cm} (230)

which for symmetric matrices, \( S \) and \( T \) leads to exactly the result obtained in Section 7.1. However, for non-symmetric matrices, the procedure eliminates the anti-symmetric portion of the matrices, and utilizes the symmetric portion no matter whether the stiffness or mass matrices are symmetric or not. Therefore, the question that needs to be answered is when are the mass and stiffness matrices symmetric.
7.4.1 Symmetric Stiffness Matrices

The derivation of the stiffness matrix shows under what circumstances the stiffness matrix is symmetric. In this chapter, the stiffness is expressed at the end effector (i.e. task space coordinates). Therefore, the stiffness in the joints and actuators need to be transformed into task space coordinates.

There are three major sources of stiffness in robotic manipulators (see Figure 16). They are:

1. Controller stiffness - the stiffness introduced by a feedback controller.
2. Actuator stiffness - the flexibility of “rigid” components at the individual joint level (e.g. gearboxes, shafts, etc.)
3. Link stiffness - flexibility of all other “rigid” components (e.g. links, bearings, joints, end effector frame, etc.)

The first two can be encompassed in a symmetric stiffness matrix at the end effector, whereas the third one in general cannot.

- **Controller Stiffness** For the controller stiffness shown in the leftmost figure,
there is a virtual spring\textsuperscript{11} that connects the link to the ground. This stiffness is easier to express in the joint space coordinates, so let us start there. Let $\theta$ be a vector of joint displacements. Assuming each joint has a stiffness due to the controller that depends only on the angle of the first link gives:

$$\tau = \mathcal{K} \Delta \theta$$  \hspace{1cm} (231)

where $\mathcal{K}$ is a diagonal matrix that contains the stiffness coefficients for each actuator and $\Delta \theta$ is a small displacement of the actuators. We now need to relate this to the task space coordinates. Using the Jacobian relationship given in Chapter 3

$$\Delta \theta = J \Delta X$$  \hspace{1cm} (232)

we can relate the torques of the joints to the end effector coordinates by substituting back into the previous equation:

$$\tau = \mathcal{K} J \Delta X$$  \hspace{1cm} (233)

where $\tau$ is a vector of actuator torques due to controller stiffness. The torques in the joints are transformed to end effector coordinates by the static Jacobian relationship (see Chapter 3):

$$\mathbf{w}^w = J^T \tau$$  \hspace{1cm} (234)

where $\mathbf{w}^w$ is the resulting wrench at the end effector due to controller stiffness. Combining these equations shows the stiffness of the manipulator due to the controller:

$$\mathbf{w}^w = \underbrace{J^T \mathcal{K} J \Delta X}_{K}$$  \hspace{1cm} (235)

where $K$ is the stiffness matrix as seen by an observer at the end effector, and $\mathcal{K}$ is the stiffness matrix at the joint level. Therefore, the stiffness at the end

\textsuperscript{11}This virtual spring is due to a feedback controller. The exact form of the controller is not important, but here assume that each joint is under independent PD control.
effector consists of the stiffness at each joint pre-multiplied by $J^T$ and post-multiplied by $J$. That is:
\[ K = J^T \mathcal{K} J \]  
(236)

which was what was presented previously. Since $\mathcal{K}$ is diagonal, the $K$ is guaranteed to be symmetric as shown below:

Proof

\[ K^T = (J^T \mathcal{K} J)^T \]
\[ = J^T \mathcal{K}^T J \]
\[ = J^T \mathcal{K} J \]
\[ = K \quad \text{ Q.E.D. } \square \]  
(237)

Therefore, controller stiffness can be modeled as a symmetric stiffness matrix.

- **Actuator Stiffness** Actuator stiffness comes from the motor and gear train if applicable. This stiffness can be very significant if a large gear train is used. The actuator stiffness also results in a diagonal matrix at the joint level, and thus can be modeled just like the controller stiffness. Therefore, actuator stiffness at the end effector is a symmetric stiffness matrix that can be accounted for in the measure.

- **Link Stiffness** Link stiffness refers to the link bending due to the internal forces in the mechanism. All links will deform under load, but with varying degrees. While the links in most applications are designed to be stiff, very large mechanisms can have links that result in a significant amount of deflection under load.

Using techniques from strength of materials, the bending of the first link could be crudely modeled as a cantilever beam in two dimensions as (see Appendix
D):

\[ k_i = \frac{3EI}{L} \]  \quad (238)

which is a diagonal stiffness matrix in the joint space and results in a symmetric stiffness matrix in the task space.

There are several problems when trying to expand this analysis.

1. If the links bend to the point where the small angle approximation no longer applies, the analysis breaks down. Another variable would need to be introduced into the analysis for the position of the end of the link. Therefore, the kinematics, or more precisely in this instance, the differential kinematics would change and would result in an equation of the form:

\[ K = J^T \mathcal{K} \hat{J} \]  \quad (239)

where \( \hat{J} \) is a matrix that relates the new variable and the output. In other words, the Jacobian relationship is no longer be valid since the kinematics which are based on a rigid model. This equation is not guaranteed to be symmetric except for special configurations.

2. The cantilever model assumes that the manipulator is in a quasi-static state. Therefore, if the manipulator is very fast, other methods from vibration analysis should be utilized. This analysis requires modeling the links with assumed modes of vibration that are different in shape than that given by a quasi-static process. This analysis then requires another coordinate that would relate the position of the end of the link. That position then needs to be related to the end effector coordinates through another matrix, i.e. the \( \hat{J} \) matrix given previously. Therefore, this type of analysis would not in general result in a symmetric stiffness matrix.
3. As was shown by Duffy [23], external forces on the manipulator cause non-symmetric stiffness matrices.

Expanding the simple cantilever model to the distal links results in the introduction of another coordinate and thus causes problems like those given in item 1 above. Therefore, the inclusion of link stiffness into the model can only be done with a crude model on the first links. However, there is a point of diminishing returns. The added effort does not give appropriate return. By adding more and more complexity to the models of stiffness does make the stiffness more accurate, but the improvement is believed to be small.

7.4.2 Symmetric Mass Matrices

Just like the stiffness case, the mass matrix also needs to be examined. Obviously, the mass of the end effector can be included without problem as it is already in the task space coordinates. The mass of the links connected to the base can easily be transferred to the end effector using the same analysis as before. The only problem is to include the mass of the distal links. If we assume that the links are all rigid, there is no need for the introduction of any other coordinates. The kinematics (and similarly, the differential kinematics) provide a relationship that allows the mass matrix to always be symmetric with the appropriate extended Jacobian like matrix. However, like the stiffnesses, the level of detail of the modeling encounters a point of diminishing returns.
CHAPTER 8

EXAMPLES OF DIFFERENTIAL KINEMATIC MEASURES USING THE GENERAL FRAMEWORK

Several of the measures in the previous chapter fail to detect some of the singularities. More specifically, using the screw weighting matrices in the $h_2$ constraint does not give a full representation of how close a pose is to a singularity. This issue is seen quite easily with an example.

First, three measures (power, input torque, and natural frequency) are computed for a 3RRR manipulator undergoing a rotational trajectory. Second, the same measures are applied to a degenerate 3RRR manipulator under the same trajectory. What is shown is that the power and input torque measures fail to detect some of the singularities in either case, whereas the natural frequency measure captures all the singularities.

8.1 Application of Measures to the 3RRR Robot

Consider the 3RRR manipulator shown in Figure 5 and repeated in Figure 17. Using the Jacobian relationships in Section 3.3, several different measures are computed for different poses. For this particular example, the leg lengths are $l_1 = l_2 = 1.1m$ and $l_3 = 0.18m$ with motor mount positions at $(0, 0)$, $(3m, 0)$ and $(1.5m, 2.6m)$. The end effector position is fixed at the center point of the workspace, $x = 1.5m$, $y = 1.3m$ and rotates around $360^\circ$.\footnote{This assumes an ideal manipulator that does not have any interference with its legs.} A suitable measure is calculated and plotted for each angular
value. Calculation of the determinant shows that singularities occur twice for this range of values at $\phi \approx 152^\circ$ and $323^\circ$.

8.1.1 Power

Figure 18 shows the power measure ($S = 1_{3 \times 3}$, $T = D$ in the velocity realm) for the manipulator as it rotates around the center point of the workspace. The square root of the eigenvalue (i.e. the power in Chapter 7 where $P = \sqrt{\lambda_{\text{min}}}$) is shown so that the singularities are easily seen. Figure 18 shows that the power does indeed go to zero at the singularities.

For this particular example, the associated eigenvector is always a screw whose axis lies perpendicular to the plane of the manipulator. This axis is through the well-known instant center of rotation.\footnote{See Appendix C for the calculations for determining the axis of rotation.} Figure 19 shows the location of the screw axis from the center of the end effector (i.e. one would add the position on the graph to
Figure 18: Power for the 3RRR Manipulator rotating around its center point

Figure 19: Corresponding instant center of rotation for the nearest singularity for Figure 18

\((x, y) = (1.5m, 1.3m)\). These plots show that the end effector tends to rotate around an axis that is located near the end effector.

For completeness, Pottmann et al.’s alternate derivation for the power via translation (noted translational power) is calculated and yields Figure 20.\(^3\) Unfortunately, this formulation is frame and scale variant and thus should not be utilized. It is included here to show how it relates to the power’s dual force domain result, i.e.

\(^3\)The instant center of rotation plots are not given for this case because the instant centers will all be at infinity and thus yield no extra information about the problem.
Figure 20: Translational power measure for 3R.RRR manipulator rotating around its center point.

input torque.

8.1.2 Input Torque

Figure 21 shows the input torque measure (i.e. $S = 1_{3 \times 3}$, $T = E$ in the force realm) over the same input. Since the input torque measure only shows the pure force wrench singularities, it does not show the singularities that are discovered in the power measure and always remains much larger than zero. In fact, the measure is very close to the translational measure shown in Figure 20. However, this measure, unlike the translational power, is frame and scale invariant. Therefore, the power and input torque measures can be combined to identify all the singularities. However, the natural frequency combines them in a physically meaningful way.

8.1.3 Lowest Linearized Natural Frequency

Figure 22 shows the natural frequency (i.e. $S = K$, $T = M_{ee}$ in the velocity realm) for the 3R.RRR robot. The mass of the end effector is 1kg and its inertia is 1kg $m^2$ (i.e. $M_{ee} = 1_{3 \times 3}$). Comparing this figure to Figure 18 shows that there is not much difference between the power and the natural frequency in this particular manipulator.
Figure 21: Square root of input torque measure for 3RRR manipulator rotating around its center point

Figure 22: Natural frequency for the 3RRR manipulator rotating around its center point

and trajectory. For this particular case, the translational portion of the eigenvalue problem is not close to the trajectory chosen. Therefore, the two graphs are very similar (close inspection shows that there is a very small difference at large values of natural frequency and power), and in fact should be since the natural frequency combines the input torque and power measures. Since the input torque measure in Figure 21 is large over the entire range, the natural frequency trends like the power measure.
8.2 Application of Measures to a Degenerate 3RRR Robot

A good example to show how the screw normed measures do not fully capture the translation is the degenerate 3RRR robot shown in Figure 23. In this robot, the end effector (point $C_2$) is no longer a triangle, but degrades to a line. Also, the motor mount positions lie on a line as well.

A robot with with $l_1 = l_2 = 1.75m$ and $l_3 = 1.5m$ and motor mounts at $(0, 0), (1.5m, 0)$ and $(3m, 0)$ is chosen. Fixing the end effector position to be at the point, $x = 1.5m, y = 1.3m$, the end effector is rotated around 360° and the associated eigenvalue measure is calculated. Calculation of the determinant of the Jacobian matrix shows that there are five angles that are singular: $\phi \approx 0°, 143°, 202°, 235°,$ and $349°$. When the end effector is at zero degrees, $\phi = 0°$, the manipulator is in a singularity that is pure translation. In fact, because of the dimensions chosen for the leg lengths, the end effector can translate large amounts at this singularity.\(^4\)

\(^4\)Again, this assumes an ideal manipulator that does not have any interference with its legs which for this manipulator would be a large problem.

\(^5\)This condition can more accurately be called a self-motion singularity. See Section 2.3 for more details.
Figure 24: Power for the degenerate 3RRR manipulator for rotation around its center point

Figure 25: Detailed view of the power for the degenerate 3RRR Manipulator

8.2.1 Power

Figure 24 shows the power ($S = 1_{3 \times 3}, T = D$ in the velocity realm) for the degenerate 3RRR manipulator. For discussion later, a detailed view from $300^\circ$ to $360^\circ$ is shown in Figure 25. Figure 26 shows the location of the instant center for rotation.

What should be evident is that in this manipulator, the power still goes to zero at the singularities no matter whether they are translational or rotational. At the angle of zero degrees, the instant center of rotation is undefined since the singularity
Figure 26: Center of rotation for the nearest singularity for the flat 3RRR robot shown in Figure 24

Figure 27: Translational power for the degenerate 3RRR manipulator

is pure translation. However, near the translational singularity, \( \phi = 0^\circ \), the instant center of rotation moves to infinity. In other words, the translation appears like a rotational degree of freedom at infinity.

Figure 27 shows the resulting translational power for the degenerate 3RRR robot utilizing the translational power optimization problem given in Equation 172. However, since the input torque gives similar results that are invariant, Figure 27 is kept here only for completeness and is not discussed further.
Figure 28: Square root of the input torque for the degenerate 3RRR manipulator

8.2.2 Input Torque

Figures 28 and 29 show the corresponding input torque measure for the degenerate 3RRR robot. The eigenvalue approaches zero at approximately $\phi = 334^\circ$. However, when comparing with Figure 24, this angle is very close to where the power is the largest for the rotational case. Further investigation of Figure 26 shows that the corresponding eigenvector is a rotation around a point very close to the end effector ($x \approx -9.6$, $y \approx -5.1$). Additionally, when the power is maximum at $\phi \approx 331.5^\circ$, the corresponding center of rotation is $x \approx 2.24$, $y \approx 0.25$, which is very rotational. The input at this same point is low, approximately 0.082. Therefore, this instance is an example where the power fails to detect a close singularity. Specifically, it ignores the translational singularity that is close at 334°.

