GRAPH STRUCTURES AND WELL-QUASI-ORDERING

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To my parents

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SUMMARY

Robertson and Seymour proved that graphs are well-quasi-ordered by the minor relation and the weak immersion relation. In other words, given infinitely many graphs, one graph contains another as a minor (or a weak immersion, respectively). An application of these theorems is that every property that is closed under deleting vertices, edges, and contracting (or "splitting off", respectively) edges can be characterized by finitely many graphs, and hence can be decided in polynomial time.

In this thesis we are concerned with the topological minor relation. We say that a graph G contains another graph H as a *topological minor* if H can be obtained from a subgraph of G by repeatedly deleting a vertex of degree two and adding an edge incident with the neighbors of the deleted vertex. Unlike the relation of minor and weak immersion, the topological minor relation does not well-quasi-order graphs in general. However, Robertson conjectured in the late 1980's that for every positive integer k, the topological minor relation well-quasi-orders graphs that do not contain a topological minor isomorphic to the path of length k with each edge duplicated.

This thesis consists of two main results. The first one is a structure theorem for excluding a fixed graph as a topological minor, which is analogous to a cornerstone result of Robertson and Seymour, who gave such a structure for graphs that exclude a fixed minor. Results for topological minors were previously obtained by Grohe and Marx and by Dvořák, but we push one of the bounds in their theorems to the optimal value. This improvement is needed for the next theorem.

The second main result is a proof of Robertson's conjecture. As a corollary, properties on certain graphs closed under deleting vertices, edges, and "suppressing" vertices of degree two can be characterized by finitely many graphs, and hence can

be decided in polynomial time.

CHAPTER I

INTRODUCTION

1.1 Basic graph notions

A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a nonempty finite set $V(G)$ and a multiset $E(G)$ of two-element multisubsets of $V(G)$. A *vertex* of G is an element of $V(G)$, and an *edge* of G is an element of $E(G)$. In fact, we allow $V(G)$ to be infinite, but we call it an *infinite graph* in this case. If $e = \{u, v\}$ is an edge of G, then we say that u, v are the *ends* of e, and e is *incident* with u and v in G. As an edge is a multiset, it is possible that u equals v. In this case, we say that e is a *loop* with end u. Furthermore, $E(G)$ is a multiset, so it is possible that different edges have the same ends. We say that two vertices x, y are *adjacent* in G if x, y are the ends of some edge of G. A vertex u is a *neighbor* of a vertex v in G if u is adjacent to v in G. For every subset X of $V(G)$, we denote the set $\{y \notin X : y$ is adjacent to a vertex in $X\}$ by $N_G(X)$. And we define $N_G[X]$ to be $N_G(X) \cup X$. When the underlying graph G is clear, we denote $N_G(X)$ and $N_G[X]$ by $N(X)$ and $N[X]$, respectively. If X consists of one vertex v, then we denote $N_G(\{v\})$ and $N_G[\{v\}]$ by $N(v)$ and $N[v]$, respectively. The *degree* of a vertex in G is the number of incident edges, counting each loop twice.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(H)$. For every nonempty subset X of $V(G)$, the *subgraph of* G *induced by* X, denoted by $G[X]$, is the graph with $V(G[X]) = X$ such that every edge of G with both ends in X is an edge of $G[X]$. A subgraph of G is *induced* if it is induced by some subset of $V(G)$. In addition, for every subset X of $V(G)$, we denote the induced subgraph $G[V(G) - X]$ by $G - X$. Similarly, for every vertex v of G, we denote $G[V(G) - \{v\}]$ by $G - v$.

Given an edge e of G, the subgraph $(V(G), E(G) - \{e\})$ is denoted by $G - e$. Two subgraphs G_1, G_2 of G are *vertex-disjoint* or *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. And G_1, G_2 are *edge-disjoint* if $E(G_1) \cap E(G_2) = \emptyset$.

We say that G is a *simple graph* if $E(G)$ is a set of two-element subsets of $V(G)$. A *complete graph* on n vertices, denoted by K_n, is the simple graph on n vertices such that each pair of distinct vertices are adjacent. A *path* on n vertices is a simple graph such that one can write the set of vertices as $\{v_1, v_2, ..., v_n\}$ such that the set of edges is $\{v_i v_{i+1} : 1 \leq i \leq n-1\}$. In this case, v_1 and v_n are the *ends* of the path, and we say that this path is from v_1 to v_n. We also say that this path is a v_1-v_n path. Similarly, for every two subsets X, Y of a graph $V(G)$, a path is a X-Y *path* in G or a *path from from X to Y* in G if it is a subgraph of G such that one end is in X and the other is in Y and no other vertex is in $X \cup Y$. A *cycle* on n vertices, where $n \geq 3$, is a simple graph that can be obtained from a path on n vertices by adding an edge with ends the ends of the path. The *length* of a path or a cycle is the number of its edges. The cycle of length 1 is the graph consisting of a vertex and a loop; the cycle of length 2 is the graph consisting of two adjacent vertices and two edges with the same ends. We say that a graph is a *forest* if it does not contain a cycle as a subgraph. A *walk* of length n in a graph G is a subgraph W of G such that $E(W)$ can be written as $\{e_1, e_2, ..., e_n\}$ such that there exists a sequence $v_1 e_1 v_2 e_2 ... e_n v_{n+1}$ such that $v_i \in V(W)$ for $1 \leq i \leq n$ and e_j is incident with v_j and v_{j+1} for $1 \leq j \leq n$, and every vertex of W is in $\{v_i : 1 \leq i \leq n+1\}$. And when $v_1 \neq v_{n+1}$, we say that v_1 and v_{n+1} are the *ends* of W, and any vertex in $V(W) - \{v_1, v_{n+1}\}$ is an *internal vertex* of W; when $v_1 = v_{n+1}$, we say that the walk is *closed*. Two non-closed walks W_1, W_2 in a graph G are *internally vertex-disjoint* or *internally disjoint* if every vertex of $V(W_1) \cap V(W_2)$ is an end of both W_1, W_2.

A graph G is *connected* if for every two vertices u, v of G, there exists a path from u to v in G. If a graph is not connected, then every maximal connected subgraph is called a *component* of the graph. A graph G is *k-connected* if $|V(G)| \geq k+1$ and there does not exist $X \subseteq V(G)$ with $|X| \leq k-1$ such that $G - X$ is not connected. A *tree* is a connected forest.

Given a graph H, an *H-minor* of a graph G is a map α with domain $V(H) \cup E(H)$ such that the following hold.

- $\alpha(h)$ is a nonempty connected subgraph of G, for every $h \in V(H)$.
- If h_1 and h_2 are distinct vertices in H, then $\alpha(h_1)$ and $\alpha(h_2)$ are disjoint.
- For each non-loop e of H with ends h_1, h_2, $\alpha(e)$ is an edge of G with one end in $\alpha(h_1)$ and one end in $\alpha(h_2)$; for each loop e of H with end h, $\alpha(e) \in E(G) - E(\alpha(h))$ with both ends in $V(\alpha(h))$.
- If e_1, e_2 are two different edges of H, then $\alpha(e_1) \neq \alpha(e_2)$.

We say that G *contains an H-minor* or G *contains H as a minor* if such a function α exists. For every $h \in V(H)$, $\alpha(h)$ is called a *branch set* of α.

Given a graph H, an *H-subdivision* (or an *H-topological minor*) in a graph G is a pair of functions (π_V, π_E) such that the following hold.

- $\pi_V : V(H) \to V(G)$ is an injective function.
- π_E maps each loop of H to a cycle in G and maps each non-loop of H to a path in G such that $\pi_E(e)$ contains $\pi_V(v)$ for every loop e with end v, and $\pi_E(e')$ has ends $\pi_V(x)$ and $\pi_V(y)$ for every non-loop $e' = xy \in E(H)$.
- $\pi_E(e) \cap \pi_E(f)$ is contained in the image of π_V for distinct edges e, f of H.
- $\pi_V(z) \notin V(\pi_E(e))$ if z is not an end of e.

We say that G *admits an H-subdivision* or G *contains H as a subdivision* if such a pair of functions (π_V, π_E) exists. We also say that G *contains H as a topological minor* in this case.

Given a graph H, a *weak H-immersion* in a graph G is a pair of functions (π_V, π_E) such that the following hold.

- $\pi_V : V(H) \to V(G)$ is an injective function.
- π_E maps each loop of H to a cycle in G and maps each non-loop of H to a path in G such that $\pi_E(e)$ contains $\pi_V(v)$ for every loop e with end v, and $\pi_E(e')$ has ends $\pi_V(x)$ and $\pi_V(y)$ for every non-loop $e' = xy \in E(H)$.
- If f_1, f_2 are two different edges in H, then $\pi_E(f_1)$ and $\pi_E(f_2)$ are edge-disjoint.

We say that G *admits a weak H-immersion* or G *contains H as a weak H-immersion* if such a pair of functions (π_V, π_E) exists.

A *strong H-immersion* (π_V, π_E) in a graph G is a weak H-immersion such that for every edge e of H and vertex v of H, $\pi_V(v) \notin V(\pi_E(e))$ unless v is an end of e. We say that G *admits a strong H-immersion* or G *contains H as a strong immersion* if such a pair of functions (π_V, π_E) exists.

Please refer to [8] for other undefined notions in the rest of the thesis.

1.2 Well-quasi-ordering

For a set S, a *relation* on S is a set of ordered pairs of elements of S. If $\preceq$ is a relation, then we write $x \preceq y$ if $(x, y) \in \preceq$. A relation $\preceq$ is *reflexive* if $x \preceq x$ for every $x \in S$; it is *transitive* if $x \preceq y$ and $y \preceq z$ implies that $x \preceq z$ for every $x, y, z \in S$. A relation $\preceq$ on a set S is a *quasi-ordering* on S if it is reflexive and transitive. A quasi-ordering on a set S is a *well-quasi-ordering* if for every infinite sequence $x_1, x_2, ...$ on S, there exist $j < j'$ such that $x_j \preceq x_{j'}$. In this case, we say that $(S, \preceq)$ is a *well-quasi-ordered set*, and S is *well-quasi-ordered by* $\preceq$. A survey of the history of the development of well-quasi-ordering theory can be found in [24].

Let $\preceq$ be a quasi-ordering on a set S. We say that $X \subseteq S$ is an *antichain* in $(S, \preceq)$ if $x \not\preceq y$ for every pair of distinct elements $x, y \in X$. For elements x, y of S, we write

$x \prec y$ if $x \preceq y$ and $x \neq y$.

The following proposition offers an alternative definition of well-quasi-ordering.

Proposition 1.2.1 ([8, Proposition 12.1.1]) *A quasi-ordering $\preceq$ on S is a well-quasi-ordering if and only if there does not exist an infinite antichain in $(S, \preceq)$ and there does not exist an infinite sequence $x_1, x_2, ...$ in S such that $x_{i+1} \prec x_i$ for every $i \geq 1$.*

Results on well-quasi-ordering not only have theoretical interest but also lead to applications in algorithms. We postpone these applications to Section 1.2.2. In Section 1.2.1, we survey some results of well-quasi-ordering graphs.

1.2.1 Well-quasi-ordering graphs

1.2.1.1 Topological minor relation

The history of well-quasi-ordering graphs can be dated to the 1940's. Vázsonyi conjectured that forests are well-quasi-ordered by the topological minor relation. This conjecture was proved by Kruskal [23] and independently by Tarkowski [49] in 1960. Then Nash-Williams [31] offered an elegant and simpler proof of this theorem.

Theorem 1.2.2 ([23, 49, 31]) *Forests are well-quasi-ordered by the topological minor relation.*

In addition, Kruskal [23] conjectured that infinite trees are well-quasi-ordered by the topological minor relation, and it was confirmed by Nash-Williams [32], and a shorter proof was provided by Kühn [25]. Indeed, they prove that infinite trees are better-quasi-ordered by the topological minor relation. Better-quasi-ordering, introduced by Nash-Williams, is a concept that is stronger than well-quasi-ordering, but we will not discuss this notion in the rest of the thesis. We refer interested readers to [32].

Theorem 1.2.3 ([32, 25]) *Infinite forests are well-quasi-ordered by the topological minor relation.*

On the other hand, Mader [27] and Fellows, Hermelin and Rosamond [14] generalized Theorem 1.2.2 as follows. A *feedback vertex set* in a graph is a subset of vertices such that every cycle contains a vertex in this set.

Theorem 1.2.4 ([27]) *Let k be a positive integer.*

1. *[27] Graphs that do not contain k disjoint cycles are well-quasi-ordered by the topological minor relation.*

2. *[14] Graphs that have a feedback vertex set of size at most k are well-quasi-ordered by the topological minor relation.*

In fact, by a result of Erdős and Pósa [13], the two statements in Theorem 1.2.4 are equivalent.

Unlike the relation of minor and weak immersion (which we will discuss later), the topological minor relation does not well-quasi-order graphs in general. Robertson in the late 1980's conjectured that the known obstruction is the only one. More precisely, he conjectured that for every positive integer k, graphs with no topological minor isomorphic to the path of length k with each edge duplicated are well-quasi-ordered by the topological minor relation. Robertson's conjecture is stronger than another conjecture of Vázsonyi: subcubic graphs are well-quasi-ordered by the topological minor relation. We say that a graph is *subcubic* if every vertex has degree at most three. We remark that this Vázsonyi's conjecture follows from the Graph Minor Theorem of Robertson and Seymour. We will address the minor relation in this section soon.

The main objective of this thesis is to prove Robertson's conjecture. This result not only solves Vázsonyi's conjecture but also generalizes Theorems 1.2.2 and 1.2.4. We will elaborate on this in Section 1.4.

1.2.1.2 Minor relation

In the 1980's, Robertson and Seymour announced a proof of the following theorem. It was first conjectured by Wagner [54], and now it is known as the Graph Minor Theorem.

Theorem 1.2.5 ([47] Graph Minor Theorem) *Graphs are well-quasi-ordered by the minor relation.*

The Graph Minor Theorem is one of the most prominent and deepest results in Graph Theory. The proof is extremely difficult and consists of around 20 papers in the Graph Minors series. The tools developed in the Graph Minors series had a significant impact in structural graph theory, and we will discuss this in subsequent sections. Moreover, the Graph Minor Theorem also confirms the conjecture of Vazsonyi on subcubic graphs, since the topological minor relation is the same as the minor relation on subcubic graphs.

Thomas [51] proved that Theorem 1.2.5 cannot be generalized to uncountable graphs. In contrast, whether countable graphs are well-quasi-ordered by the minor relation is wide open. However, Thomas [52] proved that a class of infinite graphs (countably or uncountably infinite) is well-quasi-ordered by the minor relation.

Theorem 1.2.6 ([52]) *For every finite planar graph H, (finite or infinite) graphs that do not contain H as a minor are well-quasi-ordered by the minor relation.*

The following related conjecture of Seymour is open as well. A graph H is a *proper minor* of G if G contains H as a minor with at least one non-trivial branch set.

Conjecture 1.2.7 (Seymour's Self-minor Conjecture) *Every countably infinite graph is a proper minor of itself.*

Similarly, Oporowski [33] proved that Conjecture 1.2.7 cannot be generalized to uncountable graphs. On the other hand, Pott [36] confirmed Conjecture 1.2.7 for infinite trees.

Theorem 1.2.8 ([36]) *Every infinite tree is a proper minor of itself.*

1.2.1.3 Immersion relation

In the 1960's, Nash-Williams conjectured that graphs are well-quasi-ordered by the weak immersion relation [30] and the strong immersion relation [32]. Robertson and Seymour confirmed the weak immersion conjecture in the (currently) last paper in the Graph Minors series [48] by strengthening the statement of the Graph Minors Theorem and reducing the conjecture to it. However, the strong immersion conjecture is still open. Robertson and Seymour [48] mentioned that at one time they believed they had a proof of the strong immersion conjecture, but it was very complicated, and it is unlikely that they will write it down.

Theorem 1.2.9 ([48, Theorem (1.1)]) *Graphs are well-quasi-ordered by the weak immersion relation.*

Conjecture 1.2.10 ([32]) *Graphs are well-quasi-ordered by the strong immersion relation.*

Andreae [1] proved the following special cases of Conjecture 1.2.10.

Theorem 1.2.11 ([1]) *The following classes of graphs are well-quasi-ordered by the strong immersion relation.*

1. *Simple graphs that do not contain $K_{2,3}$ as a strong immersion.*
2. *Simple graphs whose blocks are complete graphs, cycles, or complete bipartite graphs.*

A possible generalization of the immersion conjectures is to consider directed graphs. In this version, we ask for directed paths in directed graphs instead of paths in undirected graphs. However, directed graphs are not well-quasi-ordered by the weak immersion relation in general. (Consider cycles of even length whose edges are directed alternately clockwise and counterclockwise.) But Chudnovsky and Seymour proved a positive result for strong immersion.

Theorem 1.2.12 ([4]) *Directed complete graphs (i.e tournaments) are well-quasi-ordered by the strong immersion relation.*

1.2.1.4 Induced subgraph relation

In general, subgraph relation and induced subgraph relation do not well-quasi-order graphs. For example, given infinitely many cycles of different lengths, no cycle can contain another as a subgraph or an induced subgraph. However, it is possible to well-quasi-order graphs by these two relations if we restrict the problem to smaller classes of graphs. Damaschke [7], Ding [11], Petkovšek [35], Korpelainen and Lozin [22], Fellows, Hermelin and Rosamond [14], and Atminas, Brignall, Korpelainen, Lozin and Vatter [2] proved the following positive results if we exclude some graphs as induced subgraphs or restrict the problem to special classes of graphs.

Theorem 1.2.13 ([7]) *The members of the following classes of graphs are well-quasi-ordered by the induced subgraph relation.*

1. *Simple graphs that do not contain a path on four vertices as an induced subgraph are well-quasi-ordered by the induced subgraph relation.*

2. *Simple graphs that do not contain K_3 and the disjoint union of K_2 and two copies of K_1 as induced subgraphs.*

3. *Simple graphs that do not contain K_3 and the path on five vertices as induced subgraphs.*

The following results include terms we did not define. As we will not need these notions in the rest of the thesis, we refer the reader to [11, 35, 22, 14, 2] for precise definitions.

Theorem 1.2.14 ([11]) *The following classes of graphs are well-quasi-ordered by the induced subgraph relation.*

1. *Simple bipartite graphs that do not contain a path on seven vertices and two "specific bipartite graphs" as induced subgraphs.*

2. *Simple bipartite graphs that do not contain a path on eight vertices and its "bipartite complement" as induced subgraphs.*

Theorem 1.2.15 ([35]) *For every fixed k, "k-letter graphs" are well-quasi-ordered by the induced subgraph relation.*

Theorem 1.2.16 ([22]) *The members of the following classes of graphs are well-quasi-ordered by the induced subgraph relation.*

1. *Simple bipartite graphs that do not contain a path on seven vertices and a "specific graph" as induced subgraphs.*

2. *Simple bipartite graphs that do not contain a path on seven vertices and a cycle of length four as induced subgraphs.*

3. *Simple bipartite permutation graphs.*

Theorem 1.2.17 ([14]) *For every positive integer k, simple graphs that have a vertex cover of size at most k are well-quasi-ordered by the induced subgraph relation.*

Theorem 1.2.18 ([2]) *For every positive integer k, permutation graphs that do not contain P_5 or K_k as induced subgraphs are well-quasi-ordered by the induced subgraph relation.*

However, Ding [11] showed that graphs that do not contain a path on five vertices are not well-quasi-ordered by the induced subgraph relation. More examples showing that the induced subgraph relation does not well-quasi-order graphs even if we exclude some graphs as induced subgraphs can be found in [11, 22, 2]. Furthermore, Korpelainen and Lozin [21] gave more positive and negative results for well-quasi-ordering graphs by the induced subgraph relation.

On the other hand, Ding [11] proved that excluding a path as a subgraph is sufficient to make the induced subgraph relation well-quasi-order graphs. He also proved a similar result for directed graphs.

Theorem 1.2.19 ([11]) *Let k be a positive integer.*

1. *Graphs that do not contain a path of length k as a subgraph are well-quasi-ordered by the induced subgraph relation.*

2. *Directed graphs with underlying graph containing no path of length k as a subgraph are well-quasi-ordered by the induced subdigraph relation.*

We say that I is an *ideal* with respect to a quasi-ordering $\preceq$ if for every $x, y \in I$ with $x \preceq y$, $y \in I$ implies $x \in I$. Ding [11] characterized the ideals of graphs with respect to the subgraph relation such that members in the ideals are well-quasi-ordered by the induced subgraph relation.

Theorem 1.2.20 ([11]) *Let I be an ideal of graphs with respect to the subgraph relation. Then the following are equivalent.*

1. *Members of I are well-quasi-ordered by the subgraph relation.*

2. *Members of I are well-quasi-ordered by the induced subgraph relation.*

3. *There exists a positive integer k such that I does not contain any cycle of length at least k and any graph that can be obtained from a path of length at least k by attaching two leaves to each end of the path.*

1.2.1.5 Other graph containment relations

We survey more results for well-quasi-ordering graphs by other containment relations. We will not formally define these relations as we will not use these notions in the rest of the thesis.

Theorem 1.2.21 ([50]) *Graphs that do not contain K_4 as a subdivision are well-quasi-ordered by the induced minor relation.*

Thomas [50] also showed that planar graphs are not well-quasi-ordered by the induced minor relation.

Theorem 1.2.22 ([14]) *For every positive integer k, graphs that do not contain a cycle of length greater than k as a subgraph are well-quasi-ordered by the induced minor relation.*

Theorem 1.2.23 ([34]) *For every positive integer k, graphs of rank-width at most k are well-quasi-ordered by the pivot-minor relation.*

Theorem 1.2.24 ([5]) *Graphs are well-quasi-ordered by the Rao-containment relation.*

1.2.2 Applications of well-quasi-ordering

One consequence of well-quasi-ordering is that every property closed under a well-quasi-ordering can be characterized by finitely many objects. More precisely, we say that a property $\mathcal{Q}$ of graphs is *closed under a relation* $\preceq$ if for every graphs G and H with $G \preceq H$, H satisfies $\mathcal{Q}$ implies that G satisfies $\mathcal{Q}$.

Theorem 1.2.25 *Let $\mathcal{Q}$ be a property of graphs closed under a well-quasi-ordering $\preceq$. Then there exist an integer k (only depending on $\mathcal{Q}$) and graphs $H_1, H_2, ..., H_k$ such that every graph G satisfies $\mathcal{Q}$ if and only if $H_i \not\preceq G$ for every $1 \leq i \leq k$.*

Proof. Let $\mathcal{F}$ be the family of graphs that do not satisfy $\mathcal{Q}$, and let $\mathcal{M}$ be the set of minimal elements of $\mathcal{F}$ under the relation $\preceq$. Therefore, $(\mathcal{M}, \preceq)$ is an antichain. Since $\preceq$ is a well-quasi-ordering, $|\mathcal{M}|$ is finite. Consequently, there exist graphs $H_1, H_2, ..., H_{|\mathcal{M}|}$ such that a graph H is in $\mathcal{F}$ implies that $H_i \preceq H$ for some $1 \leq i \leq |\mathcal{M}|$. On the other hand, $\mathcal{Q}$ is closed under $\preceq$. So if G satisfies $\mathcal{Q}$ and $H_i \preceq G$ for some $1 \leq i \leq |\mathcal{M}|$, then H_i satisfies $\mathcal{Q}$ and is not in $\mathcal{F}$, a contradiction. In other words, a graph G satisfies $\mathcal{Q}$ if and only if $H_i \not\preceq G$ for every $1 \leq i \leq |\mathcal{M}|$. ■

Theorem 1.2.25 shows the power of well-quasi-ordering. We take a question in topological graph theory as an example. Kuratowski [26] proved that a graph can be drawn in the plane if and only if it does not contain K_5 or $K_{3,3}$ as a topological minor. Erdős asked in the 1930's whether for every surface, there exists a list of finite size such that a graph can be drawn in the surface if and only if this graph does not contain any member of the list as a topological minor. Though some progress on this problem was reported, it was wide open in general, until in the 1980's it was solved by Robertson and Seymour [39]. With a simple argument, they proved that it is sufficient to answer the same question for the minor containment, which is the following immediate corollary of the Graph Minor Theorem and Theorem 1.2.25.

Corollary 1.2.26 *For every surface Σ, there exist an integer k and graphs $H_1, H_2, ..., H_k$ such that a graph G can be drawn in Σ if and only if G does not contain H_i as a minor for every $1 \leq i \leq k$.*

Proof. For every surface Σ, whether a graph can be drawn in Σ is a minor-closed property. According to Graph Minor Theorem, the minor relation is a well-quasi-ordering on graphs. Then it follows from Theorem 1.2.25. ■

We should remark that using the Graph Minor Theorem to prove Corollary 1.2.26 is overkill. But the original proof of Robertson and Seymour is also about well-quasi-ordering.

Another application of well-quasi-ordering is in algorithms. According to Theorem 1.2.25, to test whether a graph satisfies a property that is closed under a well-quasi-ordering, it is sufficient to test whether this graph contains one of finitely many specific graphs under this well-quasi-ordering. Therefore, if we are able to test the containment in polynomial time, then a polynomial time algorithm to test the property exists.

In fact, Robertson and Seymour [43] proved that the minor containment can be tested in polynomial time.

Theorem 1.2.27 ([43]) *For a fixed graph H, whether a given graph G contains H as a minor can be decided in $O(|V(G)|^3)$-time.*

Corollary 1.2.28 *Every minor-closed property can be tested in polynomial time.*

Fellows and Langston [15] used this observation to deduce the existence of polynomial time algorithms or fixed-parameter tractable algorithms for several problems, where for some of them it was unclear whether they belong to **NP** or even whether they are decidable.

An $O(|V(G)|^{|V(H)|+3})$-time algorithm for testing topological minor containment follows from [43]. A seminal result of Grohe, Kawarabayashi, Marx and Wollan [17] shows that topological containment and weak immersion containment are fixed parameter tractable.

Theorem 1.2.29 ([17]) *For every fixed graph H, there exists a $O(|V(G)|^3)$-time algorithm to decide whether G contains H as a topological minor or a weak immersion.*

As weak immersion well-quasi-orders graphs, we have the following corollary.

Corollary 1.2.30 *Every weak immersion-closed property can be tested in polynomial time.*

In the same way, Fellows and Langston [16] used this fact to deduce polynomial time algorithms for problems that were unknown to be solvable in polynomial time, and gave more efficient algorithms for some other problems.

In this thesis, we will prove certain graphs are well-quasi-ordered by the topological minor relation. Therefore, the existence of more polynomial time algorithms for several problems follows.

1.3 Excluding minors and topological minors

One step to prove our result on well-quasi-ordering is to investigate the structure of graphs that do not contain a fixed graph as a topological minor. This information plays an important role in structural graph theory and algorithm design.

In this section, we survey theorems for excluding minors and topological minors. We will formally state these results, and the missing definitions will be given later. (Please refer to Section 2.1 for the definitions of segregations and arrangements and Section 3.1 for the definitions of tree decompositions, width, and adhesion.) Given a tree decomposition $(T, \mathcal{X})$ of a graph G, the *torso* at a node t of T is the graph obtained from $G[X_t]$ by adding edges such that every pair of vertices in $X_t \cap X_{t'}$ are adjacent, for every neighbor t' of t in T. A *path decomposition* is a tree decomposition $(T, \mathcal{X})$ in which T is a path.

The cornerstone of the Graph Minors project of Robertson and Seymour is the following excluded minor theorem.

Theorem 1.3.1 ([45, Theorem (1.3)]) *Let L be a graph. Then there exist numbers κ, ρ, ξ such that every graph G with no L-minor has a tree decomposition $(T, \mathcal{X})$ such that for every $t \in V(T)$, there exists a segregation $\mathcal{S}_1 \cup \mathcal{S}_2$ with $|\mathcal{S}_2| \leq \kappa$ of a graph that is obtained from the torso at t by deleting at most ξ vertices having a proper arrangement on a surface in which L cannot be drawn, where each member of $\mathcal{S}_1$ consists of an edge and each member (S, Ω) of $\mathcal{S}_2$ satisfies that S has a path* *decomposition of width at most ρ such that every bag contains one vertex in $\bar{\Omega}$.*

The first theorem for excluding topological minors was obtained by Grohe and Marx [18], and they applied this to obtain polynomial time algorithms for some problems, such as isomorphism test for graphs excluding a fixed graph as a topological minor.

Theorem 1.3.2 ([18, Theorem 4.1]) *For every $h \in \mathbb{N}$, there exist constants a, ξ, D, b such that the following holds. Let H be a graph on h vertices. Then for every graph G with no H-subdivision, there exists a tree decomposition $(T, \mathcal{X})$ of G of adhesion at most a such that for every $t \in V(T)$, the torso at t either contains at most ξ vertices of degree at least D or does not contain a K_b-minor.*

By combining with Theorem 1.3.1, Theorem 1.3.2 can be stated as follows.

Theorem 1.3.3 ([18, Corollary 4.4]) *For every $h \in \mathbb{N}$, there exist constants $g = O(|V(H)|^4), a, \xi, D, \kappa, \rho$ such that the following holds. Let H be a graph on h vertices. Then for every graph G with no H-subdivision, there exists a tree decomposition $(T, \mathcal{X})$ of G of adhesion at most a such that for every $t \in V(T)$, either*

1. *the torso at t contains at most ξ vertices of degree at least D, or*
2. *there exists a segregation $\mathcal{S}_1 \cup \mathcal{S}_2$ with $|\mathcal{S}_2| \leq \kappa$ of a graph that is obtained from the torso at t by deleting at most ξ vertices having a proper arrangement on a surface of genus at most g, where each member of $\mathcal{S}_1$ consists of an edge and each member (S, Ω) of $\mathcal{S}_2$ satisfies that S has a path decomposition of width at most ρ such that every bag contains one vertex in $\bar{\Omega}$.*

Let H be a graph and Σ a surface in which H can be embedded. We define $\mathrm{mf}(H, \Sigma)$ as the minimum of $|S|$, over all embeddings of H in Σ and all sets S of faces of the embedded graph such that every vertex of H of degree at least four is incident with a face in S. When H cannot be embedded in Σ, we define $\mathrm{mf}(H, \Sigma)$ to be infinity. Dvořák strengthened the result by giving neat information for the surface as follows.

Theorem 1.3.4 ([12, Theorem 3]) *For every graph H, there exist constants $g = O(|V(H)|^2), D, \kappa, \rho, \xi, n$ and m with the following property. Every graph G with no H-subdivision has a tree decomposition $(T, \mathcal{X})$ such that for every $t \in V(T)$, there exists a subset Z_t of $V(G_t)$ of size at most ξ, where G_t is the torso at t, such that the graph $G'_t = G_t - Z_t$ satisfies one of the following conditions:*

1. *the maximum degree of G'_t is less than D, or*
2. *there exists a segregation $\mathcal{S}_1 \cup \mathcal{S}_2$ with $|\mathcal{S}_2| \leq \kappa$ of G'_t having a proper arrangement on some surface Σ of genus at most g, where each member of $\mathcal{S}_1$ consists of an edge and each member (S, Ω) of $\mathcal{S}_2$ satisfies that S has a path decomposition of width at most ρ such that every bag contains a vertex in $\bar{\Omega}$, such that either*
   - *(a) H cannot be drawn in Σ, or*
   - *(b) H can be drawn in Σ and $\mathrm{mf}(H, \Sigma) \geq 2$, and there exist $\mathcal{S}'_1 \subseteq \mathcal{S}_1$, $\mathcal{S}'_2 \subseteq \mathcal{S}_2$ with $|\mathcal{S}'_2| \leq \mathrm{mf}(H, \Sigma) - 1$, and a segregation $\mathcal{S}'$ of $\bigcup_{(S,\Omega) \in \mathcal{S}'_1} S$ such that each vertex of G'_t with degree at least D belongs to a member of $\mathcal{S}' \cup \mathcal{S}'_2$, and each member (S, Ω) of $\mathcal{S}'$ is arranged in a disk and satisfies that $|\bar{\Omega}| \leq n$ and can be drawn in the plane such that the vertices in $\bar{\Omega}$ are the vertices incident with the infinite face in the order Ω and there exist $C \subseteq V(S)$ with $|C| < \mathrm{mf}(H, \mathbb{S})$ and $Z_S \subseteq V(S)$ with $|Z| \leq \xi$ such that for every vertex v of $S - Z_S$ with degree greater than D, there exists a curve joining v with a vertex in C intersecting S at most m vertices, where $\mathbb{S}$ is the sphere.*

One objective of this thesis is to further strengthen Dvořák's theorem. We will elaborate on this in Section 1.4.

1.4 Main results

This thesis consists of two main results. The first one is a general structure theorem for excluding a fixed graph as a topological minor. The second one is a proof of a conjecture of Robertson on well-quasi-ordering by the topological minor relation.

1.4.1 Structure theorem for excluding topological minors

As we pointed out in Section 1.3, Dvořák improved the structure theorem of Grohe and Marx for excluding topological minors by providing more information when the torso can be nearly drawn in the surface. Our objective is to further strengthen Dvořák's theorem by improving the bounds on D. That is, we would like to prove that the maximum degree of the torso is less than the maximum degree of the graph which we exclude, which is clearly best possible. However, we are not able to extend the theorems verbatim; our theorem gives a structure relative to a tangle. (Tangles, vortices and segregations are defined in Section 2.1. And recall that the function mf is defined prior to Theorem 1.3.4.) More precisely, we prove the following.

Theorem 1.4.1 *Let $d \geq 4, h$ be positive integers. Then there exist $\theta, \kappa, \rho, \xi, g \geq 0$ satisfying the following property. If H is a graph of maximum degree d on h vertices, and a graph G does not admit an H-subdivision, then for every tangle $\mathcal{T}$ in G of order at least θ, there exists $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that either*

1. *for every $v \in V(G) - Z$ of degree at least d in G, there exists $(A, B) \in \mathcal{T} - Z$ of order at most $d-1$ such that $v \in V(A) - V(B)$, or*

2. *there exists a $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ with $|\mathcal{S}_2| \leq \kappa$, having a proper arrangement in some surface Σ of genus at most g such that every society (S_1, Ω_1) in $\mathcal{S}_1$ satisfies that $|\overline{\Omega_1}| \leq 3$, and every society (S_2, Ω_2) in $\mathcal{S}_2$ is an ρ-vortex, and satisfies the following property: either*

   (a) H cannot be drawn in Σ, or

(b) H *can be drawn in* Σ *and* $\text{mf}(H, \Sigma) \geq 2$, *and there exists* $\mathcal{S}_2' \subseteq \mathcal{S}_2$ *with* $|\mathcal{S}_2'| \leq \text{mf}(H, \Sigma) - 1$ *such that every vertex* v *of* $G - Z$ *with no* $(A, B) \in \mathcal{T} - Z$ *of order less than* d *and* $v \in V(A) - V(B)$ *is in* $S - \bar{\Omega}$ *for some* $(S, \Omega) \in \mathcal{S}_2'$.

The following corollary will be applied to prove our second main theorem on well-quasi-ordering. It is the special case of Theorem 1.4.1 for a graph H that can be drawn in any surface Σ with $\text{mf}(H, \Sigma) = 1$, so Statement 2 of Theorem 1.4.1 does not happen.

Corollary 1.4.2 *Let* $d \geq 4, h$ *be positive integers. Then there exist* θ *and* ξ *such that for every graph* H *of order* h *and of maximum degree* d *that can be drawn in the plane such that every vertex of degree at least four is incident with the infinite face, and for every graph* G, *either* G *admits an* H*-subdivision, or for every tangle* $\mathcal{T}$ *of order at least* θ *in* G, *there exists* $Z \subseteq V(G)$ *with* $|Z| \leq \xi$ *such that for every vertex* $v \in V(G) - Z$, *there exists* $(A, B) \in \mathcal{T} - Z$ *of order at most* $d - 1$ *such that* $v \in V(A) - V(B)$.

1.4.2 Well-quasi-ordering by the topological minor relation

Unlike the relation of minor and weak immersion, the relation of topological minor does not well-quasi-order graphs in general. For every positive integer k, we say that a graph is a *Robertson chain* of length k if it can be obtained by doubling the edges of the path of length k, and we say that the ends of the Robertson chain are the ends of the original path. Let G_k be the graph obtained by adding four vertices of degree one to the Robertson chain of length k, where each of the ends is adjacent to two new vertices. Then there do not exist $i \neq j$ such that G_i contains G_j as a topological minor. There are many different infinite sequences of graphs showing that the topological minor does not well-quasi-order graphs. But topological minors of arbitrarily long Robertson chain can be found in each such sequence. In the late 1980's, Robertson conjectured that the Robertson chain is the only obstruction.

Conjecture 1.4.3 (Robertson's conjecture) *For every positive integer k, the topological minor relation well-quasi-orders the graphs with no topological minor isomorphic to the Robertson chain of length k.*

Our second main theorem is a proof of Robertson's conjecture (Conjecture 1.4.3).

Theorem 1.4.4 *For every positive integer k, the topological minor relation well-quasi-orders the graphs that do not contain a topological minor isomorphic to the Robertson chain of length k.*

We remark that Theorem 1.4.4 generalizes Kruskal's theorem (Theorem 1.2.2) and Mader's theorem (Theorem 1.2.4). On the other hand, subcubic graphs do not contain a topological minor isomorphic to the Robertson chain of length two. So Theorem 1.4.4 implies that topological minor relation well-quasi-orders subcubic graphs. As the topological minor relation is the same as the minor and the weak immersion relation on subcubic graphs, it implies the Graph Minor Theorem and the weak immersion theorem for subcubic graphs and confirms Vázsonyi's conjecture. But we should note that our proof of Theorem 1.4.4 uses the weak immersion theorem for subcubic graphs as a black box, so we do not offer a new proof of Vázsonyi's conjecture.

As we mentioned in Section 1.2.2, if a property $\mathcal{Q}$ is closed under the topological minor relation, then for every positive integer k, there exist finitely many graphs $H_1, H_2, ..., H_n$ (only depending on $\mathcal{Q}$ and k) such that every graph G that does not contain a topological minor isomorphic to the Robertson chain of length k satisfies $\mathcal{Q}$ if and only if G does not contain H_i as a topological minor for every $1 \leq i \leq n$. Recall that testing topological minor containment is polynomial time decidable by Theorem 1.2.29. Hence, for every fixed positive integer k, testing any topological minor-closed property in the class of graphs that do not contain a topological minor isomorphic to the Robertson chain of length k can be done in polynomial time.

In fact, we will prove a stronger version of Theorem 1.4.4 in this thesis.

Theorem 1.4.5 *Let $(S, \preceq)$ be a well-quasi-ordered set, and let k be a positive integer. For every $i \in \mathbb{N}$, let G_i be a graph, and let $\phi_i : V(G_i) \to S$. Then there exist $1 \leq i < j$ such that there exists a G_i-topological minor (π_V, π_E) in G_j such that $\phi_i(v) \preceq \phi_j(\pi_V(v))$ for every $v \in V(G_i)$.*

Theorem 1.4.5 not only implies Theorem 1.4.4 but also implies the following stronger theorem. This implication follows from the fact that one can delete a bounded number of vertices to kill all topological minors isomorphic to the Robertson chain of length k. So the following theorem is deduced from Theorem 1.4.5 by appropriately labelling the neighbors of those deleted vertices. The detailed proof of this implication is left to the reader.

Theorem 1.4.6 *For every nonnegative integers k, ℓ, the graphs that contain at most ℓ different topological minors isomorphic to a Robertson chain of length k are well-quasi-ordered by the topological minor relation.*

1.5 Sketch of the proofs

In this section, we sketch our proofs of Theorems 1.4.1 and 1.4.5 and explain the organization of this thesis.

Chapter 2 is dedicated to the proof of Theorem 1.4.1. We will review some basic notions, such as tangles, developed in the Graph Minors series in Section 2.1. In Section 2.2 we prove an Erdős-Pósa type result for "spiders": either there exist many disjoint paths from a given set to the tangle, or no such path exists upon the deletion of bounded number of vertices. This theorem plays an important role in the proofs of our two main theorems. Then in Section 2.3, we further show that once the mentioned disjoint paths exist, we can choose them to be in a "nice order." In Section 2.4, we will review some theorems in the Graph Minors series about graphs drawn on surfaces. This is a preparation for the proof of Theorem 1.4.1. Finally, we complete the proof of Theorem 1.4.1 in Section 2.5 by proving that the mentioned disjoint paths in a "nice order" would help us construct a topological minor of the given graph, and hence such paths cannot exist and the graph has the desired structure.

The rest of the thesis is dedicated to the proof of Theorem 1.4.5. We shall use the "minimal bad sequence" argument. To make it work, we need a "nice" tree decomposition of the graphs. One of the "nice" properties (informally speaking) is that the subgraph induced by the subtree rooted at a node contains the subgraph induced by the subtree rooted at "every" descendant of the previous node as a topological minor. The key idea to obtain this nice property is to convert the vertex-cuts realized by the bags of a tree decomposition into "edge-cuts." This is the first main objective of the proof.

In Chapter 3, we introduce a new notion of tree decompositions, which we call *strongly lean tree decompositions*. This generalizes an old notion called lean tree decomposition, which was introduced in [53] and could be found in [8] and was used by other researchers, such as [3, 6, 37], since then. Briefly speaking, strongly lean tree decomposition realizes as many vertex-cuts as possible by its bags in some sense. The existence of a strongly tree decomposition plays an important role in our proof of Theorem 1.4.5, and it is of interest on its own. In Section 3.2, we show a nice relation between a strongly lean tree decomposition and tangles and the vertex-cuts separating two distinct tangles.

Then in Chapter 4, we give a sufficient condition for the existence of a topological minor isomorphic to a sufficiently long Robertson chain in Section 4.1; we prove that if the sufficient condition does not hold, then many vertex-cuts realized by a strongly lean tree decomposition are indeed "edge-cuts" in Section 4.2. This achieves our first main objective.

Then we formally develop tools for proving well-quasi-ordering in Chapter 5. This motivates our next main objective.

The other "nice" property (informally speaking) that we need for proving well-quasi-ordering is that as long as the subgraphs induced by the subtree rooted at the children of the roots form a well-quasi-ordered set, then the set of the graphs is well-quasi-ordered as well. This condition is satisfied by the tree decomposition of bounded width. To deal with graphs of large tree-width, we develop a structure theorem for excluding Robertson chains with respect to tangles to show that such a "nice" star decomposition exists in Chapter 6. This is our second main objective.

Corollary 1.4.2 shows that every graph with no topological minor isomorphic to a long Robertson chain is more or less a graph whose every vertex has at most three neighbors. We hope to show that this graph is subcubic, and hence the theorem for weak immersion of Robertson and Seymour can apply. However, parallel edges are obstructions for this plan.

In Chapter 6, we investigate how to overcome this obstruction and prove a stronger structure theorem for excluding Robertson chain. Section 6.1 is a step toward removing parallel edges. In Section 6.2, we prove that every graph with no topological minor of a sufficiently long Robertson chain is either more or less a subcubic graph or "nicely" "nearly embeddable" in a surface of bounded genus with bounded number of vortices. We give more structure information for the vortices in Section 6.3 and prove that every graph without topological minor isomorphic to a sufficiently long Robertson chain has a star decomposition such that the root bag induces a subcubic graph or a graph that can be "nicely nearly embedded" in a surface of bounded genus with bounded number of "nice" vortices.

In Section 7.1, we prove that the mentioned star decomposition satisfies the "nice" property we expected. This accomplishes our second objective.

The final objective of the proof is to show that every graph with no topological minor isomorphic to a sufficiently long Robertson chain has a tree decomposition satisfying the above two properties. This is the purpose of the rest of Chapter 7. Section 7.2 is a preparation. More precisely, we prove that the second nice property is preserved under some minor modifications of the mentioned star decomposition. In Section 7.3, we prove that a tree decomposition with the above two nice properties exists and complete the proof of Theorem 1.4.5.

CHAPTER II

EXCLUDING SUBDIVISIONS OF BOUNDED DEGREE GRAPHS

2.1 Tangles and minors

In this section, we review some theorems about tangles and graph minors.

A *separation* of a graph G is an ordered pair (A, B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$, and the *order* of (A, B) is $|V(A) \cap V(B)|$. A *tangle* $\mathcal{T}$ in G of *order* θ is a set of separations of G, each of order less than θ such that

(T1) for every separation (A, B) of G of order less than θ, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$;

(T2) if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$, then $A_1 \cup A_2 \cup A_3 \neq G$;

(T3) if $(A, B) \in \mathcal{T}$, then $V(A) \neq V(G)$.

The notion of tangle was first defined by Roberson and Seymour in [40]. (T1), (T2) and (T3) are called the first, second and third tangle axioms, respectively.

Recall that given a graph H, an *H-minor* of a graph G is a map α with domain $V(H) \cup E(H)$ such that the following hold.

- $\alpha(h)$ is a nonempty connected subgraph of G, for every $h \in V(H)$.
- If h_1 and h_2 are distinct vertices of H, then $\alpha(h_1)$ and $\alpha(h_2)$ are disjoint.
- For each non-loop e of H with ends h_1, h_2, $\alpha(e)$ is an edge of G with one end in $\alpha(h_1)$ and one end in $\alpha(h_2)$; for each loop e of H with end h, $\alpha(e) \in E(G) - E(\alpha(h))$ with both ends in $V(\alpha(h))$.

- If e_1, e_2 are two different edges of H, then $\alpha(e_1) \neq \alpha(e_2)$.

We say that G *contains an H-minor* if such a function α exists. For every $h \in V(H)$, $\alpha(h)$ is called a *branch set* of α. A tangle $\mathcal{T}$ in G *controls* an H-minor α if α is an H-minor such that there does not exist $(A, B) \in \mathcal{T}$ of order less than $|V(H)|$ and $h \in V(H)$ such that $V(\alpha(h)) \subseteq V(A)$.

The following theorem offers a way to obtain a tangle in a graph from a minor.

Theorem 2.1.1 ([40, Theorem (6.1)]) *Let G and H be graphs. Let $\mathcal{T}'$ be a tangle in H of order $\theta \geq 2$. If G admits an H-minor, and $\mathcal{T}$ is the set of separations (A, B) of G of order less than θ such that there exists $(A', B') \in \mathcal{T}'$ with $E(A') = E(A) \cap \alpha(E(H))$, then $\mathcal{T}$ is a tangle in G of order θ.*

The tangle $\mathcal{T}$ in Theorem 2.1.1 is called the *tangle induced by $\mathcal{T}'$*. We say that $\mathcal{T}'$ is *conformal* with a tangle $\mathcal{T}''$ in G if $\mathcal{T} \subseteq \mathcal{T}''$.

A *society* is a pair (S, Ω), where S is a graph and Ω is a cyclic permutation of a subset $\bar{\Omega}$ of $V(S)$. Let ρ be a nonnegative integer. A society (S, Ω) is a *ρ-vortex* if for all distinct $u, v \in \bar{\Omega}$, there do not exist $\rho + 1$ mutually disjoint paths in S between $I \cup \{u\}$ and $J \cup \{v\}$, where I is the set of vertices in $\bar{\Omega}$ after u and before v in the natural order, and J is the set of vertices in $\bar{\Omega}$ after v and before u.

A *segregation* of a graph G is a set $\mathcal{S}$ of societies such that the following hold.

- S is a subgraph of G for every $(S, \Omega) \in \mathcal{S}$, and $\bigcup\{S : (S, \Omega) \in \mathcal{S}\} = G$.
- For every distinct (S, Ω) and $(S', \Omega') \in \mathcal{S}$, $V(S \cap S') \subseteq \bar{\Omega} \cap \overline{\Omega}'$ and $E(S \cap S') = \emptyset$.

We write $V(\mathcal{S}) = \bigcup\{\bar{\Omega} : (S, \Omega) \in \mathcal{S}\}$. If $\mathcal{T}$ is a tangle in G, a segregation $\mathcal{S}$ of G is *$\mathcal{T}$-central* if for every $(S, \Omega) \in \mathcal{S}$, there is no $(A, B) \in \mathcal{T}$ of order at most half of the order of $\mathcal{T}$ with $B \subseteq S$.

A *surface* is a nonnull compact connected 2-manifold without boundary. Let Σ be a surface and $\mathcal{S} = \{(S_1, \Omega_1), ..., (S_k, \Omega_k)\}$ a segregation of G. An *arrangement* of $\mathcal{S}$ in Σ is a function α with domain $\mathcal{S} \cup V(\mathcal{S})$, such that the following hold.

- For $1 \le i \le k$, $\alpha(S_i, \Omega_i)$ is a closed disk $\Delta_i \subseteq \Sigma$, and $\alpha(x) \in \partial\Delta_i$ for each $x \in \overline{\Omega_i}$.

- For $1 \le i \le k$, if $x \in \Delta_i \cap \Delta_j$, then $x = \alpha(v)$ for some $v \in \overline{\Omega_i} \cap \overline{\Omega_j}$.

- For all distinct $x, y \in V(\mathcal{S})$, $\alpha(x) \neq \alpha(y)$.

- For $1 \le i \le k$, Ω_i is mapped by α to the natural order of $\alpha(\overline{\Omega_i})$ determined by $\partial\Delta_i$.

An arrangement is *proper* if $\Delta_i \cap \Delta_j = \emptyset$ for all $1 \le i < j \le k$ such that $|\overline{\Omega_i}|, |\overline{\Omega_j}| > 3$.

Theorem 2.1.2 ([45, Theorem (3.1)]) *For any graph L, there are integers $\kappa, \rho, \xi \ge 0$ and $\theta \ge \xi$ with the following property. Let $\mathcal{T}$ be a tangle of order at least θ in a graph G, controlling no L-minor of G. Then there exist $Z \subseteq V(G)$ with $|Z| \le \xi$, and a $\mathcal{T} - Z$-central segregation of $G - Z$ that has a proper arrangement in some surface in which L cannot be drawn, there are at most κ members (S, Ω) in the segregation satisfying $|\bar{\Omega}| > 3$, and each such member is a ρ-vortex.*

Recall that given a graph H, an *H-subdivision* (or an *H-topological minor*) in a graph G is a pair of functions (π_V, π_E) such that the following hold.

- $\pi_V : V(H) \to V(G)$ is an injective function.

- π_E maps each loop of H to a cycle in G and maps each non-loop of H to a path in G such that $\pi_E(e)$ contains $\pi_V(v)$ for every loop e with end v, and $\pi_E(e')$ has ends $\pi_V(x)$ and $\pi_V(y)$ for every non-loop $e' = xy \in E(H)$.

- $\pi_E(e) \cap \pi_E(f)$ is contained in the image of π_V for distinct edges e, f of H.

- $\pi_V(z) \notin V(\pi_E(e))$ if z is not an end of e.

We say that G *admits an H-subdivision* if such a pair of functions (π_V, π_E) exists.

2.2 Finding disjoint spiders

First, we introduce a lemma proved by Robertson and Seymour [43].

Lemma 2.2.1 ([43, Theorem (5.4)]) *Let G be a graph, and let Z be a subset of $V(G)$ with $|Z| = \xi$. Let $k \geq \lceil \frac{3}{2}\xi \rceil$, and let α be a K_k-minor in G. If there is no separation (A, B) of G of order less than $|Z|$ such that $Z \subseteq V(A)$ and $A \cap \alpha(h) = \emptyset$ for some $h \in V(K_k)$, then for every partition $(Z_1, ..., Z_n)$ of Z into non-empty subsets, there are n connected graphs $T_1, ..., T_n$ of G, mutually disjoint and such that $V(T_i) \cap Z = Z_i$ for $1 \leq i \leq n$.*

A *d-spider with head v* is a tree such that every vertex other than v in the tree has degree at most 2, and the degree of v is d. A *leaf* is a vertex of degree one. Let G be a graph, and let S, Y be subsets of $V(G)$. A *d-spider from S to Y* is a d-spider with head $v \in S$ whose leaves are in Y.

Let G be a graph and $\mathcal{T}$ a tangle in G. We say that a subset X of $V(G)$ is *free* if there exists no $(A, B) \in \mathcal{T}$ of order less than $|X|$ such that $X \subseteq V(A)$.

Lemma 2.2.2 *Let G be a graph and H be a graph on h vertices of maximum degree d. Let $t \geq \lceil \frac{3hd}{2} \rceil$. Let $\mathcal{T}$ be a tangle of order at least hd in G that controls a K_t-minor. Let $v_1, v_2, ..., v_h$ be distinct vertices of G. If there exist pairwise disjoint sets $X_1, X_2, ..., X_h$ such that for $1 \leq i \leq h$ the set X_i consists of the vertex v_i and $d-1$ of its neighbors and $\bigcup_{i=1}^{h} X_i$ is free with respect to $\mathcal{T}$, then G has an H-subdivision.*

Proof. Let $Z = \bigcup_{i=1}^{h} X_i$, and let α be a K_t-minor controlled by $\mathcal{T}$. Suppose that there exists a separation (A, B) of G of order less than $|Z|$ such that $Z \subseteq V(A)$ and $A \cap \alpha(v) = \emptyset$ for some $v \in V(K_t)$. By the first tangle axiom, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$. Since Z is free, $(B, A) \in \mathcal{T}$. But it is a contradiction since $t \geq hd$ and $\mathcal{T}$ controls α. Therefore, there does not exist a separation (A, B) of G of order less than $|Z|$ such that $Z \subseteq V(A)$ and $A \cap \alpha(v) = \emptyset$ for some $v \in V(K_t)$.

Denote $V(H)$ by $\{u_1, u_2, ..., u_h\}$ and $E(H)$ by $\{e_1, e_2, ..., e_{|E(H)|}\}$. Since the maximum degree of H is at most d, there exist $Z_0 \subseteq Z$ and a partition $(Z_1, Z_2, ..., Z_{|E(H)|})$ of $Z - Z_0$ such that for every $1 \leq \ell \leq |E(H)|$, Z_ℓ consists of two distinct vertices where one is in X_i and one is in X_j, where the ends of e_ℓ are u_i and u_j. By Lemma 2.2.1, there exist $|E(H)|$ pairwise disjoint paths in $G' - Z_0$ connecting the two vertices of each part of $(Z_1, Z_2, ..., Z_{|E(H)|})$. This creates a subdivision of H. ■

Theorem 2.2.3 ([29, Theorem 6]) *Let G be a graph and $\mathcal{T}$ a tangle in G of order θ. Let $\{X_j \subseteq V(G) : j \in J\}$ be a family of subsets of $V(G)$ indexed by J. Let d, k be an integer with $\theta \geq (k+d)^{d+1} + d$. If $|X_j| = d$ for every $j \in J$, then there exists a set $J' \subseteq J$ satisfying the following.*

1. *For all $j \neq j' \in J'$, X_j and $X_{j'}$ are disjoint.*

2. *$\bigcup_{j \in J'} X_j$ is free.*

3. *If $|\bigcup_{j \in J'} X_j| \leq k$, then there exists Z with $|Z| \leq (k+d)^{d+1}$ satisfying that for all $j \in J'$, either $X_j \cap Z \neq \emptyset$, or X_j is not free in $\mathcal{T} - Z$.*

Theorem 2.2.4 *Let h and d be positive integers. Let G be a graph, and let S be a subset of vertices of degree at least d in G. Let $\mathcal{T}$ be a tangle in G of order θ. If $\theta \geq (hd)^{d+1} + d$, then either*

1. *there exist h vertices $v_1, v_2, ..., v_h \in S$ and h pairwise disjoint subsets $X_1, X_2, ..., X_h$ of $V(G)$, where X_i consists of v_i and $d-1$ neighbors of v_i for each $1 \leq i \leq h$, such that $\bigcup_{i=1}^{h} X_i$ is free in $\mathcal{T}$, or*

2. *there exists a set $C \subseteq V(G)$ with $|C| \leq (hd)^{d+1}$ such that for every $v \in S - C$, there exists $(A, B) \in \mathcal{T} - C$ of order less than d such that $v \in V(A) - V(B)$.*

Proof. Let $\{X_j : j \in J\}$ be the collection of the d-element subsets consisting of one vertex v_j in S and $d-1$ of its neighbors. Applying Theorem 2.2.3 by further taking

$k=(h-1)d$, then there exists $J' \subseteq J$ such that $X_j \cap X_{j'} = \emptyset$ for every distinct j, j' in J', and $\bigcup_{j\in J'} X_j$ is free. Furthermore, if $|\bigcup_{j\in J'} X_j| \leq (h-1)d$, there exists $C \subseteq V(G)$ with $|C| \leq (hd)^{d+1}$ satisfying that for all $j \in J'$, either $X_j \cap C \neq \emptyset$, or X_j is not free in $\mathcal{T} - C$.

Observe that if $|\bigcup_{j\in J'} X_j| > (h-1)d$, then $|J'| \geq h$ and the first statement holds. So we assume that $|\bigcup_{j\in J'} X_j| \leq (h-1)d$, and we shall prove that the second statement of this theorem holds. Let $v \in S - C$. Suppose that there does not exist $(A,B) \in \mathcal{T} - C$ of order less than d such that $v \in V(A) - V(B)$. Let U be the collection of those X_j that are disjoint from C and consist of v and $d-1$ neighbors of v. For every member X_j of U, we define the *rank* of X_j to be the minimum order of a separation $(A,B) \in \mathcal{T} - C$ such that $X_j \subseteq V(A)$. As no member of U is free, the rank of each member of U is at most $d-1$. Let r be the maximum rank of a member of U, and let X be a member of U of rank r. Let $(A,B) \in \mathcal{T} - C$ of order r such that $X \subseteq V(A)$, and subject to that, $|V(B) - V(A)|$ is as small as possible. By the assumption, $v \in V(A) \cap V(B)$ and $r \leq d-1$. On the other hand, there exist r disjoint paths from $X - \{v\}$ to $V(B)$, as v is adjacent to all vertices in $X - \{v\}$. We denote these r disjoint paths by $P_1, P_2, ..., P_r$, and denote the end of P_i in $X - \{v\}$ by u_i for $1 \leq i \leq r$. As $v \in V(A) \cap V(B)$ and $|V(A) \cap V(B)| = r$, $v \in V(P_i)$ for some $1 \leq i \leq r$. Without loss of generality, we may assume that $v \in V(P_r)$. In addition, v is adjacent to a vertex u in $V(B) - V(A)$, otherwise, the rank of X is smaller than r. As $(X - \{u_r\}) \cup \{u\}$ is a member of U, its rank is at most r. Let $(A', B') \in \mathcal{T} - C$ be a separation of order at most r such that $(X - \{u_r\}) \cup \{u\} \subseteq V(A')$. $X \subseteq V(A \cup A')$ and $u \in (V(B) - V(A)) - (V(B \cap B') - V(A \cup A'))$, so the order of $(A \cup A', B \cap B')$ is at least $r+1$ by the choice of (A,B). It implies that the order of $(A \cap A', B \cup B')$ is at most $r-1$. Notice that $v \in V(A') \cap V(B')$ by the assumption, so $((A \cap A') - \{v\}, (B \cup B') - \{v\})$ is a separation of $G - \{v\}$ of order less than $r-1$. But $P_1, P_2, ..., P_{r-1}$ are $r-1$ disjoint paths from $V(A \cap A') - \{v\}$ to $V(B \cup B') - \{v\}$

in $G - \{v\}$, a contradiction. This proves the second statement. ■

We need the following variation of Theorem 2.2.4. An edge-version of this theorem was proved in [28] and [29, Theorem 6].

Theorem 2.2.5 *Let G be a graph, and let X, Y be disjoint subsets of $V(G)$. Let h, d be nonnegative integers. Then either there exist h disjoint d-spiders from X to Y, or there exists $C \subseteq V(G)$ with $|C| \leq \frac{3}{2}(hd)^{d+1} + \frac{d}{2} + 1$ such that every d-spider from X to Y intersects C.*

Proof. Note that for every subset C of Y such that $|Y - C| \leq d - 1$, every d-spider from X to C intersects C. So we may assume that $|Y| \geq \frac{3}{2}((hd)^{d+1} + d)$, otherwise we are done. Let G' be the graph obtained from G by adding edges such that Y induces a clique in G'. As every clique of size k contains a tangle of order $\lfloor 2k/3 \rfloor$, $G'[Y]$ contains a tangle of order $(hd)^{d+1} + d$. And Y is a minor of G', so G' contains a tangle $\mathcal{T}$ of order $(hd)^{d+1} + d$ induced by $G'[Y]$ by Theorem 2.1.1 such that $Y \subseteq V(B)$ for every $(A, B) \in \mathcal{T}$. Let $\{X_j : j \in J\}$ be the collection of d-element subsets of $V(G)$ such that every X_j consisting of one vertex x in X and $d - 1$ neighbors of x. By Theorem 2.2.3, there exists $J' \subseteq J$ such that $X_j \cap X_{j'} = \emptyset$ for every distinct j, j' in J', and $\bigcup_{j \in J'} X_j$ is free. Furthermore, if $|\bigcup_{j \in J'} X_j| \leq (h-1)d$, there exists $C \subseteq V(G)$ with $|C| \leq (hd)^{d+1}$ satisfying that for all $j \in J'$, either $X_j \cap C \neq \emptyset$, or X_j is not free in $\mathcal{T} - Z$.

First, assume that $|\bigcup_{j \in J'} X_j| > (h-1)d$, so $|J'| \geq h$. Let $\{1, 2, ..., h\} \subseteq J'$, and let x_j be a vertex in $X_j \cap X$ for $1 \leq j \leq h$. Suppose that there do not exist dh disjoint paths from $\bigcup_{j=1}^{h} X_j$ to Y in G'. Then there exists a separation (A, B) of G' of order less than dh such that $\bigcup_{j=1}^{h} X_j \subseteq V(A)$ and $Y \subseteq V(B)$. Since $Y \subseteq V(B)$, we know that $(A, B) \in \mathcal{T}$. But this implies that $\bigcup_{j=1}^{h} X_j$ is not free, a contradiction. Hence, there exist dh disjoint paths from $\bigcup_{j=1}^{h} X_j$ in G'. That is, there exist h disjoint

d-spiders from x_j to Y in G'. We are done in this case since every d-spider from X to Y in G' contains a d-spider from X to Y in G as a subgraph.

So we may assume that $|\bigcup_{j \in J'} X_j| \leq (h-1)d$, there exists $C \subseteq V(G)$ with $|C| \leq (hd)^{d+1}$ satisfying that for all $j \in J'$, either $X_j \cap C \neq \emptyset$, or X_j is not free in $\mathcal{T} - C$. Let $v \in V(G) - C$, and let D be a d-spider from v to Y in G. Note that D is also a d-spider from v to Y in G'. Suppose that D is disjoint from C. So D contains some X_j such that $v \in X_j$ and $X_j \cap C = \emptyset$. Since X_j is not free in $\mathcal{T} - C$, there exists $(A, B) \in \mathcal{T} - C$ of order less than d such that $X_j \subseteq V(A)$ and $Y - C \subseteq V(B)$. This is a contradiction since there exist d disjoint paths in D from $V(A)$ to $V(B)$. This proves that D intersects C. ■

2.3 Taming spiders

We say that (S, Ω, Ω_0) is a *neighborhood* if S is a graph and Ω, Ω_0 are cyclic permutations with $\bar{\Omega}, \overline{\Omega_0} \subseteq V(S)$, respectively. A neighborhood (S, Ω, Ω_0) is *rural* if S has a drawing Γ on the plane without crossing and there are disks $\Delta_0 \subseteq \Delta$ such that

- Γ uses no point outside Δ and none in the interior of Δ_0, and
- $\bar{\Omega}$ are the vertices in $\Gamma \cap \partial\Delta$, and $\overline{\Omega_0}$ are the vertices in $\Gamma \cap \Delta_0$, and
- the cyclic permutations of $\bar{\Omega}$ and $\overline{\Omega_0}$ coincide with the natural cyclic order on Δ and Δ_0.

In this case, we say that $(\Gamma, \Delta, \Delta_0)$ is a *presentation* of (S, Ω, Ω_0). For a fixed presentation $(\Gamma, \Delta, \Delta_0)$ of a neighborhood (S, Ω, Ω_0) and an integer $s \geq 0$, an *s-nest* for $(\Gamma, \Delta, \Delta_0)$ is a sequence $(C_1, C_2, ..., C_s)$ of pairwise disjoint cycles of S such that $\Delta_0 \subseteq \Delta_1 \subseteq ... \subseteq \Delta_s \subseteq \Delta$, where Δ_i is the closed disk in the plane bounded by C_i.

If (S, Ω, Ω_0) is a neighborhood and (S_0, Ω_0) is a society, then $(S \cup S_0, \Omega)$ is a society and we call this society the *composition* of the society (S_0, Ω_0) with the neighborhood

(S, Ω, Ω_0). A society (S, Ω) is *s-nested* if it is the composition of a society with a rural neighborhood that has an s-nest for some presentation of it.

A subgraph F of a rural neighborhood (S, Ω, Ω_0) is *perpendicular* to an s-nest $(C_1, C_2, ..., C_s)$ if for every component P of F

- P is a path with one end in $\bar{\Omega}$ and the other in $\overline{\Omega_0}$, and
- $P \cap C_i$ is a path for all $i = 1, 2, ..., s$.

We shall use the following theorem, which was proved in [20], to prove the main theorem of this section. We present a simplified restatement of it.

Theorem 2.3.1 ([20, Theorem 10.3]) *For every three positive integers s, k, c, there exists an integer $s'(s, k, c)$ such that for every s'-nested society (S, Ω) that is a composition of a society (S_0, Ω_0) with a rural neighborhood with a s'-nest, and for every union of c pairwise disjoint k-spiders F_0 from $V(S_0) - \overline{\Omega_0}$ to $\bar{\Omega}$, there exists a union of c pairwise disjoint k-spiders F in (S, Ω) from the set of the heads of F_0 to the set of leaves of F_0 such that (S, Ω) can be expressed as a composition of some society with a rural neighborhood (S', Ω, Ω') that has a presentation with an s-nest $(C_1, C_2, ..., C_s)$ such that $S' \cap F$ is perpendicular to $(C_1, C_2, ..., C_s)$.*

Now, we are ready to prove the main theorem of this section. It adds Conclusion 4 into conclusions of Theorem 2.3.1 at the cost of requiring many more spiders to begin with.

