

**MINIMUM PRINCIPLE OF THE TEMPERATURE IN
COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH
APPLICATION TO THE EXISTENCE THEORY**

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The Academic Faculty

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Weizhe Zhang

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Approved by:

Professor Ronghua Pan,
Committee Chair
School of Mathematics
Georgia Institute of Technology

Professor Ronghua Pan, Advisor
School of Mathematics
Georgia Institute of Technology

Professor Wilfrid Gangbo
School of Mathematics
Georgia Institute of Technology

Professor Andrzej Swiech
School of Mathematics
Georgia Institute of Technology

Professor Chongchun Zeng
School of Mathematics
Georgia Institute of Technology

Professor Yanni Zeng
Department of Mathematics
University of Alabama at Birmingham

Date Approved: May 07 2014

To my family.

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SUMMARY

This thesis is on the Navier-Stokes equations which model the motion of compressible viscous fluid. A minimum principle on the temperate variable is established. Under the thermo-insulated boundary conditions and some reasonable assumptions on the solution, the minimum of the temperature does not increase. To our best knowledge, that's the first result on the minimum principle of the temperature variable in the compressible Navier-Stokes equation.

As an application of the minimum principle, global in time existence of the weak solution (as defined in [51]) for the Navier-Stokes equations is established when the viscosities and heat conductivity are power functions of the temperature. In this model the temperature is coupled with density which may have vacuum or concentration and the heat conductivity has possible degeneracy. However the temperature is proved to obey the minimum principle, which secured the dissipative mechanism of the system, and paved the road to the existence theory.

CHAPTER I

INTRODUCTION

Navier-Stokes equations, named after Claude-Louis Navier and George Gabriel Stokes, describe the motion of fluid substances. The motions of fluid substances obey the conservation of mass, balance of momentum (equivalent to the Newton's Second Law), and balance of energy (First Law of Thermodynamics). In the continuum fluid mechanics, the fluid is modeled by macroscopic state variables, such as the density ρ , fluid velocity \mathbf{u} , and temperature θ . e is the specific internal energy, κ is the heat conductivity, P is the pressure, and \mathbb{S} is the stress tensor. Under these notions, the Navier-Stokes equations read as:

$$\begin{aligned}\rho_t + \operatorname{div}_x(\rho\mathbf{u}) &= 0, \\ (\rho\mathbf{u})_t + \operatorname{div}_x(\rho\mathbf{u} \otimes \mathbf{u}) &= -\nabla_x P + \operatorname{div}_x \mathbb{S}, \\ (\rho e)_t + \operatorname{div}_x(\rho e\mathbf{u}) &= \operatorname{div}_x(\kappa \nabla_x \theta) + \mathbb{S} : \nabla_x \mathbf{u} - P \operatorname{div}_x \mathbf{u}.\end{aligned}\tag{1}$$

Mathematical studies of the Navier-Stokes equations were initiated hundreds years ago. While numerous results related to this system were obtained, many basic questions of this system for large data and time in three dimension are still unknown. In this chapter, these physical laws are treated as the forms of balance laws. The integral forms of these balance laws motivate the definitions of weak solution, particularly those defined in [43, 51].

1.1 State Equations

Generally, the thermodynamic pressure is a function of density and temperature,

$$P = P(\rho, \theta),\tag{2}$$

so does the internal energy from the caloric equation

$$e = e(\rho, \theta), \quad (3)$$

and the specific entropy

$$s = s(\rho, \theta), \quad (4)$$

these thermodynamic functions are related through the Gibbs' equation

$$\theta Ds(\rho, \theta) = De(\rho, \theta) + P(\rho, \theta)D\left(\frac{1}{\rho}\right). \quad (5)$$

The symbol D in equation (5) stands for the differential with respect to the variables ρ and θ .

The Gibbs' equation is equivalent to the Maxwell's relation,

$$\frac{\partial e(\rho, \theta)}{\partial \rho} = \frac{1}{\rho^2} \left(P(\rho, \theta) - \theta \frac{\partial P(\rho, \theta)}{\partial \theta} \right). \quad (6)$$

We assume that the macroscopic motion of the fluid is so large, that at any time the fluid is in the thermodynamic equilibrium, i.e., the temperature, pressure and other thermodynamic variables are well-defined. Although some analysis of the equations are done in the whole domain if the boundary is "far away" to produce effects on the motion of the fluid, but in this thesis we consider the problem of large time and large data on a bounded domain with real physical boundary conditions.

1.2 *Balance laws*

Macroscopic quantities are used to describe the fluid in the classical continuum mechanics, they obey the basic physical principles, which are usually in the form of balance laws,

$$\int_B d(t_2, x) dx - \int_B d(t_1, x) dx + \int_{t_1}^{t_2} \int_{\partial B} \mathbf{F}(t, x) \cdot \mathbf{n} dS_x dt = \int_{t_1}^{t_2} \int_B \sigma(t, x) dx dt \quad (7)$$

for any $t_1 \leq t_2$ and any subset $B \subset \Omega$. Here the function d is the volumetric density of an observable property, e.g., density, temperature, velocity, etc. \mathbf{F} stands

for the corresponding flux, \mathbf{n} is the outer normal vector of ∂B , and σ denotes the rate of the production of d per unit volume. The Lions-Feireisl's definition of weak solution[43, 51] is one of the most natural mathematical formulation of balance law in continuum mechanics.

Introducing a new test function $\phi(t, x)$ in $\mathbb{R} \times \mathbb{R}^3$, the equation (7) could be transformed to

$$\begin{aligned} & \int_B d(t_2, x)\phi(t_2, x)dx - \int_B d(t_1, x)\phi(t_1, x)dx + \int_{t_1}^{t_2} \int_{\partial B} \mathbf{F}(t, x) \cdot \mathbf{n}\phi(t, x)dS_x dt \\ &= \int_{t_1}^{t_2} \int_B \sigma(t, x)\phi(t, x)dx dt. \end{aligned} \quad (8)$$

The left hand side of equation (8) could be rewritten as

$$\begin{aligned} & \int_B d(t_2, x)\phi(t_2, x)dx - \int_B d(t_1, x)\phi(t_1, x)dx + \int_{t_1}^{t_2} \int_{\partial B} \mathbf{F}(t, x) \cdot \mathbf{n}\phi(t, x)dS_x dt \\ &= \int_{t_1}^{t_2} \int_B (\partial_t d(t, x) + \operatorname{div}_x \mathbf{F}(t, x))\phi(t, x)dx dt \\ &= \int_{t_1}^{t_2} \int_B (d(t, x)\partial_t \phi(t, x) + \mathbf{F}(t, x)\nabla_x \phi(t, x))dx dt. \end{aligned} \quad (9)$$

So we know that when all the quantities are continuously differentiable, equation (7) is equivalent to the traditional form

$$\partial_t d(t, x) + \operatorname{div}_x \mathbf{F}(t, x) = \sigma(t, x). \quad (10)$$

While in the equation (7), the source term σ needs not to be an integrable function. As a fact, the weak form would be well-defined if σ is only a signed Borel measure, i.e. $\sigma = \sigma^+ - \sigma^-$, where $\sigma^+, \sigma^- \in M([0, T] \times \bar{\Omega})$ are non-negative Borel measures defined on the compact set $[0, T] \times \bar{\Omega}$. On the other hand we need to incorporate the initial data $d(0, \cdot) = d_0$ and boundary condition $F_b = \mathbf{F} \cdot \mathbf{n}|_{\partial\Omega}$, so the general form of the balance law will be

$$\begin{aligned} & \langle \sigma, \phi \rangle_{[M, C]([0, T] \times \bar{\Omega})} + \int_{t_1}^{t_2} \int_{\Omega} (d(t, x)\partial_t \phi(t, x) + \mathbf{F}(t, x)\nabla_x \phi(t, x))dx dt \\ &= \int_0^T \int_{\partial\Omega} F_b(t, x)\phi(t, x)dS_x dt - \int_{\Omega} d_0(x)\phi(0, x)dx. \end{aligned} \quad (11)$$

for any test function $\phi \in C_c^\infty([0, T] \times \bar{\Omega})$. In the case when σ may not be absolutely continuous with respect to the Lebesgue measure, we can still consider the two one-side limits, which is very important for the initial values. The left instantaneous value of d is defined in the following way,

$$\begin{aligned} & \langle d(\tau-, \cdot), \phi \rangle_{[M, C](\Omega)} \\ &= \int_{\Omega} d_0(x) \phi_0(x) dx + \int_0^\tau \mathbf{F}(t, x) \nabla_x \phi + \lim_{\delta \rightarrow 0^+} \langle \sigma, \varphi_\delta \phi \rangle_{[M, C](Q)} \end{aligned} \quad (12)$$

for any test function $\phi \in C_c^\infty(\Omega)$, where $\varphi(t)$ is non-increasing and

$$\varphi_\delta(t) \in C^1(\mathbb{R}) \text{ and } \varphi_\delta(t) = \begin{cases} 1 & \text{if } t < \tau - \delta, \\ 0 & \text{if } t \geq \tau. \end{cases} \quad (13)$$

Similarly, we can define the right instantaneous value of d ,

$$\langle d(\tau+, \cdot), \phi \rangle_{[M, C](\Omega)} = \int_{\Omega} d_0(x) \phi_0(x) dx + \int_0^\tau \mathbf{F}(t, x) \nabla_x \phi + \lim_{\delta \rightarrow 0^+} \langle \sigma, \varphi_\delta \phi \rangle_{[M, C](Q)} \quad (14)$$

for any test function $\phi \in C_c^\infty(\Omega)$, where $\varphi(t)$ is non-increasing and

$$\varphi_\delta(t) \in C^1(\mathbb{R}) \text{ and } \varphi_\delta(t) = \begin{cases} 1 & \text{if } t < \tau, \\ 0 & \text{if } t \geq \tau + \delta. \end{cases} \quad (15)$$

In the case when the map $\tau \mapsto d(\tau, \cdot)$ is absolutely continuous with respect to the Lebesgue measure, we know that two instantaneous values will coincide.

In the case when d is the entropy, the identity of the balance law is hard to obtain, based on the theory in [43, 51], hence an inequality of the balance laws is used to define the solutions, which is weaker than the equality form. The inequalities still obey the basic physical law, such as the second law of thermodynamics:

$$\int_0^T \int_{\Omega} d(t, x) \partial_t \phi(t, x) + \mathbf{F}(t, x) \cdot \nabla_x \phi(t, x) dx \leq 0 \quad (16)$$

for any non-negative test function $\phi \in C_c^\infty([0, T] \times \bar{\Omega})$, the corresponding differential form is

$$\partial_t d + \operatorname{div}_x(\mathbf{F}) \geq 0. \quad (17)$$

1.3 Fields equations

1.3.1 Continuity Equation

The fluid density $\rho = \rho(t, x)$ is a fundamental state variable describing the distribution of mass. The integral

$$M(B) = \int_B \rho(t, x) dx \quad (18)$$

is the total mass of the fluid contained in a set $B \subset \Omega$ at the time t . Generally the density function could be any non-negative function in Ω , more practically we can assume that ρ is absolutely continuous with respect to the standard Lebesgue measure on \mathbb{R}^3 .

The general balance law of the density from the physical law of mass conservation takes the form

$$\int_B \rho(t_2, x) dx - \int_B \rho(t_1, x) dx + \int_{t_1}^{t_2} \int_{\partial B} \rho(t, x) \mathbf{u}(t, x) \cdot \mathbf{n} dS_x dt = 0 \quad (19)$$

for any subset $B \subset \Omega$, where the flux $\mathbf{u} = \mathbf{u}(t, x)$ is the velocity field which determines the motion of the fluid. The corresponding differentiable form is

$$\partial_t \rho(t, x) + \operatorname{div}_x(\rho(t, x) \mathbf{u}(t, x)) = 0 \quad \text{in } (0, T) \times \Omega. \quad (20)$$

The natural boundary condition will be the impermeability of the boundary $\partial\Omega$, i.e.,

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0. \quad (21)$$

To introduce the concept of renormalized equation, by multiplying equation (20) by $B(\rho) + \rho B'(\rho)$, we get

$$\partial_t(\rho B(\rho)) + \operatorname{div}_x(\rho B(\rho) \mathbf{u}) + b(\rho) \operatorname{div}_x \mathbf{u} = 0, \quad (22)$$

where

$$B(\rho) = B(1) + \int_1^\rho \frac{b(z)}{z^2} dz. \quad (23)$$

We will consider the renormalized equation for any bounded continuous function b .

The corresponding weak form of the renormalized solution is

$$\int_0^T (\rho B(\rho) \partial_t \phi + \rho B(\rho) \mathbf{u} \cdot \nabla_x \phi - b(\rho) \operatorname{div}_x \mathbf{u} \phi) dx dt = - \int_{\Omega} \rho_0 B(\rho_0) \phi(0, \cdot) dx. \quad (24)$$

Equation (24) needs to be satisfied for any test function $\phi \in C_c([0, T], \Omega)$ and any bounded continuous function b . Although the equation still represents the physical law of mass conservation, choosing different b will give different requirements on the integrability of ρ and \mathbf{u} . For example, when $B = 1$, we need some integrability of $\operatorname{div}_x \mathbf{u}$. The concept of renormalized solution dates back to some earlier work on conservation laws [1, 34]. It plays an important role in our existence theory.

1.3.2 Momentum Equations

Momentum equations are from the Newton's second law. The corresponding flux is in the form of $\rho \mathbf{u} \otimes \mathbf{u} - \mathbb{T}$, the Cauchy tensor is defined by the property of the fluid. For example, the Newtonian fluid obeys the Stokes's law such that

$$\mathbb{T} = \mathbb{S} - P\mathbb{I} \quad (25)$$

where \mathbb{I} is the 3×3 identity matrix, and the stress tensor S is determined by the Newton's rheological law

$$\mathbb{S} = \mu(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I}) + \eta \operatorname{div}_x \mathbf{u} \mathbb{I}, \quad (26)$$

Here μ is called the shear viscosity coefficient, and η is called the bulk viscosity coefficient. The fluid obeying Newton's rheological law is called Newtonian fluid, while the Non-Newtonian fluid obeys the other type of rheological laws. Lots of common fluids are Newtonian. Considering

$$\mathbb{S} : \nabla_x \mathbf{u} = \frac{\mu}{2} |\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I}|^2 + \eta |\operatorname{div}_x \mathbf{u}|^2, \quad (27)$$

we will require that both μ and η are no-negative. In the case that both of them are constantly zero, the fluid will be inviscid. In this thesis, these two coefficients will

depend on the temperature. The general balance law of linear momentum is

$$\begin{aligned} & \int_0^T \int_{\Omega} \rho \mathbf{u} \cdot \partial_t \varphi + \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + P \operatorname{div}_x \varphi dx dt \\ &= \int_0^T \int_{\Omega} \mathbb{S} : \nabla_x \varphi dx dt - \int_{\Omega} (\rho \mathbf{u})_0 \cdot \varphi(0, \cdot) dx \end{aligned} \quad (28)$$

for any test function

$$\varphi \in C_c^1([0, T] \times \bar{\Omega}; \mathbb{R}^3). \quad (29)$$

The choice of the test function is from the non-slip boundary condition, say

$$\mathbf{u}|_{\Omega} = 0. \quad (30)$$

The corresponding differential form of the balance law of linear momentum is

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x P = \operatorname{div}_x \mathbb{S}. \quad (31)$$

In the literature of mathematical fluid dynamics, when μ and η are constants, the following notations are also used:

$$\begin{aligned} \lambda &= -\frac{2}{3}\mu + \eta \\ \mathbb{S} &= 2\mu D(\mathbf{u}) + \lambda \operatorname{div}_x \mathbf{u} \mathbb{I}, \end{aligned} \quad (32)$$

where

$$D(\mathbf{u}) = \frac{1}{2}(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T). \quad (33)$$

For any $\mathbf{u} \in H_0^1(\Omega, \mathbb{R}^3)$ (*i.e.* $\mathbf{u}|_{\partial\Omega} = 0$), we have

$$\|\mathbf{u}\|_{H^1} \leq C \|D(\mathbf{u})\|_{L^2}, \quad (34)$$

and

$$\|\mathbf{u}\|_{H^1} \leq C (\|\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I}\|_{L^2} + \|\operatorname{div}_x \mathbf{u}\|_{L^2}). \quad (35)$$

These two inequalities could be used to show the ellipticity of the momentum equations, when the viscosities are not degenerate.

1.3.3 Energy Equation

The equation of temperature or specific internal energy will be the balance law of the internal energy, i.e., the first law of thermodynamics:

$$\partial_t(\rho e) + \operatorname{div}_x(\rho e \mathbf{u}) + \operatorname{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - P \operatorname{div}_x \mathbf{u}. \quad (36)$$

Here \mathbf{q} is the heat flux, based on the Fourier's law,

$$\mathbf{q} = -\kappa \nabla_x \theta. \quad (37)$$

Based on the frame work of [43, 51], it's possible to obtain the equation of the total energy, which is the first law of thermodynamic for the whole region.

The specific total energy is

$$E(t) = \frac{|\rho \mathbf{u}|^2}{2\rho}(t) + \rho e(\rho, \theta)(t). \quad (38)$$

On the other hand, multiplying the momentum equations by \mathbf{u} , we have

$$\partial_t \left(\frac{1}{2} \rho |\mathbf{u}|^2 \right) + \operatorname{div}_x \left(\frac{1}{2} \rho |\mathbf{u}|^2 \mathbf{u} \right) = \operatorname{div}_x (\mathbb{T} \mathbf{u}) - \mathbb{T} : \nabla_x \mathbf{u}. \quad (39)$$

So the total energy balance equation is

$$\partial_t E + \operatorname{div}_x (E \mathbf{u}) + \operatorname{div}_x (\mathbf{q} - \mathbb{S} \mathbf{u} + P \mathbf{u}) = 0. \quad (40)$$

And the identity we want to use as the definition of weak solution is

$$\int_0^T \int_{\Omega} E(t) dx \partial_t \varphi dt = \varphi(0) \int_{\Omega} E_0 dx \quad (41)$$

for test function satisfying

$$\varphi \in C_c^1[0, T]. \quad (42)$$

1.3.4 Entropy Production

Combining the Gibbs' equation (5) and the balance of internal energy, we have the equation for specific entropy,

$$\partial_t(\rho s) + \operatorname{div}_x(\rho s \mathbf{u}) + \operatorname{div}_x\left(\frac{\mathbf{q}}{\theta}\right) = \sigma, \quad (43)$$

where σ is the entropy product rate,

$$\sigma = \frac{1}{\theta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \theta}{\theta} \right). \quad (44)$$

The second law of thermodynamics requires the entropy production rate of any admissible thermodynamic process must be non-negative, so we know that the viscosity coefficients μ, η and heat conductivity κ are non-negative.

Based on the framework of [43, 51], we can not verify the entropy equation with identity. Instead we may obtain an entropy inequality, which could still be viewed as the second law of thermodynamics, the weak form of the entropy balance inequality is

$$\begin{aligned} & \int_0^T \int_{\Omega} \rho s (\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi) + \frac{\nabla_x \theta \cdot \nabla_x \varphi}{\theta} dx dt \\ & = -\langle \sigma, \varphi \rangle - \int_{\Omega} (\rho s)_0 \varphi(0, \cdot) dx, \end{aligned} \quad (45)$$

where $\sigma \in \mathcal{M}^+[0, T) \times \bar{\Omega}$ satisfies

$$\sigma \geq \frac{1}{\theta} \left(\mathbb{S} : \nabla_x \mathbf{u} + \frac{\kappa |\nabla_x \theta|^2}{\theta} \right), \quad (46)$$

and

$$\varphi \in C_c^1([0, T) \times \bar{\Omega}). \quad (47)$$

1.4 Renormalization on the temperature equation

In [43], the temperature equation could also be renormalized, which motivates the admissible condition defined later. Defining

$$Q_h(\theta) = \int_1^{\theta} h(z) dz, \quad (48)$$

we have the weak form of the following inequality:

$$\begin{aligned}
& \int_{\Omega} \rho Q_h(\theta)(t, x) dx - 2 \int_s^t \int_{\Omega} \mu Q'_h(\theta) D(u)^2 dx d\tau \\
& - \int_s^t \int_{\Omega} \lambda Q'_h(\theta) |\operatorname{div}_x \mathbf{u}|^2 dx d\tau + \int_s^t \int_{\Omega} \kappa Q''_h(\theta) |\nabla_x \theta|^2 dx d\tau \\
& \leq -R \int_s^t \int_{\Omega} \rho \theta Q'_h(\theta) \operatorname{div}_x \mathbf{u} dx d\tau + \int_{\Omega} \rho Q_h(\theta)(s, x) dx.
\end{aligned} \tag{49}$$

As we can not verify it is an equation, we have only inequality like the entropy inequality. Inequality(49) is in the context of [43] where special constitution relation was used. For different constitution relations, the right hand side of (49) will be different.

1.5 Minimum Principle of the temperature variable

It's well-known that linear heat equation with thermo-insulated boundary condition

$$\begin{aligned}
\theta_t &= \Delta \theta, \\
\nabla_x \theta \cdot \mathbf{n}|_{\Omega} &= 0
\end{aligned} \tag{50}$$

has a minimum principle: if the initial data

$$\theta_0(x) \geq \underline{\theta} > 0, \tag{51}$$

then the solution

$$\theta(x, t) \geq \underline{\theta} > 0. \tag{52}$$

In this thesis, we consider the following initial boundary value problem.

$$\begin{aligned}
\rho_t + \operatorname{div}_x(\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) &= -\nabla_x P + \operatorname{div}_x \mathbb{S}, \\
(\rho e)_t + \operatorname{div}_x(\rho e \mathbf{u}) &= \operatorname{div}_x(\kappa \nabla_x \theta) + \mathbb{S} : \nabla_x \mathbf{u} - P \operatorname{div}_x \mathbf{u}. \\
\mathbf{u}|_{\partial \Omega} &= 0 \\
\nabla_x \theta \cdot \mathbf{n}|_{\partial \Omega} &= 0
\end{aligned} \tag{53}$$

The first part of this thesis is the minimum principle of the temperature variable in the compressible Navier-Stokes equation. There are several obstacles for obtaining this result. First, the temperature is in a system instead of a single equation. Second, in the system the pressure may do work so that the specific internal energy may decrease. For example, the Euler equation have no minimum principle or maximal principle on the temperature variable. Third, we have little information about the density, which appears on the left hand side of the equation with the temperature variable. In a more complicate model, dark radiation will affect the temperature too.

1.5.1 Positive lower bounds

As a weaker version of the minimum principle, Mellet and Vassuer [100] obtained a positive lower bound of the temperature for the so-called admissible solution of (53).

The admissible solution is defined as the triples $(\rho, \mathbf{u}, \theta)$, such that

$$\begin{aligned} & \int_{\Omega} \rho \phi(\theta)(t, x) dx - \int_s^t \int_{\Omega} 2\mu \phi'(\theta) |D(\mathbf{u})|^2 dx d\tau - \int_s^t \int_{\Omega} \lambda \phi'(\theta) |\operatorname{div}_x \mathbf{u}|^2 \\ & + \int_s^t \int_{\Omega} \kappa \phi''(\theta) |\nabla_x \theta|^2 dx d\tau \\ & \leq -R \int_s^t \int_{\Omega} \rho \theta \phi'(\theta) \operatorname{div}_x \mathbf{u} dx d\tau + \int_{\Omega} \rho \phi(\theta)(s, x) dx \end{aligned} \quad (54)$$

for functions ϕ in the form of

$$\phi(\theta) = \left[\log\left(\frac{C}{\theta + \epsilon}\right) \right]_+, \quad (55)$$

where C and ϵ are two positive numbers. The left hand side of equation (54) will be defined as a functional, which is determined by the region where temperature is very low, while the region where temperature is high does not affect this functional. So the analysis does not require the higher integrability of the temperature, which is not available in the framework of weak solution in high dimensions. The admissible solution is closely related to the concept of renormalized solution of the temperature equation (48). The proof of the lower bound is inspired by De Giorgi's proof of Hölder regularity for the solutions of elliptic equation with discontinuous coefficients.

From the point of view of thermodynamics, the lower bound of the absolutely temperature coincides with the third law of thermodynamics:

It is impossible for any process, no matter how idealized, to reduce the entropy of a system to its absolute-zero value in a finite number of operations.

We remark that for defining the admissible condition, the balance of momentum equation is not used, we only need to use the continuity equation and the second law of thermodynamics. The definition of admissible solution here is mainly for the constitution relation of ideal gas

$$P = R\rho\theta. \tag{56}$$

For other constitution relations, one can modify the definition accordingly [3]. In [100], the viscosities and heat conductivity need to follow

$$\begin{aligned} 2\mu + 3\lambda &\geq \nu(\theta) \geq 0, \\ \nu(\theta) &\geq C\theta, \\ \kappa(\theta) &\geq \underline{\kappa} > 0. \end{aligned} \tag{57}$$

We can see that the case when viscosities and heat conductivity are power functions of θ is not covered in [100], which will be covered later in this thesis.

1.5.2 Positive lower bounds in one dimensional system

In the classical setting when μ and κ are constants, the existence of strong solutions to one dimensional Navier-Stokes equations has been successfully studied by many mathematicians, for both local and global theories were established, see [104],[82] and [112], for local theory, and [87] for global theory. Such results have been further generalized to nonlinear thermoviscoelasticity by [23], and [24], and to viscous heat-conductive “real gases” by [84], [103], [76], and [106]. In each case, μ is independent of θ , and heat conductivity is allowed to depend on temperature in a special way

with a positive lower bound, and balanced with corresponding constitution relations. So the lower bound of the temperature may not be necessary to obtain existence results. We refer the readers to [1], [73], and [74] for more references and some recent discussions. For the one dimensional problem, there are numerous results on the existence results. As a fact in most of these existence result, the lower bound of the temperature could be obtained with the estimates obtained. For example, the lower bound of the temperature variable is obtained in [107].

1.5.3 Minimum Principle of the temperature variable

The main result in this thesis is the following minimum principle of the temperature variable:

Theorem 1. *If $(\rho, \mathbf{u}, \theta)$ is an admissible solution of (53), i.e. for any functions ϕ in the form of*

$$\phi(\theta) = [\log(\frac{C}{\theta})]_+, \quad (58)$$

where C is a positive number,

$$\begin{aligned} & \int_{\Omega} \rho \phi(\theta)(t, x) dx - \int_0^t \int_{\Omega} 2\mu \phi'(\theta) |D(\mathbf{u})|^2 dx d\tau - \int_0^t \int_{\Omega} \lambda \phi'(\theta) |\operatorname{div}_x \mathbf{u}|^2 d\tau \\ & + \int_0^t \int_{\Omega} \kappa \phi''(\theta) |\nabla_x \theta|^2 dx d\tau \\ & \leq -R \int_0^t \int_{\Omega} \rho \theta \phi'(\theta) \operatorname{div}_x \mathbf{u} dx d\tau + \int_{\Omega} \rho \phi(\theta)(0, x) dx, \end{aligned} \quad (59)$$

and the coefficients satisfy

$$\begin{aligned} 2\mu + 3\lambda & \geq \underline{\eta} \geq 0, \\ \kappa(\theta) & \geq \underline{\kappa} > 0, \end{aligned} \quad (60)$$

where $\underline{\eta}$ is a positive constant. In addition, if

$$\begin{aligned} \rho & \in L^\infty L^{\bar{\gamma}} \text{ for some } \bar{\gamma} > 3, \\ \mathbf{u} & \in L^2 H^1, \end{aligned} \quad (61)$$

and

$$\theta_0 \geq \underline{\theta} > 0. \quad (62)$$

Then

$$\theta(x, t) \geq \underline{\theta} \text{ a.e. in } [0, T) \times \Omega. \quad (63)$$

This theorem will be proved in Chapter 2.

1.5.4 Minimum Principle of the temperature variable with degenerate coefficients

When we consider the first level approximation in kinetic theory, viscosity and heat conductivity are functions of temperature. Based on Chapman-Enskog theory, they are in the following form ,(see [11]).

$$\mu = \bar{\mu}\theta^b, \quad (64a)$$

$$\eta = \bar{\eta}\theta^b, \quad (64b)$$

$$\kappa = \bar{\kappa}\theta^b \quad b \in \left[\frac{1}{2}, \infty\right). \quad (64c)$$

Without loss of generality, we assume $\bar{\mu} = 1$, $\bar{\eta} = 1$ and $\bar{\kappa} = 1$ for simplicity. Based on [11], if the intermolecular potential varies as r^{-a} , where r is intermolecular distance, then in (64), $b = \frac{a+4}{2a}$. In particular, for Maxwellian molecules ($a = 4$), $b = 1$, while for elastic spheres ($a = \infty$) $b = \frac{1}{2}$. We also use the following constitution relation,

$$P(\theta, \rho) = R\rho\theta. \quad (65)$$

The definition of admissible solution also inherits from the one in Mellet and Vassuer [100] (equation (54)), and we need to use some power functions in the definition.

Theorem 2. *Assume $(\rho, \mathbf{u}, \theta)$ is an admissible solution of (53), i.e. for any functions ϕ in the form of*

$$\phi(\theta) = \left[\frac{1}{(\theta)^m} - C^K \right]_+, \quad (66)$$

where C is a positive number, $m > b$ is a constant, and $K = \frac{2m}{m-b}$, we have

$$\begin{aligned} & \int_{\Omega} \rho \phi(\theta)(t, x) dx - \int_0^t \int_{\Omega} \mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} dx d\tau - \int_0^t \int_{\Omega} \eta \phi'(\theta) |\operatorname{div}_x \mathbf{u}|^2 \\ & + \int_0^t \int_{\Omega} \kappa \phi''(\theta) |\nabla_x \theta|^2 dx d\tau \\ & \leq -R \int_0^t \int_{\Omega} \rho \theta \phi'(\theta) \operatorname{div}_x \mathbf{u} dx d\tau + \int_{\Omega} \rho \phi(\theta)(0, x) dx, \end{aligned} \quad (67)$$

and the coefficients satisfy

$$\mu = \bar{\mu} \theta^b, \quad (68a)$$

$$\eta = \bar{\eta} \theta^b, \quad (68b)$$

$$\kappa = \bar{\kappa} \theta^b \quad b \in (0, \infty). \quad (68c)$$

And additionally,

$$\rho \in L^\infty L^{\bar{\gamma}} \text{ for some } \bar{\gamma} > 3, \quad (69)$$

$$\mathbf{u} \in L^2 H^1,$$

and

$$\theta_0 \geq \underline{\theta} > 0. \quad (70)$$

Then

$$\theta(x, t) \geq \underline{\theta} \text{ a.e. in } [0, T) \times \Omega. \quad (71)$$

From this result, we can see that the momentum equations and the temperature equation have the uniform elliptic structures on their right-hand sides. This theorem will also be proved in Chapter 2.