In this instance, the measure finds the translational singularity at zero degrees. It, like the translational power, also does not find the other singularities that were found using the determinant. Furthermore, in this instance, the translational power and the input torque are nearly identical. However, again, the input torque is invariant while the power is not.
Figure 29: Detailed view of square root of the input torque for the degenerate 3RRR manipulator

8.2.3 Natural Frequency

The natural frequency is also computed and Figures 30 and 31 show the natural frequency over the range of \( \phi \) values. Comparing Figure 30 to Figures 24 and 28 shows the natural frequency catches all five of the singularities. For most of the trajectory, the natural frequency follows the trend of the power measure (Figure 24). This result is to be expected since the power measure captures all of the singularities as well. The major difference occurs when the input torque measure yields small values, but not zero (i.e. a singularity). When this happens the power measure may overlook the deficiency. This phenomenon can be seen by comparing Figure 31 to 25 and 29. The natural frequency solves the issue when the power measure and input torque disagree. When \( \phi \approx 334^\circ \), the natural frequency takes on the low value when the power exhibited high values and the input torque low values. As was seen before, the natural frequency takes on the low values when the power is low and the input torque is high. It effectively merges the two measures together via the stiffness and mass to give the physically meaningful natural frequency.
Figure 30: Natural frequency for the degenerate 3RRR manipulator

![Graph of natural frequency for the degenerate 3RRR manipulator](image)

Figure 31: Detailed view of natural frequency for the degenerate 3RRR manipulator

![Detailed graph of natural frequency for the degenerate 3RRR manipulator](image)

8.3 Discussion of Results

For this simple example, it is seen that the power and input torque measures do not fully capture all the singularities that are close to a specific manipulator pose. Although not shown here, all measures that use screw norms fall short in this manner. One needs to use at least one constitutive quantity which effectively combines the linear and rotational components together.

The natural frequency is a physically meaningful way to do this with both the stiffness and inertia matrices. It identifies all the singularities and weights them...
appropriately. Any measure that utilizes the constitutive equations is appropriate. It should also be noted that because for this instance all the inputs were of the same type (i.e. revolute joints), several of the measures give the same answer. For example, the kinetic energy normed power measure only differs from the natural frequency by a constant for this particular example since all the actuators are identical.\(^6\)

What should also be noted is that the input torque measure and the translational power measure are closely related, but are not the same. The formulation of the translational power excludes its inclusion into the general framework. However, the input torque measure can be put into the general framework and gives the results missing in the power measure. However, again, neither one of these measures should be utilized alone. A constitutive quantity should be used to make sure all singularities are accounted for in the analysis.

\(^6\)For the general case of mixed actuator types that is not the case.
CHAPTER 9

UNCONSTRAINED MOTION AS A SINGULARITY MEASURE

While the previous chapters focus on the mathematical aspects of the Jacobian matrix and how to determine how close one is to a singularity, this chapter examines the amount of unconstrained motion at the end-effector due to clearances in the joints. Whereas the other measures are based on differential kinematics, this method uses unconstrained motion to determine the severity of a singularity.

9.1 Motivation

From the definition of the Jacobian matrix,

\[ \dot{\theta} = (A^{-1}B) \dot{\mathbf{s}}^t = J \dot{\mathbf{s}}^t, \]  

(240)

it is seen that singularities occur when the \( A \) matrix becomes singular. However, all singularities are in actuality not created equal. If the manipulator is in a special type of (platform) singular configuration, self-motion can occur (see Section 2.3). Considering first manipulators without any clearances, poses can be classified as follows:

1. **Non-singular configuration**: No unconstrained motion occurs at the end effector.

2. **Singular configuration**

   (a) **Infinitesimal motion only**: No finite unconstrained motion at the end effector.
(b) **Self-motion:** Finite unconstrained motion occurs at the end effector.

When finite clearances are introduced into the nominal model given above, these distinctions can no longer be made. *Every* pose of the manipulator has some finite unconstrained motion. Engineering intuition suggests that the amount of unconstrained motion in the finite clearance model would be larger where the nominal model predicts self-motion. The question arises of what happens close to singular poses at non-singular configurations in the finite clearance model.

Several researchers have approximated this motion using the Jacobian matrix with a zero clearance model to predict the amount of motion at the end effector. This approach does not give the actual amount of motion at the end effector. It is a linear approximation, and, at or near singular configurations, the procedure breaks down. Therefore, it is evident that another method needs to be created to predict the motion.

Wohlhart’s method [121] most closely resembles the technique outlined in this dissertation. In that method, one leg of a manipulator is chosen to have a certain amount of translational backlash (clearance). This clearance creates a certain amount of end effector motion. Several different poses and configurations with different “degrees of shakiness” are investigated. However, the method does not take into account all the clearances of the links simultaneously.

### 9.2 Procedure to Determine Unconstrained Motion

This section describes the calculation of the unconstrained motion at the end-effector, taking finite clearances into account. The key point is to construct a virtual mechanism whose range of motion corresponds to the unconstrained motion of the original manipulator. This mechanism is denoted as the Equivalent Clearance Manipulator (ECM). Standard workspace generation techniques can then be applied.
to calculate the workspace of the ECM and thus the unconstrained motion of the original manipulator.

The process consists of several steps which are each discussed in detail in the following subsections:

1. Model the clearances of the individual passive joints.

2. Lock the actuators in the desired pose to obtain a structure.

3. Calculate the unconstrained motion of each leg (the combined clearance effect on the serial structure).

4. Calculate the unconstrained motion of the end effector (combine the constraints in the closed kinematic chains).

9.2.1 Unconstrained Joint Motion

The first step in the process is to determine the amount of clearance in a particular joint. Excluding compliant joints, all joints require some clearance to move. In this section two common joints, the revolute joint (R) and the spherical joint (S), are examined. With these two joint types many common manipulators can be analyzed. Other joints may be addressed by similar methods, but are outside the scope of this dissertation.

Figure 32 shows an exaggerated view of a passive revolute joint (R). Assuming a perfect circular hole and pin, the center of the pin is allowed to move in a circular region of radius, $r_{\text{model}}$, where $r_{\text{model}} = r_{\text{hole}} - r_{\text{pin}}$. Therefore, for the passive joint shown, the clearance model is that the center of the pin can move freely anywhere within the clearance model circle. Extending this to a spherical joint (S), the clearance model is simply a spherical region rather than a circular region.
9.2.2 Lock Actuators

Since the amount of unconstrained end effector motion is a function of the pose of the manipulator, each pose needs to be analyzed independently. The pose under consideration is analyzed by locking the actuators. Therefore, from this point forward, the analysis is performed on one particular configuration.

9.2.3 Unconstrained Leg Motion

Now that the clearances of each joint are modeled, they must be combined in an efficient manner. In this section, the clearances are propagated along the serial structure of each individual leg by looking at each leg of the manipulator independently. Several sample legs are now considered to determine how the process is completed.

A common leg type in PMs is the RRR leg: three revolute joints connected in series with the first joint actuated. Let $\varepsilon$ and $\delta$ be the clearances for the most distal and mid joints, respectively (Figure 33). As is also shown in the top of Figure 33, point $A_1$ is assumed to be fixed and the corresponding actuator locked in place (i.e. $\theta_1$ is constant). Because of joint clearances, point $B_1$ can translate by a small amount $\delta_1$ in any direction. Likewise, point $C_1$ can rotate about $B_1$, plus translate by $\varepsilon_1$ in any direction. The bottom of Figure 33 shows the resulting possible combined motion of $C_1$: a ring centered at $B_1$ and with the following range for its radius $r$:
Figure 33: Possible Motion of mounting point $C_1$

$$(l_2 - \delta_1 - \varepsilon_1) \leq r \leq (l_2 + \delta_1 + \varepsilon_1).$$  In other words, the motion of 2 clearances and 1 passive joint is transformed to one annulus. It is the similarity between the joints that allows the combination of the joint clearances in such a simple set (annulus). For aid in discussions later, let the annular region be more accurately defined as:

$$S_1 = \{ P \in \mathbb{R}^2 : |P - B_1| = r, (l_2 - \delta_1 - \varepsilon_1) \leq r \leq (l_2 + \delta_1 + \varepsilon_1) \}$$  \hspace{1cm} (241)

This definition represents all the points in the annular region shown in Figure 33.

If the leg in question were RSS instead, the process would lead to a very similar result. Instead of a ring, the range of motion of the point $C_1$ would be a spherical shell of radius $r$ given above, $(P \in \mathbb{R}^3)$. If the passive joints in a leg are dissimilar (i.e. not the same), the above short cut can not be performed, and the resulting set, $\mathcal{S}$, is more complicated.
Unconstrained End Effector Motion

Up to this point, each leg was treated independently. Now, the parallel nature of the mechanism needs to be addressed. To begin, we first examine the mechanism consisting of the simplest combination of legs. Let us combine two legs such that:

\[ C_1 = C_2 = C. \]  

(242)

This combination creates the five bar mechanism shown in Figure 4. For any valid EE pose, the attachment point of each leg, \( C_i \), must lie within its respective annulus, \( S_i \), determined in Section 9.2.3, \( C_i \in S_i \). Therefore, the amount of motion that can occur at the end effector is the intersection of the constraining annuli, \( S_1 \) and \( S_2 \). Figure 34 shows the unconstrained motion for the five bar mechanism.

For many other mechanisms, however, the attachment points do not coincide. If three RRR links are integrated in the case of the 3RRR robot shown in Figure 35, the amount of unconstrained motion is the set of all end-effector poses, \( (x, y, \varphi) \), for which the three attachment points of the end effector, \( C_i \), are each within their respective annuli: \( C_i \in S_i \). Mathematically, the unconstrained motion, \( \mathcal{M} \), can be defined as:

\[ \mathcal{M} = \{ (x, y, \varphi) : C_i \in S_i, i = 1, 2, 3 \} \]  

(243)
Figure 35: The 3RRR manipulator (repeated from Figure 1a)

This condition \((C_i \in \mathcal{S}_i)\) is analogous to the one encountered in the workspace
generation procedure by Gosselin [30], where the range of motion of the end-effector
is determined from the range of motion of each leg, which can also be expressed as a
set \(\mathcal{S}_i\) for each leg. Thus, to obtain an explicit representation of all points reachable by
the EE center for a particular EE orientation, one can now follow the same geometric
procedure as in workspace generation [30].

The amount of motion of the EE (point \(D\) in Figure 35) can be found by the
following procedure. Define

\[
\vec{t}_i = \vec{C}_i \vec{D} = \vec{D} - \vec{C}_i \quad \text{for} \quad i = 1, 2, 3
\]  

(244)

Applying the condition \(C_i \in \mathcal{S}_i\) leads to:

\[
\vec{C}_i = (\vec{D} - \vec{t}_i) \in \mathcal{S}_i \quad \text{for} \quad i = 1, 2, 3
\]  

(245)

and thus

\[
\vec{D} \in \mathcal{S}_i \oplus \vec{t}_i \quad \text{for} \quad i = 1, 2, 3
\]  

(246)
Figure 36: Unconstrained EE Motion of 3RRR manipulator for one particular EE orientation $\varphi$: the intersection of three translated rings.

where $\oplus$ is the operation of adding a vector to all elements of the set. Therefore, translating the annuli, $S_i$, by $\vec{t}_i$ and intersecting the result, gives the amount of motion of the end effector:

$$\vec{D} \in \left( (S_1 \oplus \vec{t}_1) \cap (S_2 \oplus \vec{t}_2) \cap (S_3 \oplus \vec{t}_3) \right)$$

(247)

Since $\vec{t}_i$ is a function of $\varphi$, this operation yields the unconstrained motion for one orientation of the end effector. Figure 36 shows the unconstrained motion of the end effector for a constant orientation, $\varphi = 0^\circ$.

9.2.5 Alternative Interpretation: Equivalent Clearance Mechanism

To formalize the process, one can create a virtual mechanism that captures the unconstrained motion of the original mechanism. The key to this step is noticing that the motion shown in Figure 33 is identical to the motion of a nominal (i.e. zero clearances) RP leg anchored at $B_1$ as shown in Figure 37. The limits on the length of the prismatic joint are from $(l_2 - \delta_1 - \varepsilon_1) \leq l \leq (l_2 + \delta_1 + \varepsilon_1)$. Therefore, we only
need to look at the combination of *nominal* legs. It is this *nominal* model that we call the Equivalent Clearance Manipulator (ECM).

For the 2RR and 3RRR robots given before, the Equivalent Clearance Manipulator is a *nominal* 2RP and 3RP R manipulator, respectively. These are shown in Figure 38 and Figure 39.

### 9.2.6 Generate Workspace for ECM

Since the ECM is a nominal model, well-established workspace generation techniques can be utilized to determine the amount of unconstrained motion. The specific technique utilized does not matter. Therefore *any* workspace generation technique
can be implemented. Some of the available techniques are presented by Gosselin [30] and Merlet et al. [81], but numerous others exist as well.

One word of caution should be noted here before continuing. The workspace of the ECM gives the unconstrained end effector motion for the original manipulator for the specific pose only. Since the unconstrained motion of the end effector is a function of the pose, this procedure needs to be done for each pose in question, varying the kinematic parameters of the ECM.

9.3 Numerical Results

The techniques outlined in Section 9.2 are applied here to the two sample manipulators: the 2RR and 3RRR.

9.3.1 Unconstrained EE Motion for the 2RRR Manipulator

Utilizing the procedure outlined in Section 9.2, the exact amount of unconstrained motion can be determined. A sample 2RR manipulator is chosen with parameters
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value ($m$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_0$</td>
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</tr>
<tr>
<td>$l_1$</td>
<td>1.5</td>
</tr>
<tr>
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<td>1.3</td>
</tr>
<tr>
<td>$l_3$</td>
<td>1.0</td>
</tr>
<tr>
<td>$l_4$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\delta_i$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\varepsilon_i$</td>
<td>0.05</td>
</tr>
</tbody>
</table>

**Table 4:** Parameter values for sample 5 bar manipulator

<table>
<thead>
<tr>
<th>Graph</th>
<th>Area ($m^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.0394</td>
</tr>
<tr>
<td>$b$</td>
<td>0.0394</td>
</tr>
<tr>
<td>$c$</td>
<td>0.0460</td>
</tr>
<tr>
<td>$d$</td>
<td>0.0645</td>
</tr>
<tr>
<td>$e$</td>
<td>0.1401</td>
</tr>
<tr>
<td>$f$</td>
<td>0.1164</td>
</tr>
<tr>
<td>$g$</td>
<td>0.1190</td>
</tr>
<tr>
<td>$h$</td>
<td>1.2566</td>
</tr>
</tbody>
</table>

**Table 5:** Amount of unconstrained motion (area) for sample 5 bar manipulator

given in Table 4. The resulting unconstrained regions are shown in Figure 40. In this figure, each column shows the mechanism starting at the same non-singular position (a and b) and heading to 2 different singular configurations (g and h). From Figure 40, it is easily seen that the two different singular configurations (g and h) result in different amounts of unconstrained motion. Since the end effector motion in this case is in consistent dimensions, one possible metric for the amount of unconstrained motion is the area of unconstrained motion.\(^1\) Table 5 shows the results for the area.