Theorem 2.3.2 *For every positive integers $d \geq 3, \rho, k$ and s, there exist integers $s'(k, d, s, \rho)$ and $k'(k, d, \rho)$ such that for every s'-nested society (S, Ω) that is a composition of a ρ-vortex (S_0, Ω_0) with a rural neighborhood that has an s'-nest, and for every k' pairwise disjoint d-spiders $D_1, D_2, ..., D_{k'}$ from $V(S_0) - \overline{\Omega_0}$ to $\bar{\Omega}$, there exist k pairwise disjoint d-spiders $D'_1, D'_2, ..., D'_k$ from $V(S_0)$ to $\bar{\Omega}$ such that the following hold.*

1. *(S,Ω) can be expressed as a composition of a society (S_0', Ω') with a rural neighborhood (S', Ω, Ω') that has a presentation with an s-nest $(C_1, C_2, ..., C_s)$ such that $D_i' \cap S'$ is perpendicular to $(C_1, C_2, ..., C_s)$ for every $1 \leq i \leq k$.*

2. *For every $1 \leq i \leq k$, the head of D_i' is the head of $D_{i'}$ for some $1 \leq i' \leq k'$.*

3. *For every $1 \leq i \leq k$, the leaves of D_i' are some leaves of $D_1 \cup D_2 \cup ... \cup D_{k'}$.*

4. *For every $1 \leq i \leq k$, there exists an interval I_i of $\bar{\Omega}$ containing all leaves of D_i' such that I_i is disjoint from I_j for every $j \neq i$.*

Proof. Let $s'(k,d,s,\rho) = s'_{2.3.1}(s,d,3k(\rho+1))$ and $k'(k,d,\rho) = 3k(\rho+1)$, where $s'_{2.3.1}$ is the function s' mentioned in Theorem 2.3.1. By Theorem 2.3.1, there exist $3k(\rho+1)$ pairwise disjoint d-spiders $D_1', D_2', ..., D_{k'}'$ from the set of the heads of $D_1, D_2, ..., D_{k'}$ to the union of the set of leaves of $D_1, D_2, ..., D_{k'}$ such that (S,Ω) can be expressed as a composition of some society with a rural neighborhood (S', Ω, Ω') that has a presentation with an s'-nest $(C_1, C_2, ..., C_s)$ such that $D_i' \cap S'$ is perpendicular to $(C_1, C_2, ..., C_s)$ for every $1 \leq i \leq k$. For every $1 \leq i \leq k'$, let I_i be a minimum interval of $\bar{\Omega}$ containing all leaves of D_i. Then it is sufficient to prove that there exist $1 \leq i_1 < i_2 < ... < i_k \leq k'$ such that are $I_{i_1}, I_{i_2}, ..., I_{i_k}$ are pairwise disjoint. Suppose that there do not exist such k pairwise disjoint intervals. Then the intersection graph of $I_1, I_2, ..., I_{k'}$ does not contain an independent set of size k, so it contains a clique of size at least $k'/(k-1) > 3(\rho+1)$, as every interval graph is perfect.

Let $\overline{\Omega_0} = \{v_1, v_2, ..., v_{|\overline{\Omega_0}|}\}$ in order. Since (S_0, Ω_0) is an ρ-vortex, by Theorem 8.1 in [38], there exists a path-decomposition $(t_1 t_2 ... t_{|\overline{\Omega_0}|}, \mathcal{X})$ of S_0 such that $|X_{t_i} \cap X_{t_j}| \leq \rho$ for every $1 \leq i < j \leq |\overline{\Omega_0}|$ and $v_i \in X_{t_i}$ for every $1 \leq i \leq |\overline{\Omega_0}|$. Since $S - S_0$ is a plane graph, for every $i \neq j$, if I_i intersects I_j, then there exists an integer a such that $D_i' \cap D_j' \cap X_a \cap X_{a+1} \neq \emptyset$. Let G be the graph obtained from S by adding edges such that $G[X_i \cap X_{i+1}]$ is a clique, for every $1 \leq i \leq |\overline{\Omega_0}| - 1$. Recall that the intersection graph of $I_1, I_2, ..., I_{k'}$ has a clique of size at least $3(\rho+1)$. Therefore, G contains a

$K_{3(\rho+1)}$-minor, where each branch set is D'_i for some i. Without loss of generality, we may assume that the branch set of the $K_{3(\rho+1)}$-minor is $D'_1, D'_2, ..., D'_{3(\rho+1)}$.

Observe that $D'_i \cap X_{t_j}$ is connected in G for every $1 \leq i \leq 3(\rho+1)$ and $1 \leq j \leq |\overline{\Omega_0}|$. Let G' be the graph obtained from G by deleting vertices not in $D'_1 \cup D'_2 \cup ... \cup D'_{3(\rho+1)}$ and then contracting each component of $D'_i \cap (S - S_0)$ into a vertex and contracting $D'_i \cap X_{t_j}$ into a vertex, for every $1 \leq i \leq 3(\rho+1)$ and $1 \leq j \leq |\overline{\Omega_0}|$. Note that G' contains a $K_{3(\rho+1)}$-minor, so the tree-width of G' is at least 3ρ. On the other hand, G' can be written as $G_1 \cup G_2$ such that $V(G_1 \cap G_2) \subseteq \overline{\Omega_0}$, and G_1 is an outerplanar graph that can be drawn in the plane such that the vertices of $V(G_1 \cap G_2)$ are in the boundary of a face in order, and G_2 has a path decomposition of width less ρ such that each bag contains a vertex in $V(G_1 \cap G_2)$ in order. By Lemma 8.1 in [10], G' has tree-width less than 3ρ, a contradiction. This proves the theorem. ∎

2.4 Theorems on surfaces

In this section, we recall some results about graphs embedded in surfaces.

A *surface* is a compact 2-manifold. An *O-arc* is a subset homeomorphic to a circle, and a *line* is a subset homeomorphic to $[0, 1]$. Let Σ be a surface. For every subset Δ of Σ, we denote the closure of Δ by $\bar{\Delta}$, and the boundary of Δ by $\partial\Delta$. A *drawing* Γ in Σ is a pair (U, V), where $V \subseteq U \subseteq \Sigma$, U is closed, V is finite, $U - V$ has only finitely many arc-wise connected components, called *edges*, and for every edge e, either $\bar{e}$ is a line whose set of ends in $\bar{e} \cap V$, or $\bar{e}$ is an O-arc and $|\bar{e} \cap V| = 1$. The components of $\Sigma - U$ are called *regions*. The members of V are called *vertices*. For a drawing $\Gamma = (U, V)$, we write $U = U(\Gamma), V = V(\Gamma)$, and $E(\Gamma), R(\Gamma)$ are defined to be the set of edges and the set of regions, respectively. The sets $\{v\}$, for $v \in V(\Gamma)$, the sets of edges and regions of Γ are called the *atoms* of Γ. If v is a vertex of a drawing Γ and e is an edge or a region of Γ, we say that e is *incident with* v if v is contained in the closure of e. Note that the incidence relation between $V(\Gamma)$ and $E(\Gamma)$ defines a graph, and we say that Γ is a *drawing of* G in Σ if G is defined by this incident relation. In this case, we say that G is *embeddable* in Σ, or G can be *drawn* in Σ. A drawing is 2-*cell* if Σ is connected and every region is an open disk.

Let Γ be a 2-cell drawing in a surface Σ. We say that a drawing K in Σ is a *radial drawing* of Γ if it satisfies the following conditions.

- $U(\Gamma) \cap U(K) = V(\Gamma) \subseteq V(K)$.
- Each region r of Γ contains a unique vertex of K.
- K is a drawing of a bipartite graph, and $(V(\Gamma), V(K) - V(\Gamma))$ is a bipartition of it.
- For every $v \in V(\Gamma)$, the edges of $K \cup \Gamma$ incident with v belong alternately to Γ and to K (in their cyclic order around v).

Let Σ be a surface, and let Γ be a drawing in Σ. A subset Z of Σ is Γ-*normal* if $Z \cap U(\Gamma) \subseteq V(\Gamma)$. If Σ is connected and not a sphere, we say that Γ is θ-*representative* if $|F \cap V(\Gamma)| \geq \theta$ for every non-null-homotopic Γ-normal O-arc F in Σ.

Let Σ be a surface, and let Γ be a drawing of a graph G in Σ. A *tangle* in Γ is a tangle in G. A tangle $\mathcal{T}$ in Γ of order θ is said to be *respectful (towards* Σ*)* if Σ is connected and for every Γ-normal O-arc F in Σ with $|F \cap V(\Gamma)| < \theta$, there is a closed disk $\Delta \subseteq \Sigma$ with $\partial\Delta = F$ such that $(\Gamma \cap \Delta, \Gamma \cap \overline{\Sigma - \Delta}) \in \mathcal{T}$. It is clear that Δ has to be unique, and we write $\Delta = \text{ins}(F)$; the function *ins* is called the *inside function* of $\mathcal{T}$. Assume that Γ is 2-cell, and let K be the radial drawing of Γ. If W is a closed walk of K, we define $K|W$ to be the subdrawing of K formed by the vertices and the edges in W. If the length of W is less than 2θ, then we define $\text{ins}(W)$ to be the union of $U(K|W)$ and $\text{ins}(C)$, taken over all cycles C of $K|W$. For every two atoms a, b of K, define a function $m_{\mathcal{T}}(a, b)$ as follows:

- if $a = b$, then $m_{\mathcal{T}}(a, b) = 0$;

- if $a \neq b$ and $a, b \subseteq \text{ins}(W)$ for some closed walk W of K of length less than 2θ, then $m_{\mathcal{T}}(a, b) = \min \frac{1}{2}|E(W)|$, taken over all such closed walks W;

- otherwise, $m_{\mathcal{T}}(a, b) = \theta$.

Note that K is bipartite, so $m_{\mathcal{T}}$ is integral. In addition, for every atom c of Γ, we define $a(c)$ to be an atom of K such that

- $a(c) = c$ if $c \subseteq V(\Gamma)$;

- $a(c)$ is the region of K including c if c is an edge of Γ;

- $a(c) = \{v\}$, where v is the vertex of K in c, if c is a region of Γ.

For all atoms b, c of Γ, we define $m_{\mathcal{T}}(b, c) = m_{\mathcal{T}}(a(b), a(c))$. The following is a consequence of Theorem 9.1 of [41].

Theorem 2.4.1 *Let Σ be a surface, and let Γ be a 2-cell drawing of a graph in Σ. If $\mathcal{T}$ is a respectful tangle in Γ, then $m_{\mathcal{T}}$ is a metric on the atoms of G.*

The following theorem is useful.

Theorem 2.4.2 ([42, Theorem (1.1)]) *Let Σ be a surface, and let Γ be a 2-cell drawing of a graph in Σ with $E(\Gamma) \neq \emptyset$. Let $\mathcal{T}$ be a respectful tangle of order θ in Γ, and let K be a radial drawing of Γ. Then $(A, B) \in \mathcal{T}$ if and only if for every edge e of A, there exists a cycle C of K with $V(C) \cap V(\Gamma) \subseteq V(A) \cap V(B)$ and with $e \subseteq \text{ins}(C)$.*

Theorem 2.4.3 *Let Σ be a surface, and let Γ be a 2-cell drawing of a graph in Σ with $E(\Gamma) \neq \emptyset$. Let $\mathcal{T}$ be a respectful tangle of order θ in Γ. Let $x \in V(\Gamma)$. If $(A, B) \in \mathcal{T}$ is a separation of Γ such that $x \in V(A) - V(B)$ and subject to that, A is minimal, then $m_{\mathcal{T}}(x, y) \leq |V(A) \cap V(B)|$ for every $y \in V(A)$.*

Proof. Let $y \in V(A)$ be a vertex different from x. Since $(A, B) \in \mathcal{T}$ is a separation with the minimal A such that $x \in V(A) - V(B)$, there exists a path P in A from x to y internally disjoint from $V(B)$. Let e be the edge in P incident with x. By Theorem 2.4.2, there exists a cycle C of the radial drawing K of Γ with $V(C) \cap V(\Gamma) \subseteq V(A) \cap V(B)$ and with $e \subseteq \text{ins}(C)$. So $x \in \text{ins}(C)$. If $y \notin \text{ins}(C)$, then C intersects P at an internal vertex of P. However, $V(C) \cap V(\Gamma) \subseteq V(A) \cap V(B)$. This implies that some internal vertex of P is in $V(A) \cap V(B)$, a contradiction. Hence, $y \in \text{ins}(C)$. Therefore, $m_{\mathcal{T}}(x, y) \leq |V(A) \cap V(B)|$. ■

The following theorem shows a relation between respectful tangles and representativity.

Theorem 2.4.4 ([41, Theorem (4.1)]) *Let Σ be a connected surface which is not a sphere. Let $\theta \geq 1$, and let Γ be a 2-cell drawing of a graph in Σ. If Γ is θ-representative, then there exists a unique respectful tangle in Γ of order θ.*

Theorem 2.4.5 ([41, Theorem (8.12)], [42, Theorem (1.2)]) *Let Γ be a respectful tangle of order θ, where $\theta \geq 2$, in a 2-cell drawing Γ in a connected surface Σ. If c is an atom in Γ, then there exists an edge e of Γ such that $m_{\mathcal{T}}(c, e) = \theta$.*

Let Γ be a 2-cell drawing in a surface Σ, and let $\mathcal{T}$ be a respectful tangle of order θ in Γ. Let x be an atom of Γ. A *λ-zone around x* is an open disk Δ in Σ with $x \subseteq \Delta$, such that $\partial\Delta$ is an O-arc, $\partial\Delta \subseteq \Gamma$, $m_{\mathcal{T}}(x, y) \leq \lambda$ for every atom y of G with $y \subseteq \bar{\Delta}$, and if $x \in E(G)$, then $\lambda \geq 2$. A *λ-zone* is a λ-zone around some atom.

Let Δ be a λ-zone. Note that $U(\Gamma) \cap \partial\Delta$ is a cycle, and the drawing $\Gamma' = \Gamma \cap (\Sigma - \Delta)$ is 2-cell in Σ. We say that Γ' is the *drawing obtained from Γ by clearing Δ*. We say that $\mathcal{T}'$ is a *tangle of order $\theta - 4\lambda - 2$ obtained by clearing Δ* if $\mathcal{T}'$ is a tangle in Γ' of order $\theta - 4\lambda - 2$, and

- $\mathcal{T}'$ is respectful with a metric $m_{\mathcal{T}'}$, and

- $\mathcal{T}'$ is conformal with $\mathcal{T}$, and
- if x, y are atoms of Γ and x', y' are atoms of Γ' with $x \subseteq x'$ and $y \subseteq y'$, then $m_{\mathcal{T}}(x, y) \geq m_{\mathcal{T}'}(x', y') \geq m_{\mathcal{T}}(x, y) - 4\lambda - 2$.

Theorem 2.4.6 ([42, Theorem (7.10)]) *Let Δ be a λ-zone. If $\theta \geq 4\lambda + 3$, then there exists a unique respectful tangle of order $\theta - 4\lambda - 2$ obtained by clearing Δ.*

Theorem 2.4.7 ([44, Theorem (9.2)]) *Let Γ be a 2-cell painting in a surface Σ, and let $\mathcal{T}$ be a respectful tangle in Γ of order θ. Let x be an atom of Γ, and λ an integer with $2 \leq \lambda \leq \theta - 4$. Then there exists a $(\lambda + 3)$-zone Δ around x such that $x' \subseteq \Delta$ for every atom x' of Γ with $m_{\mathcal{T}}(x, x') \leq \lambda$.*

Lemma 2.4.8 *Let Γ be a 2-cell drawing in a surface, z an atom, and $\mathcal{T}$ a respectful tangle in Γ of order θ. Let λ be a nonnegative integer, and let C be the cycle of the boundary of a λ-zone around z. If $\theta \geq \lambda + 8$, then there exists a $(\lambda + 7)$-zone Λ around z such that the cycle bounding Λ is disjoint from C.*

Proof. For every atom x of Γ, let Λ_x be a 4-zone around x containing all atoms y with $m_{\mathcal{T}}(x, y) \leq 1$, and let Δ_x be the closure of Λ_x, and let C_x be the boundary cycle of Δ_x. For every $v \in V(C)$, since every region incident with v has distance 1 from v, v is an interior point of Δ_v. Let $\Delta = \Delta' \cup \bigcup_{v \in V(C)} \Delta_v$, where Δ' is the open disk with the boundary C. So $V(C)$ are interior points of Δ. By the triangle-inequality, for every $v \in V(C)$ and for every vertex u in Δ_v, $m_{\mathcal{T}}(z, u) \leq \lambda + 4$. Therefore, there exists a $(\lambda + 7)$-zone Λ around z that contains Δ by Theorem 2.4.7. Since any vertex in C is an interior point of Δ, it is an interior point of Λ, so C is disjoint from the cycle that bounds Λ. ∎

Let Σ be a connected surface, and let $\Delta_1, ..., \Delta_t$ be pairwise disjoint closed disks in Σ. Let Γ be a drawing in Σ such that $U(\Gamma) \cap \Delta_i = V(\Gamma) \cap \partial\Delta_i$ for $1 \leq i \leq t$.

Let $Z = \bigcup_{i=1}^{t} V(\Gamma) \cap \partial\Delta_i$. We say that a partition $(Z_1, Z_2, ..., Z_p)$ of Z satisfies the *topological feasibility condition* if there exist pairwise disjoint disks $D_1, D_2, ..., D_p$ in Σ such that $D_j \cap (\bigcup_{i=1}^{t} \Delta_i) = Z_j$ for $1 \leq j \leq p$.

Theorem 2.4.9 ([42, Theorem (3.2)]) *For every connected surface Σ and all integers $t \geq 0$ and $z \geq 0$, there exists a positive integer $\theta \geq z$ such that the following is true. Let $\Delta_1, ..., \Delta_t$ be pairwise disjoint closed disks in Σ, and let Γ be a 2-cell drawing in Σ such that $U(\Gamma) \cap \Delta_i = V(\Gamma) \cap \partial\Delta_i$ for $1 \leq i \leq t$. Let $|Z| \leq z$, where $Z = \bigcup_{i=1}^{t} V(\Gamma) \cap \partial\Delta_i$, and let $(Z_1, Z_2, ..., Z_p)$ be a partition of Z satisfying the topological feasibility condition. Let $\mathcal{T}$ be a respectful tangle of order at least θ in Γ with metric $m_{\mathcal{T}}$ such that $m_{\mathcal{T}}(r_i, r_j) \geq \theta$ for $1 \leq i < j \leq t$, where r_i is the region of Γ meeting Δ_i for $1 \leq i \leq t$, and $V(\Gamma) \cap \partial\Delta_i$ is free for $1 \leq i \leq t$. Then there are mutually disjoint connected drawing $\Gamma_1, \Gamma_2, ..., \Gamma_p$ of Γ with $V(\Gamma_j) \cap Z = Z_j$ for $1 \leq j \leq p$.*

2.5 Excluding subdivision of a fixed graph

Let G be a graph and $\mathcal{T}$ a tangle in G. Given an integer k, a vertex v of G is said to be *k-free* (with respect to $\mathcal{T}$) if there is no $(A, B) \in \mathcal{T}$ of order less than k such that $v \in V(A) - V(B)$. Similarly, we say that a subgraph X of G is *k-free* (with respect to $\mathcal{T}$) if there is no $(A, B) \in \mathcal{T}$ of order less than k such that $V(X) \subseteq V(A) - V(B)$.

The *skeleton* of a proper arrangement α of a segregation $\mathcal{S}$ in Σ is the drawing $\Gamma = (U, V)$ in Σ with $V(\Gamma) = \bigcup_{v \in V(\mathcal{S})} \alpha(v)$, and $U(\Gamma)$ consists of the boundary of $\alpha(S, \Omega)$ for each $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| = 3$, and a line in $\alpha(S', \Omega')$ with ends $\overline{\Omega'}$ for each $(S', \Omega') \in \mathcal{S}$ with $|\overline{\Omega'}| = 2$. Note that we do not add anything into the skeleton for (S, Ω) with $|\bar{\Omega}| \leq 1$ or $|\bar{\Omega}| > 3$.

Lemma 2.5.1 *Let t, ρ, θ be nonnegative integers. Let G be a graph and $\mathcal{T}$ a tangle in G of order at least θ. Let α be a proper arrangement of a segregation $\mathcal{S}$ of G in a surface Σ. Let $(S, \Omega) \in \mathcal{S}$ be an ρ-vortex. Let G' be the skeleton of α and $\mathcal{T}'$ a* *respectful tangle in G' of order θ conformal with $\mathcal{T}$. If G' is 2-cell and $\theta \geq 4t+23$, then there exists a cycle C such that the following hold.*

1. *C bounds a $(t+2)$-zone Λ in G' around some vertex in $\bar{\Omega}$.*

2. *Λ contains every vertex x of G' with $m_{\mathcal{T}'}(x,y) \leq t$ for some $y \in \bar{\Omega}$.*

3. *The closure of Λ contains $\alpha(S,\Omega)$.*

4. *Let S' be the union of S and the subgraph of G' contained in the closure of Λ. Let $\overline{\Omega'} = V(C)$ and Ω' the cyclic ordering on $\bar{\Omega}$ that coincides the cyclic ordering of C. Then (S',Ω') is an $(\rho+4t+24)$-vortex.*

Proof. Let y be a vertex in $\bar{\Omega}$. By Theorem 2.4.7, there exists a $(t+4)$-zone Λ' around y in G' such that $x \subseteq \Lambda'$ for every $x \in V(G')$ with $m_{\mathcal{T}}(x,y) \leq t+1$. Since $m_{\mathcal{T}}(y',y'') \leq 1$ for every two vertices y', y'' in $\bar{\Omega}$, $x \subseteq \Lambda'$ for every $x \in V(G')$ with $m_{\mathcal{T}}(x,z) \leq t$ for some $z \in \bar{\Omega}$. Let H be the drawing obtained from G' by deleting every vertex $x \in V(G')$ with $m_{\mathcal{T}}(x,y) \leq t+1$. Note that every deleted vertex is contained in Λ', which is a disk. So H has a region f containing $\alpha(S,\Omega)$ and all deleted vertices. Since for every vertex v of H incident with f, there exists a path of length two in the radial drawing of G' containing v and a vertex of $G'-V(H)$, we know that there exists a closed walk ℓ_v of length at most $2t+4$ in the radial drawing of G' with $v, y \subseteq \text{ins}(\ell_v)$ such that v is adjacent to only one vertex in ℓ_v. We define $\mathcal{L}_v$ to be the set of all such ℓ_v's for each vertex v incident with f, and let $\mathcal{Q}_v$ be the set of $\text{ins}(W)$, where W is a union of two members of $\mathcal{L}_v$. Define L to be the graph obtained from H by deleting $\bigcup_v \bigcup_{W \in \mathcal{Q}_v} \text{ins}(W)$, where the first union runs through all vertices v incident with f. Clearly, L has only block containing vertices incident with f, and every vertex x in $G'-V(L)$ satisfies that $m_{\mathcal{T}}(x,y) \leq t+2$. Therefore, there exists a cycle C in L such that C is the boundary of a $(t+2)$-zone around y in G'.

Let S' be the union of S and the subgraph of G' contained in the inside of the closure of the disk bounded by C. Let $\overline{\Omega'} = V(C)$, and Ω' be the cyclic ordering on $\overline{\Omega'}$ that coincides the cyclic ordering of C. Since (S, Ω) is an ρ-vortex, for every two intervals I, J that partition $\bar{\Omega}$, there exists $X_{I,J} \subseteq V(S)$ with $|X| \leq \rho$ such that there exists no path from $I - X$ to $J - X$. Therefore, for every two intervals I', J' that partition $\overline{\Omega'}$, let u, v be the first vertex in I', J', respectively, under the ordering Ω', and let I'', J'' be the two intervals partitioning $\bar{\Omega}$ with the first vertex u', v', respectively, where $u \in \ell_u \cap \bar{\Omega}$ and $v \in \ell_v \cap \bar{\Omega}$. Then there does not exist a path from $I' - X'$ to $J' - X'$ in $S' - X'$, where $X' = V(\ell_u) \cup V(\ell_v) \cup X_{I'',J''}$. As $|X'| \leq \rho + 4t + 20$, (S', Ω') is an $(\rho + 4t + 20)$-vortex. ■

Lemma 2.5.2 *Let $d \geq 3$, and let $\kappa, h, h_1, h_2, ..., h_\kappa, \rho, \theta''$ be nonnegative integers. Then there exist integers $\theta_0(d, h, \rho, \kappa, \theta'')$, $\beta(d, h, \rho)$ and $f(d, h, \rho)$ such that the following holds. Suppose that*

1. *G is a graph and $\mathcal{T}$ is a tangle in G, and*

2. *τ is a proper arrangement of a $\mathcal{T}$-central segregation $\mathcal{S}$ of G in a surface Σ, and*

3. *G' is the 2-cell skeleton of τ and $\mathcal{T}'$ is a respectful tangle in G' of order θ, for some $\theta \geq \theta_0$, such that G contains G' as a minor and $\mathcal{T}'$ is conformal with $\mathcal{T}$, and*

4. *let $(S_1, \Omega_1), ..., (S_\kappa, \Omega_\kappa)$ be societies in $\mathcal{S}$, where each (S_i, Ω_i) is a ρ-vortex and contains at least one d-free vertex with respect to $\mathcal{T}$ such that for every $1 \leq i < j \leq \kappa$, and for every $x \in \overline{\Omega_i}$ and $y \in \overline{\Omega_j}$, $m_{\mathcal{T}'}(x, y) \geq 2f + 1$, and*

5. *$m_{\mathcal{T}'}(x, y) \geq f + 1$, for every $x \in \overline{\Omega_i}$ with $1 \leq i \leq \kappa$, and for every $y \in \bar{\Omega}$ with $(S, \Omega) \in \mathcal{S}$ and $|\bar{\Omega}| > 3$, and*

6. *$h_i \leq h$ for $1 \leq i \leq \kappa$.*

Then there exist $Z_1, Z_2, ..., Z_\kappa, U_1, U_2, ..., U_\kappa \subseteq V(G)$, *a subdrawing* $G'' = G' - \bigcup_{i=1}^{\kappa}(Z_i \cup U_i)$ *of* G', *a tangle* $\mathcal{T}''$ *in* G'' *of order at least* θ'' *conformal with* $\mathcal{T}'$ *obtained from* $\mathcal{T}' - \bigcup_{i=1}^{\kappa} Z_i$ *by clearing at most* κ f*-zones in* G' *such that for every* $i \in \{1, 2, ..., k\}$, *either*

1. $h_i \geq 2$, $U_i = \emptyset$ *and* $|Z_i| \leq \beta$ *such that every vertex in* $S_i - Z_i$ *is not d-free with respect to* $\mathcal{T}''$, *or*

2. $Z_i = \emptyset$, *and* U_i *is the set of vertices of* G *inside a* f*-zone* Λ_i *in* G' *around a vertex in* $\overline{\Omega_i}$ *with the boundary cycle* Y_i, *and* h_i *subsets* $A_{i,1}, A_{i,2}, ..., A_{i,h_i}$ *of* Y_i *such that the following hold.*

   (a) $V(S_i) \subseteq U_i$.

   (b) *Each* $A_{i,j}$ *has size d and* $\bigcup_{j=1}^{h_i} A_{i,j}$ *is free with respect to* $\mathcal{T}''$.

   (c) $I_j \cap I_k = \emptyset$ *for* $1 \leq j < k \leq h_i$, *where* I_j, I_k *is the minimum interval of* Y_i *containing* $A_{i,j}, A_{i,k}$, *respectively.*

   (d) *There exist* $v_{i,1}, v_{i,2}, ..., v_{i,h_i} \in U_i$ *such that there are* h_i *disjoint d-spiders contained in* Λ_i, *where each of them is from* $v_{i,j}$ *to* $A_{i,j}$.

Proof. Let s', k' be the value $s'(h, d, 2hd + 1, \rho), k'(h, d, \rho)$ mentioned in Theorem 2.3.2, respectively. Let $f(d, h, \rho) = 4 + 7s'$, $\beta(d, h, \rho) = 2(k'd)^{d+1} + 1$, and $\theta_0(d, h, \rho, \kappa) = \theta'' + \kappa(4f + \beta + 2)$. Let (S, Ω) be an arbitrary (S_i, Ω_i), and let v_S be a vertex in $\bar{\Omega}$. Let $\Lambda_{S,0}$ be a 4-zone around v_S such that $\Lambda_{S,0}$ contains all atoms y of G' with $m_{\mathcal{T}'}(v_S, y) \leq 1$ as interior points. Note that every vertex in $\bar{\Omega}$ has distance at most 1 from v_S with respect to the metric $m_{\mathcal{T}'}$, so $\Lambda_{S,0} \cap G$ contains S. Let $G_{S,0}$ be the subgraph of G consisting of the societies (S', Ω') with $\tau(S', \Omega')$ contained in the closure of $\Lambda_{S,0}$, and let $C_{S,0}$ be the boundary cycle of $\Lambda_{S,0}$. Let $(G_{S,0}, \Omega_{S,0})$ be a society, where $\overline{\Omega_{S,0}} = V(C_{S,0})$ with the cyclic ordering determined by $C_{S,0}$.

For $1 \leq i \leq s'$, let $\Lambda_{S,i}$ be a $(4+7i)$-zone around v_S such that $\partial\Lambda_{S,i} \cap \partial\Lambda_{S,i-1} = \emptyset$. Note that the existence of $\Lambda_{S,i}$ follows from Lemma 2.4.8. Let $C_{S,i}$ be the boundary cycle of $\Lambda_{S,i}$ for $1 \leq i \leq s'$. Let $\Lambda_S = \Lambda_{S,s'}$. Let G_S be the subgraph of G consisting of the societies (S', Ω') with $\tau(S', \Omega')$ contained in the closure of Λ_S, and let Ω_S be the cyclic ordering on the boundary cycle of Λ_S. So (G_S, Ω_S) is a composition of a circumscribed vortex $(G_{S,0}, \Omega_{S,0})$ with a rural neighborhood which has a presentation with a s'-nest $(C_{S,1}, C_{S,2}, ..., C_{S,s'})$.

Let $h'_i = k'$ if $h_i \neq 1$, and $h'_i = 1$ if $h_i = 1$. Let X_S be the set of d-free vertices in S with respect to $\mathcal{T}$. Note that $X_S \neq \emptyset$ by assumption. By Theorem 2.2.5, either there exist h'_i disjoint d-spiders from X_S to $\overline{\Omega_S}$, or there exists $W_S \subseteq V(G) \cap \Lambda_S$ with $|W_S| \leq 2(h'_i d)^{d+1} + 1 \leq \beta$ such that every d-spider from X_S to $\overline{\Omega_S}$ intersects W_S. When $S = S_i$ and the latter case holds and $h_i > 1$, the first statement of the theorem holds by taking $U_i = \emptyset$ and $Z_i = W_S$. When $S = S_i$ and $h_i = 1$, the former case holds by Menger's Theorem. Therefore, we assume that $S = S_i$ for some i and the former case holds.

Define Z_i to be the empty set. Let $D_{i,1}, D_{i,2}, ..., D_{i,h'_i}$ be disjoint d-spiders from X_{S_i} to $\overline{\Omega_{S_i}}$. Apply Theorem 2.3.2 by taking $(S, \Omega) = (G_{S_i}, \Omega_{S_i})$, $(S_0, \Omega_0) = (S_i, \Omega_i)$ and $D_j = D_{i,j}$ for $1 \leq j \leq h'_i$, there exist pairwise disjoint d-spiders $D'_{i,1}, D'_{i,2}, ..., D'_{i,h_i}$ from X_{S_i} to $V(C_{S_i,s'})$, an $(hd+1)$-nest $(N_{S_i,1}, ..., N_{S_i,hd+1})$ and intervals $I_{i,1}, I_{i,2}, ..., I_{i,h_i}$ of $C_{S_i,s'}$ satisfying the conclusions of Theorem 2.3.2. For every $1 \leq j \leq h_i$, since each $D'_{i,j}$ is perpendicular to $(N_{S_i,1}, ..., N_{S_i,2hd+1})$, there exists a set $A_{i,j}$ of $h_i d$ vertices in $D'_{i,j} \cap V(N_{S_i,1})$ such that there exist $h_i d$ disjoint paths from $A_{i,j}$ to $V(C_{S_i,s'})$, but there exists no path from $D'_{i,j} \cap X_{S_i}$ to $V(N_{S_i,1})$ in $D'_{i,j} - A_{i,j}$. Note that $N_{S_i,1}$ is the boundary of a f-zone. Define U_i to be the set of vertices of G inside the open disk bounded by $N_{S_i,1}$. So G'' is a subgraph of $G' - \bigcup_{j=1}^{\kappa} Z_i$ obtained by cleaning κ f-zones. By Theorem 2.4.6, there exists a tangle $\mathcal{T}''$ of G'' of order $\theta - \kappa\beta - \kappa(4f+2) \geq \theta''$ obtained from $\mathcal{T}' - \bigcup_{i=1}^{\kappa} Z_i$ by clearing κ f-zones. Therefore, $\mathcal{T}''$ is conformal with $\mathcal{T}' - \bigcup_{i=1}^{\kappa} Z_i$. Note that every G''-normal line from $A_{i,j}$ to $\overline{\Omega_{S_i}}$ intersects $V(N_{S_i,\ell})$ for each $2 \leq \ell \leq 2h_i d + 1$. Hence, $m_{\mathcal{T}''}(x, y) \geq 2h_i d + 1$ for every atom x in $A_{i,j}$ and atom $y \in \overline{\Omega_{S_i}}$. On the other hand, by the planarity, for every $1 \leq j \leq h_i$, there exists an interval $J_{i,j}$ of $N_{S_i,1}$ containing $A_{i,j}$, such that $J_{i,j} \cap J_{i,j'} = \emptyset$ for every $j' \neq j$.

Suppose that $\bigcup_{j=1}^{h_i} A_{i,j}$ is not free with respect to $\mathcal{T}''$ for some i, then there exists $(A, B) \in \mathcal{T}''$ such that $\bigcup_{j=1}^{h_i} A_{i,j} \subseteq V(A)$ with order less than dh_i. We assume that A is as small as possible, so $m_{\mathcal{T}''}(x, y) < dh$ for every atom x in A and $y \in V(A) \cap V(B)$. That is, $m_{\mathcal{T}''}(x, y) < 2dh$ for every atoms x, y in A. Therefore, $\overline{\Omega_{S_i}} \subseteq V(B) - V(A)$. However, there exist dh_i disjoint paths from $\bigcup_{j=1}^{h_i} A_{i,j}$ to $\overline{\Omega_{S_i}}$, a contradiction. So $\bigcup_{j=1}^{h_i} A_{i,j}$ is free with respect to $\mathcal{T}''$ for every i. This proves the lemma. ■

Lemma 2.5.3 *Let $d \geq 3, h$ be positive integers. Let G be a 2-cell drawing in a surface Σ, and let $\mathcal{T}$ be a respectful tangle in G. Then there exist integers $\theta(d, h, \Sigma), \phi(d, h, \Sigma)$ such that if $\mathcal{T}$ has order at least θ and G contains h d-free vertices $v_1, v_2, ..., v_h$ of degree at least d with $m_{\mathcal{T}}(v_i, v_j) > \phi$ for $1 \leq i < j \leq h$, then G admits an H-subdivision for every graph H of order h and of maximum degree d embeddable in Σ.*

Proof. Let H be a graph of order h and of maximum degree d embeddable in Σ. Let θ' be the positive integer θ mentioned in Theorem 2.4.9 by taking $t = h$ and $z = dh$. Note that $(\{v_i\}, \{v_i\})$ is a 0-vortex for every i. For $1 \leq i \leq h$, let Λ_i be the 7-zone around v_i of G mentioned in Lemma 2.5.1 such that Λ_i contains v_i and all its neighbors, and let S_i be the subgraph of G inside the closure of Λ_i and $\overline{\Omega_i} = \partial\Lambda_i \cap G$ with the cyclic order defined by the boundary cycle of Λ_i. So (S_i, Ω_i) is a 28-vortex. Let $\alpha = \alpha_{2.5.2}(d, 1, 28, h)$, $\beta = \beta_{2.5.2}(d, 1, 28)$ and $f = f_{2.5.2}(d, 1, 28)$, where $\alpha_{2.5.2}$, $\beta_{2.5.2}$ and $f_{2.5.2}$ be the numbers α, β, f mentioned in Lemma 2.5.2. Define $\phi = \theta_{2.4.9} + h(4f + 2)$, where $\theta_{2.4.9}$ is the θ mentioned in Theorem 2.4.9, and $\theta = \alpha + \phi$.

Applying Lemma 2.5.2 by taking $\kappa = h$, $h_i = 1$ for $1 \leq i \leq h$, and $\mathcal{S}$ the segregation consisting of $(S_1, \Omega_1), (S_2, \Omega_2), ..., (S_h, \Omega_h)$ and the societies in which each of them consists of exactly one edge that is not in $\bigcup_{i=1}^{h} S_i$, we obtain the desired subgraph G'' with a respectful tangle $\mathcal{T}''$, and $A_{i,1}$ for $1 \leq i \leq h$, such that every

$A_{i,1}$ is free with respect to $\mathcal{T}''$, as mentioned in the conclusion of Lemma 2.5.2. Then for every $x \in A_{i,1}$ and $y \in A_{j,1}$ for some $i \neq j$, we have that $m_{\mathcal{T}''}(x, y) \geq \theta_{2.4.9}$ by Theorem 2.4.6.

For $1 \leq i \leq h$, let Δ_i be a closed disk in Σ contained in the closure of Λ_i such that $\Delta_i \cap G'' = A_{i,1}$. Since H can be embedded in Σ, we can partition $\bigcup_{i=1}^{h} A_{i,1}$ and apply Theorem 2.4.9 to obtain a linear forest so that an H-subdivision in G can be obtained by concatenating these linear forests and h disjoint d-spiders in $S_1, S_2, ..., S_h$, where each S_i is from v_i to $A_{i,1}$, we obtain an H-subdivision in G. ■

Lemma 2.5.4 *Let ρ be an integer, G a graph, $\mathcal{T}$ a tangle in G of order at least $2\rho + 2$, and $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ a $\mathcal{T}$-central segregation of G such that $|\bar{\Omega}| \leq 3$ for every $(S, \Omega) \in \mathcal{S}_1$. If there exists a segregation $\mathcal{S}' = \mathcal{S}'_1 \cup \mathcal{S}'_2$ with $\mathcal{S}'_1 \subseteq \mathcal{S}_1$ of G such that every member of $\mathcal{S}'_2$ is a ρ-vortex, and there exists no $(A, B) \in \mathcal{T}$ of order at most $2\rho + 1$ such that $B \subseteq S$ for some $(S, \Omega) \in \mathcal{S}'_2$, then $\mathcal{S}'$ is $\mathcal{T}$-central.*

Proof. Suppose that $\mathcal{S}'$ is not $\mathcal{T}$-central. So there exist $(A, B) \in \mathcal{T}$ of order at most the half of the order of $\mathcal{T}$ and $(S, \Omega) \in \mathcal{S}'$ such that $B \subseteq S$. Since $\mathcal{S}'_1 \subseteq \mathcal{S}_1$ and $\mathcal{S}$ is $\mathcal{T}$-central, $(S, \Omega) \in \mathcal{S}'_2$. Let $\bar{\Omega} = v_1, v_2, ..., v_n$ in order, where $n = |\bar{\Omega}|$. We may assume that every v_i is adjacent to a vertex in $G - V(S)$, otherwise we may remove it from $\bar{\Omega}$. As (S, Ω) is a ρ-vortex, there exists a path decomposition $(P, \mathcal{X})$ of S of adhesion at most ρ such that the i-th bag X_i of $(P, \mathcal{X})$ contains v_i for every $1 \leq i \leq n$. For every subgraph H of S, we define (A_H, B_H) to be the separation of G with minimum order such that $A_H = H$. In particular, $(A_{S[X_i]}, B_{S[X_i]})$ has order at most $2\rho + 1$, so $(A_{S[X_i]}, B_{S[X_i]}) \in \mathcal{T}$. For $1 \leq i \leq n$, define $(A_i, B_i) = (A \cup A_{S[\bigcup_{j=1}^{i} X_j]}, B \cap B_{S[\bigcup_{j=1}^{i} X_j]})$. Note that if $v_i \in V(B)$, then $v_i \in V(A)$ since $B \subseteq S$. So the order of (A_i, B_i) is at most $|V(A) \cap V(B)| + |V(A_{S[\bigcup_{j=1}^{i} X_j]}) \cap V(B_{S[\bigcup_{j=1}^{i} X_j]}) \cap (V(B) - V(A))| \leq |V(A) \cap V(B)| + \rho$. Since the order of (A, B) is at most the half of the order of $\mathcal{T}$, and the order of $\mathcal{T}$ is greater than 2ρ, either $(A_i, B_i) \in \mathcal{T}$, or $(B_i, A_i) \in \mathcal{T}$. Let $(A_0, B_0) = (A, B)$. We shall prove that $(A_i, B_i) \in \mathcal{T}$ for $0 \leq i \leq n$ by induction on i.

When $i = 0$, $(A_0, B_0) = (A, B) \in \mathcal{T}$. Assume that $(A_i, B_i) \in \mathcal{T}$ for some i. Suppose that $(B_{i+1}, A_{i+1}) \in \mathcal{T}$. But $(A_i, B_i), (A_{S[X_{i+1}]}, B_{S[X_{i+1}]}) \in \mathcal{T}$, and $B_{i+1} \cup A_i \cup S[X_{i+1}] = G$, a contradiction. This proves that $(A_i, B_i) \in \mathcal{T}$ for every $0 \leq i \leq n$.

Furthermore, $(A_n, B_n) = (A \cup S, B \cap B_S)$. Recall that $V(B \cap B_S) \subseteq V(B) \cap \bar{\Omega} \subseteq V(A) \cap V(B)$, so $|V(B_n)| \leq |V(A) \cap V(B)|$. Hence, $(B_n, G - E(B_n))$ has order less the order of $\mathcal{T}$, so $(B_n, G - E(B_n)) \in \mathcal{T}$. However, $A_n \cup B_n = G$, a contradiction. Consequently, $\mathcal{S}'$ is $\mathcal{T}$-central. ■

Given an proper arrangement α of a segregation $\mathcal{S}$ in a surface Σ, we say that the *trunk* of α is the drawing $\Gamma = (U, V)$ in Σ, where $V(\Gamma) = \bigcup_{v \in V(\mathcal{S})} \alpha(v)$, and $U(\Gamma)$ consists of the following.

- The boundary of $\alpha(S, \Omega)$ for each $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| \geq 3$.
- The boundary of $\alpha(S, \Omega)$ for each $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| = 2$ such that there exist two edge-disjoint paths in S connecting the two vertices in $\bar{\Omega}$.
- A line in $\alpha(S, \Omega)$ with ends $\bar{\Omega}$ for each $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| = 2$ such that there do not exist two edge-disjoint paths in S connecting the two vertices in $\bar{\Omega}$.

Note that we do not add anything into the trunk for (S, Ω) with $|\bar{\Omega}| \leq 1$.

The notion of trunk is very similar with the skeleton, and it will be used in other chapters of this thesis. We will prove the following general lemma for skeletons and trunks. We say a graph is *weakly subcubic* if every vertex is adjacent to at most three neighbors.

Lemma 2.5.5 *For a positive nondecreasing function ϕ, integers $\rho, \lambda, \kappa, k, \theta^*, d, s$ with $d \geq 4$, and every collection of graphs $\mathcal{F}$ on at most s vertices, there exist integers θ, ρ^* such that the following is true. Assume that a graph G has a tangle $\mathcal{T}$ and a $\mathcal{T}$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ that has a proper arrangement τ in a surface Σ such that the following hold.*

1. $|\bar{\Omega}| \leq 3$ *for every* $(S, \Omega) \in \mathcal{S}_1$.

2. $|\mathcal{S}_2| \leq \kappa$.

3. (S, Ω) *is a* ρ*-vortex for every* $(S, \Omega) \in \mathcal{S}_2$.

4. *Let* G' *be the skeleton of* $\mathcal{S}$ *or the trunk of* $\mathcal{S}$. G' *is* 2*-cell embedded in* Σ *and has a respectful tangle* $\mathcal{T}'$ *of order at least* θ *conformal with* $\mathcal{T}$.

5. *There exist* k λ*-zones* $\Lambda_1, \Lambda_2, ..., \Lambda_k$ *in* G' *with respect to the metric* $m_{\mathcal{T}'}$ *such that every* d*-free subgraph of* G' *with respect to* $\mathcal{T}'$ *isomorphic to a member of* $\mathcal{F}$ *is contained in* $(\bigcup_{i=1}^{k} \Lambda_i) \cap G'$.

6. *If* G' *is the trunk of* $\mathcal{S}$, *then the following hold.*

    (a) G' *is weakly subcubic.*

    (b) $S \cap S' = \emptyset$ *for different members* $(S, \Omega), (S', \Omega') \in \mathcal{S}_1$ *with* $|\bar{\Omega}| = |\overline{\Omega'}| = 3$.

    (c) *For every* $(S, \Omega) \in \mathcal{S}_2$, *there exists a cycle in* S *passing through all vertices in* $\bar{\Omega}$ *in order.*

    (d) *For every edge in a graph in* $\mathcal{F}$, *there exists another edge that has the same ends.*

Then there exists a $\mathcal{T}$*-central segregation* $\mathcal{S}^* = \mathcal{S}_1^* \cup \mathcal{S}_2^*$ *properly arranged in* Σ *such that the following hold.*

1. $\mathcal{S}_1^* \subseteq \mathcal{S}_1$; *in particular,* $|\bar{\Omega}| \leq 3$ *for every* $(S, \Omega) \in \mathcal{S}_1^*$.

2. $|\mathcal{S}_2^*| \leq \kappa + k$ *and* $\bigcup_{(S,\Omega) \in \mathcal{S}_2} S \subseteq \bigcup_{(S,\Omega) \in \mathcal{S}_2^*} S$.

3. *There exists an integer* ρ' *with* $\rho' \leq \rho^*$ *such that* (S, Ω) *is a* ρ'*-vortex for every* $(S, \Omega) \in \mathcal{S}_2^*$.

4. *Let G^* be the skeleton of $\mathcal{S}^*$ or the trunk of $\mathcal{S}^*$, respectively, if G' is the skeleton of $\mathcal{S}$ or the trunk of $\mathcal{S}$, respectively. G^* is 2-cell embedded in Σ and has a respectful tangle $\mathcal{T}^*$ of order at least $\theta^* + \phi(\rho^*) + 2\rho^*$ conformal with $\mathcal{T}$.*

5. *For every $(S, \Omega) \in \mathcal{S}_2^*$, there exists a cycle passing through all vertices in $\overline{\Omega^*}$ in order.*

6. *If G^* is the trunk of $\mathcal{S}^*$, then it is weakly subcubic.*

7. *There is no d-free subgraph of G^* with respect to $\mathcal{T}^*$ isomorphic to a member of $\mathcal{F}$.*

8. *$m_{\mathcal{T}^*}(x, y) \geq \phi(\rho')$ for every atoms x, y of G^* with $x \in S_x, y \in S_y$ for different members $(S_x, \Omega_x), (S_y, \Omega_y) \in \mathcal{S}_2^*$,*

Proof. Note that each society that consists of a single vertex is a 0-vortex. So by Lemma 2.5.1, for each Λ_i, we can find a $(\lambda + 6)$-zone Λ_i' containing Λ_i such that $(G \cap \Lambda_i, \Omega)$ is a $(4\lambda + 24)$-vortex, where Ω is a cyclic ordering on $V(G) \cap \partial\Lambda_i$ consistent with the cyclic ordering of the cycle bounding Λ_i. Therefore, we can replace Λ_i by Λ_i' so that we may assume that every Λ_i is a λ'-zone and the subgraph of G inside the disk Λ_i is a λ'-vortex (S, Ω) for some constant λ' only depending on λ, and there exists a cycle of length at least two passing through all vertices in $\bar{\Omega}$ in order. Similarly, for each $(S, \Omega) \in \mathcal{S}_2$, there exists a 7-zone Λ_S containing the disk $\tau(S, \Omega)$, and the subgraph of G inside this disk is a $(\rho + 52)$-vortex by Lemma 2.5.1.

Let $\mathcal{C} = \{\Lambda_i, \Lambda_S : 1 \leq i \leq k, (S, \Omega) \in \mathcal{S}_2\}$, and let λ'' be the minimum t such that every member of $\mathcal{C}$ is a t-zone. For each member Λ of $\mathcal{C}$, let $S_\Lambda = G \cap \Lambda$, and let $\overline{\Omega_\Lambda} = V(G) \cap \partial\Lambda$ ordered by the cyclic ordering given by the cycle bounding Λ. Let M be the maximum depth of $(S_\Lambda, \Omega_\Lambda)$ for all members Λ of $\mathcal{C}$. Note that $|\mathcal{C}| \leq k + \kappa$, $M = \max\{\lambda', \rho + 52\}$, and $\lambda'' \leq \max\{\lambda', 7\}$. Then we consecutively test whether there exist two atoms of G' in different members of $\mathcal{C}$ with distance less than $\phi(M+2)+(4\lambda''+10)|\mathcal{C}|+2$ under the metric $m_{\mathcal{T}'}$, and if such two nearby vortices exist, then we do the following. Find a minimum number t such that the $(t+6)$-zone Λ mentioned in the conclusion of Lemma 2.5.1 containing these two nearby members of $\mathcal{C}$, and remove these two members from $\mathcal{C}$ and add Λ into $\mathcal{C}$, and then we update M and λ''. Since $|\mathcal{C}|$ decreases in each step, this process will terminate within $\kappa+k$ steps. Furthermore, when the process terminates, each member of $\mathcal{C}$ defines a M-vortex, where M only depends on κ, k, λ and ρ, and the distance between two members of $\mathcal{C}$ is at least $\phi(M+2)+(4\lambda''+10)|\mathcal{C}|+2$ under the metric $m_{\mathcal{T}'}$. Clearly, there exists an integer ρ^* (only depends on κ, k, λ, ρ) such that $M+2 \leq \rho^*$. We define $\theta = 2\rho^*(\theta^* + \phi(\rho^*) + 2\rho^*) + 4\lambda'' + 16$.

We may assume that every edge of G' whose both ends inside Λ is in Λ for every $\Lambda \in \mathcal{C}$. For every $\Lambda \in \mathcal{C}$, let Λ' be the minimal closed disk in Σ containing Λ and $\tau(S,\Omega)$ for every $(S,\Omega) \in \mathcal{S}_1$ with $|\bar{\Omega} \cap \Lambda| \geq 2$. Clearly, Λ' is a $(\lambda''+2)$-zone, and $(S_{\Lambda'}, \Omega_{\Lambda'})$ is a $(M+2)$-vortex, and every two atoms of G' in different members of $\mathcal{C}$ has distance at least $\phi(M+2)+(4\lambda''+10)|\mathcal{C}|$. If G' is the skeleton of $\mathcal{S}$, then define $(S'_\Lambda, \Omega'_\Lambda)$ to be $(S_\Lambda, \Omega_\Lambda)$ for every $\Lambda \in \mathcal{C}$. Now assume that G' is the trunk of $\mathcal{S}$. Recall that G' is weakly subcubic and $S \cap S' = \emptyset$ for different members $(S,\Omega), (S',\Omega') \in \mathcal{S}_1$ with $|\bar{\Omega}| = |\overline{\Omega'}| = 3$ in this case. Observe that there is no $(S,\Omega) \in \mathcal{S}_1$ with $S \not\subseteq G \cap \Lambda'$ and $|\bar{\Omega} \cap \Lambda'| \geq 2$ unless $|\bar{\Omega}| \leq 2$, since $S \cap S' = \emptyset$ for different members $(S,\Omega), (S',\Omega') \in \mathcal{S}_1$ with $|\bar{\Omega}| = |\overline{\Omega'}| = 3$. We replace Λ' by the minimal disk that contains Λ' and $\tau(S,\Omega)$ for every $(S,\Omega) \in \mathcal{S}_1$ with $|\bar{\Omega}| \leq 2$. Then there is no $(S,\Omega) \in \mathcal{S}_1$ with $S \not\subseteq G \cap \Lambda'$ and $|\bar{\Omega} \cap \Lambda'| \geq 2$. Then we define $(S'_\Lambda, \Omega'_\Lambda)$ to be $(S_{\Lambda'}, \Omega_{\Lambda'})$ for every $\Lambda \in \mathcal{C}$.

Define a new segregation $\mathcal{S}^* = \mathcal{S}^*_1 \cup \mathcal{S}^*_2$ of G by letting $\mathcal{S}^*_2 = \{(S'_\Lambda, \Omega'_\Lambda) : \Lambda \in \mathcal{C}\}$ and $\mathcal{S}^*_1 = \{(S,\Omega) \in \mathcal{S}_1 : V(S) \not\subseteq \bigcup_{\Lambda \in \mathcal{C}} V(S'_\Lambda)\}$. Let G^* be the skeleton of $\mathcal{S}^*$. Observe that for every integer t and separation (A,B) of G' or G^* of order t, there exists a separation (A',B') of G of order at most $2\rho^* t$ such that $A \subseteq A'$ and $B \subseteq B'$, since every member of $\mathcal{S}_2$ or $\mathcal{S}^*_2$ has depth at most ρ^*. Similarly, for every G^*-normal O-arc in Σ that intersects G^* at most t vertices, there exists a G'-normal O-arc in Σ that intersects G' at most $2\rho^* t$ vertices. Therefore, there exists a tangle $\mathcal{T}^*$ in G^* of order at least $\theta/(2\rho^*) \geq \theta^* + \phi(\rho^*) + 2\rho^*$ conformal with $\mathcal{T}$ and $\mathcal{T}'$, and $\mathcal{T}^*$ is respectful. On the other hand, $\mathcal{T}^*$ can be obtained from $\mathcal{T}'$ by clearing at most $|\mathcal{C}|$ $(\lambda''+2)$-zones, so $m_{\mathcal{T}^*}(x,y) \geq m_{\mathcal{T}'}(x,y) - |\mathcal{C}|(4\lambda''+2) \geq \phi(M+2)$ by Theorem 2.4.6. Therefore, Conclusions 1-4 and 8 hold.

Recall that every member in $\mathcal{S}_2^*$ is a society obtained by applying Lemma 2.5.1, so Conclusion 5 holds. This implies that G' contains G^* as a subdivision. So if G' is the trunk of $\mathcal{S}$, then G^* is weakly subcubic as G' is. This proves Conclusion 6. In fact, G^* is a subgraph of G' if G' is the skeleton of $\mathcal{S}$. So Conclusion 7 holds in this case. But when G^* is the trunk of $\mathcal{S}^*$, there do not exist vertices x, y of G^* such that there are multiple edges between x, y in G^* but not in G'; otherwise, there exists a society $(S, \Omega) \in \mathcal{S}_1^*$ such that $S \not\subseteq G \cap \Lambda'$ and $|\bar{\Omega} \cap \Lambda'| \geq 2$, where Λ' is the λ''-zone corresponding to the vortex containing x, y, a contradiction. But for every edge in a graph in $\mathcal{F}$, there exists another edge with the same ends in this case, so no subgraph of G^* that is not a subgraph of G' but is isomorphic to a graph in $\mathcal{F}$. Hence, Conclusion 7 holds.

It remains to prove that $\mathcal{S}^*$ is a $\mathcal{T}$-central segregation of G. Since $\mathcal{T}'$ has order at least θ and is conformal with $\mathcal{T}$, the order of $\mathcal{T}$ is at least θ. Since $\mathcal{S}_1^* \subseteq \mathcal{S}_1$ and $\mathcal{S}$ is $\mathcal{T}$-central, by Lemma 2.5.4, it is sufficient to show that there is no $(A, B) \in \mathcal{T}$ of order at most $2\rho^*+1$ such that $B \subseteq S$ for some $(S, \Omega) \in \mathcal{S}_2^*$. Suppose that such (A, B) exists, then there exist $(A', B') \in \mathcal{T}^*$, where B' contains at most $2\rho^*+1$ vertices, since $\mathcal{T}^*$ is a tangle of order at least $2\rho^*+1$ conformal with $\mathcal{T}$. However, it implies that $(G^* - E(B'), B') \in \mathcal{T}^*$, a contradiction. Hence $\mathcal{S}^*$ is $\mathcal{T}$-central. ■

A segregation $\mathcal{S}$ of G is *maximal* if there exists no segregation $\mathcal{S}'$ such that $\{(S, \Omega) \in \mathcal{S} : |\bar{\Omega}| > 3\} = \{(S', \Omega') \in \mathcal{S}' : |\overline{\Omega'}| > 3\}$ and for every $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| \leq 3$, there exists $(S', \Omega') \in \mathcal{S}'$ with $|\overline{\Omega'}| \leq 3$ such that $S' \subseteq S$, and the containment is strict for at least one society. Furthermore, if H is a triangle-free graph and the skeleton of a maximal segregation $\mathcal{S}$ of G admits an H-subdivision, then G admits an H-subdivision. Note that if a segregation $\mathcal{S}$ of G is maximal, then G contains the skeleton of $\mathcal{S}$ as a minor.

The following theorem is a stronger form of the structure theorem for excluding minors in [45].

Theorem 2.5.6 ([12, Theorem 7]) *For every graph L, there exists an integer κ such that for any nondecreasing positive function ϕ, there exist integers θ, ξ, ρ with the following property. Let $\mathcal{T}$ be a tangle of order at least θ in a graph G controlling no L-minor of G. Then there exist $Z \subseteq V(G)$ with size at most ξ and a maximal $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ properly arranged in a surface Σ in which L cannot be drawn, where every $(S, \Omega) \in \mathcal{S}_1$ has the property that $|\bar{\Omega}| \leq 3$, and $|\mathcal{S}_2| \leq \kappa$ and every member in $\mathcal{S}_2$ is a p-vortex for some $p \leq \rho$. Furthermore, the skeleton G' of $\mathcal{S}$ is 2-cell embedded in Σ with a respectful tangle $\mathcal{T}'$ of order at least $\phi(p)$ conformal with $\mathcal{T} - Z$, and if x and y are two vertices in G' incident with two different members in $\mathcal{S}_2$, then $m_{\mathcal{T}'}(x, y) \geq \phi(p)$.*

Let us recall that the function mf was defined prior to Theorem 1.3.4. A graph H has a *nice* embedding in Σ if H can be 2-cell embedded in Σ and it has a set F of faces such that every vertex of H of degree at least 4 is incident with exactly one face in F, and $|F| = \mathrm{mf}(H, \Sigma)$.

Lemma 2.5.7 ([12, Lemma 12]) *Let H be a graph of maximum degree d that can be embedded in a surface Σ. Then there exists a triangle-free graph H' of maximum degree d admitting an H-subdivision such that $\mathrm{mf}(H', \Sigma) = \mathrm{mf}(H, \Sigma)$ and H' has a nice embedding in Σ.*

Recall that a vertex v in a graph G is d-free with respect to a tangle $\mathcal{T}$ in G if there does not exist a separation $(A, B) \in \mathcal{T}$ of order less than d such that $v \in V(A) - V(B)$.

Now, we are ready to prove Theorem 1.4.1, which we restate.

Theorem 2.5.8 *Let $d \geq 4, h$ be positive integers. Then there exist $\theta, \kappa, \rho, \xi, g \geq 0$ satisfying the following property. If H is a graph of maximum degree d on h vertices, and a graph G does not admit an H-subdivision, then for every tangle $\mathcal{T}$ in G of order at least θ, there exists $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that either*

1. *no vertex of $G - Z$ is d-free with respect to $\mathcal{T} - Z$, or*
2. *there exist a $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ with $|\mathcal{S}_2| \leq \kappa$, having a proper arrangement in some surface Σ of genus at most g such that every society (S_1, Ω_1) in $\mathcal{S}_1$ satisfies that $|\overline{\Omega_1}| \leq 3$, and every society (S_2, Ω_2) in $\mathcal{S}_2$ is an ρ-vortex, and satisfies the following property: either*
   - (a) *H cannot be drawn in Σ, or*
   - (b) *H can be drawn in Σ and $\mathrm{mf}(H, \Sigma) \geq 2$, and there exists $\mathcal{S}_2' \subseteq \mathcal{S}_2$ with $|\mathcal{S}_2'| \leq \mathrm{mf}(H, \Sigma) - 1$ such that every d-free vertex of $G - Z$ with respect to $\mathcal{T} - Z$ is in $S - \bar{\Omega}$ for some $(S, \Omega) \in \mathcal{S}_2'$.*

Proof. Note that there are only finitely many graphs of maximum degree d on h vertices, and there are only finitely many surfaces in which H can be drawn but $K_{\lceil \frac{3}{2}dh \rceil}$ cannot. So there exists h^* such that for every graph H on h vertices of maximum degree d and surface in which H can be drawn but $K_{\lceil \frac{3}{2}dh \rceil}$ cannot, the graph H' mentioned in Lemma 2.5.7 has at most h^* vertices.

Let $\theta_{2.5.2}, \alpha_{2.5.2}, \beta_{2.5.2}, f_{2.5.2}$ be the functions α, β, f mentioned in Lemma 2.5.2, respectively. Let ϕ' be the maximum $\phi_{2.5.3}(d, h^*, \Sigma)$ among all surfaces Σ in which $K_{\lceil \frac{3}{2}dh \rceil}$ cannot be drawn, where $\phi_{2.5.3}$ is the number ϕ mentioned in Lemma 2.5.3. Let $\theta_{2.4.9}$ be the maximum of θ mentioned in Theorem 2.4.9 by taking all surfaces in which $K_{\lceil \frac{3}{2}dh \rceil}$ cannot be drawn, and $t = h^*$, $z = dh^*$. Let $\kappa_{2.5.6}$ be the number κ

mentioned in Theorem 2.5.6 by taking $L = K_{\lceil \frac{3}{2}dh \rceil}$. Then let $\theta_{2.5.6}$, $\xi_{2.5.6}$, $\rho_{2.5.6}$ be the number θ, ξ, ρ mentioned in Theorem 2.5.6, respectively, by further taking $\phi(x)$ the constant function $2d+1$. Let $\theta_{2.5.3}$ be the θ mentioned in Lemma 2.5.3. Let $\theta_{2.5.5}$ and $\rho_{2.5.5}$ be the number θ and ρ^* obtained by applying Lemma 2.5.5 by taking ϕ to be the function such that $\phi(x) = 2f_{2.5.2}(d, h^*, x)$, $\rho = \rho_{2.5.6}$, $\lambda = d + \phi' + 11$, $\kappa = \kappa_{2.5.6}$, $k = h^* + \kappa_{2.5.6}$, $\theta^* = \theta_{2.5.2}(d, h^*, \rho_{2.5.6}, \kappa_{2.5.6}, (dh^* + h^* + 1)(\theta_{2.4.9} + 1))$, $d = d$, $s = 1$ and $\mathcal{F}$ be the family of graphs that contains exactly one vertex with no edges.

Let $\theta_{2.2.4}$ be the number θ mentioned in Theorem 2.2.4. Let $\xi = \max\{\xi_{2.5.6} + (\kappa_{2.5.6}+h^*)\beta_{2.5.2}(d, h^*), (hd)^{d+1}\}$, $\theta = 2\rho_{\kappa_{2.5.6}+h^*}(\theta_{2.5.5}+\theta_{2.5.3}+\theta_{2.5.6})+\xi$, $\kappa = \kappa_{2.5.6}+h^*$, $\rho = \rho_{\kappa_{2.5.6}} + h^*$, and let g be the minimum genus of a surface in which $K_{\lceil \frac{3}{2}dh \rceil}$ cannot be drawn. Let $\mathcal{T}$ be a tangle of order at least θ in G.

We may assume that G contains at least h vertices of degree d, otherwise the first statement holds by letting Z be the set of vertices of degree at least d. We first assume that $\mathcal{T}$ controls a $K_{\lceil \frac{3}{2}dh \rceil}$-minor. By Lemma 2.2.2 and Theorem 2.2.4, since G does not admit an H-subdivision, there exists a set of vertices Z of G with $|Z| \leq \xi$ such that for every vertex v of $G - Z$ of degree at least d in G, there exists a separation $(A, B) \in \mathcal{T}$ such that $v \in V(A) - V(B)$ and every d-spider with head v to $V(B)$ intersects Z. In other words, there exists a separation (A_v, B_v) of $G - Z$ of order at most $d-1$ such that $v \in V(A_v) - V(B_v)$ and $B_v \supseteq B$. By the tangle axioms, $(A_v, B_v) \in \mathcal{T} - Z$. Therefore, the first statement holds. So we may assume that $\mathcal{T}$ does not control a $K_{\lceil \frac{3}{2}dh \rceil}$-minor.

By Theorem 2.5.6, there exist a surface Σ in which $K_{\lceil \frac{3}{2}dh \rceil}$ cannot be drawn, $Z \subseteq V(G)$ with $|Z| \leq \xi_{2.5.6}$, and a maximal $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ with $|\mathcal{S}_2| \leq \kappa_{2.5.6}$, having a proper arrangement τ in Σ such that every society (S, Ω) in $\mathcal{S}_1$ satisfies that $|\bar{\Omega}| \leq 3$, and every society in $\mathcal{S}_2$ is a $\rho_{2.5.6}$-vortex, and the skeleton G' of $\mathcal{S}$ is 2-cell embedded in Σ and has a respectful tangle $\mathcal{T}'$ of order at least $\phi(\rho_{2.5.6})$ conformal with $\mathcal{T} - Z$, and if x, y are two vertices in G' incident with two different members in $\mathcal{S}_2$, then $m_{\mathcal{T}'}(x,y) \geq \phi(\rho_{2.5.6})$. If H cannot be drawn in Σ, then Statement 2(a) holds, so we may assume that H can be drawn in Σ.