1.6 Existence Problem

With the help of the minimum principle, we can consider the model of the degenerate viscosities and heat conductivity with constitution relation similar to [51] Here we assume that x is the Eulerian coordinates in a bounded domain $\Omega \in C^{2,v}$, where $v \in (0, 1)$ and $t \in (0, T)$, here T could be any finite number and is not necessarily small.

1.6.1 Constitution relations

In the Navier-Stokes System, the thermodynamic functions p, e, s are determined in terms of the independent variables ρ, u and θ . Here ρ, θ characterize the state of the fluid completely, while u describes the motion of the fluid. Like [43, 51], we consider the part of the pressure and radiation effect of the gas, we assume that the constitution relation is in the following form:

$$\text{specific internal energy } e = \theta + \frac{\theta^4}{\rho};$$

$$\text{pressure } P = \rho\theta + \frac{\theta^4}{3} + \rho^\gamma;$$

$$\text{entropy } s = \log \theta - \log \rho + \frac{4\theta^3}{3\rho}.$$

For technical reason, we also assume that the constant

$$\gamma > 3. \tag{72}$$

Particularly, in the pressure equation, $\rho\theta$ is the pressure law for the ideal gas, and the ρ^γ is the isentropic part, and θ^4 is from the radiation effects. One should notice that the term ρ^γ does not show in the temperature equation.

We will deal with the following system (73).

$$\begin{aligned} \rho_t + \operatorname{div}_x(\rho \mathbf{u}) &= 0, \\ (\rho \mathbf{u})_t + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) &= -\nabla_x P + \operatorname{div}_x \mathbb{S}, \\ (\rho e)_t + \operatorname{div}_x(\rho e \mathbf{u}) &= \operatorname{div}_x(\kappa \nabla_x \theta) + \mathbb{S} : \nabla_x \mathbf{u} - \left(\rho\theta + \frac{\theta^4}{3}\right) \operatorname{div}_x \mathbf{u}. \end{aligned} \tag{73}$$

We consider the initial-boundary problem in Ω , with the initial condition:

$$\rho(0, x) = \rho_0(x), \quad \mathbf{u}(0, x) = \mathbf{u}_0(x), \quad \theta(0, x) = \theta_0(x), \tag{74}$$

and the boundary condition

$$\begin{aligned}\mathbf{u}|_{\partial\Omega} &= 0, \\ \nabla_x\theta \cdot n|_{\partial\Omega} &= 0.\end{aligned}\tag{75}$$

1.6.2 Temperature dependent coefficients

In the Navier-Stokes System, the viscosities μ, η and heat conductivity κ are functions of ρ and θ . As discussed in the previous section, they are in the form of:

$$\mu = \bar{\mu}\theta^b, \tag{76a}$$

$$\eta = \bar{\eta}\theta^b, \tag{76b}$$

$$\kappa = \bar{\kappa}\theta^b \quad b \in \left[\frac{1}{2}, \infty\right). \tag{76c}$$

1.6.3 Feireisl's weak solution

The following concepts of weak solution was introduced in [43, 51], the continuity equation are renormalized and the temperature equation is replaced by entropy inequality.

1.6.3.1 Continuity equation

$$\begin{aligned}& \int_0^T \int_{\Omega} \rho B(\rho) (\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi) dx dt \\ &= \int_0^T \int_{\Omega} b(\rho) \operatorname{div}_x \mathbf{u} \varphi dx dt - \int_{\Omega} \rho_0 B(\rho_0) \varphi(0, \cdot) dx,\end{aligned}\tag{77}$$

where

$$\begin{aligned}b &\in L^\infty \cap C[0, \infty), \\ B(\rho) &= B(1) + \int_1^\rho \frac{b(z)}{z^2} dz, \\ \varphi &\in C_c^1([0, T) \times \bar{\Omega}).\end{aligned}\tag{78}$$

1.6.3.2 *Balance of linear momentum*

$$\begin{aligned} & \int_0^T \int_{\Omega} \rho \mathbf{u} \cdot \partial_t \varphi + \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + P \operatorname{div}_x \varphi dx dt \\ &= \int_0^T \int_{\Omega} \mathbb{S} : \nabla_x \varphi dx dt - \int_{\Omega} (\rho \mathbf{u})_0 \cdot \varphi(0, \cdot) dx, \end{aligned} \quad (79)$$

where

$$\varphi \in C_c^1([0, T] \times \overline{\Omega}; \mathbb{R}^3) \quad (80)$$

with

$$\varphi|_{\partial\Omega} = 0. \quad (81)$$

1.6.3.3 *Conservation of total energy*

$$\int_0^T \partial_t \varphi \int_{\Omega} E(t) dx dt = \varphi(0) \int_{\Omega} E_0 dx. \quad (82)$$

The specific total energy is

$$E(t) = \frac{|\rho \mathbf{u}|^2}{2\rho}(t) + \frac{\rho^\gamma}{\gamma - 1}(t) + \rho\theta(t). \quad (83)$$

Equation (82) is valid for any test function

$$\varphi \in C_c^1[0, T]. \quad (84)$$

1.6.3.4 *Entropy production*

$$\begin{aligned} & \int_0^T \int_{\Omega} \rho s (\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi) + \frac{\nabla_x \theta \cdot \nabla_x \varphi}{\theta} dx dt \\ &= -\langle \sigma, \varphi \rangle - \int_{\Omega} (\rho s)_0 \varphi(0, \cdot) dx dt, \end{aligned} \quad (85)$$

where $\sigma \in \mathcal{M}^+([0, T] \times \overline{\Omega})$ satisfies

$$\sigma \geq \frac{1}{\theta} (\mathbb{S} : \nabla_x \mathbf{u} + \frac{\kappa |\nabla_x \theta|^2}{\theta}), \quad (86)$$

and

$$\varphi \in C_c^1([0, T] \times \overline{\Omega}). \quad (87)$$

1.6.4 Previous work on existence

There are huge literatures on the existence results of the compressible Navier-Stokes equations. Chapter 3 is mainly based on the following results.

1.6.4.1 Isentropic Navier-Stokes system

If the pressure of the fluid is only a function of density, we have a new system, which is called isentropic Navier-Stokes system:

$$\begin{aligned}
 \rho_t + \operatorname{div}_x(\rho \mathbf{u}) &= 0, \\
 (\rho \mathbf{u})_t + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) &= \nabla_x P(\rho) + \operatorname{div}_x \mathbb{S}. \\
 \mathbf{u}|_{\partial\Omega} &= 0 \\
 \nabla_x \theta \cdot \mathbf{n}|_{\partial\Omega} &= 0
 \end{aligned} \tag{88}$$

Lions [95, 96] obtained the weak solution of the isentropic system, while the concept of renormalized solution of the continuity equation and some delicate estimates of commutators involving Riesz transforms and variable coefficient operators were used. In the context of bounded domain, let $\Omega \in \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0, 1)$. Assume that the constitution relation is defined in section 1.6.1, and $P(\rho) = \rho^\gamma$ where $\gamma > 3$. Then for any $T > 0$, the isentropic system (88) has a weak solution.

Feireisl, etc [54] improved the requirement of $\gamma > 3$ in the above result to the case

$$\gamma > \frac{3}{2}. \tag{89}$$

1.6.4.2 Non-isentropic Navier-Stokes system

Later Feireisl, etc[43] considered the case of the non-isentropic system, with energy equality are replaced by an entropy inequality. Some earlier result, says [49], needs the viscosities to be constant. While some later result in [51], viscosities are allowed to be non-constant functions, specifically

$$C_1(1 + \theta^b) < \mu, \eta, \kappa < C_2(1 + \theta^b), \quad (90)$$

where C_1 and C_2 are two positive constants. The dependence of temperature in equation (64) implies that the operator $\operatorname{div}_x(\kappa(\theta)\nabla_x\theta)$ has the possible degeneracy when the temperature is close to 0.

1.6.5 Initial Data

We assume that the initial data satisfy the following conditions

1. The initial density ρ_0 is a non-negative measurable function such that

$$\int_{\Omega} \rho_0 dx = M_0 > 0, \quad (91)$$

$$\|\rho_0\|_{L^\gamma} \leq C. \quad (92)$$

2. The initial momentum satisfies a compatibility condition

$$(\rho\mathbf{u})_0 = 0 \text{ a.e. on the set } \{x \in \Omega \mid \rho_0(x) = 0\}. \quad (93)$$

3. The initial temperature is determined by a measurable function θ_0 satisfying

$$\theta_0 \geq \underline{\theta} > 0 \text{ a.e. in } \Omega, \quad (94)$$

$$(\rho s)_0 = \rho_0 s(\rho_0, \theta_0) \in L^1. \quad (95)$$

4. The initial energy is finite

$$E_0 = \int_{\Omega} \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0} + \frac{\rho_0^\gamma}{\gamma - 1} + \rho_0\theta_0 + \theta_0^4 dx < \infty. \quad (96)$$

1.6.6 Existence

Based on the positive lower bound obtained by the minimum principle, we have the following existence theorem.

Theorem 3. *Let $\Omega \in \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0, 1)$. Assume that*

1. The constitution relation is defined in section 1.6.1, and $\gamma > 3$;

2. The diffusion coefficients is defined in section 1.6.2 and

$$\begin{aligned}\mu &= \theta^b, \\ \eta &= \theta^b, \\ \kappa &= \theta^b, \text{ for } b \in [0, 3).\end{aligned}\tag{97}$$

3. Initial Data is defined in section 1.6.5.

For any $T > 0$, the Navier-Stokes-Fourier system has a Feireisl's weak solution defined in section 1.6.3. In addition

$$\theta(t, x) \geq \underline{\theta} > 0,\tag{98}$$

where $\underline{\theta}$ is the lower bound of the initial temperature.

This theorem will be proved in Chapter III.

CHAPTER II

MINIMUM PRINCIPLE OF THE TEMPERATURE VARIABLE

In this chapter, we will prove the minimum principle of the temperature for the admissible solution of the Navier-Stokes equations.

$$\begin{aligned}
 \rho_t + \operatorname{div}_x(\rho \mathbf{u}) &= 0, \\
 (\rho \mathbf{u})_t + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) &= -\nabla_x P + \operatorname{div}_x \mathbb{S}, \\
 (\rho e)_t + \operatorname{div}_x(\rho e \mathbf{u}) &= \operatorname{div}_x(\kappa \nabla_x \theta) + \mathbb{S} : \nabla_x \mathbf{u} - P \operatorname{div}_x \mathbf{u}. \tag{99} \\
 \mathbf{u}|_{\partial\Omega} &= 0 \\
 \nabla_x \theta \cdot \mathbf{n}|_{\partial\Omega} &= 0
 \end{aligned}$$

The key technique is still the iteration methods from De Giorgi, but we got the minimum principle locally in time and then push the bound to any time by continuity argument. We first specify some recursion relations, and the conditions for the sequence of functionals $U_k(t)$ to converge to 0. Minimum principle for various cases are proved in details in the rest of this chapter.

2.0.7 Recursion relations

Lemma 1. *If U_k is a nonnegative sequence such that*

$$U_{k+1} \leq D_1 z^{\beta_1 k} U_k^{\beta_2}, k \in \mathbb{N} \tag{100}$$

where D_1 is a positive constant independent of β_1, β_2, k, z , $\beta_2 > 1$, $\beta_1 > 0$ and $z > 1$ is a constant. If U_0 is small enough, such that

$$\log U_0 < -D_2 \leq -\max_{k \geq 0} \frac{\log D_1 + \beta_1 k \log z}{\frac{\beta_2 - 1}{2} \left(\frac{\beta_2 + 1}{2}\right)^k}. \tag{101}$$

Then we have

$$\lim_{k \rightarrow \infty} U_k = 0. \quad (102)$$

Proof. Let's define

$$x_k = \log U_k. \quad (103)$$

Then we have the recursion relation

$$x_{k+1} \leq \log D_1 + \beta_1 k \log z + \beta_2 x_k. \quad (104)$$

We want to choose x_k going to negative infinity with geometric speed, so we want to show

$$x_k \leq x_0 \left(\frac{\beta_2 + 1}{2} \right)^k. \quad (105)$$

We will prove equation (105) by induction. For $k = 0$, the equation (105) is obvious.

Now we assume

$$x_l \leq x_0 \left(\frac{\beta_2 + 1}{2} \right)^l. \quad (106)$$

Notice the condition we specified in equation (101),

$$x_0 < -D_2 \leq -\max_{k \geq 0} \frac{\log D_1 + \beta_1 k \log z}{\frac{\beta_2 - 1}{2} \left(\frac{\beta_2 + 1}{2} \right)^k}, \quad (107)$$

where constant D_2 depends only D_1 , β_1 , z and β_2 . Then

$$\begin{aligned} x_{l+1} &\leq x_0 \left(\frac{\beta_2 + 1}{2} \right)^l \beta_2 + \log D_1 + \beta_1 l \log z \\ &\leq x_0 \left(\frac{\beta_2 + 1}{2} \right)^{l+1} + \frac{\beta_2 - 1}{2} x_0 \left(\frac{\beta_2 + 1}{2} \right)^l + \log D_1 + \beta_1 l \log z \\ &\leq x_0 \left(\frac{\beta_2 + 1}{2} \right)^{l+1}. \end{aligned} \quad (108)$$

So equation (105) is true for any positive integer k . Letting $k \rightarrow \infty$ we finish the proof of lemma 1. One can notice that the constant $D_2(D_1, \beta_1, \beta_2, z)$ in the requirement of U_0 could be chosen continuously with respect to z . \square

Then we have the following version.

Lemma 2. *If U_k is a nonnegative sequence such that*

$$U_{k+1} \leq D_1 z^{\beta_1 k} U_k^{\beta_2} M^{\beta_3}, k \in \mathbb{N} \quad (109)$$

where D_1 is a positive constant independent of constants $\beta_1 > 0, \beta_2 > 1, \beta_3 > 0, k, M, z$. $z > 1, M > 1$. Then there exists a constant D_3 which is independent of M , and if

$$U_0 \leq D_3 \left(\frac{1}{M} \right)^{\frac{\beta_3}{\beta_2 - 1}}, \quad (110)$$

we have

$$\lim_{k \rightarrow \infty} U_k = 0. \quad (111)$$

Proof. We define

$$y_k = M^{\frac{\beta_3}{\beta_2 - 1}} U_k, \quad (112)$$

then we have

$$y_{k+1} \leq D_1 z^{\beta_1 k} y_k^{\beta_2}. \quad (113)$$

Thus, this lemma follows from Lemma 1 □

The following lemmas are direct consequence of Lemma 2:

Lemma 3. *If U_0 is a small enough positive number, and $\gamma_1, \gamma_3 \geq 0, \gamma_2, \gamma_4 > 1$, and $z > 1$ is a constant. $C_{21} C_{22}$ are two constants independent of M*

$$U_k \leq C_{21} \left(\frac{z^k}{M} \right)^{\gamma_1} U_{k-1}^{\gamma_2} + C_{22} \left(\frac{1}{M} \right)^{\gamma_3} U_{k-1}^{\gamma_4}, k \in \mathbb{N} \quad (114)$$

then $\lim_{k \rightarrow \infty} U_k = 0$.

Lemma 4. *If U_0 is a small enough positive number, $M > 0$ is a fixed constant, and $\gamma_1, \gamma_3, \gamma_5 \geq 0, \gamma_2, \gamma_4 > 1$, and $z > 1$ is a constant. $C_{21} C_{22}$ are two constant independent of M*

$$U_k \leq C_{21} \left(\frac{z^k}{M} \right)^{\gamma_1} U_{k-1}^{\gamma_2} M^{\gamma_5} + C_{22} \left(\frac{1}{M} \right)^{\gamma_3} U_{k-1}^{\gamma_4} M^{\gamma_5}, k \in \mathbb{N} \quad (115)$$

then $\lim_{k \rightarrow \infty} U_k = 0$.

2.0.8 Embedding Inequalities

We use $L^p L^q$ to denote $L^p(0, T; L^q(\Omega))$ and denote $Q = [0, T] \times \Omega$. Similar to [100], we will use the following inequalities.

Lemma 5. *If $0 < \int_{\Omega} \rho dx < C$ and $\|\rho\|_{L^\infty L^r} \leq C$ where $r > 1$, then on Q .*

$$\|\psi\|_{L^2 L^6} \leq C(\|\rho\psi\|_{L^\infty L^1} + \|\nabla_x \psi\|_{L^2 L^2}), \quad (116)$$

$$\|\psi\|_{L^2 L^2} \leq C(\|\rho\psi\|_{L^\infty L^1} + \|\nabla_x \psi\|_{L^2 L^2}). \quad (117)$$

Lemma 5 is a special case of the following inequality.

Lemma 6. *There exists a constant C (depending only on Ω , T , $\int_{\Omega} \rho_0 dx$ and $\|\rho\|_{L^\infty L^r}$) such that for every $F \geq 0$, $\rho F \in L^\infty([0, T], L^1(\Omega))$, and $\nabla_x F \in L^2([0, T], L^2(\Omega))$, then*

$$\|F\|_{L^2([0, T], L^6(\Omega))} \leq C \left(\|\rho F\|_{L^\infty([0, T], L^1(\Omega))} + \|\nabla_x F\|_{L^2([0, T], L^2(\Omega))} \right). \quad (118)$$

Proof. There exists two constants ϵ, D_4 independent of t such that

$$|\Omega(t)| = |\{x | \rho(t, x) > \epsilon\}| \geq D_4. \quad (119)$$

Indeed we have

$$\int_{\Omega} \rho_0 dx = \int_{\Omega} \rho dx \leq \epsilon |\Omega| + \|\rho\|_{L^\infty L^r} |\Omega(t)|^{\frac{r-1}{r}}. \quad (120)$$

For ϵ small enough, we can define

$$D_4 = \left(\frac{\int_{\Omega} \rho_0 dx - \epsilon |\Omega|}{\|\rho\|_{L^\infty L^r}} \right)^{\frac{r}{r-1}} > 0. \quad (121)$$

Then for a fixed time t ,

$$\begin{aligned}
\|F(t, x)\|_{L^6(\Omega)} &\leq \left\| F - \frac{1}{|\Omega|} \int_{\Omega} F(t, y) dy \right\|_{L^6(\Omega)} + |\Omega|^{-\frac{5}{6}} \int_{\Omega(t)} F(t, y) dy \\
&\leq \|\nabla_x F\|_{L^2(\Omega)} + |\Omega|^{-\frac{5}{6}} \frac{\Omega}{\Omega(t)} \int_{\Omega(t)} F(t, x) dx \\
&\quad + |\Omega(t)|^{-\frac{5}{6}} \left| \frac{\Omega}{\Omega(t)} \int_{\Omega(t)} F(t, x) dx - \int_{\Omega} F(t, y) dy \right|.
\end{aligned} \tag{122}$$

$$|\Omega|^{-\frac{5}{6}} \frac{\Omega}{\Omega(t)} \int_{\Omega(t)} F(t, x) dx \leq \frac{|\Omega|^{\frac{1}{6}}}{\epsilon C_1} \int_{\Omega} \rho F(t, x) dx. \tag{123}$$

For the last term, we have

$$\begin{aligned}
&\left| \frac{\Omega}{\Omega(t)} \int_{\Omega(t)} F(t, x) dx - \int_{\Omega} F(t, y) dy \right| \\
&\leq \frac{1}{C_1} \int_{\Omega(t) \times \Omega} |F(t, x) - F(t, y)| dx dy \\
&\leq \frac{1}{C_1} \int_{\Omega \times \Omega} |F(t, x) - F(t, y)| dx dy \\
&\leq C \|\nabla_x F\|_{L^2(\Omega)}.
\end{aligned} \tag{124}$$

Then (118) follows. □

2.1 Proof of the Minimum Principles

In this section, we will prove theorem 1. We first define the functional $U_k(t)$ from the admissible condition is defined. Using some interpolation estimates, we give some useful to control on the functional $U_k(t)$. In the end, the minimum principle is obtained by a continuity argument.

2.1.1 Definition of $U_k(t)$

Recall that in this section, we have the constitution relation

$$P = R\rho\theta \tag{125}$$

$(\rho, \mathbf{u}, \theta)$ is an admissible solution of (53), i.e. for any functions ϕ in the form of

$$\phi(\theta) = [\log(\frac{C}{\theta})]_+, \tag{126}$$

where C is a positive number,

$$\begin{aligned}
& \int_{\Omega} \rho\phi(\theta)(t, x)dx - \int_0^t \int_{\Omega} 2\mu\phi'(\theta)|D(\mathbf{u})|^2 dx d\tau - \int_0^t \int_{\Omega} \lambda\phi'(\theta)|\operatorname{div}_x \mathbf{u}|^2 dx d\tau \\
& + \int_0^t \int_{\Omega} \kappa\phi''(\theta)|\nabla_x \theta|^2 dx d\tau \\
& \leq -R \int_0^t \int_{\Omega} \rho\theta\phi'(\theta)\operatorname{div}_x \mathbf{u} dx d\tau + \int_{\Omega} \rho\phi(\theta)(0, x)dx,
\end{aligned} \tag{127}$$

and the coefficients satisfy

$$\begin{aligned}
2\mu + 3\lambda & \geq \underline{\eta} \geq 0, \\
\kappa(\theta) & \geq \underline{\kappa} > 0,
\end{aligned} \tag{128}$$

where $\underline{\eta}$ is a positive constant. In addition, if

$$\begin{aligned}
\rho & \in L^\infty L^{\bar{\gamma}} \text{ for some } \bar{\gamma} > 3, \\
\mathbf{u} & \in L^2 H^1,
\end{aligned} \tag{129}$$

and

$$\theta_0 \geq \underline{\theta} > 0. \tag{130}$$

In this section, T is a fixed time and we will use the following definition of U_k :

$$\begin{aligned}
& U_k(t) \\
& = \sup_{0 < \tau < t} \int_{\Omega} \rho\phi_k(\theta)(\tau, x)dx - \int_0^t \int_{\Omega} 2\mu\phi'_k(\theta)|D(\mathbf{u})|^2 dx d\tau - \int_0^t \int_{\Omega} \lambda\phi'_k(\theta)|\operatorname{div}_x \mathbf{u}|^2 dx d\tau \\
& + \int_0^t \int_{\Omega} \kappa\phi''_k(\theta)|\nabla_x \theta|^2 dx d\tau.
\end{aligned} \tag{131}$$

By the assumptions on the coefficients in equation (60)

$$\begin{aligned}
U_k(t) & \geq \int_{\Omega} \rho\phi_k(\theta)(t, x)dx + \int_0^t \int_{\Omega} \frac{\eta 1_{\{\theta < C_k\}}}{\theta} |\operatorname{div}_x \mathbf{u}|^2 dx d\tau \\
& + \int_0^t \int_{\Omega} \frac{\kappa(\theta) 1_{\{\theta < C_k\}}}{\theta^2} |\nabla_x \theta|^2 dx d\tau.
\end{aligned} \tag{132}$$

By the internal energy equation and the conditions of μ and κ , we have, if $C_k \leq \underline{\theta}$,

$$\begin{aligned}
U_k(t) &\leq \left| \int_0^t \int_{\Omega} R\rho\theta 1_{\{\theta < C_k\}} \operatorname{div}_x \mathbf{u} dx d\tau \right| \\
&\leq \frac{\eta}{4} \int_0^t \int_{\Omega} \frac{1_{\{\theta < C_k\}}}{\theta} |D(\mathbf{u})|^2 dx d\tau \\
&\quad + D_5 \int_0^t \int_{\Omega} \rho^2 \theta^3 1_{\{\theta < C_k\}} dx d\tau,
\end{aligned} \tag{133}$$

$$\begin{aligned}
U_k(t) &\leq 4D_5 \int_0^t \int_{\Omega} \rho^2 \theta^3 1_{\{\theta < C_k\}} dx d\tau \\
&\leq 4D_5 \|\rho\|_{L^\infty L^3}^2 \int_0^t \left(\int_{\Omega} 1_{\{\theta < C_k\}} dx \right)^{\frac{1}{3}} d\tau \\
&\leq D_6(T) \int_0^t \left(\int_{\Omega} 1_{\{\theta < C_k\}} dx \right)^{\frac{1}{3}} d\tau.
\end{aligned} \tag{134}$$

The $D_6(T)$ is a fixed constant which will be used in the continuity argument.

2.1.2 Interpolation Estimates

Lemma 7. *When $t \in [0, T)$, there exists constant $D_7(T, z)$, and $\gamma_1 > 1$, $\gamma_2 > 1$, for any $z > 1$, we have*

$$U_{k+1}(t) \leq D_7(T, z) \frac{1}{\left(\frac{M}{z^k}\right)^{\gamma_1}} U_k(t)^{\gamma_2}. \tag{135}$$

Proof. Let's assume that $\underline{\theta} < 1$, a scaling argument could deal with the other cases.

Define the sequence of C_k by

$$\log C_0 = -M = \log \underline{\theta}, \tag{136}$$

$$\log C_{k+1} = \log C_k - \frac{M}{z^{k+1}}. \tag{137}$$

Then we have

$$\log C_\infty = -\frac{M}{1 - \frac{1}{z}} = \frac{1}{1 - \frac{1}{z}} \log \underline{\theta}. \tag{138}$$

On the other hand, we have in the region Ω_k

$$\phi_{k-1} \geq \frac{M}{z^k}. \tag{139}$$

Using Hölder inequality, we have the following estimate,

$$\begin{aligned}
& \left(\int_0^t \left(\int_{\Omega} (\rho^{\alpha_1} \phi_{k-1}^{\beta_1})^{q_1} dx \right)^{\frac{p_1}{q_1}} d\tau \right)^{\frac{1}{p_1}} \\
&= \left(\int_0^t \left(\int_{\Omega} \rho^{\alpha_1 q_1} \phi_{k-1}^{\alpha_1 q_1} \cdot \phi_{k-1}^{(\beta_1 - \alpha_1) q_1} dx \right)^{\frac{p_1}{q_1}} d\tau \right)^{\frac{1}{p_1}} \\
&\leq \|\rho \phi_{k-1}\|_{L^\infty L^1}^{\alpha_1} \|\phi_{k-1}\|_{L^2 L^6}^{(\beta_1 - \alpha_1)},
\end{aligned} \tag{140}$$

where the parameters satisfy

$$\begin{aligned}
\frac{(\beta_1 - \alpha_1) q_1}{6} + \alpha_1 q_1 &= 1, \\
\frac{(\beta_1 - \alpha_1) p_1}{2} &= 1.
\end{aligned} \tag{141}$$

By the definition of ϕ_k , and denoting

$$\Omega_k = \overline{\{(x, t) \mid \theta < C_k\}}. \tag{142}$$

we have

$$\begin{aligned}
& \int_0^t \int_{\Omega} \rho^2 1_{\Omega_k} dx d\tau \\
&\leq C \int_0^t \int_{\Omega} \rho^2 \frac{\phi_{k-1}^{\beta_1}}{\left(\frac{M}{z^k}\right)^{\beta_1}} dx d\tau \\
&\leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \left(\int_0^t \left(\int_{\Omega} (\rho^{\alpha_1} \phi_{k-1}^{\beta_1})^{q_1} dx \right)^{\frac{p_1}{q_1}} d\tau \right)^{\frac{1}{p_1}} \\
&\leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \|\rho \phi_{k-1}\|_{L^\infty L^1}^{\alpha_1} \|\phi_{k-1}\|_{L^2 L^6}^{(\beta_1 - \alpha_1)} \\
&\leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} (U_{k-1}^{\alpha_1 + (\beta_1 - \alpha_1)} + U_{k-1}^{\alpha_1 + \frac{(\beta_1 - \alpha_1)}{2}}),
\end{aligned} \tag{143}$$

where the positive indices need to satisfy

$$\begin{aligned}
\beta_1 &> \alpha_1 \\
\frac{1}{p_1} + \frac{1}{p_2} &= 1, \\
\frac{1}{q_1} + \frac{1}{q_2} &= 1, \\
q_2 &\leq \frac{\bar{\gamma}}{2 - \alpha_1}, \\
\beta_1 &> 1, \\
\alpha_1 + \frac{(\beta_1 - \alpha_1)}{2} &> 1
\end{aligned} \tag{144}$$

and the relation of p_1, q_1 and α, β is

$$\begin{aligned}\frac{(\beta_1 - \alpha_1)}{6} + \alpha_1 &= \frac{1}{q_1}, \\ \frac{(\beta_1 - \alpha_1)}{2} &= \frac{1}{p_1}.\end{aligned}\tag{145}$$

Denoting $x = \frac{(\beta_1 - \alpha_1)}{6}$, $y = \alpha_1$ and considering the following stronger requirements:

$$\begin{aligned}0 < x + \left(1 - \frac{1}{\bar{\gamma}}\right)y &\leq 1 - \frac{2}{\bar{\gamma}}, \\ 3x + y &> 1, \\ x &< \frac{1}{3}\end{aligned}\tag{146}$$

with the condition $\bar{\gamma} > 3$, we see we just need to choose $x < \frac{1}{3}$ but very close to $\frac{1}{3}$, and y a positive number small enough. With the choice of parameters, we can find $\gamma_1 > 1$, $\gamma_2 > 1$ and $D_7(T, z)$ for any $z > 1$. \square

2.1.3 Continuity Argument

In this section, we will fix T , and show

$$\lim_{k \rightarrow \infty} U_k(t) = 0,\tag{147}$$

which will imply that

$$\log \theta \geq \log C_\infty = \frac{1}{1 - \frac{1}{z}} \log \varrho.\tag{148}$$

when z is chosen properly. We will use two key estimates

$$U_k(t) \leq D_6(T) \int_0^t \left(\int_\Omega 1_{\{\theta < C_k\}} dx \right)^{\frac{1}{3}} d\tau,\tag{149}$$

$$U_{k+1}(t) \leq D_7(T, z) \frac{1}{\left(\frac{M}{z^k}\right)^{\gamma_1}} U_k(t)^{\gamma_2},\tag{150}$$

and choose $D_8(T, z)$ is a constant such that if $A_0 \leq \frac{1}{D_8(T, z)}$ and

$$A_{k+1} \leq D_7(T, z) \frac{1}{\left(\frac{M}{z^k}\right)^{\gamma_1}} A_k^{\gamma_2},\tag{151}$$

we have $\lim_{k \rightarrow \infty} A_k = 0$. As a fact, $D_8(T, z)$ could be chosen as

$$D_8(T, z) = \max_{k \geq 0} \frac{\log D_7(T, z) + \gamma_1 k \log z}{\frac{\gamma_2 - 1}{2} \left(\frac{\gamma_2 + 1}{2}\right)^k}.\tag{152}$$

Now we consider the following function

$$LG(t) = - \sup_{0 < \tau < t} \log \left\| \frac{1}{\theta(\tau)} \right\|_{L^\infty(\Omega)} \quad (153)$$

Notice that LG is monotone decreasing. Define

$$t_1 = \sup_{LG(t) > \log C_\infty} \{t\}, \quad (154)$$

$$t_2 = \inf_{LG(t) < \log C_\infty} \{t\}, \quad (155)$$

i.e. for any $\epsilon > 0$ small enough, there exists a $\delta(\epsilon) > 0$, such that

$$LG(t_2 + \epsilon) < \log C_\infty - \delta(\epsilon), \quad (156)$$

$$LG(t_1 - \epsilon) > \log C_\infty + \delta(\epsilon). \quad (157)$$

Obviously, $t_1 \leq t_2$. As a fact, we can choose z so that they are the same.