From Figure 40 and Table 5, it is seen that the unconstrained motion for $e$ is actually larger than $g$. This shows that the amount of unconstrained motion at a non-singular pose can be larger than that at a singular pose. This result is important for the correct design and use of a PM. More importantly, $h$ is an order of magnitude

\(^1\) There are numerous metrics one can use, but the area measure was chosen for its ease of use and its intuitiveness.
Figure 40: Unconstrained EE Motion while approaching two different singular configurations
Figure 41: Unconstrained EE Motion of 3RRR manipulator for a non-singular configuration

worse than \( g \). This result is consistent with intuition, since at \( g \), the nominal model predicts self-motion. The area measure then changes the impractical binary self-motion measure\(^2\) to a physically meaningful measure that can be used at any pose of the manipulator.

9.3.2 Unconstrained EE Motion for the 3RRR Mechanism

Following the same procedure outlined above, the unconstrained end effector motion of a 3RRR mechanism (Figure 35) is found. Applying the procedure for shifting the sets by \( \ell_i \), several plots are created for different values of \( \phi \). These plots are discretized to show the unconstrained motion for different orientations of the end effector. Figures 41 and 42 show the unconstrained motion for different poses of the same manipulator, one non-singular, the other singular.

Unlike the previous example, the dimensions of the end effector motion are

\(^2\)The existing measures for self-motion calculate a pose to either have self-motion or it does not and therefore are not conducive to measuring closeness to singularities.
Figure 42: Unconstrained EE Motion of 3RRR manipulator for a singular configuration

in mixed dimensions (i.e. meters and degrees). Therefore, the volume of the unconstrained motion would not be an adequate choice for a metric. A suitable metric defined on $SE(2)$ needs to be used [16]. However, trends can be observed. The amount of unconstrained angular motion for approximately the same amount of $xy$ motion at the singular configuration (Figure 42) is much larger in this case than at the non-singular configuration (Figure 41). Note the different scale on the $\varphi$ axis.

Figure 43 shows a different manipulator (motor locations and leg lengths) in a pose where the nominal model shows self-motion. When clearances are added to the model, the mechanism shows a circular motion of the end effector at a pose near the singular pose. This figure provides a good indication of when the manipulator would start sliding into self-motion in practice.
Figure 43: Unconstrained EE Motion of degenerate 3R RR manipulator for a singular configuration (not all area shown for clarity)

<table>
<thead>
<tr>
<th>Manipulator</th>
<th>Original Manipulator</th>
<th>Modeled as</th>
<th>Equivalent Clearance Manipulator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gough-Stewart Platform</td>
<td>6 UPS</td>
<td>6 SPS</td>
<td>6 SPS</td>
</tr>
<tr>
<td>Paradex</td>
<td>6 PUS</td>
<td>6 PSS</td>
<td>6 SPS</td>
</tr>
<tr>
<td>Hexa</td>
<td>6 RUS</td>
<td>6 RSS</td>
<td>6 SPS</td>
</tr>
<tr>
<td>Hexaglide</td>
<td>6 PUS</td>
<td>6 PSS</td>
<td>6 SPS</td>
</tr>
<tr>
<td>3RRR</td>
<td>3 RRR</td>
<td>3 RRR</td>
<td>3 RPR</td>
</tr>
<tr>
<td>5 Bar</td>
<td>2 RR</td>
<td>2 RR</td>
<td>2 RR</td>
</tr>
</tbody>
</table>

Table 6: Equivalent Clearance Manipulators for some common manipulators

9.3.3 Other Mechanisms

If one assumes a universal (or Hooke’s) joint is approximated by a spherical joint, more complex mechanisms can be analyzed. Table 6 contains the Equivalent Clearance Manipulator for many common manipulators.

Note that four common mechanisms, the Gough-Stewart, Paradex, Hexa, and Hexaglide, all result in a 6 SPS mechanism, which is again a Gough-Stewart platform.
Since workspace generation techniques have been implemented extensively for the Gough-Stewart platform, these methods can be readily applied to calculate the unconstrained motion of these common mechanisms.

\section{Summary}

This chapter examines the unconstrained motion of many parallel robotic manipulators (with revolute or spherical passive joints) due to clearances in those joints. By modeling the clearances, an equivalent clearance manipulator is created. Standard workspace generation techniques are utilized on the virtual manipulator and the unconstrained end effector motion is found.

The information of the amount of unconstrained motion can be used for many purposes. It was observed through examples, that the amount of unconstrained motion of a non-singular pose could be larger than at a singular pose. The motion becomes important in the design of manipulators, especially in teach-repeat mode. Additionally, by taking into account the clearances, a classification of a manipulator’s pose can be created that is independent of whether or not the manipulator is singular. Therefore, dangerous positions can be determined and avoided.
CHAPTER 10

MEASURES OF REDUNDANT ACTUATION

This chapter outlines how to expand the frameworks given previously to determine how efficient redundant actuation is for eliminating singularities. The redundant problem becomes very straightforward to consider. The measures utilized for the redundant case are an extension of the non-redundant case.

This chapter outlines how the general framework in Chapters 6 and 7 for determining closeness to singularities can naturally be extended to determine how effective redundant actuation is at eliminating singularities. The chapter also shows how the amount of unconstrained motion developed in Chapter 9 can be expanded as well.

10.1 Extension of General Framework to Redundancy

The power of the general framework outlined in Chapters 6 and 7 is that the generalization to the redundant case is very easy. Recall the problem formulation in the velocity domain is:

\[
\begin{align*}
\text{min or max} \quad & F(\dot{\theta}) = \dot{\theta}^T S \dot{\theta} \\
\text{subject to} \quad & h_1 = \dot{\theta} - JS^t = 0 \\
& h_2 = S^{tT} TS^t - 1 = 0 \quad (248)
\end{align*}
\]

The only difference is in the redundant case is the \( S, T, \) and \( J \) matrices may change, which are discussed separately below.
10.1.1 Change in $J$ Due to Redundancy

The Jacobian relationship changes from being square to being $m \times n$ where $m > n$ and $m - n$ is the degree of redundancy. Therefore the relationship:

$$\dot{\theta} = J\dot{s}$$

(249)

still holds, and has no issues with the general formulation given previously.

10.1.2 Change in $S$ Due to Redundancy

The objective function, $F$, in Equation 248 consists of the matrix, $S$, pre- and post-multiplied by the joint velocities. Therefore the weighting matrix, $S$, must be of size $m \times m$ to make the units work out.

1. Power: For the power measure, the $S$ matrix now becomes:

$$S = 1_{m \times m}$$

(250)

The physical meaning of $F$ remains the same as before. Namely, $F$ represents the sum of the squares of the actuator velocities, which can also be interpreted as the square of the power between the rows of the Jacobian and the minimal twist at the end effector. However, now there is one more joint to be utilized in the calculation.

2. Velocity Potential Energy: Much like the case of power, the stiffness matrix remains the same. Now, the $S$ matrix is:

$$S = \mathcal{K}$$

(251)

Since there are $m$ legs, the $\mathcal{K}$ becomes $m \times m$. Additionally, the physical meaning of $F$ is the same as well. $F$ is the sum of the square of the joint stiffnesses.
3. Kinetic Energy: Likewise, the kinetic energy of the input looks at the mass of each leg. Thus:

\[ S = M \]  \hspace{1cm} (252)

is again a \( m \times m \) matrix of motor inertias.

### 10.1.3 Change in \( T \) Due to Redundancy

The same analysis needs to be performed for the constraint equation. However, the constraint deals with the twist in end effector coordinates. Adding redundancy does not change the space in which the end effector operates. Therefore, no changes need to be made to the \( T \) matrix.

### 10.1.4 Summary of Changes in General Framework Weighting Matrices

Although only the velocity domain measures were discussed above, the results apply to the force domain measures as well. Since the same type of matrices are used in the force domain, the analysis is just as valid there.

From these examples, it is shown that the measures outlined using optimization can be applied by only changing the weighting matrices to reflect the change in the number of inputs. The mathematical derivation of the measures does not need to change because nowhere in the analysis did the size of the matrix need to be assumed. The only assumptions made are that the weighting matrices are symmetric and positive definite in the velocity domain and that the \( S \) matrix is invertible in the force domain. By adding redundancy, these assumptions are not violated. Therefore, the general framework can be expanded for redundant manipulators, and the same procedure of finding the eigenvalue and the corresponding eigenvector can be utilized.
10.2 Extension of Amount of Unconstrained Motion to Redundancy

The extension of the amount of unconstrained motion is also straightforward. The method outlined in Chapter 9 can be applied to a redundant robot as well. Recall the procedure involved four steps which are:

1. Model the clearances of the individual passive joints.
2. Lock the actuators in the desired pose to obtain a structure.
3. Calculate the unconstrained motion of each leg (the combined clearance effect on the serial structure).
4. Calculate the unconstrained motion of the end effector (combine the constraints in the closed kinematic chains).

Each of these steps does not have issues with redundancy. In effect, adding a redundant leg adds another constraint to the closed kinematic chain. The five bar mechanism used in Chapter 9 can be used as an example. Adding a leg gives the mechanism schematically shown in Figure 44. Figure 45 shows the corresponding clearance circles and Figure 46 shows the Equivalent Clearance Mechanism (ECM). For this manipulator, the corresponding ECM is a redundant 3RP manipulator. Finding the workspace of the mechanism in Figure 46 then is a straightforward analysis of the intersection of 3 annuli.

For the 3RRR robot the process is the same with the redundant version being the 4RRR. The redundant leg of the 4RRR is treated like all the others, but now the angle of the end effector needs to be taken into account. Also, since the end effector works in a mixed dimensional space, an appropriate metric needs to be defined in order to combine the rotation and translation at the end effector.
10.3 Other Methods for Measuring Redundancy Effectiveness

Some methods have already been established that measure effectiveness of singularity elimination. In particular, the work by Chan and Ebert-Uphoff [15] should be discussed more fully. In this paper, the authors expand upon the nullspace of the Jacobian matrix and how it relates to redundancy. From the Jacobian static
relationship, the nullspace of the Jacobian, denoted $\mathbf{J}^{t,\text{defic}}$ and called deficiency twist, can be thought of as the direction of instantaneous unconstrained end effector motion. An additional leg provides a force and moment at the end effector. Taking the power between the force applied and the deficiency twist gives a measure of the effectiveness of the redundancy to provide the missing force-moment combination. To differentiate it from the power measure introduced via the eigenvalue problem, the Chan and Ebert-Uphoff power is called the Modified Power Product and denoted by $\mathcal{P}_{\text{mod}}$. In mathematical terms:

$$\mathcal{P}_{\text{mod}}^2 = \mathbf{J}^{t,\text{defic}}^T \mathbf{J}_{\text{red}}^T \mathbf{J}_{\text{red}} \mathbf{J}^{t,\text{defic}}$$  \hspace{1cm} (253)

where $\mathbf{J}^{t,\text{defic}} \in \mathcal{N}(\mathbf{J})$

$$\mathbf{J}^{t,\text{defic}}^T \mathbf{T} \mathbf{J}^{t,\text{defic}} = 1$$

Therefore, it is similar to the power measure outlined previously, but does not perform the optimization. The twist is forced to be in the nullspace of the Jacobian matrix and can be normalized in any of the ways outlined previously. However, if the $\mathbf{D}$ matrix is utilized as is done in [15], care must be taken that the nullspace of the Jacobian matrix is not a translation. As was shown in Chapter 8, this
normalization can give erroneous results in this situation. Using the mass instead as the normalization matrix as proposed by this dissertation eliminates this problem, and then the measure takes on some similarity to the kinetic energy normed power measure.

There are a couple of drawbacks with this procedure. It does not work at poses away from singularities since the nullspace of the Jacobian is no longer defined. It also does not find the twist that is minimal against all the forces. It assumes the minimal twist is in the nullspace of the Jacobian. This may or may not be a good idea depending on the application. In other words, it treats the redundant leg differently than the others.

An extension of the procedure to overcome the problem with only being defined at a singularity is to deal with the redundancy as a whole manipulator. We could take the minimum of all the combinations of the square Jacobian matrix. This procedure, however, does not work since the nullspace for each sub-Jacobian is different.

Even with its problems, one can use the modified measure (using $T = M$) to help in the retrofit of existing manipulators to eliminate singularities. The optimization problem then would follow

$$\max_{\mathbf{x}} \quad \mathcal{P}_{\text{mod}} = \mathbf{s}_{\text{red}}^T(\mathbf{x}) \mathbf{s}^{t,\text{defic}}$$

where

$$\mathbf{s}^{t,\text{defic}} \in \mathcal{N}(\mathbf{J})$$

$$\mathbf{s}^{t,\text{defic}}^T \mathbf{T} \mathbf{s}^{t,\text{defic}} = 1$$

where $\mathbf{s}_{\text{red}}^w(\mathbf{x})$ is dependent upon the kinematic parameters, $\mathbf{x}$, of the redundant leg. One should also note that this can also be done with the generalized framework explained in Chapter 7.
Figure 47: Singularity surface for a 3RRR robot with $l_1 = l_2 = 1.1$ and $l_3 = .18$ and motor positions at $(0, 0), (3, 0)$ and $(1.5, 2.6)$

10.4 Application of Differential Kinematic Measures to a 4RRR Robot

To show how these measures are used, they are applied to the 3RRR robot shown in Figure 5. The singularity surface of the original manipulator is generated, and an additional leg is added to see how the additional leg will transform the singularity surface. Using the results of Dasgupta and Mruthyunjaya [19], the dimension of the singularity surface is decreased by the degree of redundancy (i.e. $m - n$).

The first step is to determine the singularity surface. This is done by using the Jacobian matrix, and finding where the determinant of the Jacobian matrix is zero. The results are shown in Figure 47. If one projects the surface onto the $xy$ plane, a contour plot can be generated and can also be seen in Figure 47. There is a one to one correspondence between the $xy$ position and the angle that is singular.\footnote{This correspondence is not general and is only applicable for this particular manipulator.} Because of this special correspondence, only the $xy$ plane is utilized in subsequent plots.

The measures outlined above are computed for the redundant manipulator along the singularity surface of the non-redundant manipulator to observe the effectiveness
of the redundant leg on the singularity surface. The redundant leg is chosen to be
attached at the end effector triangle half way between legs 1 and 2 with $l_1 = l_2 = 1.1m$
and $l_3 = 0.09m$ and motor mount position at $(1.5m, -0.4m)$ as schematically shown
in Figure 48.

10.4.1 Non-redundant Case

Figure 49 shows *any* measure along the singularity surface before the redundant leg
is added. As can easily be seen, any measure is zero at all poses on the singularity
surface. This result is by definition since all the measures are zero at a singularity.