On the other hand, we may assume that $G-Z$ contains d-free vertices with respect to $\mathcal{T}-Z$, otherwise Statement 1 holds. Note that every vertex in $\bigcup_{(S,\Omega)\in\mathcal{S}_1} V(S) - V(G')$ is not d-free with respect to $\mathcal{T}-Z$ since $d \geq 4$. If v is in $(G-Z)\cap G'$ but not d-free respect to $\mathcal{T}'$, then there exists a separation $(A',B') \in \mathcal{T}'$ of order less than d such that $v \in V(A') - V(B')$. We choose (A',B') such that A' is as small as possible. Note $m_{\mathcal{T}'}(v,x) \leq d$ for every $x \in V(A)$ by Theorem 2.4.3. Suppose that there is no vertex $x \in V(S)$ with $(S,\Omega) \in \mathcal{S}_2$ and $m_{\mathcal{T}'}(v,x) \leq d$. Then there exists $(A,B) \in \mathcal{T}-Z$ of order less than d such that $V(A) = \bigcup_{(S,\Omega)\in\mathcal{S}, V(S)\subseteq V(A)} \bar{\Omega}$ and $V(A)\cap V(B) = V(A')\cap V(B')$. So v is not d-free with respect to $\mathcal{T}-Z$. Therefore, if v is a vertex in $(G-Z)\cap G'$ that is d-free with respect to $\mathcal{T}-Z$ but not d-free with respect to $\mathcal{T}'$, then $m_{\mathcal{T}'}(v,x) \leq d$ for some $x \in V(S)$ with $(S,\Omega) \in \mathcal{S}_2$. By Theorem 2.4.7 and Lemma 2.4.8, for every $(S,\Omega) \in \mathcal{S}_2$, there exists a $(d+11)$-zone Λ_S with respect to $\mathcal{T}'$ around a vertex in $\bar{\Omega}$ containing every atom y with $m_{\mathcal{T}'}(x,y) \leq d+1$ as an interior point for all such x. Thus every vertex of $(G-Z)\cap G'$ that is d-free with respect to $\mathcal{T}-Z$ but not d-free with respect to $\mathcal{T}'$ is in $\bigcup_{(S,\Omega)\in\mathcal{S}_2} \Lambda_S$.

Let H' be the graph that has a nice embedding mentioned in Lemma 2.5.7, and let $|V(H')| = h'$. Note that $h' \leq h^*$. By Lemma 2.5.3, there do not exist h' d-free vertices such that every pair of them has distance at least ϕ' under the metric $m_{\mathcal{T}'}$, otherwise, G contains an H-subdivision. So by Theorem 2.4.7 and Lemma 2.4.8, there exist integers $0 \leq k \leq h^*$, d-free vertices $v_1, v_2, ..., v_k$ of G' with respect to $\mathcal{T}'$, and $(\phi'+10)$-zones $\Lambda_1, \Lambda_2, ..., \Lambda_k$ around $v_1, v_2, ..., v_k$, respectively, such that every d-free vertex of G' with respect to $\mathcal{T}'$ is in the interior of $\bigcup_{i=1}^k \Lambda_i$. Then every d-free vertex in $G-Z$ with respect to $\mathcal{T}-Z$ is a vertex of G', and it is in the interior of $\bigcup_{i=1}^k \Lambda_i \cup \bigcup_{(S,\Omega)\in\mathcal{S}_2} \Lambda_S$.

Then let $\mathcal{S}^* = \mathcal{S}_1^* \cup \mathcal{S}_2^*$, $\mathcal{T}^*$ and G^* be the $\mathcal{S}^*$, $\mathcal{T}^*$ and G^*, respectively, mentioned in the conclusion of Lemma 2.5.5 by taking ϕ to be the function such that $\phi(x) = 2f_{2.5.2}(d, h^*, x)$, $\rho = \rho_{2.5.6}$, $\lambda = d + \phi' + 11$, $\kappa = \kappa_{2.5.6}$, $k = h^* + \kappa_{2.5.6}$, $\theta^* = \theta_{2.5.2}(d, h^*, \rho_{2.5.6}, \kappa_{2.5.6}, (dh^* + h^* + 1)(\theta_{2.4.9} + 1))$, $d = d$, $s = 1$ and $\mathcal{F}$ be the family of graphs that contains exactly one vertex with no edges, and further taking $G = G - Z$, $\mathcal{T} = \mathcal{T} - Z$, $\mathcal{S} = \mathcal{S}$, $\tau = \tau$, $\Sigma = \Sigma$, and G' to be the skeleton of $\mathcal{S}$.

Let κ' be the number of members of $\mathcal{S}_2^*$ containing d-free vertices with respect to $\mathcal{T} - Z$. Let $Z_1, Z_2, ..., Z_{\kappa'}, U_1, U_2, ..., U_{\kappa'}$ be the sets obtained by applying Lemma 2.5.2 by taking $h_i = h^*$ for every i, $G = G - Z$, $G' = G^*$ and $(S_1, \Omega_1), (S_2, \Omega_2), ...$ as the vortices in $\mathcal{S}_2^*$ containing d-free vertices with respect to $\mathcal{T} - Z$. Define $\mathcal{S}_2^{*\prime} \subseteq \mathcal{S}_2^*$ consisting of the members in which $U_i \neq \emptyset$. We replace Z by $Z \cup \bigcup_{1 \leq i \leq \kappa'} Z_i$. Note that $|Z| \leq \xi$. If $|\mathcal{S}_2^{*\prime}| = 0$, then there do not exist d-free vertices of $G - Z$ with respect to $\mathcal{T} - Z$, so Statement 1 holds. If $\mathrm{mf}(H, \Sigma) \geq 2$ and $|\mathcal{S}_2^{*\prime}| \leq \mathrm{mf}(H, \Sigma) - 1$, then Statement 2(b) holds. So we may assume that $|\mathcal{S}_2^{*\prime}| \geq \mathrm{mf}(H, \Sigma)$.

Let G'' be the graph and $\mathcal{T}''$ the tangle in G'' of order at least $2f_{2.5.2}(d, h^*, \rho_{2.5.5}) + (dh^* + h^* + 1)(\theta_{2.4.9} + 1)$ conformal with $\mathcal{T}^*$ mentioned in the conclusion of Lemma 2.5.2. For $1 \leq i \leq |\mathcal{S}_2^{*\prime}|$ and $1 \leq j \leq h^*$, let Y_i and $A_{i,j}$ be the sets mentioned in conclusion 2(b) of Lemma 2.5.2. By Lemma 2.4.5, there exist edges $e_1, e_2, ..., e_{dh^*+h^*}$ of G'' and a vertex $x \in A_{1,1}$ such that $m_{\mathcal{T}''}(x, e_1) = m_{\mathcal{T}''}(e_{i-1}, e_i) = \theta_{2.4.9} + 1 < m_{\mathcal{T}''}(x, e_i)$ for every $2 \leq i \leq dh^* + h^*$, and $m_{\mathcal{T}''}(e_i, e_j) \geq \theta_{2.4.9} + 1$, and the set of the ends of each e_i is free for $1 \leq i \leq dh^* + h^*$. Observe that $m_{\mathcal{T}''}(y, e_\ell) \geq \theta_{2.4.9} + 1$ for every $y \in \bigcup_{i=1}^{|\mathcal{S}_2^{*\prime}|} \bigcup_{j=1}^{h^*} A_{i,j}$ and $1 \leq \ell \leq dh^* + h^*$. For $1 \leq i \leq |\mathcal{S}_2^{*\prime}|$, define Δ_i to be a disk in Σ contained in the disk bounded by Y_i such that $\Delta_i \cap G'' = \bigcup_{j=1}^{h^*} A_{i,j}$. For $1 \leq i \leq dh^* + h^*$, define $\Delta_{|\mathcal{S}_2^{*\prime}|+i}$ to be a disk in Σ such that $\Delta_i \cap G''$ is the set of the ends of e_i. Since H' has a nice embedding in Σ, we can embed H' into Σ such that the vertices of degree at least 4 of H' is incident with $\mathrm{mf}(H', \Sigma)$ faces. Consequently, G'' admits an H'-subdivision by concatenating pairwise disjoint d-spiders from some vertices in some members of $\mathcal{S}_2^{*\prime}$ to $\bigcup_{i=1}^{\mathrm{mf}(H',\Sigma)} \bigcup_{j=1}^{h^*} A_{i,j}$ and a disjoint union of 3-spiders and a linear forest obtained by applying Theorem 2.4.9 by appropriately partitioning $\bigcup_{i=1}^{\mathrm{mf}(H',\Sigma)} \bigcup_{j=1}^{h^*} A_{i,j} \cup \bigcup_{i=1}^{|E(H')|+|V(H')|} \{a_i, b_i\}$, where a_i, b_i are the ends of e_i. This implies that G admits an H-subdivision, a contradiction. ∎

CHAPTER III

STRONGLY LEAN TREE-DECOMPOSITIONS

3.1 Strongly lean tree decompositions

We say that $(T, \mathcal{X})$ is a *tree decomposition* of a graph G if the following hold.

- T is a tree, and $\mathcal{X} = \{X_t : t \in V(T)\}$, where X_t is called a *bag* of t and it is a subset of $V(G)$ for every $t \in V(T)$.
- $\bigcup_{t \in V(T)} X_t = V(G)$.
- For every edge of G, some bag contains all the ends of this edge.
- For every vertex $u \in V(G)$, the nodes whose bags contain u induce a subtree of T.

For every edge $e = uv$ of T, we define C_e to be $X_u \cap X_v$ and define the *index* of e to be $|C_e|$. The *adhesion* of $(T, \mathcal{X})$ is the largest index of an edge of T. The *width* of $(T, \mathcal{X})$ is $\max\{|X_t| : t \in V(T)\} - 1$. The *tree-width*, denoted by $\mathrm{tw}(G)$, of a graph G is the minimum width of a tree decomposition of G. For every subtree L of T, we denote $\bigcup_{t \in V(L)} X_t$ by X_L.

Let $(T, \mathcal{X})$ be a tree decomposition of G. For every nonnegative integer k, let $V_k = \{t \in V(T) : |X_t| \leq k\}$. A *$k$-cell* of $(T, \mathcal{X})$ is a component of $T - V_k$, and the *volume* of a k-cell C is the size of X_C. The *k-signature* of $(T, \mathcal{X})$ is the sequence $(a_n, a_{n-1}, ..., a_1)$, denoted by s_k, where n is the number of vertices of G, and a_i is the number of k-cells of volume i, for every $1 \leq i \leq n$. Given an integer w, we define the *w-masked signature* of $(T, \mathcal{X})$ to be the sequence $(s_w, s_{w-1}, ..., s_0)$. And we call the n-masked signature as the *signature*. In this thesis, k-signatures and w-masked signatures are compared by the lexicographical order. Observe that for every

$k \geq w+1$ and every tree decomposition of G of width w, its k-signature is the sequence of n zeros. Therefore, the width of a tree decomposition with the lexicographically minimum signature equals the tree-width of G. Thus we have

Theorem 3.1.1 *For every $k \geq \mathrm{tw}(G) + 1$, every tree decomposition of a graph G with minimum k-masked signature has width* $\mathrm{tw}(G)$.

A tree decomposition $(T, \mathcal{X})$ of a graph G is *lean* if for every nodes t_1, t_2 of T, and for every $Z_1 \subseteq X_{t_1}$, $Z_2 \subseteq X_{t_2}$ with $|Z_1| = |Z_2|$, either there exist $|Z_1|$ disjoint paths in G from Z_1 to Z_2, or there exists an edge e on the t_1-t_2 path in T such that $|C_e| < |Z_1|$. We remark that t_1, t_2 may not be distinct, and Z_1, Z_2 may not be disjoint.

Theorem 3.1.2 ([53]) *Every graph of tree-width w has a lean tree decomposition of width w.*

A *separation* (A, B) of a graph G is a pair of edge-disjoint subgraphs of G such that $G = A \cup B$. The *order* of a separation (A, B) is $|V(A) \cap V(B)|$.

Let w be an integer. We say that a tree decomposition $(T, \mathcal{X})$ of a graph G is *w-strongly lean* if the following hold.

(SL1) For every nodes t_1, t_2 of T, and for every $Z_1 \subseteq X_{t_1}$, $Z_2 \subseteq X_{t_2}$ with $|Z_1| = |Z_2| \leq w$, if there exists a separation (A, B) of order at most $|Z_1|$ such that $Z_1 \subseteq V(A)$, $Z_2 \subseteq V(B)$, and $Z_1 \neq V(A) \cap V(B) \neq Z_2$, then there exists an internal node t of the t_1-t_2 path in T such that $|X_t| \leq |V(A) \cap V(B)|$, $Z_1 \neq X_t \neq Z_2$, and no component of $G - X_t$ includes vertices of both Z_1 and Z_2.

(SL2) For every subsets Z_1, Z_2 of $V(G)$ with $|Z_1| = |Z_2| \leq w$ such that there exists a $|Z_1|$-cell C with $Z_1 \cup Z_2 \subseteq X_C$, there does not exist a separation (A, B) of G of order at most $|Z_1|$ such that $Z_1 \subseteq V(A)$, $Z_2 \subseteq V(B)$, and $Z_1 \neq V(A) \cap V(B) \neq Z_2$.

In fact, (SL2) implies (SL1) except the case that $|Z_1| = |X_{t_1}| = w$ or $|Z_2| = |X_{t_2}| = w$. Observe that every w-strongly lean tree decomposition is w'-strongly lean if $w \geq w'$. We say that a tree decomposition of G is *strongly lean* if it is $|V(G)|$-strongly lean.

Proposition 3.1.3 *Every strongly lean tree decomposition of a graph is lean.*

Proof. Let $(T, \mathcal{X})$ be a strongly lean tree decomposition of a graph G, and let Z_1, Z_2 be two distinct subsets of $V(G)$ of the same size such that there exist two nodes t_1, t_2 of T with $Z_1 \subseteq X_{t_1}$ and $Z_2 \subseteq X_{t_2}$. Assume that there exist no $|Z_1|$ disjoint Z_1-Z_2 paths. By Menger's Theorem, there exists a separation (A, B) of order less than $|Z_1|$ separating Z_1 and Z_2. Since $(T, \mathcal{X})$ is strongly lean, there exists an internal node t of the t_1-t_2 path such that $|X_t| \leq |V(A) \cap V(B)| < |Z_1|$, and $Z_1 - X_t$ and $Z_2 - X_t$ are in different components of $G - X_t$. This implies that there exists an edge e on the t_1-t_2 path incident with t such that $|C_e| \leq |X_t| < |Z_1|$. So $(T, \mathcal{X})$ is lean. ∎

Lemma 3.1.4 *Let w be a nonnegative integer. Let $(T, \mathcal{X})$ be a tree decomposition of G with minimum w-masked signature. If $x, y \in V(T)$ with $|X_x \cap X_y| \leq w$ are adjacent, then $X_x \subseteq X_y$ or $X_y \subseteq X_x$.*

Proof. Suppose that $X_x - X_y \neq \emptyset \neq X_y - X_x$. Then adding a new node t between x, y and defining X_t to be $X_x \cap X_y$ will decrease the w-masked signature, a contradiction. So either $X_x \subseteq X_y$ or $X_y \subseteq X_x$. ∎

Theorem 3.1.5 *For every integer w, every tree decomposition of a graph G with minimum w-masked signature is w-strongly lean.*

Proof. Fix w as an integer. Let $(T, \mathcal{X})$ be a tree decomposition of G with minimum w-masked signature. Suppose that $(T, \mathcal{X})$ is not w-strongly lean. If (SL1) does not hold, then let Z_1,Z_2 be distinct subsets of $V(G)$ and t_1, t_2 be two nodes of T such that $|Z_1| = |Z_2| \leq w$ and $Z_1 \subseteq X_{t_1}, Z_2 \subseteq X_{t_2}$, and there exists a separation (A, B) of

G of order at most $|Z_1|$ such that $Z_1 \subseteq V(A), Z_2 \subseteq V(B)$ with $Z_1 \neq V(A) \cap V(B) \neq Z_2$, but there is no internal node t of the t_1-t_2 path such that $Z_1 \neq X_t \neq Z_2$ and $|X_t| \leq |V(A) \cap V(B)|$. Since we may assume that the vertices t_1, t_2 are as close as possible, no internal node t of the t_1-t_2 path satisfies that $X_t = Z_1$ or Z_2. Therefore, the size of the bag of every internal node on the t_1-t_2 path in T is greater than $|V(A) \cap V(B)|$. If (SL2) does not hold, then let Z_1, Z_2 be subsets of $V(G)$ with $|Z_1| = |Z_2| \leq w$ and let C be a $|Z_1|$-cell such that X_C contains $Z_1 \cup Z_2$, but there exists a separation (A, B) of G of order at most $|Z_1|$ such that $Z_1 \subseteq V(A), Z_2 \subseteq V(B)$, and $Z_1 \neq V(A) \cap V(B) \neq Z_2$; let $t_1 = t_2$ be a node in C. In both cases, we may assume that (A, B) is such a separation of the minimum order, so there exist $|V(A) \cap V(B)|$ disjoint paths $P_1, P_2, ..., P_{|V(A)\cap V(B)|}$ from Z_1 to Z_2 in G.

We construct a new tree decomposition $(T^*, \mathcal{X}^*)$ as follows. Let T' and T'' be two copies of T, and for every node t in T, let t' and t'' be the copy of t in T' and T'', respectively. Let T^* be the tree obtained from $T' \cup T''$ by adding a new node t^* and two edges $t'_2 t^*$ and $t^* t''_1$. For every $1 \leq i \leq |V(A) \cap V(B)|$, let v_i be the vertex in $V(P_i) \cap V(A) \cap V(B)$. Define $X^*_{t^*} = V(A) \cap V(B)$, and for every $t \in V(T)$, define $X^*_{t'} = (X_t \cap V(A)) \cup \{v_i : X_t \cap V(P_i) \cap V(B) \neq \emptyset\}$, and $X^*_{t''} = (X_t \cap V(B)) \cup \{v_i : X_t \cap V(P_i) \cap V(A) \neq \emptyset\}$. It is easy to check that $(T^*, \mathcal{X}^*)$ is a tree decomposition of G. (In fact, it is the same construction in [53].) We shall prove that the w-masked signature of $(T^*, \mathcal{X}^*)$ is strictly smaller than the w-masked signature of $(T, \mathcal{X})$.

Claim 1: $|X^*_{t'}| \leq |X_t|$ and $|X^*_{t''}| \leq |X_t|$ for every $t \in V(T)$. If $|X^*_{t'}| = |X_t|$, then $|X^*_{t''}| \leq |V(A) \cap V(B)|$; if $|X^*_{t''}| = |X_t|$, then $|X^*_{t'}| \leq |V(A) \cap V(B)|$.

Proof of Claim 1: If $u \in X^*_{t'} - X_t$, then $u \in V(A) \cap V(B)$, and P_u contains a vertex in X_t but not in $X^*_{t'}$, where P_u is the member in $\{P_1, P_2, ..., P_{|V(A)\cap V(B)|}\}$ containing u. As $P_1, P_2, ..., P_{|V(A)\cap V(B)|}$ are pairwise disjoint, $|X^*_{t'}| \leq |X_t|$ for every $t \in V(T)$. Similarly, $|X^*_{t''}| \leq |X_t|$ for every $t \in V(T)$. In the same way, if $|X^*_{t'}| = |X_t|$, then $X^*_{t''} \subseteq \bigcup_{j=1}^{|V(A)\cap V(B)|} V(P_j)$ and $|X^*_{t''} \cap V(P_j)| \leq 1$ for every $1 \leq j \leq |V(A) \cap V(B)|$.

Hence, if $|X^*_{t'}| = |X_t|$, then $|X^*_{t''}| \leq |V(A) \cap V(B)|$. Similarly, if $|X^*_{t''}| = |X_t|$, then $|X^*_{t'}| \leq |V(A) \cap V(B)|$. $\square$

Claim 2: For every integer k such that $w \geq k \geq |V(A) \cap V(B)|$, the k-signature of $(T^*, \mathcal{X}^*)$ is not larger than the k-signature of $(T, \mathcal{X})$.

Proof of Claim 2: Let k be an integer such that $w \geq k \geq |V(A) \cap V(B)|$, and let K^* be a k-cell of $(T^*, \mathcal{X}^*)$. Since $|X^*_{t^*}| = |V(A) \cap V(B)| \leq k$, K^* is contained in T' or T''. Let K be the subtree of T consisting of the originals of the nodes of K^*. Claim 1 implies that the size of the bag of each node in K is strictly greater than k, so K is contained in a k-cell of $(T, \mathcal{X})$, denoted by $\bar{K}$. To prove this claim, it is sufficient to prove that either the volume of K^* equals the volume of $\bar{K}$ and no other k-cell L^* of $(T^*, \mathcal{X}^*)$ satisfies that $\bar{L} = \bar{K}$, or for every k-cell L^* of $(T^*, \mathcal{X}^*)$ with $\bar{L} = \bar{K}$, the volume of L^* is smaller than the volume of $\bar{K}$.

Without loss of generality, we may assume that K^* is contained T'. If $K \neq \bar{K}$, then there exists $t \in \bar{K}$ such that $|X^*_{t'}| \leq k < |X_t|$. In this case, either X_t contains a vertex not in $V(B) \cap (\bigcup_{i=1}^{|V(A)\cap V(B)|} V(P_i))$, or X_t contains two vertices in $V(B) \cap V(P_i)$ for some i. So the volume of K^* is smaller than the volume of $\bar{K}$. If $K = \bar{K}$, then by Claim 1, there does not exist another k-cell L^* of $(T^*, \mathcal{X}^*)$ such that $\bar{L} = \bar{K}$ since the bag of every node in L^* has volume at most $|V(A) \cap V(B)| \leq k$. This proves Claim 2. $\square$

Claim 3: The $|V(A) \cap V(B)|$-signature of $(T^*, \mathcal{X}^*)$ is smaller than the $|V(A) \cap V(B)|$-signature of $(T, \mathcal{X})$.

Proof of Claim 3: Suppose that there exists a $|V(A) \cap V(B)|$-cell C' of $(T, \mathcal{X})$ containing t_1, t_2 or an internal node of the t_1-t_2 path in T. Note that C' contains every internal node of the t_1-t_2 path in T, since the bag of each internal node has size greater than $|V(A) \cap V(B)|$. Similarly, for each $i = 1, 2$, if $|X_{t_i}| > |V(A) \cap V(B)|$, then t_i is in C'. Clearly, $V(\bigcup_{i=1}^{|Z_1|} P_i) - (Z_1 \cup Z_2) \subseteq X_{C'}$. We claim that $(Z_1 \cup Z_2) \subseteq X_{C'}$. For every $z \in Z_1 \cup Z_2$, let e_z be the edge incident with z in $\bigcup_{i=1}^{|Z_1|} P_i$. Observe that there exists an internal node of the t_1-t_2 path in T whose bag contains the both ends of e_z, so $z \in X_{C'}$. This proves that $V(\bigcup_{i=1}^{|Z_1|} P_i) \subseteq X_{C'}$. Therefore, $V(A)$ contains at least two vertices in P_i and $V(B)$ contains two vertices in P_j for some $i, j \in \{1, 2, ..., |V(A) \cap V(B)|\}$, so no $|V(A) \cap V(B)|$-cell K^* of $(T^*, \mathcal{X}^*)$ has $\bar{K} = C'$ such that the volume of K^* equals the volume of C'. Hence we prove this claim if such C' exists. So (SL2) holds.

Therefore, we may assume that such C' does not exist and (SL1) does not hold. That is, there exists no internal node of the t_1-t_2 path and $|X_{t_1}| = |X_{t_2}| = |V(A) \cap V(B)|$. The former implies that either $t_1 = t_2$, or t_1 is adjacent to t_2 in T; the latter implies that $X_{t_1} = Z_1$ and $X_{t_2} = Z_2$. Since $Z_1 \neq Z_2$, $t_1 \neq t_2$, so t_1 is adjacent to t_2. Note that $|X_{t_1} \cap X_{t_2}| \leq |X_{t_1}| = |Z_1| \leq w$. By Lemma 3.1.4, either $Z_1 \subseteq Z_2$, or $Z_2 \subseteq Z_1$, a contradiction. This proves the claim. □

By Claims 2 and 3, the w-masked signature of $(T^*, \mathcal{X}^*)$ is smaller than the w-masked signature of $(T, \mathcal{X})$, a contradiction. Consequently, $(T, \mathcal{X})$ is w-strongly lean. ■

A *rooted tree* is a directed graph whose underlying graph is a tree such that all but one node has in-degree one. The vertex in a rooted tree with in-degree not one is called the *root*. It is easy to see that the root has in-degree zero. For every non-root node v, the tail u of the edge with head v is the *parent* of v, and we say that v is a *child* of u in this case. If there exists a directed path from a node x to a node y, then x is an *ancestor* of y, and y is a *descendant* of x. A *rooted tree decomposition* (T, X) is a tree decomposition such that T is a rooted tree.

Let $(T, \mathcal{X})$ be a rooted tree-decomposition of a graph G. We say that a node t and a component C of $T - t$ not containing the root of T are a *bad pair* if $G[X_C - X_t]$ is not connected.

Theorem 3.1.6 *Let G be a graph, and let w be a nonnegative integer. Assume that*

$(T, \mathcal{X})$ is the rooted tree-decomposition of G with the following properties.

1. *It has minimum w-masked signature.*

2. *Subject to 1, the signature is as small as possible.*

3. *Subject to 1 and 2, the number of bad pairs of $(T, \mathcal{X})$ is as small as possible.*

4. *Subject to 1, 2 and 3, $|V(T)|$ is as small as possible.*

Then the following hold.

1. *$G[X_C - X_t]$ is connected, for every node t of T and for every component C of $T - t$ not containing the root of T.*

2. *If t' is the parent of t in T such that $X_t \subseteq X_{t'}$, then t has degree at most two in T.*

3. *For every edge $e = tt'$ of T and for every component C of $T - e$, every vertex in $X_t \cap X_{t'}$ is adjacent to a vertex in $X_C - (X_t \cap X_{t'})$ in G.*

Proof. Note that for every adjacent $x, y \in V(T)$, $X_x \neq X_y$ by Property 4 since otherwise we can contract the edge xy of T without increasing the w-masked signature, signature and the number of bad pairs. On the other hand, if x, y are adjacent nodes in T, then either $X_x \subseteq X_y$ or $X_y \subseteq X_x$, otherwise, subdividing the edge xy and assign the bag $X_x \cap X_y$ to the new vertex either decreases the w-masked signature or decreases the signature without increasing the w-masked signature, contradicting Property 2.

Suppose that the first statement does not hold. So there exists a bad pair (t, C) of $(T, \mathcal{X})$ such that $G[X_C - X_t]$ is disconnected. We assume that $|X_t|$ is as small as possible. Let $Y_1, Y_2, ..., Y_k$ be the components of $G[X_C - X_t]$. Let t' be the node in C adjacent to t in T. Recall that either $X_{t'} \subset X_t$ or $X_t \subset X_{t'}$.

First, we assume that $X_{t'} \subset X_t$. Observe that $X_C - X_{t'}$ does not contain a vertex in $X_t - X_{t'}$. So Y_i and Y_j are different components of $G[X_C - X_{t'}]$ for $i \neq j$. By the minimality of $|X_t|$, there exist components $C_1, C_2, ..., C_k$ of $T - \{t'\}$ such that $Y_i \subseteq X_{C_i}$ for $1 \leq i \leq k$. Let $t_1, t_2, ...$ be the children of t'. Define T' to be the rooted tree obtained from $T - \{t'\}$ by adding nodes $t'_1, t'_2, ..., t'_k$ and directed edges tt'_i and $t'_i t_i$ for $1 \leq i \leq k$, and define $X'_{t'_i} = X_{t'}$ for $1 \leq i \leq k$, and define $X'_s = X_s$ for other nodes s of T'. Then the signature of $(T', \mathcal{X}')$ is the same as $(T, \mathcal{X})$, but the number of bad pairs is smaller, a contradiction.

Now, we assume that $X_t \subset X_{t'}$. Define T'' to be the tree obtained from $T - C$ by adding k copies $T_1, T_2, ..., T_k$ of C and directed edges from t to the root of each copy of t'. Define $X''_s = X_s \cap (Y_i \cup X_t)$ if $s \in V(T_i)$ for some $1 \leq i \leq k$, and define $X''_s = X_s$ for other nodes s of T''. Then $(T'', \mathcal{X}'')$ is a rooted tree decomposition of G contradicting the minimality of $(T, \mathcal{X})$. Therefore, $(T, \mathcal{X})$ has no bad pair, and the first statement holds.

Now, we prove the second statement. Suppose that tt' is an edge such that $X_t \subseteq X_{t'}$, but t has degree at least three in T. Let C be the component of $T - t'$ containing t. By the first statement, $G[X_C - X_{t'}]$ is connected. But $G[X_C - X_{t'}]$ is a subgraph of $G - X_t$. As the degree of t is at least three in T, some component C' of $T - t$ not containing t' satisfies that $X_{C'} \subseteq X_{t'}$. But it implies that $X_{C'} \subseteq X_t$. Then deleting C' from T will not increase the w-masked signature and the signature and will not create a bad pair, but will reduce the number of nodes of T, a contradiction. So the second statement holds.

Finally, we prove the third statement. Let $e = tt'$ be an edge of T. Let C_t and $C_{t'}$ be the components of $T - e$ containing t and t', respectively. Let $v \in X_t \cap X_{t'}$. If v is not adjacent to any vertex in $X_{C_{t'}} - X_t$, then we remove v from $X_{t''}$ for every node t'' in $C_{t'}$ such that $v \in X_{t''}$. Note that $v \in X_{t'}$ and $t' \in C_{t'}$, so either this process reduces the w-masked signature, or it keeps the w-masked signature but reduces the signature. So v is adjacent to some vertex in $X_{C_{t'}} - X_t$. Similarly, every vertex in $X_t \cap X_{t'}$ is adjacent to a vertex in $X_{C_t} - (X_t \cap X_{t'})$. This proves the theorem. ■

We say that a tree decomposition is *branching* if it is rooted and satisfies the conclusions of Theorem 3.1.6.

3.2 Tangles and strongly lean tree-decompositions

Lemma 3.2.1 *Let G be a graph and let h be a nonnegative integer. Let $(T, \mathcal{X})$ be an h-strongly lean tree-decomposition of G. Let $\theta \leq h$ be a nonnegative integer, and let L be a node or a k-cell of T, for some $k \geq \theta$. Let $\mathcal{T}$ be the collection of separations (A, B) of G of order less than $\theta+1$ such that $|V(B) \cap X_L| \geq \max\{|V(A) \cap X_L|, |V(A) \cap V(B)|\}$. If $|X_L| \geq 3\theta + 1$, then $\mathcal{T}$ is a tangle in G of order $\theta + 1$.*

Proof. We shall show that $\mathcal{T}$ satisfies the three tangle axioms. Suppose that (T1) does not hold, so there exists a separation (A, B) of order less than $\theta + 1$ such that $|V(A) \cap X_L| \leq |V(B) \cap X_L| \leq |V(A) \cap V(B)| - 1$. But it implies that $|X_L| \leq 2|V(A) \cap V(B)| - 2 \leq 2\theta - 2$, a contradiction. So $\mathcal{T}$ satisfies (T1).

We claim that $|V(A) \cap X_L| < |V(A) \cap V(B)|$ for every $(A, B) \in \mathcal{T}$, unless $V(A) \cap V(B) = V(A) \cap X_L$. Suppose that $(A, B) \in \mathcal{T}$ such that $|V(A) \cap X_L| \geq |V(A) \cap V(B)|$ and $V(A) \cap V(B) \neq V(A) \cap X_L$. Note that $|X_L| \geq 3\theta + 1$, so $|V(B) \cap X_L| \geq |V(A) \cap V(B)| + 1$. Let Z_1 be a subset of $X_L \cap V(A)$ of size $|V(A) \cap V(B)|$ and Z_2 a subset of $X_L \cap V(B)$ of size $|V(A) \cap V(B)|$ such that $Z_1 \neq V(A) \cap V(B) \neq Z_2$. Since $|Z_1| \leq \theta$, L is either a node or contained in a $|Z_1|$-cell of $(T, \mathcal{X})$. But $\theta \leq h$ and $Z_1 \neq V(A) \cap V(B) \neq Z_2$, it is impossible as $(T, \mathcal{X})$ is h-strongly lean. So $|V(A) \cap X_L| < |V(A) \cap V(B)|$, unless $V(A) \cap V(B) = V(A) \cap X_L$. Therefore, $|V(A) \cap X_L| \leq |V(A) \cap V(B)|$ for every $(A, B) \in \mathcal{T}$.

If $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ are in $\mathcal{T}$, then $|V(A_1 \cup A_2 \cup A_3) \cap X_L| \leq \sum_{i=1}^{3} |V(A_i) \cap V(B_i)| \leq 3\theta < |X_L| \leq |V(G)|$. Therefore, $\mathcal{T}$ satisfies (T2). Similarly, if $(A, B) \in \mathcal{T}$, then $|V(A) \cap X_L| < |X_L|$, so $V(A) \neq V(G)$. In other words, $\mathcal{T}$ satisfies (T3). ■

We call the tangle defined in Lemma 3.2.1 the *tangle induced by* X_L.

A *tie-breaker* in a graph G is a function λ which maps each separation of G to some member of a linearly ordered set $(\Lambda, \leq)$ in such a way that for all separations $(A, B), (C, D)$ of G,

- $\lambda(A, B) = \lambda(C, D)$ if and only if $(A, B) = (C, D)$ or $(A, B) = (D, C)$,
- either $\lambda(A \cup C, B \cap D) \leq \lambda(A, B)$ or $\lambda(A \cap C, B \cup D) < \lambda(C, D)$,
- if $|V(A \cap B)| < |V(C \cap D)|$, then $\lambda(A, B) < \lambda(C, D)$.

Let G be a graph and $e \in E(G)$. For each $x \in V(G) \cup E(G)$, let $\nu(x) > 0$ be a real number, such that the numbers $\nu(x)$ for $x \in V(G) \cup E(G)$ are rationally independent. For each separation (A, B) of G, define

$$\lambda'(A, B) = (|V(A) \cap V(B)|, \sum_{x \in (V(G) \cup E(G)) - (V(A) \cup E(A))} \nu(x), \sum_{x \in V(A) \cap V(B)} \nu(x)).$$

And for each separation (A, B) of G with $e \in E(A)$, we define $\lambda(A, B) = \lambda'(A, B)$, and if (A, B) is a separation with $e \in E(B)$, then we define $\lambda(A, B) = \lambda'(B, A)$. By Theorem 6.1 of [47], λ is a tie-breaker, and we call it the *tie-breaker defined by* e, ν. In the rest of this section, λ always denotes this tie-breaker. We call $\lambda(A, B)$ the λ-*order* of a separation (A, B). Note that λ' may not be a tie-breaker.

Let $\mathcal{T}, \mathcal{T}'$ be two tangles in a graph G. We say that a separation (A, B) of G *distinguishes* $\mathcal{T}$ from $\mathcal{T}'$ if $(A, B) \in \mathcal{T} - \mathcal{T}'$. A separation (A, B) of G is the $(\mathcal{T}, \mathcal{T}')$-*distinction* if (A, B) is the separation of minimum λ-order that distinguishes $\mathcal{T}$ from $\mathcal{T}'$. Note that if (A, B) is the $(\mathcal{T}, \mathcal{T}')$-distinction and the order of (A, B) is less than the order of $\mathcal{T}'$, then (B, A) is the $(\mathcal{T}', \mathcal{T})$-distinction. A separation (A, B) of G is a $(\mathcal{T}, \mathcal{T}')$-*moat* if (A, B) is a separation of the minimum λ'-order that distinguishes $\mathcal{T}$ from $\mathcal{T}'$. Here are some properties of $(\mathcal{T}, \mathcal{T}')$-moats.

Lemma 3.2.2 *Let* $\mathcal{T}, \mathcal{T}'$ *be different tangles in* G*.*

1. *If $\mathcal{T} \not\subseteq \mathcal{T}'$, then the $(\mathcal{T}, \mathcal{T}')$-moat is unique.*

2. *If (A, B) is the $(\mathcal{T}, \mathcal{T}')$-distinction and the order of (A, B) is less than the order of $\mathcal{T}'$, then either (A, B) is the $(\mathcal{T}, \mathcal{T}')$-moat, or (B, A) is the $(\mathcal{T}', \mathcal{T})$-moat.*

Proof. We first prove the first statement. Let (A, B) and (C, D) be $(\mathcal{T}, \mathcal{T}')$-moats. So the λ'-orders of (A, B) and (C, D) are the same. If the order of $(A \cup C, B \cap D)$ is no more than the order of (A, B), then $A \cup C = A = C$ and $V(A) \cap V(B) = V(C) \cap V(D)$, so $(A, B) = (C, D)$. Hence, we may assume that the order of $(A \cup C, B \cap D)$ is strictly greater than the order of (A, B). Then the order of $(A \cap C, B \cup D)$ is less than the order of (C, D). But $(A \cap C, B \cup D) \in \mathcal{T} - \mathcal{T}'$, a contradiction. This proves the uniqueness of the $(\mathcal{T}, \mathcal{T}')$-moat.

Now we prove the second statement. Let (A, B) be the $(\mathcal{T}, \mathcal{T}')$-distinction. We may assume that (A, B) is not the $(\mathcal{T}, \mathcal{T}')$-moat, for otherwise we are done. So $\lambda'(A, B) \neq \lambda(A, B)$ and hence the edge e in the definition of λ is in B. Observe that $(B, A) \in \mathcal{T}' - \mathcal{T}$, since the order of (A, B) is less than the order of $\mathcal{T}'$ and $(A, B) \notin \mathcal{T}'$. Let (C, D) be the $(\mathcal{T}', \mathcal{T})$-moat. So $\lambda'(C, D) \leq \lambda'(B, A) = \lambda(A, B)$. This implies that the order of (C, D) is less than the order of $\mathcal{T}$, so $(D, C) \in \mathcal{T} - \mathcal{T}'$. Since (A, B) is the $(\mathcal{T}, \mathcal{T}')$-distinction, $\lambda(A, B) \leq \lambda(D, C) = \lambda(C, D)$. If $\lambda'(C, D) = \lambda(C, D)$, then $\lambda(A, B) = \lambda(C, D)$, so $(B, A) = (C, D)$ as λ is a tie-breaker. So we may assume that $\lambda'(C, D) \neq \lambda(C, D)$. Then $\lambda'(D, C) = \lambda(C, D)$ and $e \in E(D)$. The order of $(A \cup D, B \cap C)$ is not less than the order of (C, D), so the order of $(A \cap D, B \cup C)$ is no more than the order of (A, B). As $\lambda'(B, A) = \lambda(A, B) \leq \lambda(A \cap D, B \cup C) = \lambda'(B \cup C, A \cap D)$, we know that $B \cup C \subseteq B$. That is, $C \subseteq B$. Furthermore, $A \cap B - D = \emptyset$. Since (C, D) is the $(\mathcal{T}', \mathcal{T})$-moat, $\lambda'(C, D) \leq \lambda'(B, A)$, so $B \subseteq C$ and $C \cap D - A = \emptyset$. Consequently, $(B, A) = (C, D)$. This proves the second statement. ∎

It is easy to show that for every tangles $\mathcal{T}, \mathcal{T}', \mathcal{T}''$, the $(\mathcal{T}, \mathcal{T}')$-distinction does not cross the $(\mathcal{T}, \mathcal{T}'')$-distinction. However, the $(\mathcal{T}, \mathcal{T}')$-moat might cross the $(\mathcal{T}, \mathcal{T}'')$-moat. The following lemma shows a relation between a strongly lean tree decomposition, tangles, and moats. In particular, the moats between two tangles induced by some cells of a strongly lean tree decomposition do not cross.

Lemma 3.2.3 *Let G be a graph and let h be a nonnegative integer. Let $(T, \mathcal{X})$ be an h-strongly lean tree-decomposition of G. Let $\theta \leq h$ be a nonnegative integer, and let each L_1, L_2 be a node or a k-cell of T, for some $k \geq \theta$, with $|X_{L_1}| \geq 3\theta + 1$ and $|X_{L_2}| \geq 3\theta + 1$. Denote the tangle induced by L_1 and L_2 by $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. If L_1 and L_2 are contained in different θ-cells, then $\mathcal{T}_1 \neq \mathcal{T}_2$, and there exists a node t^* of T such that $X_{t^*} = V(A) \cap V(B)$, where (A, B) is the $(\mathcal{T}_1, \mathcal{T}_2)$-moat. In particular, there exists a node t' of T such that $X_{t'} = V(A') \cap V(B')$, where (A', B') is the $(\mathcal{T}_1, \mathcal{T}_2)$-distinction.*

Proof. Since L_1 and L_2 are contained in different θ-cells, there exists a node t with $|X_t| \leq \theta$ such that L_1 and L_2 are contained in different components of $T - \{t\}$.

Let (A, B) be the $(\mathcal{T}_1, \mathcal{T}_2)$-moat. So $|V(B) \cap X_{L_1}| \geq |V(A) \cap V(B)|$ and $|V(A) \cap X_{L_2}| \geq |V(A) \cap V(B)|$. Therefore, there exist Y_1, Y_2 of size $|V(A) \cap V(B)|$ such that $Y_i \subseteq X_{L_i}$, $Y_1 \subseteq V(B)$, $Y_2 \subseteq V(A)$, and $Y_1 \neq V(A) \cap V(B) \neq Y_2$. Note that $|X_t| \geq |V(A) \cap V(B)|$ for every node t on the path in T from L_1 to L_2, otherwise, there exists a separation of order less than $|V(A) \cap V(B)|$ distinguishing $\mathcal{T}_1$ and $\mathcal{T}_2$. So L_1, L_2 are in the same $(|V(A) \cap V(B)| - 1)$-cell. Note that $(T, \mathcal{X})$ is h-strongly lean and $h \geq |V(A) \cap V(B)|$, so there do not exist a separation of order at most $|V(A) \cap V(B)| - 1$ separating Y_1 and Y_2. Hence, there exist $|V(A) \cap V(B)|$ disjoint paths from Y_1 to Y_2. Let u, v be the closest pair of nodes or k-cells for some $k \geq \theta$ of T such that there exist $U \subseteq X_u \cap V(A)$ and $V \subseteq X_v \cap V(B)$ with $|U| = |V| = |V(A) \cap V(B)|$ such that $U \neq V(A) \cap V(B) \neq V$ and there exist $|V(A) \cap V(B)|$ disjoint paths from U to V. Note that u, v exist since L_2, L_1 are candidates.

As (A, B) is a separation separating U and V, u and v cannot be contained in the same $|V(A) \cap V(B)|$-cell. So $u \neq v$ if both u, v are nodes of T or both are some k-cells of $(T, \mathcal{X})$ for some $k \geq \theta$; $u \notin v$ if u is a node and v is a k-cell; $v \notin u$ if v is a node and u is a k-cell. Furthermore, there exists an internal node t^* of the u-v path in T with $|X_t| = |V(A) \cap V(B)|$. Suppose that $X_{t^*} \neq V(A) \cap V(B)$. If either $X_{t^*} \subseteq V(A)$ or $X_{t^*} \subseteq V(B)$, then we obtain a pair closer than u, v, a contradiction. So $X_{t^*} \nsubseteq V(A)$ and $X_{t^*} \nsubseteq V(B)$. Let $(C, D) \in \mathcal{T}_1$ be a separation such that $X_{t^*} = V(C) \cap V(D)$. Note that t^* is an internal node of the L_1-L_2 path in T, so $(C, D) \in \mathcal{T}_1 - \mathcal{T}_2$. Observe that $|(V(A \cap C)) \cap V(B \cup D)| \geq |V(A) \cap V(B)|$, otherwise, $(A \cap C, B \cup D) \in \mathcal{T}_1 - \mathcal{T}_2$ has order less than (A, B). Therefore, $|V(A \cup C) \cap V(B \cap D)| \leq |V(C) \cap V(D)| = |V(A) \cap V(B)|$. But $\lambda'(A \cup C, B \cap D) < \lambda'(A, B)$, a contradiction. This proves that $X_{t^*} = V(A) \cap V(B)$.

Furthermore, as the $(\mathcal{T}_1, \mathcal{T}_2)$-distinction is either the $(\mathcal{T}_1, \mathcal{T}_2)$-moat or the $(\mathcal{T}_2, \mathcal{T}_1)$-moat, there exists $t' \in V(T)$ such that $X_{t'} = V(C') \cap V(D')$, where (C', D') is the $(\mathcal{T}_1, \mathcal{T}_2)$-distinction. ∎

CHAPTER IV

ROBERTSON CHAINS AND EDGE-CUTS

4.1 Looking for Robertson chains

Let w be a nonnegative integer. We say that a tree decomposition $(T, \mathcal{X})$ of a graph G is *w-linked* if for every nodes t_1, t_2 of T with $|X_{t_1}| = |X_{t_2}| \leq w$, either there exist $|X_{t_1}|$ disjoint paths from X_{t_1} to X_{t_2} in G, or there exists an internal node t of the t_1-t_2 path of T with $|X_t| < |X_{t_1}|$. We say that $(T, \mathcal{X})$ is *w-strongly linked* if for every two nodes t_1, t_2 of T with $|X_{t_1}| = |X_{t_2}| \leq w$, if there exists a vertex-cut other than X_{t_1} and X_{t_2} of size at most $|X_{t_1}|$ separating X_{t_1} from X_{t_2}, then there exists one of the form X_t, where t is on the t_1-t_2 path in T. And $(T, \mathcal{X})$ is *linked* (*strongly linked*, respectively) if it is $|V(G)|$-linked ($|V(G)|$-strongly linked, respectively). Clearly, every w-strongly lean tree decomposition is w-strongly linked, and every w-strongly linked tree decomposition is w-linked. Note that if $(T, \mathcal{X})$ has adhesion w and is w-linked (and w-strongly linked, respectively), then $(T', \mathcal{X}')$ is linked (and strongly linked, respectively), where $(T', \mathcal{X}')$ is the tree decomposition obtained from $(T, \mathcal{X})$ by subdividing every edge of T and assigning the intersection of the bags of the ends of the edge to the new node corresponding to the edge.

Let G be a graph and $(T, \mathcal{X})$ a tree decomposition of G. Let k be a positive integer, and let t_1, t_2 be two nodes of T such that $|X_{t_1}| = |X_{t_2}| = k$. Let $P_1, P_2, ..., P_k$ be disjoint paths from X_{t_1} to X_{t_2} in G, and let $v_{i,j}$ be the vertex in $X_{t_i} \cap P_j$, for $i = 1, 2$ and $1 \leq j \leq k$. Denote $\{P_1, P_2, ..., P_k\}$ by $\mathcal{P}$. Let s be an integer such that $1 \leq s \leq k$. Assume that $E(P_s) \neq \emptyset$. Let B_1 and B_2 be the block of $G - \bigcup_{i \neq s} P_i$ containing the edge of P_s incident with $v_{1,s}$ and $v_{2,s}$, respectively. Let $Q_{\mathcal{P},s}$ be the path of blocks of $G - \bigcup_{i \neq s} P_i$ from B_1 to B_2. If none of the blocks in $Q_{\mathcal{P},s}$ is an edge, then we define $L_{\mathcal{P},s} = R_{\mathcal{P},s} = Q_{\mathcal{P},s}$. If some block in $Q_{\mathcal{P},s}$ is an edge, then let Q' be the set of blocks in $Q_{\mathcal{P},s}$ that are single edges, and define $L_{\mathcal{P},s}$ (and $R_{\mathcal{P},s}$, respectively) to be the component of $Q_{\mathcal{P},s} - E(Q')$ containing $v_{1,s}$ (and $v_{2,s}$, respectively). (So $L_{\mathcal{P},s}$ and $R_{\mathcal{P},s}$ is a vertex if B_1 or B_2, respectively, is a single edge.) Let G' be the subgraph of G induced by $X_{t_1} \cup X_{t_2}$ and the bags of nodes in the component of $T - \{t_1, t_2\}$ containing an internal node of the t_1-t_2 path. A *right jump from* $v_{1,s}$ (and *left jump from* $v_{2,s}$, respectively) is a path in G' from $L_{\mathcal{P},s}$ (and $R_{\mathcal{P},s}$, respectively) to $\bigcup_{i \neq s} P_i$ internally disjoint from $\bigcup_{i=1}^{k} P_i$. A right jump from $v_{1,s}$ (and a left jump from $v_{2,s}$, respectively) is *ambiguous* if its both ends are in X_{t_1} (and in X_{t_2}, respectively); otherwise, it is *unambiguous*.

Lemma 4.1.1 *Let w, r, k be positive integers. Let $(T, \mathcal{X})$ be a w-linked tree decomposition of a graph G. Let Z be a subset of $V(G)$. Let $t_1, t_2, ..., t_r$ be nodes on a path in T with $Z \subseteq X_{t_i}$ such that $X_{t_i} - Z$ are pairwise disjoint and have the same size which is at most $w - |Z|$ for $1 \leq i \leq r$. Let $P_{i,1}, P_{i,2}, ..., P_{i,|X_{t_1}|}$ be disjoint paths from X_{t_i} to $X_{t_{i+1}}$ for each $1 \leq i \leq r-1$ such that $V(P_{i,j}) \cap X_{t_{i+1}} = V(P_{i+1,j}) \cap X_{t_{i+1}}$ for every $1 \leq i \leq r-1$. Let $v_{i,j}$ be the vertex in $X_{t_i} \cap P_{i,j}$ for every $1 \leq i \leq r$ and $1 \leq j \leq |X_{t_1}|$. Assume that there exists an integer s with $1 \leq s \leq |X_{t_1}|$ such that for every $1 \leq i \leq r-1$, $v_{i,s} \notin Z$ and either there exist two edge-disjoint paths from $v_{i,s}$ to $v_{i+1,s}$ internally disjoint from $X_{t_i} \cup X_{t_{i+1}}$ in $G - Z$, or there exist a right jump from $v_{i,s}$ and a left jump from $v_{i+1,s}$. Assume that for every $2 \leq i \leq r-1$, when there do not exist two edge-disjoint paths from $v_{i,s}$ to $v_{i-1,s}$ internally disjoint from $X_{t_{i-1}} \cup X_{t_i}$ in $G - Z$ and two edge-disjoint paths from $v_{i,s}$ to $v_{i+1,s}$ internally disjoint from $X_{t_i} \cup X_{t_{i+1}}$ in $G - Z$, there exist a left jump and a right jump from $v_{i,s}$ in $G - Z$ such that these two jumps intersect in at most one vertex. If $r \geq k^2|X_{t_1}|^{2k} + 2k + 1$, then $G - Z$ contains a topological minor isomorphic to the Robertson chain of length k.*

Proof. Without loss of generality, we may assume that Z is the empty set, otherwise we delete Z from G. Let $\mathcal{P} = \{P_1, P_2, ..., P_{|X_{t_1}|}\}$. Let s be the integer mentioned in the statement of this lemma. For every $1 \leq i \leq r-1$, define $Q_{\mathcal{P},s,i}, L_{\mathcal{P},s,i}, R_{\mathcal{P},s,i}$ to be the $Q_{\mathcal{P},s}, L_{\mathcal{P},s}, R_{\mathcal{P},s}$ mentioned above but between X_{t_i} and $X_{t_{i+1}}$, respectively. For every $1 \leq i \leq r-1$, we say that i is of *type* (s,s) if there exist two edge-disjoint paths from $v_{i,s}$ to $v_{i+1,s}$ internally disjoint from $X_{t_i} \cup X_{t_{i+1}}$. For each i with $1 \leq i \leq r-1$ such that i is not of type (s,s), we let $J_{R,i}$ and $J_{L,i+1}$ be a right jump from $v_{i,s}$ and a left jump from $v_{i+1,s}$, respectively. We may assume that $J_{R,i}$ and $J_{L,i}$ intersect in at most one vertex when both of them are defined. Note that $V(J_{R,i}) \cap V(J_{L,i+1}) \subseteq \bigcup_{j \neq s} P_j$ and $|V(J_{R,i}) \cap V(J_{L,i+1})| \leq 1$, since $L_{\mathcal{P},s,i} \neq R_{\mathcal{P},s,i}$ in this case. We say that i is of *type* (a,b) if the end of $J_{R,i}$ not in $L_{\mathcal{P},s,i}$ is on P_a and the end of $J_{L,i+1}$ not in $R_{\mathcal{P},s,i}$ is on P_b. Note that if i is of type (a,b) for some a,b such that $a \neq s$ or $b \neq s$, then $a \neq s \neq b$, since $L_{\mathcal{P},s,i} \neq R_{\mathcal{P},s,i}$.

For every $1 \leq i \leq r-k+1$, we say that the *k-type* of i is the sequence $(b_i, b_{i+1}, ..., b_{i+k-1})$, where b_j is the type of j for every $i \leq j \leq i+k-1$. Observe that there are at most $|X_{t_1}|^{2k}$ possible k-types. Since $r-k-1 \geq k^2|X_{t_1}|^{2k}+k$, there exist $1 \leq i_0 < i_1 < ... < i_k \leq k|X_{t_1}|^{2k}+1$ such that $i_j \equiv i_0 \pmod{k}$ for $1 \leq j \leq k$ and the k-types of $i_0, i_1, ..., i_k$ are the same. We shall construct a topological minor isomorphic to the Robertson chain of length k.

We claim that for every nonnegative integer $0 \leq m \leq k$, there exists $s_m = i'_m + m$, where $i'_m \in \{i_0, i_1, ..., i_m\}$ (so $s_m - i_0 \equiv m \pmod{k}$), such that G contains a topological minor S_m isomorphic to the Robertson chain of length at least m satisfying the following properties.

- S_m is contained in the subgraph of G induced by $X_{t_{s_m}} \cup \bigcup_{t \in V(T')} X_t$, where T' is the component of $T - \{t_{s_m}\}$ containing t_1.
- S_m intersects $X_{t_{s_m}}$ in exactly one or two vertices.
- If S_m intersects $X_{t_{s_m}}$ in two vertices, then J_{L,s_m} is defined, and these two vertices

are the ends of J_{L,s_m}. (So $v_{s_m,s}$ is in S_m.)

- If S_m intersects $X_{t_{s_m}}$ in exactly one vertex, then this vertex is $v_{s_m,s}$.

We prove this claim by induction on m. When $m = 0$, the statement is obviously true by choosing $S_0 = \{v_{i_0,s}\}$. We assume that $m > 0$, and that the claim holds for $m - 1$.

First, assume that s_{m-1} is of type (s, s). Note that S_{m-1} contains $v_{s_{m-1},s}$. Then, we set $s_m = s_{m-1}+1$ and $i'_m = i'_{m-1}$, and we define S_m to be the graph obtained from S_{m-1} by adding the two edge-disjoint paths from $v_{s_{m-1},s}$ to $v_{s_{m-1}+1,s}$ internally disjoint from $X_{t_{s_{m-1}}} \cup X_{t_{s_{m-1}+1}}$. It is clear that S_m satisfies all properties mentioned in the claim. So we may assume that s_{m-1} is of type (a, b), where $a \neq s \neq b$. In this case, we set $i'_m = i_{w+1}$, where w is the number such that $i'_{m-1} = i_w$, and let $s_m = i'_m + m$. Note that the type of $s_m - 1$ is the same as the type of s_{m-1}, and $J_{R,s_{m-1}}, J_{L,s_{m-1}+1}, J_{R,s_m-1}, J_{L,s_m}$ are defined. Let $u_{s_{m-1}}, u_{s_{m-1}+1}, u_{s_m-1}, u_{s_m}$ be the end of $J_{R,s_{m-1}}, J_{L,s_{m-1}+1}, J_{R,s_m-1}, J_{L,s_m}$ contained in $L_{\mathcal{P},s,s_{m-1}}, R_{\mathcal{P},s,s_{m-1}+1}, L_{\mathcal{P},s,s_m-1}, R_{\mathcal{P},s,s_m}$, respectively. Since every block in $L_{\mathcal{P},s,s_{m-1}} \cup R_{\mathcal{P},s,s_{m-1}+1} \cup L_{\mathcal{P},s,s_m-1} \cup R_{\mathcal{P},s,s_m}$ is not a single edge, we can define paths $Y_{m-1}, Z_{m-1}, Y_m, Y'_{m-1}, Y'_m$ internally disjoint from $\bigcup_{j \neq s} P_j$ and paths W_L, W_R, W such that

- Y_{m-1} and Y_m are paths from $u_{s_{m-1}}$ to $u_{s_{m-1}+1}$ containing $\{v_{s_{m-1},s}, v_{s_{m-1}+1,s}\}$, and from u_{s_m-1} to u_{s_m} containing $\{v_{s_m-1,s}, v_{s_m,s}\}$, respectively, and

- Y'_{m-1} and Y'_m are paths from $u_{s_{m-1}}$ to $v_{s_{m-1}+1,s}$ containing $v_{s_{m-1},s}$, and from $v_{s_m-1,s}$ to u_{s_m} containing $v_{s_m,s}$, respectively, and

- if $v_{s_{m-1},s}$ is not an end of $J_{R,s_{m-1}}$, then Z_{m-1} is a path from $u_{s_{m-1}}$ to $u_{s_{m-1}+1}$ containing $v_{s_{m-1}+1,s}$ but not containing $v_{s_{m-1},s}$, and

- W_R is the subpath of P_a connecting one end of $J_{R,s_{m-1}}$ and one end of J_{R,s_m-1}, and W_L is the subpath of P_b connecting one end of $J_{L,s_{m-1}+1}$ and one end of

J_{L,s_m}, and

- if $a = b$, then W is the subpath of P_a connecting one end of $J_{R,s_{m-1}}$ and one end of J_{L,s_m}.

Now, we are ready to construct S_m. First, assume that $a \neq b$ and S_{m-1} intersects $X_{t_{s_{m-1}}}$ at two vertices. So one of the two vertex is $v_{s_{m-1},s}$, and $J_{R,s_{m-1}}$ intersects S_{m-1} at most one vertex. If $J_{R,s_{m-1}}$ intersects S_{m-1}, but $v_{s_{m-1},s}$ is not the common vertex, then define S_m to be the graph obtained from S_{m-1} by adding $J_{R,s_{m-1}} \cup Z_{m-1} \cup J_{L,s_{m-1}+1} \cup W_L \cup J_{L,s_m} \cup Y_m \cup J_{R,s_m-1} \cup W_R$; otherwise, define S_m to be the graph obtained from S_{m-1} by adding $J_{R,s_{m-1}} \cup Y_{m-1} \cup J_{L,s_{m-1}+1} \cup W_L \cup J_{L,s_m} \cup Y_m \cup J_{R,s_m-1} \cup W_R$.

Second, we assume that $a \neq b$ and S_{m-1} intersects $X_{t_{s_{m-1}}}$ exactly one vertex, then this vertex is $v_{s_{m-1},s}$ by the induction hypothesis, and we define S_m to be the graph obtained from S_{m-1} by adding $J_{R,s_{m-1}} \cup Y_{m-1} \cup J_{L,s_{m-1}+1} \cup W_L \cup J_{L,s_m} \cup Y_m \cup J_{R,s_m-1} \cup W_R$. Clearly, S_m satisfies all properties mentioned in the claim.

Finally, we assume that $a = b$. Let P' be the subpath of P_s connecting $v_{s_{m-1}+1,s}$ and $v_{s_m-1,s}$. If S_{m-1} intersects $X_{t_{s_{m-1}}}$ two vertices, and $J_{R,s_{m-1}}$ intersects S_{m-1}, but $v_{s_{m-1},s}$ is not the common vertex, then we define S_m to be the graph obtained from S_{m-1} by adding $J_{R,s_{m-1}} \cup Z_{m-1} \cup P' \cup Y'_m \cup J_{L,s_m} \cup W$; otherwise, we define S_m to be the graph obtained from S_{m-1} by adding $J_{R,s_{m-1}} \cup Y'_{m-1} \cup P' \cup Y'_m \cup J_{L,s_m} \cup W$. It is also clear that S_m satisfies all properties mentioned in the claim in this case. This completes the proof. ■

4.2 Looking for edge-cuts

Lemma 4.2.1 *Let k be a positive integer. Let G be a graph and $(T, \mathcal{X})$ a k-strongly linked tree decomposition of G. Let t_1, t_2 be two nodes of T such that $|X_{t_1}| = |X_{t_2}| = k$. Let $P_1, P_2, ..., P_k$ be disjoint paths from X_{t_1} to X_{t_2} in G, and let $v_{i,j}$ be the vertex in X_{t_i} that is an end of P_j, for $i = 1, 2$ and $1 \leq j \leq k$. Fix $1 \leq s \leq k$, and define $Q_{\mathcal{P},s}$*

as above. If $v_{1,s} \neq v_{2,s}$ and $Q_{\mathcal{P},s}$ contains only one block, and either there exists no unambiguous right jump from $v_{1,s}$ or there exists no unambiguous left jump from $v_{2,s}$, then there exist two nodes t_3, t_4 of T on the t_1-t_2 path in T such that $X_{t_3} - X_{t_4} = \{v_{1,s}\}$ and $X_{t_4} - X_{t_3} = \{v_{2,s}\}$.

Proof. Without loss of generality, we may assume that there exists no unambiguous right jump from $v_{1,s}$ by symmetry. Note that every vertex-cut C separating X_{t_1} and X_{t_2} of size k must contain one vertex on each P_i. Furthermore, $C \cap P_s \subseteq \{v_{1,s}, v_{2,s}\}$, since $Q_{\mathcal{P},s}$ is a block in $G - \bigcup_{j \neq s} P_j$. Let t_3 be a node on the t_1-t_2 path in T such that $|X_{t_3}| = k$ and $v_{1,s} \in X_{t_3}$, and we assume that t_3 is as close to t_2 as possible. (Note that t_1 is a possible candidate of t_3.) Similarly, let t_4 be a node on the t_1-t_2 path in T such that $|X_{t_4}| = k$ and $v_{2,s} \in X_{t_4}$, and we assume that t_4 is as close to t_1 as possible.

Suppose that $X_{t_3} - X_{t_4}$ contains a vertex other than $v_{1,s}$. Since there exists no unambiguous right jump from $v_{1,s}$, $(X_{t_3} - \{v_{1,s}\}) \cup \{v_{2,s}\}$ is a vertex-cut of size k other than X_{t_3} and X_{t_4} separating X_{t_3} and X_{t_4}. As $(T, \mathcal{X})$ is k-strongly linked, there exists an internal node t of the t_3-t_4 path such that X_t has size k and separates X_{t_3} and X_{t_4}. But X_t contains $v_{1,s}$ or $v_{2,s}$, contradicting the choice of t_3 or t_4. This proves that $X_{t_3} - X_{t_4} = \{v_{1,s}\}$. Similarly, $X_{t_4} - X_{t_3} = \{v_{2,s}\}$. ■

Lemma 4.2.2 *Let k be a positive integer. Let G be a graph and $(T, \mathcal{X})$ a k-strongly linked tree decomposition of G. Let t_1, t_2 be two nodes of T such that $|X_{t_1}| = |X_{t_2}| = k$. Let $P_1, P_2, ..., P_k$ be disjoint paths from X_{t_1} to X_{t_2} in G, and let $v_{i,j}$ be the vertex in X_{t_i} that is an end of P_j, for $i = 1, 2$ and $1 \leq j \leq k$. Fix $1 \leq s \leq k$, and define $Q_{\mathcal{P},s}, L_{\mathcal{P},s}, R_{\mathcal{P},s}$ as above. Assume that $v_{1,s} \neq v_{2,s}$ and $L_{\mathcal{P},s} \neq R_{\mathcal{P},s}$. Let $e_{\mathcal{P},s,1}$ and $e_{\mathcal{P},s,2}$ be the block consisting of an edge incident with $L_{\mathcal{P},s}$ and $R_{\mathcal{P},s}$ in $Q_{\mathcal{P},s}$, respectively. If there exists no unambiguous right (and left, respectively) jump from $v_{1,s}$ (and $v_{2,s}$, respectively), then for every vertex v in $L_{\mathcal{P},s} \cup e_{\mathcal{P},s,1}$ (and $R_{\mathcal{P},s} \cup e_{\mathcal{P},s,2}$, respectively) contained in two blocks of $Q_{\mathcal{P},s}$, there exists a node t on the t_1-t_2 path in T such that $v \in X_t$ and $|X_t| = k$.*

Proof. By symmetry, we may assume that there exists no unambiguous right jump from $v_{1,s}$. We prove this lemma by induction on the sum of the lengths of the members in $\mathcal{P}$. Let v be a vertex in $L_{\mathcal{P},s} \cup \{e_{\mathcal{P},s,1}\}$ contained in two blocks of $Q_{\mathcal{P},s}$. Let $Z = (X_{t_1} - \{v_{1,s}\}) \cup \{v\}$. Note that Z separates X_{t_1} and X_{t_2}. Then such t exists when X_{t_1} and X_{t_2} differs only one vertex as $(T, \mathcal{X})$ is k-strongly linked. This builds the induction basis. Suppose that there exists no node t on the t_1-t_2 path in T whose bag contains v. Then there exists a node t' with $|X_{t'}| = k$ on the t_1-t_2 path such that the $X_{t'}$ crosses Z. But induction hypothesis implies that such t exists on the t'-t_2 path, a contradiction. ■

Let G be a graph on n vertices. We say that a rooted tree decomposition $(T, \mathcal{X})$ of G is *unimpeded* if for every pair of nodes t_1, t_2 of T with $|X_{t_1}| = |X_{t_2}|$, where t_1 is an ancestor of t_2, either there exist $|X_{t_1}|$ disjoint path from $|X_{t_1}|$ to $|X_{t_2}|$ in G, or there exists an internal node t of the t_1-t_2 path of T with $|X_t| < |X_{t_1}|$. Clearly, every linked rooted tree decomposition is unimpeded. We say that a node y of T is a *successor* of a node x if $|X_x| = |X_y|$, and $|X_t| > |X_x|$ for every internal node t of the directed path from x to y in T. In this case, we say that x is the *precursor* of y. A node t is a *start* if it is not a successor of any node. As $(T, \mathcal{X})$ is unimpeded, for each node x and a successor y of x, there exists a set $\mathcal{P}_{x,y}$ of $|X_x|$ disjoint paths from X_x to X_y in G. We call $\mathcal{P}_{x,y}$ a *set of foundation paths*. Note that $\mathcal{P}_{x,y}$ may not be unique for the same x, y. Fix an ordering of the vertices in X_x, then the vertices in X_y are ordered in the way that the i-th vertex in X_y and the i-th vertex in X_x are on the same path in $\mathcal{P}_{x,y}$. As long as $\mathcal{P}_{x,y}$ are chosen for each precursor-successor pair (x, y), we define $\mathcal{P}$ to be the union of all $\mathcal{P}_{x,y}$. Therefore, as long as $\mathcal{P}$ are defined and the vertices in each start are ordered by an ordering O, the vertices in every bag are ordered. We denote this ordering by $\mathcal{O}_{\mathcal{P},O}$.