Lemma 8. *There is a $z_i \in (2, 3)$, such that*

$$t_2 = t_1 \quad (158)$$

Proof. If for any $z_i \in (2, 3)$, there exists $t_1(z_i) < t_2(z_i)$, then by the monotonicity of LG , for any $t \in (t_1, t_2)$

$$LG(t) = \frac{1}{1 - \frac{1}{z_i}} \log \underline{\theta}, \quad (159)$$

We want to show that $(t_1(z_a), t_2(z_a))$ and $(t_1(z_b), t_2(z_b))$ are two non-intersecting intervals for $z_a \neq z_b \in (2, 3)$.

Indeed, if $z_a \neq z_b$, and if $\tau \in (t_1(z_a), t_2(z_a))$ and $\tau \in (t_1(z_b), t_2(z_b))$, then

$$LG(\tau) = \frac{1}{1 - \frac{1}{z_a}} \log \underline{\theta} = \frac{1}{1 - \frac{1}{z_b}} \log \underline{\theta}, \quad (160)$$

which is impossible. So $(t_1(z_a), t_2(z_a))$ and $(t_1(z_b), t_2(z_b))$ are two non-intersecting intervals. So the total length of all these non-intersecting intervals is not larger than T ,

$$\sum_{2 < z_i < 3} (t_2(z_i) - t_1(z_i)) \leq T. \quad (161)$$

Since z_i is from the uncountable set $(2, 3)$, equation (161) is impossible. So we have the existence of z_i in Lemma 8. \square

For such $z_i \in (2, 3)$, set

$$D_9(T, z_i) = (D_6(T))|\Omega|^{\frac{1}{3}}(D_8(T, z_i)). \quad (162)$$

Now we are ready to use the continuity argument to extend the bound. Firstly, we want to show

$$t_1 > 0. \quad (163)$$

Go back to the estimate:

$$U_k(t) \leq D_6(T) \int_0^t \left(\int_{\Omega} 1_{\{\theta < C_k\}} dx \right)^{\frac{1}{3}} d\tau. \quad (164)$$

So if $t \leq \frac{1}{D_9(T, z_i)}$, we have

$$U_k \leq \frac{1}{D_8(T, z_i)}, \quad (165)$$

then $\lim_{k \rightarrow \infty} U_k = 0$, so

$$t_1 \geq \frac{1}{D_9(T, z_i)}. \quad (166)$$

Secondly, we want show that $t_1 = T$, i.e.

$$\log \theta(t, x) \geq \frac{1}{1 - \frac{1}{z}} \log \underline{\theta} = \log C_{\infty} \text{ for } t \in [0, T]. \quad (167)$$

We can define

$$s_2 = t_1 - \frac{1}{2D_9(T, z_i)}. \quad (168)$$

By the definition of t_1 , we have $LG(s_2) > \log C_{\infty}$, so when k large enough, say when $k > k_1(s_2)$, we have

$$LG(s_2) > \log C_{k_1(s_2)} \geq \log C_k, \quad (169)$$

for $k > k_1(s_2)$,

$$U_k \leq D_6(T) \int_{s_2}^t \left(\int_{\Omega} 1_{\{\theta < C_k\}} dx \right)^{\frac{1}{3}} d\tau. \quad (170)$$

As long as $t < s_2 + \frac{1}{D_8(T, z_i)}$, we have $\lim_{k \rightarrow \infty} U_k = 0$. So

$$t_2 > s_2 + \frac{1}{D_9(T, z_i)}. \quad (171)$$

This contradicts to $t_2 = t_1$.

Replacing (2, 3) in Lemma 8 to $(n, n+1)$, we can also choose an increasing sequence of z_i , such that

$$z_i \in (n, n+1) \quad (172)$$

So $\lim_{i \rightarrow \infty} z_i = \infty$. Letting $z_i \rightarrow \infty$, we get the minimum principle of the temperature.

2.2 Minimum Principle with degenerate heat conductivity

In this section, we will prove Theorem 2, we have the following condition:

$(\rho, \mathbf{u}, \theta)$ is an admissible solution of (53), i.e. for any functions ϕ in the form of

$$\phi(\theta) = \left[\frac{1}{(\theta)^m} - C^K \right]_+, \quad (173)$$

where C is a positive number, $m > b$ is a constant, and $K = \frac{2m}{m-b}$, we have

$$\begin{aligned} & \int_{\Omega} \rho \phi(\theta)(t, x) dx - \int_0^t \int_{\Omega} \mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} dx d\tau - \int_0^t \int_{\Omega} \eta \phi'(\theta) |\operatorname{div}_x \mathbf{u}|^2 \\ & + \int_0^t \int_{\Omega} \kappa \phi''(\theta) |\nabla_x \theta|^2 dx d\tau \\ & \leq -R \int_0^t \int_{\Omega} \rho \theta \phi'(\theta) \operatorname{div}_x \mathbf{u} dx d\tau + \int_{\Omega} \rho \phi(\theta)(0, x) dx, \end{aligned} \quad (174)$$

and the coefficients satisfy

$$\mu = \bar{\mu} \theta^b, \quad (175a)$$

$$\eta = \bar{\eta} \theta^b, \quad (175b)$$

$$\kappa = \bar{\kappa} \theta^b \quad b \in (0, \infty). \quad (175c)$$

And additionally,

$$\begin{aligned} \rho &\in L^\infty L^{\bar{\gamma}} \text{ for some } \bar{\gamma} > 3, \\ \mathbf{u} &\in L^2 H^1, \end{aligned} \tag{176}$$

and

$$\theta_0 \geq \underline{\theta} > 0. \tag{177}$$

2.2.1 Construction of auxiliary functions

We will define the constants in the auxiliary functions of the temperature related to cut-off functions of the negative power function. Let $m > b$ is a constant, and M is a number large enough. The size of M and the relation between m and b will be specified later. The ratio

$$K = \frac{2m}{m-b} \tag{178}$$

will be used later, one can see that $K > 1$. Without the lose of generality, we consider the case of $\underline{\theta}$ small enough, the other cases could be obtained by scaling arguments. For the iteration technique, we need to introduce a sequence C_k where

$$C_0 = M, \tag{179}$$

$$C_{k+1} = C_k + \frac{M}{z^{k+1}}. \tag{180}$$

So we have

$$\lim_{k \rightarrow \infty} C_k = \frac{1}{1 - \frac{1}{z}} M. \tag{181}$$

Then define

$$\psi_k(\theta) = \left[\frac{1}{(\theta)^{\frac{m}{K}}} - C_k \right]_+, \tag{182}$$

$$\phi_k(\theta) = \left[\frac{1}{(\theta)^m} - C_k^K \right]_+. \tag{183}$$

We can see that $\psi_k(\theta)$ and $\phi_k(\theta)$ have the same supports (closure of the region where these functions are positive), which we denote as

$$\Omega_k = \overline{\{(x, t) | \theta^m < 1/C_k^K\}}. \quad (184)$$

Here we have some relations between ψ_k and ϕ_k ,

$$\begin{aligned} \psi_k(\theta) &\leq \phi_k(\theta), \\ \psi_k(\theta)^K &\leq \phi_k(\theta). \end{aligned} \quad (185)$$

On the other hand, we have in the region Ω_k

$$\psi_{k-1} \geq \frac{M}{z^k}. \quad (186)$$

2.2.2 Definition of functional $U_k(t)$

Firstly, we will define the admissible condition. Fix T , for ϕ_k defined in (183), $(\rho, \mathbf{u}, \theta)$ is an admissible solution of (53) if

$$\begin{aligned} &\int_{\Omega} \rho \phi_k(\theta)(t, x) dx + \int_0^t \int_{\Omega} \frac{m \mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}}{\theta^{m+1}} 1_{\Omega_k} dx d\tau \\ &+ \int_0^t \int_{\Omega} \frac{m(m+1)\kappa |\nabla_x \theta|^2}{\theta^{m+2}} 1_{\Omega_k} dx d\tau \\ &\leq \int_0^t \int_{\Omega} \frac{m}{\theta^{m+1}} \rho \theta 1_{\Omega_k} \operatorname{div}_x \mathbf{u} dx d\tau + \int_{\Omega} \rho \phi_k(\theta)(0, x) dx. \end{aligned} \quad (187)$$

So the functional U_k is defined as

$$\begin{aligned} U_k(t) &= \sup_{0 \leq \tau \leq t} \int_{\Omega} \rho \phi_k(\theta)(\tau, x) dx \\ &+ \int_0^t \int_{\Omega} \frac{m \mathbb{S}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}}{\theta^{m+1}} 1_{\Omega_k} dx d\tau \\ &+ \int_0^t \int_{\Omega} \frac{m(m+1)\kappa |\nabla_x \theta|^2}{\theta^{m+2}} 1_{\Omega_k} dx d\tau. \end{aligned} \quad (188)$$

We have the following bounds from the definition of U_k :

$$\begin{aligned} \|\rho \phi_k\|_{L^\infty L^1} &\leq C U_k, \\ \|\nabla_x \psi_k\|_{L^2 L^2} &\leq C U_k^{\frac{1}{2}}. \end{aligned} \quad (189)$$

Using Cauchy-Schwartz inequality, we can obtain the bound of U_k :

$$U_k \leq C \int_0^T \int_{\Omega} \frac{\rho\theta}{\theta^{m+1}} |\operatorname{div}_x \mathbf{u}| 1_{\Omega_k} dx d\tau + \int_{\Omega} \rho_0 \phi_k(\theta_0) dx, \quad (190)$$

Choose M such that

$$\left(\frac{1}{M}\right)^{\frac{2}{m-b}} = \underline{\theta}, \quad (191)$$

where $\underline{\theta}$ is the lower bound of the initial data. By this choice, the last term in equation (190) will vanish, so we have the following estimates:

$$U_k \leq C \int_0^t \int_{\Omega} \frac{\rho\theta}{\theta^{m+1}} |\operatorname{div}_x \mathbf{u}| 1_{\Omega_k} dx d\tau. \quad (192)$$

We need to control the right hand side by some power of U_{k-1} , so that we can use the De Giorgi iteration technique to get $\lim_{k \rightarrow \infty} U_k = 0$. So the main task is the interpolations of the right hand-side terms. Using Hölder inequality, we have the following estimate,

$$\begin{aligned} & \left(\int_0^t \left(\int_{\Omega} (\rho^{\alpha_1} \psi_{k-1}^{\beta_1 K})^{q_1} dx \right)^{\frac{p_1}{q_1}} d\tau \right)^{\frac{1}{p_1}} \\ &= \left(\int_0^t \left(\int_{\Omega} \rho^{\alpha_1 q_1} \psi_{k-1}^{\alpha_1 q_1 K} \cdot \psi_{k-1}^{(\beta_1 - \alpha_1) q_1 K} dx \right)^{\frac{p_1}{q_1}} d\tau \right)^{\frac{1}{p_1}} \\ &\leq \|\rho \psi_{k-1}^K\|_{L^\infty L^1}^{\alpha_1} \|\psi_{k-1}\|_{L^2 L^6}^{(\beta_1 - \alpha_1) K}, \end{aligned} \quad (193)$$

where the parameters satisfy

$$\begin{aligned} \frac{(\beta_1 - \alpha_1) q_1 K}{6} + \alpha_1 q_1 &= 1, \\ \frac{(\beta_1 - \alpha_1) p_1 K}{2} &= 1. \end{aligned} \quad (194)$$

The $\|\psi_{k-1}\|_{L^2 L^6}$ could be controlled by Lemma 5.

2.2.3 Estimates for Iterations I

In this section, we will finish the estimates needed for Lemma 3 in Chapter 2. In the case when $b < \frac{1}{2}$ and $\gamma > 3$. The term in equation (192) could be controlled in this way:

$$\begin{aligned}
& \int_0^t \int_{\Omega} \frac{\rho\theta}{\theta^{m+1}} |\operatorname{div}_x \mathbf{u}| 1_{\Omega_k} dx d\tau \\
& \leq \delta_1 \int_0^t \int_{\Omega} \frac{\theta^b |\nabla_x \mathbf{u}|^2}{\theta^{m+1}} dx d\tau + C(\delta_1) \int_0^t \int_{\Omega} \rho^2 \theta^{-m+1-b} 1_{\Omega_k} dx d\tau
\end{aligned} \tag{195}$$

By choosing δ_1 small enough and we just need to control the second term in each equation.

Then we can choose m so that $m - 1 + b \leq 0$. Recall that in this case we have m is in between b and $1 - b$, so we have

$$\begin{aligned}
& \int_0^t \int_{\Omega} \rho^2 \theta^{-m+1-b} 1_{\Omega_k} dx d\tau \\
& \leq C \int_0^t \int_{\Omega} \rho^2 \frac{\varphi_{k-1}^{\beta_1 K}}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} dx d\tau \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \left(\int_0^t \left(\int_{\Omega} (\rho^{\alpha_1} \varphi_{k-1}^{\beta_1 K})^{q_1} dx \right)^{\frac{p_1}{q_1}} dt \right)^{\frac{1}{p_1}} \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \|\rho \varphi_{k-1}^K\|_{L^\infty L^1}^{\alpha_1} \|\varphi_{k-1}\|_{L^2 L^6}^{(\beta_1 - \alpha_1)K} \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \left(U_{k-1}^{\alpha_1 + (\beta_1 - \alpha_1)K} + U_{k-1}^{\alpha_1 + \frac{(\beta_1 - \alpha_1)K}{2}} \right)
\end{aligned} \tag{196}$$

where the positive indices need to satisfy

$$\begin{aligned}
& \beta_1 > \alpha_1 \\
& \frac{1}{p_1} + \frac{1}{p_2} = 1, \\
& \frac{1}{q_1} + \frac{1}{q_2} = 1, \\
& q_2 \leq \frac{\bar{\gamma}}{2 - \alpha_1}, \\
& \beta_1 K > 0, \\
& \alpha_1 + \frac{(\beta_1 - \alpha_1)K}{2} > 1,
\end{aligned} \tag{197}$$

the relation of p, q and α, β is

$$\begin{aligned}
& \frac{(\beta_1 - \alpha_1)K}{6} + \alpha_1 = \frac{1}{q_1}, \\
& \frac{(\beta_1 - \alpha_1)K}{2} = \frac{1}{p_1}.
\end{aligned} \tag{198}$$

If we denote $x = \frac{(\beta_1 - \alpha_1)K}{6}$, $y = \alpha_1$. We can construct the following set of inequalities (199):

$$\begin{aligned} 0 < x + \left(1 - \frac{1}{\bar{\gamma}}\right)y &\leq 1 - \frac{2}{\bar{\gamma}}, \\ 3x + y &> 1, \\ x &< \frac{1}{3}. \end{aligned} \tag{199}$$

The solution of (199) satisfies the inequalities in equation (197).

With the condition $\bar{\gamma} > 3$, we can see that inequalities (199) have solutions. In fact we just need to choose $x < \frac{1}{3}$ but very close to $\frac{1}{3}$, and y a positive number small enough.

2.2.4 Estimates for Iterations II

In this section, we will finish the estimates needed for Lemma 3, in the case when $b \geq \frac{1}{2}$ and $\bar{\gamma} > 3$. We can do similar interpolations estimates.

As the case of $b < \frac{1}{2}$, we have when M satisfies

$$\left(\frac{1}{M}\right)^{\frac{2}{m-b}} \geq \theta, \tag{200}$$

$$U_k \leq C \int_0^t \int_{\Omega} \rho^2 \theta^{-m-b+1} 1_{\Omega_k} dx d\tau, \tag{201}$$

the negative power of θ could be bounded in the following way:

$$\frac{1}{\theta^{m+b-1}} 1_{\Omega_k} \leq C \left(\varphi_{k-1}^{\frac{m+b-1}{K}} + M^{\frac{m+b-1}{K}} \right). \tag{202}$$

So we have the estimates

$$\begin{aligned} &\int_0^t \int_{\Omega} \rho^2 \theta^{-m-b+1} 1_{\Omega_k} \\ &\leq C \int_0^t \int_{\Omega} \rho^2 \frac{\varphi_{k-1}^{\beta_1 K + \frac{m-1+b}{K}}}{\left(\frac{M}{2k}\right)^{\beta_1 K}} + C \int_0^t \int_{\Omega} \rho^2 \frac{\varphi_{k-1}^{\beta_2 K}}{\left(\frac{M}{2k}\right)^{\beta_2 K} M^{-\frac{m-1+b}{K}}}. \end{aligned} \tag{203}$$

Let's define $K_1 = \left(K + \frac{2(m+b-1)}{\beta_1(m-b)}\right) = \left(K + \frac{(m+b-1)}{\beta_1 m/K}\right)$, so the first term in equation could be estimated as

$$\begin{aligned}
& \int_0^t \int_{\Omega} \rho^2 \frac{\varphi_{k-1}^{\beta_1 K + \frac{m-1+b}{K}}}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \\
&= \int_0^t \int_{\Omega} \rho^2 \frac{\varphi_{k-1}^{\beta_1 K_1}}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} dx d\tau \\
&\leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \left(\int_0^t \left(\int_{\Omega} (\rho^{\alpha_1} \varphi_{k-1}^{\beta_1 K_1})^{q_1} dx \right)^{\frac{p_1}{q_1}} dt \right)^{\frac{1}{p_1}} \\
&\leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \|\rho \varphi_1^K\|_{L^\infty L^1}^{\alpha_1} \|\varphi_{k-1}\|_{L^2 L^6}^{(\beta_1 - \alpha_1) K_1} \\
&\leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \left(U_{k-1}^{\alpha_1 + (\beta_1 - \alpha_1) K_1} + U_{k-1}^{\alpha_1 + \frac{(\beta_1 - \alpha_1) K_1}{2}} \right),
\end{aligned} \tag{204}$$

where the positive index need to satisfy

$$\begin{aligned}
\frac{(\beta_1 - \alpha_1) K_1}{6} + \alpha_1 &= \frac{1}{q_1}, \\
\frac{(\beta_1 - \alpha_1) K_1}{2} &= \frac{1}{p_1}, \\
\frac{1}{p_1} + \frac{1}{p_2} &= 1, \\
\frac{1}{q_1} + \frac{1}{q_2} &= 1, \\
q_4 &\leq \frac{\bar{\gamma}}{2 - \alpha_1}, \\
\alpha_1 + \frac{(\beta_1 - \alpha_1) K_1}{2} &> 1.
\end{aligned} \tag{205}$$

While we can choose the parameters similar as the case of $b < \frac{1}{2}$. If $x = \frac{(\beta_1 - \alpha_1) K_1}{6}$, $y = \alpha_1$, all the requirement reduce to

$$\begin{aligned}
0 < x + \left(1 - \frac{1}{\bar{\gamma}}\right) y &\leq 1 - \frac{2}{\bar{\gamma}}, \\
3x + y &> 1, \\
x &< \frac{1}{3}.
\end{aligned} \tag{206}$$

Similar as the case of $b < \frac{1}{2}$, we need to choose $x < \frac{1}{3}$ but x is very close to $\frac{1}{3}$, and y a positive number small enough. Here we need more conditions on m and b later as

we need to keep use the same m, K and b for the second term of this case,

$$\begin{aligned}
& \int_0^t \int_{\Omega} \rho^2 \frac{\varphi_{k-1}^{\beta_2 K}}{\left(\frac{M}{z^k}\right)^{\beta_2 K} M^{-\frac{m-1+b}{K}}} \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_2 K} M^{-\frac{m-1+b}{K}}} \|\rho^{2-\alpha_2}\|_{L^{p_4} L^{q_4}} \left(\int_0^t \left(\int_{\Omega} (\rho^{\alpha_2} \varphi_{k-1}^{\beta_2 K})^{q_3} dx \right)^{\frac{p_3}{q_3}} dt \right)^{\frac{1}{p_3}} \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_2 K} M^{-\frac{m-1+b}{K}}} \|\rho^{2-\alpha_2}\|_{L^{p_4} L^{q_4}} \|\rho \varphi^K\|_{L^{\infty} L^1}^{\alpha_2} \|\varphi_{k-1}\|_{L^2 L^6}^{(\beta_2 - \alpha_2)K} \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_2 K} M^{-\frac{m-1+b}{K}}} \|\rho^{2-\alpha_2}\|_{L^{p_4} L^{q_4}} \left(U_{k-1}^{\alpha_2 + (\beta_2 - \alpha_2)K} + U_{k-1}^{\alpha_2 + \frac{(\beta_2 - \alpha_2)K}{2}} \right),
\end{aligned} \tag{207}$$

where the positive index need to satisfy

$$\begin{aligned}
& \frac{(\beta_2 - \alpha_2)K}{6} + \alpha_1 = \frac{1}{q_4}, \\
& \frac{(\beta_2 - \alpha_2)K}{2} = \frac{1}{p_4}. \\
& \frac{1}{p_3} + \frac{1}{p_4} = 1, \\
& \frac{1}{q_3} + \frac{1}{q_4} = 1, \\
& q_4 \leq \frac{\bar{\gamma}}{2 - \alpha_2}, \\
& \beta_2 K > 0, \\
& \alpha_2 + \frac{(\beta_2 - \alpha_2)K}{2} > 1.
\end{aligned} \tag{208}$$

If $x = \frac{(\beta_1 - \alpha_1)K}{6}$, $y = \alpha_1$, all the requirement reduce to

$$\begin{aligned}
& 0 < x + \left(1 - \frac{1}{\bar{\gamma}}\right)y \leq 1 - \frac{2}{\bar{\gamma}}, \\
& 3x + y > 1, \\
& x < \frac{1}{3}.
\end{aligned} \tag{209}$$

Similar as case 1, we need to choose $x < \frac{1}{3}$ but x is very close to $\frac{1}{3}$, and y a positive number small enough. In sum from equation, we have

$$\begin{aligned}
U_k & \leq C(2^k)^{\beta_1 K} \left(U_{k-1}^{\alpha_1 + (\beta_1 - \alpha_1)K_1} + U_{k-1}^{\alpha_1 + \frac{(\beta_1 - \alpha_1)K_1}{2}} \right. \\
& \quad \left. + U_{k-1}^{\alpha_2 + (\beta_2 - \alpha_1)K} + U_{k-1}^{\alpha_2 + \frac{(\beta_2 - \alpha_2)K}{2}} \right) M^{\beta_3}.
\end{aligned} \tag{210}$$

Similar continuity argument could be employed to show that

$$\theta \geq \underline{\theta} \text{ in } Q. \quad (211)$$

2.3 Minimum principle of the temperature with dark radiation

In this section, we will proposition of the problem introduced in Chapter 1, section 1.6 , where the radiation effect is also considered.

Proposition 1. *If the temperature variable in the approximate solution obtained following [51] before taking $\delta \rightarrow 0$ satisfying the following condition,*

$$\begin{aligned} & \int_{\Omega} \rho \phi_k(\theta)(t, x) dx + \int_{\Omega} F_1(\theta) 1_{\Omega_k} dx + \int_0^t \int_{\Omega} \frac{\delta m}{(\theta)^{m+3}} 1_{\Omega_k} dx d\tau \\ & + \int_0^t \int_{\Omega} \frac{m S_{\delta}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}}{(\theta)^{m+1}} 1_{\Omega_k} dx d\tau + \int_0^t \int_{\Omega} \frac{m(m+1) \kappa_{\delta} |\nabla_x \theta|^2}{(\theta)^{m+2}} 1_{\Omega_k} dx d\tau \\ & \leq \int_0^t \int_{\Omega} \frac{m}{(\theta)^{m+1}} (\operatorname{div}_x(\theta^4 \mathbf{u}) + \rho \theta + \frac{\theta^4}{3}) 1_{\Omega_k} dx d\tau \\ & + \int_{\Omega} \rho \phi_k(\theta)(0, x) dx + \int_{\Omega} F_1(\theta(0)) 1_{\Omega_k} dx, \end{aligned} \quad (212)$$

where $F_1 = \int_{\theta}^{(M)^{-\frac{m}{K}}} \frac{ms^4}{(s)^{m+1}} ds$, and

$$\psi_k(\theta) = \left[\frac{1}{(\theta)^{\frac{m}{K}}} - C_k \right]_+, \quad (213)$$

$$\phi_k(\theta) = \left[\frac{1}{(\theta)^m} - C_k^K \right]_+. \quad (214)$$

And the conditions of the initial data and constitution relations are the same as those in Theorem 3. Then

$$\theta(t, x) \geq \underline{\theta} > 0, \quad (215)$$

Remark. *The lower bound obtained by the minimum principle is independent of δ . With the lower bound, one can follow the framework in to get the Theorem 3 on the existence . The proof is sketched in the appendix A.*

2.3.1 Construction of auxiliary functions

We will use the same auxiliary functions used in the previous section, and also consider the case when $\underline{\theta} < 1$. Let $m > b$ is a constant, and M is a number large enough. The size of M and the relation between m and b will be specified later. The ratio

$$K = \frac{2m}{m-b} \quad (216)$$

will also be used later, one can see that $K > 1$. For the iteration technique, we need to introduce a sequence C_k where

$$C_0 = M, \text{ and } M = \underline{\theta}^{\frac{m-b}{2}} \quad (217)$$

and

$$C_{k+1} = C_k + \frac{M}{z^{k+1}}. \quad (218)$$

So we have

$$\lim_{k \rightarrow \infty} C_k = \frac{1}{1 - \frac{1}{z}} M. \quad (219)$$

We can see that $\psi_k(\theta)$ and $\phi_k(\theta)$ have the same supports (closure of the region where these functions are positive), which we denote as

$$\Omega_k = \overline{\{(x, t) | \theta^m < 1/C_k^K\}}. \quad (220)$$

Here we have some relations between ψ_k and ϕ_k ,

$$\begin{aligned} \psi_k(\theta) &\leq \phi_k(\theta), \\ \psi_k(\theta)^K &\leq \phi_k(\theta). \end{aligned} \quad (221)$$

On the other hand, we have in the region Ω_{k-1}

$$\psi_k \geq \frac{M}{z^k}. \quad (222)$$

2.3.2 Definition of functional $U_k(t)$

Firstly, we will define a functional from the admissible condition, we have the following inequality from the limit of the admissible condition from the previous section

We assume that in this section,

$$m > b > 0 \quad (223)$$

then by the assumption of the Ω_k , and the definition of F_1 , see that when $k \geq 1$, there exists a positive constant C so that

$$F_1 1_{\Omega_k} \geq C 1_{\Omega_k} (Mz)^{-\frac{m}{K}}. \quad (224)$$

So the functional U_k is defined as

$$\begin{aligned} & U_k(t) \\ &= \sup_{0 \leq \tau \leq t} \int_{\Omega} \rho \phi_k(\theta)(\tau, x) dx + \sup_{0 \leq \tau \leq t} \int_{\Omega} F_1(\theta) 1_{\Omega_k} dx + \int_0^t \int_{\Omega} \frac{\delta m}{(\theta^2)^{m+1}} 1_{\Omega_k} dx d\tau \\ &+ \int_0^t \int_{\Omega} \frac{m S_{\delta}(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}}{(\theta)^{m+1}} 1_{\Omega_k} dt d\tau + \int_0^t \int_{\Omega} \frac{m(m+1) \kappa_{\delta} |\nabla_x \theta|^2}{(\theta)^{m+2}} 1_{\Omega_k} dx d\tau. \end{aligned} \quad (225)$$

And we have the following from the definition, when $0 < m < 4$,

$$\begin{aligned} \|\rho \phi_k\|_{L^{\infty} L^1} &\leq C U_k \\ \|\nabla_x \psi_k\|_{L^2 L^2} &\leq C U_k^{\frac{1}{2}} \\ \|1_{\Omega_k}\|_{L^{\infty} L^1} &\leq C (Mz)^{-\frac{m}{K}} U_k \end{aligned} \quad (226)$$

Using Cauchy-Schwartz inequality, we can obtain the bound of U_k

$$U_k(t) \leq C \int_0^t \int_{\Omega} \frac{\rho \theta + \theta^4}{(\theta)^{m+1}} |\operatorname{div}_x \mathbf{u}| 1_{\Omega_k} dx d\tau + \int_{\Omega} \rho_0 \phi_{k,\epsilon}(\theta_0) dx + \int_{\Omega} F_1(\theta(0)) 1_{\Omega_k} dx. \quad (227)$$

the last two terms would be zero when we choose

$$\left(\frac{1}{M}\right)^{\frac{2}{m-b}} = \underline{\theta}. \quad (228)$$

where $\underline{\theta}$ is the lower bound of the initial data. By this choice, the later two terms vanish, so we have the following estimates,

$$U_k \leq C \int_0^t \int_{\Omega} \frac{\rho\theta + \theta^4}{(\theta)^{m+1}} |\operatorname{div}_x \mathbf{u}| 1_{\Omega_k} dx d\tau. \quad (229)$$

We need to control the right hand side term by some power of U_{k-1} , so that we can use the De Giorgi iteration technique to get $\lim_{k \rightarrow \infty} U_k = 0$. So the main task is the interpolations of the right hand-side terms. Using Hölder inequality, we have the following estimate,

$$\begin{aligned} & \left(\int_0^t \left(\int_{\Omega} (\rho^{\alpha_1} \psi^{\beta_1 K})^{q_1} dx \right)^{\frac{p_1}{q_1}} d\tau \right)^{\frac{1}{p_1}} \\ &= \left(\int_0^t \left(\int_{\Omega} \rho^{\alpha_1 q_1} \psi^{\alpha_1 q_1 K} \cdot \psi^{(\beta_1 - \alpha_1) q_1 K} dx \right)^{\frac{p_1}{q_1}} d\tau \right)^{\frac{1}{p_1}} \\ &\leq \|\rho \psi^K\|_{L^\infty L^1}^{\alpha_1} \|\psi\|_{L^2 L^6}^{(\beta_1 - \alpha_1) K}. \end{aligned} \quad (230)$$

where the parameters satisfy

$$\begin{aligned} \frac{(\beta_1 - \alpha_1) q_1 K}{6} + \alpha_1 q_1 &= 1, \\ \frac{(\beta_1 - \alpha_1) p_1 K}{2} &= 1. \end{aligned} \quad (231)$$

2.3.3 Estimates for Iterations I

In this section, we will finish the estimates needed for lemma 3, in the case when $b < \frac{1}{2}$ and $\gamma > 3$. The first term in equation (229) could be controlled in this way,

$$\begin{aligned} & \int_0^t \int_{\Omega} \frac{\rho\theta}{(\theta)^{m+1}} |\operatorname{div}_x \mathbf{u}| 1_{\Omega_k} dx d\tau \\ & \leq \delta_1 \int_0^t \int_{\Omega} \frac{\theta^b |\nabla_x \mathbf{u}|^2}{\theta^{m+1}} dx d\tau + C(\delta_1) \int_0^t \int_{\Omega} \rho^2 \theta^{-m+1-b} 1_{\Omega_k} dx d\tau. \end{aligned} \quad (232)$$

the second term will be controlled similarly,

$$\begin{aligned}
& \int_0^t \int_{\Omega} \frac{\theta^4}{(\theta)^{m+1}} |\operatorname{div}_x \mathbf{u}| 1_{\Omega_k} dx d\tau \\
& \leq \delta_1 \int_0^t \int_{\Omega} \frac{\theta^b |\nabla_x \mathbf{u}|^2}{\theta^{m+1}} dx d\tau + C(\delta_1) \int_0^t \int_{\Omega} \theta^{7-m-b} 1_{\Omega_k} dx d\tau.
\end{aligned} \tag{233}$$

Choose δ_1 small and we just need to control the second term in each equation.