10.4.2 Power Measure

Using the eigenvalue formulation for the power ($S = I_{6 \times 6}$, $T = D$), it becomes easy
to plot the value of the power eigenvalue on the singularity surface and it is shown
Figure 49: Any measure on the singularity surface without a redundant leg

Figure 50: Power on the singularity surface with a redundant leg

in Figure 50. The colors on the figure follow the color spectrum (red, orange, yellow, green, blue, indigo, violet) from large power (red) to small power (violet). From this figure, it is seen that the power product over the singular surface is reduced from a 2-dimensional surface to a 1-dimensional line (violet area). This fact is anticipated and predicted by Dasgupta and Mruthyunjaya [19].
Figure 51: Power (kinetic energy normed) on the singularity surface with a redundant leg

10.4.3 Power Measure (Kinetic Energy Normed)

For comparisons later, the kinetic energy normed power product is displayed in Figure 51. One can see in this figure that the same general trend is observed, but the actual values change with the different norm. As expected, Figure 50 is similar to Figure 51. Although these measures are closely related, they are not identical.

10.4.4 Modified Power Product

Figure 52 shows the power product of the deficiency twist (normalized via the screw norm) with the wrench provided by a unit torque input. The input torque is normalized with the mass of the end effector (i.e. $T = M_{ee}$). This choice makes it very close to the kinetic energy normed power measure in the generalized framework.

It is seen that the modified power product has the same resulting singularity line that is seen in Figures 50 and 51. Since the modified power computes the power only with the nullspace of the Jacobian matrix, it is artificially inflated. However, for this particular instance of adding a redundant leg to an existing manipulator, the measure may be useful.
Figure 52: Modified power product on the singularity surface with a redundant leg

10.4.5 Natural Frequency

Figure 53 shows the linearized natural frequency for the same redundant robot as given above. This plot again shows that the singular surface has been transformed to a line consistent with that shown in Figures 50, 51 and 52.

To show that the natural frequency via the eigenvalue method with only the mass of the end effector is close to that of the others, the natural frequency is computed using the dynamic equations of motion. The complete derivation of the equations of motion can be found in Appendix A.

Figure 54 shows the result from this analysis. It also shows that the singularity curve that is left is the same, and is also very “close” to that shown in Figure 53. However, the amount of computation to get to Figure 54 is much more intensive than that of Figure 53.

10.4.6 Determinant of $J^T J$

Figure 55 shows the determinant of the $J^T J$ matrix along the singular surface. Figure 55(a) shows the exact value, while Figure 55(b) shows the $\frac{1}{8}$ power to enhance contrast. This plot shows again that the singularity surface has transformed itself
**Figure 53:** Natural frequency measure along the singularity surface with a redundant leg

**Figure 54:** Natural frequency along the singularity surface with a redundant leg using dynamics
to a singularity line evident in Figure 55(b). However, one should note that the determinant value is very flat and does not explicitly show how effective the reduction was in other areas.


Figure 55: Determinant of $J^T J$ along the singularity surface with a redundant leg

10.4.7 Discussion of Results

From Figures 50 through 55, we can see that all the measures correctly identify the remaining singularity surface which in this case is a line. Also, the generalized framework allows one to see how the different measures are weighted, but yet show similar results. One should note though, given the specific choice of kinematic parameters, the translational singularity does not appear in the analysis (i.e. all
singularities inside the workspace are rotational). This translational singularity causes problems with the power (and not shown here, the stiffness) measures. Additionally, the natural frequency measure is the square of the linearized natural frequency of the manipulator which has a good physical meaning. What also should be evident is that the determinant measure is not that useful in this instance. All values of the determinant are very small and it is very difficult to discern a good value from a bad value. What remains to be seen is how and why the specific leg configuration should be chosen. The optimization of the manipulator for a specific workspace is discussed in the following chapter.
CHAPTER 11

OPTIMIZATION OF A REDUNDANT MANIPULATOR

The generalized framework for the Jacobian type measures allows for optimization of the kinematic parameters for optimum design. However, like any optimization problem, the results are highly dependent upon the choice of the objective function and the constraints. For our case, the objective function and constraints are chosen using engineering judgment and optimized using standard techniques.

The purpose of this chapter is to show how the measures from the generalized framework can be used with optimization. The 4RRR shown in Figure 56 is used as an example of how the optimization can be applied. For ease of computation, not all of the kinematic parameters are utilized since the computational complexity would increase dramatically.

11.1 Problem Definition

The first step is to determine the objective function of the optimization. For purposes of this dissertation, it is assumed that the manipulator should maintain the highest performance throughout the desired workspace. This requirement is equivalent to finding the lowest value of the measure in the desired workspace and maximizing it. Therefore, the optimization is to maximize over the workspace the minimum of the measure (e.g. power, stiffness, lowest linearized natural frequency) outlined in the generalized framework in Chapter 7. For this study, the square of the lowest linearized natural frequency, $\lambda_{\text{min}}$, is chosen as the measure. Therefore, the resulting
minimization problem is:

\[
\text{Maximize} \quad F(\mathbf{x}) = \min_{\mathcal{W}} \lambda_{\text{min}}
\]  

(255)

where \( \mathbf{x} \) is a vector of the kinematic parameters, \( \lambda_{\text{min}} \) is the square of the lowest linearized natural frequency and \( \mathcal{W} \) is the desired workspace.

A challenge is that the measure needs to be computed over the entire desired workspace, \( \mathcal{W} \), of the manipulator. For this dissertation, the workspace was divided into 100 increments in both the positional and rotational spaces, resulting in \( 10^6 \) increments. By discretizing the workspace, there is the potential to miss the minimum lowest linearized natural frequency. However, it is assumed that the measure will be sufficiently smooth that this condition does not arise.

The objective function given previously also assumes that all the points in the desired workspace are feasible (i.e. inside the actual workspace). This may not always be true. There are three ways that this can be addressed.

1. To ensure that the desired workspace points are feasible, the natural frequency
could be set to zero for any poses that violate this constraint. However, many numerical routines use a numerical calculation of the gradient to determine the optimal solution. By setting the natural frequency to zero at these points, the gradient can become very large, or in some cases zero, which can lead to suboptimal solutions.

2. Another way of approaching the problem is to set the actual workspace as a constraint. This constraint would include checking the inverse kinematics for a feasible solution.

3. A third way is to modify the objective function to include the actual workspace of the manipulator. Since the objective function already checks all poses, a penalty function is subtracted from the lowest linearized natural frequency for each desired pose that is outside of the manipulator’s actual workspace. Therefore, the numerical calculation of the gradient is still well defined. Also, modifying the objective function saves on calculations. The drawbacks are that the objective function has mixed units and the final solution could still have non-feasible poses inside the desired workspace. Therefore, the final solution must be verified not to have any points outside the actual workspace. If this is verified at the end, the penalty function has no bearing on the outcome of the optimization routine and the mixed units will not be a problem.

For the actual implementation, the third solution is used. Namely, a penalty function is subtracted from the eigenvalue if there are poses that are inside the desired workspace, but outside the manipulator’s actual workspace. In mathematical terms, the objective function, $F$, becomes:

$$\text{Maximize } F(x) = \min_{w} \lambda_{\text{min}} - w$$  \hspace{1cm} (256)

where $w$ is the penalty function. The workspace penalty function is chosen as the sum of the number of poses inside the desired workspace that could not be reached.
by the manipulator. It should be stressed that at the optimum, the penalty function should be zero.

The second step in the process is to determine those kinematic parameters to vary. To keep the problem simple, the motor mount positions are fixed symmetrically on a 2 unit square (i.e. (0,0), (2,0), (2,2), and (0,2)) as shown in Figure 56. Also, for ease of fabrication and repair, all legs are chosen to be identical, i.e. \( l_{1i} = l_1 \), \( l_{2i} = l_2 \), and \( l_{3i} = l_3 \) for \( i = 1, \ldots, 4 \). Also for this study, the desired workspace is set as a 1 unit square centered inside the 2 unit motor mount positions (i.e. corners of the workspace in \( xy \) space at (0.5, 0.5), (1.5, 0.5), (1.5,1.5) and (0.5,1.5)). The angle of the end effector is desired to have \( \pm 45^\circ \) of angular motion. Because there is 360\(^\circ\) of travel available for the end effector, the portion in which the end effector is designed to operate must be determined. For this study, the beginning angle of the end effector, denoted \( \phi_0 \), is a design parameter.

Therefore, for this particular problem there are only four variables, the three different leg lengths, and the initial angle of the end effector:

\[
\mathbf{x} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ \phi_0 \end{bmatrix} \quad (257)
\]

The end effector is chosen to be a square with constant density. The mass of the end effector changes depending upon the kinematic parameter, \( l_3 \). The stiffnesses of all the joints are assumed to be the same, and thus set to 1 since the stiffnesses only changes the output by a constant value.

From a practical standpoint, the leg lengths should be within certain bounds. In other words, if the leg lengths are very large in relation to the motor mount positions, the manipulator would require a large amount of floorspace. Also from a numerical calculation standpoint, there needs to be bounds on the leg lengths to keep
the optimization routine from possibly getting into an infinite loop. To solve these issues, the legs are constrained to stay positive and less than some “large” value, in this case 10 units. These constraints are not active in the final solution, but are included here for completeness.

Mathematically, all these constraints and requirements result in the full optimization problem:

\[
\text{Maximize} \quad f(\mathbf{x}) = \min_{\mathbf{w}} \lambda_{\min} - \mathbf{w} \\
\text{Subject to} \quad l_i > 0 \\
\quad l_i < 10 \\
\text{where} \quad \mathbf{x} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ \phi_0 \end{bmatrix} \\
\quad w = \sum w_i \\
\quad w_i = \begin{cases} 
0 & \text{if the pose is reachable} \\
1 & \text{otherwise}
\end{cases} \\
\lambda_{\min} = \min (eig(S, T)) \\
\quad S = k \mathbf{1}_{3 \times 3} \\
\quad T = M_{ee} = \begin{bmatrix} m & m \\ m & I_n \end{bmatrix}
\]

where as stated the $\lambda_{\min}$ is found from the eigenvalue problem discussed in Chapter 7 and is performed at all points inside the desired workspace. Again, it is assumed that the end effector is square and had constant density, but its size, and thus its mass, $m$ and inertia, $I_n$, are functions of the last link length, $l_3$. Therefore, as $l_3$ increase in size, the mass changes accordingly.
11.2 Implementation Techniques

11.2.1 Trial and Error

A first try at the optimization is performed by a simple trial and error technique using engineering judgment. As a first guess, the results from a study from Bush [13] of a non-redundant manipulator are used. In this study, the objective function examined the size of the feasible workspace from a force transmission point of view. However, the manipulator had several singularities inside of the workspace that had to be accounted for. For the study in this dissertation, a similar tact is taken. A MATLAB routine that outputs a visual picture of the manipulator for a specific kinematic parameter is utilized. The routine also outputs the forces required for static equilibrium for a given arbitrary wrench on the end effector. Also as a check, the determinant is calculated as well. Engineering judgment is used to change the leg lengths and starting end effector angle.

11.2.2 Numerical Routine

Since Equation 258 uses the standard optimization framework, the implementation becomes a simple matter of embedding it into a standard optimization routine. The MATLAB *fmincon* routine is utilized for the optimization and uses a Sequential Quadratic Programming (SQR) technique to find the constrained optimum.\(^1\) The SQR technique finds the solution to the constrained optimization problem by transforming it into a quadratic optimization problem (the QP subproblem) by use of Lagrange multipliers. This subproblem is iteratively performed until the optimum is found. In effect, it is a generalization of Newton’s method for constrained optimization. Since the gradient and the Hessian are both required for this type of optimization, the routine numerically estimates these quantities which is the reason

\(^1\)For a complete description of the routine used, consult the MATLAB website at http://www.mathworks.com/access/helpdesk/help/toolbox/optim/fmincon.shtml. A more palatable introduction can be found in Reklaitis et al. [103].
the penalty function is included in the objective function.

### 11.3 Numerical Results

#### 11.3.1 Trial and Error

Using trial and error and engineering judgment, good values for the kinematic parameters are:

\[
x = \begin{bmatrix}
  l_1 \\
  l_2 \\
  l_3 \\
  \phi_0
\end{bmatrix} = \begin{bmatrix}
  1 \\
  1 \\
  0.2 \\
  0^\circ
\end{bmatrix}
\]

Calculations show the minimal natural frequency measure is 11.32. An example of a hardware implementation of this robot are included in Appendix E.

Figure 57 graphically shows the results from the trial and error technique. Since there are three output variables, \(x, y, \phi\), each subfigure shows the measure for a specified angle. The upper left is the beginning angular value, \(\phi_0 = 0^\circ\) and progresses left to right and downward to the final angle of approximately 45\(^\circ\). The purple box denotes the desired workspace. All poses inside the actual workspace have a color associated with them, and thus, all the white area is unreachable. The colorbar on the bottom shows the value of the natural frequency value. Violet denotes singularities, while red shows a very large measure.

As is evident from the plots, there are no singularities inside the desired workspace of the manipulator. The measure also shows very consistent results over the different angular values. However, the very large values for the measure (i.e. the red portions of the plots) occur only at larger values of \(\phi \geq 26^\circ\) and only occur outside the desired workspace.
Figure 57: Natural frequency measure using trial and error method over the workspace
11.3.2 Optimization Solution

The problem that was defined in Equation (258) is embedded into MATLAB and optimized with the \textit{fmincon} routine as discussed previously. The initial guess was utilized from the trial and error technique:

\[
\mathbf{x}_0 = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ \phi_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0.2 \\ 0^\circ \end{pmatrix}.
\]

The resulting kinematic parameters from the optimization are:

\[
\mathbf{x}^* = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ \phi_0 \end{pmatrix} = \begin{pmatrix} 0.7979 \\ 1.3444 \\ 0.2597 \\ 46.15^\circ \end{pmatrix}.
\]

and the minimum natural frequency is 25.48.

Several different starting points were attempted, and the results above were the best. Verification of the optimization routine shows that the penalty function at this optimum value is indeed zero. Therefore, the minimized objective function is truly a maximum over all poses of the desired workspace. Also, as hoped, the lower and upper bounds on the leg lengths are not active for this manipulator.

Figure 58 shows the results in the same format (including colorbar) for the optimum design. It is clear from the figure, that there are no singularities inside the desired workspace. What is even more striking, is that the desired workspace has very large measure values over all angular values. However, the value of the measure does vary significantly inside the workspace. In other words, the minimum value of the measure, \(F(\mathbf{x})\), is larger than all those in the trial and error, but they vary much more.
Figure 58: Natural frequency measure using the MATLAB fmincon optimization routine over the workspace
Figure 59: Example 4RRR manipulator with rotated motor mount positions

11.3.3 Refinement of the Optimization Solution

Figure 58 shows that the actual workspace is a rotated square, and thus, it was wondered that if rotating the motor mount positions by 45° as shown in Figure 59 would yield a better result. All other parameters are kept the same. Ideally, the motor mount positions would be used as design parameters, but are only rotated here to keep the computational time on the optimization to an acceptable level.