Let G be a graph. Let $(T, \mathcal{X})$ be an unimpeded lean rooted tree decomposition of G, and $\mathcal{P}$ a set of foundation paths, and an ordering O of the starts. Let t_1, t_2, t_3

be three distinct nodes in T of degree two in T such that there exists a directed path in T passing through t_1, t_2, t_3 in order or passing through t_3, t_2, t_1 in order. Assume that $|X_{t_1}| = |X_{t_2}| = |X_{t_3}| \leq |X_t|$ for every node t in the path in T from t_1 to t_3. For every $1 \leq i \leq 3$ and $1 \leq j \leq |X_{t_i}|$, let $v_{i,j}$ be the j-th vertex in X_{t_i}. Let T' and T'' be the components of $T - \{t_1, t_2, t_3\}$ such that T' is adjacent to t_1 and t_2, and T'' is adjacent to t_2 and t_3, respectively. Assume that there exists an integer s for some $1 \leq s \leq |X_{t_1}|$ such that the following hold.

- $v_{1,s} = v_{2,s}$, but $X_{t_1} \neq X_{t_2}$.
- $X_{t_2} - X_{t_3} = \{v_{2,s}\} \neq \{v_{3,s}\} = X_{t_3} - X_{t_2}$.
- No path other than the edges with both ends in X_{t_1} in $G[\bigcup_{t \in V(T')} X_t]$ is from $v_{1,s}$ to a vertex in a path in $\mathcal{P}$ not containing $v_{1,s}$.
- There does not exist a path in G from $\bigcup_{t \in V(T'')} X_t - X_{t_3}$ to $X_{t_2} - X_{t_1}$.
- If there is a path in $G[\bigcup_{t \in V(T'')} X_t]$ from $v_{3,s}$ to $X_{t_3} - \{v_{3,s}\}$, then t_1 is an ancestor of t_3.

Let y_1, y_2 be the nodes in T' adjacent to t_1 and t_2, respectively, and let y_3, y_4 be the nodes in T'' adjacent to t_2 and t_3, respectively. Define T^* to be the rooted tree such that $T^* = (T - \{t_1y_1, y_2t_2, t_2y_3, y_4t_3\}) \cup \{t_1y_3, y_4t_2, t_2y_1, y_2t_3\}$, and the orientation of edges of T^* are the same as in T if they are also edges of T. For each subset S of X_{t_3}, let y_S be the node in T with $S \cap X_{t_2} \subseteq X_{y_S}$ closest to the root of T. If t_1 is an ancestor of t_3, then for each $S \subseteq X_{t_3}$, we redefine the parent of each node $s \in V(T'')$ with $X_s = S$ adjacent to a node in the path in T from t_1 to t_3 to be y_S. Define $\mathcal{X}^* = (X^*_t : t \in V(T^*))$ such that the following hold.

- $X^*_{t_2} = \{v_{1,j}, v_{3,s} : j \neq s\}$.
- If $t \in V(T')$ and $v_{1,s} \in X_t$, then $X^*_t = (X_t - \{v_{1,s}\}) \cup \{v_{3,s}\}$; if $t \in V(T')$ and $v_{1,s} \notin X_t$, then $X^*_t = X_t$.

- If $t \in V(T'')$ and t is not a descendant of a node s with $X_s \subseteq X_{t_3}$ in T, then $X_t^* = (X_t - \{v_{2,j} \in X_t : j \neq s\}) \cup \{v_{1,j} : v_{2,j} \in X_t, j \neq s\}$; if $t \in V(T'')$ and t is a descendant of a node s with $X_s \subseteq X_{t_3}$, then $X_t^* = X_t$.

- If $t \in V(T^*) - (V(T') \cup V(T'') \cup \{t_2\})$, then $X_t^* = X_t$.

The following lemma shows that $(T^*, \mathcal{X}^*)$ is a tree decomposition of G. We say that $(T^*, \mathcal{X}^*)$ is the tree decomposition obtained from $(T, \mathcal{X})$ by *swapping subtrees between* t_1, t_2, t_3.

Lemma 4.2.3 *Let $(T^*, \mathcal{X}^*)$ be defined as above. If no vertex in $X_{t_1} \cup X_{t_3}$ is an end of a foundation path, and there do not exist $t \in V(T)$ and a component C of $T - t$ not containing the root of T such that $G[X_C - X_t]$ is disconnected, then $(T^*, \mathcal{X}^*)$ is a tree decomposition of G having the same signature as $(T, \mathcal{X})$, and there do not exist t' and a component C' of $T^* - t'$ not containing the root of T^* such that $G[X^*_{C'} - X^*_{t'}]$ is disconnected.*

Proof. Let e be an edge of G. We shall show that there exists $t \in V(T^*)$ such that $e \subseteq X_t^*$. Since $(T, \mathcal{X})$ is a tree decomposition of G, there exists $t \in V(T)$ such that $e \subseteq X_t$. Clearly, $e \subseteq X_t^*$ if $t \not\in V(T') \cup V(T'') \cup \{t_2\}$. So we may assume that t can only be chosen in $V(T') \cup V(T'') \cup \{t_2\}$. Furthermore, we are done if e is not incident with a vertex in X_{t_2}. Note that $t \not\in V(T'')$, since there does not exist a path in G from $\bigcup_{t \in V(T'')} X_t - X_{t_3}$ to $X_{t_2} - X_{t_1}$. In addition, $e \subseteq X_t^*$ if both ends of e are in $X_{t_2} - \{v_{2,s}\}$. $v_{1,s} = v_{2,s}$ is not an end of e, since no path other than the edges with both ends in X_{t_1} in $G[\bigcup_{t \in V(T')} X_t]$ is from $v_{1,s}$ to a vertex in a path in $\mathcal{P}$ not containing $v_{1,s}$, and there do not exist $t \in V(T)$ and a component C of $T - t$ not containing the root of T such that $G[X_C - X_t]$ is disconnected. So $t \in V(T')$ and e is not incident with $v_{1,s}$. Hence $e \subseteq X_t^*$. This proves that for every $e \in E(G)$, there exists $t \in V(T^*)$ such that $e \subseteq X_t^*$. Similarly, for every $v \in V(G)$, there exists $t \in V(T^*)$ such that $v \in X_t^*$.

Let v be a vertex of G, and let $S^* = \{t \in V(T^*) : v \in X_t^*\}$. We shall prove that S^* induces a subtree of T^*. Since $(T, \mathcal{X})$ is a tree decomposition of G, the set $\{t \in V(T) : v \in X_t\}$, denoted by S_v, induces a subtree of T. We are done if $v \notin \bigcup_{i=1}^{3} X_{t_i}$. If $v = v_{1,s}$, then $S^* = S_v - (V(T') \cup \{t_2\})$ induces a subtree in T^*. If $v \in X_{t_1} - \{v_{1,s}\}$, then $S^* = S_v \cup S' \cup \{t_2\}$ induces a subtree in T^*, where S' is a subtree of $V(T'')$ containing the path from y_3 to y_4. Similarly, S^* induces a subtree in T^* if $v \in X_{t_2} - \{v_{2,s}\}$. If $v = v_{3,s}$, then $S^* = S_v \cup S''$ induces a subtree in T^*, where S'' is a subtree of $V(T')$ containing the path from y_1 to y_2. This proves that $(T, \mathcal{X})$ is a tree decomposition of G.

It is obvious that $|X_t^*| = |X_t|$ for every $t \in V(T)$. So if $k \geq |X_{t_1}|$, then every k-cell L^* of $(T^*, \mathcal{X}^*)$ is a k-cell L of $(T, \mathcal{X})$ disjoint from $\{t_1, t_2, t_3\}$, and it is easy to see that the volume of L^* is the same as the volume of L. Since there exist foundation paths passing through vertices in $X_{t_1} \cup X_{t_2} \cup X_{t_3}$, $|X_t| \geq |X_{t_1}|$ for every node t in the path in T from t_1 to t_3. If $k < |X_{t_1}|$, then for every k-cell L^* of $(T^*, \mathcal{X}^*)$, either L^* is disjoint from $\{t_1, t_2, t_3\}$ and is a k-cell of $(T, \mathcal{X})$ with the same volume, or L^* contains $\{t_1, t_2, t_3\}$ and there uniquely exists a k-cell L of $(T, \mathcal{X})$ with $V(L) = V(L^*)$ having the same volume as L^*. Therefore, the signature of $(T^*, \mathcal{X}^*)$ is the same as $(T, \mathcal{X})$.

Now we assume that no vertex in $X_{t_1} \cup X_{t_3}$ is an end of a foundation path, and there do not exist $t \in V(T)$ and a component C of $T - t$ not containing the root of T such that $G[X_C - X_t]$ is disconnected. Suppose that there exist t' and a component C' of $T^* - t'$ not containing the root of T^* such that $G[X_{C'}^* - X_{t'}^*]$ is disconnected. Clearly, t' is an internal node of the path in T^* from t_1 to t_3 and C contains the node in $\{t_1, t_3\}$ that is further from the root of T^*. However, no vertex in $X_{t_1} \cup X_{t_3}$ is an end of a foundation path, so every vertex in $X_{t_1} \cup X_{t_3}$ is incident with an edge with the other end in a bag of its descendant. Then it is not hard to see that such t', C' do not exist since there do not exist $t \in V(T)$ and a component C of $T - t$ not containing the root of T such that $G[X_C - X_t]$ is disconnected. ■

Let $(T, \mathcal{X})$ be a branching unimpeded rooted tree decomposition of a graph G. For every non-root node t of T, let C_t be the component of $T - t$ containing the root. For every node t of T, we say that a vertex v in X_t *corresponds to an edge at* t if t is not a start, and there exists an edge e incident with v with the other end in $\bigcup_{t' \in C_t} X_{t'} - X_t$, and every path in $G[\bigcup_{t' \in C_t} X_{t'}]$ from $\{v\}$ to $X_t - (\{v\} \cup X_p)$ passing through a vertex in $\bigcup_{t' \in C_t} X_{t'}$ contains e, where p is the precursor of t. For every positive integer j, we say that a node t in T is j-*good* with respect to $\mathcal{O}_{\mathcal{P},O}$ if the i-th vertex (ordered by $\mathcal{O}_{\mathcal{P},O}$) in X_t corresponds to an edge at t for every $1 \leq i \leq j$. The *goodness* of a node t with respect to $\mathcal{O}_{\mathcal{P},O}$ is the maximum j such that t is j-good. For every $1 \leq k \leq n$, we define the k-*badness of* $(T, \mathcal{X})$ *with respect to* $\mathcal{O}_{\mathcal{P},O}$ to be the sequence $b_k = (a_0, a_1, ..., a_{k-1})$, where for every i, a_i is the number of $t \in V(T)$ of goodness i with $|X_t| = k$ having a child. The *badness of* $(T, \mathcal{X})$ *with respect to* $\mathcal{O}_{\mathcal{P},O}$ is the sequence $(b_1, b_2, ..., b_n)$, and the *badness of* $(T, \mathcal{X})$ is the minimum badness with respect to an ordering $\mathcal{O}_{\mathcal{P},O}$.

Lemma 4.2.4 *Let G be a graph and h a nonnegative integer. Then there exist a branching h-strongly lean rooted tree decomposition $(T, \mathcal{X})$, a set of foundation paths $\mathcal{P}$, and an ordering O of the starts satisfying the following. Assume that t_1, t_2 are two distinct nodes of T such that t_1 is a descendant of t_2 with the following properties:*

1. *$|X_{t_1}| = |X_{t_2}|$.*
2. *Every vertex in $X_{t_1} \cup X_{t_2}$ is not an end of a foundation path.*
3. *t_1 and t_2 have the same goodness, say $s - 1$. And $1 \leq s \leq |X_{t_1}|$.*
4. *For every internal node t of the path in T from t_1 to t_2, either $|X_t| > |X_{t_1}|$, or $|X_t| = |X_{t_1}|$ and the goodness of t is at most $s - 1$.*

Let $v_{1,s}$ and $v_{2,s}$ be the s-th vertex of X_{t_1} and X_{t_2}, respectively. Then either

1. *$v_{1,s} = v_{2,s}$, or*

2. *there exist two edge-disjoint paths in G from $v_{1,s}$ to $v_{2,s}$ internally disjoint from $X_{t_1} \cup X_{t_2}$, or*

3. *there exist a right jump from $v_{1,s}$ disjoint from $X_{t_1} \cap X_{t_2}$ between t_1, t_2 and an unambiguous left jump from $v_{2,s}$ disjoint from $X_{t_1} \cap X_{t_2}$ between t_1, t_2.*

Furthermore, if $h \geq \mathrm{tw}(G)$, then $(T, \mathcal{X})$ has width $\mathrm{tw}(G)$.

Proof. Let $(T, \mathcal{X})$ be a tree decomposition mentioned in the condition of Theorem 3.1.6, and subject to that, with the minimum badness. So $(T, \mathcal{X})$ is branching and h-strongly lean, and if $h \geq \mathrm{tw}(G)$, then $(T, \mathcal{X})$ has width $\mathrm{tw}(G)$. Let $\mathcal{P} = \{P_1, P_2, ..., P_{|X_{t_1}|}\}$ and O be a set of foundation paths and an ordering of starts such that $\mathcal{O}_{\mathcal{P},O}$ minimizes the badness, respectively. Suppose that there exist nodes t_1, t_2 of T satisfying the condition of this lemma but do not satisfy the conclusion of this lemma. That is, $v_{1,s} \neq v_{2,s}$ and there do not exist two edge-disjoint paths in G from $v_{1,s}$ to $v_{2,s}$ internally disjoint from $X_{t_1} \cup X_{t_2}$, and either there does not exist a right jump from $v_{1,s}$ disjoint from $X_{t_1} \cap X_{t_2}$ between t_1, t_2, or there does not exist an unambiguous left jump from $v_{2,s}$ disjoint from $X_{t_1} \cap X_{t_2}$ between t_1, t_2. We first assume that there does not exist a right jump from $v_{1,s}$ disjoint from $X_{t_1} \cap X_{t_2}$ between t_1, t_2.

We define $Q_{\mathcal{P},s}$, $L_{\mathcal{P},s}$, $R_{\mathcal{P},s}$, $e_{\mathcal{P},s,1}$ and $e_{\mathcal{P},s,2}$ between t_1 and t_2 as before. Since there do not exist two edge disjoint path from $v_{1,s}$ to $v_{2,s}$ in G, $L_{\mathcal{P},s} \neq R_{\mathcal{P},s}$. Let $u_0 = v_{1,s}$ and let $S_L = \{u_1, u_2, ..., u_\ell\}$ be the set of vertices in $L_{\mathcal{P},s} \cup \{e_{\mathcal{P},s,1}\}$ contained in two blocks on $Q_{\mathcal{P},s}$, where $u_0, u_1, ..., u_\ell$ are appeared on P_s in this order. By Lemmas 4.2.1 and 4.2.2, there exist nodes $x_0 = t_2, x_1, x_2, ..., x_{2\ell}$ of T (not necessarily pairwise distinct) that have bags of size equal to $|X_{t_2}|$ such that $u_i \in X_{x_{2i}} \cap X_{x_{2i+1}}$ and $X_{x_{2j-1}} - X_{x_{2j}} = \{u_{j-1}\}$ and $X_{x_{2j}} - X_{x_{2j-1}} = \{u_j\}$ for $0 \leq i \leq \ell$ and $1 \leq j \leq \ell$. Clearly, there exists a directed path in T contains $x_0, x_1, x_2, ..., x_{2\ell}$ in this order.

Since the size of the bag of each node of the path in T from t_1 to t_2 is at least $|X_{t_1}|$, and $|X_j| = |X_{t_1}|$ for every $0 \le j \le 2\ell$, we know that the degree of x_j is two in T for every $0 \le j \le 2\ell$ as $(T, \mathcal{X})$ is branching. If there exists a node t in the component of $T - \{x_{2i-1}, x_{2i}\}$ between x_{2i-1} and x_{2i} satisfying that $X_t \cap (X_{x_{2i}} - X_{x_{2i-2}} - \{u_i\}) \neq \emptyset$ and t has a descendant in the component of $T - t$ not containing x_{2i-1} or x_{2i}, then the bags in the subtree rooted at t intersect the bags of the component of $T - \{x_{2i-1}, x_{2i}\}$ between x_{2i-1} and x_{2i} only at a subset of $X_{x_{2i}}$, since there exists no right jump from $v_{1,s}$ disjoint from $X_{t_1} \cap X_{t_2}$. Therefore, we can move the subtree rooted at t out of the component of $T - \{x_{2i-1}, x_{2i}\}$ between x_{2i-1} and x_{2i}. Hence, we may assume that X_t is disjoint from $X_{x_{2i}} - X_{x_{2i-2}} - \{u_i\}$ for every node t in the component of $T - \{x_{2i-1}, x_{2i}\}$ between x_{2i-1} and x_{2i} having a descendant in the component $T - t$ not containing x_{2i-1} or x_{2i}.

Let $(T^{(0)}, \mathcal{X}^{(0)}) = (T, \mathcal{X})$, and let $t^{(0)} = x_0$. For $1 \le i \le \ell$, define $(T^{(i)}, \mathcal{X}^{(i)})$ to be the rooted tree decomposition of G obtained from $(T^{(i-1)}, \mathcal{X}^{(i-1)})$ by swapping subtrees between $t^{(i-1)}, x_{2i-1}, x_{2i}$, and define $t^{(i)} = x_{2i-1}$. By Lemma 4.2.3, every $(T^{(i)}, \mathcal{X}^{(i)})$ has the same signature and is branching for $0 \le i \le \ell$.

Now, we prove that the k-badness of $(T^{(i)}, \mathcal{X}^{(i)})$ is not larger than the k-badness of $(T, \mathcal{X})$ for every $k \le |X_{t_1}|$ by induction on i. When $i = 0$, there is nothing to prove. We assume that $i > 0$. Let $k \le |X_{t_1}|$, and let t be a node in $T^{(i)}$. It is clear that the goodness of t in $(T^{(i)}, \mathcal{X}^{(i)})$ is the same as the goodness of t in $(T^{(i-1)}, \mathcal{X}^{(i-1)})$ if $t \notin T'^{(i-1)} \cup T''^{(i-1)} \cup \{x_{2i-1}\}$. If $t \in T''^{(i-1)}$ and t has a child, then either X_t is disjoint from $X_{x_{2i}} - X_{x_{2i-2}} - \{u_i\}$, or t is an internal node of x_{2i-1}-x_{2i} path in T. In the later case, $|X_t| > |X_{t_1}|$ since deleting $X_{x_{2i-1}} \cap X_{x_{2i}}$ from the subgraph of G induced by the bags between x_{2i-1} and x_{2i} is a block. So, the goodness of t does not affect the k-badness. If $t \in T'^{(i-1)}$, then $X_t^{(i-1)} - X_t^{(i)} \subseteq \{u_{i-1}\}$, but t_1 is not s-good and $u_1, u_2, ..., u_{\ell-1}$ do not correspond to edges, so the goodness of t in $(T^{(i)}, \mathcal{X}^{(i)})$ is not smaller than the goodness of t at $(T^{(i-1)}, \mathcal{X}^{(i-1)})$. Similarly, the goodness of x_{2i-1} in $(T^{(i)}, \mathcal{X}^{(i)})$ is not smaller than the goodness of x_{2i-1} in $(T^{(i-1)}, \mathcal{X}^{(i-1)})$. Therefore, the

k-badness of $(T_i, \mathcal{X}_i)$ is not larger than the k-badness of $(T, \mathcal{X})$ for every $k \leq |X_{t_1}|$.

On the other hand, the goodness of $t^{(\ell)} = x_{2\ell-1}$ in $(T^{(\ell)}, \mathcal{X}^{(\ell)})$ is greater than the goodness of $x_{2\ell-1}$ in $(T^{(\ell-1)}, \mathcal{X}^{(\ell-1)})$. Consequently, $(T^{(i)}, \mathcal{X}^{(i)})$ has smaller badness than $(T, \mathcal{X})$ for some $1 \leq i \leq \ell$, a contradiction. This proves that there exists a right jump from $v_{1,s}$ disjoint from $X_{t_1} \cap X_{t_2}$.

Hence, there does not exist an unambiguous left jump from $v_{2,s}$ disjoint from $X_{t_1} \cap X_{t_2}$ between t_1, t_2. Since $(T, \mathcal{X})$ is branching, there does not exist ambiguous left jump from $v_{2,s}$, so there do not exist left jump from $v_{2,s}$ between t_1, t_2. But the same argument shows that swapping subtrees decreases the badness, a contradiction. This completes the proof. ■

Let $(T, \mathcal{X})$ be a rooted tree decomposition. Let Z be a subset of $V(G)$. A node t of T *corresponds to an edge-cut modulo* Z if $Z \subseteq X_t$ and for every vertex v in $X_t - Z$, there exists an edge e incident with v whose other end is in $G[C - (X_t \cup Z)]$, and every path in $G[C]$ from $\{v\}$ to $X_t - \{v\}$ passing through a vertex in C contains e, where C is the union of bags of the component of $T - t$ containing the root of T. Let T_Z be the maximal subtree of T such that every bag of T_Z contains Z, and let $\mathcal{X}_Z = \{X_t - Z : t \in V(T_Z)\}$. For every nonnegative integer s, let $D_{Z,s}$ be the set of nodes of T corresponding to edge-cuts modulo Z whose bags are of size $s + |Z|$, and let $D'_{Z,s}$ be the set of edges of T whose tails in $D_{Z,s}$. The (Z, s)-*depth* of $(T, \mathcal{X})$ is the maximum h such that there exists a directed path P in $T_Z - D'_Z$ such that P contains h nodes that have pairwise disjoint bags of size s in $\mathcal{X}_Z$.

Theorem 4.2.5 *Let k, h be positive integers. Then there exists $f(k, h)$ such that every graph G that does not contain a topological minor isomorphic to the Robertson chain of length k has a branching h-strongly lean rooted tree decomposition (T, X) such that its (Z, s)-depth is at most $f(k, h)$ for every $Z \subseteq V(G)$ and nonnegative integer $s \leq h$. Furthermore, if $h \geq \mathrm{tw}(G)$, then $(T, \mathcal{X})$ has width* $\mathrm{tw}(G)$.

Proof. Let $(T, \mathcal{X})$ be a branching h-strongly lean rooted tree decomposition of G, $\mathcal{P}$ the set of foundation paths and the ordering O of starts such that Statements of Lemma 4.2.4 hold. Fix a nonnegative integer $s \leq h$ and a subset Z of $V(G)$. Let T_Z be the maximal subtree of T such that $X_t \supseteq Z$ for every node t of T_Z, and let $\mathcal{X}^Z = \{X_t^Z = X_t - Z : t \in V(T_Z)\}$. Note that O and $\mathcal{P}$ define a natural ordering of the vertices in X_t^Z for every $t \in V(T_Z)$, and we can define the goodness of t in T_Z as before.

Let $D_{Z,s}$ be the set of nodes of T corresponding to edge-cuts modulo Z, and let $D'_{Z,s}$ be the set of edges of T whose tails are in $D_{Z,s}$. Let $d_{1,0} = k$. For every $1 \leq a \leq h$ and $0 \leq b \leq a-1$, and define $d_{a,a-1} = (k^2a^{2k} + k + 3)(\sum_{i=1}^{a-1} \sum_{j=0}^{i-1} d_{i,j} + 1)$ if $a \geq 2$, and define $d_{a,b} = (k^2a^{2k} + k + 3)(\sum_{i=1}^{a} \sum_{j=b+1}^{i-1} d_{i,j} + 1)$ if $b \neq a-1$. We claim that for every $1 \leq a \leq h$ and $0 \leq b \leq a-1$, every directed path in $T_Z - D'_{Z,s}$ contains at most $d_{a,b}$ nodes t such that X_t^Z are pairwise disjoint and have size a and their goodness is b.

We prove the claim by induction on the lexicographic order of $(a, a-b)$. When $(a, a-b) = (1, 1)$, $d_{a,b} = k$. If some directed path in $T_Z - D'_{Z,s}$ contains $k+1$ nodes t such that X_t^Z are pairwise disjoint and have size 1 of goodness 0, then there exists a path of at least $k+1$ blocks, where none of them is a single edge, so G contains a topological minor isomorphic to the Robertson chain of length k, a contradiction. This builds the induction basis. Assume that $(a, a-b)$ is the smallest pair such that the claim does not hold. Suppose that P is a directed path in $T_Z - D'_{Z,s}$ such that it contains a set R of $d_{a,b}+1$ nodes t such that X_t^Z are pairwise disjoint and have size a of goodness b. Let S be a maximum set of nodes t in P such that X_t^Z are pairwise disjoint and for every node x in S either $|X_x^Z| < a$, or $|X_x^Z| = a$ but the goodness is less than b. By the induction hypothesis, $|S| \leq \sum_{i=1}^{a} \sum_{j=0}^{i-1} d_{i,j}$ if $b = a-1$, and $|S| \leq \sum_{i=1}^{a} \sum_{j=b+1}^{i-1} d_{i,j}$ otherwise. So there exists at least $k^2a^{2k} + k + 3$ nodes in R contained in a component of $T_Z - S$. Since S is a maximum set, there exist

$k^2a^{2k}+k+1$ nodes $t_1, t_2, ..., t_{k^2a^{2k}+k+1}$ in R such that for every $1 \leq i \leq k^2a^{2k}+k$, there does not exist a node t on the t_i-t_{i+1} path in T_Z such that either $|X_t^Z| < a$ or $|X_t^Z| = a$ but the goodness of t is greater than b. By Lemma 4.2.4, there exist desired edge-disjoint paths and jumps, and together with Lemma 4.1.1, G contains a topological minor isomorphic to the Robertson chain of length k, a contradiction. This proves the claim.

Consequently, every path in $T_Z - D'_{Z,s}$ contains at most $\sum_{i=1}^{s}\sum_{j=0}^{i-1} d_{i,j}$ nodes t such that X_t^Z are pairwise disjoint and of size at most s. Note that $(T, \mathcal{X})$ is h-strongly lean and of adhesion h, so for every node t of T with $|X_t| > h$ that has the parent, there exists a node t' of T such that $|X_{t'}| \leq h$ and $X_{t'} \subseteq X_t$. Then the theorem follows from taking $f(k,h) = \sum_{i=1}^{h}\sum_{j=0}^{i-1} d_{i,j} + 1$. ■

Theorem 4.2.6 *Let k, w be positive integers. Then there exists $f(k,w)$ such that every connected graph G of tree-width at most w that does not contain a topological minor isomorphic to the Robertson chain of length k has a strongly lean rooted tree decomposition such that for every $Z \subseteq V(G)$ and nonnegative integer s, the (Z,s)-depth is at most $f(k,w)$.*

Proof. This theorem follows from Theorem 4.2.5 by taking $h = w$. ■

CHAPTER V

WELL-BEHAVED SETS OF FRAMES

5.1 Well-behaved sets of frames

We say that $(S, \preceq)$ is a *well-quasi-ordered set* if $\preceq$ is a well-quasi-ordering on S. Note that if $(S_1, \preceq_1)$ and $(S_2, \preceq_2)$ are two well-quasi-ordered sets, then $S_1 \times S_2$ is well-quasi-ordered by $\preceq_3$, where $(s_1, s_2) \preceq_3 (s_1', s_2')$ if and only if $s_1 \preceq s_1'$ and $s_2 \preceq s_2'$. We call $(S_1 \times S_2, \preceq_3)$ the *well-quasi-ordered set obtained from* $(S_1, \preceq_1), (S_2, \preceq_2)$ *by Cartesian product*, and denote it by $(S_1 \times S_2, \preceq_1 \times \preceq_2)$. For every two sets A, B, we define $A \uplus B$ to be the union of A and a disjoint copy of B. Then $S_1 \uplus S_2$ is well-quasi-ordered by $\preceq_4$, where $s \preceq_4 s'$ if and only if either $s, s' \in S_1$ and $s \preceq_1 s'$, or $s, s' \in S_2$ and $s \preceq_2 s'$. We call $(S_1 \uplus S_2, \preceq_4)$ the *well-quasi-ordered set obtained from* $(S_1, \preceq_1), (S_2, \preceq_2)$ *by disjoint union.*

The following theorem was proved by Higman and gave another way to obtain another well-quasi-ordered set from a well-quasi-ordered set.

Theorem 5.1.1 ([19]) *Let* $(S, \preceq)$ *be a well-quasi-ordered set. For every finite sequences* $A = a_1, a_2, ..., a_n$ *and* $B = b_1, b_2, ..., b_m$ *of* S*, we say that* $A \preceq' B$ *if there exists* $1 \leq i_1 < i_2 < ... < i_n \leq m$ *such that* $a_j \preceq b_{i_j}$ *for every* $1 \leq j \leq n$*. Then the finite sequences of* S *are well-quasi-ordered by* $\preceq'$*.*

We call the new well-quasi-ordered set mentioned in Theorem 5.1.1 the *well-quasi-ordered set obtained from* S *by Higman's lemma.*

A *march* in a graph is either the empty set or a sequence of vertices of the graph without repeated entries. We say that (G, γ, σ) is a *rooted graph* if G is a graph, γ is a march in G, and σ is a sequence whose entries are 0 and 1 with the same length as γ. In this case, we say that γ is the *root march* and σ is the *type* of γ.

Let $(S, \preceq)$ be a well-quasi-ordered set. We say that $(G, \gamma, \sigma, \phi)$ is an $(S, \preceq)$-*labelled graph* if (G, γ, σ) is a rooted graph, and ϕ is a function from $V(G)$ to S. In this case, we call ϕ the *labelling function*. Given two $(S, \preceq)$-labelled graphs $(G_1, \gamma_1, \sigma_1, \phi_1)$ and $(G_2, \gamma_2, \sigma_2, \phi_2)$, we say that $(G_2, \gamma_2, \sigma_2, \phi_2)$ contains $(G_1, \gamma_1, \sigma_1, \phi_1)$ as an $(S, \preceq)$-*labelled topological minor* if the following hold.

- There exists a G_1-topological minor (π_V, π_E) in G_2.
- γ_1 and γ_2 contain the same number of entries, say m. And π_V maps the i-th entry of γ_1 to the i-th entry of γ_2 for every i.
- $\sigma_1 = \sigma_2$.
- For every $v \in V(G_1)$, $\phi_1(v) \preceq \phi_2(\pi_V(v))$.

Let G be a graph. A *location* $\mathcal{L}$ in G is a set of separations of G such that $A \subseteq B'$ for every distinct separations (A, B) and (A', B') in $\mathcal{L}$. The *order* of $\mathcal{L}$ is the maximum order of a separation in $\mathcal{L}$. We say that $\mathcal{L}$ is *ordered* if for every $(A, B) \in \mathcal{L}$, there is an associated ordering of the vertices in $V(A) \cap V(B)$ such that if $(A', B'), (A'', B'')$ are members in $\mathcal{L}$ with $V(A') \cap V(B') = V(A'') \cap V(B'')$, then the ordering associated with these two separations are the same.

We say that $\mathcal{L}$ is *rooted* if exactly one separation (A^*, B^*) in $\mathcal{L}$ is specified, and we call the specified separation (A^*, B^*) the *root* of $\mathcal{L}$. Let $\mathcal{L}$ be a rooted location with root (A^*, B^*) in a graph G. An *edge-extension* τ is a function that maps each $(A, B) \in \mathcal{L}$ to a subset of $V(A) \cap V(B)$ such that the following hold.

- Every vertex in $\tau(A, B)$ is incident with an edge with one end in $V(D) - V(C)$, and the edges in G with one end in $\tau(A, B)$ and one end in $V(D) - V(C)$ form a matching, denoted by $M_{\tau,A}$, where $(C, D) = (A, B)$ if $(A, B) \neq (A^*, B^*)$, and $(C, D) = (B^*, A^*)$ otherwise.

- For every vertex $v \in V(G)$, if $v \in \tau(A,B) \cap V(A^* \cap B^*)$ for some $(A,B) \in \mathcal{L}$, then $v \in \tau(A^*, B^*)$.

In this case, we say that $(\mathcal{L}, \tau)$ is an *extended location*. For each $(A,B) \in \mathcal{L}$ and each $v \in V(A) \cap V(B)$, we define the *partner* v' of v with respect to (A,B) and τ to be the other end of the edge in $M_{\tau,A}$ incident with v if $v \in \tau(A,B)$, and define $v' = v$ if $v \notin \tau(A,B)$. We say that $(\mathcal{L}, \tau)$ is *ordered* if $\mathcal{L}$ is ordered. Assume that $\mathcal{L}$ is ordered. For every $(A,B) \in \mathcal{L}$, define $G_{\tau,A}$ to be $G[V(A)] \cup M_{\tau,A}$ if $(A,B) \neq (A^*, B^*)$, and $G[V(B^*)] \cup M_{\tau,A^*}$ otherwise; define $\gamma_{\tau,A}$ to be the march whose entries are the partners of the vertices in $V(A) \cap V(B)$ with respect to (A,B) and τ such that the i-th entry of $\gamma_{\tau,A}$ is the partner of the i-th vertex in $V(A) \cap V(B)$ according to the ordering associated with (A,B), for every $1 \leq i \leq |V(A) \cap V(B)|$; define $\sigma_{\tau,A}$ to be the sequence of length $|V(A) \cap V(B)|$ such that the i-th entry is 1 if the i-th vertex of $V(A) \cap V(B)$ belongs $\tau(A,B)$, and 0 otherwise, for every $1 \leq i \leq |V(A) \cap V(B)|$. The *periphery* of $(\mathcal{L}, \tau)$ in G, denoted by $\mathcal{P}(\mathcal{L}, \tau)$, is the collection of rooted graphs $\{(G_{\tau,A}, \gamma_{\tau,A}, \sigma_{\tau,A}) : (A,B) \in \mathcal{L} - \{(A^*, B^*)\}\}$. If a function $\phi : V(G) \to S$ for some well-quasi-ordered set $(S, \preceq)$ is defined, then the *ϕ-periphery* of $(\mathcal{L}, \tau)$, denoted by $\mathcal{P}_\phi(\mathcal{L}, \tau)$, is the collection of $(S, \preceq)$-labelled graphs $\{(G_{\tau,A}, \gamma_{\tau,A}, \sigma_{\tau,A}, \phi|_{G_{\tau,A}}) : (G_{\tau,A}, \gamma_{\tau,A}, \sigma_{\tau,A}) \in \mathcal{P}(\mathcal{L}, \tau)\}$.

Let $(\mathcal{L}, \tau)$ be an extended location in a graph G with root (A^*, B^*). Let $\mathcal{L}' \subseteq \mathcal{L} - \{(A^*, B^*)\}$. The *condensation of* $(\mathcal{L}, \tau)$ *over* $\mathcal{L}'$ is the rooted graph, denoted by $\text{Con}(\mathcal{L}, \tau, \mathcal{L}')$, obtained from $(G_{\tau,A^*}, \gamma_{\tau,A^*}, \sigma_{\tau,A^*})$ by identifying the vertices of $V(A) - V(B)$ into a vertex v_A for each $(A,B) \in \mathcal{L}'$ with $V(A) - V(B) \neq \emptyset$, and adding an isolated vertex v_A for each $(A,B) \in \mathcal{L}'$ with $V(A) \subseteq V(B)$. We define $(\mathcal{L}, \tau)/\mathcal{L}'$ to be the location $(\mathcal{L} - \mathcal{L}') \cup \{(A', B') : (A,B) \in \mathcal{L}'\}$ in $\text{Con}(\mathcal{L}, \tau, \mathcal{L}')$, where each A', B' is the graph obtained from A, B, respectively, by contracting $V(C) - V(D)$ into a vertex v_C for each $(C,D) \in \mathcal{L}'$ with $V(C) - V(D) \neq \emptyset$ and adding a vertex v_C for each $(C,D) \in \mathcal{L}'$ with $V(C) \subseteq V(D)$.

Let $(\mathcal{L}, \tau)$ and $(\mathcal{L}', \tau')$ be extended locations in graphs G, G' with roots (A^*, B^*)

and (A', B'), respectively. Let $(S, \preceq)$ be a well-quasi-ordered set, and let ϕ, ϕ' be $(S, \preceq)$-labelling functions on G, G', respectively. Let $(S', \preceq')$ be a well-quasi-ordered set, and let $\psi : (\mathcal{L}, \tau) \to S'$ and $\psi' : (\mathcal{L}', \tau') \to S'$ be functions. We say that $(\mathcal{L}', \tau', \phi', \psi')$ *simulates* $(\mathcal{L}, \tau, \phi, \psi)$ if the following hold.

- There exists an injection $\iota : \mathcal{P}_\phi(\mathcal{L}, \tau) \to \mathcal{P}_{\phi'}(\mathcal{L}', \tau')$ such that $\iota(x)$ contains x as an $(S, \preceq)$-topological minor.

- There exists an injection ζ from $\mathcal{L} - \{(A^*, B^*)\}$ to $\mathcal{L}' - \{(A', B')\}$ such that $\zeta(A, B) = (C, D)$ if $\iota(G_{\tau,A}, \gamma_{\tau,A}, \sigma_{\tau,A}) = (G'_{\tau',C}, \gamma_{\tau',C}, \sigma_{\tau',C})$.

- $\psi(x) \preceq' \psi'(\zeta(x))$ for every $x \in \mathcal{L} - \{(A^*, B^*)\}$.

- Let I be the image of ζ. $(\text{Con}(\mathcal{L}', \tau', I)_{\tau',A'}, \gamma_{\tau',A'}, \sigma_{\tau',A'}, \phi'|_{V(\text{Con}(\mathcal{L}',\tau',I)_{\tau',A'})})$ contains $(\text{Con}(\mathcal{L}, \tau, \mathcal{L} - \{(A^*, B^*)\})_{\tau,A^*}, \gamma_{\tau,A^*}, \sigma_{\tau,A^*}, \phi|_{V(\text{Con}(\mathcal{L},\tau,\mathcal{L}-\{(A^*,B^*)\})_{\tau,A^*})})$ as an $(S, \preceq)$-labelled topological minor, realized by a pair of functions (π_V, π_E) such that for every $(A, B) \in \mathcal{L} - (A^*, B^*)$, let $(C, D) = \zeta(A, B)$ and the following hold.

  - $\pi_V(v_A) = v_C$.
  - $\pi_V(v)$ is the i-th vertex of $\zeta(A, B)$ if v is the i-th vertex in $A \cap B$ and $\sigma(i) = 0$.
  - If $V(A) - V(B) \neq \emptyset$, then $\pi_E(v_A x_i)$ contains the i-th vertex in $V(C) \cap V(D)$, where x_i is the i-th vertex in $A \cap B$.

Lemma 5.1.2 *Let $(\mathcal{L}, \tau)$ and $(\mathcal{L}', \tau')$ be extended locations in graphs G, G' with roots (A^*, B^*) and (A', B'), respectively. Let $(S, \preceq)$ be a well-quasi-ordered set, and let ϕ, ϕ' be $(S, \preceq)$-labelling functions on G, G', respectively. If there exist a well-quasi-ordered set $(S', \preceq')$ and functions $\psi : \mathcal{L} - \{(A^*, B^*)\} \to S'$, $\psi' : \mathcal{L}' - \{(A', B')\} \to S'$, such that $(\mathcal{L}', \tau', \phi', \psi')$ simulates $(\mathcal{L}, \tau, \phi, \psi)$, then $(G'_{\tau',A'}, \gamma_{\tau',A'}, \sigma_{\tau',A'}, \phi'|_{V(G'_{\tau',A'})})$ contains $(G_{\tau,A^*}, \gamma_{\tau,A^*}, \sigma_{\tau,A^*}, \phi|_{V(G_{\tau,A^*})})$ as an $(S, \preceq)$-topological minor.*

Proof. Let $(S', \preceq')$ be a well-quasi-ordered set, $\psi : \mathcal{L} - \{(A^*, B^*)\} \to S'$ and $\psi' : \mathcal{L}' - \{(A', B')\} \to S'$ functions, such that $(\mathcal{L}', \tau', \phi', \psi')$ simulates $(\mathcal{L}, \tau, \phi, \psi)$. And we define $\iota, \zeta, I, (\pi_V, \pi_E)$ as mentioned in the definition of the simulation relation. For every $(A, B) \in \mathcal{L} - \{(A^*, B^*)\}$, we define (π_V^A, π_E^A) to be a pair of function that realizes that $\iota((G_{\tau,A}, \gamma_{\tau,A}, \sigma_{\tau,A}, \phi|_{V(G_{\tau,A})}))$ contains $(G_{\tau,A}, \gamma_{\tau,A}, \sigma_{\tau,A}, \phi|_{V(G_{\tau,A})})$ as an $(S, \preceq)$-topological minor. We shall define a pair of functions (π'_V, π'_E) to realize that $(G'_{\tau',A'}, \gamma_{\tau',A'}, \sigma_{\tau',A'}, \phi'|_{V(G'_{\tau',A'})})$ contains $(G_{\tau,A^*}, \gamma_{\tau,A^*}, \sigma_{\tau,A^*}, \phi|_{V(G_{\tau,A^*})})$ as an $(S, \preceq)$-topological minor.

First, we define π'_V. Let $v \in V(G_{\tau,A^*})$. If $v \in V(A) - V(B)$ for some $(A, B) \in \mathcal{L} - \{(A^*, B^*)\}$, then define $\pi'_V(v) = \pi_V^A(v)$. If $v \in V(G_{\tau,A^*}) - \bigcup_{(A,B)\in\mathcal{L}-\{(A^*,B^*)\}} V(A)$, then define $\pi'_V(v) = \pi_V(v)$. If $v \in V(A) \cap V(B)$ for some $(A, B) \in \mathcal{L}$ such that v is the ℓ-th vertex of $V(A) \cap V(B)$ and the ℓ-th entry of $\sigma_{\tau,A}$ is 1, then pick such an (A, B) and define $\pi'_V(v) = \pi_V^A(v)$. Otherwise, define $\pi'_V(v) = \pi_V(v)$.

Second, we define π'_E. Let $e \in E(G_{\tau,A^*})$, and denote the ends of e by x, y. For each $v \in \{x, y\}$, if there exist $(A, B) \in \mathcal{L}$ and $(C, D) \in \mathcal{L}'$ such that $\pi'_V(v) \in V(C)$ and $\iota((G_{\tau,A}, \gamma_{\tau,A}, \sigma_{\tau,A}, \phi|_{V(G_{\tau,A})})) = (G'_{\tau',C}, \gamma_{\tau',C}, \sigma_{\tau',C}, \phi'|_{V(G'_{\tau',C})})$, then we define (A_v, B_v) to be such an (A, B). Observe that for each $v \in \{x, y\}$, if A_v is not defined, then $\pi'_V(v) = \pi_V(v)$. If both A_x, A_y are defined and $e \in E(G_{\tau,A_x}) \cup E(G_{\tau,A_y})$, then define $\pi'_E(e)$ to be the path from $\pi'_V(x)$ to $\pi'_V(y)$ contained in $\pi_E^{A_x}(e) \cup \pi_E^{A_y}(e)$. If both A_x, A_y are defined but $e \notin E(G_{\tau,A_x}) \cup E(G_{\tau,A_y})$, then $x \in V(A_x) - V(A_y)$ and $y \in V(A_y) - V(A_x)$, and $v \in V(A_v) \cap V(B_v) - \tau(A_v, B_v)$ for each $v \in \{x, y\}$, so $\pi'_V(x) = \pi_V(x)$ and $\pi'_V(y) = \pi_V(y)$, and we define $\pi'_E(e) = \pi_E(e)$. If exactly one of A_x, A_y is defined, say A_x, and $e \in E(G_{\tau,A_x})$, then define $\pi'_E(e)$ to be $\pi_E^{A_x}(e) \cup \pi_E(e)$. If either none of A_x, A_y is defined, or exactly one of A_x, A_y is defined, say A_x, with $e \notin E(G_{\tau,A_x})$, then $\pi'_V(x) = \pi_V(x)$ and $\pi'_V(y) = \pi_V(y)$, and we define $\pi'_E(e)$ to be $\pi_E(e)$.

Clearly, π'_E maps different edges to internally disjoint paths. Therefore, (π'_V, π'_E)

realizes that $(G'_{\tau',A'}, \gamma_{\tau',A'}, \sigma_{\tau',A'}, \phi'|_{V(G'_{\tau',A'})})$ contains $(G_{\tau,A^*}, \gamma_{\tau,A^*}, \sigma_{\tau,A^*}, \phi|_{V(G_{\tau,A^*})})$ as an $(S, \preceq)$-topological minor. ■

We say that $(H, \mu, \mathcal{Q})$ is a *frame* if H is a graph, μ is a collection of marches in H, and $\mathcal{Q}$ is a set of properties of locations. We say that an ordered location $\mathcal{L}$ in a graph G *fits* a frame $(H, \mu, \mathcal{Q})$ if the following hold.

- $H = G[V(\bigcap_{(A,B)\in\mathcal{L}} B)]$, and
- for every $(A, B) \in \mathcal{L}$, the march whose entries are the vertices in $V(A) \cap V(B)$ ordered by the ordering associated with (A, B) is in μ, and
- $\mathcal{L}$ satisfies all properties in $\mathcal{Q}$.

The *order* of $(H, \mu, \mathcal{Q})$ is the maximum length of a march in μ.

We say that a collection of frames $\mathcal{F}$ is *well-behaved* if the following statement holds. For every well-quasi-ordered sets $(S, \preceq), (S', \preceq')$, and for every infinite sequence of graphs $G_1, G_2, ...$, functions $\phi_i : V(G_i) \to S$, ordered extended locations $(\mathcal{L}_i, \tau_i)$ in G_i with root (A_i, B_i), and functions $\psi_i : \mathcal{L} - \{(A_i, B_i)\} \to S'$ for each $i \geq 1$, such that $\mathcal{L}_i$ fits a frame $(H_i, \mu_i, \mathcal{Q}_i)$ in $\mathcal{F}$, and if $\bigcup_{i\geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i)$ is well-quasi-ordered by the $(S, \preceq)$-labelled topological minor relation, then there exist $j' > j \geq 1$ such that $(\mathcal{L}_{j'}, \tau_{j'}, \phi_{j'}, \psi_{j'})$ simulates $(\mathcal{L}_j, \tau_j, \phi_j, \psi_j)$.

Lemma 5.1.3 *Let $(S, \preceq), (S', \preceq')$ be well-quasi-ordered sets, and for $i = 1, 2$, let G_i be a graph and $\phi_i : V(G_i) \to S$ be a function. Let $(S'', \preceq'')$ be the well-quasi-ordered set obtained from $(S', \preceq')$ by Higman's lemma. For $i = 1, 2$, let $(\mathcal{L}_i, \tau)$ be an extended location in G_i with root (A_i^*, B_i^*), and let $\psi_i : \mathcal{L}_i - \{(A_i^*, B_i^*)\} \to S'$ be a function. For $i = 1, 2$, let $(\mathcal{L}'_i, \tau'_i)$ be the extended location with root (A_i^*, B_i^*) and $\psi'_i : \mathcal{L}' - \{(A^*, B^*)\} \to S''$ a function obtained from $(\mathcal{L}_i, \tau_i)$ by repeated doing the following operations whenever there exist two separations $(A_1, B_1), (A_2, B_2) \in \mathcal{L}_i - \{(A_i^*, B_i^*)\}$ with $V(A_1) \subseteq V(B_1), V(A_2) \subseteq V(B_2)$ and $V(A_1) \cap V(B_1) = V(A_2) \cap V(B_2)$.*

- *Removing* $(A_1, B_1), (A_2, B_2)$ *from* $\mathcal{L}_i$ *and adding* $(A_1 \cup A_2, B_1 \cap B_2)$ *into* $\mathcal{L}_i$*.*

- *Define* $\tau_i(A_1 \cup A_2, B_1 \cap B_2) = \tau_i(A_1, B_1) \cup \tau(A_2, B_2)$*.*

- *The order associated with* $(A_1 \cup A_2, B_1 \cap B_2)$ *is the same as the order of* (A_1, B_1)*.*

- $\psi_i(A_1 \cup A_2, B_1 \cap B_2)$ *be the sequence obtained from* $\psi_i(A_1, B_1)$ *by concatenating* $\psi_i(A_2, B_2)$*.*

If $(\mathcal{L}'_2, \tau'_2, \phi_2, \psi'_2)$ *simulates* $(\mathcal{L}'_1, \tau'_1, \phi_1, \psi'_1)$*, then* $(\mathcal{L}_2, \tau_2, \phi_2, \psi_2)$ *simulates* $(\mathcal{L}_1.\tau_1, \phi_1, \psi_1)$*.*

Proof. Let $\iota', \eta', (\pi'_V, \pi'_E)$ be the functions $\iota, \eta, (\pi_V, \pi_E)$ mentioned in the definition of the simulation relation that witness that $(\mathcal{L}'_2, \tau'_2, \phi_2, \psi'_2)$ simulates $(\mathcal{L}'_1, \tau'_1, \phi_1, \psi'_1)$. Then it is easy to extend those functions to $\iota, \eta, (\pi_V, \pi_E)$ to witness that $(\mathcal{L}_2, \tau_2, \phi_2, \psi_2)$ simulates $(\mathcal{L}_1.\tau_1, \phi_1, \psi_1)$ by splitting off those separations in $\mathcal{L}'_1, \mathcal{L}'_2$ merged from the separations in $\mathcal{L}_1, \mathcal{L}_2$. ■

Let $(H, \mu, \mathcal{Q})$ be a frame, and let $Z \subseteq V(H)$. Define $\mu - Z$ to be the set of marches in $H - Z$ such that for every $M \in \mu - Z$, there exists $M' \in \mu$ such that M is the subsequence of M' consisting of the entries not in Z.

Let $(\mathcal{L}, \tau)$ be an extended location in a graph G with root (A^*, B^*), and let $Z \subseteq \bigcap_{(A,B)\in\mathcal{L}} V(B)$. Then we define $\mathcal{L} - Z$ to be the rooted location $\{(A - Z, B - Z) : (A, B) \in \mathcal{L}\}$ in $G - Z$ with root $(A^* - Z, B^* - Z)$, and define $\tau - Z$ to be a function from $\mathcal{L} - Z$ such that $(\tau - Z)(A - Z, B - Z) = \tau(A, B) - (Z \cup \{v : N(v) \subseteq V(A) \cup Z\})$ for every $(A, B) \in \mathcal{L} - \{(A^*, B^*)\}$ and $(\tau - Z)(A^* - Z, B^* - Z) = \tau(A^*, B^*) - Z$. If $\mathcal{L}$ is ordered, then the ordering associated with each $(A - Z, B - Z) \in \mathcal{L} - Z$ is the same as the ordering associated with $(A, B) \in \mathcal{L}$, but removing the terms in $V(A) \cap V(B) \cap Z$, for every $(A, B) \in \mathcal{L}$.

Lemma 5.1.4 *Let* $\mathcal{F}$ *be a well-behaved collection of frames, and let* k *be a positive integer. Let* $\mathcal{F}_k$ *be the collection of frames such that for every frame* $(H, \mu, \mathcal{Q})$ *in* $\mathcal{F}_k$*,* *there exists $Z \subseteq V(H)$ with $|Z| \leq k$ such that $(H - Z, \mu - Z, \mathcal{Q}') \in \mathcal{F}$ for some $\mathcal{Q}'$, and $(\mathcal{L} - Z, \tau - Z)$ fits $(H - Z, \mu - Z, \mathcal{Q}')$ for every ordered location $(\mathcal{L}, \tau)$ that fits $(H, \mu, \mathcal{Q})$. If there exists an integer o such that the order of $\mathcal{F}$ is at most o, then $\mathcal{F}_k$ is well-behaved.*

Proof. Let $(S, \preceq), (S', \preceq')$ be well-quasi-ordered sets. Let $G_1, G_2, ...$ be an infinite sequence of graphs. And for every $i \geq 1$, let $(\mathcal{L}_i, \tau_i)$ be an ordered extended location in G_i with root (A_i, B_i), and let $\phi_i : V(G_i) \to S$, $\psi_i : \mathcal{L}_i - \{(A_i, B_i)\} \to S'$. Assume that $\mathcal{L}_i$ fits a frame $(H_i, \mu_i, \mathcal{Q}_i)$ in $\mathcal{F}_k$, and $\bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i)$ is well-quasi-ordered by the $(S, \preceq)$-labelled topological minor relation. As k is finite, we may assume that $(H_i, \mu_i, \mathcal{Q}_i) \in \mathcal{F}_k - \mathcal{F}_{k-1}$, where $\mathcal{F}_0 = \mathcal{F}$. We may assume that no two distinct separations $(X, Y), (X', Y') \in \mathcal{L}_i - \{(A_i, B_i)\}$ for some i such that $V(X) \subseteq V(Y)$, $V(X') \subseteq V(Y')$ and $V(X) \cap V(Y) = V(X') \cap V(Y')$ by Lemma 5.1.3.

For every $i \geq 1$, let $Z_i = \{z_{i,1}, z_{i,2}, ..., z_{i,k}\} \subseteq V(H_i)$ such that $(H_i - Z_i, \mu_i - Z_i, \mathcal{Q}_i') \in \mathcal{F}$ for some $\mathcal{Q}_i'$. As k is finite, we may assume that $G_{j'}[Z_{j'}]$ contains $G_j[Z_j]$ as a subgraph for every $1 \leq j < j'$, and for every $1 \leq \ell \leq k$, the sequence $(\phi_1(z_{1,\ell}), \phi_2(z_{2,\ell}), ...)$ is non-decreasing. Define $(S_1, \preceq_1)$ to be the well-quasi-ordered set $(S \times \mathbb{Z}^k, \preceq \times \leq^k)$. For every $1 \leq \ell \leq k$ and $i \geq 1$ and for every $v \in V(G_i)$, let $m_{i,\ell}(v)$ be the number of edges with end v and $z_{i,\ell}$ in G_i. For every $i \geq 1$, define $\phi_i' : V(G_i) - Z_i \to S_1$ by setting $\phi_i'(v) = (\phi_i(v), m_{i,1}(v), m_{i,2}(v), ..., m_{i,k}(v))$ for every $v \in \bigcap_{(A,B) \in \mathcal{L}_i - \{(A_i^*, B_i^*)\}} V(B) - Z_i$, and $\phi_i'(v) = (\phi_i(v), 0, 0, ..., 0)$ otherwise.

If $A - Z_i = \emptyset$ for some $(A, B) \in \mathcal{L}_i$, then $V(A) \subseteq Z_i$. Recall that we assume that no two distinct separations $(X, Y), (X', Y') \in \mathcal{L}_i - \{(A_i, B_i)\}$ for some i such that $V(X) \subseteq V(Y)$, $V(X') \subseteq V(Y')$ and $V(X) \cap V(Y) = V(X') \cap V(Y')$. So there are at most 2^k separations $(A, B) \in \mathcal{L}_i$ with $A - Z_i = \emptyset$ for every $i \geq 1$. Let $(S_1, \preceq_1)$ be the well-quasi-ordered set obtained from $(S', \preceq')$ and $(\{*\}, =)$ by disjoint union. Let S^* be $\bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i)$, and let $\preceq^*$ be the $(S, \preceq)$-topological minor relation. Let $(S_2, \preceq_2)$ be the well-quasi-ordered set obtained from $(S^*, \preceq^*)$ and $(\{*\}, =)$ by disjoint union. Let W' be the set of marches whose entries are in $\{1,2,...,k\}$, and let $(W,=)$ be the well-quasi-ordered set obtained from $(W',=)$ and $(\{*\},=)$ by disjoint union. Define $(S_3,\preceq_3)$ to be the well-quasi-ordered set $({S_1}^{2^k} \times {S_2}^{2^k} \times \{0,1,2,...,k\}^{o+k} \times (\mathbb{N}\cup\{0\})^{o+k} \times W^{2^k}, {\preceq_1}^{2^k} \times {\preceq_2}^{2^k} \times =^{o+k} \times \leq^{o+k} \times =^{2^k})$. For every $i \geq 1$, define $\psi'_i : (\mathcal{L}_i - Z_i, \tau_i - Z_i) - \{(A_i - Z_i, B_i - Z_i)\} \to S_3$ to be the function such that for every $(A,B) \in \mathcal{L}_i - \{(A_i^*, B_i^*)\}$ the following hold.

- If $A - Z_i \neq \emptyset$, then let $\psi'_i(A - Z_i, B - Z_i) = (\Psi_i(A,B), G_{A,B}, T_A, D_A, *, ..., *)$ such that the following hold
  - $\Psi_i(A,B) = (\psi_i(A,B), ..., \psi_i(A,B)) \in S_1^{2^k}$.
  - $G_{A,B} = (G_{i_{\tau_i,A}}, \gamma_{i_{\tau_i,A}}, \sigma_{i_{\tau_i,A}}, \phi_i|_{G_{i_{\tau_i,A}}}), ..., (G_{i_{\tau_i,A}}, \gamma_{i_{\tau_i,A}}, \sigma_{i_{\tau_i,A}}, \phi_i|_{G_{i_{\tau_i,A}}})) \in S_2^{2^k}$.
  - $T_A = (t_1, t_2, ..., t_{o+k})$, where for every $1 \leq \ell \leq o+k$, t_ℓ is 0 if no member of Z_i is the ℓ-th vertex of $V(A) \cap V(B)$, and t_ℓ is the index ℓ' such that $z_{i,\ell'}$ is the ℓ-th vertex of $V(A) \cap V(B)$ otherwise.
  - $D_A = (d_1, d_2, ..., d_{o+k})$, where for every $1 \leq \ell \leq o+k$, d_ℓ is the number of edges between the ℓ-th vertex in $V(A) \cap V(B)$ and $V(A) - V(B)$.
- If $A - Z_i = \emptyset$, then $(A - Z_i, B - Z_i) = (\emptyset, V(G_i) - Z_i)$, and we define $\psi'_i(A - Z_i, B - Z_i) = (x_1, x_2, ..., x_{2^k}, y_1, y_2, ..., y_{2^k}, 0, ..., 0, r_1, r_2, ..., r_{2^k})$, where for each $1 \leq \ell \leq 2^k$, x_ℓ, y_ℓ, r_ℓ are defined as the following.
  - Let $p_A = \sum_{1 \leq s \leq k} a_s 2^{s-1}$, where $a_s = 1$ if $z_{i,s} \in V(A) \cap V(B)$, and $a_s = 0$ otherwise, for $1 \leq s \leq k$.
  - If there exists (A,B) with $p_A = \ell$, then $x_\ell = \psi_i(A,B)$, $y_\ell = (G_{i_{\tau_i,A}}, \gamma_{i_{\tau_i,A}}, \sigma_{i_{\tau_i,A}}, \phi_i|_{V(G_{i_{\tau_i,A}})})$, and r_ℓ is the march of length $|V(A) \cap V(B)|$ such that the s-th entry is the number s' such that $z_{i,s'}$ is the s-th vertex in $V(A) \cap V(B)$ for $1 \leq s \leq |V(A) \cap V(B)|$.
  - If there does not exist (A,B) with $p_A = \ell$, then $x_\ell = y_\ell = r_\ell = *$.

By assumption, for every $i \geq 1$, there exist a frame $(H - Z, \mu - Z, \mathcal{Q}'_i) \in \mathcal{F}$ that is fitted by $(\mathcal{L}_i - Z_i, \tau_i - Z_i)$. Since $\mathcal{F}$ is well-behaved, there exist $j' > j \geq 1$ such that $(\mathcal{L}_{j'} - Z_{j'}, \tau_{j'} - Z_{j'}, \phi'_{j'}, \psi'_{j'})$ simulates $(\mathcal{L}_j - Z_j, \tau_j - Z_j, \phi'_j, \psi'_j)$. Let $\iota, \zeta, (\pi_V, \pi_E)$ be the functions mentioned in the definition of simulation relation that realizes the above simulation. According to the definition of ψ'_j and $\psi'_{j'}$, we know that if $(A, B) \in \mathcal{L}_j - \{(A^*_j, B^*_j)\}$ such that $V(A) \subseteq Z_j$, then there exists a separation $(A', B') \in \mathcal{L}_{j'} - \{(A^*_{j'}, B^*_{j'})\}$ with $V(A') \subseteq Z_{j'}$ such that $|V(A) \cap V(B)| = |V(A') \cap V(B')|$, and for every $1 \leq \ell \leq k$, the ℓ-th vertex in $V(A) \cap V(B)$ is $z_{j,s}$ for some s if and only of the ℓ-th vertex in $V(A') \cap V(B')$ is $z_{j',s}$. Define $\zeta' : \mathcal{L}_j - \{(A^*_j, B^*_j)\} \to \mathcal{L}_{j'} - \{(A^*_{j'}, B^*_{j'})\}$ such that $\zeta'(A, B) = \zeta(A - Z_j, B - Z_j)$ if $A - Z_i \neq \emptyset$, and $\zeta'(A, B) = (A', B')$ if $A - Z_i = \emptyset$, where (A', B') is mentioned above. Define $\iota' : \mathcal{P}_{\phi_j}(\mathcal{L}_j, \tau_j) \to \mathcal{P}_{\phi_{j'}}(\mathcal{L}_{j'}, \tau_{j'})$ such that it is consistent with ζ'. Let I' be the image of ζ'. According to the definition of ϕ'_j and $\phi'_{j'}$, and the definition of d_ℓ in the definition of ψ'_j and $\psi'_{j'}$, we know that if for every ℓ and for every $v \in V(\text{Con}(L_j, \tau_j, I') - Z_j)$ such that $z_{j,\ell}$ is adjacent to v in $\text{Con}(\mathcal{L}_j, \tau_j, I')$, then $\pi_V(v) \in N(z_{j',\ell})$ and the number of edges between $z_{j,\ell}$ and v is at most the number of edges between $z_{j',\ell}$ and $\pi_V(v)$. Finally, define $\pi'_V : V(\text{Con}(\mathcal{L}, \tau, \mathcal{L} - \{(A^*, B^*)\})_{\tau, A^*}) \to V(\text{Con}(\mathcal{L}', \tau', I)_{\tau', A'})$ to be the function such that the following hold.

- $\pi'_V(v_A) = v_{A'}$, for every $(A, B) \in \mathcal{L}_j$ with $V(A) \subseteq V(B) \cap Z_j$.
- $\pi'_V(z_{j,\ell}) = z_{j',\ell}$ for every $1 \leq \ell \leq k$.
- $\pi'_V(v) = \pi_V(v)$ otherwise.

Define π'_E to be the function with domain $E(\text{Con}(\mathcal{L}, \tau, \mathcal{L} - \{(A^*, B^*)\})_{\tau, A^*})$ such that the following hold.

- $\pi'_E(e) = \pi_E(e)$ if e is not incident with any member of Z_j.
- $\pi'_E(e)$ is an edge with ends $z_{j',\ell}$ and $\pi'_V(v)$, if the ends of e are $z_{j,\ell}$ and v.

- $\pi'_E(e) \neq \pi'_E(e')$ if e and e' are two distinct edges incident with some members of Z_j with the same ends.

Clearly, functions $\iota', \zeta', (\pi'_V, \pi'_E)$ are functions that realize that $(\mathcal{L}_{j'}, \tau_{j'})$ simulates $(\mathcal{L}_j, \tau_j)$. ■

We say that the *center-essential property* is the property of rooted locations such that if a rooted location $\mathcal{L}$ with root (A^*, B^*) satisfies it, then v is adjacent to a vertex in $V(B) - V(A)$, for every $(A, B) \in \mathcal{L} - \{(A^*, B^*)\}$ and every $v \in V(A) \cap V(B)$, and v is adjacent to a vertex in $V(A^*) - V(B^*)$ for every $v \in V(A^*) \cap V(B^*)$. A frame $(H, \mu, \mathcal{Q})$ is *center-essential* if $\mathcal{Q}$ includes the center-essential property.

Lemma 5.1.5 *Let $\mathcal{Q}$ be the set of property that only includes the center-essential property. Let k be an nonnegative integer, and let $\mathcal{F}$ be the set of center-essential frames $(H, \mu, \mathcal{Q})$ with $|V(H)| \leq k$ and $\mu \subseteq 2^{V(H)}$. Then $\mathcal{F}$ is well-behaved.*

Proof. First assume that $k = 0$. So every location $\mathcal{L}$ that fits a frame in $\mathcal{F}$ has order 0. Therefore, the separation in $\mathcal{L}$ can be encoded as a march with the entries in its periphery. Then this lemma follows from Theorem 5.1.1. Note that $\mathcal{Q}$ is satisfied by every separation of order 0. Hence, the case for general k immediately follows from Lemma 5.1.4. ■

Let $\mathcal{L}$ be a rooted location in a graph G with root (A^*, B^*). A *tree refinement* of a location $\mathcal{L}$ is a rooted tree decomposition $(T, \mathcal{X})$ of G together with a set of marches $\{m_e : e \in E(T)\}$ such that the following hold.