Then we can choose m so that $m - 1 + b \leq 0$, recall that in this case we have m is in between b and $1 - b$, so we have

$$\begin{aligned}
& \int_0^T \int_{\Omega} \rho^2 \theta^{-m+1-b} 1_{\Omega_k} dx d\tau \\
& \leq C \int_0^T \int_{\Omega} \rho^2 \frac{\psi_{k-1}^{\beta_1 K}}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} dx d\tau \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \left(\int_0^T \left(\int_{\Omega} (\rho^{\alpha_1} \psi_{k-1}^{\beta_1 K})^{q_1} dx \right)^{\frac{p_1}{q_1}} dt \right)^{\frac{1}{p_1}} \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \|\rho \psi_{k-1}^K\|_{L^{\infty} L^1}^{\alpha_1} \|\psi_{k-1}\|_{L^2 L^6}^{(\beta_1 - \alpha_1)K} \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \|\rho \phi_{k-1}\|_{L^{\infty} L^1}^{\alpha_1} \|\psi_{k-1}\|_{L^2 L^6}^{(\beta_1 - \alpha_1)K} \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \left(U_{k-1}^{\alpha_1 + (\beta_1 - \alpha_1)K} + U_{k-1}^{\alpha_1 + \frac{(\beta_1 - \alpha_1)K}{2}} \right).
\end{aligned} \tag{234}$$

where the positive index need to satisfy

$$\begin{aligned}
& \frac{(\beta_1 - \alpha_1)q_1 K}{6} + \alpha_1 q_1 = 1, \\
& \frac{(\beta_1 - \alpha_1)p_1 K}{2} = 1, \\
& \frac{1}{p_1} + \frac{1}{p_2} = 1, \\
& \frac{1}{q_1} + \frac{1}{q_2} = 1, \\
& q_2 \leq \frac{\gamma}{2 - \alpha_1}, \\
& \beta_1 K > 1, \\
& \alpha_1 + \frac{(\beta_1 - \alpha_1)K}{2} > 1.
\end{aligned} \tag{235}$$

The relation of p, q and α, β is

$$\begin{aligned}\frac{(\beta_1 - \alpha_1)K}{6} + \alpha_1 &= \frac{1}{q_1}, \\ \frac{(\beta_1 - \alpha_1)K}{2} &= \frac{1}{p_1}.\end{aligned}\tag{236}$$

Let's denote $x = \frac{(\beta_1 - \alpha_1)K}{6}$, $y = \alpha_1$. We can construct the following set of inequalities (237).

$$\begin{aligned}0 < x + \left(1 - \frac{1}{\gamma}\right)y &\leq 1 - \frac{2}{\gamma}, \\ 3x + y &> 1, \\ x &< \frac{1}{3}.\end{aligned}\tag{237}$$

And the solution of (237) satisfies the inequalities in equation (235).

With the condition $\gamma > 3$, we can see that inequalities (237) have solutions. In fact we just need to choose $x < \frac{1}{3}$ but very close to $\frac{1}{3}$, and y a positive number small enough.

For the other term, we can use an similar estimate from Hölder inequality

Lemma 9. *For any function $f \in L^2L^6 \cap L^\infty L^1$, we have $f \in L^{\frac{8}{3}}L^{\frac{8}{3}}$, in addition*

$$\|f\|_{L^{\frac{8}{3}}L^{\frac{8}{3}}}^{\frac{8}{3}} \leq \|f\|_{L^2L^6}^2 \|f\|_{L^\infty L^1}^{\frac{2}{3}}.\tag{238}$$

Then for the other term

$$\begin{aligned}&\int_0^T \int_\Omega \theta^{6-m-b} 1_{\Omega_k} dx d\tau \\ &\leq C(M)^{-\frac{7-m-b}{m}} \int_0^T \int_\Omega 1_{\Omega_k} dx d\tau \\ &\leq C(M)^{-\frac{7-m-b}{m}} \int_0^T \int_\Omega 1_{\Omega_k}^{\frac{8}{3}} dx d\tau \\ &\leq C(M)^{-\frac{7-m-b}{m}} \|1_{\Omega_k}\|_{L^2L^6}^2 \|1_{\Omega_k}\|_{L^\infty L^1}^{\frac{2}{3}}.\end{aligned}\tag{239}$$

Then the two terms in the product could be controlled by some power of U_{k-1}

$$\|1_{\Omega_k}\|_{L^2L^6}^2 \leq \left\| \frac{\psi_{k-1}}{\frac{M}{2^k}} \right\|_{L^2L^6}^2 \leq \frac{1}{\left(\frac{M}{z^k}\right)^2} U_{k-1} \quad (240)$$

$$\|1_{\Omega_k}\|_{L^\infty L^1}^{\frac{2}{3}} \leq C(Mz)^{\frac{2m}{3K}} U_{k-1}^{\frac{2}{3}} \quad (241)$$

In sum from equation (234)(239)(240)(241), we have

$$U_k \leq C U_{k-1}^{\frac{5}{3}} (Mz)^{\beta_3} + C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} (U_{k-1}^{\alpha_1 + (\beta_1 - \alpha_1)K} + U_{k-1}^{\alpha_1 + \frac{(\beta_1 - \alpha_1)K}{2}}) \quad (242)$$

where β_3 is a constant, so we have if T is small enough, say $T \leq T_\epsilon$ we have

$$\lim_{k \rightarrow \infty} U_k = 0 \quad (243)$$

By the continuity argument, we have

$$\theta \geq \left(\frac{1}{M}\right)^{\frac{2}{m-b}} \frac{1}{1 - \frac{1}{z}} \quad (244)$$

for a sequence of z_i , where $\lim_{i \rightarrow \infty} z_i = \infty$. Let $z \rightarrow \infty$, we have

$$\theta \geq \left(\frac{1}{M}\right)^{\frac{2}{m-b}} \quad (245)$$

2.3.4 Estimates for Iterations II

In this section, we will finish the estimates needed for lemma 3, in the case when $\frac{1}{2} \leq b < 3$ and $\gamma > 3$. We can do similar interpolations estimates, but the index for M is no longer the same sign as the first case. As case 1, we will show the same type of lower bound locally in time in $[0, T_\epsilon]$, and then the bound will be still global by the continuity argument. As the case 1, we have when M satisfies

$$\left(\frac{1}{M}\right)^{\frac{2}{m-b}} = \underline{\theta}, \quad (246)$$

$$U_k(t) \leq C \int_0^t \int_\Omega \rho^2 \theta^{-m-b+1} 1_{\Omega_k} + C \int_0^t \int_\Omega \theta^{7-m-b} 1_{\Omega_k} dx d\tau. \quad (247)$$

the negative power of θ could be bounded in the following way,

$$\frac{1}{\theta^{m+b-1}} 1_{\Omega_k} \leq C \left(\psi_{k-1}^{\frac{m+b-1}{K}} + M^{\frac{m+b-1}{K}} \right). \quad (248)$$

so we have the estimates

$$\begin{aligned}
& \int_0^t \int_{\Omega} \rho^2 \theta^{-m-b+1} 1_{\Omega_k} \\
& \leq C \int_0^t \int_{\Omega} \rho^2 \frac{\psi_{k-1}^{\beta_1 K + \frac{m-1+b}{K}}}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} + C \int_0^t \int_{\Omega} \rho^2 \frac{\psi_{k-1}^{\beta_2 K}}{\left(\frac{M}{z^k}\right)^{\beta_2 K} M^{-\frac{m-1+b}{K}}}.
\end{aligned} \tag{249}$$

We can define $K_1 = \left(K + \frac{2(m+b-1)}{\beta_1(m-b)}\right) = \left(K + \frac{(m+b-1)}{\beta_1 m/K}\right)$, so the first term in equation (249) could be estimated as

$$\begin{aligned}
& \int_0^t \int_{\Omega} \rho^2 \frac{\psi_{k-1}^{\beta_1 K + \frac{m-1+b}{K}}}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \\
& = \int_0^t \int_{\Omega} \rho^2 \frac{\psi_{k-1}^{\beta_1 K_1}}{\left(\frac{M}{z^k}\right)^{\beta_1 K_1}} dx d\tau \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \left(\int_0^t \left(\int_{\Omega} (\rho^{\alpha_1} \psi_{k-1}^{\beta_1 K_1})^{q_1} dx \right)^{\frac{p_1}{q_1}} dt \right)^{\frac{1}{p_1}} \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} \|\rho \psi^{K_1}\|_{L^{\infty} L^1}^{\alpha_1} \|\psi_{k-1}\|_{L^2 L^6}^{(\beta_1 - \alpha_1) K_1} \\
& \leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_1 K}} \|\rho^{2-\alpha_1}\|_{L^{p_2} L^{q_2}} (U_{k-1}^{\alpha_1 + (\beta_1 - \alpha_1) K_1} + U_{k-1}^{\alpha_1 + \frac{(\beta_1 - \alpha_1) K_1}{2}}).
\end{aligned} \tag{250}$$

where the positive index need to satisfy

$$\begin{aligned}
& \frac{(\beta_1 - \alpha_1) K_1}{6} + \alpha_1 = \frac{1}{q_4}, \\
& \frac{(\beta_1 - \alpha_1) K_1}{2} = \frac{1}{p_4}. \\
& \frac{1}{p_1} + \frac{1}{p_2} = 1, \\
& \frac{1}{q_1} + \frac{1}{q_2} = 1, \\
& q_2 \leq \frac{\gamma}{2 - \alpha_1}, \\
& \alpha_1 + \frac{(\beta_1 - \alpha_1) K_1}{2} > 1.
\end{aligned} \tag{251}$$

While we can choose the parameters similar the case of $b < \frac{1}{2}$. If $x = \frac{(\beta_1 - \alpha_1) K_1}{6}$, $y = \alpha_1$, all the requirement reduce to

$$\begin{aligned}
0 < x + \left(1 - \frac{1}{\gamma}\right)y &\leq 1 - \frac{2}{\gamma}, \\
3x + y &> 1, \\
x &< \frac{1}{3}.
\end{aligned} \tag{252}$$

Similar as the case of $b < \frac{1}{2}$, we need to choose $x < \frac{1}{3}$ but x is very close to $\frac{1}{3}$, and y a positive number small enough. And we need more conditions on m and b later as we need to keep use the same m, K and b for the second term of this case,

$$\begin{aligned}
&\int_0^t \int_{\Omega} \rho^2 \frac{\psi_{k-1}^{\beta_2 K}}{\left(\frac{M}{z^k}\right)^{\beta_2 K} M^{-\frac{m-1+b}{K}}} \\
&\leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_2 K} M^{-\frac{m-1+b}{K}}} \|\rho^{2-\alpha_2}\|_{L^{p_4} L^{q_4}} \left(\int_0^t \left(\int_{\Omega} (\rho^{\alpha_2} \psi_{k-1}^{\beta_2 K})^{q_3} dx \right)^{\frac{p_3}{q_3}} dt \right)^{\frac{1}{p_3}} \\
&\leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_2 K} M^{-\frac{m-1+b}{K}}} \|\rho^{2-\alpha_2}\|_{L^{p_4} L^{q_4}} \|\rho \psi^K\|_{L^{\infty} L^1}^{\alpha_2} \|\psi_{k-1}\|_{L^2 L^6}^{(\beta_2 - \alpha_2)K} \\
&\leq C \frac{1}{\left(\frac{M}{z^k}\right)^{\beta_2 K} M^{-\frac{m-1+b}{K}}} \|\rho^{2-\alpha_2}\|_{L^{p_4} L^{q_4}} (U_{k-1}^{\alpha_2 + (\beta_2 - \alpha_2)K} + U_{k-1}^{\alpha_2 + \frac{(\beta_2 - \alpha_2)K}{2}}).
\end{aligned} \tag{253}$$

where the positive index need to satisfy

$$\begin{aligned}
\frac{(\beta_2 - \alpha_2)K}{6} + \alpha_1 &= \frac{1}{q_4}, \\
\frac{(\beta_2 - \alpha_2)K}{2} &= \frac{1}{p_4}. \\
\frac{1}{p_3} + \frac{1}{p_4} &= 1, \\
\frac{1}{q_3} + \frac{1}{q_4} &= 1, \\
q_4 &\leq \frac{\gamma}{2 - \alpha_2}, \\
\beta_2 K &> 1, \\
\alpha_2 + \frac{(\beta_2 - \alpha_2)K}{2} &> 1.
\end{aligned} \tag{254}$$

On the other hand

$$\begin{aligned}
& \int_0^t \int_{\Omega} \theta^{7-m-b} 1_{\Omega_k} dx d\tau \\
& \leq C \int_0^t \int_{\Omega} 1_{\Omega_k} dx d\tau \\
& \leq C \int_0^t \int_{\Omega} 1_{\Omega_k}^{\frac{8}{3}} dx d\tau \\
& \leq C \|1_{\Omega_k}\|_{L^2 L^6}^2 \|1_{\Omega_k}\|_{L^\infty L^1}^{\frac{2}{3}}.
\end{aligned} \tag{255}$$

Then the two terms in the product could be controlled by some power of U_{k-1} ,

$$\|1_{\Omega_k}\|_{L^2 L^6}^2 \leq \left\| \frac{\psi_{k-1}}{\frac{M}{2^k}} \right\|_{L^2 L^6}^2 \leq \frac{1}{\left(\frac{M}{z^k}\right)^2} U_{k-1} \tag{256}$$

$$\|1_{\Omega_k}\|_{L^\infty L^1}^{\frac{2}{3}} \leq C(Mz)^{\frac{2m}{3K}} U_{k-1}^{\frac{2}{3}} \tag{257}$$

In sum from equation (250) (253)(255)(256)(257), we have

$$\begin{aligned}
U_k & \leq C U_{k-1}^{\frac{5}{3}} (Mz)^{\beta_3} \\
& \quad + C(z^k)^{\beta_1 K} (U_{k-1}^{\alpha_1 + (\beta_1 - \alpha_1)K_1} + U_{k-1}^{\alpha_1 + \frac{(\beta_1 - \alpha_1)K_1}{2}} \\
& \quad + U_{k-1}^{\alpha_2 + (\beta_2 - \alpha_1)K} + U_{k-1}^{\alpha_2 + \frac{(\beta_2 - \alpha_2)K}{2}}) M^{\beta_3}.
\end{aligned} \tag{258}$$

Similar continuity argument with the help of Lemma 4 will give us

$$\theta \geq \underline{\theta} \text{ in } Q. \tag{259}$$

CHAPTER III

EXISTENCE WITH DEGENERATE TRANSPORT COEFFICIENTS

In this chapter, we sketch the proof of the existence theorem, which is mainly from [51]. Based on Proposition 1, the approximate admissible condition before letting $\delta \rightarrow 0$ could be used to obtain the minimum principle. One can notice that the lower bound obtained by the minimum principle is independent of δ . The proof followed the framework of [51]. First, design the Feado-Galerkin approximation. The continuity equation is regularized by artificial viscosity term while the artificial pressure part is also introduced. In the same time, the transport coefficients are modified. Results about the existence of the approximation system were obtained in [51]. The basic idea is using fixed point theorem on the momentum equations, with which we can find at least a local solution in $[0, T_{\max}]$. The properties of the solution operators in the approximate continuity equation and approximate energy equation are used. Uniform estimates of $\|u_n\|_{X_n}$ independent of T_{\max} could be derived by approximate, mass conservation, energy conservation and Helmholtz free energy. By the uniform estimates and bootstrap, we know the approximate problem actually has a solution in $[0, T]$.

Second, we need to let $n \rightarrow \infty$ to get first level approximation. Based on uniform estimates independent of n , we can get the weak convergence of most terms, which is not enough. The strong convergence of ρ from the parabolic estimates and strong convergence of θ came from a technique using Div-Curl lemma. From the first level, we have entropy inequality instead of entropy equation. The new ingredient is the generalized admissible conditions.

Third, we need to let $\epsilon \rightarrow 0$ to erase the artificial viscosity. Strong convergence of θ_n could be derived by Div-Curl lemma with uniform estimates independent of ϵ , strong convergence of ρ_n comes from the weak convergence of pressure. The artificial pressure introduced in [51] played an important role. The generalized admissible conditions should be checked in this step, with negative powers of θ , this admissible condition is stronger than entropy condition in the region when θ is very small.

Fourth, with the generalized admissible condition derived in the third step, one can get a strictly positive lower bound of θ , independent of δ . With this positive lower bound, the technique of [51] would be able to be employed to let $\delta \rightarrow 0$. While $\gamma > 3$, the lemma of [34](lemma 15) would be enough to show ρ is a renormalized solution based only on δ -independent estimates. Still one needs to get $\|\rho\|_{L^{\gamma+\nu}L^{\gamma+\nu}}$ estimates and equality related to effective flux. One of the difference here is we allowed the growth rate of θ in $\kappa(\theta)$ lower than that of [51].

3.1 Preliminary

3.1.1 Notations

We list use some standard notations here:

1. Use $\overline{F(U)}$ to denote the L^1 weak limit of $F(U_n)$;
2. R_{ij} to denote the double Riesz transform;
3. Use L^pL^q to denote $L^p(0, T; L^q(\Omega))$;
4. use Q to denote $[0, T] \times \Omega$.

3.1.2 Some lemmas from analysis

Here we list some useful lemmas most of which could be found in Feireisl's book [43]. Lemma 10 is about the weak limit of lower-semicontinuous functions and convex functions, and also important tool for finding the L^∞ limit of a sequence. Particularly,

the strong limit of the density function were obtained in this approach, in the cases of both $\epsilon \rightarrow 0$ and $\delta \rightarrow 0$.

Lemma 10. *Let $O \subset \mathbb{R}^m$ be a measurable set, and $\{v_n\}_{n=1}^\infty$ a sequence of functions in $L^1(O; \mathbb{R}^m)$, $v_n \rightharpoonup v$ weakly in $L^1(O, \mathbb{R}^m)$. Let $\phi : \mathbb{R}^m \rightarrow (-\infty, \infty)$, $\phi(v_n) \in L^1(O)$ for any n , and ϕ is lower semi-continuous function such that*

$$\phi(v_n) \rightarrow \overline{\phi(v)} \text{ weakly in } L^1(O). \quad (260)$$

Then

$$\phi(v) \leq \overline{\phi(v)} \text{ a.e. on } O. \quad (261)$$

If moreover, ϕ is strictly convex on an open set $U \subset \mathbb{R}^n$, and

$$\phi(v) = \overline{\phi(v)} \text{ a.e. on } O, \quad (262)$$

then, by extracting a subsequence, we have

$$v_n \rightarrow v \text{ for a.e. } y \in \{y \in O \mid v(y) \in U\}. \quad (263)$$

Lemma 11 (weak convergence and monotonicity). *Let $I \subset \mathbb{R}$ be an interval, $B \subset \mathbb{R}^N$ a domain, and $(F, G) \in C(I) \times C(I)$ a couple of non-negative non-decreasing functions. Assume that $\rho_n \in L^1(B; I)$, and*

$$\begin{aligned} F(\rho_n) &\rightharpoonup \overline{F(\rho)}, \\ G(\rho_n) &\rightharpoonup \overline{G(\rho)}, \\ F(\rho_n)G(\rho_n) &\rightharpoonup \overline{F(\rho)G(\rho)} \end{aligned} \quad (264)$$

weakly in $L^1(B; I)$, then

1. $\overline{F(\rho)} \overline{G(\rho)} \leq \overline{F(\rho)G(\rho)}$.

2. If $G \in C(\mathbb{R})$, $G(\mathbb{R}) = \mathbb{R}$, G is strictly increasing; $F \in C(\mathbb{R})$, F is non-decreasing, and $\overline{F(\rho)} \overline{G(\rho)} = \overline{F(\rho)G(\rho)}$, then

$$\overline{F(\rho)} = F(G^{-1}(\overline{G(\rho)})). \quad (265)$$

3. In particular, if $G(z) = z$, then

$$\overline{F(\rho)} = F(\rho). \quad (266)$$

Lemma 12 is the famous Div-Curl lemma [101, 113], this lemma is the key tool to take the limit of temperature.

Lemma 12. *Let $Q \subset \mathbb{R}^N$ be an open set. Assume*

$$\mathbf{U}_n \rightharpoonup \mathbf{U} \text{ weakly in } L^p(Q, \mathbb{R}^N), \quad (267)$$

$$\mathbf{V}_n \rightharpoonup \mathbf{V} \text{ weakly in } L^q(Q, \mathbb{R}^N), \quad (268)$$

and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1. \quad (269)$$

In addition, let $\operatorname{div} \mathbf{U}_n$, and $\operatorname{curl} \mathbf{U}_n$ are pre-compact in $W^{-1,s}(Q)$, and $W^{-1,s}(Q, \mathbb{R}^N)$ respectively where $s > 1$. Then

$$\mathbf{U}_n \mathbf{V}_n \rightharpoonup \mathbf{U} \mathbf{V} \text{ weakly in } L^r(Q). \quad (270)$$

Lemma 13. *Suppose X is a reflexive Banach space, Y is a Banach space. $X \mapsto Y$, Y' is separable and dense in X' . Suppose $\{f_n\}$ satisfies:*

$$\left\{ \begin{array}{l} f_n \in L^\infty(0, T, X) \cap C^0([0, T], Y); \\ \|f_n\|_{L^\infty(0, T, X)} \leq C \quad \forall n \geq 1; \\ \forall \varphi \in Y', \langle \varphi, f_n(t) \rangle_{Y' \times Y} \text{ is uniformly continuous with respect to } t \\ \text{and uniformly with respect to } n, \end{array} \right. \quad (271)$$

then f_n is relative compact in $C^0([0, T], X_{\text{weak}})$.

One can check the condition could be replaced by

there exists a $p \in (1, \infty]$

$$f_n \in L^\infty([0, T], X) \quad \partial_t f_n \in L^p([0, T], Y), \quad (272)$$

$$\|f\|_{L^\infty([0, T], X)} + \|\partial_t f_n\|_{L^p([0, T], Y)} \leq C \text{ uniformly.}$$

Lemma 14. *Let Ω be a $C^{2,\nu}$ domain, and $1 < s < 3$ a constant, and*

$$f_n \in L^\infty(0, T, L^p(\Omega)),$$

where $p > \frac{3s}{4s-3}$. If

$$f_n \rightarrow f \text{ in } C^0([0, T], L^p_{weak}),$$

then

$$f_n \rightarrow f \text{ in } C^0([0, T], W^{-1,s}).$$

Finally, we cite an important lemma of DiPerna and P.L. Lions [34], which states that the weak solution is actually the renormalized solution if the density and velocity have some integrability condition.

Lemma 15. *[DiPerna, Lions [34]]*

Let ρ, u be a weak solution of the continuity equation, and $\rho \in L^\infty L^\gamma$, $\mathbf{u} \in L^{p_1} W^{1,p_1}$, where

$$\frac{1}{p_1} + \frac{1}{\gamma} \leq 1, \tag{273}$$

then \mathbf{u} is a renormalized solution specified in section 1.6.3.1.

3.2 The Faedo-Galerkin approximation

Following the framework of Feireisl [51], we define the approximation scheme in the following way. First, the equation of continuity is regularized by means of an artificial viscosity term, with homogeneous Neumann boundary condition.

1. Approximate continuity equation

$$\begin{aligned} \rho_t + \operatorname{div}_x(\rho \mathbf{u}) &= \epsilon \Delta \rho, \\ \nabla_x \rho \cdot n|_{\partial\Omega} &= 0, \end{aligned} \tag{274}$$

where ϵ is a positive small constant, and n is the unit normal outward vector.

2. Approximate momentum equations

$$\begin{aligned} \partial_t(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) &= \nabla_x(\rho^\gamma + \rho\theta + \frac{\theta^4}{3} + \delta(\rho^\beta)) \\ &+ \epsilon \nabla_x \rho \nabla_x \mathbf{u} + \operatorname{div}_x \mathbb{S}_\delta, \end{aligned} \quad (275)$$

where the approximate stress tensor \mathbb{S}_δ is defined as

$$\mathbb{S}_\delta = (\bar{\mu}\theta^b + \delta\theta)(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\operatorname{div}_x \mathbf{u} \mathbb{I}) + \eta \operatorname{div}_x \mathbf{u} \mathbb{I}, \quad (276)$$

where \mathbb{I} is the identity 3×3 matrix. Approximate momentum equations are true for any test function

$$\psi \in C_c^1([0, T], X_n), \quad (277)$$

where X_n is a n -dimensional functional space. Here, we use the space spanned by the first n eigenfunctions of the Laplace operator.

3. Approximate temperature equation

$$\begin{aligned} &\partial_t(\rho\theta + \theta^4) + \operatorname{div}_x((\rho\theta + \theta^4)\mathbf{u}) \\ &= \operatorname{div}_x(\kappa_\delta \nabla_x \theta) + \mathbb{S}_\delta : \nabla_x \mathbf{u} - (\rho\theta + \frac{\theta^4}{3})\operatorname{div}_x \mathbf{u} \\ &+ \epsilon \delta(\beta \rho^{\beta-2}) |\nabla_x \rho|^2 + \epsilon(\gamma \rho^{\gamma-2}) |\nabla_x \rho|^2 + \frac{\delta}{\theta^2} - \epsilon \theta^5, \end{aligned} \quad (278)$$

where e and \mathcal{K}_δ are defined as

$$\begin{aligned} \rho e(\rho, \theta) &= \rho\theta + \theta^4, \\ \mathcal{K}_\delta &= \int_1^\theta \kappa_\delta dz, \\ \kappa_\delta &= \kappa(\theta) + \delta(\theta^\beta + \frac{1}{\theta}). \end{aligned} \quad (279)$$

3.2.0.1 On the initial data of the first level approximate equations

We assume the initial data is regularized in the following way.

1.

$$0 < \delta \leq \rho_{0,\delta}(x) \leq \delta^{-\frac{1}{\beta}}, \nabla_x \rho_{0,\delta}(x) \cdot \mathbf{n}|_{\partial\Omega} = 0. \quad (280)$$

2. $\rho_{0,\delta} \rightarrow \rho_0$ in L^β .

3. For the momentum if the initial momentum is $\mathbf{m}_0 = \rho_0 \mathbf{u}_0$

$$\bar{\mathbf{m}}_\delta^i = \begin{cases} \mathbf{m}_0^i \sqrt{\frac{\rho_{0,\delta}}{\rho_0}} & \text{if } \rho_0 > 0, \\ 0 & \text{if } \rho_0 = 0. \end{cases} \quad (281)$$

So

$$\int_\Omega \frac{|\bar{\mathbf{m}}_\delta^i|^2}{\rho_{0,\delta}} dx < \infty$$

uniformly with respect to δ , so there exists h_δ^i

$$\left\| \frac{\bar{\mathbf{m}}_\delta^i}{\sqrt{\rho_{0,\delta}}} - h_\delta^i \right\|_{L^2(\Omega)} < \delta, \quad i = 1, 2, 3. \quad (282)$$

Take $\mathbf{m}_\delta^i = h_\delta^i \sqrt{\rho_{0,\delta}}$, $i = 1, 2, 3$ then

$$\frac{\mathbf{m}_\delta^i}{\rho_{0,\delta}} \text{ are bound in } L^1, \quad (283)$$

$$\mathbf{m}_\delta^i \rightarrow \mathbf{m}_i \text{ in } L^1(\Omega) \text{ as } \delta \rightarrow 0. \quad (284)$$

$$\mathbf{u}_{0,\delta} = \frac{\mathbf{m}_\delta}{\rho_{0,\delta}} \quad (285)$$

4.

$$0 < \delta \leq \theta_{0,\delta}(x) \leq \delta^{-\frac{1}{\beta}}, \quad \nabla_x \theta_{0,\delta}(x) \cdot \mathbf{n}|_{\partial\Omega} = 0. \quad (286)$$

5. $\theta_{0,\delta} \rightarrow \theta_0$ in L^4 .