As a first guess, the results from Section 11.3.2 are used. The results of the optimization are:

\[
x^* = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ \phi_0 \end{bmatrix} = \begin{bmatrix} 0.8629 \\ 1.5064 \\ 0.1321 \\ 44.37^\circ \end{bmatrix}.
\]

The minimum natural frequency is 23.15. Also, the workspace penalty function at the optimum is zero. Figure 60 shows the results from the optimization routine.
The results of this minimization are not as good as the first optimization solution but are close. However, in this solution, the workspace boundaries are far from the desired workspace boundaries, so in cases where the desired workspace may change, the rotated solution may be a better result.

11.3.4 Interpretation of Results

Using the natural frequency as a metric for designing redundant actuation gives interesting results. For both the standard and rotated optimization routines, the second link, $l_2$, is larger than the first link, $l_1$. The starting angle for the two also are around $45^\circ$. What is interesting to note is that the results are consistent with that of the commercial ABB Flexpicker.\footnote{For full specification on the Flexpicker, see //http://www.abb.com.} While this result cannot be generalized for all robots, the overall procedure can.

11.4 General Comments about Optimization

This chapter shows that the generalized framework measures could be utilized in an optimization routine. The 4RRR manipulator was chosen so that the results could be easily plotted and visually seen. The number of design parameters was kept to four for numerical ease.

For this study, the objective function was chosen to maximize the minimum lowest natural frequency inside the desired workspace. It should be mentioned that the use of other objective functions can be useful. For example, if one would minimize the variation of the natural frequency around some nominal value, the resulting manipulator could be thought of as having linear dynamics and thus linear control techniques utilized. Also, the use of other measures would also be appropriate in some circumstances. The unconstrained motion could also be used, although a different optimization routine would have to be used, because this measure is discontinuous.
Figure 60: Natural frequency measure using the MATLAB fmincon optimization routine over the workspace with rotated motor mount positions
This numerical routine could be expanded for larger degree of freedom manipulators and number of design parameters. A complete analysis would contain all kinematic parameters: leg lengths, motor mount positions, controller stiffness, end effector mass, etc. With \( n \) degree of freedom manipulators, the optimization routine would have to search a \( n \) dimensional space which would be very time consuming. Additionally, the visualization of the results would be very difficult to perform. However, there is nothing in the analysis that would prohibit optimization on these higher degree of freedom manipulators, only the time required for the optimization.

To utilize the techniques given in this dissertation to a general manipulator, these steps should be followed:

- **Pick appropriate performance measure** Any performance measure that captures all the of singularities would be a good choice. The natural frequency is a good choice in that it gives a familiar physical quantity.

- **Choose objective function** The choice of objective function is as important as the performance measure. A natural choice is to maximize the performance measure over the workspace. Others could include to have the largest connected or uniform actual workspace.

- **Establish optimization parameters** The choice of optimization parameters affect the numerical complexity of the resulting optimization. In general, the leg lengths and the angle of the end effector are a good starting point.

- **Determine constraints** The constraints in the routine follow from the parameters. There needs to be constraints on the size of all kinematic parameters. The actual workspace must also be accounted for in either the objective function or as a constraint.

- **Place in appropriate optimization routine** After setting up in the standard format, the problem can be easily placed into any numerical scheme.
• **Verify results of optimization** As with any optimization routine, the results should be verified.
CHAPTER 12

CONCLUSIONS

Parallel manipulators suffer from platform singularities where the manipulator loses stiffness in one or more directions. These directions are the nullspace of the Jacobian matrix that relate the input velocities (joint coordinates) and output velocities (end effector coordinates) to each other. Although this deficiency has been studied at quite a length, it has not been fully understood due to many researchers overlooking the mixed dimensions of the Jacobian matrix.

This dissertation has taken a step back to re-examine the problem from the beginning to fully understand the problems of singularities. By fully understanding them, appropriate measures and techniques for analysis of closeness to singularities can be performed. These measures then can be easily expanded to determine how to effectively eliminate singularities through redundant actuation.

12.1 Contributions of This Research

This dissertation examined singularities and their elimination from a physical symptom standpoint. While most other researchers focus on singularities from a mathematical point of view, this research presents a physical view of singularities, their problems, and their elimination through redundant actuation. More specifically, this dissertation provides:

1. An investigation of the physical problems with platform singularities by examining degenerated accuracy, large internal forces, and loss of solution tree.

2. A general framework for the analysis of closeness to singularities using the techniques from constrained optimization. This framework contains an exhaustive
list of different weighting matrices so that they are easily compared. Also, the framework allows for new measures to be formed. Some of the most promising new measures include the lowest linearized natural frequency, the power, and the input torque measures. This framework is important not only for the design of redundant actuation, but also allows computation of how dangerous a pose near a singularity is.

3. A new way to determine the unconstrained motion of a parallel manipulator with joint clearances at and near singularities. The method bridges the gap between the current established methods of clearance propagation and the field of singularity identification. The procedure changes a complex clearance propagation problem into a standard workspace generation problem which can be solved using existing techniques.

4. An effective frame invariant cost function for optimization of redundant manipulators. Using the constrained optimization technique, a sample manipulator is designed by using maximizing the minimum lowest linearized natural frequency. Such a manipulator has never been designed using frame invariant techniques.

12.2 Future Research

As with any research, the questions that arise from the study outnumber the questions that were answered. These questions can be grouped into some general categories:

1. Singularity Understanding

   (a) The topology of a parallel manipulator is currently still being studied, as even the simplest parallel manipulator, the four bar mechanism, has changing topology with a change in kinematic parameters. This is in contrast to serial manipulators with \( n \) revolute joints whose topology is that of a \( n \)-torus. The author believes that there can be great strides in
designing parallel manipulators if the topology of the manipulator is better explored. Another possibility is the use of differential forms.

2. Singularity Measures:

(a) The input torque measure and the translational power are very similar in one case. These measures provide different values in other situations. A better theoretical understanding of the relationship between these two measures would be beneficial to show how the translational power can be described in the generalized framework. A rigorous mathematical study would be needed, but was excluded here due to time and space constraints.

(b) The use of the flipper matrix is invariant under coordinate transformations. However, the physical meaning behind using this weighting matrix as a constraint is not understood. More investigation into the meaning of the flipper matrix in this context would help to fully understand all the measures in the tables.

(c) In the analysis of the natural frequency as a measure for singularities, it is noted that the dynamic equations of motion and the optimization framework give the same eigenvalue problem. Further investigation of the relationship between the optimization formulation and the dynamic equations of motion may provide some better understanding of the natural frequency and how it relates to singularities.

(d) The general framework for closeness to singularities and the measures for singularity elimination do not fully explain all measures that are used today. Some further study into the other measures may lead to a better understanding of singularities in general.

(e) A more exhaustive look at the advantages of in-leg actuation versus additional leg to quantify the trade-off between range of motion and inertia.
would be beneficial to designers. To do so, the lowest linearized natural frequency over a desired workspace can be used as an effective measure to penalize those manipulators with high inertia. The generalized framework would need to be expanded to include the mass at the joints which is feasible but is not done in this dissertation.

(f) In a broader context, the use of differential forms and/or differential geometry for the use of measures could be an even more general framework that would include the determinant. However, the analysis would require an extensive knowledge of differential geometry.

3. Optimization

(a) Only one optimization routine is used to show the feasibility of the measures for use as a cost function. A more exhaustive search into optimization routines to determine general trends would be beneficial to designers of parallel manipulators.

(b) The optimization routine section gives an example of how to design a manipulator for the largest minimum eigenvalue over the workspace. A study into the optimization of the workspace to have a uniform natural frequency may be very beneficial from a controls standpoint. Doing this would effectively change the highly nonlinear parallel manipulator into a linear one where more advanced control strategies could be tried. This has broader applications into the control of any parallel manipulator, not only redundantly actuated ones.

(c) In a more global scheme, establishing a mathematical theory to expand the general framework to the entire workspace, rather than individual poses would be beneficial. Therefore, the optimization routine would not need to check every pose of the manipulator.
4. Prototype Implementation

(a) Due to time constraints, all the links in the robot are fabricated out of wood. To clean up the implementation, metal links that have less compliance would be beneficial to the operation. A more robust bearing system would also benefit the control of the manipulator by decreasing the non-linear frictional effects.

(b) Also due to time restrictions, the prototype is made from the results from the trial and error solution. It would be very interesting to build the optimal manipulator and test its performance versus the trial and error solution.

(c) The control of the prototype uses a simple independent PD control on all the joints. This technique is adequate for this study, but much more investigation into the control could be completed that addresses the problem with chatter.

(d) An investigation of the measured lowest linearized natural frequency in comparison to the calculated measure would hopefully give further reason to use the lowest linearized natural frequency as a measure.
APPENDIX A

DYNAMIC EQUATIONS OF MOTION FOR PARALLEL MANIPULATORS

The dynamics of parallel manipulators are in general very complex. However, the equations of motion can be derived using the techniques from multi-body dynamics. For parallel manipulators, the equations can be efficiently derived using techniques from analytical mechanics.

This dissertation’s thrust is not the dynamic analysis of parallel manipulators, but rather the use of the equations to better understand how to eliminate singularities. Several references exist in literature that further explain the dynamics of parallel manipulators. The two major parallel manipulator textbook references, Tsai [116] and Merlet [80], do not give a general procedure for all parallel robots. Merlet reviews the dynamics of a specialized Gough-Stewart platform\(^1\) using Newton-Euler’s method while Tsai covers the Gough-Stewart platform and the University of Maryland manipulator\(^2\) using both Newton-Euler and Virtual Work techniques. A wonderful review of all the techniques from analytical mechanics that can be used to formulate the equations of motion for parallel manipulators are found in the dissertation by Kozak [56] or in less detail in the corresponding journal article [58].

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\(^1\)In Merlet’s designation, the specialized Gough-Stewart platforms are denoted as a SSM, TSSM and MSSM manipulators.
\(^2\)This manipulator is based upon the delta robot configuration.
A.1 Dynamics of 3RRR Robot

This section overviews the derivation of the dynamics of the 3RRR robot. Lagrange’s formulation with constrained generalized coordinates are utilized for ease of derivation. Use of constrained coordinates makes the derivation of the equations easier, but requires much more computation in use. However, since the equations are numerically computed, this trade-off is acceptable.

The notation used is shown in Figure 5 and again in Figure 61. Lagrange’s equation for constrained generalized coordinates states:

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = \mathbf{B}(q)^T \lambda + \mathbf{Q},
\]  

(262)

where \( \mathbf{Q} \) is a vector of generalized coordinates, \( \mathbf{B} \) is a matrix of constraints, \( \lambda \) is a vector of Lagrange multipliers, \( \mathbf{Q} \) is a vector of generalized forces, and \( \mathcal{L} \) is the Lagrangian and is defined by:

\[
\mathcal{L} = T - V
\]  

(263)
where \( T \) is the kinetic energy and \( V \) is the potential energy. For this particular case, the generalized coordinates are chosen to be:

\[
\mathbf{q} = \begin{bmatrix} x \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad \text{where} \quad x = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}, \quad \mathbf{p}_i = \begin{bmatrix} \dot{x}_{m_2i} \\ \dot{y}_{m_2i} \\ \dot{\eta}_i \end{bmatrix}, \quad i = 1, 2, 3
\]  

(264)

Note the use of the indices vastly simplifies the derivation and is utilized throughout the derivation. The kinetic energy, \( T \), is:

\[
T = \sum_{i=1}^{3} (T_{1i} + T_{2i}) + T_{ee}, \quad \text{(265)}
\]

where \( T_{1i} \) is the kinetic energy of the first link, \( T_{2i} \) is the kinetic energy of the distal link, and \( T_{ee} \) is the kinetic energy of the end effector including its payload (if applicable). Expanding these terms gives:

\[
T_{1i} = \frac{1}{2} I_{1i}\dot{\theta}_i^2
\]  

(266)

\[
T_{2i} = \frac{1}{2} m_{2i}(\dot{x}_{m_2i}^2 + \dot{y}_{m_2i}^2) + \frac{1}{2} I_{2i}\dot{\eta}_i^2
\]  

(267)

\[
T_{ee} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I\dot{\phi}^2
\]  

(268)

where \( I_{1i} \) is the inertia around the base point of the first link, \( I_{2i} \) is the inertia around the center of mass of the distal link, and \( I \) is the inertia of the end effector around the point \((x, y)\). These relationships can also be written in matrix form as:

\[
T = \frac{1}{2} \mathbf{q}^T \mathbf{M} \mathbf{q},
\]  

(269)

where \( \mathbf{M} \) is

\[
\mathbf{M} = \begin{bmatrix}
M_{ee} & 0 & 0 & 0 \\
0 & M_1 & 0 & 0 \\
0 & 0 & M_2 & 0 \\
0 & 0 & 0 & M_3 \\
\end{bmatrix}
\]  

(270)
and
\[
M_{ee} = \begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & I
\end{bmatrix}, \quad \text{and} \quad M_i = \begin{bmatrix}
I_{1i} & 0 & 0 & 0 \\
0 & m_{2i} & 0 & 0 \\
0 & 0 & m_{2i} & 0 \\
0 & 0 & 0 & I_{2i}
\end{bmatrix}.
\] (271)

Likewise, assuming that gravity acts in the positive \textit{y} direction, the potential energy terms may be written as
\[
V = \sum_{i=1}^{3} (V_{1i} + V_{2i}) + V_{ee}.
\] (272)

where
\[
V_{1i} = m_{1i} \frac{l_{1i}}{2} \sin \theta_i
\] (273)
\[
V_{2i} = m_{2i} g y_{m2i}
\] (274)
\[
V_{ee} = m g y
\] (275)

Having defined the kinetic and potential energy functions, we apply the left hand side of Equation 262 which gives:
\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = M \ddot{q} + g(q)
\] (276)

where \( M \) is defined in Equation 271 and \( g(q) \) is defined as:
\[
g(q) = \begin{bmatrix}
g_{ee} \\
g_1 \\
g_2 \\
g_3
\end{bmatrix}, \quad \text{where} \quad g_{ee} = \begin{bmatrix}
m \\
0
\end{bmatrix}, \quad \text{and} \quad g_i = \begin{bmatrix}
m_{1i} \frac{l_{1i}}{2} \cos \theta_i \\
0 \\
m_{2i} g \\
0
\end{bmatrix}
\] (277)