- For every $(A, B) \in \mathcal{L}$, there exist a leaf t_A in T and a neighbor t'_A of t_A in T such that $V(A) = X_{t_A}$ and $V(A) \cap V(B) = X_{t_A} \cap X_{t'_A}$.
- t_{A^*} is the root of T.
- For every edge $e = xy$ of T, the entries of m_e are the vertices in $X_x \cap X_y$.

Note that for every edge $e = pc \in E(T)$, where p is the parent of c, we write T_p, T_c as the component of $T - e$ containing p, c, respectively, and there exists a separation (A_e, B_e) of G such that $V(A_e) = \bigcup_{t' \in V(T_c)} X_{t'}$ and $V(B_e) = \bigcup_{t' \in V(T_p)} X_{t'}$. So for every $t \in V(T) - \{t_A : (A, B) \in \mathcal{L}\}$, there exists an ordered rooted location $\mathcal{L}^t = \{(A_e, B_e), (B_{p_t t}, A_{p_t t}) : e$ is incident with t but not incident with $p_t\}$ with root $(B_{p_t t}, A_{p_t t})$, where p_t is the parent of t, and each (A_e, B_e) is associated with an ordering given by m_e. Let $\mathcal{F}$ be a collection of frames. We say that a tree refinement $((T, \mathcal{X}), \{m_e : e \in E(T)\})$ is *over* $\mathcal{F}$ if for every $t \in V(T) - \{t_A : (A, B) \in \mathcal{L}\}$, there exists a frame $(H, \mu, \mathcal{Q}) \in \mathcal{F}$ such that $\mathcal{L}^t$ fits $(H, \mu, \mathcal{Q})$.

The *height* of a rooted tree T is the maximum number of vertices of a path from the root to a leaf. And the *height* of a tree refinement $((T, \mathcal{X}), \{m_e : e \in E(T)\})$ is the height of T.

Lemma 5.1.6 *Let $\mathcal{F}$ be a well-behaved family of center-essential frames, and let k be a positive integer. Let $\mathcal{F}'$ be a family of frames such that every extended location that fits a frame in $\mathcal{F}'$ has a tree refinement $(T, \mathcal{X})$ over $\mathcal{F}$ of height at most k. Then $\mathcal{F}'$ is well-behaved.*

Proof. We shall prove that $\mathcal{F}'$ is well-behaved by induction on k. Note that $\mathcal{F}' = \mathcal{F}$ when $k \leq 2$, so the base case holds. We assume that this lemma holds for every smaller k. Let $(S, \preceq), (S', \preceq')$ be well-quasi-ordered sets. Let $G_1, G_2, ...$ be an infinite sequence of graphs. And for every $i \geq 1$, let $\phi_i : V(G_i) \rightarrow S$ and let $(\mathcal{L}_i, \tau_i)$ be an ordered extended location in G_i with root (A_i, B_i). For every $i \geq 1$, let $\psi_i : \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i) \rightarrow S'$ be a function. Assume that $\mathcal{L}_i$ fits a center-essential frame $(H_i, \mu_i, \mathcal{Q}_i)$ in $\mathcal{F}'$, and $\bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i)$ is well-quasi-ordered by the $(S, \preceq)$-labelled topological minor relation.

For $i \geq 1$, let $((T_i, \mathcal{X}_i), \{m_e : e \in E(T_i)\})$ be a tree refinement of $\mathcal{L}_i$ over $\mathcal{F}$ of height at most k. Note that the root of T_i is t_{A_i}. For every $i \geq 1$ and a non-leaf node t in T_i, we define T^t to be the subtree of T_i rooted at t, and define τ^t to be the function with domain $\mathcal{L}^t$ such that $\tau^t(A,B) = V(A) \cap V(B) \cap \bigcup_{(A',B') \in \mathcal{L}_i, t_{A'} \in V(T^t)} \tau_i(A',B')$ for every $(A,B) \in \mathcal{L}^t$. Note that $(\mathcal{L}^t, \tau^t)$ is an extended ordered rooted location since $\mathcal{L}^t$ satisfies the center-essential property.

For every $i \geq 1$, let r_i be the neighbor of t_{A_i}, and for every child c of r_i, let $(T_i', \mathcal{X}_i')$ be obtained from $(T_i, \mathcal{X}_i)$ by contracting $T^c - \{t_A : (A,B) \in \mathcal{L}_i\}$ into a new node c^*. Note that $\mathcal{L}^{c^*}$ fits a frame in $\mathcal{F}_{k-1}$ for every such c^*. By the induction hypothesis, $\bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}^{r_i}, \tau^{r_i})$ is well-quasi-ordered by the $(S, \preceq)$-labelled topological minor relation. Furthermore, if distinct $x, y \in \bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}^{r_i}, \tau^{r_i})$ such that $x \leq y$ under the $(S, \preceq)$-topological minor relation, then there exist functions $\iota_{x,y}, \zeta_{x,y}$ as the functions ι, ζ mentioned in the definition of the simulation relation respect to x, y. As $\mathcal{F}$ is well-behaved, there exist $j' > j \geq 1$, such that $(\mathcal{L}^{r_{j'}}, \tau^{r_{j'}}, \phi_{j'}, \emptyset)$ simulates $(\mathcal{L}^{r_j}, \tau^{r_j}, \phi_j, \emptyset)$ with the functions $\iota' : \mathcal{P}_{\phi_j}(\mathcal{L}^{r_j}, \tau^{r_j}) \to \mathcal{P}_{\phi_{j'}}(\mathcal{L}^{r_{j'}}, \tau^{r_{j'}})$ and $\zeta' : \mathcal{L}^{r_j} - \{(A_j, B_j)\} \to \mathcal{L}^{r_{j'}} - \{(A_{j'}, B_{j'})\}$ as the functions ι, ζ mentioned in the definition of the simulation relation. By Lemma 5.1.2, it is sufficient to define functions $\iota : \mathcal{P}_{\phi_j}(\mathcal{L}_j, \tau_j) \to \mathcal{P}_{\phi_{j'}}(\mathcal{L}_{j'}, \tau_{j'})$ and $\zeta : \mathcal{L}_j - \{(A_j, B_j)\} \to \mathcal{L}_{j'} - \{(A_{j'}, B_{j'})\}$ as in the definition of the simulation relation. For every $w \in \mathcal{L}_j - \{(A_j, B_j)\}$, let t_w be child of r_j such that $w \in \mathcal{L}^{t_w^*}$, and let x_w be the graph in $\mathcal{P}_{\phi_j}(\mathcal{L}^{r_j}, \tau^{r_j})$ corresponding to t_w. Then we define $\zeta(w) = \zeta_{x_w, \zeta'(x_w)}(w)$ for every $w \in \mathcal{L}_j - \{(A_j, B_j)\}$. And we define ι similarly. This completes the proof. ∎

Let $(H, \mu, \mathcal{Q})$ be a frame, and let $((T, \mathcal{X}), \{m_e : e \in E(T)\})$ be a tree refinement of an ordered rooted location $\mathcal{L}$ that fits $(H, \mu, \mathcal{Q})$. Let $T' = T - \{t_A : (A,B) \in \mathcal{L}\}$, and let $\mathcal{X}' = \{X_t : t \in V(T')\}$. For every $Z \subseteq V(H)$ and nonnegative integer s, we define the (Z,s)-*depth of* $((T, \mathcal{X}), \{m_e : e \in E(T)\})$ to be the (Z,s)-depth of $(T', \mathcal{X}')$. The *adhesion* of $((T, \mathcal{X}), \{m_e : e \in E(T)\})$ is the adhesion of $(T', \mathcal{X}')$. We say that $((T, \mathcal{X}), \{m_e : e \in E(T)\})$ is *unimpeded* if for every two non-root nodes a, d of T such that a is an ancestor of d and $|X_{p_a} \cap X_a| = |X_{p_d} \cap X_d|$, where p_a and p_d is the parent of a, d, respectively, there exist $|X_{p_a} \cap X_a|$ disjoint paths from $X_{p_a} \cap X_a$ to $X_{p_d} \cap X_d$

such that the ends of the i-th path are the i-th entry of $m_{p_a a}$ and the i-th entry of $m_{p_d d}$, for every $1 \le i \le |X_{p_a} \cap X_a|$.

Let G be an infinite graph and $I \subseteq V(G)$. We say that I is *rich* in G if no infinite subset of I is an independent set. Let T be a rooted tree, $h \ge 0$ an integer, and $\phi : E(T) \to \{1, 2, ..., h\}$ a function. For $v, w \in V(T)$, where v is not the root of T, we say that v *precedes* w *(respective to* ϕ*)* if v is an ancestor of w and $\phi(e_v) = \phi(e_w)$, where e_v and e_w is the edge of T incident with v, w and its parent, respectively, and $\phi(e) \ge \phi(e_v)$ for every edge e on the v-w path in T.

Theorem 5.1.7 ([37]) *Let $T_1, T_2, ...$ be a countable sequence of disjoint rooted trees, and let h be a positive integer. For each $i \ge 1$, let $\phi_i : E(T_i) \to \{0, 1, ..., h\}$ be some function. Let M be an infinite graph with $V(M) = V(T_1 \cup T_2 \cup ...)$ such that for $i' > i \ge 1$, if $u \in V(T_i)$ is adjacent to $w \in V(T_{i'})$ in M, and $v \in V(T_{i'})$ precedes w with respect to $\phi_{i'}$, then u is adjacent to v in M. If the set of the roots of $T_1, T_2, ...$ is an independent set of M, then there is an infinite independent set X of M such that $|X \cap V(T_i)| \le 1$ for each $i \ge 1$ and such that the set of the children of the members of X is rich in M.*

The following theorem is the main theorem in this section.

Theorem 5.1.8 *Let $\mathcal{F}$ be a well-behaved family of center-essential frames. Let d, h be integers. Let $\mathcal{F}'$ be the family of center-essential frames such that every location that fits a frame $(H, \mu, \mathcal{Q})$ in $\mathcal{F}'$ has an unimpeded tree refinement over $\mathcal{F}$ of adhesion at most h and of (Z, s)-depth at most d, for every $Z \subseteq V(H)$ and $0 \le s \le h$. Then $\mathcal{F}'$ is well-behaved.*

Proof. For every $0 \le k \le h$, define $\mathcal{F}_k$ to be the family of center-essential frames $(H, \mu, \mathcal{Q})$ such that every location that fits $(H, \mu, \mathcal{Q})$ has a tree refinement $((T, \mathcal{X}), \{m_e : e \in E(T)\})$ over $\mathcal{F}$ of adhesion at most h and of (Z, s)-depth at most d for every

$Z \subseteq V(H)$ and $0 \leq s \leq h$, and there exists $X \subseteq V(H)$ with $|X| \geq k$ such that $X \subseteq X_t$ for every $t \in V(T) - \{t_A : (A,B) \in \mathcal{L}\}$. So $\mathcal{F}' = \mathcal{F}_0$. We shall prove that $\mathcal{F}_k$ is well-behaved for every nonnegative integer k by induction on $h+1-k$. Note that this theorem follows from this claim.

When $h+1-k \leq 0$, the $T - \{t_A : (A,B) \in \mathcal{L}\}$ contains only one node. So $(H, \mu, \mathcal{Q}) \in \mathcal{F}$, and $\mathcal{F}_k \subseteq \mathcal{F}$ is well-behaved for every $k \geq h+1$. We assume that $\mathcal{F}_j$ is well-behaved for every $j \geq k+1$. Let $(H, \mu, \mathcal{Q})$ be a frame in $\mathcal{F}_k$, and let $\mathcal{L}$ be a location that fits $(H, \mu, \mathcal{Q})$. Let $((T, \mathcal{X}), \{m_e : e \in E(T)\})$ be an unimpeded tree refinement of $\mathcal{L}$ over $\mathcal{F}$ of adhesion at most h and of (Z,s)-depth at most d for every $Z \subseteq V(H)$ and $0 \leq s \leq h$. Let $T' = T - \{t_A : (A,B) \in \mathcal{L}\}$, and let $\mathcal{X}' = \{X_t : t \in V(T')\}$. Let X be the maximum subset of $V(H)$ such that X is contained in every bag of $(T', \mathcal{X}')$. Let D be the set of nodes t of T' such that t corresponds to an edge-cut modulo X, and let D' be the set of edges of T' whose child-end is in D. Let C be a component of $T' - D'$. Since the (X,s)-depth of (T', X') is at most d for every $0 \leq s \leq h$, the vertices of C can be partitioned into at most d parts, where each part is a disjoint union of subtrees of T' such that every bag of each subtree T'' contains a vertex in the bag of the root of T'' but not in X. We denote this partition of C by $\mathcal{P}_C$. Since the adhesion of $(T', \mathcal{X}')$ is at most h, each subtree U in $\mathcal{P}_C$ can be partitioned into at most $2^h - 1$ parts, where each part is a disjoint union of subtrees of U such that every bag of each subtree U' contains the same non-empty subset X_U of the bag of the root of U, where $X \subset X_{U'}$. We denote this partition of U by $\mathcal{P}_{C,U}$. So C has a partition of at most $d(h+1)2^h$ disjoint union of subtrees of C, where every bag of each component U' of each part contains the same non-empty set $X_{U'}$ with $X \subset X_{U'}$. So the subframe induced by each component of each part of $\mathcal{P}_{C,U}$ is in $\mathcal{F}_j$ for some $j > k$. Let $\mathcal{F}^*$ be the family of center-essential frames such that every location that fits a frame in $\mathcal{F}^*$ has an unimpeded tree refinement over $\mathcal{F}_{k+1}$ of height at most $d(h+1)2^h$. Hence, the subframe induced by C is in $\mathcal{F}^*$. By the induction hypothesis and Lemma 5.1.6, $\mathcal{F}^*$ is well-behaved. Let $(T^*, \mathcal{X}^*)$ be the tree decomposition obtained from $(T, \mathcal{X})$ by contracting each component of $T' - D'$ to a new node. Then $((T^*, \mathcal{X}^*), \{m_e : e \in E(T^*)\})$ is an unimpeded tree refinement over $\mathcal{F}^*$. This proves that every location that fits a frame in $\mathcal{F}_k$ has an unimpeded tree refinement over $\mathcal{F}^*$ of adhesion at most h such that every bag contains a common set X of size at least k and every edge not incident with a leaf corresponds to an edge-cut modulo X.

Now, we shall prove that $\mathcal{F}_k$ is well-behaved. Let $(S, \preceq), (S', \preceq')$ be well-quasi-ordered sets. Let $G_1, G_2, ...$ be an infinite sequence of graphs. And for every $i \geq 1$, let $\phi_i : V(G_i) \to S$ and let $(\mathcal{L}_i, \tau_i)$ be an ordered extended location in G_i. For each $i \geq 1$, let (A_i, B_i) be the root of $\mathcal{L}_i$. For every $i \geq 1$, let $\psi_i : \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i) \to S'$ be a function. Assume that $\mathcal{L}_i$ fits a frame $(H_i, \mu_i, \mathcal{Q}_i)$ in $\mathcal{F}_k$, and $\bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i)$ is well-quasi-ordered by the $(S, \preceq)$-labelled topological minor relation.

For $i \geq 1$, let $((T_i, \mathcal{X}_i), \{m_e : e \in E(T_i)\})$ be an unimpeded tree refinement of $\mathcal{L}_i$ over $\mathcal{F}^*$ of adhesion at most h such that every bag contains a common set of size k. For every $t \in V(T_i) - \{t_A : (A, B) \in \mathcal{L}_i\}$, define $\mathcal{L}'_t$ to be the rooted location $\{(A, B), (B_{p_t t}, A_{p_t t}) : t_A \text{ is a descendant of } t \text{ in } T_i\}$ with root $(B_{p_t t}, A_{p_t t})$, where p_t is the parent of t; define τ'_t to be a function with domain $\mathcal{L}'_t$ such that $\tau'_t(B_{p_t t}, A_{p_t t}) = V(A_{p_t t}) \cap V(B_{p_t t}) \cap (\bigcup_{(A,B) \in \mathcal{L}'_t - \{(A_{p_t t}, B_{p_t t})\}} \tau_i(A, B))$ and $\tau'_t(A, B) = \tau_i(A, B)$ for every $(A, B) \in \mathcal{L}_i - \{(A_{p_t t}, B_{p_t t})\}$; we make $\mathcal{L}'_t$ an ordered location by assigning the ordering to $(B_{p_t t}, A_{p_t t})$ according to $m_{p_t t}$ and to each other separation (A, B) the same ordering as it in $\mathcal{L}_i$. Note that the periphery of $\mathcal{L}'_t$ is a subset of the periphery of $\mathcal{L}_i$. For every $(A, B) \in \mathcal{L}_i$, define $\mathcal{L}'_{t_A}$ to be the rooted location $\{(B, A)\}$ with root (B, A), and define $\tau'_t(B, A) = \tau_i(A, B)$, and assign the ordering of (A, B) in $\mathcal{L}_i$ to (B, A) in $\mathcal{L}'_{t_A}$. Note that the periphery of $\mathcal{L}'_{t_A}$ is empty.

Let M be the infinite graph with $V(M) = V(T_1 \cup T_2 \cup ...)$ such that for every $i' > i \geq 1$, $u \in V(T_i)$ and $w \in V(T_{i'})$ in M, u is adjacent to w if and only if

$(\mathcal{L}'_w, \tau'_w, \phi_{i'}, \psi_{i'})$ simulates $(\mathcal{L}'_u, \tau'_u, \phi_i, \psi_i)$. Since $\mathcal{P}(\mathcal{L}'_{t_A}, \tau'_{t_A})$ is empty for every $i \geq 1$ and $(A, B) \in \mathcal{L}_i$, and $\bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i)$ is well-quasi-ordered by the $(S, \preceq)$-labelled topological minor relation, we know that $\{t_A : i \geq 1, (A, B) \in \mathcal{L}_i - \{(A^*_i, B^*_i)\}\}$ is rich in M. For every $i \geq 1$, define $\Phi_i : E(T_i) \to \{0, 1, ..., h\}$ such that $\Phi_i(pc) = |X_{i,p} \cap X_{i,c}|$ for every edge pc of T_i.

Now we claim that for $i' > i \geq 1$, if $u \in V(T_i)$ is adjacent to $w \in V(T_{i'})$ in M, and $v \in V(T_{i'})$ precedes w with respect to $\Phi_{i'}$, then u is adjacent to v in M. Let p_v and p_w be the parent of v and w, respectively. Since $(T_{i'}, \mathcal{X}_{i'})$ is unimpeded, there are $\Phi_{i'}(p_v v)$ disjoint paths in $G_{i'_{\tau'_v, A_{p_v v}}}$ from $\gamma_{\tau'_{i'}, A_{p_v v}}$ to $\gamma_{\tau'_{i'}, A_{p_w w}}$ such that the ℓ-entry of $\gamma_{\tau'_{i'}, A_{p_v v}}$ and the ℓ-th entry of $\gamma_{\tau'_{i'}, A_{p_w w}}$ are the ends of the ℓ-th path, for $1 \leq \ell \leq \Phi_{i'}(p_v v)$. In addition, the entries of $\gamma_{\tau'_{i'}, A_{p_v v}}$ and the entries of $\gamma_{\tau'_{i'}, A_{p_w w}}$ are corresponding to an edge-cut in $G_{i'}$ under modulo of the same set. So the claim follows.

For $i \geq 1$, let $T'_i = T_i - \{r_i\}$, where r_i is the root of T_i. Note that r_i is a leaf, so T'_i is a rooted tree with root the neighbor of r_i. Suppose that no $j' > j \geq 1$ such that $(\mathcal{L}_{j'}, \tau_{j'}, \phi_{j'}, \psi_{j'})$ simulates $(\mathcal{L}_j, \tau_j, \phi_j, \psi_j)$. Then the set of the roots of $T'_1, T'_2, ...$ is an independent set in $M - \{r_1, r_2, ...\}$. By Theorem 5.1.7, there exists an infinite independent set Y of $M - \{r_1, r_2, ...\}$ such that $|Y \cap V(T'_i)| \leq 1$ for every $i \geq 1$ and such that the set of the children of the members of Y is rich in $M - \{r_1, r_2, ...\}$. As $\{t_A : i \geq 1, (A, B) \in \mathcal{L}_i - \{(A^*_i, B^*_i)\}\}$ is rich in M, $|Y \cap \{t_A : i \geq 1, (A, B) \in \mathcal{L}_i - \{(A^*_i, B^*_i)\}\}|$ is finite. So we may assume that Y is disjoint from $\{t_A : i \geq 1, (A, B) \in \mathcal{L}_i - \{(A^*_i, B^*_i)\}\}$. However, it is impossible, by Lemma 5.1.2, since $\mathcal{F}^*$ is well-behaved, a contradiction. So there exist $j' > j \geq 1$ such that $(\mathcal{L}_{j'}, \tau_{j'}, \phi_{j'}, \psi_{j'})$ simulates $(\mathcal{L}_j, \tau_j, \phi_j, \psi_j)$. This proves that $\mathcal{F}'_k$ is well-behaved. ■

We are ready to prove Robertson's conjecture for the graphs of bounded tree-width.

Theorem 5.1.9 *Let w be a positive integer. Then for every well-quasi-ordered set*

$(S, \preceq)$, the graphs of tree-width at most w are well-quasi-ordered by the $(S, \preceq)$-topological minor relation.

Proof. Let $G_1, G_2, ...$ be an infinite sequence of graphs of tree-width at most w. For each $i \geq 1$, let $\mathcal{L}_i = \{(\emptyset, G_i)\}$ be the location in G_i. Note that $(\mathcal{L}_i, \emptyset)$ is an extended ordered rooted location in G_i. Then this theorem follows from Theorem 4.2.6, Lemma 5.1.5 and Theorem 5.1.8. ■

CHAPTER VI

EXCLUDING ROBERTSON CHAINS

6.1 Faithful locations and 3-separations

Let $\mathcal{T}$ be a tangle in a graph of G of order θ, and let Z be a subset of $V(G)$ of size less than θ. Then $\mathcal{T} - Z$ is the set of separations (A, B) of $G - Z$ of order less than $\theta - |Z|$ such that $(A', B') \in \mathcal{T}$, where $V(A') = V(A) \cup Z$ and $V(B') = V(B) \cup Z$. By Theorem 6.2 in [40], $\mathcal{T} - Z$ is a tangle in $G - Z$ of order $\theta - |Z|$.

Let G be a graph and $\mathcal{T}$ a tangle in G. We say that a vertex v of G is *k-separable from* $\mathcal{T}$ if there exists $(A, B) \in \mathcal{T}$ of order at most k such that $v \in V(A) - V(B)$. We say that G is *weakly subcubic* if every vertex is adjacent to at most three vertices. The degree of a vertex v in G, denoted by $\deg_G(v)$, is the number of edges incident with v.

Let $\mathcal{L}$ be a location of order at most three in a graph G. The *contour* of $\mathcal{L}$, denoted by $C(L)$, is the graph obtained from $\bigcap_{(A,B)\in\mathcal{L}} B$ by adding a cycle C_A on $V(A) \cap V(B)$ for each $(A, B) \in \mathcal{T}$ with $V(A) \cap V(B) \neq \emptyset$. In this case, we say that C_A *represents* (A, B). Note that if $|V(A) \cap V(B)| = 1$, then C_A consists of a loop; if $|V(A) \cap V(B)| = 2$, then C_A consists of two parallel edges. In addition, for every separation (X', Y') of $C(\mathcal{L})$, there exists a separation (X, Y) of G such that X and Y is obtained from $C(\mathcal{L})[V(X')]$ and $C(\mathcal{L})[V(Y')] - E(C(\mathcal{L})[V(X')]$, respectively, by replacing C_A by A for each $(A, B) \in \mathcal{L}$ with $V(A) \cap V(B) \neq \emptyset$. We define $\mathcal{T}_{\mathcal{L}}$ to be the set of separations of $C(\mathcal{L})$ consisting of (X', Y') for which $(X, Y) \in \mathcal{T}$. Clearly, $\mathcal{T}_{\mathcal{L}}$ is a tangle in $C(\mathcal{L})$ of the same order as $\mathcal{T}$.

Let G be a graph and $\mathcal{T}$ a tangle in G. Let $\mathcal{L} \subseteq \mathcal{T}$ be a location in G. We say that $\mathcal{L} \subseteq \mathcal{T}$ is *faithful* if the following hold.

(F1) For every $(A, B) \in \mathcal{L}$, the order of (A, B) is at most three.

(F2) C_A and $C_{A'}$ are disjoint for every two distinct separations (A, B) and (A', B') in $\mathcal{L}$ of order three.

(F3) For every $(A, B) \in \mathcal{L}$ with $V(A) \cap V(B) \neq \emptyset$, there does not exist $(A', B') \in \mathcal{L}$ such that $V(A) \cap V(B) \subseteq V(A') \cap V(B')$. If there are at least two edges of $C(\mathcal{L})$ with the same ends u, v, then these two edges are the only two edges between u, v, and $\{u, v\} = V(C_A)$ for some $(A, B) \in \mathcal{L}$ with $|V(A) \cap V(B)| = 2$.

(F4) If v is a vertex of $C(\mathcal{L})$ such that there exists a separation $(A, B) \in \mathcal{T}_{\mathcal{L}}$ of order at most two with $v \in V(A) - V(B)$, then v is adjacent to exactly two other vertices in $C(\mathcal{L})$, and v is incident with an edge xv that is not in $C_{A'}$ for every $(A', B') \in \mathcal{L}$, and there is only one edge between x, v; for every $u \in V(A) \cap V(B)$, u is adjacent to at most one vertex in A.

(F5) $C(\mathcal{L})$ is weakly subcubic.

(F6) For every C_A in $C(\mathcal{L})$ with $|V(C_A)| \geq 2$ for some $(A, B) \in \mathcal{L}$ and for every two different vertices u, v of C_A, there exist two edge-disjoint paths with ends u, v in A. Furthermore, if $|V(C_A)| = 3$, say $V(C_A) = \{u, v, w\}$, then there exist two edge-disjoint paths in A, where one is from u to v, and the other is from u to w.

Notice that if $\mathcal{L}$ is faithful and G does not contain a topological minor isomorphic to the Robertson chain of length k, then $C(\mathcal{L})$ does not contain a topological minor isomorphic to the Robertson chain of length k.

We say that a vertex in a graph is a 4^+-*vertex* if it is adjacent to at least four vertices.

Lemma 6.1.1 *Let G be a graph, and let $\mathcal{T}$ be a tangle in G. If every vertex v of G is 3-separable from $\mathcal{T}$, then there exists a faithful location $\mathcal{L} \subseteq \mathcal{T}$ in G.*

Proof. For every vertex v of G 2-separable from $\mathcal{T}$, let $(A_v, B_v) \in \mathcal{T}$ be of order at most two such that $v \in V(A_v) - V(B_v)$ and A_v is connected, and subject to that, A_v is maximal. Let $\mathcal{L}_1 = \{(A_v, B_v) : v \text{ is 2-separable from } \mathcal{T}\}$. It is clear that $\mathcal{L}_1$ is a location in G. For $j = 1, 2$, let $\mathcal{L}_{1,j} = \{(A_v, B_v) \in \mathcal{L}_1 : |V(A_v) \cap V(B_v)| = j\}$. For each $(A_v, B_v) \in \mathcal{L}_{1,2}$, let $Q = Q_1Q_2...Q_t$ be the path of blocks of A_v such that Q_1 contains one vertex u_0 in $V(A_v) \cap V(B_v)$ and Q_t contains the other vertex u_t in $V(A_v) \cap V(B_v)$. In this case, let u_i be the vertex in $Q_i \cap Q_{i+1}$ for $1 \leq i \leq t-1$, and $I_v = \{1 \leq j \leq t : Q_j \text{ is an edge}\}$. For every $j \in \{1, 2, ..., t\} - I_v$, let $(A_{v,j}, B_{v,j}) \in \mathcal{T}$ such that $V(A_{v,j}) \cap V(B_{v,j}) = \{u_{j-1}, u_j\}$ and $Q_j \subseteq A_{v,j}$. For each maximal subset J of consecutive integers in $[t] - I_v$, let $(A_{v,J}, B_{v,J}) \in \mathcal{T}$ such that $A_{v,J} = \bigcup_{j \in J} A_{v,j}$ and $B_{v,J} = \bigcap_{j \in J} B_{v,J}$. Let $\mathcal{L}_2 = \mathcal{L}_{1,1} \cup \{(A_{v,J}, B_{v,J}) : (A_v, B_v) \in \mathcal{L}_{1,2}, J \text{ is a maximal subset of consecutive integers in } [t] - I_v\}$. Since each $A_{v,J}$ is contained in A_v, $\mathcal{L}_2$ is a location. Observe that $\mathcal{L}_2$ satisfies (F6).

Let H_2 be the contour of $\mathcal{L}_2$. As every separation in $\mathcal{L}_2$ has order at most two, every vertex of H_2 is 3-separable from $\mathcal{T}_{\mathcal{L}_2}$. Note that every 2-separable vertex v in H_2 is adjacent to at most two vertices in H_2 and incident with one edge xv that is not in C_A for every $(A, B) \in \mathcal{L}_2$, and there is only one edge between x and v. Furthermore, if (A, B) is a separation of order two in $\mathcal{T}_{\mathcal{L}_2}$, then every vertex in $V(A) \cap V(B)$ is adjacent to at most one vertex in A.

We say that a separation (A, B) *mix-separates* a vertex v of H_2 from $\mathcal{T}_{\mathcal{L}_2}$ if $(A, B) \in \mathcal{T}_{\mathcal{L}_2}$ such that every vertex in $V(A) \cap V(B)$ is adjacent to a vertex in $V(A) - V(B)$, and either $v \in V(A) - V(B)$, or $v \in V(A) \cap V(B)$ and v is adjacent to at most one vertex in $V(B) - V(A)$. Note that every vertex v of H_2 that is adjacent to at least three vertices in H_2 is not mix-separable from $\mathcal{T}_{\mathcal{L}_2}$ by a separation of order at most two. For every 4^+-vertex v of H_2, let (M_v, N_v) be a separation of order three mix-separating v from $\mathcal{T}_{\mathcal{L}_2}$ such that (M_v, N_v) mix-separates as many 4^+-vertices of H_2 from $\mathcal{T}_{\mathcal{L}_2}$ as possible, and subject to that, $|V(M_v) - V(N_v)|$ is as small as possible, and subject to that, $|E(M_v)|$ is as small as possible. Observe that $(M_v, N_v) \in \mathcal{T}_{\mathcal{L}_2}$, and every vertex in $V(M_v) \cap V(N_v)$ is adjacent to a vertex in $M_v - N_v$.

Let $\mathcal{S} = \{(M_v, N_v) : v$ is a 4^+-vertex in $H_2\}$.

Claim 1: $\mathcal{S}$ is a location in H_2.

Proof of Claim 1: Note that if there exists a pair of 4^+-vertices u, v in H_2 such that $M_u \subset M_v$, then either (M_v, N_v) mix-separates u from $\mathcal{T}_{\mathcal{L}_2}$ and mix-separates more 4^+-vertices from $\mathcal{T}_{\mathcal{L}_2}$ than (M_u, N_u), or (M_u, N_u) mix-separates v from $\mathcal{T}_{\mathcal{L}_2}$ and mix-separates the same number of 4^+-vertices from $\mathcal{T}_{\mathcal{L}_2}$ as (M_v, N_v) but $M_u - N_u \subset M_v - N_v$. Both cases contradict our choices of (M_u, N_u) or (M_v, N_v). Similarly, unless $(M_u, N_u) = (M_v, N_v)$, it is impossible that $V(M_u) \cap V(N_u) = V(M_v) \cap V(N_v)$, otherwise, $(M_u \cup M_v, N_u \cap N_v)$ or $(M_u \cap M_v, N_u \cup N_v)$ is a better choices than (M_u, N_u) or (M_v, N_v).

Suppose that $\mathcal{S}$ is not a location in H_2. So there exists a pair of 4^+-vertices u, v in H_2 such that $V(M_v) \cap V(N_v) \cap (V(M_u) - V(N_u)) \neq \emptyset \neq V(M_u) \cap V(N_u) \cap (V(M_v) - V(N_v))$.

If $V(M_u) \cap V(N_u) \cap V(M_v) \cap V(N_v) \neq \emptyset$, then $|V(M_u) \cap V(N_u) \cap (V(M_v) - V(N_v))| = |V(M_v) \cap V(N_v) \cap (V(M_u) - V(N_u))| = |V(M_u) \cap V(N_u) \cap V(M_v) \cap V(N_v)| = 1$. However, $(M_u \cup M_v, N_u \cap N_v)$ has order three and mix-separates every 4^+-vertex in H_2 mix-separated by (M_u, N_u) or (M_v, N_v) from $\mathcal{T}_{\mathcal{L}_2}$. By the choices of (M_u, N_u) and (M_v, N_v), we know that $(M_u \cup M_v, N_u \cap N_v), (M_u, N_u), (M_v, N_v)$ mix-separate the same 4^+-vertices from $\mathcal{T}_{\mathcal{L}_2}$. If some vertex w in $V(M_u \cap M_v) \cap V(N_u \cup N_v)$ is not adjacent to any vertex in $V(M_u \cap M_v) - V(N_u \cup N_v)$, then there exists a separation $(A, B) \in \mathcal{T}_{\mathcal{L}_2}$ of order two such that $A = (M_u \cap M_v) - \{w\}$, so u, v are the vertices in $V(M_u \cap M_v) \cap V(N_u \cup N_v) - \{w\}$, and one of u, v, say u, is adjacent to exactly one vertex in $A = M_u \cap M_v - \{w\}$. This implies that $u \in V(M_u) \cap V(N_u)$ or $u \in V(M_v) \cap V(N_v)$. But $(M_u, N_u), (M_v, N_v)$ mix-separate the same 4^+-vertices from $\mathcal{T}_{\mathcal{L}_2}$, so either u is adjacent to at most one vertex in $V(N_u) - V(M_u)$ or at most one vertex in $V(N_v) - V(M_v)$. In either case, u is adjacent to at most three vertices in H_2, a contradiction. So every vertex in $V(M_u \cap M_v) \cap V(N_u \cup N_v)$ is adjacent with some vertex in $V(M_u \cap M_v) - V(N_u \cup N_v)$. However, it means that $(M_u \cap M_v, N_u \cup N_v)$ is a mix-separation separating the same number of 4^+-vertices as (M_u, N_u), but $|V(M_u \cap M_v) - V(N_u \cup N_v)| < |V(M_u) - V(N_u)|$, a contradiction. Therefore, $V(M_u) \cap V(N_u) \cap V(M_v) \cap V(N_v) = \emptyset$.

If one of $|V(M_u) \cap V(N_u) \cap (V(M_v) - V(N_v))|$ and $|V(M_v) \cap V(N_v) \cap (V(M_u) - V(N_u))|$ equals 1, and the other equals 2, then using a similar argument in the last paragraph, either $(M_u \cup M_v, N_u \cap N_v)$ or $(M_u \cap M_v, N_u \cup N_v)$ is a better choice than (M_u, N_u) or (M_v, N_v), a contradiction. If $|V(M_u) \cap V(N_u) \cap (V(M_v) - V(N_v))| = |V(M_v) \cap V(N_v) \cap (V(M_u) - V(N_u))| = 2$, then $(M_u \cup M_v, N_u \cap N_v) \in \mathcal{T}_{\mathcal{L}_2}$ has order two, so $u \in V(M_u) \cap V(N_u)$. But it implies that u is adjacent to at most two vertices in H_2, a contradiction. Hence, $|V(M_u) \cap V(N_u) \cap (V(M_v) - V(N_v))| = |V(M_v) \cap V(N_v) \cap (V(M_u) - V(N_u))| = 1$. In this case, $(M_u \cap M_v, N_u \cup N_v) \in \mathcal{T}_{\mathcal{L}_2}$ has order two, so $u \notin V(M_v) - V(N_v)$ and $v \notin V(M_u) - V(N_u)$. (i.e. $u \in V(N_v)$ and $v \in V(N_u)$.) Furthermore, if $u \in V(M_v)$, then $u \in V(M_v) \cap V(N_v)$ and u is adjacent to at most one vertex in $M_u \cap M_v$, and hence u is adjacent to at most one vertex in $M_v \cup N_u$; similarly, if $v \in V(N_u)$, then v is adjacent to at most one vertex in $M_u \cup N_v$. By a similar as before, every vertex in $V(M_u \cap N_v) \cap V(M_v \cup N_u)$ is adjacent to a vertex in $V(M_u \cap N_v) - V(M_v \cup N_u)$, since u is a 4^+-vertex. However, $(M_u \cap N_v, M_v \cup N_u) \in \mathcal{T}_{\mathcal{L}_2}$ has order three mixed-separating u, and it mixed-separates the same number of 4^+-vertices of H_2 as (M_u, N_u), since v is a 4^+-vertex. But $|V(M_u \cap N_v) - V(M_v \cup N_u)| < |V(M_u) - V(N_u)|$, a contradiction. This proves that $\mathcal{S}$ is a location in H_2. □

Claim 2: For every distinct $(M_u, N_u), (M_v, N_v)$ in $\mathcal{S}$, $V(M_u) \cap V(N_u) \cap V(M_v) \cap V(N_v) = \emptyset$.

Proof of Claim 2: Suppose to the contrary that there exists a vertex $w \in V(M_u) \cap V(N_u) \cap V(M_v) \cap V(N_v)$. Assume that w is a 4^+-vertex in H_2. Without loss of generality, we may assume that $(M_w, N_w) \neq (M_u, N_u)$ by symmetry. Since $\mathcal{S}$ is a location and $w \in V(M_u)$, we have that $w \in V(N_w)$, so $w \in V(M_w) \cap V(N_w)$. This implies that w is adjacent to at most one vertex in $V(N_w) - V(M_w)$. In particular, since $M_u \subseteq N_w$ and $M_w \subseteq N_u$, w is adjacent to exactly one vertex x in $V(M_u) - V(N_u)$. Note that $N_{H_2}(w) \subseteq V(M_w) \cup \{x\}$. Recall that we proved that $V(M_u) \cap V(N_u) \neq V(M_w) \cap V(N_w)$ in the proof of Claim 1, so some vertex in $V(M_u) \cap V(N_u)$ is not in M_w and hence not adjacent to w. Furthermore, since w is a 4^+-vertex in H_2, w is adjacent to at least two vertices in $V(N_u) - V(M_u)$. Since (M_u, N_u) is a better choice than $(M_u - \{w\}, H_2[V(N_u) \cup \{x\}] - E(M_u))$, either some vertex in $V(M_u - \{w\}) \cap V(N_u \cup \{x\})$ is not adjacent to any vertex in $V(M_u - \{w\}) - (V(N_u) - \{x\})$, or some vertex y in $V(M_u) \cap V(N_u) - \{w\}$ is adjacent to w. The former case is impossible, otherwise u is in $V(A)$ for some $(A, B) \in \mathcal{T}_{\mathcal{L}_2}$ of order two, but it leads to a contradiction since u is a 4^+-vertex mix-separated by (M_u, N_u). So the later case happens, and it implies that $V(M_u) \cap V(N_u) \cap V(M_w) \cap V(N_w) = \{y, w\}$. Since y is in $V(M_u) \cap V(N_u) \cap V(M_w) \cap V(N_w)$, it is adjacent to one vertex in $V(M_u) - V(N_u)$ and one vertex in $V(M_w) - V(N_w)$. Therefore, $N_{H_2}(y) \not\subseteq V(M_u) \cup V(M_w)$, otherwise, u is mix-separated from $\mathcal{T}_{\mathcal{L}_2}$ by $(M_u \cup M_w, N_u \cap N_w - \{y, w\})$, which has order two. In particular, y is a 4^+-vertex in H_2. But y is adjacent to one vertex in $V(M_u) - V(N_u)$ and one vertex in $V(M_w) - V(N_w)$, so (M_y, N_y) equals (M_u, N_u) or (M_w, N_w). However, it implies that $N_{H_2}(y) \subseteq V(M_u) \cup V(M_w)$, a contraction. This proves that every vertex in $V(M_u) \cap V(N_u) \cap V(M_v) \cap V(N_v)$ has degree at most three in H_2.

If w is adjacent to at most one vertex in either M_u or M_v, say M_u, then $(M_u - \{w\}, N_u \cup \{x\})$ is a better choice than (M_u, N_u) as in the last paragraph, where x is the neighbor of w in M_u. So w is incident with at least two edges in both M_u and M_v. Since w is adjacent to at most three vertex in H_2, $|V(M_u) \cap V(N_u) \cap V(M_v) \cap V(N_v)| = 2$, and w is adjacent to the vertex y in $V(M_u) \cap V(N_u) \cap V(M_v) \cap V(N_v)$ other than w. But y is adjacent to at most three vertices in H_2, so u is mix-separable by a separation of order two, a contradiction. This proves that for every distinct $(M_u, N_u), (M_v, N_v)$ in $\mathcal{S}$, $V(M_u) \cap V(N_u) \cap V(M_v) \cap V(N_v) = \emptyset$. $\square$

For each $(M_v, N_v) \in \mathcal{S}$, let T_v be the minimal connected subgraph of the tree of blocks of M_v such that T_v contains the blocks of M_v intersecting $V(M_v) \cap V(N_v)$. Let $\mathcal{Q}_v$ be the set of blocks in T_v that are single edges. When $\mathcal{Q}_v \neq \emptyset$, let $\mathcal{L}_v$ be the set consisting of the separations (A, B) of order at most three in $\mathcal{T}_{\mathcal{L}_2}$ such that each A equals a component of $M_v - \bigcup_{Q \in \mathcal{Q}_v} E(Q)$. Note that $\mathcal{L}_v$ contains at most one separation of order three. Define $\mathcal{S}' = \{(M_v, N_v) \in \mathcal{S} : Q_v = \emptyset\} \cup \bigcup_{Q_v \neq \emptyset} \mathcal{L}_v$. Clearly, $\mathcal{S}'$ is a location in H_2 satisfying (F1), (F2), (F4) and (F6) by Claims 1 and 2. By replacing C_A by A in H_2 for each $(A, B) \in \mathcal{L}_2$, $\mathcal{S}'$ defines a location $\mathcal{S}''$ in G satisfying (F1), (F2), (F4) and (F6). If $\mathcal{S}''$ contains distinct separations $(A, B), (A', B')$ such that $V(A) \cap V(B) \neq \emptyset$ and $V(A) \cap V(B) \subseteq V(A') \cap V(B')$, then we remove (A, B) and (A', B') from $\mathcal{S}''$ and add $(A \cup A', B \cap B')$ into $\mathcal{S}''$. We repeat this process until no such pair of separations exist. For each pair of vertices u, v of $C(\mathcal{S}'')$ such that there are at least two edges between them, if there exists $(A, B) \in \mathcal{S}''$ such that $\{u, v\} \subseteq V(A) \cap V(B)$, then we replace (A, B) by $(A \cup E(G[\{u, v\}]), B - E(G[\{u, v\}]))$; otherwise, we add a new separation $(G[\{u, v\}], C(\mathcal{S}'') - E(G[\{u, v\}]))$ into $\mathcal{S}''$. We define $\mathcal{S}'''$ to be the resulting location. Then $\mathcal{S}'''$ is a location in G satisfying (F1)-(F4) and (F6).

To prove that $\mathcal{S}'''$ is faithful, it is sufficient to prove that $C(\mathcal{S}''')$ is weakly subcubic. Let $H = C(\mathcal{S}''')$. Suppose that there exists a 4^+-vertex v of H. Since $\mathcal{S}'''$ satisfies (F6), v is a 4^+-vertex of H_2. (Note that $V(H) \subseteq V(H_2)$.) Since $v \in V(H)$, there exists $(M_v, N_v) \in \mathcal{L}_2$ such that $v \in V(M_v) \cap V(N_v)$ and v is adjacent to at most one vertex in $V(N_v) - V(M_v)$. But this implies that v is adjacent to at most three vertices in H, a contradiction. So H is weakly subcubic. This proves the lemma. $\blacksquare$

Let $\mathcal{L}$ be a location in a graph G. If $Z \subseteq V(C(\mathcal{L}))$, then we define the location $\mathcal{L} - Z$ of $G - Z$ to be $\{(A - Z, B - Z) : (A, B) \in \mathcal{L}\}$. Note that if $\mathcal{T}$ is a tangle in

G of order greater than the sum of $|Z|$ and the order of $\mathcal{L}$ such that $\mathcal{L} \subseteq \mathcal{T}$, then $\mathcal{L} - Z \subseteq \mathcal{T} - Z$.

Let $\mathcal{L}$ be a location in a graph G. We say that C_A is a *thick cycle* in $C(\mathcal{L})$ if $(A, B) \in \mathcal{L}$ with $|V(A) \cap V(B)| = 2$. Furthermore, if $\mathcal{L}$ is faithful and C_A is thick, then there exists a topological minor in A isomorphic to a Robertson chain with ends u, v, where $V(A) \cap V(B) = \{u, v\}$. In this case, we say that the *level* of C_A is the maximum t such that A contains a topological minor in A isomorphic to the Robertson chain of length t with ends u, v. We say that a thick cycle C_A of $C(\mathcal{L})$ is *pendant* if

- for every $v \in V(C_A)$, $v \notin V(C_{A'})$ for every $(A', B') \in \mathcal{L}$ other than (A, B), and
- at least one vertex of C_A is adjacent to exactly two vertices of G.

The *level* of a faithful location $\mathcal{L}$ is the minimum level of a non-pendant thick cycle of $C(\mathcal{L})$.

Lemma 6.1.2 *Let G be a graph, and let $\mathcal{T}$ be a tangle in G of order θ. Let t be a positive integer, and let $\mathcal{L} \subseteq \mathcal{T}$ be a faithful location in G of level at least t. If there exists $Z \subseteq V(C(\mathcal{L}))$ with $|Z| < (\theta - 6)/3$ such that for every thick cycle C_A in $C(\mathcal{L})$ with $V(C_A) \cap Z = \emptyset$ there exists $(A', B') \in \mathcal{T}_{\mathcal{L}} - Z$ of order at most three such that $V(C_A) \subseteq V(A') - V(B')$, then there exist $Z' \subseteq V(C(\mathcal{L}))$ with $|Z'| \leq 3|Z|$ and a faithful location $\mathcal{L}^* \subseteq \mathcal{T} - Z'$ in $G - Z'$ of level at least $t + 1$.*

Proof. Let $H = C(\mathcal{L})$. Since $\mathcal{L}$ is faithful, H weakly subcubic. Let $Z' = Z \cup \bigcup(V(C_A) : (A, B) \in \mathcal{L}, V(C_A) \cap Z \neq \emptyset, |V(A) \cap V(B)| = 3)$, so $|Z'| \leq 3|Z|$. Note that $\mathcal{L} - Z'$ is a location in G satisfying (F1),(F2), (F5) and (F6), since $\mathcal{L}$ satisfies (F2) and for every separation $(A, B) \in \mathcal{L}$ of order three, either $V(A) \cap V(B) \cap Z' = \emptyset$, or $V(A) \cap V(B) \subseteq Z'$.

Let $\mathcal{L}' \subseteq \mathcal{T}_{\mathcal{L}} - Z'$ be the set of separations (A, B) of $H - Z'$ of order at most two such that $V(A) - V(B)$ is nonempty and every vertex in $V(A) \cap V(B)$ is adjacent to a vertex in $V(A) - V(B)$, and subject to that, $|V(A) \cap V(B)|$ is as small as possible, and subject to that, A is maximal. Let $(A, B) \in \mathcal{L}'$, and let $v \in V(A) \cap V(B)$. If v is adjacent to at least two vertices in A, then v is adjacent to at most one vertex of $V(B) - V(A)$ in $H - Z'$, but this means that there exists $(A', B') \in \mathcal{T}_{\mathcal{L}} - Z'$ of order at most two such that $A' \supseteq A$ and either (A', B') has order smaller than (A, B), or $A' \supset A$, a contradiction. So v is only adjacent to one vertex in A. For every separation $(A, B) \in \mathcal{L}'$ of order two, see the path of blocks of A connecting the blocks that contain each vertex in $V(A) \cap V(B)$. Let $\mathcal{L}''$ be the separations of $H - Z'$ in $\mathcal{T}_{\mathcal{L}} - Z'$ of order two corresponding to each component of each A obtained by deleting the edges in those blocks in the mentioned path of blocks for which each of them consists of a single edge. For each $(X, Y) \in \mathcal{L}''$, let (A_X, B_X) be the separation of $G - Z'$ obtained from (X, Y) by replacing $C_A - Z'$ by $A - Z'$ for each $(A, B) \in \mathcal{L}$ with $V(A) \cap V(B) \neq \emptyset$. Note that $\mathcal{L}''$ is a location in $G - Z'$. Let $\mathcal{L}''' = \{(A_X, B_X) : (X, Y) \in \mathcal{L}''\} \cup \{(C, D) \in \mathcal{L} - Z' : C \subseteq \bigcap_{(A,B) \in \mathcal{L}''} B\}$ be a location in $G - Z'$. Observe that every separation in $\mathcal{L}'''$ has order at most three. If there exist two distinct $(A, B), (A', B') \in \mathcal{L}'''$ such that $\emptyset \neq V(A) \cap V(B) \subseteq V(A') \cap V(B')$, then we delete $(A, B), (A', B')$ from $\mathcal{L}'''$ and add $(A \cup A', B \cap B')$ into $\mathcal{L}'''$. We repeat this process until there does not such pair of separations. Hence, $\mathcal{L}'''$ satisfies (F1)-(F3), (F5), and (F6).

Let $H''' = C(\mathcal{L}''')$. Let C_A be a thick cycle in H''' but not a thick cycle in H. If C_A does not contain any vertex v that is in $V(A') \cap V(B')$ for some $(A', B') \in \mathcal{L}'$, then it is pendant. If C_A contains a vertex v that is in $V(A') \cap V(B')$ for some $(A', B') \in \mathcal{L}'$, then A contains a thick cycle $C_{A''}$ for some $(A'', B'') \in \mathcal{L}'$ with $v \in V(C_{A''})$, since v is adjacent to exactly one vertex in A'. In this case, A contains a block other than $C_{A''}$ intersecting $C_{A''}$, since C_A is not a thick cycle in H'. By the assumption, either $C_{A''}$ is pendant or $C_{A''}$ has level at least t, so either C_A is pendant or C_A has level at least $t + 1$. Furthermore, if there exists a vertex u in H''' and a separation $(A, B) \in \mathcal{T}_{\mathcal{L}'''}$

of order at most two such that $u \in V(A) - V(B)$, then u is adjacent to at most two vertices in H''', and u is incident with an edge ux, for some $x \in V(H''')$, that is not in $C_{A'}$ for every $(A', B') \in \mathcal{L}'''$ with $|V(A') \cap V(B')| = 2$, and there is only one edge between u and x. In addition, if w is in $V(A) \cap V(B)$ for some such an (A, B), then w is adjacent to at most one vertex in A. Therefore, $\mathcal{L}'''$ is a faithful location in $G - Z'$. We call a non-pendant thick cycle in H''' as a *bad* cycle. In other words, $\mathcal{L}'''$ is faithful location in $G - Z'$ of level at least t, and every thick cycle of level t is bad.

We say that a thick cycle C_A of H''' is *mix-separated* from $\mathcal{T}_{\mathcal{L}'''}$ by (A', B') if $(A', B') \in \mathcal{T}_{\mathcal{L}'''}$ such that every vertex in $V(A') \cap V(B')$ is adjacent to a vertex in $V(A) - V(B)$, and for each vertex $v \in V(C_A)$, either $v \in V(A') - V(B')$, or $v \in V(A') \cap V(B')$ and v is adjacent to at most one vertex in $V(B) - V(A)$. Recall that every non-pendant thick cycle in H''' of level at most t is a thick cycle in $H - Z'$. So by the assumption, for every thick cycle C_A in $H - Z'$ there exists $(X'_A, Y'_A) \in \mathcal{T}_{\mathcal{L}} - Z'$ of order at most three such that $V(C_A) \subseteq V(X'_A) - V(Y'_A)$. But this implies that for every bad thick cycle C_A in H''', there exists $(X_A, Y_A) \in \mathcal{T}_{\mathcal{L}'''}$ such that $V(C_A) \subseteq V(X_A) - V(Y_A)$, so (X_A, Y_A) mix-separates C_A from $\mathcal{T}_{\mathcal{L}'''}$. On the other hand, every bad thick cycle in H''' is not mix-separable from $\mathcal{T}_{\mathcal{L}'''}$ by a separation of order at most two. For each bad cycle C_A of H''', we pick (X_A, Y_A) to be a separation of order three that mix-separates C_A from $\mathcal{T}_{\mathcal{L}'''}$, and subject to that, (X_A, Y_A) mix-separates as many bad cycles from $\mathcal{T}_{\mathcal{L}'''}$ as possible, and subject to that, $|V(X_A) - V(Y_A)|$ is as small as possible, and subject to that $|E(X_A)|$ is as small as possible. Let $\mathcal{S} = \{(X_A, Y_A) : C_A$ is a bad cycle in $H'''\}$.

Claim 1: $\mathcal{S}$ is a location in H'''.

Proof of Claim 1: Suppose that $\mathcal{S}$ is not a location in H'''. Note that for every pair of different bad cycles $C_A, C_{A'}$, it is impossible that $X_A \subseteq X_{A'}$ or $V(X_A) \cap V(Y_A) = V(X_{A'}) \cap V(Y_{A'})$ by our choices of (X_A, Y_A) and $(X_{A'}, Y_{A'})$. So there exists a pair of bad cycles $C_A, C_{A'}$ of H''' such that $V(X_A) \cap V(Y_A) \cap (V(X_{A'}) - V(Y_{A'})) \neq \emptyset \neq$

$V(X_{A'}) \cap V(Y_{A'}) \cap (V(X_A) - V(Y_A))$. If $V(X_A) \cap V(Y_A) \cap V(X_{A'}) \cap V(Y_{A'}) \neq \emptyset$, then as in the proof of Claim 1 in Lemma 6.1.1, either $(X_A \cup X_{A'}, Y_A \cap Y_{A'})$ mix-separates more bad cycles than (X_A, Y_A) or $(X_{A'}, Y_{A'})$, or the three mix-separations $(X_A \cap X_{A'}, Y_A \cup Y_{A'})$, (X_A, Y_A) and $(X_{A'}, Y_{A'})$ mix-separate the same bad cycles, but $|V(X_A \cap X_{A'}) - V(Y_A \cup Y_{A'})| < |V(X_A) - V(Y_A)|$. Therefore, $V(X_A) \cap V(Y_A) \cap V(X_{A'}) \cap V(Y_{A'}) = \emptyset$. Similarly, it is impossible that one of $|V(X_A) \cap V(Y_A) \cap (V(X_{A'}) - V(Y_{A'}))|$ and $|V(X_{A'}) \cap V(Y_{A'}) \cap (V(X_A) - V(Y_A))|$ equals one and the other equals two. In addition, since C_A is not mix-separable by a separation of order two, $|V(X_A) \cap V(Y_A) \cap (V(X_{A'}) - V(Y_{A'}))| = |V(X_{A'}) \cap V(Y_{A'}) \cap (V(X_A) - V(Y_A))| = 1$. However, in this case, $(X_A \cap Y_{A'}, X_{A'} \cup Y_A)$ mix-separates the same non-pendant edges as (X_A, Y_A) (since $\mathcal{L}'''$ satisfies (F4)), but $|V(X_A \cap Y_{A'}) - V(X_{A'} \cup Y_A)| < |V(X_A) - V(Y_A)|$, a contradiction. This proves that $\mathcal{S}$ is a location in H'''. $\square$

Claim 2: For every $(X_A, Y_A) \in \mathcal{S}$ and for every $v \in V(X_A) \cap V(Y_A)$, v is adjacent to exactly one vertex in $V(Y_A) - V(X_A)$. Furthermore, $|N_{H'''}(V(X_A))| = 3$ for every $(X_A, Y_A) \in \mathcal{S}$.

Proof of Claim 2: Since C_A is not 2-mix-separable from $\mathcal{T}_{\mathcal{L}'''}$, and H''' is subcubic, v is adjacent to at least one vertex in $V(Y_A) - V(X_A)$ and $|N_{H'''}(V(X_A))| \geq 3$. So it is sufficient to prove the first statement. We may assume that v is adjacent to at least two vertices in $V(Y_A) - V(X_A)$, otherwise we are done. So v is adjacent to at most one vertex in $V(X_A)$, and this implies that this vertex, denoted by u, exists, and $u \in V(X_A) - V(Y_A)$. By our choice of (X_A, Y_A), $(X_A - \{v\}, Y_A \cup \{u\} \cup E(H'''[\{u, v\}]))$ mix-separates more non-pendant thick cycles than (X_A, Y_A). But it leads to a contradiction since v is the only neighbor of u not in $X_A - \{v\}$. Therefore, v is adjacent to exactly one vertex in $V(Y_A) - V(X_A)$. $\square$

Claim 3: For every distinct $(X_A, Y_A), (X_{A'}, Y_{A'}) \in \mathcal{S}$, $V(X_A) \cap V(Y_A) \cap V(X_{A'}) \cap V(Y_{A'}) = \emptyset$.

Proof of Claim 3: Suppose that $V(X_A) \cap V(Y_A) \cap V(X_{A'}) \cap V(Y_{A'}) \neq \emptyset$, then

$1 \leq |V(X_A) \cap V(Y_A) \cap V(X_{A'}) \cap V(Y_{A'})| \leq 2$. Let $v \in V(X_A) \cap V(Y_A) \cap V(X_{A'}) \cap V(Y_{A'})$. Since $\mathcal{S}$ is a location and by Claim 2, v is adjacent to exactly one vertex in $V(Y_A) - V(X_A) \supseteq V(X_{A'}) - V(X_A)$ and exactly one vertex in $V(Y_{A'}) - V(X_{A'}) \supseteq V(X_A) - V(X_{A'})$. So either $\deg_{H'''}(v) = 2$, or $|V(X_e) \cap V(Y_e) \cap V(X_f) \cap V(Y_f)| = 2$ and v is adjacent to the other vertex w in $V(X_e) \cap V(Y_e) \cap V(X_f) \cap V(Y_f)$. When $\deg_{H''}(v) = 2$, v is not in any non-pendant thick cycle, otherwise, this non-pendant thick cycle is 2-mix-separable from $\mathcal{T}_{\mathcal{L}'''}$. But this implies that $(X_A - \{v\}, Y_A \cup \{u\} \cup E(H'''[\{u, v\}]))$ is a better choice than (X_A, Y_A), where u is the vertex in $X_A - X_{A'}$ adjacent to v, a contradiction. Therefore, $V(X_A) \cap V(Y_A) \cap V(X_{A'}) \cap V(Y_{A'}) = \{v, w\}$, and v is adjacent to w. Similarly, w is adjacent to one vertex in $V(X_A) - V(X_{A'})$ and one vertex in $V(X_{A'}) - V(X_A)$. However, it implies that C_A is 2-mix-separated by $(X_e \cup X_f, (Y_e \cap Y_f) - \{v, w\})$ from $\mathcal{T}_{\mathcal{L}'''}$, a contradiction. $\Box$

For every bad thick cycle C_A in H''' and every vertex v in $V(X_A) \cap V(Y_A)$, if there exists a thick cycle $C_{A'}$ containing u, v, where u is the vertex in $V(Y_A) - V(X_A)$ adjacent to v, such that u is adjacent to at most two vertices in H''' and u is not in any thick cycle other than $C_{A'}$, then we define (X'_A, Y'_A) to be $(X_A \cup \{u\} \cup E(H'''[\{u, v\}]), Y_A - \{v\})$; otherwise we define $(X'_A, Y'_A) = (X_A, Y_A)$. In particular, $(X'_A, Y'_A) \neq (X_A, Y_A)$ if $C_{A'}$ is thick but not bad. Define $\mathcal{S}' = \{(X'_A, Y'_A) : (X_A, Y_A) \in \mathcal{S}\}$. Then $\mathcal{S}'$ satisfies Claims 1,2,3.

Claim 4: If $(X'_A, Y'_A) \in \mathcal{S}'$ and $v \in V(X'_A) \cap V(Y'_A)$, then there does not exist a thick cycle in H''' containing v and a vertex in $V(Y'_A) - V(X'_A)$.

Proof of Claim 4: Suppose that there exists a thick cycle $C_{A'}$ in H''' containing v and a vertex in $V(Y_A) - V(X_A)$. By the construction, $C_{A'}$ is bad. Since $v \in V(X'_A) \cap V(Y'_A)$, we know that $v \notin V(X'_{A'}) \cap V(Y'_{A'})$ by Claim 2. But $v \in V(X'_A) \subseteq V(Y'_{A'})$, so $v \notin V(X'_{A'})$. However, it is impossible since $C_{A'}$ contains v, a contradiction. $\Box$

Finally, for every $(X'_A, Y'_A) \in \mathcal{S}'$, see the minimal subtree of block trees containing the blocks that contain some vertices in $V(X'_A) \cap V(Y'_A)$. Define $\mathcal{L}_A$ to be the set of separations (X, Y), where each X is a component of X'_A deleting the edges in each block of the mentioned block tree consisting of a single edge. Then, we define $\mathcal{L}^* = \bigcup_{(X'_A, Y'_A) \in \mathcal{S}'} \mathcal{L}_A \cup \{(C_A, H''' - E(C_A)) : (A, B) \in \mathcal{L}''', C_A \subseteq \bigcap_{(X'_{A'}, Y'_{A'}) \in \mathcal{S}'} Y'_{A'}\}$. Note that $\mathcal{L}^*$ is a location in H'''. For each $(X, Y) \in \mathcal{L}^*$, let X^* and Y^* be the subgraph of $G - Z'$ obtained from X and Y, respectively, by replacing C_A by A for each C_A contained in X^* and Y^*, respectively. Let $\mathcal{L}^{**} = \{(X^*, Y^*) : (X, Y) \in \mathcal{L}^*\}$. By Claims 1-4. $\mathcal{L}^{**}$ is a faithful location in $G - Z'$. Since no bad thick cycle in H''' is in $C(\mathcal{L}^{**})$, the level of $\mathcal{L}^{**}$ is at least $t + 1$. This proves the lemma. ■

6.2 Separating thick cycles

Let G be a graph and $\mathcal{T}$ a tangle in G. Let H be a graph and α an H-minor in G. We say that $\mathcal{T}$ *controls* α if there is no $(A, B) \in \mathcal{T}$ of order less than $|V(H)|$ and $v \in V(H)$ such that $V(\alpha(v)) \subseteq V(A)$. And we say that a set $X \subseteq V(G)$ is *free* with respect to $\mathcal{T}$ if there exists no $(A, B) \in \mathcal{T}$ of order less than $|X|$ such that $X \subseteq V(A)$.

Lemma 6.2.1 *Let k be a positive integer, and let G be a graph such that every vertex is adjacent to at most three vertices. Let $\mathcal{T}$ be a tangle in G of order at least $6k$ controlling a K_{6k}-minor α. Assume that there exist k pairs of adjacent vertices x_i, y_i for $1 \leq i \leq k$ such that $|N(\{x_i, y_i\})| = 4$ and there are at least two parallel edges with ends x_i, y_i for each $1 \leq i \leq k$. If $N(\{x_i, y_i\})$ are pairwise disjoint for $1 \leq i \leq k$ and $\bigcup_{i=1}^{k} N(\{x_i, y_i\})$ is free, then G contains a topological minor isomorphic to the Robertson chain of length $2k - 1$.*

Proof. Let $Z = \bigcup_{i=1}^{k} N(\{x_i, y_i\})$. Suppose that there exists a separation (A, B) of G of order less than $|Z|$ such that $Z \subseteq V(A)$ and $A \cap \alpha(h) = \emptyset$ for some $h \in V(K_{6k})$. Since Z is free, $(A, B) \notin \mathcal{T}$. But $V(\alpha(h)) \subseteq V(B)$ and $\mathcal{T}$ controls α, so $(B, A) \notin \mathcal{T}$. It contradicts the first tangle axiom.

For $1 \leq i \leq k$, let the neighbors of x_i other than y_i be $v_{i,1}, v_{i,2}$, and let the neighbors of y_i other than x_i be $v_{i,3}, v_{i,4}$. For $1 \leq i \leq k-1$, let $Z_{i,1} = \{v_{i,3}, v_{i+1,1}\}$ and $Z_{i,2} =$

$\{v_{i,4}, v_{i+1,2}\}$. Let $Z_0 = \{v_{1,1}, v_{1,2}, v_{k,3}, v_{k,4}\}$. So $\{Z_{i,j}, Z_0 : 1 \leq i \leq k-1, j = 1, 2\}$ is a partition of Z. By Lemma 2.2.1, there exists pairwise disjoint paths connecting the two vertices in each $Z_{i,j}$. Note that these paths do not intersect $\{x_i, y_i : 1 \leq i \leq k\}$. These paths together with two edges between x_i, y_i gives a topological minor of the Robertson chain of length $2k - 1$, where the branch vertices are $\{x_i, y_i : 1 \leq i \leq k\}$.
■

Lemma 6.2.2 *Let k be a positive integer, and let G be a graph. Let S be a matching in G such that every end of an edge in S is adjacent to at most three vertices in G. Let $\mathcal{T}$ be a tangle in G of order θ. If $\theta > (4k + 3)^5 + 4$, then either*

1. *there exist k edges $e_1, e_2, ..., e_k$ in S with $|N(e_i)| = 4$ for every $1 \leq i \leq k$ such that $N(e_i)$ are pairwise disjoint for $1 \leq i \leq k$ and $\bigcup_{i=1}^k N(e_i)$ is free, or*
2. *there exists $Z \subseteq V(G)$ with $|Z| \leq (4k + 3)^5$ such that for every edge e in S, either e has an end in Z, or there exists $(A, B) \in \mathcal{T} - Z$ of order at most three such that the ends of e are in $V(A) - V(B)$.*

Proof. For every edge e in S such that $|N(e)| = 4$, let $X_e = N(e)$. By Theorem 2.2.3, either there exist k edges $e_1, e_2, ..., e_k$ in S with $|X_{e_i}| = 4$ for each $1 \leq i \leq k$ such that $N(e_i)$ are pairwise disjoint for $1 \leq i \leq k$ and $\bigcup_{i=1}^k X_{e_i}$ is free, or there exists $Z \subseteq V(G)$ with $|Z| \leq (4k+3)^5$ satisfying that for every edge e in S with $|N(e)| = 4$, either $X_e \cap Z \neq \emptyset$ or X_e is not free in $\mathcal{T} - Z$. Observe that Statement 1 holds for the former case. So we may assume that the latter case holds.