After defining the mollified initial data, we have the following initial data for the approximate equations:

$$\rho(0, x) = \rho_{0,\delta}(x), \quad (287)$$

$$\mathbf{u}(0, x) = \mathbf{u}_{0,\delta}(x), \quad (288)$$

$$\theta(0, x) = \theta_{0,\delta}(x). \quad (289)$$

3.3 Existence of solution of the approximate equations

In this section, we sketch the proof of the existence of the approximate equations which is from [51].

3.3.1 On the approximate continuity equation

The following result can be obtained in the same way as [51].

Lemma 16. *Let the initial data satisfies the conditions in subsection 3.2.0.1, then there exists a mapping $\mathbf{S} \rightarrow \mathbf{S}(\mathbf{u})$*

$$\mathbf{S} : C([0, T]; [C^2(\bar{\Omega})]^3) \mapsto C([0, T]; C^{2+\nu}(\bar{\Omega})) \quad (290)$$

with the following properties:

1. $\rho = \mathbf{S}(\mathbf{u})$ is the unique solution of equation (274).

2. For all $t > 0$,

$$\underline{\rho} \exp\left(-\int_0^t \|\operatorname{div}_x \mathbf{u}\|_{L^\infty} ds\right) \leq \mathbf{S}(\mathbf{u}) \leq \bar{\rho} \exp\left(\int_0^t \|\operatorname{div}_x \mathbf{u}\|_{L^\infty} ds\right). \quad (291)$$

3. \mathbf{S} is Lipschitz in the sense of

$$\|\mathbf{S}(\mathbf{u}^1) - \mathbf{S}(\mathbf{u}^2)\|_{C([0, T], W_0^{1,2}(\Omega))} \leq C(r_1, T) \|\mathbf{u}^1 - \mathbf{u}^2\|_{C([0, T], W_0^{1,2}(\Omega))} \quad (292)$$

in the region

$$M_{r_1} = \{\mathbf{u} \in C([0, T], W_0^{1,2}(\Omega)) \mid \|\mathbf{u}\|_{L^\infty(Q)} + \|\nabla_x \mathbf{u}\|_{L^\infty(Q)} \leq r_1\}. \quad (293)$$

3.3.2 On the approximate internal energy equation

Similar as [51], the following result holds for temperature.

Lemma 17. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0, 1)$, let $\mathbf{u} \in C([0, T]; X_n)$ be a given vector field and $\rho = \rho_{\mathbf{u}}$ be the unique solution of (274). If*

the initial data is in the class given in subsection 3.2.0.1, the equation (278) has a unique strong solution $\theta = \theta_{\mathbf{u}}$ such that

$$\begin{aligned} \partial_t \theta &\in L^2((0, T) \times \Omega), \Delta K_\delta(\theta) \in L^2((0, T) \times \Omega), \\ \theta &\in L^\infty((0, T), W^{1,2} \cap L^\infty(\Omega)), \\ \frac{1}{\theta} &\in L^\infty((0, T) \times \Omega). \end{aligned} \tag{294}$$

The approximate internal energy equation can be viewed as a quasilinear parabolic problem for the unknown θ . We will use the case where

$$\begin{aligned} e_\delta(\rho, \theta) &= \theta + \frac{\theta^4}{\rho} \\ e_M(\rho, \theta) &= \theta \\ p_M(\rho, \theta) &= \rho\theta + \rho^\gamma \end{aligned}$$

Comparison principle. A comparison principle in the class of strong (super, sub) solutions of problem (278) is established in [51]. We recall that a function θ is termed a super (sub) solution if it satisfies (278) with “=” sign replaced by “ \geq ” (“ \leq ”).

Lemma 18. *Given the quantities*

$$\mathbf{u} \in C([0, T]; X_n), \varrho \in C([0, T]; C^2(\bar{\Omega})), \tag{295}$$

$$\partial_t \varrho \in C([0, T] \times \bar{\Omega}), \inf_{(0, T) \times \Omega} \varrho > 0,$$

assume that θ_{sub} and θ_{sup} are respectively a sub and super-solution to problem (3.55–3.57) belonging to the class

$$\left\{ \begin{array}{l} \theta_{sub}, \theta_{sup} \in L^2(0, T; W^{1,2}(\Omega)), \partial_t \theta_{sub}, \partial_t \theta_{sup} \in L^2((0, T) \times \Omega), \\ \Delta \mathcal{K}_\delta(\theta_{sub}), \Delta \mathcal{K}_\delta(\theta_{sup}) \in L^2((0, T) \times \Omega), \end{array} \right\}, \tag{296}$$

$$\left\{ \begin{array}{l} 0 < \text{ess inf}_{(0, T) \times \Omega} \theta_{sub} \leq \text{ess sup}_{(0, T) \times \Omega} \theta_{sub} < \infty, \\ 0 < \text{ess inf}_{(0, T) \times \Omega} \theta_{sup} \leq \text{ess sup}_{(0, T) \times \Omega} \theta_{sup} < \infty, \end{array} \right\} \tag{297}$$

and satisfying

$$\theta_{sub}(0, \cdot) \leq \theta_{sup}(0, \cdot) \text{ a.e. in } \Omega. \quad (298)$$

Then

$$\theta_{sub}(t, x) \leq \theta_{sup}(t, x) \text{ a.e. in } (0, T) \times \Omega.$$

Proof. By taking the difference of θ_{sub} and θ_{sup} , we compute

$$\begin{aligned} & \text{sgn}^+ \left(\varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup}) \right) \left[\left(\partial_t \left(\varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup}) \right) \right. \right. \\ & \quad \left. \left. + \nabla_x \left(\varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup}) \right) \cdot \mathbf{u} \right) \right] \\ & \quad + \Delta_x \left(\mathcal{K}_\delta(\theta_{sup}) - \mathcal{K}_\delta(\theta_{sub}) \right) \text{sgn}^+ \left(\varrho e(\varrho, \theta_{sub}) - \varrho e(\varrho, \theta_{sup}) \right) \\ & \leq |F(t, x, \theta_{sub}) - F(t, x, \theta_{sup})| \text{sgn}^+ \left(\varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup}) \right), \end{aligned} \quad (299)$$

where we denote

$$\text{sgn}^+(z) = \begin{cases} 0 & \text{if } z \leq 0, \\ 1 & \text{if } z > 0, \end{cases}$$

and where we use the notation.

$$\begin{aligned} F(t, x, \theta) &= \mathbb{S}_\delta(\theta, \nabla_x \mathbf{u}(t, x)) : \nabla_x \mathbf{u}(t, x) + ((\varepsilon \delta (\beta \varrho^{\beta-2}) + \varepsilon \gamma \varrho^{\gamma-2}) |\nabla_x \varrho|^2)(t, x) \\ & \quad - \varrho(t, x) e_\delta(\varrho(t, x), \theta) \text{div}_x \mathbf{u}(t, x) \\ & \quad - p(\varrho(t, x), \theta) \text{div}_x \mathbf{u}(t, x) + \delta \frac{1}{\theta^2} - \varepsilon \theta^5. \end{aligned}$$

When ρ and \mathbf{u} are treated as fixed functions, $F = F(t, x, \theta)$ is Lipschitz with respect to θ .

Denoting by $|z|^+ = \max\{z, 0\}$ the positive part, we have

$$\partial_t |w|^+ = \text{sgn}^+(w) \partial_t w, \quad \nabla_x |w|^+ = \text{sgn}^+(w) \nabla_x w \text{ a.e. in } (0, T) \times \Omega$$

for any $w \in W^{1,2}((0, T) \times \Omega)$, in particular,

$$\begin{aligned} & \text{sgn}^+ \left(\varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup}) \right) \\ & \quad \times \left[\left(\partial_t \left(\varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup}) \right) \right) + \nabla_x \left(\varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup}) \right) \cdot \mathbf{u} \right] \quad (300) \\ & = \partial_t \left| \varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup}) \right|^+ + \nabla_x \left| \varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup}) \right|^+ \cdot \mathbf{u}. \end{aligned}$$

Moreover, observing e_δ and \mathcal{K}_δ are increasing functions of θ , we have

$$\operatorname{sgn}^+\left(\varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup})\right) = \operatorname{sgn}^+\left(\mathcal{K}_\delta(\theta_{sub}) - \mathcal{K}_\delta(\theta_{sup})\right).$$

Seeing that

$$\int_{\Omega} \Delta_x w \operatorname{sgn}^+(w) \, dx \leq 0 \text{ whenever } w \in W^{2,2}(\Omega), \nabla_x w \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

we integrate (300) to deduce

$$\begin{aligned} & \int_{\Omega} \left| \varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup}) \right|^+(\tau) \, dx \\ & \leq c \int_0^\tau \int_{\Omega} (1 + |\operatorname{div}_x \mathbf{u}|) \left| \varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup}) \right|^+ \, dx \, dt \end{aligned}$$

for any $\tau > 0$. Here we have used Lipschitz continuity of $F(t, x, \cdot)$ and the fact that $|\theta_{sub} - \theta_{sup}| \operatorname{sgn}^+[\varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup})] \leq c |\varrho e_\delta(\varrho, \theta_{sub}) - \varrho e_\delta(\varrho, \theta_{sup})|^+$. With the monotonicity of e_δ with respect to θ and Gronwall's lemma, we get the comparison principle. \square

Corollary 3.3.1. *For given data ϱ , \mathbf{u} satisfying (295), and a measurable function $\theta_{0,\delta}$ such that*

$$0 < \theta_{sub,0} = \operatorname{ess\,inf}_{\Omega} \theta_{0,\delta} \leq \operatorname{ess\,sup}_{\Omega} \theta_{0,\delta} = \theta_{sup,0} < \infty, \quad (301)$$

problem (278) admits at most one (strong) solution θ in the class specified in (296)–(297).

Another application of Lemma 18 gives rise to uniform bounds on the function θ in terms of the data.

Corollary 3.3.2. *Let ϱ , \mathbf{u} belong to the regularity class (295), and let $\theta_{0,\delta}$ satisfy (301). Suppose that θ is a (strong) solution of problem (278) belonging to the regularity class (296).*

Then there exist two constants θ_{sub} , θ_{sup} depending only on the quantities

$$\|\mathbf{u}\|_{C([0,T];X_n)}, \|\varrho\|_{C^1([0,T]\times\Omega)}, \quad (302)$$

satisfying

$$0 < \theta_{sub} \leq \theta_{sub,0} \leq \theta_{sup,0} \leq \theta_{sup}, \quad (303)$$

and

$$\theta_{sub} \leq \theta(t, x) \leq \theta_{sup} \text{ for a.e. } (t, x) \in (0, T) \times \Omega. \quad (304)$$

Proof. It is a routine matter to check that a constant function θ_{sub} is a subsolution of since

$$\begin{aligned} \frac{\delta}{\theta_{sub}^2} \geq & \left[\varepsilon \theta_{sub}^5 + p_M(\varrho, \theta_{sub}) \operatorname{div}_x \mathbf{u} + a \theta_{sub}^4 \operatorname{div}_x \mathbf{u} \right. \\ & + \varrho \frac{\partial e_M(\varrho, \theta_{sub})}{\partial \varrho} \left(\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho \right) + \left(e_M(\varrho, \theta_{sub}) + a \theta_{sub}^4 + \delta \theta_{sub} \right) \left(\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) \right) \\ & \left. - \mathbb{S}_\delta(\theta_{sub}, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - (\varepsilon \delta (\beta \varrho^{\beta-2}) + \varepsilon \gamma \varrho^{\gamma-2}) |\nabla_x \varrho|^2 \right]. \end{aligned} \quad (305)$$

Notice that all quantities on the right-hand side of (305) are bounded in terms of $\|\varrho\|_{C^1([0,T] \times \bar{\Omega})}$ and $\|\mathbf{u}\|_{C([0,T]; X_n)}$ provided, say that $0 < \theta_{sub} < 1$. Note that all norms are equivalent when restricted to the finite-dimensional space X_n .

Consequently, a direct application of the comparison principle established in Lemma 18 yields the left inequality in (304).

Following step by step, with obvious modifications, the above procedure, the upper bound claimed in (304) can be established by the help of the dominated term $-\varepsilon \theta^5$ in (278). \square

A priori estimates. We shall derive a priori estimates satisfied by any strong solution of problem (278).

Lemma 19. *Let the data ϱ, \mathbf{u} belong to the regularity class (295), and let $\theta_{0,\delta} \in W^{1,2}(\Omega)$ satisfy (301).*

Then any strong solution θ of problem (278) belonging to the class (296–297)

satisfies the estimate

$$\begin{aligned} & \operatorname{ess\,sup}_{t \in (0, T)} \|\theta\|_{W^{1,2}(\Omega)}^2 + \int_0^T \left(\|\partial_t \theta\|_{L^2(\Omega)}^2 + \|\Delta_x \mathcal{K}_\delta(\theta)\|_{L^2(\Omega)}^2 \right) dt \\ & \leq h \left(\|\varrho\|_{C^1([0, T] \times \bar{\Omega})}, \|\mathbf{u}\|_{C([0, T]; X_n)}, \left(\inf_{(0, T) \times \Omega} \varrho \right)^{-1}, \|\theta_{0, \delta}\|_{W^{1,2}(\Omega)} \right), \end{aligned} \quad (306)$$

where h is bounded on bounded sets.

Proof. Note that relation (306) represents the standard energy estimates for problem (278). These are easily deduced via multiplying equation (3.55) by θ and integrating the resulting expression by parts in order to obtain

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \varrho \frac{\partial e_\delta}{\partial \theta}(\varrho, \theta) \partial_t \theta^2 \, dx - \int_{\Omega} \varrho e_\delta(\varrho, \theta) \nabla_x \theta \cdot \mathbf{u} \, dx + \int_{\Omega} \kappa_\delta(\theta) |\nabla_x \theta|^2 \, dx \\ & = \int_{\Omega} F_1(t, x) \theta \, dx, \end{aligned} \quad (307)$$

where

$$\begin{aligned} F_1 = & - \frac{\partial(\varrho e_\delta)}{\partial \varrho}(\varrho, \theta) \partial_t \varrho + \mathbb{S}_\delta(\theta, \nabla_x u) : \nabla_x \mathbf{u} \\ & + (\varepsilon \delta (\beta \varrho^{\beta-2}) + \varepsilon \gamma \varrho^{\gamma-2}) |\nabla_x \varrho|^2 - p(\varrho, \theta) \operatorname{div}_x \mathbf{u} + \delta \frac{1}{\varrho^2} - \varepsilon \theta^5. \end{aligned}$$

In view of the uniform bounds already proved in (304), the function F_1 is bounded in $L^\infty((0, T) \times \Omega)$ in terms of the data.

Similarly, multiplying (278) by $\partial_t \mathcal{K}_\delta(\theta)$ gives rise to

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla_x \mathcal{K}_\delta(\theta)|^2 \, dx + \int_{\Omega} \varrho \kappa_\delta(\theta) \frac{\partial e_\delta}{\partial \theta}(\varrho, \theta) |\partial_t \theta|^2 \, dx \\ & + \int_{\Omega} \varrho \frac{\partial e_\delta}{\partial \theta}(\varrho, \theta) \partial_t \theta \nabla_x \mathcal{K}_\delta(\theta) \cdot \mathbf{u} \, dx = \int_{\Omega} F_2(t, x) \partial_t \theta \, dx \end{aligned}$$

where

$$\begin{aligned} F_2 = & - \kappa_\delta(\theta) \left(\partial_\varrho[\varrho e_\delta](\varrho, \theta) \partial_t \varrho - \partial_\varrho[\varrho e_\delta](\varrho, \theta) \nabla_x \varrho \cdot \mathbf{u} \right. \\ & \left. - \varrho e_\delta(\varrho, \theta) \operatorname{div}_x \mathbf{u} \right) + \mathbb{S}_\delta(\theta, \nabla_x u) : \nabla_x \mathbf{u} + (\varepsilon \delta (\beta \varrho^{\beta-2}) + \varepsilon \gamma \varrho^{\gamma-2}) |\nabla_x \varrho|^2 \\ & - p(\varrho, \theta) \operatorname{div}_x \mathbf{u} + \delta \frac{1}{\varrho^2} - \varepsilon \theta^5 \end{aligned}$$

is bounded in $L^\infty((0, T) \times \Omega)$ in terms of the data.

Taking the sum of (307), (308), and using Young's inequality and Gronwall's lemma, we conclude that

$$\begin{aligned} & \operatorname{ess\,sup}_{t \in (0, T)} \|\nabla_x \mathcal{K}_\delta(\theta)\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \int_0^T \|\partial_t \theta\|_{L^2(\Omega)}^2 \, dt \\ & \leq h \left(\|\varrho\|_{C^1([0, T] \times \bar{\Omega})}, \|\mathbf{u}\|_{C([0, T; X_n])}, \left(\inf_{(0, T) \times \Omega} \varrho \right)^{-1}, \|\theta_0\|_{W^{1,2}(\Omega)} \right). \end{aligned}$$

Finally, evaluating $\Delta_x \mathcal{K}_\delta(\theta)$ by internal energy equation, we get (306). \square

Existence for the approximate internal energy equation. One possible way to construct the approximations to problem is described in [51].

In fact, the a priori bounds (304), (306) imply compactness of solutions in the space $C([0, T]; W^{1,2}(\Omega))$, in particular, any accumulation point of a family of strong solutions is another solution of the same problem. Under these circumstances, showing existence is a routine matter. Regularizing the data ϱ, \mathbf{u} with respect to the time variable, and approximating the quantities $\mu, \eta, \kappa_\delta, e, p$ by smooth ones as the case may be, we can construct a family of approximate solutions to problem (278) via the classical results for quasilinear parabolic equations. Then we pass to the limit in a suitable sequence of approximate solutions to recover the (unique) solution.

1. Let $\nu \in (0, 1)$ be the same parameter as in the Lemma 16 for approximate equation. To begin, we extend $\varrho \in C([0, T]; C^{2,\nu}(\bar{\Omega})) \cap C^1([0, T]; C^{0,\nu}(\bar{\Omega}))$, $\mathbf{u} \in C([0, T]; X_n)$, continuously to $\varrho \in C(\mathbb{R}; C^{2,\nu}(\bar{\Omega})) \cap C^1(\mathbb{R}; C^{0,\nu}(\bar{\Omega}))$, $\operatorname{supp} \varrho \subset (-2T, 2T) \times \bar{\Omega}$, $\mathbf{u} \in C(\mathbb{R}, X_n)$, $\operatorname{supp} \mathbf{u} \subset (-2T, 2T) \times \bar{\Omega}$. We approximate \mathcal{Q}_δ by smooth functions \mathcal{Q}_ω on $[0, T] \times \bar{\Omega}$ and we take a sequence of initial conditions

$$C^{2,\nu}(\bar{\Omega}) \ni \theta_{0,\omega} \rightarrow \theta_{0,\delta} \text{ in } W^{1,2}(\Omega) \cap L^\infty(\Omega)$$

such that $\inf_{x \in \Omega} \theta_{0,\omega}(x) > \underline{\theta}_0$ uniformly with respect to $\omega \rightarrow 0+$, where $\theta_{sub,0}$ is a positive constant.

2. We denote

$$E_M(\varrho, \theta) = \varrho e_M(\varrho, \theta)$$

and set

$$\begin{aligned} E_{\delta, \omega}(\varrho, \theta) &= [\langle E_M \rangle]^\omega(\varrho, \theta_\omega) + a\theta_\omega^4 + \delta\varrho\theta, \\ \{\partial_\theta E\}_{\delta, \omega}(\varrho, \theta) &= [\langle \partial_\theta E_M \rangle]^\omega(\varrho, \theta) + 4a \frac{\theta^4}{\sqrt{\theta^2 + \omega^2}} + \delta\varrho, \\ \kappa_{\delta, \omega}(\theta) &= [\langle \kappa_M \rangle]^\omega(\theta_\omega) + [\langle \kappa_R \rangle]^\omega(\theta_\omega) + \delta \left(\theta_\omega^\beta + \frac{1}{\sqrt{\theta^2 + \omega^2}} \right), \\ \mathcal{K}_{\delta, \omega}(\theta) &= \int_1^\theta \kappa_{\delta, \omega}(\tau) \, d\tau, \\ p_\omega(\varrho, \theta) &= [\langle p_M \rangle]^\omega(\varrho, \theta_\omega) + \frac{a}{3}\theta_\omega^4, \\ G(t, x) &= \left((\beta\varrho^{\beta-2}) |\nabla_x \varrho|^2 \right)(t, x), \quad G_\omega(t, x) = G^\omega(t, x), \\ \mathbb{S}_{\delta, \omega}(\theta, \nabla_x \mathbf{u}^\omega) &= \langle \mu \rangle^\omega(\theta_\omega) \left(\nabla_x \mathbf{u}^\omega + \nabla_x^T \mathbf{u}^\omega - \frac{2}{3} \operatorname{div}_x \mathbf{u}^\omega \mathbb{I} \right) + \langle \eta \rangle^\omega(\theta_\omega) \operatorname{div}_x \mathbf{u}^\omega \mathbb{I}, \end{aligned} \tag{308}$$

where

$$\begin{aligned} \theta_\omega = \theta_\omega(\theta) &= \frac{\sqrt{\theta^2 + \omega^2}}{1 + \omega\sqrt{\theta^2 + \omega^2}}, \\ \langle a \rangle(z) &= \begin{cases} a(z) & \text{if } z \in (0, \infty)^3 \\ \inf_{z \in (0, \infty)^3} a(z) & \text{otherwise} \end{cases} \end{aligned}$$

The operator $b \mapsto b^\omega$, $\omega > 0$ is the standard regularizing operator, that applies to all independent variables in the case of functions $\langle E_M \rangle$, $\langle \partial_\theta E_M \rangle$, $\langle p \rangle$, $\langle \mu \rangle$, $\langle \eta \rangle$, $\langle \kappa_M \rangle$, and to the variable t in the case of functions $\varrho(t, x)$, $\mathbf{u}(t, x)$, $G(t, x)$. Notice that by the constitution relations,

$$\kappa_{\delta, \omega}(\theta) \geq \underline{\kappa}_M > 0, \quad \{\partial_\theta E\}_{\delta, \omega}(\varrho, \theta) > \delta \underline{\varrho} > 0 \tag{309}$$

for all $(\varrho, \theta) \in \mathbb{R}^2$, where $\underline{\varrho} = \inf_{(0, T) \times \Omega} \varrho$.

3. We will find a solution of problem (278), as a limit of the sequence $\{\theta_\omega\}_{\omega>0}$ of solutions to the equation

$$\begin{aligned}
& \{\partial_\theta E\}_{\delta,\omega}(\varrho^\omega, \theta) \partial_t \theta + \operatorname{div}_x \left(E_{\delta,\omega}(\varrho^\omega, \theta) \mathbf{u} \right) - \Delta_x \mathcal{K}_{\delta,\omega}(\theta) \\
& = -\partial_\varrho E_{\delta,\omega}(\varrho^\omega, \theta) \partial_t \varrho^\omega + \mathbb{S}_{\delta,\omega}(\nabla_x \mathbf{u}^\omega, \theta) : \nabla_x \mathbf{u}^\omega \\
& \quad + \varepsilon \delta G_\omega - p_\omega(\varrho^\omega, \theta) - \frac{\delta}{\theta^2 + \omega^2} + \varepsilon \theta_\omega^5, \\
& \nabla_x \theta \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \theta(0, x) = \theta_{0,\omega}(x).
\end{aligned} \tag{310}$$

In the approximate equation, θ satisfies the following quasilinear parabolic equation:

$$\begin{aligned}
\partial_t \theta - \sum_{i,j=1}^3 a_{ij}(t, x, \theta) \partial_{x_i} \partial_{x_j} \theta + b(t, x, \theta, \nabla_x \theta) &= 0 \quad \text{in } (0, T) \times \Omega, \\
\left(\sum_{i,j=1}^3 a_{ij} \partial_{x_j} \theta n_i + \psi \right) \Big|_{(0,T) \times \partial\Omega} &= 0, \\
\theta|_{\{0\} \times \Omega} &= \theta_{0,\omega}(x),
\end{aligned} \tag{311}$$

where

$$a_{ij}(t, x, \theta) = \frac{\kappa_{\delta,\omega}(\theta)}{[\partial_\theta E]_{\delta,\omega}(\varrho^\omega(t, x), \theta)} \delta_{ij}, \quad i, j = 1, 2, 3, \quad \psi = 0 \tag{312}$$

and

$$\begin{aligned}
b(t, x, \theta, \mathbf{z}) &= \frac{1}{\{\partial_\theta E\}_{\delta,\omega}(\varrho^\omega(t, x), \theta)} \\
& \times \left[-\kappa'_{\delta,\omega}(\theta) |\mathbf{z}|^2 + \partial_\varrho E_{\delta,\omega}(\varrho^\omega(t, x), \theta) \partial_t \varrho^\omega(t, x) \right. \\
& \quad + \partial_\varrho E_{\delta,\omega}(\varrho^\omega(t, x), \theta) (\nabla_x \varrho^\omega \cdot \mathbf{u}^\omega)(t, x) \\
& \quad - \mathbb{S}_{\delta,\omega}(\nabla_x \mathbf{u}^\omega(t, x), \theta) : \nabla_x \mathbf{u}^\omega(t, x) \\
& \quad + \partial_\theta E_{\delta,\omega}(\varrho^\omega(t, x), \theta) (\mathbf{z} \cdot \mathbf{u}^\omega)(t, x) \\
& \quad + E_{\delta,\omega}(\varrho^\omega(t, x), \theta) \operatorname{div}_x \mathbf{u}^\omega + p_\omega(\varrho^\omega(t, x), \theta) \operatorname{div}_x \mathbf{u}^\omega(t, x) \\
& \quad \left. - \varepsilon \delta G_\omega(t, x) + \frac{\delta}{\theta^2 + \omega^2} - \varepsilon \theta_\omega^5(\theta) \right].
\end{aligned} \tag{313}$$

In accordance with the properties of mollifiers and the theory of general parabolic equations, problem (310) admits a (unique) solution $\theta = \theta_\omega$ which belongs to class

$$\theta_\omega \in C([0, T]; C^{2,\nu}(\bar{\Omega})) \cap C^1([0, T] \times \bar{\Omega}), \quad \partial_t \theta_\omega \in C^{0,\nu/2}([0, T]; C(\bar{\Omega})).$$

3.3.3 On the approximate momentum equation

Then we can consider the approximate momentum equations in the following ways,

$$\begin{aligned} \mathbf{u}(t) &= J \left[\rho(\tau), \int_0^\tau M(t, \rho(t), \theta(t), \mathbf{u}(t)) dt + (\rho \mathbf{u})_0^* \right] \\ &= S[\mathbf{u}](\tau), \quad \tau \in [0, T], \end{aligned} \quad (314)$$

where

$$\begin{aligned} (\rho \mathbf{u})_0^* &\in X_0^*, \\ \langle (\rho \mathbf{u})_0^*, \varphi \rangle &= \int_\Omega (\rho \mathbf{u})_0 \cdot \varphi dx \quad \text{for all } \varphi \in X_n, \\ M(t, \rho, \theta, \mathbf{u}) &\in X_n^*, \end{aligned} \quad (315)$$

$$\begin{aligned} &\langle M(t, \rho, \theta, \mathbf{u}) : \varphi \rangle \\ &= \int_\Omega \left(\rho[\mathbf{u} \otimes \mathbf{u}] : \nabla_x \varphi + \left(\rho\theta + \frac{1}{3}\theta^4 + \delta(\rho^\beta + \rho^2) + \rho^r \right) \text{div}_x \varphi \right) dx \\ &\quad - \int_\Omega (\epsilon \nabla_x \rho \cdot \nabla_x \mathbf{u}) \cdot \varphi + \mathbb{S}_\delta : \nabla_x \varphi dx \quad \text{for all } \varphi \in x_n^* \end{aligned} \quad (316)$$

and

$$\begin{aligned} J[\rho, \cdot] : X_n^* &\rightarrow X_n^*, \quad \text{such that} \\ \int_\Omega \rho J[\rho, \xi] \cdot \varphi dx &= \langle \xi : \varphi \rangle, \quad \text{for all } \xi \in X_n^*, \varphi \in X_n. \end{aligned} \quad (317)$$

If $\|\mathbf{u}\|_{C([0,T], X_n)} \leq R$, then

$$\begin{aligned} &\left\| J \left[\rho(\tau), \int_0^\tau M(t, \rho(t), \theta(t), \mathbf{u}(t)) dt + (\rho \mathbf{u})_0^* \right] \right\|_{x_n} \\ &\leq C \frac{\bar{\rho}_0}{\underline{\rho}_0} \exp(2R\tau) \|\mathbf{u}_{0,\delta,n}\|_{x_n} + \tau h(R) \\ &\leq R, \end{aligned} \quad (318)$$

when $R \geq 2c_0 \frac{\bar{\rho}_0}{\rho_0} \|\mathbf{u}_{0,\delta,n}\|_{x_n}$ and τ is small enough.