The remaining portion that needs to be completed is the right hand side of Equation 262. Since there are 15 generalized coordinates and the manipulator only has 3 DOF, there are 12 constraint equations that must be simultaneously satisfied. The constraint equations also come into Lagrange’s equation through the Lagrange multipliers. The constraints are typically written in the velocity domain,
and therefore, are nothing more than the Jacobian relationship derived in Section 3.3. However, this relationship requires the use of all the generalized coordinates. Using an expanded version of the vector loop closure equations, gives:

\[
x_{0i} + l_{1i} \cos \theta_i + l_{2i} \cos \eta_i + l_{3i} \cos \alpha_i - x = 0 \quad (278)
\]

\[
y_{0i} + l_{1i} \sin \theta_i + l_{2i} \sin \eta_i + l_{3i} \sin \alpha_i - y = 0 \quad (279)
\]

\[
x_{0i} + l_{1i} \cos \theta_i + \frac{l_{2i}}{2} \cos \eta_i - x_{m2i} = 0 \quad (280)
\]

\[
y_{0i} + l_{1i} \sin \theta_i + \frac{l_{2i}}{2} \sin \eta_i - y_{m2i} = 0 \quad (281)
\]

for \(i = 1, 2, 3\). Differentiating these equations gives the 12 velocity relationships which are written compactly in the form:

\[
B(q) \dot{q} = 0 \quad (282)
\]

where

\[
B(q) = \begin{bmatrix}
B_{ee,1} & B_{leg,1} & 0 & 0 \\
B_{ee,2} & B_{leg} & 0 & 0 \\
B_{ee,3} & 0 & B_{leg,3} & 0
\end{bmatrix} ,
\]

(283)

and

\[
B_{ee,i} = \begin{bmatrix}
-1 & 0 & -l_{3i} \sin \alpha_i \\
0 & -1 & l_{3i} \cos \alpha_i \\
0 & 0 & 0
\end{bmatrix} 
\]

(284)

and

\[
B_{leg,i} = \begin{bmatrix}
-l_{1i} \sin \theta_i & 0 & 0 & -l_{2i} \sin \eta_i \\
l_{1i} \cos \theta_i & 0 & 0 & l_{2i} \cos \eta_i \\
-l_{1i} \sin \theta_i & -1 & 0 & -\frac{l_{2i}}{2} \sin \eta_i \\
l_{1i} \cos \theta_i & 0 & -1 & \frac{l_{2i}}{2} \cos \eta_i
\end{bmatrix} .
\]

(285)

The last piece of the puzzle is the generalized forces. Each stationary joint has an actuator on it and thus provides a force. Additionally, the end effector has a load.
Thus, the generalized forces can be written as:

\[
Q = \begin{bmatrix}
\tau_{ee} \\
\tau_1 \\
\tau_2 \\
\tau_3 
\end{bmatrix}, \quad \text{where} \quad \tau_{ee} = \begin{bmatrix}
0 \\
0 \\
0 
\end{bmatrix}, \quad \text{and} \quad \tau_i = \begin{bmatrix}
\tau_i \\
0 \\
0 
\end{bmatrix}.
\]  

(286)

Putting all of the answers together, one comes up with the constrained equations of motion:

\[
M\ddot{q} + g(q) = B(q)^T \lambda + Q
\]  

(287)

while satisfying the constraints:

\[
B(q)\dot{q} = 0
\]  

(288)

with all the matrices defined as above.

### A.2 General Equations and Linearization

In general, the equations of motion for a parallel manipulator are in the form:

\[
M(p)\ddot{p} + v(p, \dot{p}) + g(p) - B^T(p)\lambda = \tau
\]

\[
c(p) = 0
\]  

(289)

where

\begin{align*}
p & : \text{ vector of generalized coordinates} \\
M(p) & : \text{ generalized mass matrix} \\
v(p, \dot{p}) & : \text{ vector of nonlinear terms} \\
g(p) & : \text{ vector of gravitational (potential force) terms} \\
B^T(p) & : \text{ matrix of constraints} \\
\lambda & : \text{ Lagrange multipliers} \\
c(p) & : \text{ constraints on generalized coordinates to make the set solvable}
\end{align*}
Applying the assumption that gravity terms are accounted for in the control gives:

\[
M(p)\ddot{p} + v(p, \dot{p}) - B^T(p)\lambda - \tau(p, \dot{p}) = 0
\]
\[
c(p) = 0
\]  

(290)

In general these equations are highly nonlinear and are difficult to work with. However, under the assumptions that:

1. No external forces or moments act on the end effector, including gravity.

2. The speed of the manipulator is low so that centripetal and Coriolis terms are negligible (i.e. quasi-static).

3. The controller flexibility dominates the other flexibility effects.

the equations can be linearized around a particular pose. Assuming independent joint PD control and applying Taylor’s theorem to the manipulator, the linearized equations become:

\[
M_s \begin{bmatrix} \Delta \ddot{p} \\ \Delta \ddot{\lambda} \end{bmatrix} + C_s \begin{bmatrix} \Delta \dot{p} \\ \Delta \dot{\lambda} \end{bmatrix} + K_s \begin{bmatrix} \Delta p \\ \Delta \lambda \end{bmatrix} = 0
\]  

(291)

where

\[
M_s = \begin{bmatrix} \tilde{M} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{M} = M(p)\bigg|_{p=p_0} = \left( \sum_{i=1}^{N} J_i^T M_i(p_i) J_i \right)_{p=p_0}
\]  

(292)

\[
C_s = \begin{bmatrix} \tilde{C} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{C} = (J^T K_d J)_{p=p_0}
\]  

(293)

\[
K_s = \begin{bmatrix} \tilde{K} & -\tilde{B}^T \\ \tilde{B} & 0 \end{bmatrix} \quad \text{and} \quad \tilde{K} = (J^T K_p J)_{p=p_0}
\]  

(294)

where \( J \) is the Jacobian matrix, \( K_d \) is the diagonal derivative gain matrix, and \( K_p \) is the proportional gain matrix. From these linearized equations, a linearized natural frequency (and damping ratio) are calculated.
APPENDIX B

SIMULINK DIAGRAM FOR DYNAMICS OF 3RRR ROBOT
three_rrr_dynamics2

/home/phil/matlab/research/dynamics/three_rrr_dynamics2.mdl

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APPENDIX C

BASICS OF SCREW THEORY

This appendix is designed to help the reader understand what screws are and how to generate the screw axis. This appendix does not go into detail of screws, but outlines what screws are, how to find them, and what purpose they serve. This appendix follows the notation and thought process from Alexou [2]. For a more detailed theory on screws see the original treatise by Ball [5].

C.1 The Basics
C.1.1 Twists

Consider a rigid body. One can write the linear velocity of a point, $P$, on the rigid body relative to a $xyz$ coordinate system as:

$$
v_p = \begin{pmatrix} v_{px} \\ v_{py} \\ v_{pz} \end{pmatrix}
$$

and the body’s angular velocity as:

$$\omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}
$$

A velocity twist is defined as a vector that contains both the linear velocity and angular velocity components. Therefore, the twist, $\mathbf{S}^t$ is:
\[ \$^t = \begin{pmatrix} v \\ \omega \end{pmatrix} \]  \hspace{1cm} (297)

Before we continue, there are some things about the twist that should be noted. If one looks at the units on the entries in the vector, it can easily be seen that the vector has mixed units. Therefore, taking the norm of the vector results in a mismatch of the dimensions. In addition, the vector is 6 dimensional.

As one might suspect, similar reasoning can be extended to the \textit{local} (spatial) acceleration to achieve the acceleration twist. In other words:

\[ \$^a = \begin{pmatrix} a \\ \alpha \end{pmatrix} \]  \hspace{1cm} (298)

where \( a \) and \( \alpha \) are the local linear and angular acceleration of the point, \( P \). The concept of acceleration twists is not examined further, but is brought up for reference.

**C.1.2 What About Screws?**

The definition of twists by itself is not that interesting. It appears we have not made any progress in explaining the motion of the rigid body. However, it is shown that all the vectors on the rigid body (assuming a rigid body) can be written as one twist vector. This special vector is the screw. It states that the motion of the body can be described as a translation along an axis and a rotation around the same axis. The term screw is much like its physical counterpart. When turning a “real” screw, the translation and rotation happen simultaneously. Neither motion is present without the other. The question becomes: how do we find this axis and its rotation?

**C.1.3 Finding the Screw Axis**

Suppose we know the linear and angular velocity of the body at the point, \( P \), \( v_P \), and \( \omega \), respectively. The linear velocity of any arbitrary point, \( Q \) can be found using
the equation:

$$\mathbf{v}_Q = \mathbf{v}_P + \omega \times \mathbf{r}_{Q/P}$$  \hspace{1cm} (299)

Now we want point $Q$ to be on the screw axis. We then establish the velocity of $Q$ to be:

$$\mathbf{v}_Q = \rho \omega$$  \hspace{1cm} (300)

where $\rho$ is called the pitch. In terms of a twist, it is:

$$\mathbf{s}' = \begin{pmatrix} \rho \omega \\ \omega \end{pmatrix}$$  \hspace{1cm} (301)

The pitch has dimensions of length. Additionally, the pitch again parallels that of the “real” screw. It is the ratio between linear velocity and angular velocity, much like linear position and angular position. Substituting Equation (300) into Equation (299) gives:

$$\rho \omega = \mathbf{v}_P + \omega \times \mathbf{r}_{Q/P}$$  \hspace{1cm} (302)

We have defined the pitch and $\mathbf{r}_{Q/P}$, but have not shown how to find them. If we dot product $\omega$ on both sides of Equation (302) we can find $\rho$. In a more mathematical sense, we are finding the projection of the velocity vector onto the screw axis. Performing this operation gives:

$$\omega \cdot \rho \omega = \omega \cdot \mathbf{v}_P + \omega \cdot (\omega \times \mathbf{r}_{Q/P})$$  \hspace{1cm} (303)

However, one should note that the second term on the right hand side is zero since the cross product gives a vector that is perpendicular to $\omega$. Also, the dot product of $\omega$ with itself gives the square of the magnitude, $\omega^2$. Therefore:
\[ \rho \omega^2 = \omega \cdot v_P \]  

(304)

Solving for \( \rho \) gives:

\[ \rho = \frac{\omega \cdot v_P}{\omega^2} \]  

(305)

Finding the vector \( r_{Q/P} \) is a little tougher. If we take the cross product of \( \omega \) with Equation (299), then we get:

\[ \omega \times v_Q = \omega \times v_P + \omega \times (\omega \times r_{Q/P}) \]  

(306)

What we are finding is the projection of the velocity vector on a plane that is perpendicular to the screw axis. Noting that \( v_Q \) is along the axis of rotation (see Equation (300)) gives:

\[ 0 = \omega \times v_P + \omega \times (\omega \times r_{Q/P}) \]  

(307)

There are an infinity number of \( r_{Q/P} \) that can solve the equation. Now if we recall a theory from Calculus that:

\[ a \times (b \times c) = b(a \cdot c) - c(a \cdot b) \]  

(308)

Equation (307) becomes:

\[ 0 = \omega \times v_P + \omega (\omega \cdot r_{Q/P}) - r_{Q/P} (\omega \cdot \omega) \]  

(309)

If we take the shortest distance (i.e. the perpendicular distance to the \( \omega \) vector), \( r_\perp \), then the second term is equal to zero, and the equation becomes:

\[ 0 = \omega \times v_P - \omega^2 r_\perp \]  

(310)
\[ \boldsymbol{\omega} \times \boldsymbol{v}_P = \omega^2 \boldsymbol{r}_\perp \]  \hspace{1cm} (311)

\[ \boldsymbol{r}_\perp = \frac{\boldsymbol{\omega} \times \boldsymbol{v}_P}{\omega^2} \]  \hspace{1cm} (312)

Therefore the most general vector to get to the screw axis from point \( P \) can be found by adding, \( \boldsymbol{r}_\perp \) to any point on the rotation axis. Therefore:

\[ \boldsymbol{r}_{Q/P} = \boldsymbol{r}_\perp + \mu \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \]  \hspace{1cm} (313)

where \( \mu \) can be any arbitrary real constant.

## C.2 Screws

### C.2.1 Standard Form

We now can use the pitch and screw axis for something useful. The twist at any point of the rigid body can be written as:

\[ \mathbf{s}^t = \begin{pmatrix} \boldsymbol{d} \times \boldsymbol{\omega} + \rho \boldsymbol{\omega} \\ \omega \end{pmatrix} \]  \hspace{1cm} (314)

where \( \boldsymbol{d} \) is the vector from to the point on the rigid body from the screw axis. The screw is a twist vector of a rigid body. The first three components are the linear velocity at any point due to the angular velocity around the screw axis, plus that component of linear velocity along the screw axis. The next three components are the angular velocity. This designation is the standard form for a screw in axis coordinates.

If we normalize the \( \boldsymbol{\omega} \) vector to a unit vector, we can come up with a “unit” screw. In other words, let:

\[ \mathbf{s} = \frac{\boldsymbol{\omega}}{||\boldsymbol{\omega}||} \]  \hspace{1cm} (315)
Therefore,

\[
\hat{\mathbf{s}} = \begin{pmatrix} \mathbf{d} \times \mathbf{s} + \rho \mathbf{s} \\ \mathbf{s} \end{pmatrix}
\] (316)

It is strongly stressed, that this definition is not the same as the two norm of the screw vector. As was stated previously, the two norm cannot be taken of a vector with mixed units.

### C.2.2 Uses in Robotics

The reason this designation is its usefulness is in robotics and gear design. The instantaneous motion of the links of a manipulator can be described as a screw. Two great examples are prismatic and revolute joints. For revolute joints, the screw axis is known (i.e. at the joint), and the motion is pure rotation (no translational component) can be described as:

\[
\mathbf{s}' = \dot{\theta} \hat{\mathbf{s}}
\] (317)

where

\[
\dot{\mathbf{s}} = \begin{pmatrix} \mathbf{d} \times \mathbf{s} \\ \mathbf{s} \end{pmatrix}
\] (318)

\[
\mathbf{s} = \text{axis of rotation}
\]

Similarly, the prismatic joint has no rotation and only translation. Therefore,

\[
\mathbf{s}' = \dot{d} \hat{\mathbf{s}}
\] (319)

where
\[ \dot{s} = \begin{pmatrix} s \\ 0 \end{pmatrix} \]  

(320)

where \( \dot{d} \) and \( \dot{\theta} \) are the angular and linear velocities of the joints, respectively. The screws can then be used to form the Jacobian matrix.