Let e be an edge in S. Note that we are done if some end of e is in Z, so we may assume that the both ends are not in Z. If $|N(e)| \leq 3$, then clearly there exists $(A, B) \in \mathcal{T} - Z$ of order at most three such that the both ends are in $V(A)$. So we may assume that $|N(e)| \geq 4$. But each end of e is adjacent to at most three vertices, so $|N(e)| = 4$. Therefore, X_e is not free and there exists $(A, B) \in \mathcal{T} - Z$ of order at most three such that $X_e \subseteq V(A)$. We choose such (A, B) such that $|V(A) \cap V(B)|$ is as small as possible, and subject to that, A is maximal. Then it is easy to see that the both ends of e are in $V(A) - V(B)$. This proves the lemma. ■

Theorem 6.2.3 *Let k, t be a positive integers. Then there exist positive integers θ, ξ such that if the following hold:*

1. *G is a graph that does not contain a topological minor isomorphic to the Robertson chain of length k.*

2. *$\mathcal{T}$ is a tangle in G of order at least θ controlling a K_{6k}-minor.*

3. *There exists a faithful location $\mathcal{L} \subseteq \mathcal{T}$ of level at least t,*

then there exists $Z^* \subseteq V(G)$ with $|Z^*| \leq \xi$ such that there exists a faithful location $L^* \subseteq \mathcal{T} - Z^*$ of $G - Z^*$ of level at least $t+1$.

Proof. Take $\xi = 12(4k+3)^5$ and $\theta = \xi + t + 6$. Recall that every separation in $\mathcal{L}$ has order at most three by (F1), so $\mathcal{T}_{\mathcal{L}}$ controls a K_{6k}-minor. By (F5), $C(\mathcal{L})$ is weakly subcubic, so there exist pairwise disjoint matchings M_1, M_2, M_3, M_4 of $C(\mathcal{L})$ such that every vertex that is in thick cycle is incident with an edge in $M_1 \cup M_2 \cup M_3 \cup M_4$ by Vizing's theorem. Apply Lemma 6.2.2 by taking $G = C(\mathcal{L})$, $\mathcal{T} = \mathcal{T}_{\mathcal{L}}$ and $S = M_i$ for each $i = 1, 2, 3, 4$ respectively, the first statement of the conclusion of Lemma 6.2.2 cannot hold by Lemma 6.2.1. So the second statement of the conclusion of Lemma 6.2.2 happens for each $i = 1, 2, 3, 4$. In other words, for every $1 \leq i \leq 4$, there exists $Z_i \subseteq V(C(\mathcal{L}))$ with $|Z_i| \leq (4k+3)^5$ such that for every $e \in M_i$ whose both ends are disjoint from Z_i, there exists $(A, B) \in \mathcal{T}_{\mathcal{L}} - Z_i$ of order at most three such that both ends of e are in $V(A) - V(B)$. Let $Z = \bigcup_{i=1}^{4} Z_i$. Therefore, for every thick cycle C in $C(\mathcal{L})$ whose vertices are disjoint from Z, there exists $(A', B') \in \mathcal{T}_{\mathcal{L}} - Z$ of order at most three such that $V(C) \subseteq V(A') - V(B')$. By Lemma 6.1.2, there exists $Z^* \subseteq V(C(\mathcal{L}))$ with $|Z^*| \leq 12(4k+3)^5 = \xi$ and a faithful location $\mathcal{L}^* \subseteq \mathcal{T} - Z^*$ of $G - Z^*$ of level at least $t+1$. ■

Given a society (S, Ω), we say that the vertices in $\bar{\Omega}$ are the *pegs* of (S, Ω). Recall that trunk was defined prior to Theorem 2.5.5.

Theorem 6.2.4 *For any graph L and positive nondecreasing function ϕ, there are integers $\kappa, \rho, \xi \geq 0$ and $\theta \geq 1$ with the following property. Let $\mathcal{T}$ be a tangle of order at least θ in a graph G controlling no L-minor of G. Then there exists $Z \subseteq V(G)$ with $|Z| \leq \xi$, and a $\mathcal{T} - Z$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ which has a proper arrangement in some surface Σ in which L cannot be drawn such that the following hold.*

1. *$|\bar{\Omega}| \leq 3$ for every $(S, \Omega) \in \mathcal{S}_1$, and $|\mathcal{S}_2| \leq \kappa$, and there exists p with $p \leq \rho$ such that every member in $\mathcal{S}_2$ is a p-vortex.*

2. *For each $(S, \Omega) \in \mathcal{S}_2$, G contains a cycle passing the pegs of (S, Ω) in the order Ω.*

3. *The trunk of the segregation is 2-cell embedded in Σ with a respectful tangle $\mathcal{T}'$ of order at least $\phi(p)$ conformal with $\mathcal{T} - Z$.*

4. *For each pair of societies $(S_1, \Omega_1), (S_2, \Omega_2)$ in $\mathcal{S}_2$ and vertices $x \in \overline{\Omega_1}$ and $y \in \overline{\Omega_2}$, we have that $m_{\mathcal{T}-Z}(x, y) \geq \phi(p)$.*

Proof. By Theorem 7 in [12], there exist integers $\kappa, \theta', \xi, \rho', p'$ with $p' \leq \rho'$ such that if $\mathcal{T}$ is a tangle of order at least θ' in a graph G controlling no L-minor of G, then there exist $Z \subseteq V(G)$ with $|Z| \leq \xi$, a $(\mathcal{T} - Z)$-central segregation $\mathcal{S}' = \mathcal{S}'_1 \cup \mathcal{S}'_2$ of $G - Z$ properly arranged in a surface Σ in which L cannot be drawn and satisfies the first conclusion of this theorem; furthermore, the trunk G' of $\mathcal{S}$ is 2-cell embedded in Σ with a respectful tangle $\mathcal{T}_1$ of order at least $\phi(p') + 2p' + 48$ conformal with $\mathcal{T} - Z$, and for every two vertices x, y in G' incident with two different members in $\mathcal{S}_2$, then $m_{\mathcal{T}'}(x, y) \geq \phi(p') + 2p' + 48$. Taking $\theta = \theta' + 23$, $\rho = \rho' + 24$ and $p = p' + 24$. Apply Lemma 2.5.1 by further taking $t = 0$, we know that for every $(S, \Omega) \in \mathcal{S}'_2$, there exist a cycle C_S and a 6-zone Λ_S satisfying the conclusions of Lemma 2.5.1 and a p-vortex (S', Ω') mentioned in Conclusion 4 of Lemma 2.5.1. Let $\mathcal{S}_2 = \{(S', \Omega') : (S, \Omega) \in \mathcal{S}_2\}$ and let $\mathcal{S}_1 = \{(S'' - \bigcup_{(S,\Omega)\in\mathcal{S}_2'} E(S'), \Omega'') : (S'', \Omega'') \in \mathcal{S}_1', \bar{\Omega}'' \not\subseteq \Lambda\}$. Note that $\mathcal{S}_1 \cup \mathcal{S}_2$ is $\mathcal{T} - Z$-central by the same argument in the proof of Theorem 2.5.5. Then $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ is desired. ■

Theorem 6.2.5 ([42, Theorem 3.2]) *For every connected surface Σ and all integers $t \geq 0$ and $z \geq 0$, there exists an integer $\theta \geq 1$ such that the following is true. Let $\Delta_1, ..., \Delta_t$ be mutually disjoint closed discs in Σ, and let Γ be a 2-cell drawing in Σ such that $U(\Gamma) \cap \Delta_i = V(\Gamma) \cap \partial\Delta_i$ for $1 \leq i \leq t$. Let $|Z| \leq z$, where $Z = \bigcup_{1\leq i\leq t} V(\Gamma) \cap \partial\Delta_i$, and let $(Z_j : 1 \leq j \leq p)$ be a partition of Z satisfying the "topological feasibility condition." Let $\mathcal{T}$ be a respectful tangle of order at least θ in Γ such that $m_{\mathcal{T}}(r_i, r_j) \geq \theta$ for $1 \leq i < j \leq t$, where r_i is the region of Γ meeting Δ_i for $1 \leq i \leq t$, and $V(\Gamma) \cap \partial\Delta_i$ is free for $1 \leq i \leq t$. Then there are mutually disjoint, connected subdrawings $H_1, H_2, ..., H_r$ of Γ with $V(H_j) \cap Z = Z_j$ for $1 \leq j \leq p$.*

Let $\mathcal{S}$ be a segregation of a graph G. A society (S, Ω) in $\mathcal{S}$ in *thick* if $2 \leq |\bar{\Omega}| \leq 3$ and for every pair of disjoint subsets X, Y of $\bar{\Omega}$, there exist two edge-disjoint paths from X to Y, and when $|X| = 1$ and $|Y| = 2$, the vertex in X is the only common end of these two paths. The *level* of a thick society with two pegs is the maximum k such that the society contains a topological minor isomorphic to the Robertson chain of length k with the two pegs as the ends. A thick society (S, Ω) with $|\bar{\Omega}| = 2$ is *pendant* if the following hold.

- For every other society (S', Ω') in $\mathcal{S}$, where $\bar{\Omega} \cap \bar{\Omega}' \neq \emptyset$, $|\Omega'| \leq 2$ and (S', Ω') is not thick.
- One vertex in $\bar{\Omega}$ is in at most one other society in $\mathcal{S}$, and the other vertex is in at most two other societies in $\mathcal{S}$.

The *level* of a segregation is the minimum level of a non-pendant thick society with two pegs.

Lemma 6.2.6 *For every positive integers k, there exists an integer κ such that for every integer θ' and nonnegative nondecreasing function ψ, there exist positive integers ξ, ρ, θ such that if G is a graph that does not contain a topological minor isomorphic to the Robertson chain of length k, and $\mathcal{T}$ is a tangle of order at least θ in G not controlling a K_{6k}-minor, then there exist $Z \subseteq V(G)$ with $|Z| \leq \xi$ and a $(\mathcal{T}-Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ such that the following hold:*

1. *$\mathcal{S}$ can be properly arranged in a surface Σ in which K_{6k} cannot be drawn.*
2. *Every society in $\mathcal{S}_1$ has at most three pegs, and $|\mathcal{S}_2| \leq \kappa$.*
3. *There exists an integer p with $p \leq \rho$ such that for every society (S, Ω) in $\mathcal{S}_2$, (S, Ω) is a p-vortex, and S contains a cycle passing the pegs in the order Ω.*
4. *The trunk of $\mathcal{S}$ is 2-cell embedded in Σ and has a tangle $\mathcal{T}'$ of order at least $\psi(p) + \theta'$ conformal with $\mathcal{T} - Z$.*
5. *For every two vertices x, y that are pegs of different members of $\mathcal{S}_2$, $m_{\mathcal{T}'}(x, y) \geq \psi(p)$.*
6. *Every society in $\mathcal{S}_1$ with exactly two or three pegs is thick unless it consists of a single edge.*
7. *If v is a vertex of the trunk of $\mathcal{S}$ such that there exists $(A, B) \in \mathcal{T}'$ of order at most two and $v \in V(A) - V(B)$, then v is in at most two societies $(S_1, \Omega_1), (S_2, \Omega_2)$ in $\mathcal{S}$, and each $(S_1, \Omega_1), (S_2, \Omega_2)$ is in $\mathcal{S}_1$ and has at most two pegs, and one of them is not thick; every vertex $u \in V(A) \cap V(B)$ is adjacent to at most one vertex in A.*
8. *The trunk of $\mathcal{S}$ is weakly subcubic.*

9. Every thick society in $\mathcal{S}$ with exactly two pegs is pendant.

Proof. Fix k. We shall prove by induction on t that there exists an integer κ_1 such that for every positive integers t, θ' and every nonnegative nondecreasing function ψ, there exist positive integers ξ_t, ρ_t, θ_t such that if G is a graph that does not contain a topological minor isomorphic to the Robertson chain of length k, and $\mathcal{T}$ is a tangle of order at least θ_t in G not controlling a K_{6k}-minor, then there exist $Z_t \subseteq V(G)$ with $|Z_t| \leq \xi_t$ and a $(\mathcal{T} - Z_t)$-central segregation $\mathcal{S}^t = \mathcal{S}_1^t \cup \mathcal{S}_2^t$ such that Conclusion 1-8 hold, but replace κ by $\kappa_1 + (t-1)k$ in Conclusion 2, and the level of $\mathcal{S}^t$ is at least t. Note that either Statement 9 holds (i.e. the level is infinite), or G contains a topological minor of Robertson chain of length t. So this implies the lemma by taking $\kappa = \kappa_1 + (k-1)k$, $\xi = \max\{\xi_t : 1 \leq t \leq k\}$, $\rho = \max\{\rho_t : 1 \leq t \leq k\}$ and $\theta = \max\{\theta_t : 1 \leq t \leq k\}$.

We first prove the base case for $t = 1$. Let θ be a large number that we will determine later. Let G be a graph that does not contain a topological minor isomorphic to the Robertson chain of length k, and let $\mathcal{T}$ be a tangle in G of order θ. Let ξ_0 and θ_0 be the numbers such that if $\theta > \theta_0$, then there exist $Z_0 \subseteq V(G)$ with $|Z_0| \leq \xi_0$ and a faithful location $\mathcal{L}$ contained in $\mathcal{T} - Z_0$. These numbers ξ_0 and θ_0 exist by Corollary 1.4.2 and Lemma 6.1.1. Let $\mathcal{T}'$ be the tangle in $C(\mathcal{L})$ induced by $\mathcal{L}$. Note that $\mathcal{L}$ is faithful and $\mathcal{T} - Z_0$ does not control a K_{6k}-minor, so $\mathcal{T}'$ does not control a K_{6k}-minor.

By Theorem 7 in [12], there exist integers $\kappa_1, \theta_1, \xi_1', \rho_1', p_1'$ with $p_1' \leq \rho_1'$ such that there exist $Z_1' \subseteq V(C(\mathcal{L}))$ with $|Z_1'| \leq \xi_1$, a $(\mathcal{T}' - Z_1')$-central segregation $\mathcal{S}' = \mathcal{S}_1' \cup \mathcal{S}_2'$ of $C(\mathcal{L}) - Z_1'$ properly arranged in a surface Σ in which K_{6k} cannot be drawn and satisfies the Conclusion 1,2 of this lemma and every member of $\mathcal{S}_2'$ is a p_1'-vortex; furthermore, the trunk G' of $\mathcal{S}'$ is 2-cell embedded in Σ with a respectful tangle $\mathcal{T}_1$ of order at least $\psi(p_1') + \theta' + 2p_1' + 48$ conformal with $\mathcal{T}' - Z_1'$, and for every two vertices x, y in G' incident with two different members in $\mathcal{S}_2'$, then $m_{\mathcal{T}'}(x, y) \geq \psi(p_1') + \theta' + 2p_1' + 48$. Taking $\theta = \theta' + 23$, $\rho = \rho' + 24$ and $p = p' + 24$. Apply Lemma

2.5.1 by further taking $t = 0$, we know that for every $(S, \Omega) \in \mathcal{S}_2'$, there exist a cycle C_S and a 6-zone Λ_S satisfying the conclusions of Lemma 2.5.1 and a p-vortex (S', Ω') mentioned in Conclusion 4 of Lemma 2.5.1. Let $\mathcal{S}_2^1 = \{(S, \Omega) : (S, \Omega) \in \mathcal{S}_2'\}$ and let $\mathcal{S}_1^1 = \{(S'' - \bigcup_{(S,\Omega)\in\mathcal{S}_2'} E(S'), \Omega'') : (S'', \Omega'') \in \mathcal{S}_1', \bar{\Omega}'' \not\subseteq \Lambda\}$. Note that $\mathcal{S}_1^1 \cup \mathcal{S}_2^1$ is $\mathcal{T} - Z$-central by the same argument in the proof of Lemma 2.5.5. Then $\mathcal{S}^1 = \mathcal{S}_1^1 \cup \mathcal{S}_2^1$ satisfies Conclusions 1-5. We now determine θ. We define θ to be $\theta_0 + \theta_1$. And we let $Z_1 = Z_0 \cup Z_1'$ and $\xi_1 = \xi_0 + \xi_1'$.

For each society (S, Ω) in $\mathcal{S}^1$, we see the minimal subtree of the block-tree of S containing the blocks containing the pegs, and then make each of those blocks in the subtree consisting of a single edge into a new society and merge the component of the graph obtained from S by deleting those blocks into a new society. We replace $\mathcal{S}^1$ by the resulting segregation. Then $\mathcal{S}^1$ satisfies Conclusions 1-7. Note that $C(\mathcal{L})$ is weakly subcubic, so $\mathcal{S}^1$ satisfies Conclusion 8 as well. Observe that Conclusion 6 implies that the level of $\mathcal{S}^1$ is at least one. Therefore, we prove the base case of our claim.

Now we assume that our claim holds for a positive integer t and prove the case for $t + 1$. Let ϕ_0 be the maximum number θ mentioned in Theorem 6.2.5 by taking $t = 4k$, $z = 4k$, and Σ any surface in which K_{6k} cannot be drawn. Notice that ϕ only depends on k. Let $w(x)$ be the function θ obtained from applying Lemma 2.5.5 by taking $\phi = \psi$, $\rho = x$, $\lambda = \phi_0 + 2k + 15$, $\kappa = \kappa_0 + (t - 1)k$, $k = k$, $\theta^* = \theta'$, $d = d$, $s = 2$, and $\mathcal{F}$ the family of graphs consisting of the graph consisting of two vertices and two parallel edges. Apply induction hypothesis by taking $k = k$, $\theta' = \theta'$, and $\psi(x) = w(x) + \phi_0 + 2k + 15$, we obtain ξ_t, ρ_t, θ_t that satisfy our claim. For every integer x, let $\theta_{t+1}'(x)$ and $\rho_{t+1}'(x)$ be the number θ and ρ^* mentioned in the conclusion of Lemma 2.5.5 by taking $\phi = \psi$, $\rho = x$, $\lambda = \phi_0 + 2k + 15$, $\kappa = \kappa_1 + (t - 1)k$, $k = k$, $\theta^* = \theta'$, $d = 4$, $s = 2$, and $\mathcal{F}$ be the set consisting of the graph that consists of two vertices and two parallel edges between them. We define $\xi_{t+1} = \xi_t$,

$\theta_{t+1} = \max\{\theta'_{t+1}(x) : 0 \leq x \leq \rho_t\} + \xi_t$ and $\rho_{t+1} = \max\{\rho'_{t+1}(x) : 0 \leq x \leq \rho_t\}$.

Let G be a graph with a tangle of order θ with $\theta \geq \theta_{t+1}$ such that G does not contain a topological minor isomorphic to the Robertson chain of length k, and $\mathcal{T}$ does not control a K_{6k}-minor. By induction, there exists $Z_t \subseteq V(G)$ with $|Z_t| \leq \xi_t$, $p_t \leq \rho_t$, and a $(\mathcal{T} - Z_t)$-central segregation $\mathcal{S}^t = \mathcal{S}_1^t \cup \mathcal{S}_2^t$ such that Conclusions 1-8 hold, and $\mathcal{S}^t$ has level at least t. Let $\mathcal{T}_t$ be the tangle $\mathcal{T}'$ of order $w(p_t)+\phi_0+2k+15+\theta'$ mentioned in Conclusion 4. Let H_t be the trunk of $\mathcal{S}^t$.

We say that a pair of two vertices of H_t is a *twin pair* if they are the two pegs of a thick society with two pegs in $\mathcal{S}^t$. We say that a twin pair is 3-*separable* if there exists $(A, B) \in \mathcal{T}_t$ such that the two vertices in the pair are in $V(A) - V(B)$.

Suppose that there exist $k + 1$ non-3-separable twin pairs $W_1, W_2, ..., W_{k+1}$ in H_t such that every vertex in a pair has distance at least $\phi_0 + 2(k + 1) + 6$ to any vertex in another pair under the metric $m_{\mathcal{T}_t}$. For $1 \leq i \leq k + 1$, let Δ_i be a closed disk in Σ such that $\partial\Delta_i \cap H_t = N_{H_t}[W_i]$ and $\Delta_i \cap H_t = H_t[N_{H_t}[W_i]]$. Let $W = \bigcup_{i=1}^{k+1} W_i$ and $H'_t = H_t - W$. So $\mathcal{T}_t - W$ is a tangle in H'_t conformal with $\mathcal{T}_t$ of order at least ϕ_0. Since every vertex in some W_i has distance at least $\phi_0 + 2(k+1) + 6$ to any vertex in W_j for $i \neq j$ under the metric $m_{\mathcal{T}_t}$, the distance between every vertex in $N_{H_t}[W_i]$ and every vertex in $N_{H_t}[W_j]$ for $1 \leq i < j \leq k + 1$ is at least $\phi_0 + 4$, and $N_{H_t}[W_i]$ is free in $\mathcal{T}_t - W$ for each $1 \leq i \leq k + 1$. Then by Theorem 6.2.5, H_t contains a topological minor isomorphic to the Robertson chain of length k. Since $\mathcal{S}^t$ satisfies Conclusion 6, it implies that G contains a topological minor isomorphic to the Robertson chain of length k, a contradiction. Therefore, there exist an integer ℓ with $0 \leq \ell \leq k$ and ℓ $(\phi_0 + 2k + 15)$-zones $\Lambda_1, ..., \Lambda_\ell$ of H_t such that every non-3-separable twin pair is contained in the interior of some Λ_i.

Observe that $\mathcal{T}'$ has order at least $w(p_t)$. So we can apply Lemma 2.5.5 by taking $\phi = \psi$, $\rho = p_t$, $\lambda = \phi_0+2k+15$, $\kappa = \kappa_1+(t-1)k$, $k = k$, $\theta^* = \theta'$, $d = 4$, $s = 2$, and $\mathcal{F}$ be the set consisting of the graph that consists of two vertices and two parallel edges between them, and then further taking $G = G - Z_t$, $\mathcal{T} = \mathcal{T} - Z_t$, $\mathcal{S} = \mathcal{S}^t$ to obtain a segregation $\mathcal{S}^{t*} = \mathcal{S}_1^{t*} \cup \mathcal{S}_2^{t*}$ mentioned in the conclusion of Lemma 2.5.5. That is, $\mathcal{S}^{t*} = \mathcal{S}_1^{t*} \cup \mathcal{S}_2^{t*}$ is a $(\mathcal{T} - Z_t)$-central segregation of $G - Z_t$ such that the following hold.

- $\mathcal{S}_1^{t*} \subseteq \mathcal{S}_1^t$.
- Each member of $\mathcal{S}_2^{t*}$ is a ρ_{t+1}-vortex.
- $|\mathcal{S}_2^{t*}| \leq \kappa_1 + tk$.
- $\bigcup_{(S,\Omega) \in \mathcal{S}_2^t} S \subseteq \bigcup_{(S,\Omega) \in \mathcal{S}_2^{t*}} S$.
- There exists a cycle passing though $\bar{\Omega}$ in order Ω for every $(S, \Omega) \in \mathcal{S}_2^{t*}$.
- The trunk of $\mathcal{S}^{t*}$ is 2-cell embedded in Σ with a respectful tangle $\mathcal{T}_t^*$ of order at least η conformal with $\mathcal{T} - Z_t$.
- Every two pegs of different members of $\mathcal{S}_2^{t*}$ have distance at least η under the metric $m_{\mathcal{T}_t^*}$.

Observe that $\mathcal{S}^{t*}$ satisfies conclusions 1-6 and 8 of this lemma, and every twin pair is 3-separable in $\mathcal{T}_t^*$. Note that we can further assume that there does not exist $(S, \Omega) \in \mathcal{S}_2^{t*}$ and $(A, B) \in \mathcal{T} - Z_t$ of order at most two such that A consists of the union of some member of $\mathcal{S}_1^{t*}$ and $V(A) \cap V(B) \subseteq \bar{\Omega}$, since we can replace S by $S \cup A$ without violating Conclusion 1-6 and 8 and creating non-3-separable twin pairs.

Now we determine C and C'. Define $C' = \eta + \phi + 2k + 15 + f(k, \eta)$ and $C = C' + \rho_1 + \kappa_1 + \xi_1$.

Let H_t^* be the trunk of $\mathcal{S}^{t*}$. For every vertex v in H_t^* that is 2-separable from $\mathcal{T}_t^*$, let $(A_v, B_v) \in \mathcal{T}_t^*$ of order at most two such that $v \in V(A_v) - V(B_v)$, and subject to that, A_v is as large as possible. By our assumption, at most one vertex in $V(A_v) \cap V(B_v)$ is in a peg of a member of $\mathcal{S}_2^{t*}$, for every v such that (A_v, B_v) is defined. On the other hand, the collection of these (A_v, B_v) is a location. Therefore, by merging members of $\mathcal{S}_1^{t*}$ contained in some A_v into a new society, we may assume no vertex in H_t^* is 2-separable from $\mathcal{T}_t^*$, but it might not satisfy Conclusion 6 anymore. Then for each society (S, Ω) in $\mathcal{S}_1^{t*}$ with two pegs, we see the minimal subtree of the block-tree of S containing the blocks containing a peg, and then making each block consisting of a single edge as a new society and making each component of the graph obtained from S by deleting those edges into a new society. This operation makes $\mathcal{S}^{t*}$ satisfies Conclusions 1-8 and without creating non-3-separable twin pairs. In addition, if this operation creates a new non-pendant thick society with two pegs, then this new society contains a non-pendant old thick society with two pegs, and there exist two edge-disjoint paths from one peg of the new society to one peg of the old society and two edge-disjoint paths from the other peg of the new society to the other peg of the old society (these paths can be trivial), since H_t^* is weakly subcubic and by the maximality of A_v. Therefore, the level of H^{t*} is at least t.

For each $(S, \Omega) \in \mathcal{S}_1^{t*}$, let (A_S, B_S) be the separation in $\mathcal{T}_t^*$ such that $A_S = S$ and $B_S = H_t^*[V(S) - \bar{\Omega}]$. Consequently, $\{(A_S, B_S) : (S, \Omega) \in \mathcal{S}_1^{t*}\}$ is a faithful location, denoted by $\mathcal{L}^*$, in H_t^* of level at least t such that for every thick cycle C_A in $C(\mathcal{L}^*)$, there exists $(A', B') \in \mathcal{T}_{\mathcal{L}^*}$ of order at most three such that $V(C_A) \subseteq V(A') - V(B')$. By Lemma 6.1.2, there exists a faithful location $\mathcal{L}^{**} \subseteq \mathcal{T}_t^*$ in H_t^* of level at least $t+1$.

Note that if $(A, B) \in \mathcal{L}^{**}$ such that at least two vertices in $V(A) \cap V(B)$ are in $\bigcup_{(S,\Omega)\in\mathcal{S}_2^{t*}} V(S)$, then there exists $(S_A, \Omega_{S_A}) \in \mathcal{S}_2^{t*}$ such that $V(A) \cap V(B) \cap \bigcup_{(S,\Omega)\in\mathcal{S}_2^{t*}} V(S) \subseteq \overline{\Omega_{S_A}}$, since members of $\mathcal{S}_2^{t*}$ are far apart under the metric $m_{\mathcal{T}_t^*}$. For every $(S, \Omega) \in \mathcal{S}_2^{t*}$, let $S^* = S \cup (\bigcup_{(A,B)\in\mathcal{L}^{**}, S_A = S} S_A)$ and let $\bar{\Omega}^*$ be the set of vertices in S^* adjacent to some vertex not in S^*. Notice that there exists a cycle passing all vertices in $\bar{\Omega}^*$, and we define Ω^* to be a cyclic ordering to $\bar{\Omega}^*$ consistent with the cycle. For every $(A, B) \in \mathcal{L}^{**}$, we let Ω_A be an arbitrary ordering of $V(A) \cap V(B)$. Define $\mathcal{S}_1^{t+1} = \{(A, \Omega_A) : (A, B) \in \mathcal{L}^{**}, |V(A) \cap V(B) \cap \bigcup_{(S,\Omega)\in\mathcal{S}_2^{t*}} V(S)| \leq 1\}$ and

$\mathcal{S}_2^{t+1} = \{(S^*, \Omega^*) : (S, \Omega) \in \mathcal{S}_2^{t*}\}$. Then $\mathcal{S}^{t+1} = \mathcal{S}_1^{t+1} \cup \mathcal{S}_2^{t+1}$ is a segregation of $C(\mathcal{L})$ satisfies Conclusions 1-8 and has level at least $t+1$. Since $\mathcal{L}$ is faithful, by (F2) and (F5), we may assume that $\mathcal{S}^{t+1}$ is a $\mathcal{T} - Z_t$-central segregation of $G - Z_t$ by flipping edges from a society to another such that C_A is contained in a society in $\mathcal{S}^{t+1}$ for each $(A, B) \in \mathcal{L}$. This completes the proof. ■

6.3 Taming a vortex

Let (U, V) be a drawing in Σ. We say that disjoint cycles $C_1, C_2, ..., C_n$ in U are *concentric* if for $1 \leq i \leq n$, C_i bounds an open disk $D(C_i)$ in Σ such that $D(C_1) \supseteq ... \supseteq D(C_n)$.

Let G be a graph with a segregation $\mathcal{S}$ properly arranged by τ in Σ. We say that concentric cycles $C_1, C_2, ..., C_n$ in the trunk of $\mathcal{S}$ *enclose* $(S, \Omega) \in \mathcal{S}$ if $\tau(S, \Omega) \subseteq D(C_n)$ and $\bar{\Omega} \cap V(C_n) = \emptyset$. Furthermore, $C_1, C_2, ..., C_n$ *tightly enclose* (S, Ω) if they enclose (S, Ω), and for every $1 \leq k \leq n$ and every point $v \in \partial D(C_k)$, there exist a vertex $w \in \bar{\Omega}$ and a curve in Σ connecting v, w that intersects the trunk in at most $n-k+2$ vertices.

Lemma 13 in [9] can be restated as follows.

Lemma 6.3.1 ([9, Lemma 13]) *Let G be a graph and $\mathcal{S}$ a segregation of G properly arranged in a surface Σ by an arrangement τ. Let $(S, \Omega) \in \mathcal{S}$. If there exist cycles $C_1, C_2, ..., C_n$ in the trunk of $\mathcal{S}$ enclosing a member (S, Ω) of $\mathcal{S}$, then there are n cycles $C_1', C_2', ..., C_n'$ in the trunk of $\mathcal{S}$ that enclose (S, Ω) tightly, such that $D(C_1') \subseteq D(C_1)$.*

Let (S, Ω) be a society, and denote $\bar{\Omega}$ by $v_1, v_2, ..., v_n$ in order. A *circular decomposition* $(P, \mathcal{X})$ of a society (S, Ω) is a tuple $\mathcal{X} = (X_1, X_2, ..., X_n)$ of subsets $V(S)$ such that the following hold.

- $v_i \in X_i$ for $1 \leq i \leq n$.
- $X_1 \cup X_2 \cup ... \cup X_n = V(S)$.

- $v_i < v_j < v_k < v_\ell$ according to the cyclic ordering Ω, then $X_i \cap X_k \subseteq X_j \cup X_\ell$.

- Every edge of S has both ends in X_i for some $1 \leq i \leq n$.

The *circular adhesion* of $(P, \mathcal{X})$ is the maximum size of $X_i \cap X_{i+1}$. We say that $(P, \mathcal{X})$ is *circularly peg-linked* if the following hold.

- For every i, $X_i \cap X_{i+1} - \bar{\Omega}$ has the same size.

- There are $|X_i \cap X_{i+1} - \bar{\Omega}|$ disjoint paths from $X_{i-1} \cap X_i - \bar{\Omega}$ to $X_i \cap X_{i+1} - \bar{\Omega}$ in $S[X_i] - \bar{\Omega}$.

- $X_i \cap \bar{\Omega} = \{v_{i-1}, v_i\}$.

Note that the union of these disjoint paths from $X_{i-1} \cap X_i - \bar{\Omega}$ to $X_i \cap X_{i+1} - \bar{\Omega}$ in $S[X_i] - \bar{\Omega}$ are disjoint cycles. We call the set of these cycles the *circular linkage* of (S, Ω).

Lemma 23 in [9] can be restated as follows.

Lemma 6.3.2 ([9, Lemma 23]) *Let ξ, κ, ρ be integers. Let G be a graph such that there exists $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that $G - Z$ has a segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ with $|\mathcal{S}_2| \leq \kappa$ properly arranged in a surface Σ by an arrangement τ. Assume that the following hold.*

1. *$|\bar{\Omega}| \leq 3$ for every $(S, \Omega) \in \mathcal{S}_1$.*

2. *For every member $(S, \Omega) \in \mathcal{S}_2$, (S, Ω) is a ρ-vortex and there exist $\rho+1$ concentric cycles $C_0(S), C_1(S), ..., C_\alpha(S)$ in the skeleton of $\mathcal{S}$ tightly enclosing (S, Ω).*

3. *For every different members $(S, \Omega), (S', \Omega') \in \mathcal{S}_2$, the closure of $D(C_0(S))$ and the closure of $D(C_0(S'))$ are disjoint.*

Then there exist $Z' \subseteq V(G)$ containing Z with $|Z' - Z| \leq \xi + \kappa(2\rho + 2)$ containing Z, a segregation $\mathcal{S}' = \mathcal{S}_1' \cup \mathcal{S}_2'$ of $G - Z'$ such that the following hold.

1. *$\mathcal{S}'$ has a proper arrangement τ' in Σ, and the skeleton of $\mathcal{S}'$ is a subgraph of the skeleton of $\mathcal{S}$.*

2. *$\mathcal{S}'_1 \subseteq \mathcal{S}_1$.*

3. *For every $(S, \Omega) \in \mathcal{S}'_2$, there exists $(S', \Omega') \in \mathcal{S}_2$ such that $D(C_0(S')) \supseteq \tau'(S, \Omega) \supseteq \tau(S', \Omega')$.*

4. *$|\mathcal{S}'_2| \leq \kappa$, and every member of $\mathcal{S}'_2$ is a $(\rho + 1)$-vortex.*

5. *Every member (S, Ω) of $\mathcal{S}'_2$ has a peg-linked circular decomposition of circular adhesion at most $\rho + 1$ and a cycle C in S that is disjoint from all the cycles of the circular linkage of S and traverses $\bar{\Omega}$ in the order Ω.*

Let (S, Ω) be a society, and denote $\bar{\Omega}$ by $w_1, w_2, ..., w_n$ in order. We say that $(P, \mathcal{X})$ is a *path-cut decomposition* of (S, Ω) if P is a path $t_1 t_2 ... t_n$ on $|\bar{\Omega}|$ vertices, and $\mathcal{X}$ is a partition of $V(S)$ (but we allow some bags to be empty) such that the bag of t_i contains w_i for $1 \leq i \leq |\bar{\Omega}|$. A *circular cut-decomposition* of (S, Ω) is a path-cut decomposition $(P = v_1 v_2 ... v_n, \mathcal{X})$ of (S, Ω) such that every edge of S with ends in different bags, say X_i and X_j, is associated with a subpath of the cycle $C = v_1 v_2 ... v_n v_1$ of the ends v_i and v_j. The *adhesion-set* of an edge uv of C is a maximum set of edges of S with ends in different bags associating with subpaths containing uv such that no two edges have the same ends. The *circular-adhesion of an edge uv of C in* $(P, \mathcal{X})$ is the size of its adhesion-set. The *circular-adhesion of* $(P, \mathcal{X})$ is the maximum circular-adhesion of an edge of C in $(P, \mathcal{X})$. For every i with $1 \leq i \leq n$, the *in-adhesion-set* of v_i is the set of edges of S in the adhesion-set of $v_{i-1} v_i$ but not in the adhesion-set of $v_i v_{i+1}$; the *out-adhesion-set* of v_i is the set of edges of S in the adhesion-set of $v_i v_{i+1}$ but not in the adhesion-set of $v_{i-1} v_i$. A circular cut-decomposition $(P, \mathcal{X})$ is *circularly peg-linked* if the following hold.

- Every edge of C has the same circular-adhesion p.

- For every edge e of C, the edges in the adhesion-set of e form a matching in S.
- There exist disjoint cycles in S containing all edges in the union of the adhesion-sets such that for every such cycle C, and for every i with $1 \leq i \leq |\bar{\Omega}|$, every component of $C \cap X_i$ is a path from a vertex that is incident with exactly one edge in the in-adhesion set of v_i to a vertex that is incident with exactly one edge in the out-adhesion set of v_i. We call these disjoint cycles a *circular linkage* of (S, Ω). And one of these cycles passes the pegs in the order the same as Ω, and it contains exactly one edge in each adhesion set.

A *thick pair* in a graph G is a pair of vertices x, y of G such that there exist at least two edges whose ends are x, y. Let (S, Ω) be a society. A *thick spider* in S with head (x, y) is a subgraph of S consisting of a thick pair (x, y) in S and paths $Q_{x,1}, Q_{x,2}, Q_{y,1}, Q_{y,2}$, where $Q_{x,1}, Q_{x,2}$ are edge-disjoint paths from x to $\bar{\Omega}$, and $Q_{y,1}, Q_{y,2}$ are edge-disjoint paths from y to $\bar{\Omega}$, and $V(Q_{x,1}) \cup V(Q_{x,2})$ is disjoint from $V(Q_{y,1}) \cup V(Q_{y,2})$. We say that a thick spider is *aggressive* if there exist two disjoint cyclic subintervals of $(0, |\bar{\Omega}|]$ such that one of them contains $\bar{\Omega} \cap (V(Q_{x,1}) \cup V(Q_{x,2}))$ and the other contains $\bar{\Omega} \cap (V(Q_{y,1}) \cup V(Q_{y,2}))$. The *territory* of an aggressive thick spider with head (x, y) is a minimum cyclic subinterval $[a, b]$ of $(0, |\bar{\Omega}|]$ such that $[a, b]$ contains the ends of $Q_{x,1}, Q_{x,2}, Q_{y,1}, Q_{y,2}$ in $\bar{\Omega}$ in order. The following is an Erdős-Pósa property for aggressive thick spiders.

Lemma 6.3.3 *Let (S, Ω) be a society, and let p be a positive integer. If (S, Ω) has a circular cut-decomposition $(P, \mathcal{X})$ of circular adhesion at most p, then for every positive integer k, either there exist k aggressive thick spiders with disjoint heads in S having disjoint territories, or there exist at most $(4p + 2)(k - 1)$ vertices intersecting all aggressive thick spiders in S.*

Proof. Let k be the minimum positive integer such that there do not exist k aggressive spiders with disjoint heads in S with pairwise disjoint territory. We are done if

$k = 1$, so we may assume that $k \geq 2$. Let $K_1, K_2, ..., K_{k-1}$ be a maximal family of aggressive thick spiders with disjoint heads having disjoint territories $I_1, I_2, ..., I_{k-1}$, and subject to this, $\sum_{i=1}^{k-1}|I_i|$ is as small as possible. Denote I_i by $[a_i, b_i]$ for every $1 \leq i \leq k-1$. We say that an edge e of S is a *boundary-edge* if the ends of e are in two different bags X_α and X_β, for some different integers α, β, and there exists $1 \leq i \leq k-1$ such that $a_i \in [\alpha, \beta]$ or $b_i \in [\alpha, \beta]$. Note that there exist at most $(4p+2)(k-1)$ boundary-edges.

Observe that there does not exist an aggressive thick spider whose heads is disjoint from the heads of $K_1, ..., K_{k-1}$ and whose territory is strictly contained in I_i for some $1 \leq i \leq k-1$, by the minimality of $\sum_{i=1}^{k-1}|I_i|$. So the territory of any aggressive thick spider contains an end of I_i for some $1 \leq i \leq k-1$, unless its head intersects a head of K_j for some $1 \leq j \leq k-1$. Note that every thick spider intersects at least two different pegs of (S, Ω), so the territory of every aggressive thick spider contains at least two integers. Therefore, every aggressive thick spider either contains a boundary-edge or has the head intersects the head of some K_i. So there exist at most $4p(k-1) + 2(k-1)$ vertices intersecting all aggressive thick spiders. ■

Lemma 6.3.4 *Let Σ be a surface and Δ a closed disk in Σ. Let G_0 be a drawing in Σ such that $G_0 \cap \Delta \subseteq V(G_0)$. Let G be a graph and (S, Ω) a society such that $G = G_0 \cup S$, $V(G_0) \cap V(S) = \bar{\Omega}$, and $E(G_0) \cap E(S) = \emptyset$. Assume that there exists a cycle C in $G_0 - \bar{\Omega}$ such that the following hold.*

1. *C bounds a closed disc $D(C)$ in Σ such that $\Delta \subseteq D(C)$.*

2. *(S, Ω) has a circular cut-decomposition $(P^*, \mathcal{X}^*)$ such that S contains a cycle Q that passes all pegs in the order same as Ω, and for every bag X_i of $(P^*, \mathcal{X}^*)$, $Q \cap X_i^*$ is a subpath of Q from a vertex incident with an edge in the in-adhesion set of t_i to a vertex incident with an edge in the out-adhesion set of t_i, where $P^* = t_1 t_2 ... t_{|\bar{\Omega}|}$.*

3. *Every vertex of* $G_0 - \bar{\Omega}$ *inside* $D(C)$ *has at most three neighbors in* G_0.

4. *Every vertex in* $\bar{\Omega}$ *is adjacent to exactly one vertex in* $G - S$.

Let k *be a positive integer. Then either there exists a separation* (A, B) *of order less than* $2k$ *of* G *such that* $S \subseteq A$ *and* $C \subseteq B$, *or there exists a society* (S^*, Ω^*) *with* $S \subseteq S^* \subseteq G \cap D(C)$, *such that if there exist* $k+1$ *aggressive thick spiders in* S^* *with disjoint heads having disjoint territories, then* G *contains a topological minor isomorphic to the Robertson chain of length* $2k+1$.

Proof. Pick (S^*, Ω^*) be a society that satisfies the following properties.

- $V(S) \cap V(C) = \emptyset$.
- (S^*, Ω^*) has a circular cut-decomposition $(P^*, \mathcal{X}^*)$ such that there exists a cycle Q^* passing all pegs in the order same as Ω, and for every bag X_i^* of $(P^*, \mathcal{X}^*)$, $Q^* \cap X_i^*$ is a subpath of Q^* from a vertex incident with an edge in the in-adhesion set of the i-th node of P to a vertex incident with an edge in the out-adhesion set of the i-th node of P.
- $S \subseteq S^* \subseteq G \cap D(C)$.
- Every vertex in $\bar{\Omega}^*$ is adjacent to exactly one vertex in $G - S^*$.
- $N_G(G - S^*) \subseteq \bar{\Omega}^*$.
- Subject to the above conditions, S^* is maximal.

Note that (S^*, Ω^*) exists since (S, Ω) is a candidate. We are done if there exists a separation (A, B) of G of order less than $2k$ such that $S^* \subseteq A$ and $C \subseteq B$. So we may assume that there exist $2k$ disjoint paths from $\bar{\Omega}^*$ to $V(C)$.

We claim that for every $U \subseteq \bar{\Omega}^*$ with $|U| \leq 2k$, there exist $2k$ disjoint paths from U to $V(C)$ internally disjoint from S^*. Assume that there exist an integer ℓ with

$0 \leq \ell \leq 2k$ and a maximal set $\{P_i : 0 \leq i \leq \ell\}$ of disjoint paths from U to $V(C)$ internally disjoint from S^*, for some $0 \leq \ell \leq 2k$. The claim is proved if $\ell = 2k$, so we may assume that $\ell < 2k$. So there exists a vertex w in $U - \bigcup_{1 \leq i \leq \ell} P_i$. First assume that there does not exist a path from w to $V(C)$ internally disjoint from S^*. Let W be the set of vertices in $G - V(S^*)$ connected by a path from w internally disjoint from S^*. Note that W is nonempty since w is adjacent to a vertex not in S^*. On the other hand, W is disjoint from $V(C) \cup \bigcup_{i=0}^{\ell} V(P_i)$. Therefore, there exists Ω' with $\bar{\Omega}' \subseteq \bar{\Omega}^* - \{w\}$ such that $(G[V(S^*) \cup W], \Omega')$ is a society satisfying the above condition but contradicting the maximality of S^*. Hence, there exists a path P from w to $V(C)$ internally disjoint from S^*. Since $\{P_i : 1 \leq i \leq \ell\}$ is a maximal set, P intersects P_i for some $1 \leq i \leq \ell$. As every vertex in C is adjacent to at most three vertices in G, $V(P) \cap V(P_i) \not\subseteq V(C)$. Let $D(S^*)$ be the minimum disk in Σ such that $S^* \subseteq G \cap D(S^*)$. Since $D(C)$ is a disk, there exists a disk D^* in Σ disjoint from C with $\partial D^* \subseteq \partial D(S^*) \cup P_i \cup P$ such that D^* properly contains $D(S^*)$. Therefore, there exists Ω'' with $\bar{\Omega}'' \subseteq \bar{\Omega}^* \cup V(P) \cup V(P_i)$ such that $(G \cap D^*, \Omega'')$ is a society satisfying the above condition but contradicts the maximality of S^*. This proves the claim.

Now we assume that there exist $k+1$ aggressive thick spiders $L_1, L_2, ..., L_{k+1}$ with head disjoint heads (x_i, y_i) in (S^*, Ω^*) having disjoint territories $I_1, I_2, ..., I_{k+1}$. Without loss of generality, we may assume that $I_1, I_2, ..., I_{k+1}$ are in order. For $1 \leq i \leq k$, let Q_i^* be the shortest subpath of Q^* with one end in $V(L_i)$ and the other end in $V(L_{i+1})$ without passing any vertex in X_j for every j in the territory of L_ℓ with $\ell \in \{1, 2, ..., k+1\} - \{i, i+1\}$. Note that the existence of Q_i^* follows from the fact that Q^*, L_i, L_{i+1} contain some pegs and the fact the intersection of Q^* and any bag is a subpath. Denote the four paths in L_i by $Q_{x_i,1}, Q_{x_i,2}, Q_{y_i,1}, Q_{y_i,2}$. Without loss of generality, we may assume that Q_i^* connects one vertex in $Q_{y_i,2}$ and one vertex in $Q_{x_{i+1},1}$. For $2 \leq i \leq k+1$, let u_i be the peg contained in $Q_{x_i,2}$; for $1 \leq i \leq k$, let v_i be the peg contained in $Q_{y_i,1}$.

By our previous claim, there exist $2k$ disjoint paths $W_1, W_2, ..., W_{2k}$ from $\{u_{i+1}, v_i : 1 \leq i \leq k\}$ to $V(C)$ internally disjoint from S^*. For every $1 \leq i \leq k$, let u'_{i+1} and v'_i be the ends of those disjoint paths such that W_{2i-1} is from v_i to v'_i and W_{2i} is from u_{i+1} to u'_{i+1}. Since $D(C)$ is a disk, $\{u'_{i+1}, v'_i : 1 \leq i \leq k\}$ appears in C in the order $v'_1, u'_2, v'_2, ..., u'_k, v'_k, u'_{k+1}$. Consequently, it is obvious that $\bigcup_{i=1}^{k+1} L_i \cup \bigcup_{i=1}^{k} Q^*_i \cup \bigcup_{i=1}^{2k} W_i \cup C$ contains a topological minor isomorphic to the Robertson chain of length $2k+1$. This completes the proof. ■

Lemma 6.3.5 *Let p be an integer, and let (S, Ω) be a society having a circularly peg-linked circular cut-decomposition $(P, \mathcal{X})$ of circular adhesion p. Let $C_1, C_2, ..., C_q$ be a circular linkage in S, where $q \leq p$ and C_1 contains $\bar{\Omega}$. Then one of the following holds.*

1. *There exists another circularly peg-linked circular decomposition with the same circular linkage that has less number of edges of S between different bags and has no more thick pairs between two different bags.*

2. *For every $2 \leq i \leq q$, $1 \leq j \leq n$, if $C_i \cap X_j \neq \emptyset$, then for every component Q of $C_i \cap X_j$, there exists a path in X_j from Q to $C_1 \cap X_j$.*

Proof. Let $(P = t_1 t_2 ... t_{|\bar{\Omega}|}, \mathcal{X})$ be the circularly peg-linked circular decomposition of (S, Ω) with the circular linkage $C_1, C_2, ..., C_q$ such that the number of thick pairs between two different bags is minimum, and subject to that the number of edges between two different bags is minimum. We shall prove that the second statement holds.

Suppose that there exist i, j with $2 \leq i \leq q$, $1 \leq j \leq n$ and a component Q of $C_i \cap X_j$ such that there exists no path in X_j from Q to $C_1 \cap X_j$. Then there exist at least two components D_1, D_2 of X_i such that D_1 contains $C_1 \cap X_j$ and D_2 contains Q. Let $A = \{2 \leq k \leq q : C_k \cap D_2 \neq \emptyset\}$. Consider the edges in C_k in the in-adhesion set of t_j for every $k \in A$, each such edge is associated with a subpath of the cycle obtained from P by adding an edge connecting its ends. Let k^* be the end other than j of the shortest subpath that is associated an edge just mentioned. Note that $k^* \neq j$. Then removing D_2 from X_j and putting D_2 into X_{k^*} still keeps $(P, \mathcal{X})$ a circularly peg-linked circular cut-decomposition of circular adhesion at most p, and the number of thick pairs of S between two different bags does not increase, but the number of edges of S between two different bags decreases. ■

Lemma 6.3.6 *Let p be an integer. Let (S, Ω) be a society having a circularly peg-linked circular cut-decomposition $(P, \mathcal{X})$ of circular adhesion p. Let $C_1, C_2, ..., C_q$, be a circular linkage in S, where $q \leq p$ and C_1 contains $\bar{\Omega}$. Assume that for every $2 \leq i \leq q$, $1 \leq j \leq n$, if $C_i \cap X_j \neq \emptyset$, then for every component Q of $C_i \cap X_j$, there exists a path in X_j from Q to $C_1 \cap X_j$. Let $x \in X_i$ be a vertex such that there exists a vertex y in another bag such that (x, y) is a thick pair. Then one of the following holds.*

1. *There exists another circular-peg linked circular cut-decomposition with the same circular linkage that has less thick pairs between two different bags.*
2. *Let v_i be the peg in X_i and let t_i be the i-th vertex of P. Then either*
   (a) *there exist two edge-disjoint paths from x to v_i in X_i, or*
   (b) *there exists $x' \in X_i$ incident with an edge in the union of the in-adhesion set and the out-adhesion set of t_i such that there exist two edge-disjoint paths in X_i, where one is from x to v_i and the other is from x to x'. Furthermore, either*
      i. *$x = x'$ and x incident with an edge in the in-adhesion set and an edge in the out-adhesion set of t_i, or*
      ii. *$x \neq x'$ and one of x and x' is incident with an edge in the in-adhesion set of t_i, and the other is incident with an edge in the out-adhesion set of t_i.*

Proof. Let (x, y) be a thick pair, where $y \in X_j$ for some $j \neq i$. By symmetry, we may assume that xy is an edge in the out-adhesion set of t_i. Suppose that there do not exist two edge-disjoint paths from x to v_i in X_i. By Menger's Theorem, there exists an edge-cut $[U, V]$ of $S[X_i]$ of size exactly one such that $x \in U$ and $v_i \in V$. We choose such an $[U, V]$ such that $S[U]$ is as small as possible. Let v be the vertex in U incident with the edge between U and V. If $|U| = 1$, then $U = \{x\}$. In this case, if x is incident with an edge in the in-adhesion set of t_i, then statement 2 (i) holds; otherwise, deleting x from X_i and adding x into X_j reduces the number of thick pairs between different bags, so the first statement holds. Hence we may assume that $|U| \geq 2$. By the minimality of U, $S[U]$ is connected and there exist two edge-disjoint paths in X_i from x to v. Furthermore, this implies that for every vertex $u \in U - \{x\}$, there exist two edge-disjoint paths in X_i, where one is from x to u and the other is from x to v. So the second statement holds if $U - \{x\}$ contains a vertex incident with an edge in the in-adhesion set of t_i. Therefore, we may assume that no vertex in U is incident with an edge in the in-adhesion set of t_i. However, the existence of the circular linkage implies that x is the only vertex incident with the out-adhesion set of t_i. Therefore, we can remove U from X_i and add U into X_j to reduce the number of thick pairs between different bags, so the first statement holds. This proves the lemma. ■

Lemma 6.3.7 *Let (S, Ω) be a society, and let p be an integer. If (S, Ω) has a circularly peg-linked circular cut-decomposition of circular adhesion p with circular linkage $C_1, C_2, ..., C_q$ for some $q \leq p$ such that C_1 contains $\bar{\Omega}$ and there exists no aggressive thick spider in (S, Ω), then one of the following holds.*

1. *There exists $Z \subseteq V(S) - \bar{\Omega}$ with $|Z| \leq 6$ such that $(S - Z, \Omega)$ has a circular cut-decomposition of circular adhesion at most $p - 2$.*

2. *(S, Ω) has a circularly peg-linked circular cut-decomposition of circular adhesion*

at most $p-1$.

3. *There exists $Z \subseteq E(S) - E(S[\bar{\Omega}])$ with $|Z| \leq 3p(2p+1)$ such that no two edges in Z have the same ends, and $(S-Z, \Omega)$ has a circularly peg-linked circular cut-decomposition $(P^*, \mathcal{X}^*)$ of circular adhesion at most p with the following properties.*

   (a) *$C_1, C_2, ..., C_q$ form a circular linkage in $S-Z$ and C_1 contains $\bar{\Omega}$.*

   (b) *For every $2 \leq i \leq q$ and $1 \leq j \leq |\bar{\Omega}|$, if $C_i \cap X_j^* \neq \emptyset$, then for every component Q of $C_i \cap X_j$, there exists a path in X_j from Q to $C_1 \cap X_j^*$.*

   (c) *No thick pair between two different bags.*

Proof. Denote $\bar{\Omega}$ by $v_1, v_2, ..., v_n$ in order. Let $(P, \mathcal{X})$ be a circularly peg-linked circular cut-decomposition of (S, Ω) of circular adhesion at most p such that $C_1, C_2, ..., C_q$ be a circular linkage, where C_1 contains $\bar{\Omega}$. We further assume that the number of thick pairs between two different bags is as small as possible, and subject to that, the number of edges of S between two bags is as small as possible. Let $P = t_1 t_2 ... t_n$.

We say that a set of edges between different bags is a *long set* if deleting the ends of the edges in this set decreases the circular adhesion of (S, Ω). Clearly, there are at most p pairwise disjoint long sets.

Claim 1: There are at most $3p(2p+1)$ disjoint thick pairs between different bags.

Proof of Claim 1: Let (x, y) be a thick pair such that $x \in X_i$ and $y \in X_j$ for some $i \neq j$. By symmetry, we may assume an edge between x, y is in the out-adhesion set of t_i and the in-adhesion set of t_j. By Lemmas 6.3.5 and 6.3.6, either there exist two edge-disjoint paths from x to v_i, or there exist a vertex x' in X_i incident with an edge $e_{x'}$ in the in-adhesion set of t_i and two edge-disjoint paths in X_i, where one is from x to v_i and the other is from x to x'. Similarly, either there exist two edge-disjoint paths from y to v_j, or there exist a vertex y' in X_j incident with an edge $e_{y'}$ in the out-adhesion set of t_j and two edge-disjoint paths in X_j, where one is from y to v_j

and the other is from y to y'. Observe that at least one of x' and y' exists, otherwise, there exists an aggressive thick spider with head (x, y). Similarly, if exactly one of x', y' exists, say y', then the end of $e_{y'}$ other than y' is in X_i; otherwise, an aggressive thick spider with head (x, y) exists. So there exists a long set of size two in this case. On the same way, if x', y' exist, then there exists a long set of size at most three.

For every set M of disjoint thick pairs between different bags, we define a map ϕ from M to the collection of long sets of size at most three by mapping each thick pair to the mentioned long set of size at most three. Since there are at most p pairwise disjoint long sets, the union of the members of the image of ϕ has size at most $3p$. On the other hand, each element in the union of the members of the image of ϕ appears in at most $2p+1$ members of the image of ϕ. Therefore, there $|M| \leq 3p(2p+1)$. So there exist at most disjoint $3p(2p+1)$ thick pairs between different bags. □

Claim 2: For every thick pair between two different bags, there exists at most one edge f such that the circular linkage does not contain f, and $(S - f, \Omega)$ has a new circularly peg-linked circular cut-decomposition that has the same circular linkage but has less thick pairs between different bags than $(P, \mathcal{X})$.

Proof of Claim 2: Let (x, y) be a thick pair between two different bags, say $x \in X_i$ and $y \in X_j$ for some $i < j$ such that an edge with ends x, y is in the out-adhesion set of t_i and the in-adhesion set of t_j. Since there does not exist an aggressive spider, by symmetry, we may assume that there does not exist two edge-disjoint paths from x to v_i in X_i. Let f be the edge between U and V, where $[U, V]$ is the edge-cut of X_i of size one such that $x \in U$ and $v_i \in V$ with $|U|$ as small as possible. Let $U_i \subseteq U - \{x\}$ such that every vertex u in U_i is incident with an edge e_u in the union of the in-adhesion set and the out-adhesion set of t_i. By the minimality of U, for every $u \in U_i$, there exist two edge-disjoint paths in X_i, where one is from x to the end of f in U, and the other is from x to u. For every $u \in U_i$, let k_u be the number such that the end of e_u other than u is in X_{k_u}. Let I, J be the two cyclic intervals partitioning $\bar{\Omega} - \{i, j\}$. If

$k_u \notin I \cup J$ for every $u \in U$, then deleting U from X_i and adding U into X_j decreases the number of thick pairs between different bags, since x, y are in the same bag, a contradiction. So $I \cup J$ contains k_u for some $u \in U$.

If there exist two edge-disjoint paths in X_j from y to v_j, then there exists an aggressive thick spider with head (x, y) and leaves v_i, k_u, v_j, since $k_u \in I \cup J$ for some $u \in U$. Hence, there exists an edge-cut $[U', V']$ of X_j of size one such that $y \in U'$ and $v_j \in V'$. We let f' be the edge between U', V'. For every $u \in U'$, we define e'_u to be an edge in the union of the in-adhesion set and the out-adhesion set of t_j, and we define k'_u to be the number such that the end of e'_u other than u is in $X_{k'_u}$. The same argument shows that $I \cup J$ contains k'_u for some $u \in U'$. Note that if one of I, J contains k_u for some $u \in U$, and the other contains $k'_{u'}$ for some $u' \in U'$, then there exists a thick spider with head (x, y). So without loss of generality, we may assume that I does not contain k_u and $k'_{u'}$ for every $u \in U$ and $u' \in U'$. Furthermore, if there exist $u \in U$ and $u' \in U'$ such that $i < j < k'_{u'} < k_u$, then there exists an aggressive thick spider with head (x, y). So $i < j \leq k_u \leq k'_{u'} \leq i$ for every $u \in U$ and $u' \in U'$.

If f is not in a cycle in a circular linkage, then we delete f and move U from X_i to X_j along the interval I. This operation leads to a circularly peg-linked circular cut-decomposition that has less thick pairs between bags without changing a circular linkage, so we prove the claim. If f is in a cycle in a circular linkage, but the intersection of this cycle and X_i is a path whose end incident with an edge in the in-adhesion set of t_i is in V, then we move U from X_i to X_j along I to obtain a better circularly peg-linked circular cut-decomposition. So we may assume that f is in a cycle in circular linkage, and the intersection of this cycle and X_i is a path whose end incident with an edge in the in-adhesion set of t_i is in U. This implies that there are at least two vertices in U incident with edges in the in-adhesion set of t_i. Similarly, we may assume that f' is in a cycle in circular linkage, and the intersection of this cycle and X_j is a path whose end incident with an edge in the out-adhesion set of t_j is in

U'. And there are at least two vertices in U' incident with edges in the out-adhesion set of t_j.

Since $i < j \leq k_u \leq k'_{u'} \leq i$ for every $u \in U$ and $u' \in U'$, and $k_u \notin I$ for every $u \in U$, x is the only vertex in U incident with an edge in the out-adhesion set of t_i, otherwise, we can delete at most six vertices to reduce the circular adhesion by at least two. Similarly, y is the only vertex in U' incident with an edge in the in-adhesion set of t_j.

Then we move U from X_i to X_{k_u} along the subinterval of J between k_u and i, and move U' from X_j to $X_{k'_{u'}}$ along the subinterval of J between j and $k'_{u'}$. If there exist $u \in U$ and $u' \in U$ such that $k_u = k'_{u'}$, then this operation leads to a circularly peg-linked circular cut-decomposition with less number of thick pairs between bags. So we may assume that $i < j \leq k_u < k'_{u'} \leq i$ for every $u \in U$ and $u' \in U'$. But in this case, the edge xy traverse the society twice, so we replace the subpath associated with xy by the path from k_u to $k'_{u'}$ contained in J. So the circular adhesion is reduced by one without loss a circular linkage. This proves the claim. □

Therefore, by Claim 2, we can repeatedly delete edges to obtain a better circular decomposition. By Claim 1, the number of edges we deleted is at most $3p(2p + 1)$. This completes the proof. ■

Lemma 6.3.8 *For every positive integers x, p, there exists an integer ξ such that if S is a weakly subcubic graph and (S, Ω) is a society that has a circularly peg-linked circular cut-decomposition of circular adhesion p with circular linkage $\mathcal{C} = \{C_1, C_2, ..., C_q\}$ for some $q \leq p$, where C_1 passes through $\bar{\Omega}$, with the following property:*

1. *There exists no aggressive thick spider in (S, Ω), and*
2. *for every $2 \leq i \leq q$ and $1 \leq j \leq |\bar{\Omega}|$, if $C_i \cap X_j^* \neq \emptyset$, then for every component Q of $C_i \cap X_j$, there exists a path in X_j from Q to $C_1 \cap X_j^*$, and*
3. *no thick pair between two different bags,*

then there exist $Z \subseteq E(S) - E(S[\bar{\Omega}])$ with $|Z| \leq \xi$ and $\bar{\Omega}' \subseteq \bar{\Omega}$ such that no two edges in Z have the same ends, $|\bar{\Omega} - \bar{\Omega}'| \leq \xi$, and $(S', \bar{\Omega}')$ has a circular cut-decomposition $(P', \mathcal{X}')$, where $S' = S - Z$, such that no thick pair between two different bags, and either

1. *$(P', \mathcal{X}')$ has circular adhesion at most $p-2$, or*
2. *$(P', \mathcal{X}')$ is circularly peg-linked with circular adhesion at most $p-1$, or*
3. *$(P', \mathcal{X}')$ is circularly peg-linked with circular adhesion p having $\mathcal{C}$ as a circular linkage such that for every $X \in \mathcal{X}'$, let $\mathcal{Q}_X$ be the set of component obtained from the cycles in $\mathcal{C}$ by deleting $E(S'[X])$ and isolated vertices, then there is at most one member in $\mathcal{Q}_X$ that intersects at most x bags in $\mathcal{X}'$; furthermore, if such a component Q exists, then for every $Y \in \mathcal{X}' - \{X\}$ with $Y \cap V(Q) \neq \emptyset$, there exist at least x disjoint paths in $S'[Y]$ from Q to $\bigcup_{Q' \in \mathcal{Q}_X - \{Q\}} Q'$.*

Proof. Let $\xi = 5x^2p^2(x+4)^2$. We assume that we cannot delete ξ edges from S to decrease the circular adhesion of (S, Ω) by at least two or decrease the circular adhesion by at least one without loss a circular linkage. And we may assume that $(P, \mathcal{X})$ does not satisfy Conclusion 3. For every $X \in \mathcal{X}$, define $\mathcal{Q}_X$ to be the set of components obtained from $C_1, C_2, ..., C_q$ by deleting $E(S[X])$ and isolated vertices. Let $\mathcal{U} \subseteq \mathcal{X}$ be the set consisting of $U \in \mathcal{X}$ such that there exists $Q \in \mathcal{Q}_U$ that intersects at most x bags in $\mathcal{X}$. We are done unless $|\mathcal{U}| \geq 1$. For every $U \in \mathcal{U}$, if there exist different $Q_1, Q_2 \in \mathcal{Q}_U$ such that each of them intersects at most x bags of $\mathcal{X}$, then deleting the edges in $Q_1 \cup Q_2$ between different bags decreases $(P, \mathcal{X})$ by at least two, so Conclusion 1 holds. Note that we only delete at most $2x+2$ edges in this case, so we may assume that there is exactly one member $Q_U \in \mathcal{Q}_U$ that intersects at most x bags, for each $U \in \mathcal{U}$. Similarly, for every member of U, every other member U' of $\mathcal{U}$ must intersect Q_U, otherwise, deleting the edges in $Q_U \cup Q_{U'}$ between different bags reduces the circular adhesion by at least two. Hence, $1 \leq |\mathcal{U}| \leq x$. Furthermore, if there exists $U \in \mathcal{U}$ such that Q_U is a cycle or it is a path whose ends are in the same component of $U \cap \bigcup_{C \in \mathcal{C}} C$, then we delete the edges of Q_U between different bags to decrease the circular adhesion by at least one but a circular linkage still exists. So we may assume that Q_U is a path whose ends are in different components of $U \cap \bigcup_{C \in \mathcal{C}} C$, for every $U \in \mathcal{U}$.