Now we know that the approximate system has a solution when $t \in [0, T_{\max}]$. We still need to show $T_{\max} = T$ by uniform estimates for $t \in [0, T_{\max}]$. First we have the conservation of mass

$$\int_{\Omega} \rho(t) dx = \int_{\Omega} \rho_{0,\delta} dx. \quad (319)$$

Second we have the approximate conservation of energy

$$\begin{aligned} & \int_{\Omega} \left(\rho\theta + \theta^4 + \frac{1}{2}\rho|\mathbf{u}|^2 + \delta \left(\frac{\rho^\beta}{\beta-1} + \rho^2 \right) + \frac{\rho^r}{r-1} \right) dx \Big|_0^t \\ &= \int_0^t \int_{\Omega} \frac{\delta}{\theta^2} - \epsilon\theta^5 dx d\tau. \end{aligned} \quad (320)$$

Multiplying $\frac{1}{\theta}$ in the approximate energy equation and use

$$\begin{aligned} & (\rho \log \rho)_t + \operatorname{div}_x(\rho \log \mathbf{u}) = -\rho \operatorname{div}_x \mathbf{u} + \epsilon \delta \rho + \epsilon \delta \rho \cdot \ln \rho \\ & (\rho \log \theta)_t + \operatorname{div}_x(\rho \log \theta \mathbf{u}) = \frac{1}{\theta} ((\rho \theta)_t + \operatorname{div}_x(\rho \theta \mathbf{u})) + \epsilon \delta \rho + \epsilon \delta \rho \log \theta \\ & \theta \left(\frac{4}{3} \theta^3 \right)_t + \operatorname{div}_x(\mathbf{u} \cdot \frac{4}{3} \theta^3) = (\theta^4)_t + \operatorname{div}_x(\mathbf{u} \cdot \theta^4) + \frac{1}{3} \theta^4 \operatorname{div}_x \mathbf{u}. \end{aligned} \quad (321)$$

We have

$$\begin{aligned} & \frac{d}{dt} \left(\rho(\log \theta - \log \rho) + \frac{4}{3} \theta^3 \right) + \operatorname{div}_x \left(\rho \mathbf{u}(\log \theta - \log \rho) + \frac{4}{3} \frac{\theta^3}{\rho} \right) \\ &= \operatorname{div}_x(k \nabla_x \theta) \cdot \frac{1}{\theta} + \frac{S_\delta \cdot \nabla_x \mathbf{u}}{\delta} + \frac{1}{\theta} \left(\epsilon \delta (\beta \rho^{\beta-2} + 2) |\nabla_x \rho|^2 \right. \\ & \quad \left. + \epsilon r \rho^{r-2} |\nabla_x \rho|^2 + \frac{\delta}{\theta^2} - \epsilon \theta^5 \right) + \epsilon \delta \rho (\log \theta - \log \rho). \end{aligned} \quad (322)$$

Considering the Helmholtz free energy

$$H_{\bar{\theta}}(\rho_n, \theta_n) = \rho_n e(\rho_n, \theta_n) - \bar{\theta} s(\rho_n, \theta_n), \quad (323)$$

we have

$$\begin{aligned}
& \int_{\Omega} \frac{1}{2} \rho |\mathbf{u}|^2 + H_{\theta}(\rho_n, \theta_n) + \delta \left(\frac{\rho^{\beta}}{\beta - 1} + \rho^2 \right) + \frac{\rho^r}{r - 1}(\tau) dx \\
& \quad + \int_0^T \int_{\Omega} \sigma_{n,\delta} dx dt + \int_0^T \int_{\Omega} \epsilon \theta^5 dx dt \\
& = \int_{\Omega} \frac{1}{2} \rho |\mathbf{u}|^2 + H_{\bar{\theta}}(\rho_n, \theta_n) + \delta \left(\frac{\rho^{\beta}}{\beta - 1} + \rho^2 \right) + \frac{\rho^r}{r - 1}(0) dx \\
& \quad + \int_0^T \int_{\Omega} \frac{\delta}{\theta^2} + \epsilon \bar{\theta} \theta^4 dx dt + \epsilon \int_0^T \int_{\Omega} \frac{\bar{\theta}}{\theta} \nabla_x \rho \cdot \nabla_x \theta dx dt.
\end{aligned} \tag{324}$$

3.3.4 Coercivity of $H_{\bar{\theta}}(\rho, \theta)$

Lemma 20.

$$H_{\bar{\theta}}(\rho, \theta) \geq \frac{1}{4}(\rho\theta + \bar{\theta}\rho|s|) - \left| \left(\rho - \bar{\rho} \frac{\partial H_{2\bar{\theta}}(\bar{\rho}, 2\bar{\theta})}{\partial \rho} \right) + H_{2\bar{\theta}}(\bar{\rho}, 2\bar{\theta}) \right|. \tag{325}$$

Proof. If $s \leq 0$, the result holds. If $s > 0$

$$H_{2\bar{\theta}}(\rho, \theta) \geq (\rho - \bar{\rho}) \frac{\partial H_{2\bar{\theta}}}{\partial \rho}(\bar{\rho}, 2\bar{\theta}) + H_{2\bar{\theta}}(\bar{\rho}, 2\bar{\theta}), \tag{326}$$

where

$$\begin{aligned}
H_{\bar{\theta}}(\rho, \theta) &= \frac{1}{2} \rho e(\rho, \theta) + \frac{1}{2} H_{2\theta}(\rho, \theta) \\
&\geq \frac{1}{2} \rho e(\rho, \theta) + \frac{1}{2} \left[(\rho - \bar{\rho}) \frac{\partial H_{2\bar{\theta}}}{\partial \rho}(\bar{\rho}, 2\bar{\theta}) + H_{2\bar{\theta}}(\bar{\rho}, 2\bar{\theta}) \right].
\end{aligned} \tag{327}$$

Similarly

$$\begin{aligned}
H_{\bar{\theta}}(\rho, \theta) &= \bar{\theta} \rho s(\rho, \theta) + H_{2\theta}(\rho, \theta) \\
&\geq \bar{\theta} \rho s(\rho, \theta) + (\rho - \bar{\rho}) \frac{\partial H_{2\bar{\theta}}}{\partial \rho}(\bar{\rho}, 2\bar{\theta}) + H_{2\bar{\theta}}(\bar{\rho}, 2\bar{\theta}).
\end{aligned} \tag{328}$$

Summing up the two inequality, we get the conclusion. \square

So we have a bound on u_n independent of the T_{max} . And it gives the existence of a solution for the approximation system in the following Lemma 21.

Lemma 21 (Feireisl). *(Existence of solution of the approximation system). If the initial data is in the class given in subsection 3.2.0.1, and β is large enough, then the*

approximation system has a solution such that

$$\begin{aligned}
\rho &\in C([0, T], C^{2+\nu}(\Omega)), \quad \partial_t \rho \in C([0, T], C^\nu(\Omega)), \\
\inf_{[0, T] \times \Omega} \rho &> 0, \\
\mathbf{u} &\in C^1([0, T], X_n), \\
\theta &\in C([0, T], W^{1,2}(\Omega) \cap L^\infty([0, T] \times \Omega)) \\
\partial_t \theta, \Delta \theta &\in L^2([0, T] \times \Omega), \quad \inf_{[0, T] \times \Omega} \theta > 0.
\end{aligned} \tag{329}$$

3.4 $n \rightarrow \infty$

In this section, we will let $n \rightarrow \infty$ to get an approximate solution in a new level. Particularly, we will keep artificial viscosity and artificial pressure. We will have the limiting entropy inequality and admissible conditions as $n \rightarrow \infty$.

3.4.1 Uniformly estimates with respect to n

We recall that approximate Helmholtz free energy defined in(323)

$$H_{\bar{\theta}}(\rho_n, \theta_n) = \rho_n(e(\rho_n, \theta_n)) - \bar{\theta}s(\rho_n, \theta_n). \tag{330}$$

From the balanced laws, we have

$$\begin{aligned}
&\int_{\Omega} \frac{1}{2} \rho |\mathbf{u}_n|^2 + \frac{\rho_n^\gamma}{\gamma - 1} + \delta \frac{\rho_n^\beta}{\beta - 1} + \rho e_n(\tau, x) + H_{\bar{\theta}}(\rho_n, \theta_n) dx \Big|_0^t \\
&+ \bar{\theta} \int_0^t \int_{\Omega} \frac{(\mathbb{S}(\nabla_x \mathbf{u}_n, \theta_n) : \nabla_x \mathbf{u}_n)}{\theta_n^2} + \frac{\kappa_n |\nabla_x \theta_n|^2}{\theta_n^2} dx d\tau \\
&+ \bar{\theta} \int_0^t \int_{\Omega} \frac{\epsilon \delta}{\theta} (\beta \rho^{\beta-2} |\nabla_x \rho|^2) + \frac{\epsilon}{\theta} (\gamma \rho^{\gamma-2} |\nabla_x \rho|^2) dx d\tau \\
&+ \int_0^t \int_{\Omega} \frac{\delta \bar{\theta}}{\theta^3} + \epsilon \theta^5 + \epsilon \bar{\theta} \frac{|\nabla_x|^2}{\rho} dx d\tau \\
&= \int_0^t \int_{\Omega} \epsilon \bar{\theta} \theta^4 + \delta \frac{1}{\theta^2} dx d\tau.
\end{aligned} \tag{331}$$

We have the following estimates independent of n for the two terms on the right-hand side,

$$\int_0^t \int_{\Omega} \epsilon \bar{\theta} \theta^4 dx d\tau < C \int_0^t \int_{\Omega} 1 dx d\tau + \frac{1}{16} \epsilon \int_0^t \int_{\Omega} \bar{\theta} \theta^5 dx d\tau, \quad (332)$$

$$\int_0^t \int_{\Omega} \delta \frac{1}{\theta^2} dx d\tau < \frac{1}{16} \int_0^t \int_{\Omega} \delta \frac{\bar{\theta}}{\theta^3} dx d\tau + C \int_0^t \int_{\Omega} 1 dx d\tau. \quad (333)$$

So we have the following sets of estimates, with the help of Lemma 20

$$\|\theta\|_{L^\infty L^4} \leq C, \quad (334)$$

$$\sup \int_{\Omega} \rho |\mathbf{u}|^2 dx \leq C, \quad (335)$$

$$\|\rho\|_{L^\infty L^\gamma}, \|\rho\|_{L^\infty L^\beta} \leq C, \quad (336)$$

$$\int_0^T \int_{\Omega} \frac{\mathbb{S}(\nabla_x \mathbf{u}_n, \theta_n) : \nabla_x \mathbf{u}_n}{\theta_n} dx d\tau \leq C, \quad (337)$$

$$\int_0^T \int_{\Omega} \frac{\kappa_n |\nabla_x \theta_n|^2}{\theta_n^2} dx d\tau \leq C, \quad (338)$$

$$\bar{\theta} \int_0^t \int_{\Omega} \frac{\epsilon \delta}{\theta} (\beta \rho^{\beta-2} |\nabla_x \rho|^2) + \frac{\epsilon}{\theta} (\gamma \rho^{\gamma-2} |\nabla_x \rho|^2) dx d\tau \leq C, \quad (339)$$

$$\int_0^t \int_{\Omega} \frac{\delta \bar{\theta}}{\theta^3} dx d\tau \leq C, \quad (340)$$

$$\epsilon \int_0^t \int_{\Omega} \theta^5 dx d\tau \leq C, \quad (341)$$

$$\epsilon \bar{\theta} \int_0^t \int_{\Omega} \frac{|\nabla_x \rho|^2}{\rho} dx d\tau \leq C. \quad (342)$$

3.4.2 Weak limits of states variables

With the estimates in the previous section, we have at least weak limit of the state variables $\rho_n, \mathbf{u}_n, \theta_n$, etc. In fact

$$\{\rho_n\} \rightharpoonup \rho \text{ weakly* in } L^\infty L^\beta, \quad (343)$$

as $\{\rho_n\}$ is bounded in $L^\infty L^\beta$.

$$\{\mathbf{u}_n\} \rightharpoonup \mathbf{u} \text{ weakly in } L^2 W^{1,2}, \quad (344)$$

as $\{\mathbf{u}_n\}$ is bounded in $L^2W^{1,2}$.

By Lemma 13 and $\rho_t \in L^2L^2$, we can get

$$\{\rho_n\} \rightarrow \rho \text{ in } C_{weak}([0, T], L^\beta(\Omega)). \quad (345)$$

Similarly, we have

$$\begin{aligned} \{\theta_n\} &\rightharpoonup \theta \text{ weakly}^* \text{ in } L^\infty L^4, \\ \{\theta_n\} &\rightharpoonup \theta \text{ weakly in } L^2W^{1,2}. \end{aligned} \quad (346)$$

By the convexity, we have

$$\int_0^T \int_\Omega \frac{1}{\theta^3} dx dt \leq \limsup_{n \rightarrow \infty} \int_0^T \int_\Omega \frac{1}{\theta_n^3} dx dt. \quad (347)$$

3.4.3 Continuity equation

We also have the following bound from the free energy estimates,

$$\begin{aligned} \{\rho_n^\beta\} &\text{ bounded in } L^2W^{1,2}, \\ \{\rho_n\} &\text{ bounded in } L^2L^{3\beta}. \end{aligned} \quad (348)$$

Considering the continuity equation in the following form

$$(\partial_t - \epsilon \Delta)[\rho_n] = -\nabla_x \rho_n \cdot \mathbf{u}_n - \rho_n \operatorname{div} \mathbf{u}_n. \quad (349)$$

As long as we can show the right hand is in some L^pL^q for some $p, q > 1$, then we can employ the theory of maximal regularity for parabolic equation on this approximate continuity equation. Since $\|\rho_n \operatorname{div}_x u_n\|_{L^2L^2} \leq C$, we just need to control the other term, Choosing $G = \rho_n \log \rho_n$ in the renormalized equation, we have

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} G(\rho_n) + \epsilon \int_{\Omega} G''(\rho_n) |\nabla_x \rho_n| &= \int_{\Omega} (G(\rho_n) - G'(\rho_n) \rho_n) \operatorname{div}_x \mathbf{u}_n \\
&= \int_{\Omega} \rho_n \operatorname{div}_x \mathbf{u}_n \\
&= - \int_{\Omega} \nabla_x \rho_n \cdot \mathbf{u}_n \\
&\leq \frac{\epsilon}{2} \int_{\Omega} \frac{|\nabla_x \rho|^2}{\rho_n} + C(\epsilon) \int_{\Omega} \rho_n |\mathbf{u}_n|^2
\end{aligned} \tag{350}$$

Then we have

$$\epsilon \int_{\Omega} \frac{|\nabla_x \rho|^2}{\rho_n} \leq C, \tag{351}$$

$$\|\nabla_x \rho \cdot \mathbf{u}_n\|_{L^2 L^1} \leq \left\| \frac{\nabla_x \rho_n}{\sqrt{\rho_n}} \right\|_{L^2 L^2} \|\sqrt{\rho_n} \mathbf{u}_n\|_{L^2 L^2} \leq C(\epsilon). \tag{352}$$

On the other hand if

$$\frac{1}{p_1} = \frac{1}{2} + \frac{1}{r_1} = \frac{1}{2} + \frac{1}{2\beta} + \frac{1}{6}, \tag{353}$$

then

$$\begin{aligned}
\left\| \nabla_x \rho_n \mathbf{u}_n \right\|_{L^1 L^{p_1}} &\leq \left\| \frac{\nabla_x \rho_n}{\sqrt{\rho_n}} \right\|_{L^2, L^2} \|\sqrt{\rho_n} \mathbf{u}_n\|_{L^2, L^{r_1}} \\
&\leq \left\| \frac{\nabla_x \rho_n}{\sqrt{\rho_n}} \right\|_{L^2 L^2} \|\sqrt{\rho_n}\|_{L^2 L^2} \|\mathbf{u}_n\|_{L^2 L^6} \\
&\leq \left\| \frac{\nabla_x \rho_n}{\sqrt{\rho_n}} \right\|_{L^2 L^2} \|\sqrt{\rho_n}\|_{L^2 L^2} \|\nabla_x \mathbf{u}_n\|_{L^2 L^2} \\
&\leq C.
\end{aligned} \tag{354}$$

Then by the theory of maximal regularity of heat equation, we have $\partial_t \rho, \partial_i \partial_j \rho \in L^{p_2, q_2}$, where $p_2, q_2 > 1$, then we can get the strong convergence of ρ_n . The strong convergence of $\nabla_x \rho$ in $L^2 L^2$ is via the following two relations:

$$\int_{\Omega} \rho_n^2(\tau) dx + 2\epsilon \int_0^\tau |\nabla_x \rho_n|^2 dx dt \rightarrow \int_{\Omega} \rho_{0,\delta}^2 dx - \epsilon \int_0^\tau \rho^2 \operatorname{div}_x \mathbf{u} dx dt, \tag{355}$$

$$\int_{\Omega} \rho_{0,\delta}^2(\tau) dx + \epsilon \int_0^\tau \rho^2 \operatorname{div}_x \mathbf{u} dx dt = \int_{\Omega} \rho_{0,\delta}^2 dx - 2\epsilon \int_0^\tau |\nabla_x \rho_n|^2 dx dt. \tag{356}$$

3.4.4 Strong Convergence of temperature and internal energy equation

The basic idea to get the strong convergence of temperature is by the so called Div-Curl lemma (Lemma12). We can define the auxiliary functions as following

$$\begin{aligned} U_n &= [\rho_n s(\rho_n, \theta_n), r_n^{(1)}], \\ V_n &= [\theta_n, 0, 0, 0]. \end{aligned} \quad (357)$$

The approximate entropy equation could be rewritten as

$$\operatorname{div}_{t,x} U_n = r_n^{(2)} + r_n^{(3)}, \quad (358)$$

where

$$r_n^{(1)} = \rho_n s_\delta(\rho_\delta, \theta_n) \mathbf{u}_n - \frac{\kappa(\theta_n)}{\theta_n} \nabla_x \theta_n - \epsilon (\log \theta_n - \log \rho_n - 2) \nabla_x \rho_n, \quad (359)$$

$$\begin{aligned} r_n^{(2)} &= \frac{1}{\theta_n} [S_\delta(\theta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n + (\frac{\kappa(\theta_n)}{\theta_n} + \delta(\theta^{\beta-1} + \frac{1}{\theta_n^2}))] \\ &\quad (\gamma \rho_n^{\gamma-2} + 2) \frac{|\nabla_x \rho_n|^2}{\theta_n} + \epsilon \delta (\beta \rho_n^{\beta-2} + 2) \frac{|\nabla_x \rho_n|^2}{\theta_n} + \epsilon \frac{|\nabla_x \rho_n|^2}{\rho_n}, \end{aligned} \quad (360)$$

$$r_n^{(3)} = -\epsilon \frac{\nabla_x \rho_n \nabla_x \theta_n}{\theta_n} - \epsilon \theta_n^4. \quad (361)$$

Then we need to verify the conditions in Lemma 12,

$$\rho |s_\delta(\rho_n, \theta_n)| \leq C(\rho_n + \theta_n^3 + \rho_n |\log \rho_n| + \rho_n |\log \theta_n|). \quad (362)$$

So we have $\rho |s_\delta(\rho_n, \theta_n)|$ is bounded in $L^{\beta/3}((0, T) \times \Omega)$, in addition $\rho |s_\delta(\rho_n, \theta_n) \mathbf{u}_n|$ is bounded in $L^{p_2}((0, T) \times \Omega)$, where $1/p_2 = 1/2 + 3/\beta$,

$$\frac{\sqrt{\kappa_\delta(\theta_n)} \nabla_x \theta_n}{\theta_n} \text{ is bounded in } L^2(0, T, R^3). \quad (363)$$

The term with ϵ could be estimated as following. For p_1 is a constant slight bigger than 1:

$$\begin{aligned}
& \|\epsilon(\log \theta_n - \log \rho_n - 2)\nabla_x \rho_n\|_{L^{p_1}} \\
& \leq \int_0^T \int_{\Omega} \frac{\epsilon |\nabla_x \rho_n|^2}{\rho_n} dx dt \\
& + C \int_0^T \int_{\Omega} \rho_n^{\frac{1}{2-p_1}} (|\log \theta_n|^{\frac{2}{2-p_1}} - |\log \rho_n|^{\frac{2}{2-p_1}} + C) dx dt.
\end{aligned} \tag{364}$$

The Div-Curl lemma tells us

$$\overline{\rho s} = \rho \bar{s}. \tag{365}$$

Second, the strong convergence of θ will come from the following relations, as the terms in $\overline{\rho s_{\delta}(\rho, \theta)} = \overline{\rho s_{\delta}(\rho, \theta)} \theta$ are monotone increasing with respect to θ , so

$$\overline{\rho \log(\theta) \theta} \geq \overline{\rho \log(\theta) \theta} ; \quad \bar{\theta}^4 \geq \bar{\theta}^3 \theta. \tag{366}$$

So the inequality should be equality in equation (366), then

$$\bar{\theta}^4 = \bar{\theta}^3 \theta, \tag{367}$$

which gives the strong convergence of θ in $L^{\infty}(Q)$. We can then get the equation of total internal energy.

3.4.5 Limit of the approximate entropy equation

With the strong convergence of temperature and density, we can obtain the approximate entropy inequality in this section. Consider the following relation

$$\begin{aligned}
& \partial_t(\rho_n s_n(\rho_n, \theta_n)) + \operatorname{div}_x \left(\rho_n s_n(\rho_n, \theta_n) \mathbf{u}_n - \frac{k_{\delta}}{\theta_n} \nabla_x \theta_n \right) \\
& - \epsilon \operatorname{div}_x [[\log \theta - \log \rho_n] \cdot \nabla_x \rho_n] \\
& = \frac{1}{\theta_n} \left[S_{\delta}(\theta_n, \nabla_x \mathbf{u}_n) \cdot \nabla_x \mathbf{u}_n + \left[\frac{\kappa(\theta_n)}{\theta_n} + \frac{\delta}{2} \left(\theta^{\beta-1} + \frac{1}{\theta_n^2} \right) \right] |\nabla_x \theta_n|^2 + \delta \frac{1}{\theta_n^2} \right] \\
& + \frac{\epsilon \delta}{2\theta_n} (\beta \rho_n^{\beta-2} + 2) |\nabla_x \rho_n|^2 + \frac{\epsilon}{2\theta_n} \gamma \rho_n^{\gamma-2} |\nabla_x \rho_n|^2 + \epsilon |\nabla_x \rho_n|^2 - \epsilon \theta_n^4.
\end{aligned} \tag{368}$$

By the strong convergence of θ_n and ρ_n , we have

$$\rho_n s_n(\rho_n, \theta_n) \rightarrow \rho s(\rho, \theta) \quad \text{strongly in } L^2 L^2. \quad (369)$$

Then

$$\rho_n s_n \mathbf{u}_n \rightarrow \rho s \mathbf{u} \quad \text{weakly in } L^1 L^1, \quad (370)$$

$$\frac{\kappa(\theta_n)}{\theta_n} \rightarrow \frac{\kappa(\theta)}{\theta} \quad \text{strongly in } L^2 L^2, \quad (371)$$

$$\frac{\kappa(\theta_n)}{\theta_n} \nabla_x \theta_n \rightarrow \frac{\kappa(\theta)}{\theta} \nabla_x \theta \quad \text{weakly in } L^1 L^1, \quad (372)$$

$$\left(\theta_n^{\beta-1} + \frac{1}{\theta_n^2} \right) \nabla_x \theta_n \rightarrow \frac{1}{\beta} \nabla_x (\overline{\theta^\beta}) - \nabla_x \left(\frac{\overline{1}}{\theta} \right) \quad \text{in } L^{p_1}, \quad \text{for some } p_1 > 1, \quad (373)$$

and

$$\frac{1}{\beta} \nabla_x (\overline{\theta^\beta}) - \nabla_x \left(\frac{\overline{1}}{\theta} \right) = \theta^{\beta-1} \nabla_x \theta + \frac{1}{\theta^2} \nabla_x \theta, \quad (374)$$

$$(\log \theta_n - \log \rho_n) \cdot \nabla_x \rho_n \rightarrow (\log \theta - \log \rho) \cdot \nabla_x \rho \quad \text{weakly in } L^1 L^1. \quad (375)$$

Let $n \rightarrow \infty$, by the semi-continuity of weak limit with respect to L^2 norms, we have

$$\begin{aligned} \sqrt{\frac{\mu(\theta_n)}{\theta_n} + \delta} \left(\nabla_x \mathbf{u}_n + \nabla_x^T \mathbf{u}_n - \frac{2}{3} \operatorname{div}_x \mathbf{u}_n \right) &\rightarrow \sqrt{\frac{\mu(\theta)}{\theta} + \delta} \left(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \right) \\ &\quad \text{weakly in } L^2 L^2. \end{aligned} \quad (376)$$

$$\begin{aligned}
\sqrt{\frac{\eta(\theta_n)}{\theta_n}} \operatorname{div}_x \mathbf{u}_n &\rightarrow \sqrt{\frac{\eta(\theta)}{\theta}} \operatorname{div}_x \mathbf{u} \quad \text{weakly in } L^2 L^2, \\
\frac{\sqrt{k_\delta(\theta_n)}}{\theta_n} \nabla_x \theta_n &\rightarrow \frac{\sqrt{k_\delta(\theta)}}{\theta} \nabla_x \theta \quad \text{weakly in } L^2 L^2, \\
\sqrt{\frac{\beta \rho_n^{\beta-2} + 2}{\theta_n}} \nabla_x \rho_n &\rightarrow \sqrt{\frac{\beta \rho^{\beta-2} + 2}{\theta}} \nabla_x \rho \quad \text{weakly in } L^2 L^2, \\
\epsilon \theta_n^4 &\rightarrow \epsilon \theta^4 \quad \text{in } L^{p_1} L^{p_1} \quad \text{for some } p_1 > 1.
\end{aligned} \tag{377}$$

Then we get the entropy inequality

$$\begin{aligned}
&\int_0^T \int_\Omega \rho s(\partial_t \varphi \mathbf{u} \cdot \nabla_x \varphi) + \int_0^T \int_\Omega \left(\frac{q_\delta}{\theta_\epsilon} + \epsilon r_\epsilon \right) + \int_0^T \int_\Omega \sigma_{\epsilon, \delta} \varphi dx dt \\
&\leq - \int_\Omega (\rho s)_{0, \delta} dx + \int_0^T \int_\Omega \epsilon \theta^4 dx dt,
\end{aligned} \tag{378}$$

where

$$\begin{aligned}
q_\delta &= \kappa_\delta(\theta) \nabla_x \theta, \\
\kappa_\delta(\theta) &= \kappa_\delta + \delta \left(\theta^\rho + \frac{1}{\theta} \right), \\
S(\rho, \theta) &= \log \theta - \log \rho, \\
\sigma_{\epsilon, \delta} &= \frac{1}{\delta} \left[S_\delta : \nabla_x \mathbf{u} + \left[\frac{\kappa(\theta)}{\theta} + \frac{\delta}{2} \left(\theta^{\beta-1} + \frac{1}{\theta^2} \right) |\nabla_x \theta|^2 + \delta \frac{1}{\theta^2} \right] \right. \\
&\quad \left. \frac{\epsilon \delta}{2} (\beta \rho^{\beta-2} + 2) |\nabla_x \rho|^2 + \frac{\gamma}{2\theta} \rho^{\gamma-2} + \epsilon \frac{|\nabla_x \rho|^2}{\rho} \right], \\
r_\epsilon &= -(\log \theta - \log \rho) \nabla_x \rho,
\end{aligned} \tag{379}$$

and φ is a non-negative test function.

3.4.6 Admissible Conditions

We have the identity in equation (381) by direct calculation. Recall

$$\phi_k = \left[\frac{1}{\theta^m} - C_k^K \right]_+. \tag{380}$$

Then

$$\begin{aligned}
& (\rho\phi_k)_t + \operatorname{div}_x(\rho\phi_k\mathbf{u}) \\
&= 1_{\Omega_k} \frac{-m}{\theta^{m+1}} (\theta_t + \nabla_x \theta \cdot \mathbf{u}) + \Delta \rho \cdot \phi_k \\
&= 1_{\Omega_k} \frac{-m}{\theta^{m+1}} (-\Delta \rho \cdot \theta - (\theta^4)_t - \operatorname{div}_x(\theta^4 \mathbf{u})) \\
&\quad + 1_{\Omega_k} \frac{-m}{\theta^{m+1}} \left(\operatorname{div}_x(K_\delta \nabla_x \theta) + S_\delta : \nabla_x \mathbf{u} - (\rho\theta + \frac{\theta^4}{3}) \operatorname{div}_x \mathbf{u} \right. \\
&\quad \left. + \epsilon \delta (\beta \rho^{\beta-2} + 2) |\nabla_x \rho|^2 + \epsilon r \rho^{\gamma-2} |\nabla_x \rho|^2 + \frac{\delta}{\theta^2} - \epsilon \theta^5 \right).
\end{aligned} \tag{381}$$

Integrating the equation (381) , we have

$$\begin{aligned}
& \int_{\Omega} (\rho\phi_k)(t) dx + \int_{\Omega} F_1(\theta) 1_{\Omega_k} dx + \int_0^t \int_{\Omega} \frac{m S_\delta(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}}{\theta^{m+1}} 1_{\Omega_k} dx d\tau \\
&+ \int_0^t \int_{\Omega} \frac{m(m+1)k_\delta |\nabla_x \theta|^2}{\theta^{m+2}} 1_{\Omega_k} dx d\tau \\
&+ \int_0^t \int_{\Omega} \frac{m}{\theta^{m+1}} (\epsilon \delta (\beta \rho^{\beta-2} + 2) |\nabla_x \rho|^2 + \epsilon r \rho^{\gamma-2} |\nabla_x \rho|^2 + \frac{\delta}{\theta^2}) dx d\tau \\
&\leq \int_{\Omega} \rho\phi_k(0) dx + \int_{\Omega} F_1(\theta)(0) 1_{\Omega_k} dx \\
&+ \int_0^t \int_{\Omega} \frac{m}{\theta^{m+1}} \left(\operatorname{div}(\theta^4 \mathbf{u}) + (\rho\theta + \frac{4}{3}\theta^4) \operatorname{div} \mathbf{u} \right) 1_{\Omega_k} dx d\tau \\
&+ \int_0^t \int_{\Omega} 1_{\Omega} \frac{m}{\theta^{m+1}} \epsilon \theta^5 + \int_0^t \int_{\Omega} \left(\delta \rho \cdot \phi_k + \Delta \rho \cdot \frac{1}{\theta^m} \right) 1_{\Omega} dx d\tau,
\end{aligned} \tag{382}$$

where $F_1 = \int_{\theta}^{(M)^{-\frac{m}{K}}} \frac{ms^4}{(s)^{m+1}} ds$.