### C.3 Example

An interesting example is one of a Frisbee thrown in the horizontal plane. Figure 62 shows the case. We align the \( xyz \) coordinate system with the Frisbee as shown. Given the position and the linear and angular velocity as:

\[ \mathbf{r}_P = x\mathbf{i} + y\mathbf{j} \]

\[ \mathbf{v}_P = v\mathbf{i} \]  

(321)

\[ \omega = \omega\mathbf{k} \]

Using Equations (305) and (312) for \( \rho \) and \( \mathbf{r}_\perp \) respectively gives:
\[ \rho = \frac{\omega \hat{k} \cdot \nu \hat{i}}{\omega^2} \]

\[ \rho = 0 \]

and

\[ \mathbf{r}_\perp = \frac{\omega \hat{k} \times \nu \hat{i}}{\omega^2} \]

\[ \mathbf{r}_\perp = \frac{\omega \nu \hat{j}}{\omega^2} \]

\[ \mathbf{r}_\perp = \frac{\nu}{\omega} \hat{j} \]

One should note that \( \mathbf{r}_\perp \) gives the vector from \( P \) to \( Q \). Therefore to get from the origin to the screw axis, we just add vectors:

\[ \mathbf{r}_{\text{screw}} = \mathbf{r}_P + \mathbf{r}_\perp \]

\[ \mathbf{r}_{\text{screw}} = x \hat{i} + \left( y + \frac{\nu}{\omega} \right) \hat{j} \]

This result is very similar to instantaneous centers. The Frisbee appears to be rolling on a point up the \( y \) axis. Also, one should note that the screw axis is parallel to \( \hat{k} \) and goes through the point \( Q \) (see Figure 3).

### C.4 Summary

This appendix outlined twists and screws. The motion of any rigid body can be described as rotation around an axis as well as translation along that same axis.
Equations to find the (screw) axis were derived, and a simple example was performed. Although not performed here, a similar analysis on wrenches could be performed, but they follow the same outline as given above.
APPENDIX D

DERIVATION OF LINK STIFFNESS BASED ON CANTILEVER BEAM ASSUMPTION

The stiffness of the first link for a planar revolute actuated manipulator can be modeled under the assumptions of a cantilever beam. Figure 63 shows a model of the link as a cantilever beam. Under load, the beam deflects from its straight position to the curved position shown in the figure. The end of the beam (or in this case, the link) deflects by an amount of [105]:

\[ y = \frac{PL^3}{3EI} \]  

(322)

where \( P \) is the force on the beam, \( L \) is the length of the beam, \( E \) is Young’s Modulus, and \( I \) is the second moment of inertia of the planar cross-sectional area. If we rearrange this equation:

\[ \frac{y}{L} = \frac{\tau}{3EI} \frac{PL}{L} \]

(323)

Using the small angle approximation, \( \frac{y}{L} \) can approximate the angle \( \theta \), and the torque required for static equilibrium is \( \tau = PL \) which gives:

\[ \theta = \frac{L}{3EI} \tau \]

(324)

Pulling the constant to the other side gives:

\[ \tau = \frac{3EI}{L} \theta \]

\[ = k\theta \]

(325)
This can then be done for every leg and a stiffness matrix could be formed in the joint space as:

\[
K = \begin{bmatrix}
k_1 \\
\vdots \\
k_n
\end{bmatrix}
\] (326)
APPENDIX E

PROTOTYPE IMPLEMENTATION

A prototype was fabricated to verify the utility of a redundantly actuated parallel manipulator and whether or not it could be controlled. The prototype was designed via the results of the trial and error optimization of the 4RRR robot configuration given in Chapter 11. This appendix is included to show how easily the redundant manipulator can be built and used. Most important is the basic controller that was utilized in the manipulator. Future research will complete the prototype and show the validity of the results.

E.1 Component Design

E.1.1 Mechanical Design

For ease of fabrication, wooden links were chosen for the legs. However, the wood is not as stiff as metal links and thus adds the nonlinear effects that were neglected in prior chapters. The entire assembly was assembled on a 3’ × 3’ × 1 1\(\frac{1}{2}\)” Medium Density Fiberboard (MDF) base. Again, the MDF was chosen for its ease of use and cost effectiveness.

For ease of use and application, a symmetric robot with two different leg lengths was chosen as was outlined in Chapter 11. The kinematic parameters were chosen with a “trial and error” technique. Simple ¼” carriage bolts serve as the revolute joints. A Parker Hannifin Compumotor BE232DJ-KFON direct drive motor with OEM770T drivers was chosen as the power plant.\(^1\) These motors were chosen for two

\(^1\)Complete specifications on the motors can be found at the Compumotor website: http://www.compumotor.com/wwwroot/literature/be_series.htm
different reasons: 1.) use with the drivers that were already purchased and 2.) budget constraints. However, the motors were more than adequate for this application and could be used for other applications if the need arises. The direct drive motor was chosen to eliminate the nonlinearity of the gearboxes and to further test out the control system that will be discussed later.

E.1.2 Electrical Design

The motors and drivers were connected to a PC with a Servo2Go I/O board. The PC ran Real Time Linux connected to MATLAB Simulink. This type of connection allows for easy control and modification to the manipulator’s control system. The motors were equipped with encoders that provided 2000 pulses per revolution of resolution. The encoders were utilized for feedback in the control system. A joystick for input was connected for use in manual mode operation. However, limits had to be made in the joystick control to ensure that the end effector would stay in its workspace.

E.1.3 Final Design

Figures 64 and 65 show pictures of the mechanical and electrical portion of the manipulator. As the pictures show, the mechanism is not aesthetically pleasing, but does perform as instructed. Further work will be on improving the link materials and design and cleaning up the electrical wiring.

E.2 Control

E.2.1 Background

The approach taken in this thesis is to use an independent PD joint position control that was suggested by Luecke and Gardner [69, 70]. In this work, an “ostrich” technique is used that ignores the dynamics of the manipulator and treats each joint as independent entity. Luecke and Gardner prove that this technique is Lyapunov stable and that the control minimizes a weighted Euclidean norm of the input torques.
Figure 64: Picture of the mechanical section of completed prototype.

Figure 65: Picture of the electrical portion of completed prototype.
While this technique may seem crude, it is effective and offers advantages over other techniques. Most of the existing schemes for controlling redundant parallel manipulators are based on the cooperating serial manipulators or robotic hands.\textsuperscript{2} However, the key difference between cooperating serial manipulators and a redundant parallel manipulator is that in the latter not all the legs are fully actuated. In other words, there are passive joints in the legs. This key difference does not allow some of the control strategies to be transferred. On the other hand, the independent joint position control is applicable to whether the manipulator is fully actuated or not.

Most of the work in redundant parallel mechanisms has been on lower degree of freedom manipulators. Kock and Schumacher [54, 55] derived a minimum norm torque and a lower bound stiffness control for the 3RR mechanism. Hui [40] also uses several independent joint control techniques to control another 3RR. Beiner [7] also controls a 3RR using standard constrained optimization technique based on the dynamics of the manipulator that minimize the joint torques on a certain trajectory, while Kumar and Gardner [60] use a continuous switching scheme. However, the simple control of Luecke and Gardner is as effective.

Some other less related work is included in that of master/slave robots [128], a partitioned actuator set control technique [28] or a hybrid force/position control [38]. However, real care needs to be taken to insure that the hybrid control strategy is invariant [65].

### E.2.2 Implemented Control

Figure 66 shows the block diagram for the control of the manipulator. The desired position, $X_{des}$, is fed through the inverse kinematics,\textsuperscript{3} IK, where it is subtracted from the actual angular output to create the error signal, $e$. The error is input into a PD controller and output through the Servo2Go I/O board, S2G. The motor drivers then

\textsuperscript{2}See Boubekri and Chakraborty for a very detailed reference list of control strategies for grasping.

\textsuperscript{3}The inverse kinematics from Chapter 3 can easily be augmented for an additional leg.
Figure 66: Control diagram of the 4RRR robot.

create the correct voltages for the motors which turn to the corresponding joint angle, \( \Theta \), which is fed back through the encoders and the Servo2Go board. The manipulator through the linkages, does the forward kinematics, FK, to give the desired output.

Figure 67 shows the implemented Simulink Control Diagram. Chatter was a problem in the actual implementation of the manipulator, and therefore, the proportional gain was given the largest value before chatter ensued. The derivative gain was then increased to a point where the overshoot on a point to point trajectory was small. However, since the mechanism is nonlinear, several trajectories were chosen and a compromise was created to have decent control over the workspace.

E.3 Future Work

The robot was given step inputs of varying sizes to determine the effectiveness of the control and it performed satisfactorily. The next steps in this implementation would be to include creating more robust links and joints as well as addressing the chatter problem. Even further, a new robot based on the results from the optimization given in Section 11.3.2 would be beneficial to check the theoretical framework.
Attention!!!!! Load variables before "building" by typing "load inputvars" into matlab

Wire Connection Instructions

Motor Initial angles

Kp - Gain

Load variables before "building" by typing "load inputvars" into matlab

Wire Connection Instructions

Motor Initial angles

Kp - Gain

Initing

Wire Connection Instructions

Motor Initial angles

Kp - Gain
APPENDIX F

ANTIPODAL CABLE THEOREM

This appendix shows how parallel manipulators, grasping, and wire driven robots are similar. More specifically, this appendix derives the Antipodal Cable Theorem from the Antipodal Grasp Theorem.

The remainder of this appendix is organized as follows: Section F.1 provides background material, including correspondences between cable robots and frictionless grasps. Section F.2, which the the core of this appendix, discusses the application of the Antipodal Grasp Theorem to certain cable robots, resulting in the Antipodal Cable Theorem. Section F.3 very briefly discusses some applications.

F.1 Background

F.1.1 Architecture Correspondences

F.1.1.1 Cable and Parallel Robots

A cable robot consists of a moving platform that is connected to a fixed base through N cables, as indicated in Figure 68(a). The length of each cable can be changed individually by actuating a motor with a spool that reels in and out the cable. The corresponding spatial parallel robot is shown in Figure 68(b), where each leg consists of a SPS (spherical-prismatic-spherical) chain with the prismatic joint being actuated and all other joints passive. As was pointed out in [84], these two robots have the same kinematics if the cables are assumed to always be in tension.
(a) Sample cable robot with $N = 5$ cables.

(b) Corresponding parallel robot with $N = 5$ legs.

(c) Corresponding frictionless grasp with $N = 5$ fingers

Figure 68: Sample correspondences for a spatial cable robot.

F.1.1.2 Corresponding Grasp with Frictionless Point Contacts

Figure 68(c) shows an object grasped by $N$ frictionless point contacts, corresponding to the cable robot in Figure 68(a). There are several characteristics to note in order for the grasping model to precisely match the cable robot model:
1. Since the contact force for frictionless point contacts is always normal to the object’s surface at the contact point, the object’s surface normals must match the direction of the corresponding cables.

2. Each finger is located at the inside of the object to be grasped in order to generate a force in the same direction as the cables in Figure 68(a).

3. The geometry of the object in Figure 68(c) is arbitrary, except for the location and normals at the contact points. Thus, without loss of generality, one can choose a polyhedron. One cannot assume the polyhedron to be convex, since even for the example in Figure 68(c) it is impossible to construct a convex polyhedron for the given conditions.

4. Unless two or more cables are attached at the same point on the moving platform, it is legitimate to assume non-singular interface points, i.e. the surface normals are always well defined. If two or more cables are attached at the same point, many of the mathematical grasping tools apply nevertheless, but caution must be applied to ensure each tool’s validity. (An alternative, as will be shown later in this appendix, is to model several cables with coincident attachment points as a single contact with friction.)

5. Although the contact model does not contain friction, i.e. only normal forces can be transmitted, the contact points cannot slide in this corresponding grasping model, to represent the fact that the corresponding cables in Figure 68(a) are rigidly attached to the mobile platform.

F.1.2 Force Transmission

As expected, the standard force transmission models for the cable robot and multi-finger grasp are also related.
**F.1.2.1 Cable Robot**

The force and moment at the end-effector of the cable robot is related to the actuator torques as follows [108]:

\[
\begin{bmatrix}
F_{ee} \\
M_{ee}
\end{bmatrix} = J^T \begin{bmatrix}
\tau_1 \\
\vdots \\
\tau_N
\end{bmatrix}, \quad \tau_i \geq 0, \quad (327)
\]

where \(J\) is the Jacobian matrix consisting of pure-force wrenches along the cables:

\[J^T = [s_1 \cdots s_N],\]

and \(s_i\) is the screw along the \(i\)th cable. The actuator torques, \(\tau_i\), can only take positive values (\(\tau_i \geq 0\)), because cables can only pull but not push.\(^1\)

**F.1.2.2 Corresponding Grasp with Frictionless Point Contacts**

For frictionless point contact the relationship between finger forces transmitted at the contact points and corresponding wrench at the object’s center-of-mass can be expressed by the grasp map, \(G\), as follows [74, 88]:

\[
\begin{bmatrix}
F_{ee} \\
M_{ee}
\end{bmatrix} = G \begin{bmatrix}
f_{c1} \\
\vdots \\
f_{cN}
\end{bmatrix}, \quad f_{ci} \geq 0, \quad (328)
\]

where \(f_{ci}\) are the (scalar) contact forces, i.e. the amount of force provided by the fingers at the contact point. \(G\) is the grasp map consisting of the pure-force screws pointing perpendicular to the surface normal of the object being grasped and away from the finger contact (see [88] for details). The finger forces, \(f_{ci}\), can only take positive values, \(f_{ci} \geq 0\), because the fingers (frictionless point contacts) can only push but not pull.

---

\(^1\)The Jacobian relationship in Equation 327 also holds for the equivalent parallel robot except that the input torques can take any value, i.e. \(\tau_i \in (-\infty, \infty)\)

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Comparing Equation (327) for the cable robot to Equation (328) for the corresponding grasping problem, it is apparent that they are identical when matching finger forces with actuator torques and the grasp map with the Jacobian transpose of the cable robot:

\[ f_{ci} = \tau_i, \quad G = J^T. \]

### F.1.3 Force Closure

Based on the definition of force closure for grasping [75, 36], an equivalent definition for cable robots can be constructed:²

**Definition: Force Closure Pose**

A cable robot is said to have **force closure** in a particular pose if and only if any arbitrary external wrench applied at the moving platform can be counteracted through appropriate tension forces in the cables. Assuming actuator torques of unlimited magnitudes force closure for a cable robot is mathematically described as

\[ \forall \mathbf{s}_{ee} \in \mathbb{R}^6 : \exists \tau_1, \ldots, \tau_N \in [0, \infty] : \mathbf{s}_{ee} = J^T [\tau_1 \ldots \tau_N]^T, \]

In the grasping area, force closure criteria have thus been developed based on convexity theory [109]. For example it is well known from convexity theory that the minimal number of frictionless finger contacts required to achieve force closure is 4 for the planar case and 7 for the spatial case [88]. Furthermore, a force closure grasp with the above number of fingers can be found for any object that is not *exceptional* [86],[88]. On the other hand, force closure cannot be achieved for exceptional objects (e.g. spheres and cylinders) with frictionless point contacts, no matter how many fingers are used [86],[88].