For every two different bags $X_1, X_2 \in \mathcal{X}$, let I_{X_1,X_2} and J_{X_1,X_2} be the two maximal intervals obtained from $\bar{\Omega}$ by deleting the indices of X_1 and X_2, where I_{X_1,X_2} contains the index of the bag whose index one larger than the index of X_1. Note that if X_1, X_2 are consecutive bags, then one of I_{X_1,X_2}, J_{X_1,X_2} is empty. For every $X \in \mathcal{X}$, $Q \in \mathcal{Q}_X$, and $Y \in \mathcal{X} - \{X\}$ with $Y \cap V(Q) \neq \emptyset$, we say that Q is *Y-in-short* (or *Y-out-short*, respectively) if Q intersects at most $x+3$ bags whose indices in $I_{X,Y}$ (or $J_{X,Y}$, respectively). Note that for every $U \in \mathcal{U}$, $Q_U \in \mathcal{Q}_U$, and $Y \in \mathcal{X} - \{U\}$ with $V(Q_U) \cap Y \neq \emptyset$, every $Q \in \mathcal{Q}_U - \{Q_U\}$ cannot be both Y-in-short and Y-out-short; otherwise, one can delete at most $3x+7$ edges to reduce the circular adhesion at least two. Similarly, there do not exist $Q_1, Q_2 \in \mathcal{Q}_U - \{Q_U\}$ such that Q_1 is Y-in-short and Q_2 is Y-out-short. We define NS_Y (and OS_Y, respectively) to be the set of vertices in Y that is not contained in Q_U but is incident with an edge in the in-adhesion (out-adhesion, respectively) set of Y contained in a Y-in-short (Y-out-short, respectively) path. As we just discussed, at least one of NS_Y and OS_Y is empty. And we define NL_Y (and OL_Y, respectively) to be the set of vertices in Y that is incident with an edge in the in-adhesion (out-adhesion, respectively) set of Y not contained in a Y-in-short (Y-out-short, respectively) path.

We say that an edge-cut $[A, B]$ of a subgraph H of S is *thin* if no thick pair in H is between A and B. For every $U \in \mathcal{U}$ and $Y \in \mathcal{X} - \{U\}$ with $V(Q_U) \cap Y \neq \emptyset$, we say that an edge-cut $[A, B]$ of $S[Y]$ is *in-admissible* (or *out-admissible*, respectively) if the following hold.

- $V(Q_U) \subseteq A$, and the peg in Y is in B.

- $[A, B]$ is thin and is of size at most x.

- no member of $\mathcal{Q}_U$ contains at least two edges between A, B.

- If a member Q of $\mathcal{Q}_U$ contains an edge between A, B, then the end of $Q \cap S[Y]$ incident with an edge in the in-adhesion (out-adhesion, respectively) set is in A.

For every $U \in \mathcal{U}$, define $\mathcal{Y}_U \subseteq \mathcal{X} - \{X\}$ to be the set of bags $Y \in \mathcal{X} - \{X\}$ such that if $V(Q_U) \cap Y \neq \emptyset$, then there exists a separation (A', B') of $S[Y]$ of order less than x such that $Q_U \cap S[Y] \subseteq A'$ and $\bigcup_{Q \in \mathcal{Q}_U - \{Q_U\}} Q \cap S[Y] \subseteq B'$. Note that $|\mathcal{Y}_U| \leq x$ for every $U \in \mathcal{U}$, and we are done if $\mathcal{Y}_U$ is empty. We shall delete edges to decrease $|\bigcup_{U \in \mathcal{U}} \mathcal{Y}_U|$.

Claim 1: If $U \in \mathcal{U}$ and $Y \in \mathcal{Y}_U$ such that OS_Y is empty, then there exists an in-admissible edge-cut $[A, B]$ such that $NL_Y \cup OL_Y \cup \{y\} \subseteq B$, where y is the peg in Y.

Proof of Claim 1: By definition of $\mathcal{Y}_U$, there exists a separation (A', B') of $S[Y]$ of order less than x such that $Q_U \cap S[Y] \subseteq A'$ and $\bigcup_{Q \in \mathcal{Q}_U - \{Q_U\}} Q \cap S[Y] \subseteq B'$. Since S is weakly subcubic, and $|N_{S[Y]}(v) - V(Q_U)| \leq 1$ for every $v \in V(Q_u)$ and $|N_{S[Y]}(v) - V(\bigcup_{Q \in \mathcal{Q}_U - \{Q_U\}} Q)| \leq 1$ for every $v \in V(\bigcup_{Q \in \mathcal{Q}_U - \{Q_U\}} Q)$, there exists an edge-cut $[A, B]$ of $S[Y]$ such that $V(Q_U) \cap Y \subseteq A$, $V(\bigcup_{Q \in \mathcal{Q}_U - \{Q_U\}} Q) \cap Y \subseteq B$, and $[A'', B'']$ has size less than x, where $[A'', B'']$ is obtained from $[A, B]$ by replacing parallel edges between $[A, B]$ by single edges. We choose such an $[A, B]$ such that the size of $[A'', B'']$ is as small as possible, and subject to that, the number of parallel edges is as small as possible. We prove the claim if $[A, B]$ is thin, so we assume that $[A, B]$ is not thin.

Let $u \in A, v \in B$ such that u, v is a thick pair. If there exists an edge-cut $[O, K]$ of $S[A]$ of size at most one such that $V(Q_U) \subseteq O$ and $u \in K$, then $[O, B \cup K]$ is an edge-cut of $S[Y]$ that has the property that $V(Q_U) \cap Y \subseteq O$ and $V(\bigcup_{Q \in \mathcal{Q}_U - \{Q_U\}} Q) \cap Y \subseteq$

$B \cup K$, and either $[O'', (B \cup K)'']$ has size smaller than $[A'', B'']$, or they have the same size but the number of thick pairs between O and $B \cup K$ is smaller than the number of thick pairs between A, B, contradicting the minimality of $[A, B]$. So there exist two edge-disjoint P_1, P_2 paths in $S[A]$ from u to $Q_U \cap S[Y]$.

If there do not exist two edge-disjoint paths in $S[B]$, where one is from v to $NL_Y \cup \{y\}$ and the other is from v to $OL_Y \cup \{y\}$, then there exists an edge-cut $[O_v, K_v]$ of $S[B]$ of size at most one such that $v \in O_v$ and $NL_Y \cup OL_Y \cup \{y\} \subseteq K_v$, since $V(\bigcup_{Q \in \mathcal{Q}_U - \{Q_U\}} Q) \subseteq B$. Note that Q is Y-out-short for every $Q \in \mathcal{Q}_U - \{Q_u\}$, since OS_Y is empty, so if Q contains an edge between $[O_v, K_v]$, then the end of $Q \cap S[Y]$ incident with an edge in the in-adhesion set is in O_v. Therefore, if there exists no thick pair (u, v) such that $u \in A$ and $v \in B$ such that there exist two edge-disjoint paths in $S[B]$, where one is from v to $NL_Y \cup \{y\}$ and the other is from v to $OL_Y \cup \{y\}$, then $[A \cup \bigcup O_v, B \cap \bigcap K_v]$, where the union and intersection run through all such thick pair (u, v), is an in-admissible edge-cut such that $NL_Y \cup OL_Y \cup \{y\} \subseteq B$. Hence, we may assume that $u \in A$ and $v \in B$ such that (u, v) is a thick pair such that there exist two edge-disjoint paths P_3, P_4 in $S[B]$, where one is from v to $NL_Y \cup \{y\}$ and the other is from v to $OL_Y \cup \{y\}$. Let y_1, y_2 be the two closest pegs that can be connected through $P_3 \cup \bigcup_{Q \in \mathcal{Q}_U - Q_U} Q$ and $P_4 \cup \bigcup_{Q \in \mathcal{Q}_U - \{Q_U\}} Q$ from v but not in a bag that is not Y but intersects Q_U, respectively, and let Y_1, Y_2 be the bag contains y_1, y_2, respectively.

We say a member $Q \in \mathcal{Q}_U - \{Q_U\}$ is *in-short* (or *out-short*, respectively) if it intersects at most $x + 3$ bags whose indices in $J_{U,Y}$ (or $I_{U,Y}$, respectively). If there exist an in-short member and an out-short member, then we can delete at most $3x+7$ edges to decrease the circular adhesion by at least two.

Let NS_U (and OS_U, respectively) be the set of vertices in U that is incident with an edge in the in-adhesion (or out-adhesion, respectively) set contained in a in-short path (or an out-short path, respectively). Note that one of NS_U and OS_U is empty.

Let ℓ, r be the ends of Q_U, where ℓ is incident with an edge in the in-adhesion set of U and r is incident with an edge in the out-adhesion set of U. Let NL_U (OL_U, respectively) be the set of vertices that are incident with an edge in the in-adhesion (out-adhesion, respectively) set of U but not in $NS_U \cup \{\ell\}$ ($OS_U \cup \{r\}$, respectively). Let p be the peg in U. Let K_U be NL_U or OL_U such that $K_U = NL_U$ implies that $NS_U = \emptyset$, and $K_U = OL_U$ implies that $OS_U = \emptyset$. Suppose that there do not exist two edge-disjoint paths in $S[U]$ from $\{\ell, r\}$ to $K_U \cup \{p\}$, where the only possible common end is p. Then there exists an edge-cut $[M, N]$ of $S[U]$ of size at most one such that $\{\ell, r\} \subseteq M$ and $K_U \cup \{p\} \subseteq N$. Since every component of $\bigcup_{C \in \mathcal{C}} C \cap S[U]$ not incident with ℓ, r has at least one end in K_U, and at least one component of $\bigcup_{C \in \mathcal{C}} C \cap S[U]$ incident with ℓ, r has an end in K_U, we know that the edge between M, N is contained in a component of $\bigcup_{C \in \mathcal{C}} C \cap S[U]$ incident with ℓ, r, and every component of $\bigcup_{C \in \mathcal{C}} C \cap S[U]$ not incident with ℓ, r is contained in N. On the other hand, every component of $\bigcup_{C \in \mathcal{C}} C \cap S[U]$ is connected to p by a path in $S[U]$, so there exists a path W in $S[U]$ connecting the components of $\bigcup_{C \in \mathcal{C}} C \cap S[U]$ containing ℓ, r and disjoint from other components of $\bigcup_{C \in \mathcal{C}} C \cap S[U]$. Therefore, deleting the edges in Q_U reduces the circular adhesion by at least one, but there exists a circular linkage by the existence of W. Hence, there exist two edge-disjoint paths P_5, P_6 in $S[U]$ from $\{\ell, r\}$ to $K_U \cup \{p\}$ whose possible common end is p. Let y_5, y_6 be the two closest pegs that can be connected through $P_5 \cup \bigcup_{Q \in \mathcal{Q}_U - Q_U} Q$ and $P_6 \cup \bigcup_{Q \in \mathcal{Q}_U - \{Q_U\}} Q$ from $\{\ell, r\}$, respectively, and let Y_5, Y_6 be the bag contains y_5, y_6, respectively. Since there exists no aggressive thick spider with head (u, v), the indices of Y_1, Y_2 and Y_5, Y_6 interlace, say $i_1 \leq i_5 \leq i_2 \leq i_6 \leq i_1$, where i_j is the index of Y_j for $j \in \{1, 2, 5, 6\}$. As we cannot delete at most $2x + 6$ edges to decrease the circular adhesion by at most two, we may assume that $i_1 \leq i_5 < i_2 \leq i_6 \leq i_1$ by symmetry. But this implies that $NS_U = \emptyset = OS_U$. Let $K'_U = NL_U$ if $K_U = OL_U$, and $K'_U = OL_U$ if $K_U = NL_U$. Let P'_5, P'_6 be two edge-disjoint paths in $S[U]$ from $\{\ell, r\}$ to $K'_U \cup \{p\}$ whose possible common end is p. Let y'_5, y'_6 be the two closest pegs that can be connected through $P'_5 \cup \bigcup_{Q \in \mathcal{Q}_U - Q_U} Q$ and $P'_6 \cup \bigcup_{Q \in \mathcal{Q}_U - \{Q_U\}} Q$ from $\{\ell, r\}$, respectively, and let Y'_5, Y'_6 be the bag contains y'_5, y'_6, respectively. Then the indices of Y_1, Y_2, Y'_5, Y'_6 cannot interlace, otherwise, we can delete at most $3x + 10$ edges to decrease the circular adhesion by at least two. Therefore, there exist an aggressive thick spider with head (u, v), a contradiction. This proves that $[A, B]$ is thin and proves this claim. $\square$

By symmetry, we have the following claim.

Claim 2: If $U \in \mathcal{U}$ and $Y \in \mathcal{Y}_U$ such that NS_Y is empty, then there exists an out-admissible edge-cut $[A, B]$ such that $NL_Y \cup OL_Y \cup \{y\} \subseteq B$, where y is the peg in Y.

Now let $U \in \mathcal{U}$ and $Y \in \mathcal{Y}_U$, and let y be the peg of Y. Let Q_0 be the member in $\mathcal{Q}_U - \{Q_U\}$ passing some pegs. We first assume that Q_0 is not Y-in-short nor Y-out-short. By symmetry, we may assume that OS_Y is empty. By Claim 1, there exists an in-admissible edge-cut $[A, B]$ such that $NL_Y \cup OL_Y\{y\} \subseteq B$. Therefore, we can delete the edges between A, B not contained in a member of $\mathcal{C}$ and move A from Y to a bag whose index in $I_{U,Y}$ along a subinterval of $I_{U,Y}$. This operation either makes a bag Y' in $\mathcal{Y}_U$ Q_0 Y'-in-short, or reduce the lexicographic order of $(|\{U \in \mathcal{X} : \mathcal{Q}_U \neq \emptyset\}|, |\mathcal{Y}_U|, p - c, d)$, where c is total number of Y''-in-short members of $\mathcal{Q}_U - \{Q_U\}$ for $Y'' \in \mathcal{Y}_U$, and d is the sum of the lengths of Y''-in-short members of $\mathcal{Q}_U - \{Q_U\}$ for $Y'' \in \mathcal{Y}_U$. Since each terms is bounded, we only delete bounded number of edges until Q_0 is Y'-in-short. When Q_0 is in-short, we can remove at most $x + 3$ vertices from $\bar{\Omega}$ and merging bags to reduce the lexicographic order of $(|\{U \in \mathcal{X} : \mathcal{Q}_U \neq \emptyset\}|, |\mathcal{Y}_U|)$. This completes the proof. ■

Lemma 6.3.9 *For every integer ρ, there exists a positive integer ξ such that the following is true. Let Σ be a surface and Δ a closed disk in Σ. Let G_0 be a drawing in Σ such that $G_0 \cap \Delta \subseteq V(G_0)$. Let G be a weakly subcubic graph and (S, Ω) a society such that $G = G_0 \cup S$, $V(G_0) \cap V(S) = \bar{\Omega}$, and $E(G_0) \cap E(S) = \emptyset$. Assume that there* *exist disjoint cycles C_1, C_2, C_3, C_4 in $G_0 - \bar{\Omega}$ such that the following hold.*

1. *Each C_i is the boundary of a closed disk $D(C_i)$ in Σ such that $\Delta \subseteq D(C_1) \subseteq D(C_2) \subseteq D(C_3) \subseteq D(C_4)$ and there does not exist a separation (A, B) of G of order at most two such that $C_1 \subseteq A$ and $C_4 \subseteq B$.*

2. *(S, Ω) has a circularly peg-linked circular cut-decomposition $(P, \mathcal{X})$ of circular adhesion ρ such that no thick pair is between different bags.*

3. *Every vertex in $\bar{\Omega}$ is adjacent in G to at most one vertex in $G - V(S^*)$.*

4. *For every bag X of $\mathcal{X}$, for every component in X of a cycle in a circular linkage, there exists a path in X from this component to the component in X of the cycle passing through $\bar{\Omega}^*$ in the same circular linkage.*

5. *S does not contain an aggressive thick spider.*

Then there exist a disk Δ^* in Σ, a society (S^*, Ω^*), and $Z \subseteq V(G) \cup E(G)$ with $|Z| \leq \xi$ such that no two edges in Z have the same ends the following hold.

1. *Δ^* is contained in the interior of $D(C_3)$, $S^* \subseteq G \cap \Delta^*$, $\bar{\Omega}^* = V(G - Z) \cap \partial\Delta^*$, $Z \subseteq V(G \cap \Delta^*) \cup (E(G \cap \Delta^*) - E(G[\bar{\Omega}^*]))$, and (S^*, Ω^*) has a circular cut-decomposition $(P^*, \mathcal{X}^*)$ of circular adhesion at most ρ, and no thick pair is between different bags, and every vertex in $\bar{\Omega}^*$ is adjacent in G to at most one vertex in $G - V(S^*)$.*

2. *One of the following holds.*

   (a) *$(P^*, \mathcal{X}^*)$ is circularly peg-linked, and for every bag $X \in \mathcal{X}^*$,*

      i. *there does not exist a separation (A, B) of $G - Z$ of order less than $|N_{G-Z}(V(G - Z) - X)|$ such that $X \subseteq V(A)$ and $C_4 \subseteq B$, and* *ii. if (A, B) is a separation of $G - Z$ order $|N_{G-Z}(V(G-Z) - X)|$ such that $X \subseteq V(A)$ and $C_4 \subseteq B$, then for every vertex v in $N_{G-Z}(V(G - Z) - X) - (V(A) \cap V(B))$, v is incident with at most one edge whose the other end not in X.*

(b) $(P^*, \mathcal{X}^*)$ is circularly peg-linked and has circular adhesion at most $\rho - 1$.

(c) $(P^*, \mathcal{X}^*)$ has circular adhesion at most $\rho - 2$.

Proof. Let ξ be two plus the twice of the number ξ mentioned in Lemma 6.3.8 by taking $p = \rho$ and $x = 2\rho + 1$, and let (S^*, Ω^*) be the society (S', Ω') and Z_E the set Z mentioned in the conclusions of Lemma 6.3.8. Let Z_V be the set of vertices in $G_0 - V(S)$ that is adjacent to a vertex in $\bar{\Omega} - \bar{\Omega}'$. Let Δ^* be a disk in Σ such that $S^* \subseteq \Delta^* \cap G - Z$ and $\partial\Delta^* \cap G - Z = \bar{\Omega}^*$. Therefore, we are able to choose Δ^*, (S^*, Ω^*) such that (S^*, Ω^*) satisfies the conclusions of Lemma 6.3.8 and Δ^* is contained in the interior of $D(C_2)$ and satisfies the first conclusion of this lemma with $Z = Z_E \cup Z_V$. We further choose Δ^*, S^*, Ω^* that satisfy the above conditions, and subject to that, Δ^* is maximal.

We are done if Conclusions 1 or 2 of Lemma 6.3.8 holds. So we may assume that Conclusion 3 of Lemma 6.3.8 holds. Let $H = G - (Z_V \cup Z_E)$. Suppose that there exists $X \in \mathcal{X}^*$ and a separation of H of order less than $|N_H(V(H) - X)|$ such that $X \subseteq V(A)$ and $C_4 \subseteq B$. We pick such a separation (A, B) of X such that $|V(A) \cap V(B)|$ is minimal. If $V(A) \cap V(B) - V(S^*) = \emptyset$, then $V(A) \cap V(C_2) = \emptyset$ since there is no separation (A', B') of G of order at most two such that $C_2 \subseteq A'$ and $C_4 \subseteq B'$, so we can put A into S^* without violating the conditions, a contradiction. So we may assume that $|V(A) \cap V(B) - V(S^*)| \geq 1$.

Let p be the peg in X. We may assume that p is adjacent to a vertex in $G - V(S^*)$, otherwise, we can remove p from $\bar{\Omega}^*$ and merge bags to obtain a new circularly peg-linked circular cut-decomposition. Let p' be the vertex in $G - V(S^*)$ adjacent to p in G. If $|V(A) \cap V(B) - V(S^*)| = 1$, then let p'' be the vertex in $V(A) \cap V(B) - V(S^*)$. By the maximality of Δ^*, there exists a path from p' to p''. So again by the maximality of (S^*, Ω^*), either $p'' = p'$, or p'' is adjacent in G to at least two vertices in $V(B) - V(A)$ and p'' is adjacent to p' in A. For the latter case, we can put $A - \{p''\}$ into S^* and obtain a contradiction to the maximality of Δ^*. Therefore, $|V(A) \cap V(B) - V(S^*)| = 1$ implies that $V(A) \cap V(B) - V(S^*) = \{p'\}$.

Let $\mathcal{C}$ be a circular linkage of $(P^*, \mathcal{X}^*)$, and let Q_0 be the cycle in $\mathcal{C}$ passing through $\bar{\Omega}^*$. Let $\mathcal{Q}$ be the set of components obtained from the members of $\mathcal{C}$ by deleting $E(H[X])$ and isolated vertices. Observe that $1 + \sum_{Q \in \mathcal{Q}} |V(A) \cap V(B) \cap V(Q)| \leq |V(A) \cap V(B) \cap (V(H) - V(S^*))| + \sum_{Q \in \mathcal{Q}} |V(A) \cap V(B) \cap V(Q)| \leq |V(A) \cap V(B)| < |N_H(V(H) - X)| \leq 1 + \sum_{Q \in \mathcal{Q}} |V(Q) \cap X|$. So there exists a nonempty $\mathcal{Q}' \subseteq \mathcal{Q}$ such that for every $Q \in \mathcal{Q}'$, $V(Q) \cap X \neq \emptyset$ and $|V(A) \cap V(B) \cap V(Q)| \leq |V(Q) \cap X| - 1$. Observe that $Q \subseteq A$ for every $Q \in \mathcal{Q}'$. Let J_A be the set of the indices such that the j-th peg of $(P^*, \mathcal{X}^*)$ is in A, and let J_B be the set of indices such that the j-th peg is not in A but intersects some members of $\mathcal{Q}'$. Recall that for every bag $Y \in \mathcal{X}^*$ and for every $Q \in \mathcal{Q}$, there is a path in $S^*[Y]$ connecting $Q \cap S^*[Y]$ and $Q_0 \cap S^*[Y]$. So for every $j \in J_B$, X_j contains a vertex in $V(A) \cap V(B)$. Therefore, $|J_B| \leq |V(A) \cap V(B)| \leq 2\rho$ and hence every member of $\mathcal{Q}'$ intersects at most 2ρ bags in $\mathcal{X}^*$. By Conclusion 3 of Lemma 6.3.8, $|\mathcal{Q}'| = 1$, and we denote the element of $\mathcal{Q}'$ by Q^*. This implies that $|V(A) \cap V(B) \cap V(H - V(S^*))| = 1$ and $|V(A) \cap V(B) \cap V(Q)| = |V(A) \cap V(B) \cap N_H(H - X)|$ for every $Q \in \mathcal{Q} - \mathcal{Q}'$. In particular, the two vertices in $V(A) \cap V(B) \cap Q_0$ are in $X_{x-1} \cup X \cup X_{x+1}$, and $J_A \subseteq \{x-1, x, x+1\}$, where x is the index of X. By Lemma 6.3.8, for every $Y \in \mathcal{X}^* - \{X\}$ with $V(Q^*) \cap Y \neq \emptyset$, there exist at least $2\rho + 1$ disjoint paths in $S^*[Y]$ from Q^* to Q_0. But $|V(A) \cap V(B)| \leq 2\rho$ and $Q^* \subseteq A$, so the index of Y is in J_A. Let v_{x-1} and v_{x+1} be the peg in X_{x-1}, X_{x+1}, respectively, and let y_{x-1}, y_{x+1} be the vertex in $G - S^*$ adjacent to v_{x-1} and v_{x+1}, respectively. Then let $Z = Z_V \cup Z_E \cup \{y_{x-1}, y_{x+1}\}$ and merge X_{x-1}, X, X_{x+1} into a new bag to obtain a new circularly peg-linked circular cut-decomposition of circular adhesion at most $\rho - 1$. Hence $(S^*, \Omega^* - \{v_{x-1}, v_x\})$ satisfies Conclusion 2(b) of this lemma. And we slightly enlarge Δ^* such that $y_{x-1}, y_{x+1} \in V(\Delta^* \cap G)$. Note that $y_{x-1}, y_{x+1} \subseteq D(C_2)$, so Δ^* can be chosen in the interior of $D(C_3)$. This proves that no separation (A, B) of H of order less than $|N_H(V(H) - X)|$ such that $X \subseteq V(A)$ and $C_4 \subseteq B$.

Similarly, if there exists a separation (A, B) of H of order $|N_H(V(H) - X)|$ such that $X \subseteq V(A)$ and $C_4 \subseteq B$, then $|V(A) \cap V(B) \cap V(Q)| \geq |V(Q) \cap X|$ for every $Q \in \mathcal{Q}$. The above arguments show that $v \in V(A) \cap V(B)$ if $V(Q) \cap X$ consists of some single vertex v for some $Q \in \mathcal{Q}$. In addition, if the peg in X is incident with an edge whose the other end in another bag, then the peg of X must be in $V(A) \cap V(B)$, otherwise, $|V(A) \cap V(B) \cap V(Q_0)| \leq 1$ unless $V(A) \cap V(B) \subseteq S^*$. Hence, if $v \in N_H(V(H) - X) - (V(A) \cap V(B))$, then v is incident with at most one edge whose the other end not in X. Furthermore, since every vertex in $N_H(V(H)-X)$ is either in a cycle in the circular linkage or the peg in X, every vertex in $N_H(V(H) - X) - (V(A) \cap V(B))$ is incident with exactly one edge whose the other end not in X. This completes the proof. ■

Theorem 6.3.10 *For every positive integers k, θ', there exist positive integers $\kappa, \xi, \rho, \theta$ such that if G is a graph that does not contain a topological minor isomorphic to the Robertson chain of length k, and $\mathcal{T}$ is a tangle of order at least θ in G not controlling a K_{6k}-minor and is contained in a tangle in G of order at least 6ρ more than $\mathcal{T}$, then there exist $Z = Z_V \cup Z_E$, where $Z_V \subseteq V(G)$ and $Z_E \subseteq E(G)$, with $|Z| \leq \xi$, and a $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ such that the following hold:*

1. *$\mathcal{S}$ can be properly arranged in a surface Σ in which K_{6k} cannot be drawn.*
2. *Every society in $\mathcal{S}_1$ has at most three pegs, and $|\mathcal{S}_2| \leq \kappa$.*
3. *The trunk of $\mathcal{S}$ is 2-cell embedded in Σ and has a tangle $\mathcal{T}'$ of order at least θ' conformal with $\mathcal{T} - Z$.*

4. *Every society in $\mathcal{S}_1$ with exactly two or three pegs is thick unless it consists of a single edge.*

5. *The trunk of $\mathcal{S}$ is weakly subcubic.*

6. *Every thick society in $\mathcal{S}$ with exactly two pegs is pendant.*

7. $Z_E \subseteq E(G[\bigcup_{(S,\Omega)\in\mathcal{S}_2} V(S)]) - E(G[\bigcup_{(S,\Omega)\in\mathcal{S}_2} \bar{\Omega}])$*, and no two edges in Z_E have the same ends.*

8. *For each member (S,Ω) in $\mathcal{S}_2$, it has a circularly peg-linked circular cut-decomposition $(P,\mathcal{X})$ of circular adhesion at most ρ such that the following hold.*

   (a) *No thick pair is between different bags.*

   (b) *For every $X \in \mathcal{X}$,*

      i. *there exists no separation $(A,B) \in \mathcal{T}-Z$ of order less than $|N_{G-Z}(V(G-Z)-X)|$ such that $X \subseteq V(A)$, and*

      ii. *if $(A,B) \in \mathcal{T}-Z$ of order $|N_{G-Z}(V(G-Z)-X)|$ such that $X \subseteq V(A)$, then for every vertex v in $N_{G-Z}(V(G-Z)-X)-(V(A)\cap V(B))$, v is incident with at most one edge whose the other end not in X.*

9. *if $(A,B) \in \mathcal{T}'$ is of order at most two, then either (A,B) has order at most one and A consists of $|V(A)\cap V(B)|$ vertices, or every vertex in A has at most two neighbors in A.*

Proof. Define κ to be the number κ mentioned in Lemma 6.2.6 by taking $k=k$. Let ξ_0, ρ_0, θ_0 be the numbers ξ, ρ, θ, respectively, mentioned in Lemma 6.2.6 by further taking $\theta' = \theta'$ and $\psi(x) = 2000\kappa k\theta' x$. Let ξ' be the number ξ obtained by applying Lemma 6.3.9 by taking $\rho = \rho_0 + 2$. Define $\rho = \rho_0 + 2$, $\xi = 2\xi_0 + 120k\xi'\kappa\rho^2$, and $\theta = \theta_0$. Let G be a graph with no topological minor isomorphic to the Robertson chain of length k, and let $\mathcal{T}$ be a tangle in G of order at least θ_0 no controlling a K_{6k}-minor. We remark that it is sufficient to deal with the case that G is weakly subcubic, since we can apply Lemma 6.1.1 to obtain a faithful location, and a segregation of G satisfying the conclusions of this lemma can be obtained from a segregation of the contour of the faithful location satisfying the conclusions of this lemma. So we may assume that G is weakly subcubic. By Lemma 6.2.6, there exist $Z_0 \subseteq V(G)$ with $|Z_0| \leq \xi_0$ and a $(\mathcal{T} - Z_0)$-central segregation $\mathcal{S}^0 = \mathcal{S}_1^0 \cup \mathcal{S}_2^0$ such that Conclusions 1-7 of this theorem and Conclusion 5 of Lemma 6.2.6 hold, if we take Z_E to be the empty set.

Let G^0 be the trunk of $\mathcal{S}^0$, and let $\mathcal{T}^0$ be the tangle in G^0 mentioned in Conclusion 3. Since Conclusion 5 of Lemma 6.2.6 holds, every $(S, \Omega) \in \mathcal{S}_2^0$ is enclosed by $100\kappa\rho_0$ concentric cycles $C_1(S), C_2(S), ..., C_{100\kappa\rho_0}(S)$ in G^0, where $C_i(S)$ bounds a $8(100\kappa\rho_0 - i + 1)$-zone in G^0 around a vertex in $\bar{\Omega}$ for each $1 \leq i \leq 100\kappa\rho_0$, such that $C_1(S) \cap C_1(S') = \emptyset$ for every $(S', \Omega') \in \mathcal{S}_2^0 - \{(S, \Omega)\}$. By Lemma 6.3.1, we may assume that those cycles tightly enclose (S, Ω). By Lemma 6.3.2, there exists $Z_1 \subseteq V(G)$ with $Z_0 \subseteq Z_1$ and $|Z_1| \leq \xi_0 + \kappa(2\rho_0 + 2)$ such that for each $(S, \Omega) \in \mathcal{S}_2^0$, there exists a $(\rho_0 + 1)$-vortex (S', Ω') that has a circularly peg-linked circular decomposition $(P, \mathcal{X})$ of adhesion $\rho_0 + 1$ and is enclosed by $C_1(S), C_2(S), ..., C_{100\kappa\rho_0 - \rho_0 - 1}$. Since G and G^0 are weakly subcubic, if there exists a vertex not in S' for some (S', Ω') adjacent to two pegs of S' in G^0, then we can add this vertex into S' and change the set of pegs and modify the circular linkage without increase the depth of the vortex. So we may assume that there does not exist a vertex not in S' adjacent to two pegs of S' in G^0. We define $\mathcal{S}_2^1 = \{(S', \Omega') : (S, \Omega) \in \mathcal{S}_2^0\}$ and $\mathcal{S}_1^1 = \{(S, \Omega) \in \mathcal{S}_1^0 : V(S) \not\subseteq \bigcup_{(S', \Omega') \in \mathcal{S}_2^1} V(S')\}$.

Since G is weakly subcubic, we can replace each vertex-cut corresponding to the adhesion set of the circular decomposition by an edge-cut of the same size. Furthermore, the old circular decomposition is circularly peg-linked, so the edges in each edge-cut we obtain form a matching. In addition, every peg of S' has at exactly two neighbors in S', so we obtain a circular cut-decomposition $(P, \{Y_1, Y_2, ..., Y_n\})$ of (S', Ω'). As $(P, \mathcal{X})$ is a circularly peg-linked circular decomposition, $(P, \{Y_1, ..., Y_n\})$ is a circularly peg-linked circular cut-decomposition.

For each $(S', \Omega') \in \mathcal{S}_2^1$, if there exists a separation $(A, B) \in G - Z_1$ of order less than $2k$ such that $S' \subseteq A$ and $C_2(S) \subseteq B$, then there exists A' such that $(A', \emptyset)$ is in a tangle obtained from $\mathcal{T}^1 - Z_1$ by clearing a zone, contradicting to the third tangle axiom. Hence, by Lemma 6.3.4, there do not exist $k+1$ aggressive thick spiders in S' with disjoint heads having disjoint territories, for each $(S', \Omega') \in \mathcal{S}_2^1$. By Lemma 6.3.3, there exists $Z_2 \subseteq V(G)$ with $Z_1 \subseteq Z_2$ and $|Z_2| \leq 2\xi_0 + \kappa(2\rho_0 + 2 + (4\rho_0 + 6)k)$ such that there exists no aggressive thick spider in S', for every $(S', \Omega') \in \mathcal{S}_2^1$.

Let G^2 be the trunk of $\mathcal{S}^2$, and let $\mathcal{T}^2$ be the tangle in G^2 conformal with $\mathcal{T}^1$ that is obtained from $\mathcal{T}^1$ by clearing κ $8(\rho_0 + 1)$-zones. Apply Lemmas 6.3.1 and 6.3.2 to each member of $\mathcal{S}_2^1$, and then convert the vertex-cut corresponding to each adhesion set to an edge-cut. Then for each $(S', \Omega') \in \mathcal{S}_2^1$, we obtain a $(\rho_0 + 2)$-vortex (S'', Ω'') that is enclosed by cycles $C_1(S), C_2(S), ..., C_{100\kappa\rho_0 - 2\rho_0 - 3}$ and has a circularly peg-linked circular cut-decomposition of circular adhesion at most $\rho_0 + 2$, such that no vertex not in S'' is adjacent to two pegs of S'' in G^2. Since G^2 is weakly subcubic and there exists a cycle passing through $\bar{\Omega}''$ in S'', we may assume that every vertex in $\bar{\Omega}''$ is adjacent to exactly one vertex in $G^2 - S''$. Define $\mathcal{S}_2^2 = \{(S'', \Omega'') : (S', \Omega') \in \mathcal{S}_2^1\}$ and $\mathcal{S}_1^2 = \{(S, \Omega) \in \mathcal{S}_1^1 : S \not\subseteq \bigcup_{(S'', \Omega'') \in \mathcal{S}_2^2} S''\}$. Observe that each $(S'', \Omega'') \in \mathcal{S}_2^2$ does not contain an aggressive thick spider by the planarity of the disk bounded by $C_1(S)$ and the fact that every thick society in $\mathcal{S}^1$ is pendant. Therefore, by Lemma 6.3.7, for every $(S'', \Omega'') \in \mathcal{S}_2^2$, there exists $Z_S \subseteq E(S'') - E(S[\bar{\Omega}''])$ with $|Z_S| \leq 3(p_0 + 2)(2p_0 + 5)$ such that $(S'' - Z_S, \Omega'')$ has a circularly peg-linked circular cut-decomposition $(P_S, \mathcal{X}_S)$ such that there are no thick pairs between different bags. Note that $\mathcal{S}^2 = \mathcal{S}_1^2 \cup \mathcal{S}_2^2$ satisfies Conclusions 1-7 and 8 (a) of this theorem. Actually, by Lemma 6.3.7, for every component that is the intersection of a cycle in the circular linkage and a bag, there exists a path in this bag connecting the component and the cycle in the circular linkage passing through the peg in the bag. Furthermore, we may assume that there does not exist $(A, B) \in \mathcal{T}^2$ of order at most two such that some peg of a member in $\mathcal{S}_2^2$ is in $V(A) - V(B)$, where $\mathcal{T}^2$ is the tangle in the trunk of G^2 conformal with $\mathcal{T}^1$ obtained from cleaning at most κ $8(\rho_0 + 2)$-zones in the trunk of G^1; otherwise, we can put A into that member of $\mathcal{S}_2^2$ and merge bags to reduce the number of pegs and keep the circular linkage without increasing the circular adhesion and without decreasing the order of the tangle in the trunk of the new segregation.

For every $(S'', \Omega'') \in \mathcal{S}_2^2$, we apply Lemma 6.3.9 by taking $\rho = \rho_0 + 2$ at most twice, we obtain a society (S_0'', Ω_0'') that is enclosed by $C_1(S), C_2(S), ..., C_{100\kappa\rho_0 - 2\rho_0 - 7}(S)$ and its circular cut-decomposition $(P_S^0, \mathcal{X}_S^0)$ is such that either $(P_S^0, \mathcal{X}_S^0)$ is circularly peg-linked and has circular adhesion at most ρ satisfying Conclusion 8 (b) of this theorem, or $(P_S^0, \mathcal{X}_S^0)$ has circular adhesion at least two less than the circular adhesion of (P_S, Ω_S). We do not modify (S_0'', Ω_0'') if the former case happens. If the latter case happens, then we do the following operations to (S_i'', Ω_i'') to obtain a new society $(S_{i+1}'', \Omega_{i+1}'')$ until it satisfies Conclusion 8 (b), starting from $i = 0$:

- Applying Lemmas 6.3.1 and 6.3.2 to delete at most $2\rho_0 + 4$ vertices and obtain a new society that is enclosed by at most $\rho_0 + 1$ less concentric cycles than (S_i'', Ω_i'') and a circularly peg-linked circular decomposition of adhesion at most one more than the (S_0'', Ω_0'').

- Converting each vertex-cut of the mentioned circular decomposition into an edge-cut to obtain a circularly peg-linked circular cut-decomposition with the same circular adhesion.

- Applying Lemma 6.3.7 to delete at most $3(\rho_0 + 1)(2\rho_0 + 3)$ edges and modify the circular cut-decomposition such that no thick pair is between different bags.

- Applying Lemma 6.3.9 at most twice to delete at most ξ' vertices and edges and sacrifice at most four concentric cycles that enclose (S_i'', Ω_i'') to obtain a society $(S_{i+1}'', \Omega_{i+1}'')$ and its circular cut-decomposition $(P_S^{i+1}, \mathcal{X}_S^{i+1})$ of adhesion at least one less than the adhesion of $(P_S^i, \mathcal{X}_S^i)$.

For each $(S'', \Omega'') \in \mathcal{S}_2^2$, we define (S^*, Ω^*) to be the society obtained from the ultimate (S_i'', Ω_i'') that satisfies Conclusion 8 (b) by adding vertices not in S_i'' but adjacent to two adjacent vertices in $\bar{\Omega}''$. Then we define $\mathcal{S}_2^* = \{(S^*, \Omega^*) : (S'', \Omega'') \in \mathcal{S}_2^2\}$ and $\mathcal{S}_1^* = \{(S, \Omega) \in \mathcal{S}_1^2 : S \not\subseteq \bigcup_{(S^*,\Omega^*)\in\mathcal{S}_2^*} S^*\}$, and we define $\mathcal{S}^* = \mathcal{S}_1^* \cup \mathcal{S}_2^*$. Consequently, $\mathcal{S}^*$ satisfies Conclusions 1-8 of this theorem.

Suppose that Conclusion 9 of this theorem does not hold, and we let (A, B) be such a separation of order two in the trunk of $\mathcal{S}^*$. Then we see the block structure of A, and merge each component of the graph obtained from A by deleting the edges in the blocks consisting of single edges in the path of blocks connecting the blocks containing $V(A) \cap V(B)$ into a new society. So we may assume that Conclusion 9 holds. This completes the proof. ■

Let G be a graph and $\mathcal{T}$ an tangle in G. We say a location $\mathcal{L}$ in G is *linked with respect to* $\mathcal{T}$ if $\mathcal{L} \subseteq \mathcal{T}$ but there do not exist $(A, B) \in \mathcal{L}$ and $(C, D) \in \mathcal{T}$ of order less than $|V(A) \cap V(B)|$ such that $A \subseteq C$ and $D \subseteq B$. The following theorem gives the structure with respect to every tangle.

Theorem 6.3.11 *Let k, η, θ be positive integers. Then there exist positive integers ξ, κ, ρ such that the following hold. If G is a graph that does not contain a topological minor isomorphic to the Robertson chain of length k, and $\mathcal{T}$ is a tangle in G of order at least θ, then there exist $Z \subseteq V(G) \cup E(G)$ with $|Z| \leq \xi$ and a linked location $\mathcal{L} \subseteq \mathcal{T} - Z$ of $G - Z$ with respect to $\mathcal{T} - Z$ such that one of the following holds.*

1. *$Z \subseteq V(G)$, and $\mathcal{L}$ is faithful, and every thick cycle in its contour is pendant.*

2. *There exist pairwise disjoint $\mathcal{L}_1, \mathcal{L}_{2,1}, \mathcal{L}_{2,2}, ..., \mathcal{L}_{2,\kappa}$ with $\mathcal{L} = \mathcal{L}_1 \cup \bigcup_{i=1}^{\kappa} \mathcal{L}_{2,i}$ such that the following hold.*

(a) *No two edges in Z have the same end. And if $e \in Z \cap E(G)$, then there exist $1 \leq i \leq \kappa$ and $(A, B), (A', B') \in \mathcal{L}_{2,i}$ such that one end of e is in $V(A) \cap V(B)$ and the other end of e is in $V(A') \cap V(B')$.*

(b) *$V(A) \cap V(B) \cap V(A') \cap V(B') = \emptyset$ for every distinct $(A, B), (A', B') \in \mathcal{L}$ unless one of them is in $\mathcal{L}_1$ and have order one.*

(c) *No two edges in $\bigcap_{(A,B)\in\mathcal{L}} B$ have the same ends.*

(d) *Every separation in $\mathcal{L}_1$ has order at most three.*

(e) *For every $(A, B) \in \mathcal{L}_1$ and disjoint subsets X, Y of $V(A) \cap V(B)$, there exist two edge-disjoint paths in A from X to Y, and when $|X| = 1$ and $|Y| = 2$, these two paths have at most one common end.*

(f) *If $(A, B) \in \mathcal{L}_1$ of order three, then every vertex in $V(A) \cap V(B)$ is incident with at most one edge in $G - Z$ whose the other end in $V(B) - V(A)$.*

(g) *If $(A, B) \in \mathcal{L}_1$ of order two, then one vertex in $V(A) \cap V(B)$ is incident with at most one edge in $G - Z$ whose the other end in $V(B) - V(A)$, and the other vertex in $V(A) \cap V(B)$ is incident with at most two edges in $G - Z$ whose the other ends in $V(B) - V(A)$, and these two edges are not parallel edges.*

(h) *If $(A, B) \in \mathcal{L}_1$ of order one, then the vertex in $V(A) \cap V(B)$ is adjacent in $G - Z$ to at most three vertices in $V(B) - V(A)$.*

(i) *For every $1 \leq i \leq \kappa$ and $(A, B) \in \mathcal{L}_{2,i}$, there exists at most one vertex in $V(A) \cap V(B)$ adjacent in $G - Z$ to a vertex not in $\bigcup_{(A,B)\in\mathcal{L}_{2,i}} V(A)$. We denote the set of such vertices by $\overline{\Omega_i}$.*

(j) *For every $1 \leq i \leq \kappa$, the subgraph of $G - Z$ induced by $\bigcup_{(A,B)\in\mathcal{L}_{2,i}} V(A)$ is a ρ-vortex with the set of pegs $\overline{\Omega_i}$ that has a circularly peg-linked circular cut-decomposition $(P_i, \mathcal{X}_i)$ of adhesion at most ρ such that $\mathcal{L}_{2,i} = \{(A, B) : A = (G - Z)[X], X \in \mathcal{X}_i\}$.*

(k) *For* $1 \leq i < j \leq \kappa$, *no vertex in* $\overline{\Omega_i}$ *is adjacent in* $G - Z$ *to a vertex in* $\overline{\Omega_j}$ *in* $G - Z$.

(l) *Let* H *be the graph obtained from the subgraph of* $G - Z$ *induced by* $V(G - Z) - (\bigcup_{i=1}^{\kappa} \bigcup_{(A,B) \in \mathcal{L}_{2,i}} (V(A) - \overline{\Omega_i}))$ *by adding a cycle passing through* $V(A) \cap V(B)$ *for every* $(A, B) \in \mathcal{L}_1$. *Then*

   i. H *is weakly subcubic.*

   ii. *There exist a surface* Σ *in which* K_{6k} *cannot be drawn and* κ *pairwise disjoint open disks* $\Delta_1, \Delta_2, ..., \Delta_\kappa$ *in* Σ *such that* H *can be drawn in* $\Sigma - \bigcup_{i=1}^{\kappa} \Delta_i$ *such that the vertices in* $\overline{\Omega_i}$ *are drawn in* $\partial \Delta_i$ *in the nature order, and every* H*-normal* O*-arc in* Σ *intersecting* $V(H)$ *in at most two vertices is a boundary of a disk in* Σ.

(m) *For every* $(A, B) \in \mathcal{L}$, *if there exists* $(C, D) \in \mathcal{T} - Z$ *of the same order as* (A, B) *such that* $A \subseteq C$ *and* $D \subseteq B$, *then every vertex in* $(V(A) \cap V(B)) - (V(C) \cap V(D))$ *is incident with at most one edges in* $G - Z$ *whose the other end in* $V(B) - V(A)$.

Proof. If $\mathcal{T}$ does not control a K_{6k}-minor, then there exist $Z \subseteq V(G) \cap E(G)$, surface Σ, and segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ mentioned in Lemma 6.2.6. Then we define $\mathcal{L}_1 = \{(S, B) : (S, \Omega) \in \mathcal{S}_1, |E(S)| \geq 2, V(B) \cap V(S) = \bar{\Omega}\}$. For each member (S_i, Ω_i) of $\mathcal{S}_2$, let $(P_i, \mathcal{X}_i)$ be a circularly peg-linked circular cut-decomposition of adhesion at most ρ of (S_i, Ω_i), and we define $\mathcal{L}_{2,i} = \{((G - Z)[X], B) : X \in \mathcal{X}_i, V(X) \cap V(B)$ consists of the vertices adjacent to a vertex not in $X\}$. Then the second conclusion of this theorem holds. So we may assume that $\mathcal{T}$ controls a K_{6k}-minor.

By Corollary 1.4.2, there exists $Z_1 \subseteq V(G)$ such that every vertex is 3-separable from $\mathcal{T} - Z_1$. If $\mathcal{T} - Z_1$ does not control a K_{6k}-minor, then we are done as in the last paragraph. So we assume that $\mathcal{T} - Z_1$ controls a K_{6k}-minor. By Lemma 6.1.1, there exists a faithful location $\mathcal{L}_1 \subseteq \mathcal{T} - Z_1$ of level at least one.

By applying Theorem 6.2.3 at most k times, there exists $Z' \subseteq V(G)$ such that either $\mathcal{T} - Z'$ does not control a K_{6k}-minor, or there exists a faithful location $\mathcal{L} \subseteq \mathcal{T} - Z'$ of $G - Z'$ of level at least k. We are done for the former case as in the last paragraph. So we may assume that the latter case happens. In this case, $G - Z'$ has no non-pendant thick edges, since G does not contain a topological minor isomorphic to the Robertson chain of length k. Therefore, the first conclusion of this theorem holds. This completes the proof. ■

CHAPTER VII

WELL-QUASI-ORDERING GRAPHS BY THE TOPOLOGICAL MINOR RELATION

7.1 Well-quasi-ordering graphs with a tangle

Theorem 7.1.1 ([48, Theorem 1.5]) *Let $(S, \preceq)$ be a well-quasi-ordered set. For every positive integer i, let G_i be a graph and let $g_i : V(G_i) \to S$ be a function. Then there exist $j > j'$ such that the following hold.*

1. *There exist an injective function $\pi_V : V(G_{j'}) \to V(G_j)$ and a function π_E that maps each edge xy of $G_{j'}$ to a path in G_j with the ends $\pi_V(x)$ and $\pi_V(y)$ such that $\pi_E(e_1)$ and $\pi_E(e_2)$ are edge-disjoint for every two different edges e_1, e_2 of $G_{j'}$.*

2. *For every $v \in V(G_{j'})$, $g_{j'}(v) \preceq g_j(\pi_V(v))$.*

Lemma 7.1.2 *Let $\mathcal{F}$ be a family of frames such that every ordered extended location $(\mathcal{L}, \tau)$ that fits a frame in $\mathcal{F}$ satisfies that $\mathcal{L}$ is faithful and every thick cycle in the contour of $\mathcal{L}$ is pendant. Then $\mathcal{F}$ is well-behaved.*

Proof. Let $(S, \preceq), (S', \preceq')$ be well-quasi-ordered sets. Let $G_1, G_2, ...$ be an infinite sequence of graphs. And for every $i \geq 1$, let $(\mathcal{L}_i, \tau_i)$ be an ordered extended location in G_i with root (A_i, B_i), and let $\phi_i : V(G_i) \to S$, $\psi_i : \mathcal{L}_i - \{(A_i, B_i)\} \to S'$. Assume that $\mathcal{L}_i$ fits a frame $(H_i, \mu_i, \mathcal{Q}_i)$ in $\mathcal{F}$, and $\bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i)$ is well-quasi-ordered by the $(S, \preceq)$-labelled topological minor relation. We denote the $(S, \preceq)$-labelled topological minor relation by $\preceq_{SUB}$.

Let $(S_1, \preceq_1)$ be the well-quasi-ordered set obtained from $(\bigcup_{i \geq 1} \mathcal{P}_i \cup \{*\}, \preceq_{SUB})$ and $(\{*\}, =)$ by disjoint union, and let $(S_2, \preceq_2)$ be the well-quasi-ordered set obtained from

$(S', \preceq')$ and $(\{*\}, =)$ by disjoint union. Let $S'' = S \times S_1 \times S_2 \times \{0,1,2,3\} \times \{0,1,2,3\} \times \{0,1,2,...,6\} \times (\mathbb{N} \cup \{0\})$, and $\preceq'' = (\preceq, \preceq_1, \preceq_2, =, =, =, \leq)$. Note that $(S'', \preceq'')$ is a well-quasi-ordered set.

For $i \geq 1$, let G'_i be the graph obtained from the contour of $\mathcal{L}_i$ by duplicating each edge of C_A for every $(A,B) \in \mathcal{L}_i$ of order at least two. For every $i \geq 1$ and $v \in V(G'_i)$, let $d'(v)$ be the pair of numbers whose first entry is the number of non-loops incident with v in G'_i and the second entry is the number of loops in G'_i incident with v. Since $\mathcal{L}_i$ is faithful, v is incident with at most six non-loops in G'_i. Define $g_i : V(G'_i) \to S''$ such that the following hold.

- $g_i(v) = (\phi_i(v), (G_{i_{\tau_i,A}}, \gamma_{i_{\tau_i,A}}, \sigma_{i_{\tau_i,A}}, \phi_i|_{G_{i_{\tau_i,A}}}), \psi_i(A), |V(A) \cap V(B)|, j, d'(v))$, if v is in $V(A) \cap V(B)$ for some $(A,B) \in \mathcal{L}_i - \{(A_i, B_i)\}$ and v is the j-th vertex in $V(A) \cap V(B)$ according to the ordering associated with (A,B),
- $g_i(v) = (\phi_i(v), *, *, |V(A) \cap V(B)|, j, d'(v))$, if v is in $V(A_i) \cap V(B_i)$ and v is the j-th vertex in $V(A_i) \cap V(B_i)$ according to the ordering associated with (A,B),
- $g_i(v) = (\phi_i(v), *, *, 0, 0, d'(v))$, otherwise.

By Theorem 7.1.1, there exist $j > j'$ such that the following hold.

- There exist an injective function $\pi_V : V(G'_{j'}) \to V(G'_j)$ and a function π_E that maps each edge xy of $G'_{j'}$ to a path in G'_j with the ends $\pi_V(x)$ and $\pi_V(y)$ such that $\pi_E(e_1)$ and $\pi_E(e_2)$ are edge-disjoint for every two different edges e_1, e_2 of $G'_{j'}$.
- For every $v \in V(G'_{j'})$, $g_{j'}(v) \preceq g_j(\pi_V(v))$.

Note that $d'(v)$ is an entry of $g_i(v)$ for every $i \geq 1$ and vertex v of G'_i, so π_V maps each vertex to a vertex that is incident with the same number of non-loops. In other words, (π_V, π_E) is a strong immersion.

Claim 1: If $(A, B) \in \mathcal{L}_{j'}$ of order two and let x, y be the two vertices of $V(A) \cap V(B)$, then there exists $(A', B') \in \mathcal{L}_j$ of order two such that $V(A') \cap V(B') = \{\pi_V(x), \pi_V(y)\}$.

Proof of Claim 1: Since every thick cycle in the contour of $\mathcal{L}_j$ or $\mathcal{L}_{j'}$ is pendant, so we may assume that y is adjacent to at most two vertices. Let (A', B') be the separation in $\mathcal{L}_j$ such that $\pi_V(x) \in V(A') \cap V(B')$. Let y' be the vertex in $V(A') \cap V(B') - \{\pi_V(x)\}$. Note that $[\{\pi_V(x), y'\}, V(G'_{j'}) - \{\pi_V(x), y'\}]$ is an edge-cut of G'_j of size less than the number of non-loops incident with $\pi_V(x)$ in G'_j. So there exists $v \in N_{G'_{j'}}(x)$ such that $\pi_V(v) = y'$. But (π_V, π_E) is a strong immersion and there exist two edges with the ends x, y, so $v = y$. This proves the claim. $\square$

The following claim is similar.

Claim 2: If $(A, B) \in \mathcal{L}_{j'}$ of order three and let x, y, z be the three vertices of $V(A) \cap V(B)$, then there exists $(A', B') \in \mathcal{L}_j$ of order three such that $V(A') \cap V(B') = \{\pi_V(x), \pi_V(y), \pi_V(z)\}$.

Then we define $\zeta : \mathcal{L}_{j'} - \{(A_{j'}, B_{j'})\} \to \mathcal{L}_j - \{(A_j, B_j)\}$ such that $\zeta(A, B) = (A', B')$ if $(A, B) \in \mathcal{L}_{j'} - \{(A_{j'}, B_{j'})\}$ of order at least two, where (A', B') is the separation mentioned in Claims 1 and 2; and $\zeta(A, B) = (A'', B'')$ if $(A, B) \in \mathcal{L}_{j'} - \{(A_{j'}, B_{j'})\}$ of order one, where (A'', B'') is the separation in $\mathcal{L}_j - \{(A_j, B_j)\}$ such that $V(A'') \cap V(B'')$ contains the image of the vertex in $V(A) \cap V(B)$ under π_V. And we define $\iota : \mathcal{P}_{\phi_{j'}}(\mathcal{L}_{j'}, \tau_{j'}) \to \mathcal{P}_{\phi_j}(\mathcal{L}_j, \tau_j)$ such that $\iota(G_{j'_{\tau_{j'}, A}}, \gamma_{j'_{\tau_{j'}, A}}, \sigma_{j'_{\tau_{j'}, A}}) = (G_{j_{\tau_j, C}}, \gamma_{j_{\tau_j, C}}, \sigma_{j_{\tau_j, C}})$ for every $(A, B) \in \mathcal{L}_{j'} - \{(A_{j'}, B_{j'})\}$, where $\zeta(A, B) = (C, D)$. According to the second and the third entry of $g_i(v)$ for each $i \geq 1$ and $v \in V(G'_i)$, we know that $\iota(x)$ contains x as an $(S, \preceq)$-labelled topological minor, and $\psi_{j'}(x) \preceq' \psi_j(\zeta(x))$ for every $x \in \mathcal{L}_{j'} - \{(A_{j'}, B_{j'})\}$.

Let I be the image of ζ. We define $\pi'_V : V(\text{Con}(\mathcal{L}_{j'}, \tau_{j'}, \mathcal{L}_{j'} - \{(A_{j'}, B_{j'})\})_{\tau_{j'}, A_{j'}}) \to V(\text{Con}(\mathcal{L}_j, \tau_j, I)_{\tau_j, A_j})$ such that $\pi'_V(v_A) = v_C$ for every $(A, B) \in \mathcal{L}_{j'} - \{(A_{j'}, B_{j'})\}$, where $(C, D) = \iota(A, B)$, and $\pi'_V(v) = \pi_V(v)$ for other vertices v. And we define π'_E from $E(\text{Con}(\mathcal{L}_{j'}, \tau_{j'}, \mathcal{L}_{j'} - \{(A_{j'}, B_{j'})\})_{\tau_j, A_{j'}})$ such that $\pi_E(v_A x_i) = v_C x'_i$ for every

$(A, B) \in \mathcal{L}_{j'} - \{(A_{j'}, B_{j'})\}$, where $(C, D) = \iota(A, B)$ and x_i, x'_i is the i-th vertex in (A, B) and (C, D), respectively, and $\pi'_E(e) = \pi_E(e)$ for other edges e. Since $\mathcal{L}_j$ and $\mathcal{L}_{j'}$ are faithful, they satisfy (F6). Therefore, $(\mathcal{L}_j, \tau_j, \phi_j, \psi_j)$ simulates $(\mathcal{L}_{j'}, \tau_{j'}, \phi_{j'}, \psi_{j'})$. This proves that $\mathcal{F}$ is a well-behaved set of frames. ■

We need a couple of notions defined in [47] to prove the next lemma. Let Σ be a surface. Every component of the boundary of Σ is a *cuff*. For every subset S of Σ, we denote the closure of S by $\bar{S}$. A *painting* Γ in a surface Σ is a triple (U, V, γ), where $U \subseteq \Sigma$ is closed, $V \subseteq U$ is finite, and

- $\partial\Sigma \subseteq U$, and $U - V$ has only finitely many arc-wise connected components, called *cells*,
- for each cell c, $\bar{c}$ is a closed disk and $|\bar{c} - c| = 2$ or 3, and $\bar{c} \cap V = \bar{c} - c \subseteq \partial\bar{c}$,
- for each cell c, if $c \cap \partial\Sigma \neq \emptyset$, then $|\bar{c} - c| = 2$, and $\bar{c} \cap \partial\Sigma$ is a line and its ends are the members of $\bar{c} - c$,
- for each cell c, $\gamma(c)$ is a march μ whose entries are $\bar{c} - c$.

We write $U(\Gamma) = U$, $V(\Gamma) = V$, $\gamma_\Gamma = \gamma$. The members of $V(\Gamma)$ are called *nodes*. If c is a cell of Γ and $1 \leq i \leq |\bar{c} - c|$, we call the i-th term of $\gamma(c)$ the i-th node of c, and we call the first node of c the *tail* of c. A cell c is a *border cell* if $c \cap \partial\Sigma \neq \emptyset$; otherwise it is *internal*. Nodes in $\partial\Sigma$ are *border nodes* and the others are *internal*. If Θ is a cuff, we say that a cell c or node n *borders* Θ if $c \cap \Theta \neq \emptyset$ or $n \in \Theta$. The components of $\Sigma - U(\Gamma)$ are the *regions* of Γ. A painting Γ in Σ is *3-connected* if

- for every Γ-normal O-arc F in Σ with $|F \cap V(\Gamma)| \leq 2$, there is a closed disc $\Delta \subseteq \Sigma$ with $\partial\Delta = F$ which includes at most one cell of Γ and with $\Delta \cap V(\Gamma) \subseteq F$,
- for every Γ-normal line F in Σ with $|F \cap V(\Gamma)| \leq 2$ and with both ends in $\partial\Sigma$ and with no other point in $\partial\Sigma$, there is a closed disc $\Delta \subseteq \Sigma$ with $F \subseteq \partial\Delta \subseteq F \cup \partial\Sigma$ which includes at most one cell of Γ and with $\Delta \cap V(\Gamma) \subseteq F$.

The *skeleton* of Γ is the subgraph with vertex-set $V(\Gamma)$ and two vertices $n_1, n_2 \in V(\Gamma)$ are adjacent in the skeleton if and only if there exists a cell c of Γ such that $n_1, n_2 \in \bar{c} - c$.

Let Γ, Γ' be two paintings in a surface Σ. A function η with domain the union of $V(\Gamma)$ and the set of cells of Γ is a *linear inflation* of Γ in Γ' if the following hold.

- $\eta(c)$ is a cell of Γ' for each cell c of Γ, and $|\overline{\eta(c)} - \eta(c)| = |\bar{c} - c|$, and for each cuff Θ, c borders Θ if and only if $\eta(c)$ does.

- $\eta(c_1) \neq \eta(c_2)$ for all distinct cells c_1, c_2 of Γ.

- For each cuff Θ, if a cell c of Γ borders Θ and we orient Θ so that the tail of c immediately precedes $c \cap \Theta$, then the tail of $\eta(c)$ immediately precedes $\eta(c) \cap \Theta$ under the same orientation of Θ.

- For every $n \in V(\Gamma)$, $\eta(n)$ is a non-null induced connected subgraph of the skeleton of Γ'.

- $\eta(n_1)$ and $\eta(n_2)$ are disjoint for distinct $n_1, n_2 \in V(\Gamma)$.

- For all $n \in V(\Gamma)$ and cell c of Γ and $1 \leq i \leq |\bar{c} - c|$, n is the i-th node of c if and only if $\eta(n)$ contains the i-th node of $\eta(e)$.

- For every border cell c' of Γ', if there does not exist a cell c of Γ such that $c' = \eta(c)$, then the nodes of c' are adjacent in $\eta(n)$ for some $n \in V(\Gamma)$.

Theorem 7.1.3 ([47, Theorem 8.1]) *Let Σ be a surface and let $(S, \preceq)$ be a well-quasi-ordered set. For each $i \geq 1$, let Γ_i be a 3-connected painting in Σ, and let f_i be a function mapping the cells of Γ_i to S. Then there exist $j > j' \geq 1$ and a linear inflation η of $\Gamma_{j'}$ in Γ_j such that $f_{j'}(c) \preceq f_j(\eta(c))$ for each cell c of $\Gamma_{j'}$.*

Lemma 7.1.4 *Let κ, ρ be nonnegative integers and Σ a surface. Let $\mathcal{F}$ be the family of frames such that every ordered extended locations $(\mathcal{L}, \tau)$ of a graph G that fits a frame in $\mathcal{F}$ satisfies the following.*

1. *There exist pairwise disjoint $\mathcal{L}_1, \mathcal{L}_{2,1}, \mathcal{L}_{2,2}, ..., \mathcal{L}_{2,\kappa}$ such that $\mathcal{L} = \mathcal{L}_1 \cup \bigcup_{i=1}^{\kappa} \mathcal{L}_{2,i}$.*

2. *$V(A) \cap V(B) \cap V(A') \cap V(B') = \emptyset$ for every distinct $(A, B), (A', B') \in \mathcal{L}$ unless one of them is in $\mathcal{L}_1$ and have order one.*

3. *No two edges in $\bigcap_{(A,B)\in\mathcal{L}} B$ have the same ends.*

4. *Every separation in $\mathcal{L}_1$ has order at most three.*

5. *For every $(A, B) \in \mathcal{L}_1$ and disjoint subsets X, Y of $V(A) \cap V(B)$, there exist two edge-disjoint paths in A from X to Y, and when $|X| = 1$ and $|Y| = 2$, these two paths have at most one common end.*

6. *If $(A, B) \in \mathcal{L}_1$ of order three, then every vertex in $V(A) \cap V(B)$ is incident with at most one edge in G whose the other end in $V(B) - V(A)$.*

7. *If $(A, B) \in \mathcal{L}_1$ of order two, then one vertex in $V(A) \cap V(B)$ is incident with at most one edge in G whose the other end in $V(B) - V(A)$, and the other vertex in $V(A) \cap V(B)$ is incident with at most two edges in G whose the other ends in $V(B) - V(A)$, and these two edges are not parallel edges.*

8. *If $(A, B) \in \mathcal{L}_1$ of order one, then the vertex in $V(A) \cap V(B)$ is adjacent in G to at most three vertices in $V(B) - V(A)$.*

9. *For every $1 \leq i \leq \kappa$ and $(A, B) \in \mathcal{L}_{2,i}$, there exists at most one vertex in $V(A) \cap V(B)$ adjacent in G to a vertex not in $\bigcup_{(A,B)\in\mathcal{L}_{2,i}} V(A)$, and this vertex is incident with at most one edge in B. We denote the set of such vertices by $\overline{\Omega_i}$.*

10. *For every* $1 \leq i \leq \kappa$*, the subgraph of* G *induced by* $\bigcup_{(A,B)\in\mathcal{L}_{2,i}} V(A)$ *is a* ρ*-vortex with the set of pegs* $\overline{\Omega_i}$ *that has a circularly peg-linked circular cut-decomposition* $(P_i, \mathcal{X}_i)$ *of adhesion at most* ρ *such that* $\mathcal{L}_{2,i} = \{(A,B) : A = (G-Z)[X], X \in \mathcal{X}_i\}$.

11. *For* $1 \leq i < j \leq \kappa$*, no vertex in* $\overline{\Omega_i}$ *is adjacent in* G *to a vertex in* $\overline{\Omega_j}$ *in* G.