We need to prove the right hand side terms are bounded by $C(m)$, a constant depends on initial data and m :

$$\begin{aligned}
& \int_0^t \int_{\Omega} \frac{m}{\theta^{m+1}} \operatorname{div}_x(\theta^4 \mathbf{u}) 1_{\Omega_k} dx d\tau \\
&= \int_0^t \int_{\Omega} \frac{m}{\theta^{m-3}} \operatorname{div}_x \mathbf{u} 1_{\Omega_k} dx d\tau + \int_0^t \int_{\Omega} \frac{m}{\theta^{m+1}} 4\theta^3 \cdot \nabla_x \theta \cdot \mathbf{u} 1_{\Omega_k} dx d\tau \\
&\leq C \left\| \frac{1}{\theta^{m-3}} 1_{\Omega_k} \right\|_{L^1 L^1} \|\nabla_x u\|_{L^2 L^2} \\
&\leq \frac{m\delta}{4} \left\| \frac{1}{\theta^{m+3}} \right\|_{L^1 L^1} + C \|\mathbf{u}\|_{L^2 H^1} \\
&\leq C(m),
\end{aligned} \tag{383}$$

$$\begin{aligned}
& \int_0^t \int_{\Omega} \delta \rho \cdot \frac{1}{\theta^m} 1_{\Omega} dx d\tau \\
&= - \int_0^t \int_{\Omega} \nabla_x \rho \cdot \frac{1}{\theta^{m+1}} \cdot \nabla_x \theta \cdot 1_{\Omega} dx d\tau \\
&\leq C(m).
\end{aligned} \tag{384}$$

With the help of the strong convergence of θ_n , letting $n \rightarrow \infty$ we can get the approximated admissible condition.

3.4.7 Momentum equations

Firstly

$$\rho_n \mathbf{u}_n \otimes \mathbf{u}_n \rightharpoonup \overline{\rho \mathbf{u} \otimes \mathbf{u}} \text{ weakly in } L^{q_1} L^{q_1}. \tag{385}$$

Here $q_1 > 1$, and

$$\rho_n \mathbf{u}_n \rightharpoonup \rho \mathbf{u} \text{ weakly } * \text{ in } L^\infty L^{p_1}. \tag{386}$$

We can also show that $t \rightarrow \int_{\Omega} \rho_n u_n \phi dx$ are equi-continuous and bounded in $C([0, T])$ for any $\phi \in \cup_{n=1}^{\infty} X_n$. Then by the Lemma 13, $\rho_n \mathbf{u}_n \rightarrow \rho \mathbf{u}$ in $C_{\text{weak}}([0, T], L^{\frac{3}{2}}(\Omega))$. Then $\rho_n \mathbf{u}_n \rightarrow \rho \mathbf{u}$ in $C_{\text{weak}}[0, T], W^{-1,2}(\Omega)$. Combining the fact $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $L^2([0, T], W^{1,2})$, we have

$$\rho_n \mathbf{u}_n \otimes \mathbf{u}_n \rightarrow \rho \mathbf{u} \otimes \mathbf{u} \tag{387}$$

in the sense of distribution.

3.4.8 The result system

Therefore, we have proved the following approximate equation in the limit of $n \rightarrow \infty$

1. Approximated continuity equation

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = \epsilon \Delta \rho \quad \text{a.e. in } (0, T) \times \Omega, \quad (388)$$

together with

$$\nabla_x \rho \cdot \mathbf{n}|_{\partial \Omega} = 0, \rho(0, \cdot) = \rho_{0, \delta}. \quad (389)$$

2. Approximated balance of momentum

$$\begin{aligned} & \int_0^T \int_{\Omega} \rho \mathbf{u} \cdot \partial_t \varphi + \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + (\rho \theta + \rho^\gamma + \delta(\rho^\beta + \rho^2)) \operatorname{div}_x \varphi \, dx dt \\ &= \int_0^T \int_{\Omega} \epsilon \nabla_x \rho \nabla_x \mathbf{u} \cdot \varphi + \mathbb{S}_\delta : \nabla_x \varphi \, dx dt - \int (\rho \mathbf{u})_0 \cdot \varphi \, dx, \\ & \varphi|_{\partial \Omega} = 0, \\ & \mathbb{S}_\delta = \mathbb{S}_\delta(\theta : \nabla_x \mathbf{u}), \\ & = (\mu(\theta) + \delta \theta)(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \operatorname{div} \mathbf{u} \mathbb{I}) + \eta(\theta) \operatorname{div} \mathbf{u} \mathbb{I}. \end{aligned} \quad (390)$$

3. Approximate entropy inequality

$$\begin{aligned} & \int_0^T \int_{\Omega} \rho s_\delta(\rho, \theta) (\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi) \, dx dt \\ & + \int_0^T \int_{\Omega} \left(\frac{k_\delta(\theta) \nabla_x \theta}{\theta} + \epsilon r \right) \nabla_x \varphi \, dx dt \\ & + \int_0^T \int_{\Omega} \sigma_{\epsilon, \delta} \varphi \, dx dt \\ & \leq - \int_{\Omega} (\rho s)_{0, \delta} \varphi(0, \cdot) \, dx + \int_0^T \int_{\Omega} \epsilon \theta^4 \varphi \, dx dt, \end{aligned} \quad (391)$$

where

$$\begin{aligned}
\mathbb{S}_\delta(\rho, \theta) &= \log \theta - \log \rho, \\
k_\delta(\rho, \theta) &= \kappa(\theta) + \delta(\rho^\beta + \frac{1}{\theta}), \\
\sigma_{\epsilon, \delta} &= \frac{1}{\theta} \left[\mathbb{S}_\delta : \nabla_x \mathbf{u} + \left(\frac{\kappa(\theta)}{\theta} + \frac{\delta}{2} (\theta^{\beta-1} + \frac{1}{\theta}) \right) |\nabla_x \theta|^2 + \frac{\delta}{\theta^2} \right. \\
&\quad \left. + \frac{\epsilon \delta}{2\theta} (\beta \rho^{\beta-2} + 2) |\nabla_x \rho|^2 + \epsilon \frac{|\nabla_x \rho|^2}{\rho} \right], \\
r &= -(\log \theta - \log \rho) \frac{\nabla_x \rho}{\theta}.
\end{aligned} \tag{392}$$

Entropy production rate could be represented by $\sum_{\epsilon, \delta}$ so that equation (391) is equality and $\sum_{\epsilon, \delta} \geq \sigma_{\epsilon, \delta}$.

4. Approximate total energy balance

$$\begin{aligned}
&\int_\Omega \frac{1}{2} \rho |\mathbf{u}|^2 + \rho \theta + \theta^4 + \delta \left(\frac{\rho^\beta}{\beta-1} + \rho^2 \right) + \frac{\rho^\gamma}{\gamma-1}(\tau) dx \Big|_0^t \\
&= \int_0^t \int_\Omega \frac{\delta}{\theta^2} - \epsilon \theta^5 dx dt \quad \text{a.e. } t \in [0, T].
\end{aligned} \tag{393}$$

5. Admissible Condition:

$$\begin{aligned}
&\int_\Omega (\rho \phi_k)(t) dx + \int_\Omega F_1(\theta) 1_{\Omega_k}(t) dx + \int_0^t \int_\Omega \frac{m \mathbb{S}_\delta(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}}{\theta^{m+1}} 1_{\Omega_k} dx d\tau \\
&+ \int_0^t \int_\Omega \frac{m(m+1) k_\delta |\nabla_x \theta|^2}{\theta^{m+2}} 1_{\Omega_k} dx d\tau \\
&+ \int_0^t \int_\Omega \frac{m}{\theta^{m+1}} (\epsilon \delta (\beta \rho^{\beta-2} + 2) |\nabla_x \rho|^2 + \epsilon r \rho^{r-2} |\nabla_x \rho|^2 + \frac{\delta}{\theta^2}) dx d\tau \\
&\leq \int_\Omega \rho \phi_k(0) dx + \int_\Omega F_1(\theta)(0) 1_{\Omega_k} dx \\
&+ \int_0^t \int_\Omega \frac{m}{\theta^{m+1}} \left(\text{div}(\theta^4 \mathbf{u}) + (\rho \theta + \frac{4}{3} \theta^4) \text{div} \mathbf{u} \right) 1_{\Omega_k} dx d\tau \\
&+ \int_0^t \int_\Omega 1_\Omega \frac{m}{\theta^{m+1}} \epsilon \theta^5 + \int_0^t \int_\Omega \left(\delta \rho \cdot \phi_k + \Delta \rho \cdot \frac{1}{\theta^m} \right) 1_\Omega dx d\tau,
\end{aligned} \tag{394}$$

where $F_1 = \int_\theta^{(M)^{-\frac{m}{K}}} \frac{ms^4}{(s)^{m+1}} ds$.

3.5 The vanishing viscosity limit

In this section, we will let $\epsilon \rightarrow 0$. At this level, we will not have uniform dissipative estimates for the density. Instead we need to use the Div-Curl lemma to obtain the strong convergence of the temperature first, and then use the technique of weak convergence on convex functions to obtain the strong convergence of the the density. As in the framework in [51], the pressure estimates, or $L^{\gamma+1}$ estimate of the density is important for the compactness. We also need to use the important relation of the so called effective flux, which was introduced in [120, 86, 95, 96] and generalized in [43, 51]. At this level, by the entropy inequities obtained in the previous section, the dissipation balance is

$$\begin{aligned}
& \int_{\Omega} \frac{1}{2} \rho_{\epsilon} |\mathbf{u}_{\epsilon}|^2 + H_{\delta, \bar{\theta}}(\rho_{\epsilon}, \theta_{\epsilon}) + \delta \left(\frac{\rho_{\epsilon}^{\beta}}{\beta-1} + \rho_{\epsilon}^2 \right) + \frac{\rho_{\epsilon}^{\gamma}}{\gamma-1} dx \\
& \quad + \bar{\theta} \Sigma_{\epsilon, \delta}[Q] + \int_0^T \int_{\Omega} \epsilon \theta^5 dx dt \\
& = \int_{\Omega} \frac{1}{2} \rho_{\epsilon} |\mathbf{u}_{\epsilon}|^2 + H_{\delta, \bar{\theta}}(\rho_{\epsilon}, \theta_{\epsilon}) + \delta \left(\frac{\rho_{\epsilon}^{\beta}}{\beta-1} + \rho_{\epsilon}^2 \right) + \frac{\rho_{\epsilon}^{\gamma}}{\gamma-1}(0) dx \\
& \quad + \int_0^t \int_{\Omega} \frac{f}{\theta_{\epsilon}^2} + \epsilon \bar{\theta} \theta^4 dx d\tau \\
& \quad \text{for a.e. } t \in [0, T),
\end{aligned} \tag{395}$$

$$\text{where } \sigma_{\epsilon, \delta} \geq \frac{1}{\theta} \left[\mathbb{S}_{\delta} : \nabla_x \mathbf{u} + \left(\frac{\kappa(\theta)}{\theta} + \frac{\delta}{2} (\theta^{\beta-1} + \frac{1}{\theta}) \right) |\nabla_x \theta|^2 + \frac{\delta}{\theta^2} \right].$$

So we have the following estimates from the dissipation balance, which is independent on ϵ .

$$\text{ess sup}_{t \in [0, T]} \int \frac{1}{2} \rho_{\epsilon} |\mathbf{u}_{\epsilon}|^2 + H_{\delta, \bar{\theta}}(\rho_{\epsilon}, \theta_{\epsilon}) + \delta \left(\frac{\rho_{\epsilon}^{\beta}}{\beta-1} + \rho_{\epsilon}^2 \right) + \frac{\rho_{\epsilon}^{\gamma}}{\gamma-1}(t) dx \leq C, \tag{396}$$

$$\begin{aligned}
& \int_0^t \int_{\Omega} \frac{1}{\theta_{\epsilon}} [\mathbb{S}_{\delta}(\theta_{\epsilon}, \nabla_x \mathbf{u}_{\epsilon}) : \nabla_x \mathbf{u}_{\epsilon}] dx dt \leq C, \\
& \int_0^t \int_{\Omega} \left[\frac{1}{\theta^2} \kappa(\theta_{\epsilon}) + \delta(\theta_{\epsilon}^{\beta-2} + \frac{1}{\theta_{\epsilon}^3}) \right] |\nabla_x \theta_{\epsilon}|^2 dx dt \leq C, \\
& \int_0^t \int_{\Omega} \frac{\delta}{\theta_{\epsilon}^3} + \epsilon \theta_{\epsilon}^5 dx dt \leq C, \\
& \epsilon \int_0^t \int_{\Omega} \frac{1}{\theta_{\epsilon}} (\beta \rho_{\epsilon}^{\beta-2} + 2) |\nabla_x \rho_{\epsilon}|^2 dx dt \leq C \\
& \int_0^t \int_{\Omega} \epsilon \frac{|\nabla_x \rho_{\epsilon}|^2}{\rho_{\epsilon}} \leq C.
\end{aligned} \tag{397}$$

Particularly, we have

$$\|\rho_{\epsilon}\|_{L^{\infty}, L^{\beta}} \leq C, \tag{398}$$

$$\|\nabla_x \mathbf{u}\|_{L^2, L^2} \leq C, \tag{399}$$

$$\|\theta_{\epsilon}\|_{L^{\infty}, L^4} \leq C. \tag{400}$$

Since these bounds are uniform with respect to ϵ , we have the weak convergence

$$\begin{aligned}
\rho_{\epsilon} &\rightharpoonup \rho \quad \text{weakly}^* \quad \text{in } L^{\infty} L^{\beta}, \\
\mathbf{u}_{\epsilon} &\rightharpoonup \mathbf{u} \quad \text{weakly} \quad \text{in } L^2 W^{1,2}, \\
\theta_{\epsilon} &\rightharpoonup \theta \quad \text{weakly}^* \quad \text{in } L^{\infty} L^4.
\end{aligned} \tag{401}$$

In the next few sections, we will reinforce these limits, with the help of better estimates, and possibly taking subsequence in the case of weak limits.

3.5.1 Continuity Equation

We can get the accurate estimates of the approximate equation, especially how the bounds depend on ϵ . In contrast of the analysis of previous section when taking limits with respect to n , where we have parabolic estimates from the maximal regularity, we

do not have the uniform higher order estimates of the density now. We can still consider the approximate continuity equation as parabolic, so multiply the approximate continuity equation by ρ_ϵ , we have

$$\begin{aligned} & \frac{1}{2} \int \rho_\epsilon^2(t) dx + \epsilon \int_0^t \int_\Omega |\nabla_x \rho_\epsilon|^2 dx d\tau \\ &= \frac{1}{2} \int_\Omega \rho_{0\epsilon}^2 dx - \frac{1}{2} \int_0^t \int_\Omega \rho_\epsilon^2 \operatorname{div}_x \mathbf{u}_\epsilon dx d\tau \\ &\leq C. \end{aligned} \tag{402}$$

Where the uniform bound $L^\infty L^4$ of the density and $\|\nabla_x \mathbf{u}\|_{L^2 L^2}$ are used, which could be obtained from the entropy balance relation with $\beta \geq 4$.

So $\{\sqrt{\epsilon} \nabla_x \rho_\epsilon\}$ are uniformly bounded in $L^2 L^2$, then we have

$$\epsilon \nabla_x \rho_\epsilon \rightarrow 0 \quad \text{in} \quad L^2 L^2. \tag{403}$$

On the other hand, we want to consider the $L^2 H^{-1}$ norm of $\partial_t \rho_\epsilon$,

$$\partial_t \rho_\epsilon = -\operatorname{div}_x(\rho_\epsilon \mathbf{u}_\epsilon) + \epsilon \Delta \rho_\epsilon, \tag{404}$$

$$\int \partial_t \rho_\epsilon \cdot \varphi = \int \rho_\epsilon \mathbf{u}_\epsilon \nabla_x \varphi - \epsilon \int \nabla_x \rho_\epsilon \cdot \nabla_x \varphi \leq C + C\sqrt{\epsilon}. \tag{405}$$

So

$$\|\partial_t \rho_\epsilon\|_{L^2 H^{-1}} \leq C. \tag{406}$$

Then by Lemma 13 and $\|\rho_\epsilon\|_{L^\infty L^\beta} \leq C$, we have

$$\rho_\epsilon \rightarrow \rho \quad \text{in} \quad C_{\text{weak}}([0, T, L^\beta(\Omega)]) \tag{407}$$

In addition, based on the fact:

$$\|\rho\|_{L^\infty L^\beta} \leq C, \tag{408}$$

and

$$\|\sqrt{\rho}\mathbf{u}\|_{L^\infty L^2} \leq C, \quad (409)$$

and Hölder inequality, we have

$$\|\rho_\epsilon \mathbf{u}_\epsilon\|_{L^\infty L^{\frac{2\beta}{\beta+1}}} \leq C. \quad (410)$$

So we have

$$\rho_\epsilon \mathbf{u}_\epsilon \rightarrow \rho \mathbf{u} \quad \text{weakly}^* \quad \text{in} \quad L^\infty L^{\frac{2\beta}{\beta+1}}. \quad (411)$$

Then we know that

$$\int_0^T \int_\Omega (\rho \partial_t \varphi + \rho \mathbf{u} \nabla_x \varphi) dx dt + \int \rho_{0,\delta} \varphi(0, \cdot) dx = 0. \quad (412)$$

for and test function φ . By extending $\mathbf{u} = 0$ outside Ω , we have

$$\rho_t + \operatorname{div}_x(\rho \mathbf{u}) = 0 \quad \text{in} \quad D'([0, T] \times \mathbb{R}^3). \quad (413)$$

This fact that the extended solution is the solution in the whole space is proved in [43].

3.5.2 Strong convergence of temperature

Without the strong convergence of density which is available in the case of taking $n \rightarrow \infty$, the convergence of temperature could still be obtained by using Div-Curl lemma to show the coincidence of the weak limits $\overline{\rho s G(\theta)} = \overline{\rho s} \overline{G(\theta)}$, for any bounded and continuous function G . In the level taking $n \rightarrow \infty$, we basically have the strong convergence of the density from the parabolic structure, and then use the Div-Curl lemma to obtain the convergence of the temperature. When we do not have the uniform parabolic structure for the density, it's only possible to obtain the convergence of the temperature before obtaining the strong convergence of the density. Similar as the previous section, we can define

$$U_\epsilon = [\rho_\epsilon s_\delta, \rho_\epsilon s_\delta \mathbf{u}_\epsilon + \frac{\kappa_\delta \nabla_x \theta_\epsilon}{\theta_\epsilon} + \epsilon \mathbf{r}_\epsilon], \quad (414)$$

$$V_\epsilon = [G(\theta), 0, 0, 0],$$

where G is a bounded globally Lipschitz function on $[0, \infty)$ and where

$$r_\epsilon^{(1)} = \rho_\epsilon s_\delta(\rho_{\delta, \theta_\epsilon}) \mathbf{u}_\epsilon - \frac{\kappa(\theta_\epsilon)}{\theta_\epsilon} \nabla_x \theta_\epsilon - \epsilon (\log \theta_\epsilon - \log \rho_\epsilon - 2) \nabla_x \rho_\epsilon. \quad (415)$$

So we have

$$\operatorname{div}_{t,x} = \Sigma_{\epsilon, \theta} - \epsilon \theta_\epsilon^4, \quad (416)$$

and

$$\operatorname{curl}_{t,x} V_\epsilon(\theta) = G'(\theta_\epsilon) \begin{pmatrix} 0 & \nabla_x \theta_\epsilon \\ \nabla_x^T \theta_\epsilon & 0 \end{pmatrix}. \quad (417)$$

which are relative compact in $W^{-1,s}$ for $s \in [1, \frac{4}{3})$, as $\{V_{\epsilon, \theta}\}$ is bounded in $L^2([0, T] \times \overline{\Omega})$

Then we need to show that $\{U_{\epsilon, \theta}\}$ is bounded in $L^p([0, T] \times \overline{\Omega})$, where $p > 1$. Similar to the previous section we have $\rho_\epsilon s_\delta$, $\rho_\epsilon s_\delta \mathbf{u}$, and $\frac{\kappa_\delta \nabla_x \theta_\epsilon}{\theta_\epsilon}$ are bounded in L^p for some $p > 1$. In the end

$$\epsilon r_\epsilon \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (418)$$

So by the Div-Curl lemma, we have

$$\overline{\rho s G(\theta)} = \overline{\bar{\rho} s G(\theta)}, \text{ for any bounded and continuous function } G, \quad (419)$$

Then we try to show

$$\rho \log \rho \rightarrow \overline{\rho \log \rho} \text{ in } C_{weak}((0, T), L^\beta). \quad (420)$$

Since we already know the fact

$$\rho \rightarrow \bar{\rho} \text{ in } C_{weak}((0, T), L^\beta), \quad (421)$$

we just need to show that

$$t \mapsto \int_{\Omega} \rho \log \rho \psi \quad (422)$$

is uniformly bounded and equi-continuous in $C([0, T])$ to claim (420), which implies $\rho \log \rho \rightarrow \overline{\rho \log \rho}$ in $C((0, T), W^{-1,2})$. We have

$$G(\theta_\epsilon) \rightarrow \overline{G(\theta)} \in L^2(0, T, W^{1,2}), \quad (423)$$

$$\log(\theta_\epsilon)(\theta_\epsilon) \rightarrow \overline{\log(\theta)(\theta)} \text{ in } L^2(0, T, W^{1,2}). \quad (424)$$

Therefore $\overline{\rho \log \rho G(\theta)} = \overline{\rho \log G(\theta)}$, which give us

$$\overline{\theta^3 G(\theta)} \rightarrow \overline{\theta^3 G(\theta)}. \quad (425)$$

By lemma 11, we have that $\theta^\epsilon \rightarrow \theta$ strongly. Now we consider the approximate entropy balance is:

$$\partial_t(\rho_\epsilon s(\rho_\epsilon, \theta_\epsilon)) + \operatorname{div}_x(\rho_\epsilon s_\delta(\rho_\epsilon, \theta_\epsilon) \mathbf{u}) + \frac{\kappa_\delta(\theta_\epsilon) \nabla_x \theta_\epsilon}{\theta_\epsilon} + \epsilon r_\epsilon = \Sigma_{\epsilon, \delta} - \epsilon \theta_\epsilon^4. \quad (426)$$

$$\operatorname{div}_{(t,x)} U_\epsilon = \Sigma_{\epsilon, \delta} - \epsilon \theta_\epsilon^4 \in L^1(Q), \quad (427)$$

Since

$$\operatorname{curl}_{(t,x)} V_\epsilon = \begin{pmatrix} 0 & \nabla_x \theta_\epsilon \\ \nabla_x^T \theta_\epsilon & 0 \end{pmatrix} \in L^2(Q), \quad (428)$$

so they are relative compact in $W^{-1,s}(Q)$ for $s \in [1, \frac{4}{3})$ as

$$\mathcal{M}^+([0, T] \times \overline{\Omega}) \hookrightarrow W^{-1,s}, \quad (429)$$

$$L^1([0, T] \times \overline{\Omega}) \hookrightarrow W^{-1,s}. \quad (430)$$

Firstly, we have $\int \epsilon \theta_\epsilon^4 \rightarrow 0$ in $L^1 L^1$. Left hand terms could be estimated similarly, i.e.

$$\theta \in L^\infty L^4, \quad (431)$$

$$\rho_\epsilon s(\rho_\epsilon, \theta_\epsilon) \in L^{p_1} \text{ for a } p_1 > 1 \quad (432)$$

$$\rho_\epsilon s(\rho_\epsilon, \theta_\epsilon) \mathbf{u}_\epsilon \in L^{q_1} \text{ for a } q_1 > 1 \text{ when } \beta \text{ is large enough,} \quad (433)$$

$$\epsilon r_\epsilon \rightarrow 0 \text{ in } L^{p_1} \text{ for a } p_1 > 1. \quad (434)$$

Then we have

$$\overline{\rho s(\rho_\epsilon, \theta_\epsilon) \theta} = \overline{\rho s(\rho_\epsilon, \theta_\epsilon) \bar{\theta}}. \quad (435)$$

By the L^2 estimates of $\nabla_x \theta$

$$\frac{\kappa_\delta(\theta_\epsilon)}{\theta_\epsilon} \nabla_x \theta_\epsilon \rightharpoonup \frac{\kappa_\delta(\theta)}{\theta} \nabla_x \theta \text{ weakly* in } \mathcal{M}(Q). \quad (436)$$

Let $\epsilon \rightarrow 0$, we have the asymptotic limit in the entropy balance

$$\begin{aligned} & \int_0^T \int_\Omega \overline{\rho s(\rho \theta)} (\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi) dx dt \\ & + \int_0^T \int_\Omega \frac{\kappa_\delta(\theta)}{\theta} \nabla_x \theta : \nabla_x \varphi dx dt + \langle \sigma_\delta, \varphi \rangle_{\langle C, \mathcal{M} \rangle(Q)} \\ & = - \int_\Omega (\rho s)_{0, \delta} \varphi(0, \cdot) dx. \end{aligned} \quad (437)$$

3.5.3 $L^{\beta+1}$ estimates of density

We need a higher order integrability of the density, or equivalently uniformly estimates on pressure. Only with the higher integrability of the density, we can use a similar method as the previous section to get the strong convergence the the density via the convex functions.

We will use the following test function in the momentum equation

$$\begin{aligned} \varphi &= \xi \zeta, \\ \zeta &\in C_c^\infty(0, T), \\ \xi &= \mathcal{B}[\rho_\epsilon - \bar{\rho}]. \end{aligned} \quad (438)$$

Here $\bar{\rho}$ is the average approximate density which is a constant

$$\bar{\rho} = \frac{1}{|\Omega|} \int_\Omega \rho_\epsilon dx, \quad (439)$$

here $\mathcal{B} = \operatorname{div}_x^{-1}$ is the Bogovskii operator. Since the ρ_ϵ satisfies the approximate continuity equation, we have

$$\partial_t \xi = \mathcal{B}(\operatorname{div}_x(\rho_\epsilon \mathbf{u}_\epsilon - \epsilon \nabla_x \rho_\epsilon)). \quad (440)$$

By the property of \mathcal{B} [51], we have

$$\|\xi\|_{W^{1,p}} \leq C(p, \Omega) \|\rho_\epsilon\|_{L^p}, \quad 1 < p < \infty, \quad (441)$$

$$\|\xi_t\|_{L^p} \leq C(p, \Omega) \|\rho_\epsilon \mathbf{u}_\epsilon + \epsilon \nabla_x \rho_\epsilon\|_{L^p}, \quad 1 < p < \infty. \quad (442)$$

Then from the momentum equations

$$\begin{aligned} & \int_0^T \int_\Omega \rho_\epsilon \mathbf{u}_\epsilon \partial_t \varphi + \rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon \nabla_x \varphi + (\rho_\epsilon \theta_\epsilon + \frac{\theta_\epsilon^4}{3} + \rho_\epsilon^\gamma + \delta(\rho_\epsilon^\beta + \rho_\epsilon^2)) \operatorname{div}_x \varphi dx dt \\ &= \int_0^T \int_\Omega \epsilon \nabla_x \rho_\epsilon \nabla_x \mathbf{u}_\epsilon \varphi + S_\delta : \nabla_x \varphi dx dt, \end{aligned} \quad (443)$$

we have

$$\int_\Omega \left[\zeta \int_0^T (\rho_\epsilon \theta_\epsilon + \frac{\theta_\epsilon^4}{3} + \rho_\epsilon^\gamma + \delta(\rho_\epsilon^\beta + \rho_\epsilon^2)) \rho_\epsilon dx \right] dt = \sum_{j=1}^6 I_j. \quad (444)$$

The integrals are estimated in the following way:

$$\begin{aligned} I_1 &= \int_\Omega \left[\zeta \bar{\rho}_\epsilon \int_0^T (\rho_\epsilon \theta_\epsilon + \frac{\theta_\epsilon^4}{3} + \rho_\epsilon^\gamma + \delta(\rho_\epsilon^\beta + \rho_\epsilon^2)) dx \right] dt \\ &\leq \frac{1}{2} \int_\Omega \left[\zeta \bar{\rho} \int_0^T (\rho_\epsilon \theta_\epsilon + \frac{\theta_\epsilon^4}{3} + \rho_\epsilon^\gamma + \delta(\rho_\epsilon^\beta + \rho_\epsilon^2)) \rho_\epsilon dx \right] dt + C, \end{aligned} \quad (445)$$

where Hölder, Young inequality are used

$$\begin{aligned} I_2 &= - \int_\Omega \left[\zeta \int_0^T \rho_\epsilon \mathbf{u}_\epsilon \partial_t \xi dx \right] dt \\ &\leq C \|\rho_\epsilon\|_{L^\infty L^3} \|\mathbf{u}_\epsilon\|_{L^6 L^6} \|\xi_t\|_{L^2 L^2} \\ &\leq C. \end{aligned} \quad (446)$$

Here the embedding and the property of \mathcal{B} was used

$$\begin{aligned}
I_3 &= - \int_0^T \zeta \int_{\Omega} \rho_{\epsilon} \mathbf{u}_{\epsilon} \otimes \mathbf{u}_{\epsilon} \nabla_x \xi dx dt \\
&\leq C \|\sqrt{\rho_{\epsilon}}\|_{L^{9/2} L^{9/2}} \|\sqrt{\rho_{\epsilon}} \mathbf{u}_{\epsilon}\|_{L^{\infty} L^2} \|\mathbf{u}_{\epsilon}\|_{L^6 L^6} \|\nabla_x \xi\|_{L^{\infty} L^{9/2}} \\
&\leq C.
\end{aligned} \tag{447}$$

Here the embedding and the property of \mathcal{B} was used

$$\begin{aligned}
I_4 &= - \int_0^T \zeta \int_{\Omega} S_{\delta}(\mathbf{u}_{\epsilon}, \rho_{\epsilon}) : \nabla_x \xi dx dt \\
&\leq C \|\theta\|_{L^{\infty} L^4} \|\nabla_x u\|_{L^2 L^2} \|\nabla_x \xi\|_{L^{\infty} L^4} \\
&\leq C,
\end{aligned} \tag{448}$$

$$\begin{aligned}
I_5 &= - \int_0^T \zeta' \int_{\Omega} \rho_{\epsilon} \mathbf{u}_{\epsilon} \xi dx dt \\
&\leq \|\sqrt{\rho}\|_{L^{\infty} L^3} \|\sqrt{\rho} u\|_{L^{\infty} L^2} \|\zeta\|_{L^{\infty} L^6} \\
&\leq C,
\end{aligned} \tag{449}$$

$$\begin{aligned}
I_6 &= \int_0^T \zeta' \int_{\Omega} \epsilon \nabla_x \rho_{\epsilon} \nabla_x \mathbf{u}_{\epsilon} \cdot \xi dx dt \\
&\leq C,
\end{aligned} \tag{450}$$

where we used L^{∞} of ζ' is bounded. In sum, we can get the estimate

Lemma 22.

$$\|\rho_{\epsilon}\|_{L^{\beta+1} L^{\beta+1}} < C. \tag{451}$$

3.5.4 Effective flux

The weak compactness identity for effective pressure at the level of ϵ is the same as [51],

Lemma 23.

$$\overline{p_{\delta}(\rho, \theta) \rho} - \left(\frac{4}{3} \mu(\theta) + \frac{4}{3} \delta \theta + \eta \right) \overline{\rho \operatorname{div}_x \mathbf{u}} = \overline{p_{\delta} \rho} - \left(\frac{4}{3} \mu(\theta) + \frac{4}{3} \delta \theta + \eta \right) \rho \operatorname{div}_x \mathbf{u}. \tag{452}$$

By the fact the ρ is a renormalized solution, so

$$\int_{\Omega} \rho \log \rho(t) dx + \int_0^t \int_{\Omega} \rho \operatorname{div} \mathbf{u} dx dt \leq \int_0^t \int_{\Omega} \rho_{0,\delta} \log(\rho_{0,\delta}) dx. \quad (453)$$

ρ_{δ} is non-decreasing with respect to ρ and by $\theta_{\epsilon} \rightarrow \theta$ a.e., we have $\overline{P(\rho, \theta)} \geq \overline{P(\rho, \theta)} \rho$.