Equivalently for a cable robot, the number of cables required to achieve a force closure pose is 4 for the planar case and 7 for the spatial case [52]. There are no

²This definition is equivalent to those already discussed by [52] and [118]
Figure 69: Convention for direction of friction cones in grasping and force triangles/tetrahedra in cable robots

exceptional geometries in the context of cable robots, i.e. those numbers of cables are always sufficient to achieve force closure.

While several research groups have used the above analogy to determine the number of cables required to achieve force closure in cable robots [52], we have not seen efforts to transfer any of the more advanced tools from grasping, e.g. theorems that analyze whether a grasp is force closure, to cable robots. An example of such a theorem and its application to cable robots is given in Section F.2.

F.2 The Antipodal Cable Theorem

To show the benefit of further connecting the research areas of grasping and cable robots, this section demonstrates how a theorem from grasping can be useful for cable robots. Subsection F.2.1 provides a convention, Subsection F.2.2 discusses the use of the planar antipodal grasp theorem and Subsection F.2.3 follows up with the spatial case.

F.2.1 Convention for Direction of Force Cones, etc.

The section deals with point contacts with friction and in the context of this appendix it is important to clearly define the “direction” of the friction cone. In literature the friction cone, which specifies the set of forces that can be generated by a finger, is always drawn such that the force point away from the contact point, as shown in
Figure 70: Equivalences of Planar Antipodal Grasp Theorem

Figure 69(a). This convention is employed throughout this appendix.

Equivalently, when examining the set of forces that can be generated by a set of coincident cable in a cable robot, the resulting half-open triangle (in 2D) or half-open tetrahedron (in 3D) is always drawn such that the forces point away from the apex, as shown in Figure 69(b) for just two coincident cables. Two different cases are shown in Figure 69(b): the two cables can coincide either at the moving platform or at the actuator location.

F.2.2 Planar Antipodal Cable Theorem

A planar cable robot must use at least 4 cables to achieve force closure. Furthermore for mechanism symmetry, kinematic simplicity, and wire tangling it is beneficial to use only two attachment points on the moving platform, i.e. two pairs of cables that coincide at the platform. An example is shown in Figure 70(a). Since the use of four cables is necessary, but not sufficient for force closure, the question arises whether there are any simple geometric force closure tests for planar cable robots with two pairs of coincident cables. The necessary tool exists in grasping literature, namely the Planar Antipodal Grasp Theorem.

Planar Antipodal Grasp Theorem [90] [88]:

A planar grasp with two point contacts with friction is force closed if and only if the line connecting the contact points lies inside both friction cones.

If the above condition is satisfied, the grasp is called an antipodal grasp. Note
that this formulation of the antipodal grasp theorem assumes that both fingers push from the outside, which is the standard case considered in grasping.

A planar point contact with friction can be modeled as a pair of point contacts without friction [88] that are applied at the same point. Thus the planar cable robot with the two coincident cable pairs shown in Figure 70(a) corresponds to the two-finger grasp with friction shown in Figure 70(b), where the boundaries of each friction cone coincide with the directions of the corresponding cables. This in turn defines the surface normal at the contact point, as the surface normal of the object to be grasped must be along the center of the friction cone. The fingers are pushing from the inside in order to generate forces in the same direction as the cables.

Figure 70(c) shows the same grasp but with reversed finger direction, to which the antipodal grasp theorem can be applied. The following corollary relates these two grasps.

**Corollary:** A two-finger planar friction grasp with fingers pushing from the inside (e.g. as shown in Figure 70(b)) is force closure if and only if the two-finger friction grasp with reversed finger directions (e.g. as shown in Figure 70(c)) is force closure.

**Proof:** Let $S_1, \ldots, S_4$ denote the wrenches corresponding to the friction boundaries of the original two-finger (from inside) grasp. By definition, this grasp is force closure, if and only if any external wrench, $S_{\text{load}}$, can be generated by a positive linear combination of those four wrenches, i.e.

$$\forall S_{\text{load}} \in \mathbb{R}^3 : \exists \alpha_i > 0 : \sum_{i=1}^{4} \alpha_i S_i = S_{\text{load}}.$$ 

If any arbitrary wrench $S_{\text{load}}$ can be generated, then the same must hold for $(-S_{\text{load}})$:

$$\forall S_{\text{load}} \in \mathbb{R}^3 : \exists \alpha_i > 0 : \sum_{i=1}^{4} \alpha_i S_i = (-S_{\text{load}}).$$

which is equivalent to

$$\forall S_{\text{load}} \in \mathbb{R}^3 : \exists \alpha_i > 0 : \sum_{i=1}^{4} \alpha_i (-S_i) = S_{\text{load}}.$$
Figure 71: Test for full constraint of this planar cable robot: Does the line segment from P to Q lie completely within the two reversed triangles? (For this case: Yes.)

This last condition is satisfied if and only if the two-finger friction grasp with reversed finger direction (from outside) is force closure, since $(-s_i)$ is the $i$th wrench of the grasp with reversed finger direction. Q.E.D. □

Combining the Antipodal Grasp Theorem with the above corollary results in:

**Planar Antipodal Cable Theorem:**

A planar cable robot with 2 pairs of cables with coincident attachment points, $P$ and $Q$, is force closed if and only if the line from $P$ to $Q$ lies completely in the two open force triangles defined by the reversed forces of the two cable pairs.

**Proof:** The set of forces the the planar cable robot with the above properties can apply to the moving platform is identical to the set of forces that the two-finger planar friction grasp with fingers from the inside in the above Corollary can exert on the object to be grasped. Thus the same geometric criterion applies for force closure in both systems.

**Example 1:** The two-pair planar cable robot in Figure 70(a) is fully-constrained if and only if the corresponding grasp in Figure 70(c) is an antipodal grasp as specified in the Antipodal Grasp Theorem. This criterion can easily be checked directly for the cable robot, as shown in Figure 71, by extending the lines of the cables and checking whether the line segment from $P$ to $Q$ lies completely within the two reversed open-ended triangles. This geometric criterion may be useful for the synthesis of planar cable robots. More importantly, it is an example of a tool from grasping that carries
**Figure 72:** Test for planar cable robot where the outer attachment points of the cables coincide pairwise: Does the line segment from P to Q lie completely within the two reversed triangles? (For this case: No.)

over to cable robots.

**Example 2:** Shown on the left in Figure 72 is a cable robot with cable pairs originating from the same point, i.e. the axes of their motors coincide. The corresponding criterion is demonstrated on the right in Figure 72: Does the line segment from P to Q lie completely in the two reversed open-ended triangles? Note that in this case points P and Q are not located on the mobile platform.

### F.2.3 Spatial Antipodal Cable Theorem

This section is based on the spatial antipodal grasp theorem, which can be formulated as follows:

**Spatial Antipodal Grasp Theorem** [90] [88]:

A spatial grasp with two soft-finger contacts is force closed if and only if the line connecting the contact points lies inside both friction cones. A grasp satisfying this geometric condition is called antipodal.

A soft-finger contact consists of two components, forces within a one-sided friction cone and a two-sided torque about the surface normal at the contact point. In contrast, a hard-finger contact only provides the forces of the friction cone, but no torque. As it turns out, only one soft-finger contact is needed for the antipodal grasp theorem.

**Corollary:** A spatial grasp with one soft-finger contact and one hard-finger contact is force closed if and only if the line connecting the contact points lies inside both...
friction cones.

**Proof:** One direction of the proof is trivial: If the geometric condition that the connecting line lies inside both friction cones is necessary for force closure with two soft fingers, it is certainly necessary also for a soft finger and hard finger contact.

Second direction: It remains to show that this geometric condition is also *sufficient* to achieve force closure using only one soft finger and one hard finger contact. Revisiting the proof of the original Antipodal Grasp Theorem in Nguyen’s thesis [89], it is apparent that the only role of the two soft finger torques is to generate a bidirectional moment about the line connecting the two contact points. Such moment is provided by even one soft finger if the surface normal at the contact point is not orthogonal to the connecting line. Since the connecting line is required to lie within the friction cone (and the friction cone is less than 180 degrees), the surface normal is guaranteed not to be orthogonal to the connecting line. Thus one of the soft finger contacts plus a hard finger contact is sufficient. Q.E.D. ∎

To transfer the above corollary to cable robots the following correspondences are important. For a cable robot, a set of three cables that coincide either at the mobile platform or at the actuator location provides a force tetrahedron similar to the friction cone of a hard of soft finger contact. The bidirectional torque component of a soft finger contact can only be provided by cables that are *not* coincident with the three-cable set and two such cables are needed. Thus the equivalent of a soft-finger plus hard-finger contact is a cable robot with 8 cables, consisting of two sets of three cables each which coincide either at the moving platform or at the actuator location, and two additional cables attached elsewhere. Following the notation in [113], this type of cable robot is denoted a 3-3-2 robot if the two additional cables also coincide and a 3-3-1-1 robot if they do not coincide.
The corollary above motivates the following Spatial Antipodal Cable Theorem:

**Spatial Antipodal Cable Theorem:**

Consider a spatial cable robot with two three-cable sets that coincide at two points, $P$ and $Q$, and two additional cables. If the following conditions are both satisfied, the cable robot pose is force closed:

1. The line connecting points $P$ and $Q$ lies inside the two reversed force tetrahedra spanned by the two three-cable sets;
2. The remaining two cable forces create moments of opposing direction along the line segment $PQ$.

Note that the above theorem only specifies that the two conditions are sufficient, but not that they are necessary for force closure.

**Proof:** Let us assume that Conditions (1) and (2) of the antipodal cable theorem are satisfied. Without loss of generality, we choose the origin of our reference frame to lie at the mid point between $P$ and $Q$ and choose its $x$-axis to point along line segment $PQ$, pointing towards $Q$. This is shown in Figure 73, where the $y$-axis points upwards and the $z$-axis points out of the page.

Based on Condition (1) forces along and close to the negative $x$-axis lie within $P$’s force tetrahedron. Specifically for some small $\epsilon > 0$ the following forces at $P$ are included in the force tetrahedron:

\[
\text{At } P: \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ \pm \epsilon \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ \pm \epsilon \end{pmatrix}.
\]
<table>
<thead>
<tr>
<th>Component Name</th>
<th>$F_{x}^{+}$</th>
<th>$F_{x}^{-}$</th>
<th>$F_{y}^{+}$</th>
<th>$F_{y}^{-}$</th>
<th>$F_{z}^{+}$</th>
<th>$F_{z}^{-}$</th>
<th>$M_{y}^{+}$</th>
<th>$M_{y}^{-}$</th>
<th>$M_{z}^{+}$</th>
<th>$M_{z}^{-}$</th>
</tr>
</thead>
<tbody>
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<td>Force $F_P$ at $P$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>Force $F_Q$ at $Q$</td>
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<td>$1$</td>
<td>$1$</td>
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<td>$1$</td>
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<td>$1$</td>
<td>$0$</td>
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<td>$0$</td>
</tr>
<tr>
<td>Resulting screw</td>
<td>$[-1]$</td>
<td>$[1]$</td>
<td>$[0]$</td>
<td>$[0]$</td>
<td>$[0]$</td>
<td>$[0]$</td>
<td>$[0]$</td>
<td>$[0]$</td>
<td>$[0]$</td>
<td>$[0]$</td>
</tr>
</tbody>
</table>

Table 7: Force combinations that positively span all forces in $\mathbb{R}^3$ and all moments about $M_y$ and $M_z$.

Likewise, forces along and close to the positive x-axis lie within $Q$'s force tetrahedron:

$$
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
1 \\
\pm \epsilon \\
0
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
\pm \epsilon
\end{bmatrix}.
$$

Table 7 lists ten combinations of these forces that yield independent positive and negative forces in the $x,y,$ and $z$-axes at origin $O$, as well as independent positive and negative moments about the $y$ and $z$-axes. ($d > 0$ denotes the distance between points $P$ and $Q$ in Table 7.) This clearly shows that the two force tetrahedra at $P$ and $Q$ are sufficient to positively span a 5-dimensional space, namely all forces in $x,y,z$ and all moments about $y,z$.

Obviously, no moment can be generated about the $x$-axis by any forces of the two force tetrahedra. However, condition (2) guarantees that the forces along the remaining two cables each provide a moment about the $x$-axis, one in positive and the other in negative direction. While each of these two cable forces may simultaneously create other forces and moments that are coupled with the desired moment about the $x$-axis, those components can be eliminated using the 5-dimensional space spanned by the combinations listed in Table 7. Thus force closure is achieved. Q.E.D. $\square$

Whether the two conditions in the Antipodal Cable Theorem are not only
sufficient but also necessary to achieve force closure, is still an open question and topic of future research.

F.3 Applications

Several observations can be drawn from the above theorems for some cable robots that have been proposed in literature, for example the WARP robot described by [113]. Using the techniques from the Spatial Antipodal Cable Theorem (SACT), it provides an intuitive explanation of why certain configurations are better than others.

The WARP manipulator described by [113] is of the 3-3-2 architecture which is of the type given in the SACT. This configuration was chosen using symmetry and some simple numerical analysis. The SACT now provides a simple necessary condition to guarantee force closure of the WARP mechanism.

They state that the 3-3-2 configuration is “superior” to the 3-3-1-1 due to better “efficiency for generating moment.” From the SACT, it is clear that the two wires need to provide a moment around the intersecting line for all configurations (Condition 2). From the geometry, the 3-3-2 is guaranteed to have larger moment generating capacity around the intersecting line. To further optimize the moment generating capability, the positions on the end effector should be placed as wide as possible from the intersecting line. Also, from Condition 1, the larger the force tetrahedra, the larger the workspace will be.

Likewise, the proposed 2-2-2-2 configuration has 4 wires intersecting at each point on the end effector. These form two force pyramids (i.e. a five sided polygon) which provide the 5 dimensional force set. However, there exists no moment along the line connecting the two points, and thus does not satisfy Condition 2 of the SACT. This result is consistent with the findings in [113].

Even though the Spatial Antipodal Cable Theorem applies to 8 wire manipulators, it can be useful for other cases. For example, it is clear that a 3-3-1 manipulator can
never achieve force closure, because no bi-directional moment about axis \( \vec{PQ} \) can be generated.
REFERENCES