Then

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{\Omega} (P(\rho_{\epsilon}, \theta_{\epsilon}) \rho_{\epsilon} - P(\rho_{\epsilon}, \theta_{\epsilon}) \rho) dx dt \\ &= \lim_{\epsilon \rightarrow 0} \int_{\Omega} (P(\rho_{\epsilon}, \theta_{\epsilon}) - P(\rho, \theta_{\epsilon})) (\rho_{\epsilon} - \rho) dx dt \\ & \quad + \lim_{\epsilon \rightarrow 0} \int_{\Omega} P(\rho, \theta_{\epsilon}) (\rho_{\epsilon} - \rho) dx dt. \end{aligned} \quad (454)$$

The first term on the right of (454) is non-negative. By the $L^{\beta+1} L^{\beta+1}$ integrability, the second term approaches 0 as $\epsilon \rightarrow 0$. So $\overline{\rho \operatorname{div}_x \mathbf{u}} \geq \rho \operatorname{div}_x \mathbf{u}$. Then

$$\overline{\rho \log \rho} = \rho \log \rho,$$

$$\rho_{\epsilon} \rightarrow \rho \text{ a.e. in } Q.$$

With the strong convergence of θ and ρ , we can verify the limit of entropy inequality, while the admissible condition could be verified by checking the right-hand are convergent. The limit of this step is summarized below.

3.5.5 Result system

1. Renormalized continuity equation

$$\begin{aligned} & \int_0^T \int_{\Omega} \rho B(\rho) (\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi) dx dt \\ &= \int_0^T \int_{\Omega} b(\rho) \operatorname{div} \mathbf{u} \varphi dx dt - \int_{\Omega} \rho_{0,\delta} B(\rho_{0,\delta}) \varphi(0, \cdot) dx, \end{aligned} \quad (455)$$

for any

$$b \in L^{\infty} \cap C[0, \infty), \quad B(\rho) = B(1) + \int_1^{\rho} \frac{b(z)}{z^2} dz$$

for any test function

$$\varphi \in C_c^{\infty}([0, T] \times \overline{\Omega}),$$

2. Approximated balance of momentum

$$\begin{aligned}
& \int_0^T \int_{\Omega} [\rho \mathbf{u} \cdot \partial_t \varphi + \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi \\
& \quad + (\rho \theta + \delta(\rho^\beta + \rho^2) + \rho^\gamma) \operatorname{div} \varphi] dx dt \\
& = \int_0^T \int_{\Omega} S_\delta : \nabla_x \varphi dx dt - \int (\rho \mathbf{u})_0 \varphi dx,
\end{aligned} \tag{456}$$

for any

$$\begin{aligned}
& \varphi \in C_c^\infty \quad \varphi|_{\partial\Omega} = 0, \\
& S_\delta = (\mu(\theta) + \delta\theta) \left(\nabla_x \mathbf{u} + \nabla_x^2 \mathbf{u} - \frac{2}{3} \operatorname{div} \mathbf{u} I \right) + \eta(\theta) \operatorname{div} \mathbf{u} I.
\end{aligned}$$

3. Approximated entropy balance

$$\begin{aligned}
& \int_0^T \int_{\Omega} \rho s(\rho, \theta) (\partial_t \varphi + \mathbf{u} \nabla_x \varphi) dx dt \\
& \quad + \int_0^T \int_{\Omega} \frac{\kappa_\delta(\theta) \nabla_x \theta}{\theta} \nabla_x \theta dx dt + \langle \sigma_\delta : \varphi \rangle_{(M,C)}(\theta) \\
& = - \int_{\Omega} (\rho s)_{0,\delta} \varphi(0, \cdot) dx,
\end{aligned} \tag{457}$$

$$\sigma_\delta \geq \frac{1}{\theta} \left[\mathbb{S}(\theta_\delta, \nabla_x \mathbf{u}_\delta) : \nabla_x \mathbf{u}_\delta + \left(\frac{\kappa(\theta)}{\theta} + \frac{\delta}{2} (\theta^{\beta-1} + \frac{1}{\theta^2}) \right) |\nabla_x \theta|^2 + \frac{\delta}{\theta^2} \right], \tag{458}$$

4. Approximated energy balance

$$\begin{aligned}
& \int \frac{1}{2} \rho |\mathbf{u}|^2 + \rho \theta + \theta^4 + \delta \left(\frac{\rho^\beta}{\beta-1} + \rho^2 \right) + \frac{\rho^\gamma}{\gamma-1} dx \Big|_0^T \\
& = \int_0^t \int_{\Omega} \frac{\delta}{\theta^2} \quad \text{a.e. } t \in [0, T],
\end{aligned} \tag{459}$$

5. Admissible Condition

$$\begin{aligned}
& \int \rho \phi_k dx + \int F_1(\theta) 1_{\Omega_k} dx + \iint \frac{m \mathbb{S}_\delta(\theta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}}{\theta^{m+1}} 1_{\Omega_k} dx d\tau \\
& \quad + \iint \frac{m(m+1) \kappa_\delta |\nabla_x \theta|^2}{\theta^{m+2}} 1_{\Omega_k} dx d\tau + \iint \frac{m\delta}{\theta^{m+3}} dx d\tau \\
& \leq \int \rho \phi_k(0) dx + \int F_1(\theta)(0) 1_{\Omega_k} dx \\
& \quad + \iint \frac{m}{\theta^{m+1}} \left(\operatorname{div}_x(\theta^4 \mathbf{u}) + (\rho \theta + \frac{4}{3} \theta^4) \operatorname{div}_x \mathbf{u} \right) 1_{\Omega_k} dx d\tau,
\end{aligned} \tag{460}$$

where $F_1 = \int_{\theta}^{(M)^{-\frac{m}{K}}} \frac{ms^4}{(s)^{m+1}} ds$.

3.6 Passing to the limit of the artificial pressure term

In this section, we will follow the framework in [51] to pass the limit with respect to δ . With the uniform estimates with respect to δ , we can use the technique similar as the limits of artificial viscosity. Pressure estimates give us higher integrability of the density ρ in Q . The strong convergence of the temperature θ is from the Div-Curl lemma. As $\gamma > 3$, we have the uniform bounds of $\|\rho\|_{L^\infty L^3}$, on the other hand, from the uniform lower bound of θ , we can get the integrability of $\nabla_x u$, so lemma 15 could be used to show that ρ is a solution of the renormalized continuity equation.

3.6.1 Uniform Estimates w.r.t δ

At this level, the dissipation balance is

$$\begin{aligned}
& \int_{\Omega} \frac{1}{2} \rho_{\delta} |\mathbf{u}_{\delta}|^2 + H_{\bar{\theta}}(\rho_{\delta}, \theta_{\delta})(t) + \delta_{\Omega} \left(\frac{1}{\beta-1} \rho_{\delta}^{\beta} + \rho_{\delta}^2 \right) + \frac{\rho_{\delta}^{\gamma}}{\gamma-1}(t) dx \\
& + \bar{\theta} \sigma_{\delta}[Q] \\
& = \int \frac{1}{2} \frac{|(\rho \mathbf{u})_0|^2}{\rho_{0\delta}} + H_{\bar{\theta}}(\rho_{0,\delta}, \theta_{0,\delta}) + \delta \left(\frac{1}{\beta-1} \rho_{0,\delta}^{\beta} + \rho_{0,\delta}^2 \right) + \frac{\rho_{0,\delta}^{\gamma}}{\gamma-1} dx \\
& + \int_0^t \int_{\Omega} \frac{\delta}{\theta_{\delta}^2} dx dt,
\end{aligned} \tag{461}$$

where $\bar{\theta}$ is from the Helmholtz free energy, and

$$\sigma_{\delta} \geq \frac{1}{\theta} \left[\mathbb{S}(\theta_{\delta}, \nabla_x \mathbf{u}_{\delta}) : \nabla_x \mathbf{u} + \left(\frac{\kappa(\theta)}{\theta} + \frac{\delta}{2} (\theta^{\beta-1} + \frac{1}{\theta^2}) \right) |\nabla_x \theta|^2 + \frac{\delta}{\theta^2} \right]. \tag{462}$$

So we have the uniform estimates

$$\begin{aligned}
& \|\sqrt{\rho_{\delta}} \mathbf{u}_{\delta}\|_{L^\infty L^2} \leq C, \\
& \|\rho_{\delta}\|_{L^\infty L^\gamma} \leq C, \\
& \|\rho_{\delta}\|_{L^\infty L^\beta} \leq C \delta^{-\frac{1}{\beta}}, \\
& \|\theta_{\delta}\|_{L^\infty L^4} \leq C.
\end{aligned} \tag{463}$$

From $\sigma_\delta(Q) \leq C$ and the minimum principle $\theta_\delta \geq \underline{\theta} > 0$,

$$\|\nabla_x \theta^q\|_{L^2 L^2} \leq C \quad \text{for any } q < \frac{b}{2},$$

$$\text{when } q = 0, \|\nabla_x \log \theta\|_{L^2 L^2} \leq C, \quad (464)$$

$$\theta^{\frac{b}{2}} \in L^2 L^6.$$

And

$$\delta \int_0^T \int_\Omega (\theta \delta^{\beta-2} + \frac{1}{\theta_\delta^3}) |\nabla_x \theta_\delta|^2 dx d\tau \leq C. \quad (465)$$

With these estimates, one has

$$\rho_\delta \rightharpoonup \rho \text{ weakly } * \text{ in } L^\infty L^\gamma, \quad (466)$$

$$\theta_\delta \rightharpoonup \rho \text{ weakly } * \text{ in } L^\infty L^4, \quad (467)$$

3.6.2 Pressure Estimates

Lemma 24. *There exists a constant C and ν such that*

$$\int_0^T \int_\Omega (\delta \rho^\beta + \rho_\delta \theta_\delta + \rho_\delta^\gamma) \rho_\delta^\nu dx dt \leq C. \quad (468)$$

Proof. When $b \geq 1$ we also have

$$\theta \in L^\infty L^4 \cap L^1 L^3,$$

$$\nabla_x \mathbf{u} \in L^2 L^2, \quad (469)$$

$$\text{so } S_\delta \in L^{p_1} \quad \text{for some } p_1 > 1.$$

The pressure estimates could be done similar as the previous section.

In the case of $b < 1$, We can use the estimates from the free energy. When $b \leq 1$ we have

$$\|\mathbf{u}\|_{L^2 W^{1, \frac{8}{5-b}}} \leq \left\| \frac{\nabla_x \mathbf{u}}{\theta^{\frac{1-b}{2}}} \right\|_{L^2 L^2} \|\theta^{\frac{1-b}{2}}\|_{L^\infty L^{\frac{8}{1-b}}} = \left\| \frac{\nabla_x \mathbf{u}}{\theta^{\frac{1-b}{2}}} \right\|_{L^2 L^2} \|\theta\|_{L^2 L^4}^{\frac{1-b}{2}} \leq C. \quad (470)$$

If h is a smooth bound function, and use $[\]^\alpha$ to denote the convolution in the time variable t with the standard family of regularizing kernels. We have the renormalized continuity equation is

$$\partial_t [h(\rho)]^\alpha + \text{div}_x [h(\rho) \mathbf{u}]^\alpha + [(\rho h'(\rho) - h(\rho)) \text{div}_x \mathbf{u}]^\alpha = 0. \quad (471)$$

We can construct the following functions as test function in the momentum equation:

$$\begin{aligned}\varphi &= [\xi]^\alpha \zeta, \\ \zeta &\in C_c^\infty(0, T), \\ [\xi]^\alpha &= \mathcal{B}[b(\rho_\delta) - \overline{h(\rho)}]^\alpha.\end{aligned}\tag{472}$$

where

$$\overline{h(\rho)} = \frac{1}{|\Omega|} \int_{\Omega} h(\rho_\delta) dx,\tag{473}$$

which is no longer a constant. Then

$$\begin{aligned}\partial_t [\xi]^\alpha &= -\mathcal{B}[\operatorname{div}_x(h(\rho_\delta)\mathbf{u})]^\alpha \\ &\quad -\mathcal{B}[(\rho h'(\rho_\delta) - h(\rho_\delta))\operatorname{div}_x\mathbf{u} - \frac{1}{|\Omega|} \int_{\Omega} (\rho_\delta h'(\rho_\delta) - h(\rho_\delta)) dx]^\alpha.\end{aligned}\tag{474}$$

□

By the property of \mathcal{B} [51], we have

$$\|[\xi]^\alpha\|_{W^{1,p}} \leq C(p, \Omega) \|[h(\rho_\delta)]^\alpha\|_{L^p}.\tag{475}$$

When $\frac{3}{2} < p < \infty$, $t \in [\alpha, T - \alpha]$, it holds

$$\|[\partial_t \xi]^\alpha\|_{W^{1,p}} \leq C(p, \Omega) \|[h(\rho_\delta)\mathbf{u}]^\alpha\|_{L^p} + \|[(\rho h'(\rho_\delta) - h(\rho_\delta))\operatorname{div}_x\mathbf{u}]^\alpha\|_{L^{\frac{3p}{3+p}}},\tag{476}$$

When $1 < p \leq \frac{3}{2}$, $t \in [\alpha, T - \alpha]$, it holds for any $1 < s < \infty$

$$\|[\partial_t \xi]^\alpha\|_{W^{1,p}} \leq C(p, \Omega) \|[h(\rho_\delta)\mathbf{u}]^\alpha\|_{L^p} + \|[(\rho h'(\rho_\delta) - h(\rho_\delta))\operatorname{div}_x\mathbf{u}]^\alpha\|_{L^s},\tag{477}$$

Then we choose

$$h(\rho_\delta) = \rho_\delta^\nu.\tag{478}$$

where ν is a small parameter to be determined in the later estimates. From the momentum equation

$$\begin{aligned}&\int_0^T \int_{\Omega} \rho_\delta \mathbf{u}_\delta \partial_t \varphi + \rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \nabla_x \varphi + (\rho_\delta \theta_\delta + \frac{\theta_\delta^4}{3} + \rho_\delta^\gamma + \delta(\rho_\delta^\beta + \rho_\delta^2)) \operatorname{div}_x \varphi dx dt \\ &= \int_0^T \int_{\Omega} S_\delta : \nabla_x \varphi dx dt,\end{aligned}\tag{479}$$

we have

$$\int_0^T \left[\int_{\Omega} \zeta(\rho_{\delta}\theta_{\delta} + \frac{\theta_{\delta}^4}{3} + \rho_{\delta}^{\gamma} + \delta(\rho_{\delta}^{\beta} + \rho_{\delta}^2)) [h(\rho_{\delta})]^{\alpha} dx \right] dt = \sum_{j=1}^6 I_j. \quad (480)$$

The integrals are estimated in the following way:

$$\begin{aligned} I_1 &= \frac{1}{|\Omega|} \int_0^T \left[\zeta \int_{\Omega} [h(\rho_{\delta})]^{\alpha} dx \int_{\Omega} (\rho_{\delta}\theta_{\delta} + \frac{\theta_{\delta}^4}{3} + \rho_{\delta}^{\gamma} + \delta(\rho_{\delta}^{\beta} + \rho_{\delta}^2)) dx \right] dt \\ &\leq \frac{1}{2} \int_0^T \int_{\Omega} \left[\zeta(\rho_{\delta}\theta_{\delta} + \frac{\theta_{\delta}^4}{3} + \rho_{\delta}^{\gamma} + \delta(\rho_{\delta}^{\beta} + \rho_{\delta}^2)) [h(\rho_{\delta})]^{\alpha} dx \right] dt + C. \end{aligned} \quad (481)$$

where Hölder, Young inequality are used,

$$\begin{aligned} I_2 &= - \int_{\Omega} \left[\zeta \int_0^T \rho_{\delta} \mathbf{u}_{\delta} \partial_t \xi dx \right] dt \\ &\leq C \|\rho_{\delta}\|_{L^{\infty}L^3} \|\mathbf{u}_{\delta}\|_{L^2L^3} \|[h(\rho_{\delta})]^{\alpha}\|_{L^2L^3}. \\ &\leq C \end{aligned} \quad (482)$$

Here the embedding and the property of \mathcal{B} was used, we have

$$\begin{aligned} I_3 &= - \int_0^T \zeta \int_{\Omega} \rho_{\delta} \mathbf{u}_{\delta} \otimes \mathbf{u}_{\delta} \nabla_x \xi dx dt \\ &\leq C \|\sqrt{\rho_{\delta}}\|_{L^6L^6} \|\sqrt{\rho_{\delta}} \mathbf{u}_{\delta}\|_{L^{\infty}L^2} \|\mathbf{u}_{\delta}\|_{L^2L^{\frac{24}{7}}} \|\nabla_x [\xi]^{\alpha}\|_{L^{\infty}L^{48}} \\ &\leq C \end{aligned} \quad (483)$$

$$\begin{aligned} I_4 &= - \int_0^T \zeta \int_{\Omega} S_{\delta}(\mathbf{u}_{\delta}, \rho_{\delta}) : \nabla_x \xi dx dt \\ &\leq C \|\theta\|_{L^{\infty}L^4} \|\nabla_x u\|_{L^2L^{\frac{8}{5}}} \|\nabla_x \xi\|_{L^{\infty}L^{p_1}} \\ &\leq C \end{aligned} \quad (484)$$

where $p_1 > 1$ could be chosen large when ν is small enough.

$$\begin{aligned} I_5 &= - \int_0^T \zeta' \int_{\Omega} \rho_{\delta} \mathbf{u}_{\delta} \xi dx dt \\ &\leq C \|\sqrt{\rho}\|_{L^{\infty}L^6} \|\sqrt{\rho} u\|_{L^{\infty}L^2} \|\xi\|_{L^{\infty}L^3} \\ &\leq C \end{aligned} \quad (485)$$

as the L^{∞} of ζ' is bounded.

3.6.3 Renormalized solution

On the other hand, we have $\rho \in L^\infty L^3$, then $\|\rho\|_{W^{-1, \frac{3}{2}}} \leq C$, by lemma 13 we have

$$\rho_\delta \rightharpoonup \rho \text{ in } C_{weak}([0, T], L^{\frac{3}{2}}) \quad (486)$$

In addition

$$\rho_\delta \mathbf{u}_\delta \rightharpoonup \rho \mathbf{u} \text{ in } L^\infty L^{\frac{3}{2}}, \quad (487)$$

$$\rho_\delta \rightharpoonup \rho \text{ weakly}^* \text{ in } L^\infty L^\gamma,$$

$$\theta_\epsilon \rightharpoonup \theta \text{ weakly}^* \text{ in } L^\infty L^4,$$

$$\mathbf{u}_\delta \rightharpoonup \mathbf{u} \text{ weakly}^* \text{ in } L^2 W^{1, \frac{8}{5}}, \quad (488)$$

$$\rho_\delta \rightharpoonup \rho \text{ in } C_{weak}([0, T], L^\gamma),$$

$$\rho_\delta \mathbf{u}_\delta \rightharpoonup \rho \mathbf{u} \text{ weakly}^* \text{ in } L^\infty L^{\frac{24}{7}}.$$

By the approximate momentum equation

$$\rho_\delta \mathbf{u}_\delta \rightharpoonup \rho \mathbf{u} \text{ in } C_{weak}([0, T], L^{\frac{24}{7}}). \quad (489)$$

By $\|\mathbf{u}_\delta\|_{L^2 W^{1, \frac{8}{5}}} \leq C$ we have

$$\rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \rightharpoonup \rho \mathbf{u} \otimes \mathbf{u} \text{ in the sence of distribution.} \quad (490)$$

$$\int_{\Omega} \rho(t, x) dx = \int_{\Omega} \rho_{0, \delta}(x) dx \text{ a.e. } t \in [0, T], \quad (491)$$

while $\rho_{0, \delta}(x) \rightarrow \rho_0$ in $L^1(\Omega)$ when $\delta \rightarrow 0$.

Similarly $(\rho \mathbf{u})_t \in L^{p_1}$ for some $p_1 > 1$, then

$$\rho_\delta \mathbf{u}_\delta \rightharpoonup \rho \mathbf{u} \text{ in } C_{weak}([0, T], L^{\frac{3}{2}}), \quad (492)$$

$$\rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \rightharpoonup \rho \mathbf{u} \otimes \mathbf{u} \text{ in } L^{q_1} \text{ for a } q_1 > 1, \quad (493)$$

So we have

$$p(\rho_\delta, \mathbf{u}_\delta) \rightharpoonup \overline{\rho\theta} + \frac{1}{3}\overline{\theta^4} + \overline{\rho^\gamma}. \quad (494)$$

By lemma 15, continuity equation is satisfied in the sense of renormalized solution defined in section 1.6.3.1.

Taking the limit in the momentum equation

$$\begin{aligned} & \int_0^t \int_\Omega \rho \mathbf{u} \cdot \partial_t \varphi + \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + (\overline{\rho\theta} + \overline{\theta^4}) \operatorname{div} \varphi dx d\tau \\ &= \int_0^t \int_\Omega \overline{\mathbb{S}(\theta, \nabla_x \mathbf{u})} : \nabla_x \varphi dx d\tau - \int_\Omega (\rho \mathbf{u})_0 \varphi(0, \cdot) dx \\ & \overline{\mathbb{S}(\theta, \nabla_x \mathbf{u})} = \overline{\mu(\theta)(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \operatorname{div} \mathbf{u} \mathbf{u} I) + \eta(\theta) \operatorname{div} \mathbf{u} I} \end{aligned} \quad (495)$$

Take limit in the approximate total energy equation, we get

$$\int_\Omega \frac{1}{2} \rho |\mathbf{u}|^2 + \overline{\rho\theta + \theta^4}(t) d\tau = \int_\Omega \frac{1}{2} \frac{(\rho \mathbf{u})_0^2}{\rho_0} + \rho_0 \theta_0 + \theta_0^4 dx \quad (496)$$

As we have the condition $\gamma > 3$, one can see that the estimates in the previous section still holds, so that we can use the Div-Curl lemma to get the strong convergence of temperature.

3.6.4 Strong limit of density

When $\gamma > 3$, we can introduce

$$T_k(z) = kT\left(\frac{1}{k}\right), \quad z \geq 0, k \geq 0, \quad (497)$$

The cutoff function $T \in C^\infty[0, \infty)$

$$T(z) = \begin{cases} z & \text{for } 0 \leq z \leq 1 \\ \text{concave on } [1, 3] & \\ 2 & \text{for } z \geq 3 \end{cases}. \quad (498)$$

One can notice that $T(z)$ is concave on $[0, \infty)$. We can use

$$\varphi(t, x) = \psi(t) \xi(x) \nabla_x \Delta^{-1} [1_\Omega, T_k(\rho_\delta)]. \quad (499)$$

as test function where $\psi \in C_c^\infty(0, T)$, $\xi \in C_c^\infty(\Omega)$. Similar to the previous section, we have

$$\begin{aligned} & \int_0^T \int_\Omega \psi \xi [(P(\rho_\delta, \theta_\delta) + \delta(\rho_\delta^\Gamma + \rho_\delta^2)) T_k(\rho_\delta) \\ & \quad - S_\delta : R[1_\Omega T[k](\rho_\delta)]] dx dt \\ & = \sum_{j=1}^6 I_{j,\delta}. \end{aligned} \tag{500}$$

$$\begin{aligned} I_{1,\delta} &= \int_0^T \int_\Omega \psi \xi (\rho_\delta \mathbf{u}_\delta \cdot R[1_\Omega T_k(\rho_\delta) \mathbf{u}_\delta] - \rho \mathbf{u}_\delta \otimes \mathbf{u}_\delta : R[1_\Omega T_k(\rho_\delta)]) dx dt, \\ I_{2,\delta} &= - \int_0^T \int_\Omega \psi \xi \rho_\delta \mathbf{u}_\delta \cdot \nabla_x \Delta^{-1} [1_\Omega (T_k(\rho_\delta) - T'_k(\rho_\delta) \rho_\delta) \operatorname{div} \mathbf{u}_\delta] dx d\tau, \\ I_{3,\delta} &= - \int_0^T \int_\Omega \psi (P(\rho_\delta \theta_\delta) + \delta(\rho_\delta^\Gamma + \rho_\delta^2)) \nabla_x \xi \cdot \nabla_x \Delta_x^{-1} [1_\Omega T_k(\rho_\delta)] dx d\tau, \\ I_{4,\delta} &= - \int_0^T \int_\Omega \psi \mathbb{S}_\delta(\theta_\delta, \nabla_x \mathbf{u}_\delta) : \nabla_x \xi \otimes \nabla_x \Delta^{-1} [1_\Omega T_k(\rho_\delta)] dx d\tau, \\ I_{5,\delta} &= - \int_0^T \int_\Omega \psi (\rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla_x \xi \otimes \nabla_x \Delta^{-1} [1_\Omega T_k(\rho_\delta)] dx dt, \\ I_{6,\delta} &= - \int_0^T \int_\Omega \partial_t \psi \xi \rho_\delta \mathbf{u}_\delta \cdot \nabla_x \Delta^{-1} [1_\Omega T_k(\rho_\delta)] dx dt. \end{aligned} \tag{501}$$

Similarly, for the limiting parts we define

$$\begin{aligned} \varphi(t, x) &= \psi(t) \xi(x) \nabla_x \Delta^{-1} [1_\Omega \overline{T_k(\rho)}] \\ \psi &\in C_c^\infty(0, T), \quad \xi \in C_c^\infty(\Omega) \end{aligned} \tag{502}$$

We have

$$\int_0^T \int_\Omega 4\xi \left[\overline{\rho\theta + \rho^r} + \frac{1}{4}\theta^4 \right] \overline{T_k(\rho)} - S_\delta(\theta, \nabla_x u) R[1_\Omega \overline{T_k(\rho)}] dx dt = \sum_{j=1}^7 I_j \tag{503}$$

These estimates will be the same as [51], so there is a relation of the effective flux:

$$\begin{aligned} & \overline{(\rho\theta + \rho^r) T_k(\rho)} - \left(\frac{4}{3} \mu(\theta) + \eta(\theta) \right) \overline{T_k(\rho) \operatorname{div} \mathbf{u}} \\ & = \overline{(\rho\theta + \rho^r) T_k(\rho)} - \left(\frac{4}{3} \mu(\theta) + \eta(\theta) \right) \overline{T_k(\rho) \operatorname{div} \mathbf{u}} \end{aligned} \tag{504}$$

By lemma(15), ρ, \mathbf{u} is the solution of renormalized equation. Define

$$\begin{aligned}
L_k(\rho) &= \int_1^\rho \frac{T_k(z)}{z^2} dz \\
\int_0^T \int_\Omega \left(\overline{\rho L_k(\rho)} \partial_t \varphi + \overline{\rho L_k(\rho)} \mathbf{u} \cdot \nabla_x \varphi - \overline{T_k(\rho) \operatorname{div} \mathbf{u}} \varphi \right) dx dt & \quad (505) \\
&= - \int_\Omega \rho_0 L_k(\rho_0) \cdot \varphi(0, \cdot) dx.
\end{aligned}$$

So $\overline{T_k(\rho) \operatorname{div} \mathbf{u}} - \overline{T_k(\rho)} \operatorname{div}_x \mathbf{u} \geq 0$ As $\gamma > 3$ we have

$$\lim_{k \rightarrow \infty} \int_0^t \int_\Omega (T_k(\rho) \operatorname{div} \mathbf{u} - \overline{T_k(\rho)} \operatorname{div} \mathbf{u}) dx dt \rightarrow 0. \quad (506)$$

So

$$\begin{aligned}
\overline{\rho \log \rho} &= \rho \log \rho, \\
\rho_\delta &\rightarrow \rho \text{ a.e. in } Q.
\end{aligned} \quad (507)$$

This completes the proof of the existence theorem.

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VITA

Weizhe Zhang was born in Yuanshi, Hebei, China on November 11 1985. At age 18, he went to the University of Science and Technology of China (USTC) in Hefei, China, and graduated with a Bachelor of Science in mathematics in June of 2008. In July of 2008, he came to Atlanta and joined the Georgia Institute of Technology to work in Theory of Mathematical Fluid Dynamics with Professor Ronghua Pan in the School of Mathematics.

Weizhe married Jingfang Liu in April 2008. Their son Andy was born in Atlanta 2009. Jingfang obtained PhD in Computational Sciences and Engineering from the School of Mathematics in 2013 under the supervision of Dr. Haomin Zhou.