12. *Let* H *be the graph obtained from the subgraph of* G *induced by* $V(\bigcap_{(A,B)\in\mathcal{L}} V(B)) - (\bigcup_{i=1}^{\kappa} \bigcup_{(A,B)\in\mathcal{L}_{2,i}} (V(A) - \overline{\Omega_i}))$ *by adding a cycle passing through* $V(A) \cap V(B)$ *for every* $(A,B) \in \mathcal{L}_1$ *of order at least two. Then*

    (a) H *is weakly subcubic.*

    (b) *There exist* κ *pairwise disjoint open disks* $\Delta_1, \Delta_2, ..., \Delta_\kappa$ *in* Σ *such that* H *can be drawn in* $\Sigma - \bigcup_{i=1}^{\kappa} \Delta_i$ *such that the following hold.*

        i. *the vertices in* $\overline{\Omega_i}$ *are drawn in* $\partial\Delta_i$ *in the nature order.*

        ii. *Every* H*-normal O-arc in* $\Sigma - \bigcup_{i=1}^{\kappa} \Delta_i$ *intersecting* $V(H)$ *in at most two vertices is a boundary of a disk* Δ *in* $\Sigma - \bigcup_{i=1}^{\kappa} \Delta_i$*, and every vertex in* $\Delta \cap H$ *is adjacent to at most two vertices in* $\Delta \cap H$.

Then $\mathcal{F}$ *is well-behaved.*

Proof. Let $(S, \preceq), (S', \preceq')$ be well-quasi-ordered sets. Let $G_1, G_2, ...$ be an infinite sequence of graphs. And for every $i \geq 1$, let $(\mathcal{L}_i, \tau_i)$ be an ordered extended location in G_i with root (A_i, B_i), and let $\phi_i : V(G_i) \to S$, $\psi_i : \mathcal{L}_i - \{(A_i, B_i)\} \to S'$. Assume that $\mathcal{L}_i$ fits a frame in $\mathcal{F}$, and $\bigcup_{i\geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i)$ is well-quasi-ordered by the $(S, \preceq)$-labelled topological minor relation. We denote the $(S, \preceq)$-labelled topological minor relation by $\preceq_{SUB}$.

For every $i \geq 1$, let H_i be the graph H and let $\overline{\Omega_{i,j}}$ and $(P_{i,j}, \mathcal{X}_{i,j})$ be the set $\overline{\Omega_j}$ and the circularly peg-linked circular cut-decomposition for each $\mathcal{L}_i$ and $1 \leq j \leq \kappa$ mentioned in the condition of the lemma, respectively. For $i \geq 1$ and $1 \leq j \leq \kappa$, we define $q_{i,j}$ to be a function from the set of edges between different bags of $(P_{i,j}, \mathcal{X}_{i,j})$ to $\{1, 2, ..., \rho\} \times \{1, 2, ..., \rho\}$ as follows.

- We fix a node $t \in P_{i,j}$. Let $C_1, C_2, ..., C_p$ be a circular linkage of $(P_{i,j}, \mathcal{X}_{i,j})$, where C_1 is the cycle passing through $\overline{\Omega_{i,j}}$ in order and $1 \leq p \leq \rho$. Let e_1 be the edge in C_1 that is incident with a vertex in the in-adhesion set of t. Define $q_{i,j}(e_1) = (1, 1)$.

- For every $2 \leq \ell \leq p$, we pick an edge e_ℓ in C_ℓ between different bags whose associated subpath contains t, and define $q_{i,j}(e_\ell) = (\ell, 1)$.

- For every $1 \leq \ell \leq p$, we traverse C_ℓ starting at e_ℓ in the direction that the next edge in C_ℓ between different bags is an edge incident with a vertex in the out-adhesion set of the node whose in-adhesion set contains a vertex incident with e_ℓ. For every $1 \leq \ell \leq p$ and $2 \leq r \leq \rho$, if e is the r-th edge between different bags whose associated subpaths contains t appeared in C_ℓ under the mentioned transversal, then we define $q_{i,j} = (\ell, r)$. We denote such edge e by $e_{\ell,r}$.

- For every $1 \leq \ell \leq p$ and edge e in $C_\ell - \{e_\ell, e_{\ell,2}, e_{\ell,3}, ...\}$ between different bags, if r is the largest number such that e appears later than $e_{\ell,r}$, then we define $q_{i,j}(e) = (\ell, r)$.

For every $i \geq 1$ and $1 \leq j \leq \kappa$, let $\mathcal{L}_{i,1}$ and $\mathcal{L}_{i,2,j}$ be the corresponding locations $\mathcal{L}_1, \mathcal{L}_{2,j}$ mentioned in the condition of this lemma. For every $i \geq 1$ and $1 \leq j \leq \kappa$, we define $q'_{i,j} : \bigcup_{(A,B) \in \mathcal{L}_{i,2,j}} V(A) \cap V(B) \rightarrow (\{1, 2, ..., \rho\} \cup \{*\})^4$ such that

- if $v \in \overline{\Omega_{i,j}}$, then $q'_{i,j}(v) = (*, *, *, *)$;

- if $v \notin \overline{\Omega_{i,j}}$ and v is incident with an edge e in the in-adhesion set of the node whose bags contains v, then the first two entries of $q'_{i,j}(v)$ equal $q_{i,j}(e)$, otherwise the first two entries of $q'_{i,j}(v)$ equal $(*, *)$;

- if $v \notin \overline{\Omega_{i,j}}$ and v is incident with an edge e in the out-adhesion set of the node whose bags contains v, then the last two entries of $q'_{i,j}(v)$ equal $q_{i,j}(e)$, otherwise the last two entries of $q'_{i,j}(v)$ equal $(*, *)$;

For every $i \geq 1$, we define h_i to be a function from $\bigcup_{j=1}^{\kappa} \overline{\Omega_{i,j}}$ to $\mathbb{N} \times (\{1, 2, ..., \rho\} \cup \{*\})^{4\rho}$ such that if $v \in \overline{\Omega_{i,j}}$, then the following hold.

- Let (A, B) be the separation in $\mathcal{L}_i$ corresponding to the bag containing v. The first entry of $h_i(v)$ is $|V(A) \cap V(B)|$.

- For every $1 \leq \ell \leq |V(A) \cap V(B)|$, the $4\ell - 2$-th, $4\ell - 1$-th, 4ℓ-th and $4\ell + 1$-th entries of $h_i(v)$ equal $q'_{i,j}(u)$, where u is the ℓ-th vertex in $V(A) \cap V(B)$.

- We arbitrarily order the edges e whose associated subpaths contain the node whose bag contains v but not incident with a vertex in $V(A) \cap V(B)$. The rest of entries of $h_i(v)$ are $(q_{i,j}(e_1), q_{i,j}(e_1), q_{i,j}(e_2), q_{i,j}(e_2), ...)$, where $e_1, e_2, ...$ are the mentioned edges ordered by the mentioned ordering.

Let $(S_1, \preceq_1)$ be the well-quasi-ordered set obtained from $(\mathbb{N} \times (\{1, 2, ..., \rho\} \cup \{*\})^{4\rho}, (=, =, ..., =))$ and $(\{*\}, =)$ by disjoint union. Let $(S_2, \preceq_2)$ be the well-quasi-ordered set obtained from $(S, \preceq)$ and $(S_1, \preceq_1)$ by Cartesian production. For $i \geq 1$, define $\phi'_i : V(H_i) \to S_2$ such that

- if $v \in \overline{\Omega_{i,j}}$ for some $1 \leq j \leq \kappa$, then $\phi'_i(v) = (\phi_i(v), h_i(v))$;

- if $v \notin \overline{\Omega_{i,j}}$ for some $1 \leq j \leq \kappa$, then $\phi'_i(v) = (\phi_i(v), *)$.

Let $S_3 = \bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i) \times S' \times S_2$ and $\preceq_3 = (\preceq_{SUB}, \preceq', \preceq_2)$. For $i \geq 1$, define $\phi''_i : V(H_i) \to S_3$ such that $\phi''_i(v) = ((G_{i_{\tau_i,A}}, \gamma_{i_{\tau_i,A}}, \sigma_{i_{\tau_i,A}}, \phi|_{V(G_{i_{\tau_i,A}})}), \psi_i(A, B), \phi'_i(v))$ for every $v \in V(H_i)$, where (A, B) is the separation in $\mathcal{L}_i$ such that $v \in V(A) \cap V(B)$.

Let $(S_4, \preceq_4)$ be the well-quasi-ordered set obtained from $(S_3, \preceq_3)$ and $(\{*\}, =)$ by disjoint union. Let $(S_5, \preceq_5) = (S_4^3, \preceq_4^3)$. Let H'_i be the graph obtained from H_i by subdivided every edge in $(\bigcap_{(A,B)\in\mathcal{L}} B) \cap H_i$ once. We denote the vertex obtained by subdivided the edge e by v_e. Note that H_i' can also be drawn in $\Sigma - \bigcup_{j=1}^{\kappa} \Delta_j$. We construct a painting Γ_i in $\Sigma - \bigcup_{i=1}^{\kappa} \Delta_i$ and define f_i to be a function from the set of cells of Γ_i to S_5 as follows.

- $V(\Gamma_i) = \{v_e : e \in E(\bigcap_{(A,B)\in\mathcal{L}} B \cap H_i)\} \cup \bigcup_{j=1}^{\kappa} \overline{\Omega_{i,j}}$.

- For every $(A,B) \in \mathcal{L}_1$ of order three, add an internal cell c such that $\bar{c} - c = \{v_e : e$ is incident with a vertex in $V(A) \cap V(B)\}$, and define $\gamma(c)$ to be the march on $\bar{c} - c$ such that the j-th node of c corresponds to the edge incident with the j-th vertex in $V(A) \cap V(B)$, for $1 \leq j \leq 3$; define $f_i(c)$ such that its j-th entry is $h_i(v_j)$, where v_j is the j-th node of $V(A) \cap V(B)$, for $1 \leq j \leq 3$. We denote the cell c by $c_{i,A}$.

- For every $(A,B) \in \mathcal{L}_1$ of order two, we let x, y be the vertices in $V(A) \cap V(B)$. By the assumption, at least one of x, y, say x, is incident in G with exactly one edge whose the other end in $V(B) - V(A)$. Add an internal cell c such that $\bar{c} - c = \{v_e : e$ is incident with x or $y\}$, and define $\gamma(c)$ to be the march on $\bar{c} - c$ such that the first node of c corresponds to the edge incident x and the rest correspond to the edges incident with y; define $f_i(c) = (h_i(x), h_i(y), h_i(y))$. We denote the cell c by $c_{i,A}$.

- For every vertex $v \in V(H_i)$, add an internal cell c such that $\bar{c} - c = \{v_e : e$ is incident with $v\}$, and define $\gamma(c)$ to be a march on $\bar{c} - c$; define $f_i(c) = (h_i(v), h_i(v), h_i(v))$. We denote the cell c by c_v.

- For every $1 \leq j \leq \kappa$ and vertex v in $\overline{\Omega_{i,j}}$, add an internal cell c such that $\bar{c} - c = \{v, v_e : e$ is incident with $v\}$, and define $\gamma(c)$ to be the march on $\bar{c} - c$ such that the first node of c is v; define $f_i(c) = (h_i(v), h_i(v'), *)$, where v' is the end of e other than v. We denote the cell c by $c_{i,A}$, where $(A,B) \in \mathcal{L}_i$ is the separation such that $v \in V(A) \cap V(B)$.

- For every $1 \leq j \leq \kappa$ and every two consecutive vertex u, v in $\overline{\Omega_{i,j}}$ ordered by the natural ordering on $\overline{\Omega_{i,j}}$, add a border cell c such that $\bar{c} - c = \{u, v\}$, and define $\gamma(c)$ to be the march on $\bar{c} - c$ whose ordering consistent with the natural ordering; define $f_i(c) = (h_i(u), h_i(v), *)$.

Claim 1: If there exist $j > j' \geq 1$ and a linear inflation η of $\Gamma_{j'}$ in Γ_j such that $f_{j'}(c) \preceq_5 f_j(\eta(c))$ for each cell c of $\Gamma_{j'}$, then $(\mathcal{L}_j, \tau_j, \phi_j, \psi_j)$ simulates $(\mathcal{L}_{j'}, \tau_{j'}, \phi_{j'}, \psi_{j'})$.

Proof of Claim 1: Define $\zeta : \mathcal{L}_{j'} - \{(A_{j'}, B_{j'})\} \to \mathcal{L}_j - \{(A_j, B_j)\}$ such that $\zeta(A, B) = (C, D)$ if $(A, B) \in \mathcal{L}_{j'}$, $(C, D) \in \mathcal{L}_j$, and $\eta(c_{j',A}) = c_{j,C}$. And we define $\iota : \mathcal{P}_{\phi_{j'}}(\mathcal{L}_{j'}, \tau_{j'}) \to \mathcal{P}_{\phi_j}(\mathcal{L}_j, \tau_j)$ such that $\iota(G_{j'_{\tau_{j'},A}}, \gamma_{j'_{\tau_{j'},A}}, \sigma_{j'_{\tau_{j'},A}}) = (G_{j_{\tau_j,C}}, \gamma_{j_{\tau_j,C}}, \sigma_{j_{\tau_j,C}})$ if $\zeta(A, B) = (C, D)$.

For $i \geq 1$, let $L_i = \bigcap_{(A,B) \in \mathcal{L}_i} B$. Define $\pi_V : V(L_{j'}) \to V(L_j)$ such that for every $x \in V(H_{j'})$ the following hold.

- If x the ℓ-th vertex in $V(A) \cap V(B)$ for some $(A, B) \in \mathcal{L}_{j'}$ and $1 \leq \ell \leq |V(A) \cap V(B)|$, then $\pi_V(x)$ is the ℓ-th vertex of $V(C) \cap V(D)$, where $(C, D) \in \mathcal{L}_j$ with $\zeta(A, B) = (C, D)$.
- Otherwise, $\pi_V(x) = x'$, where $\eta(c_{j',x}) = c_{j,x'}$.

Define π_E to be a function from $E(L_{j'})$ to paths in G_j such that for every $e \in E(L_{j'})$, $\pi_E(e)$ is the path in G_j obtained from the path in $\eta(v_e)$ connecting the nodes in $\eta(v_e) \cap \eta(c_1)$ and $\eta(v_e) \cap \eta(c_2)$ by replacing the edges in L_j but not in G_j by a path in G_j with the same ends, where c_1, c_2 are the cells in $L_{j'}$ such that corresponds to the ends of e. Note that it is possible to replace those edges in L_j by paths in G_j since conditions 5 and 10 hold. Furthermore, we can choose those paths such that $\pi_E(e_1)$ is internally disjoint from $\pi_E(e_2)$ if e_1, e_2 are distinct edges of $L_{j'}$, and $\pi_V(v) \notin \pi_E(e)$ if $v \in V(L_{j'})$ is not an end of $e \in E(L_{j'})$. According to the definitions of $f_{j'}$ and f_j, it is not hard to see that $(\mathcal{L}_j, \tau_j, \phi_j, \psi_j)$ simulates $(\mathcal{L}_{j'}, \tau_{j'}, \phi_{j'}, \psi_{j'})$. $\square$

Let F be a Γ_i-normal O-arc in $\Sigma - \bigcup_{j=1}^{\kappa} \Delta_j$ with $|F \cap V(\Gamma)| \leq 2$. Then there exists a H_i-normal O-arc F' in $\Sigma - \bigcup_{j=1}^{\kappa} \Delta_j$ with $|F' \cap V(H_i)| \leq 2$. By condition 12(b)(ii), F' bounds a disk Δ' in $\Sigma - \bigcup_{j=1}^{\kappa} \Delta_j$ and every vertex in $\Delta' \cap H_i$ is adjacent to at most two vertices in $\Delta' \cap H_i$. So there exists an open disk Δ_F in $\Sigma - \bigcup_{j=1}^{\kappa} \Delta_j$ with $\partial \Delta_F = F$ such that the intersection of Δ_F and the skeleton of Γ_i is a path. We say that such a O-arc F is *maximal* if $\Delta_F \cap \Gamma_i$ is maximal.

Let $(S_6, \preceq_6)$ be the well-quasi-ordered set obtained from $(S_5, \preceq_5)$ by Higman's lemma. For $i \geq 1$, let $\mathcal{C}_i$ be a maximal set of maximal Γ_i-normal O-arcs in $\Sigma - \bigcup_{j=1}^{\kappa} \Delta_j$. Note that if F_1, F_2 are members of $\mathcal{C}_i$, then $\Delta_{F_1} \cap \Delta_{F_2} \cap \Gamma_i = \emptyset$ since F_1, F_2 are maximal. Define Γ_i' to be the painting in $\Sigma - \bigcup_{j=1}^{\kappa} \Delta_j$ obtained from Γ_i by deleting $\Delta_F \cap \Gamma_i$ and adding a cell c_F with nodes $\partial \Delta_F \cap \Gamma_i$ for every $F \in \mathcal{C}_i$. Define f_i' to be the function from the set of cells of Γ_i' to S_6 such that $f_i'(c_F)$ is the sequence of entries in S_5 such that the ℓ-th entry is the f_i-value of the cell of Γ_i corresponding to the ℓ-th edge of the maximal path in the intersection of the closure of Δ_F and the skeleton of Γ_i, for every ℓ.

Let F be a Γ_i'-normal line in $\Sigma - \bigcup_{j=1}^{\kappa} \Delta_j$ with $|F \cap V(\Gamma_i')| \leq 2$ and with both ends in $\bigcup_{j=1}^{\kappa} \partial \Delta_j$ and with no other point in $\bigcup_{j=1}^{\kappa} \partial \Delta_j$. Then there exists $1 \leq \ell \leq \kappa$ such that $F \cap V(\Gamma_i') \subseteq \partial \Delta_\ell$, and there exists a closed disk $\Delta_F \subseteq \Sigma - \bigcup_{j=1}^{\kappa} \Delta_j$ with $F \subseteq \partial \Delta_F \subseteq F \cup \partial \Delta_\ell$. So there exists a cyclic interval I_F of $\overline{\Omega_{i,\ell}}$ such that every vertex in $\Delta_F \cap \overline{\Omega_{i,\ell}}$ is in I_F. We say that F is *minimal* if $\Delta_F \cap \Gamma_i'$ contains at least two cells of Γ_i' and subject to that, I_F is minimal. For $i \geq 1$, let $\mathcal{C}_i'$ be the set of such minimal lines F. Note that if F is minimal, then $\Delta_F \cap \Gamma_i'$ is a 3-connected painting in Σ. Furthermore, the interiors of I_{F_1} and I_{F_2} are disjoint for distinct $F_1, F_2 \in \mathcal{C}_i'$, since F_1, F_2 are minimal and every node of Γ_i' is in three cells of Γ_i' incident with distinct sets of nodes.

Let S_7 be the set of 3-connected paintings Γ in Σ with a function f_Γ mapping the cells of Γ to S_6, and let $\preceq_7$ be the relation on S_7 such that for every $x, y \in S_7$, $x \preceq_7 y$

if and only if there exists a linear inflation η of x in y such that $f_x(c) \preceq_6 f_y(\eta(c))$ for each cell c of x. By Theorem 7.1.3, $(S_7, \preceq_7)$ is a well-quasi-ordered set. For $i \geq 1$, define Γ_i'' to be the painting in Σ obtained from Γ_i' by deleting the intersection of the interior of Δ_F and Γ_i' and adding a cell c_F incident with the nodes $\Delta_F \cap V(\Gamma_i')$, for each $F \in \mathcal{C}_i'$. Let $(S_8, \preceq_8)$ (and $(S_9, \preceq_9)$, respectively) be the well-quasi-ordered set obtained from $(S_6, \preceq_6)$ (and $(S_7, \preceq_7)$, respectively) and $(\{*\}, =)$ by disjoint union. Let $(S_{10}, \preceq_{10}) = (S_8 \times S_9, \preceq_8 \times \preceq_9)$. For $i \geq 1$, define f_i'' to be the function from the set of cells of Γ_i'' to S_{10} such that $f_i''(c_F) = (*, \Delta_F \cap \Gamma_i')$ for every $F \in \mathcal{C}_i'$, and $f_i''(c) = (f_i'(c), *)$ for other cell c of Γ_i''.

Let F'' be a Γ_i''-normal line in $\Sigma - \bigcup_{j=1}^{\kappa} \Delta_j$ with $|F'' \cap V(\Gamma_i'')| \leq 2$ and with both ends in $\bigcup_{j=1}^{\kappa} \partial\Delta_j$ and with no other point in $\bigcup_{j=1}^{\kappa} \partial\Delta_j$. Then there exists $1 \leq \ell \leq \kappa$ such that $F'' \cap V(\Gamma_i'') \subseteq \partial\Delta_\ell$, and there exists a closed disk $\Delta_{F''} \subseteq \Sigma - \bigcup_{j=1}^{\kappa} \Delta_j$ with $F'' \subseteq \partial\Delta_{F''} \subseteq F'' \cup \partial\Delta_\ell$. So there exists a cyclic interval $I_{F''}$ of $\overline{\Omega_{i,\ell}}$ such that every vertex in $\Delta_{F''} \cap \overline{\Omega_{i,\ell}}$ is in $I_{F''}$. We say that such a F'' is *maximal* if I_F is maximal. For $i \geq 1$, let C_i'' be the set of such maximal line F''. For $i \geq 1$, define Γ_i^* to be the painting in Σ obtained from Γ_i'' by deleting the intersection of the interior of $\Delta_{F''}$ and Γ_i'' and adding a cell $c_{F''}$ incident with the nodes $\Delta_{F''} \cap V(\Gamma_i'')$, for each $F'' \in \mathcal{C}_i''$. Observe that Γ_i^* is a 3-connected painting in Σ.

Note that the skeleton of $\Delta_{F''} \cap \Gamma_i''$ is a path. Let $(S_{11}, \preceq_{11})$ be the set of well-quasi-ordered set obtained from $(S_{10}, \preceq_{10})$ by Higman's lemma. Let $(S_{12}, \preceq_{12})$ (and $(S_{13}, \preceq_{13})$, respectively) be the well-quasi-ordered set obtained from $(S_{10}, \preceq_{10})$ (and $(S_{11}, \preceq_{11})$, respectively) and $(\{*\}, =)$ by disjoint union. Let $(S_{14}, \preceq_{14}) = (S_{12} \times S_{13}, \preceq_{12} \times \preceq_{13})$. For $i \geq 1$, define f_i^* to be a function from the set of cells of Γ_i^* to S_{14} such that $f_i^*(c_{F''}) = (*, x)$ for $F'' \in \mathcal{C}_i''$, where x is the sequence whose ℓ-th entry is the f_i''-value of the cell of Γ_i'' corresponding to the ℓ-th edge in the path that is the skeleton of $\Gamma_i'' \cap \Delta_{F''}$, and $f_i^*(c) = (f_i''(c), *)$ for other cell c of Γ_i''.

Therefore, by Theorem 7.1.3, there exist $j > j' \geq 1$ and a linear inflation η^* of

$\Gamma^*_{j'}$ in Γ^*_j such that $f^*_{j'}(c) \preceq_{14} f^*_j(\eta^*(c))$ for each cell c of $\Gamma^*_{j'}$. This implies that there exist a linear inflation η of $\Gamma_{j'}$ in Γ_j such that $f_{j'}(c) \preceq_5 f_j(\eta(c))$ for each cell c of $\Gamma_{j'}$. So $\mathcal{F}$ is well-behaved by Claim 1. ■

Lemma 7.1.5 *Let $\mathcal{F}$ be a well-behaved collection of frames. Let $\mathcal{F}_k$ be the collection of frames such that for every frame $(H, \mu, \mathcal{Q})$ in $\mathcal{F}_k$, there exists $Z \subseteq E(H)$ with $|Z| \leq k$ such that the following hold.*

1. *For each end of each edge e in Z, there uniquely exists a march in μ such that the end is an entry of this march. And the two ends of e are entries of different marches in μ.*

2. *If v is an end of an edge e in Z, then v is incident with an edge of H not in Z.*

3. *$(H - Z, \mu, \mathcal{Q}') \in \mathcal{F}$ for some $\mathcal{Q}'$.*

4. *$(\mathcal{L} - Z, \tau)$ fits $(H - Z, \mu, \mathcal{Q}')$ for every ordered location $(\mathcal{L}, \tau)$ that fits $(H, \mu, \mathcal{Q})$.*

If there exists an integer o such that the order of $\mathcal{F}$ is at most o, then $\mathcal{F}_k$ is well-behaved.

Proof. It is sufficient to prove the case that $k = 1$ since the general case follows from induction on k. Let $(S, \preceq), (S', \preceq')$ be well-quasi-ordered sets. Let $G_1, G_2, ...$ be an infinite sequence of graphs. And for every $i \geq 1$, let $(\mathcal{L}_i, \tau_i)$ be an ordered extended location in G_i with root (A_i, B_i), and let $\phi_i : V(G_i) \rightarrow S$, $\psi_i : \mathcal{L}_i - \{(A_i, B_i)\} \rightarrow S'$. Assume that $\mathcal{L}_i$ fits a frame $(H_i, \mu_i, \mathcal{Q}_i)$ in $\mathcal{F}$, and $\bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i)$ is well-quasi-ordered by the $(S, \preceq)$-labelled topological minor relation.

For $i \geq 1$, let Z_i be the set Z mentioned in the condition, and let e_i be the edge in Z_i with ends u_i, v_i. Let $(S_1, \preceq')$ be the well-quasi-ordered set obtained from $(S, \preceq)$ and $(\{0, 1, 2\}, =)$ by Cartesian product. Define $\phi'_i : V(G_i) \rightarrow S_1$ such that $\phi'_i(u_i) = (\phi_i(u_i), 1)$, $\phi'_i(v_i) = (\phi_i(v_i), 2)$, and $\phi'_i(v) = (\phi_i(v), 0)$ for $v \in V(G_i) - \{u_i, v_i\}$.

Note that u_i, v_i are incident with edges of H other than e_i, so $u_i, v_i \not\in \tau(A, B)$ for every $(A, B) \in \mathcal{L}_i$. Since the order of $\mathcal{L}_i$ is at most o, we know that $\bigcup_{i \geq 1} \mathcal{P}_{\phi'_i}(\mathcal{L}_i, \tau_i)$ is well-quasi-ordered by the $(S_1, \preceq_1)$-labelled topological minor relation. Since $(\mathcal{L} - Z, \tau)$ fits a frame $(H - Z, \mu, \mathcal{Q}')$ in $\mathcal{F}$ for some $\mathcal{Q}'$, there exist $j > j' \geq 1$ such that $(\mathcal{L}_j - Z_j, \tau_j, \phi'_j, \psi_j)$ simulates $(\mathcal{L}_{j'} - Z_{j'}, \tau_{j'}, \psi_{j'})$. Let π_V be the function π_V mentioned in the definition of the simulation relation. According to the definitions of ϕ'_j and $\phi'_{j'}$, $\pi_V(u_{j'}) = u_j$, and $\pi_V(v_{j'}) = v_j$. Therefore, $(\mathcal{L}_j, \tau_j, \phi_j, \psi_j)$ simulates $(\mathcal{L}_{j'}, \tau_{j'}, \phi_{j'}, \psi_{j'})$. This proves that $\mathcal{F}_k$ is well-behaved. ∎

Theorem 7.1.6 *For every positive integer k, there exist a well-behaved family of frames $\mathcal{F}$ and an integer θ such that for every graph G containing no topological minor isomorphic to the Robertson chain of length k and every tangle $\mathcal{T}$ in G of order at least θ, there exists a location $\mathcal{L} \subseteq \mathcal{T}$ such that $(\mathcal{L}, \tau)$ fits a frame in $\mathcal{F}$ for every edge-extension τ.*

Proof. Lemmas 7.1.2 and 7.1.4 imply that there exist a well-behaved family $\mathcal{F}'$, integer ξ, and $Z \subseteq V(G) \cup E(G)$ with $|Z| \leq \xi$ and edges in Z satisfying the condition mentioned in Lemma 7.1.5 such that $\mathcal{L} - Z \subseteq \mathcal{T} - Z$ and $(\mathcal{L} - Z, \tau - Z)$ fits a frame in $\mathcal{F}'$. Then this theorem follows from Lemmas 5.1.4 and 7.1.5. ∎

7.2 Well-quasi-ordering locations after some minor alterations

Lemma 7.2.1 *Let $\mathcal{F}$ be a well-behaved family of frames, and let k be a nonnegative integer. Let $\mathcal{F}_k$ be a family of frames such that for every location $\mathcal{L}$ that fits a frame in $\mathcal{F}_k$, there exist a location $\mathcal{L}'$ that fits a frame in $\mathcal{F}$ and a location $\mathcal{L}'' \subseteq \mathcal{L}'$ such that for every $(A, B) \in \mathcal{L}''$, there exist separations $(A_1, B_1), (A_2, B_2)$ of order at most k such that $A = A_1 \cup A_2$, $B = B_1 \cap B_2 - \{v : N[v] \subseteq V(A)\}$, and $\mathcal{L} = (\mathcal{L}' - \mathcal{L}'') \cup \{(A_1, B_1), (A_2, B_2) : (A, B) \in \mathcal{L}''\}$. If there exists an integer o such that every frame in $\mathcal{F}$ has order at most o, then $\mathcal{F}_k$ is well-behaved.*

Proof. Let $(S,\preceq),(S',\preceq')$ be well-quasi-ordered sets. Let $G_1,G_2,...$ be an infinite sequence of graphs. And for every $i\geq 1$, let $(\mathcal{L}_i,\tau_i)$ be an ordered extended location in G_i with root (A_i^*,B_i^*), and let $\phi_i:V(G_i)\to S$, $\psi_i:\mathcal{P}(\mathcal{L}_i,\tau_i)\to S'$ be a function. Assume that $\mathcal{L}_i$ fits a frame $(H_i,\mu_i,\mathcal{Q}_i)$ in $\mathcal{F}_k$ for every $i\geq 1$, and $\bigcup_{i\geq 1}\mathcal{P}_{\phi_i}(\mathcal{L}_i,\tau_i)$ is well-quasi-ordered by the $(S,\preceq)$-labelled topological minor relation. For each $i\geq 1$, let $\mathcal{L}_i'$ be a location in G_i that fits a frame in $\mathcal{F}$ and let $\mathcal{L}_i''\subseteq\mathcal{L}_i'$ such that for every $(A,B)\in\mathcal{L}_i''$, there exist separations $(A_1,B_1),(A_2,B_2)$ of order at most k such that $A=A_1\cup A_2$ and $B=B_1\cap B_2-\{v:N[v]\subseteq V(A)\}$, and $\mathcal{L}_i=(\mathcal{L}_i'-\mathcal{L}_i'')\cup\{(A_1,B_1),(A_2,B_2):(A,B)\in\mathcal{L}_i''\}$. If $(A_i^*,B_i^*)\in\mathcal{L}_i'-\mathcal{L}_i''$, then define the root of $\mathcal{L}_i'$ to be (A_i^*,B_i^*); otherwise, there exists a separation $(A,B)\in\mathcal{L}_i''$ such that (A_i^*,B_i^*) is (A_1,B_1) or (A_2,B_2), and we define the root of $\mathcal{L}_i'$ to be this (A,B). We write the root of $\mathcal{L}_i'$ as (A_i',B_i'). For each $(A,B)\in\mathcal{L}_i'-\mathcal{L}_i''$, we assign the same ordering associated with it in $\mathcal{L}_i$ to it; for each $(A,B)\in\mathcal{L}_i''$, we assign an arbitrary ordering to it and assign an ordering to vertices in $V(A_1\cup A_2)\cap V(B_1\cap B_2)$ consistent with the ordering of (A,B) we just gave. Define τ_i' to be the function with domain $\mathcal{L}_i'$ such that the following hold.

- $\tau_i'(A,B)=(\tau_i(A_1,B_1)\cup\tau_i(A_2,B_2))-\{v:N[v]\subseteq V(A)\}$ if $(A,B)\in\mathcal{L}_i''-\{(A_i',B_i')\}$.
- $\tau_i'(A,B)=\tau_i(A,B)$ if $(A,B)\in\mathcal{L}_i'-(\{\mathcal{L}_i''\cup\{(A_i',B_i')\})$.
- $\tau_i'(A_i',B_i')=\tau_i(A_i^*,B_i^*)$ if $(A_i',B_i')=(A_i^*,B_i^*)$.
- If $(A_i^*,B_i^*)=(A_{ij}',B_{ij}')$ for some $j=1,2$, say $j=1$, then $\tau_i'(A_i',B_i')=\tau_i(A_{i1}',B_{i1}')-(\{v:v\text{ is adjacent to a vertex in }V(A_{i2}')-V(B_{i2}')\}\cup\{v:N[v]\subseteq V(A_i')\})$.

Let $S^*=\bigcup_{i\geq 1}\mathcal{P}_{\phi_i}(\mathcal{L}_i,\tau_i)$, and let $\preceq^*$ be the $(S,\preceq)$-labelled topological minor relation. Let $(S_0,\preceq_0)=(\{1,2,...,o+k,*\},=)\times((S^*,\preceq^*)\uplus(\{*\},=))$. For $i\geq 1$, define $\phi_i':V(G_i)\to S_0$ such that the following hold.

- If $(A_i^*, B_i^*) = (A_i', B_i')$, then $\phi_i'(v) = (*, *)$ for every $v \in V(G)$.

- If $(A_i^*, B_i^*) \neq (A_i', B_i')$, then let (A_i'', B_i'') be the separation in $\{(A_{i1}', B_{i1}'), (A_{i2}', B_{i2}')\} - \{(A_i^*, B_i^*)\}$. Then if v is the j-th vertex of $V(A_i'') \cap V(B_i'')$ for some $1 \leq j \leq |V(A_i'') \cap V(B_i'')|$, then $\phi_i'(v) = (j, (G_{i_{\tau_i, A_i''}}, \gamma_{i_{\tau_i, A_i''}}, \sigma_{i_{\tau_i, A_i''}}, \phi_i|_{V(G_{i_{\tau_i, A_i''}})}))$; if $v \notin V(A_i'') \cap V(B_i'')$, then $\phi_i'(v) = (*, *)$.

Let $(S_1, \preceq_1) = (S^*, \preceq^*) \uplus (\{*\}, =)$, $(S_2, \preceq_2) = (S', \preceq') \uplus (\{*\}, =)$, and $(S_3, \preceq_3) = (\{1, 2, ..., o+k, *\}, =)$. Let $(S'', \preceq'') = (S_1^2 \times S_2^2 \times S_3^{o+k}, \preceq_1^2 \times \preceq_2^2 \times \preceq_3^{o+k})$. Note that $(S'', \preceq'')$ is a well-quasi-ordered set. For $i \geq 1$, define $\psi_i' : \mathcal{P}_{\phi_i'}(\mathcal{L}_i', \tau_i') \to S''$ such that the following hold.

- If $(A, B) \in \mathcal{L}_i' - \mathcal{L}_i''$, then $\psi_i'(G_{i_{\tau_i', A}}, \gamma_{i_{\tau_i', A}}, \sigma_{i_{\tau_i', A}}, \phi_i|_{V(G_{i_{\tau_i', A}})}) = ((G_{i_{\tau_i', A}}, \gamma_{i_{\tau_i', A}}, \sigma_{i_{\tau_i', A}}, \phi_i|_{V(G_{i_{\tau_i', A}})}), *, \psi_i(G_{i_{\tau_i', A}}, \gamma_{i_{\tau_i', A}}, \sigma_{i_{\tau_i', A}}, \phi_i|_{V(G_{i_{\tau_i', A}})}), *, 1, *, 2, *, ..., o+k, *)$.

- If $(A, B) \in \mathcal{L}_i''$, then $\psi_i'(G_{i_{\tau_i', A}}, \gamma_{i_{\tau_i', A}}, \sigma_{i_{\tau_i', A}}, \phi_i|_{V(G_{i_{\tau_i', A}})}) = ((G_{i_{\tau_i', A_1}}, \gamma_{i_{\tau_i', A_1}}, \sigma_{i_{\tau_i', A_1}}, \phi_i|_{V(G_{i_{\tau_i', A_1}})}), (G_{i_{\tau_i', A_2}}, \gamma_{i_{\tau_i', A_2}}, \sigma_{i_{\tau_i', A_2}}, \phi_i|_{V(G_{i_{\tau_i', A_2}})}), \psi_i(G_{i_{\tau_i', A_1}}, \gamma_{i_{\tau_i', A_1}}, \sigma_{i_{\tau_i', A_1}}, \phi_i|_{V(G_{i_{\tau_i', A_1}})}), \psi_i(G_{i_{\tau_i', A_2}}, \gamma_{i_{\tau_i', A_2}}, \sigma_{i_{\tau_i', A_2}}, \phi_i|_{V(G_{i_{\tau_i', A_2}})}), a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}, ..., a_{o+k,1}, a_{o+k,2})$, where for $1 \leq j \leq o+k$ and $1 \leq t \leq 2$,

  - if the j-th vertex v of $V(A_1 \cup A_2) \cap V(B_1 \cap B_2)$ is in $V(A_t) \cap V(B_t)$, then $a_{j,t} = \ell$, where ℓ is the number such that v is the ℓ-th vertex of $V(A_t) \cap V(B_t)$, and
  - $a_{j,t} = *$, otherwise.

Note that $\bigcup_{i \geq 1} \mathcal{P}_{\phi_i'}(\mathcal{L}_i', \tau_i')$ is well-quasi-ordered by the $(S_0, \preceq_0)$-labelled topological minor relation, since each of these graphs is either obtained from one graph in S^* by adding some labels on some vertices in the root march, or obtained from two graphs in S^* by adding some labels on some vertices in their root marches and removing some vertices in the root march and identifying some vertices in their root marches, where the vertices we identify allow us obtain the witness of the new topological minor containment from the previous two by merging. Since $\mathcal{F}$ is well-behaved, there exist $j' > j \geq 1$ such that $(\mathcal{L}'_{j'}, \tau'_{j'}, \phi'_{j'}, \psi'_{j'})$ simulates $(\mathcal{L}'_j, \tau'_j, \phi'_j, \psi'_j)$. Let $\iota', \zeta', (\pi'_V, \pi'_E)$ be the functions $\iota, \zeta, (\pi_V, \pi_E)$ mentioned in the definition of the simulation relation. Then by using the information of ι', ζ' and (π'_V, π'_E), we can define $\iota, \zeta, (\pi_V, \pi_E)$ to be functions to realize that $(\mathcal{L}_{j'}, \tau_{j'}, \phi_{j'}, \psi_{j'})$ simulates $(\mathcal{L}_j, \tau_j, \phi_j, \psi_j)$. This proves that $\mathcal{F}_k$ is well-behaved. ∎

Lemma 7.2.2 *Let $\mathcal{F}$ be a well-behaved family of frames, and let k be a nonnegative integer. Let $\mathcal{F}'$ be a family of frames such that for every location $\mathcal{L}$ that fits a frame in $\mathcal{F}'$, there exists a separation $(A, B) \in \mathcal{L}$ of order at most k such that $(\mathcal{L} - \{(A, B)\}) \cup \{(C, D) : C \subseteq A, D \supseteq B\}$ fits a frame in $\mathcal{F}$. Then $\mathcal{F}'$ is well-behaved.*

Proof. Let $(S, \preceq), (S', \preceq')$ be well-quasi-ordered sets. Let $G_1, G_2, ...$ be an infinite sequence of graphs. And for every $i \geq 1$, let $\phi_i : V(G_i) \to S$, $\psi_i : \mathcal{P}(\mathcal{L}_i, \tau_i) \to S'$ be functions, and let $(\mathcal{L}_i, \tau_i)$ be an ordered extended location in G_i with root (A_i^*, B_i^*). Assume that $\mathcal{L}_i$ fits a frame $(H_i, \mu_i, \mathcal{Q}_i)$ in $\mathcal{F}'$, and $\bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i)$ is well-quasi-ordered by the $(S, \preceq)$-labelled topological minor relation. For each $i \geq 1$, let (A_i, B_i) be a separation of order at most k such that $(\mathcal{L}_i - \{(A_i, B_i)\}) \cup \{(C, D) : C \subseteq A_i, D \supseteq B_i\}$ fits a frame in $\mathcal{F}$. We denote $(\mathcal{L}_i - \{(A_i, B_i)\}) \cup \{(C, D) : C \subseteq A_i, D \supseteq B_i\}$ by $\mathcal{L}'_i$, and we define the root of $\mathcal{L}'_i$ to be (A_i^*, B_i^*) if $(A_i, B_i) \neq (A_i^*, B_i^*)$, otherwise, define the root of $\mathcal{L}'_i$ to be $(\emptyset, G_i)$. We assign an ordering to each separation in $\mathcal{L}_i \cap \mathcal{L}'_i$ as the ordering associated with it in $\mathcal{L}_i$, and assign an arbitrary ordering to each separation in $\mathcal{L}'_i - \mathcal{L}_i$. Define G'_i to be the graph $\text{Con}(\mathcal{L}'_i, \tau', \mathcal{L}'_i - \mathcal{L}_i)$. Let $\mathcal{L}''_i$ be the location in G'_i obtained from $\mathcal{L}'_i$ by contracting $V(C) - V(D)$ for each $(C, D) \in \mathcal{L}'_i - \mathcal{L}_i$ with $V(C) - V(D) \neq \emptyset$ and adding a vertex for each $(C, D) \in \mathcal{L}'_i - \mathcal{L}_i$ with $V(C) \subseteq V(D)$. Define τ''_i to be the function τ_i restricted on $\mathcal{L}''_i$.

Let $S^* = \bigcup_{i \geq 1} \mathcal{P}_{\phi_i}(\mathcal{L}_i, \tau_i)$ and $\preceq^*$ the $(S, \preceq)$-labelled topological minor relation, and let $(S^{**}, \preceq^{**}) = (S^*, \preceq^*) \uplus (\{*\}, =)$. Let $(S_1, \preceq_1) = (S \times \{-1, 0, 1, 2, ..., k\} \times S_2 \times$

$S^{**}, \preceq \times = \times \preceq_2 \times \preceq^{**})$, where $(S_2, \preceq_2) = (S', \preceq') \uplus (\{*\}, =)$. For $i \geq 1$, define $\phi'_i : V(G'_i) \to S_1$ to be the function by setting the following.

- $\phi'_i(v) = (\phi_i(v), j, \psi_i(A_i, B_i), (G_{i_{\tau_i, A_i}}, \gamma_{\tau_i, A_i}, \sigma_{\tau_i, A_i}), \phi_i|_{V(G_{i_{\tau_i, A_i}})})$ if $v \in V(A_i) \cap V(B_i)$ and v is the j-th vertex in $V(A_i) \cap V(B_i)$ for some $1 \leq j \leq |V(A_i) \cap V(B_i)|$.
- $\phi'_i(v) = (\phi_i(v), 0, *, *)$ if $v \in V(G_i) \cap V(G'_i) - (V(A_i) \cap V(B_i))$.
- $\phi'_i(v) = (\phi_i(v), -1, *, *)$, otherwise.

Note that $\bigcup_{i \geq 1} \mathcal{P}_{\phi'_i}(\mathcal{L}''_i, \tau''_i)$ is well-quasi-ordered by the $(S_1, \preceq_1)$-labelled topological minor relation. For $i \geq 1$, define ψ'_i to be the function ψ_i restricted on the periphery of $(\mathcal{L}''_i, \tau''_i)$. Since $\mathcal{F}$ is well-behaved, there exist $j' > j \geq 1$ such that $(\mathcal{L}''_{j'}, \tau''_{j'}, \phi'_{j'}, \psi'_{j'})$ simulates $(\mathcal{L}''_j, \tau''_j, \phi'_j, \psi'_j)$. Let $\iota', \zeta', (\pi'_V, \pi'_E)$ be the functions that realize the above simulation relation. Define ζ to be the function with domain $\mathcal{L}_j$ such that $\zeta((A, B)) = \zeta'((A', B'))$ if $(A, B) \in \mathcal{L}_j$ and there exists some (A', B') in the domain of ι' such that $A' = A$, and $\zeta((A_j, B_j)) = (A_{j'}, B_{j'})$. And we define ι to be the function with domain $\mathcal{P}_{\phi_j}(\mathcal{L}_j, \tau_j)$ consistent with ζ. According to the labels on $V(A_j) \cap V(B_j)$ and $V(A_{j'}) \cap V(B_{j'})$ given by ϕ'_j and $\phi'_{j'}$, we know that $|V(A_j) \cap V(B_j)| = |V(A_{j'}) \cap V(B_{j'})|$, and $\pi_V(V(A_j) \cap V(B_j)) = V(A_{j'}) \cap V(B_{j'})$, and $(G_{j'_{\tau_{j'}, A_{j'}}}, \gamma_{\tau_{j'}, A_{j'}}, \sigma_{\tau_{j'}, A_{j'}}, \phi_{j'}|_{V(G_{j'_{\tau_{j'}, A_{j'}}})})$ contains $(G_{j_{\tau_j, A_j}}, \gamma_{\tau_j, A_j}, \sigma_{\tau_j, A_j}, \phi_j|_{V(G_{j_{\tau_j, A_j}})})$ as an $(S_1, \preceq_1)$-topological minor. Then we define (π_V, π_E) to be a pair of functions obtained from (π'_V, π'_E) by contracting $V(A_j) - V(B_j)$ into a new vertex and contracting $V(A_{j'}) - V(B_{j'})$ into a new vertex. Therefore, ι, ζ and (π_V, π_E) are functions that realize that $(\mathcal{L}_{j'}, \tau_{j'}, \phi_{j'}, \psi_{j'})$ simulates $(\mathcal{L}_j, \tau_j, \phi_j, \psi_j)$. This proves that $\mathcal{F}'$ is well-behaved. ■

Lemma 7.2.3 *Let $\mathcal{F}$ be a well-behaved family of center-essential frames, and let k be a nonnegative integer. Let $\mathcal{F}_k$ be a family of frames such that for every rooted location $\mathcal{L}$ that fits a frame in $\mathcal{F}_k$, there exists a rooted location $\mathcal{L}'$ that fits a frame in $\mathcal{F}$ and a separation $(A_{\mathcal{L}'}, B_{\mathcal{L}'}) \in \mathcal{L}'$ of order at most k such that every vertex in $V(A_{\mathcal{L}'}) \cap V(B_{\mathcal{L}'})$*

is adjacent to a vertex in $V(A_{\mathcal{L}'}) - V(B_{\mathcal{L}'})$ *and a vertex in* $V(B_{\mathcal{L}'}) - V(A_{\mathcal{L}'})$*, and* $\mathcal{L} = \{(A_{\mathcal{L}'}, B_{\mathcal{L}'}), (C \cap B_{\mathcal{L}'}, D \cup A_{\mathcal{L}'}) : (C, D) \in \mathcal{L}'\}$*. Then* $\mathcal{F}_k$ *is a well-behaved family of center-essential frames.*

Proof. We first prove that every frame in $\mathcal{F}_k$ is center-essential. Let $\mathcal{L}$ be a rooted location fits a frame in $\mathcal{F}_k$, and let $\mathcal{L}'$, $(A_{\mathcal{L}'}, B_{\mathcal{L}'})$ be the rooted location and separation mentioned in the assumption. By assumption, every vertex in $V(A_{\mathcal{L}'}) \cap V(B_{\mathcal{L}'})$ is adjacent to a vertex in $V(B_{\mathcal{L}'}) - V(A_{\mathcal{L}'})$ and a vertex in $V(A_{\mathcal{L}'}) - V(B_{\mathcal{L}'})$. So every vertex v in $V(C \cap B_{\mathcal{L}'}) \cap V(D \cup A_{\mathcal{L}'})$ is either in $V(C) \cap V(D)$ or in $V(A_{\mathcal{L}'}) \cap V(B_{\mathcal{L}'})$. For the former, v is adjacent to a vertex in $V(D) - V(C) \subseteq V(D \cup A_{\mathcal{L}'}) - V(C \cap B_{\mathcal{L}'})$, since $\mathcal{L}'$ fits a center-essential frame. For the latter, v is adjacent to a vertex in $V(A_{\mathcal{L}'}) - V(B_{\mathcal{L}'}) \subseteq V(D \cup A_{\mathcal{L}'}) - V(C \cap B_{\mathcal{L}'})$. Similarly, if (C, D) is the root of $\mathcal{L}$, then every vertex in $V(C) \cap V(D)$ is adjacent to a vertex in $V(C) - V(D)$. This proves that every frame in $\mathcal{F}_k$ is center-essential.

We now prove that $\mathcal{F}_k$ is well-behaved. Let $\mathcal{L}$ be a rooted location that fits a frame in $\mathcal{F}_k$. So there exists a rooted location $\mathcal{L}'$ that fits a frame in $\mathcal{F}$ and a separation $(A_{\mathcal{L}'}, B_{\mathcal{L}'})$ of order at most k such that $\mathcal{L} = \{(A_{\mathcal{L}'}, B_{\mathcal{L}'}), (C \cap B_{\mathcal{L}'}, D \cup A_{\mathcal{L}'}) : (C, D) \in \mathcal{L}'$, either $C \cap B_{\mathcal{L}'} \not\subseteq A_{\mathcal{L}'}$, or $D \cup A_{\mathcal{L}'} \not\supseteq B_{\mathcal{L}'}\}$. Let $\mathcal{L}_1$ be the location $\{(C \cap B_{\mathcal{L}'}, D \cup A_{\mathcal{L}'}), (C \cap A_{\mathcal{L}'}, D \cup B_{\mathcal{L}'}) : (C, D) \in \mathcal{L}'\}$. By Lemma 7.2.1, $\mathcal{L}_1$ fits a frame in a well-behaved family. Then $\mathcal{F}_k$ is well-behaved by Lemma 7.2.2. ■

7.3 Well-quasi-ordering graphs with several tangles

Let $(T, \mathcal{X})$ be a tree-decomposition of a graph G. Let R be a nonempty subtree of T, and let r be a node in R with $N_T(r) \subseteq V(R)$. For every $t \in V(T) - \{r\}$, let (A_t, B_t) be a separation of G such that $X_t = V(A_t) \cap V(B_t)$ and $V(B_t) = \bigcup_{t' \in V(W_t)} X_{t'}$, where W_t is the union of $\{t\}$ and the component of $T - \{t\}$ containing r. Let $\mathcal{L}$ be a location in G such that for every $t \in N_T(R)$, there uniquely exists $(C, D) \in \mathcal{L}$ such that $A_t \subseteq C$ and $D \subseteq B_t$. For every $t \in N_T(R)$, let n_t be the node in R adjacent to

t in T. For every $(C, D) \in \mathcal{L}$, let $T_{(C,D)}$ be a copy of R, and let $r_{(C,D)}$ and $n_{t,(C,D)}$ be the copy of r and n_t in $T_{(C,D)}$, respectively, for every $t \in N_T(R)$. Define T' to be the tree such that $V(T') = \{t^*, t_{(C,D)} : (C, D) \in \mathcal{L}\} \cup (V(T) - R) \cup \bigcup_{(C,D)\in\mathcal{L}} T_{(C,D)}$ and $E(T') = \{t^*t_{(C,D)}, t_{(C,D)}r_{(C,D)} : (C, D) \in \mathcal{L}\} \cup \{tn_{t,(C,D)} : t \in N_T(R), A_t \subseteq C, D \subseteq B_t\}$. Define $\mathcal{X}' = (X'_t : t \in V(T')\}$ such that

- $X'_{t^*} = \bigcap_{D\in\mathcal{L}} D$, and
- $X'_{t_{(C,D)}} = V(C) \cap V(D)$ for every $(C, D) \in \mathcal{L}$, and
- $X'_{r_{(C,D)}} = (X_r \cap V(C)) \cup (V(C) \cap V(D))$ for every $(C, D) \in \mathcal{L}$, and
- $X'_t = V(A_t \cap C) \cap V(B_t \cup D)$ for every $(C, D) \in \mathcal{L}$ and $t \in V(T_{(C,D)}) - \{r_{(C,D)}\}$, and
- $X'_t = X_t$ for other t.

Lemma 7.3.1 *Let $(T, \mathcal{X})$ be a tree-decomposition of a graph G. If $(T', \mathcal{X}')$ is defined as above, then (T', X') is a tree-decomposition of G.*

Proof. First, we prove that the ends of each edge f of G is contained in $X'_{t'}$ for some $t' \in V(T')$. Since $(T, \mathcal{X})$ is a tree-decomposition of G, there exists $t \in V(T)$ such that $f \subseteq X_t$. If $t \not\in V(R)$, then $f \subseteq X'_t$. If $f \subseteq V(D)$ for every $(C, D) \in \mathcal{L}$, then $f \subseteq X'_{t^*}$. So we may assume that $t \in V(R)$ and $f \in V(C)$ for some $(C, D) \in \mathcal{L}$. If $t = r$, then $f \subseteq X_r \cap V(C) \subseteq X'_{r_{(C,D)}}$; if $t \neq r$, then $f \subseteq V(A_t) \cap V(B_t) \cap V(C) \subseteq X'_{t''}$, where t'' is the copy of t in $T_{(C,D)}$. This proves that some bag of (T', X') contains the ends of f. Similarly, $\bigcup_{t\in V(T')} X'_t = V(G)$.

Second, we prove that for every $v \in V(G)$, the nodes of T' whose bags contain v induce a subtree. Let $T_v = \{t \in V(T) : v \in X_t\}$. Observe that for every $(C, D) \in \mathcal{L}$, either $v \in V(C)$ and every $t' \in T_{(C,D)}$ that is a copy of a node in T_v satisfies that $v \in X'_{t'}$, or $v \not\in V(C)$ and every $t' \in T_{(C,D)}$ satisfies that $v \not\in X'_{t'}$. Furthermore, if

$t' \in T_{(C,D)}$ that is a copy of a node s in $T - T_v$ such that $v \in X'_{t'}$, then $v \in V(C) \cap V(D)$ and $v \in X'_{t_{(C,D)}}$. Since $A_s \subseteq A_{s'}$ if s' is an internal node of the path in T from r to s, we know that $T'[\{t \in V(T_{(C,D)}) : v \in X'_t\}]$ is connected for each $(C, D) \in \mathcal{L}$. In addition, if there exist $t' \in V(T_{(C,D)})$ and $t'' \in V(T_{(C',D')})$ for distinct $(C, D), (C', D') \in \mathcal{L}$ such that $v \in X'_{t'} \cap X'_{t''}$, then $v \in V(C) \subseteq V(D'')$ for every D'' such that $(C'', D'') \in \mathcal{L} - \{(C, D)\}$ and $v \in V(C') \subseteq V(D'')$ for every $(C'', D'') \in \mathcal{L} - \{(C', D')\}$. Therefore, $v \in X'_{t^*} \cap X'_{t_{(C,D)}} \cap X'_{t_{(C',D')}}$ and $T'[\{t \in \bigcup_{(C,D)\in\mathcal{L}} V(T_{(C,D)}) \cup \{t^*\} : v \in X'_t\}]$ is connected.

Then we prove that $T'[\{t \in V(T') : v \in X'_t\}] \cap M$ is connected for every component M of $T' - \{t^*\}$. For each $(C, D) \in \mathcal{L}$, let $K_{(C,D)}$ be the union of the components of $T - V(R)$ adjacent to a vertex in $T_{(C,D)}$ in T'. Let $s_1 \in K_{(C,D)}$ with $v \in X'_{s_1} = X_{s_1}$. So $v \in V(A_{s_1}) \subseteq V(C)$. Therefore, every node t that is a copy of T_v satisfies that $v \in X'_t$. Let s_2 be in $V(T_{(C,D)}) \cup V(K_{(C,D)})$, and let $s' = s_2$ if s_2 is in $V(K_{(C,D)})$, and let s' be the original of s_2 otherwise. Hence, if $v \in X'_{s_2}$ and $v \in X'_{s'}$, then $v \in X'_t$ for every node t in the path in T from s_1 to s_2. So we may assume that $v \in X'_{s_2}$ but $v \notin X'_{s'}$. Since $T'[\{t \in \bigcup_{(C,D)\in\mathcal{L}} V(T_{(C,D)}) \cup \{t^*\} : v \in X'_t\}]$ is connected, we may assume that no copy of a node in T_v is in $T_{(C,D)}$. However, it implies that s_2 is in the same component of $K_{(C,D)}$ as s_1. This proves that $T'[\{t \in V(T') : v \in X'_t\}] \cap M$ is connected for every component M of $T' - \{t^*\}$.

Finally, let s_1, s_2 be two nodes in different components of $T' - \{t^*\}$ such that $v \in X'_{s_1} \cap X'_{s_2}$, say s_i is in the component of $T' - \{t^*\}$ containing $t_{(C_i,D_i)}$ for $(C_i, D_i) \in \mathcal{L}$ and $i = 1, 2$. Note that $v \in V(C_1) \cap V(C_2)$. Hence, $v \in \bigcap_{(C,D)\in\mathcal{L}} V(D) = X'_{t^*}$ as $\mathcal{L}$ is a location in G. This completes the proof. ∎

Let $\mathcal{S}$ be a set of tangles in a graph G. We say that a location $\mathcal{L}$ *θ-isolates $\mathcal{T}$ from $\mathcal{S}$ with respect to a tie-breaker λ* if $\theta \geq 1$, $\mathcal{L} \subseteq \mathcal{T}$ and has order less than θ, and for every $(C, D) \in \mathcal{L}$ and for every tangle $\mathcal{T}' \in \mathcal{S}$ in G of order at least θ with $(D, C) \in \mathcal{T}'$, we have that $A \subseteq C$ and $D \subseteq B$, where (A, B) is the $(\mathcal{T}, \mathcal{T}')$-distinction with respect to λ.

Theorem 7.3.2 *Let h, θ be positive integers, where $h \geq \theta$. Let $(T, \mathcal{X})$ be a h-strongly lean branching rooted tree-decomposition of a graph G. Assume that there exists an edge f of G whose ends are in the bag of the root of T. Let ν be a function that maps $V(G) \cup E(G)$ to positive real numbers, and let λ be the tie-breaker defined by f and ν. Let $\mathcal{S} = \{\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n\}$ be the set of the tangles in G defined by the θ-cells $L_1, L_2, ..., L_n$ of $(T, \mathcal{X})$ with size at least $3\theta + 1$. Let $\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_n$ be locations in G of order at most $\theta/2$ such that $\mathcal{L}_j$ θ-isolates $\mathcal{T}_j$ from $\mathcal{S}$ with respect to the tie-breaker λ, for every $1 \leq j \leq n$. Then there exists $I \subseteq \{1, 2, ..., n\}$ such that for every $1 \leq j \leq n$, there uniquely exists $i \in I$ such that $\mathcal{L}_i \subseteq \mathcal{T}_j$.*

Furthermore, assume that $\phi : \{1, 2, ..., n\} \to I$ is the function such that $\phi(a) = b$ if and only if $\mathcal{L}_b \subseteq \mathcal{T}_a$ for every $1 \leq a \leq n$ and $b \in I$. Then there do not exist $i, j, k \in \{1, 2, ..., n\}$ such that $\phi(i) = \phi(j) \neq \phi(k)$ but there exists a path in T from L_i to L_j passing through L_k.

Proof. We prove this lemma by using a similar idea as the proof of Theorem 4.2 in [46]. For every edge e of T with tail u and head v, let T_u and T_v be the component of $T - e$ containing u, v, respectively, and we define (A_e, B_e) to be the separation such that $V(A_e) = \bigcup_{t \in V(T_u)} X_t$ and $V(B_e) = \bigcup_{t \in V(T_v)} X_t$ and subject to that, B is maximal. By Theorem 3.2.3, for every two L_i, L_j with $i \neq j$, there exists a node $t_{i,j}$ of T such that $X_{t_{i,j}} = V(A_{i,j}) \cap V(B_{i,j})$, where $(A_{i,j}, B_{i,j})$ is the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction, and hence there exists an edge $e_{i,j}$ with tail $t_{i,j}$ such that $V(A_{e_{i,j}}) \cap V(B_{e_{i,j}}) = V(A_{i,j}) \cap V(B_{i,j})$. Note that $(T, \mathcal{X})$ is branching, so $B_{i,j}$ is connected. Therefore, for every L_i, L_j with $i \neq j$, there exists an edge $e^*_{i,j}$ of T in the path in T from L_i to L_j such that $(A_{e^*_{i,j}}, B_{e^*_{i,j}})$ is the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction or the $(\mathcal{T}_j, \mathcal{T}_i)$-distinction. Let T' be the tree obtained from T by contracting each component of $T - \{e^*_{i,j} : 1 \leq i < j \leq n\}$ into a node. Note that each component of $T - \{e^*_{i,j} : 1 \leq i < j \leq n\}$ contains at most one L_k for $1 \leq k \leq n$.

Let t_1, t_2, t_3 be three nodes of T' such that each t_1, t_2, t_3 contains $L_{i_1}, L_{i_2}, L_{i,3}$ for some $i_1, i_2, i_3 \in \{1, 2, ..., n\}$, and there exists a path P in T' from t_1 to t_3 passing through t_2. Assume that $\mathcal{L}_{i_1} \subseteq \mathcal{T}_{i_1} \cap \mathcal{T}_{i_3}$. We claim that $\mathcal{L}_{i_1} \subseteq \mathcal{T}_{i_2}$. Suppose that $\mathcal{L}_{i_1} \not\subseteq \mathcal{T}_{i_2}$. So there exists $(C, D) \in \mathcal{L}_{i_1} - \mathcal{T}_{i_2}$. Since $\mathcal{L}_{i_1}$ θ-isolates $\mathcal{T}_{i_1}$, we know that $A_{i_1,i_2} \subseteq C$ and $D \subseteq B_{i_1,i_2}$. Note that P passes through $t_1, e^*_{i_1,i_2}, t_2, t_3$ in order. Since $(A_{i_1,i_2}, B_{i_1,i_2}) \in \mathcal{T}_{i_1}$, $(B_{i_1,i_2}, A_{i_1,i_2}) \in \mathcal{T}_{i_2} \cap \mathcal{T}_{i_3}$. But $(C, D) \in \mathcal{L}_{i_1} \subseteq \mathcal{T}_{i_3}$, so $(C \cup B_{i_1,i_2}, D \cap A_{i_1,i_2}) \in \mathcal{T}_{i_3}$, as the order of $(C \cup B_{i_1,i_2}, D \cap A_{i_1,i_2})$ is at most θ, which is less than the order $\theta + 1$ of $\mathcal{T}_{i_3}$. However, $G = C \cup D \subseteq C \cup B_{i_1,i_2}$, a contradiction. This proves the claim.

For every $t \in V(T')$, if t contains L_i for some $1 \leq i \leq n$, then we define T_t to be the minimal subtree of T' containing every $t' \in V(T')$ in which t' contains L_j for some $1 \leq j \leq n$ and $\mathcal{L}_i \subseteq \mathcal{T}_j$; if t does not contain L_i for every $1 \leq i \leq n$, then we define $T_t = \{t\}$. Clearly, $t \in V(T_t)$ for every $t \in V(T')$. Note that every edge of T' is an edge of T. Define μ to be a function from $E(T')$ to positive real numbers such that $\mu(e) = \lambda(A_e, B_e)$ for every $e \in E(T')$. Observe that μ is a linear order of $E(T)$. We claim that for every node t of T' and every edge e of T' between T_t and $T' - V(T_t)$, then $\mu(e) \leq \mu(e')$ for every edge e' in the path in T' from t to an end of e. Suppose to the contrary that $\mu(e) > \mu(e')$. Note that t contains L_i for some $1 \leq i \leq n$, otherwise $e' = e$. By the minimality of T_t, the end of e in T_t, denoted by s, contains L_k for some $1 \leq k \leq n$. Let j be a number such that (A_e, B_e) or (B_e, A_e) is the $(\mathcal{T}_k, \mathcal{T}_j)$-distinction, and we denote the node containing L_j by t_j. We may assume that e' is the edge from t to an end of e in T' such that $\mu(e')$ is minimum. Hence, $\mu(e') < \mu(e'')$ for every edge e'' in the path in T' from t_j to t. In other words, $(A_{e'}, B_{e'})$ or $(B_{e'}, A_{e'})$ is the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction, say $(A_{e'}, B_{e'})$. Since $\mathcal{L}_i$ θ-isolates $\mathcal{T}_i$, there exists $(C, D) \in \mathcal{L}_i$ such that $A_{e'} \subseteq C$ and $D \subseteq B_{e'}$. But there exists a path in T' passing t, e', s in order, so $(B_{e'}, A_{e'}) \in \mathcal{T}_k$. However, $\mathcal{L}_i \subseteq \mathcal{T}_k$, so $(C, D) \in \mathcal{T}_k$. Therefore, $(B_{e'} \cup C, A_{e'} \cap D) \in \mathcal{T}_k$. But $G = D \cup C \subseteq B_{e'} \cup C$, a contradiction. This proves the claim.

Therefore, by Theorem 4.1 in [46], there exists $S \subseteq V(T')$ such that $\{V(T_s) : s \in S\}$ is a partition of $V(T')$. Define I to be the set of indices i such that L_i is contained in some node s in S. This completes the proof. ■

Let G be a graph and $\mathcal{T}$ an tangle in G. Recall that we say a location $\mathcal{L}$ in G is *linked with respect to* $\mathcal{T}$ if $\mathcal{L} \subseteq \mathcal{T}$ but there do not exist $(A, B) \in \mathcal{L}$ and $(C, D) \in \mathcal{T}$ of order less than $|V(A) \cap V(B)|$ such that $A \subseteq C$ and $D \subseteq B$.

Let $(T, \mathcal{X})$ be a tree-decomposition of a graph G. We say that a bag X_t *realizes* a separation (A, B) of G if $X_t = V(A) \cap V(B)$.

Theorem 7.3.3 *For every positive integer k, there exists a well-behaved family of frames $\mathcal{F}$ such that if G does not contain a topological minor isomorphic to the Robertson chain of length k, and every vertex of G is incident with a loop, then the extended location $(\{(\emptyset, G)\}, \tau)$ fits a frame in $\mathcal{F}$, where τ is the edge-extension of the location $\{(\emptyset, G)\}$.*

Proof. Let θ be a positive integer that is larger than four times the number θ in Theorem 6.3.11, and let $\theta' = \theta^2$ and $h = 3\theta' + 1$. Let $(T, \mathcal{X})$ be the branching h-strongly lean rooted tree-decomposition mentioned in the proof of Theorem 4.2.5. That is, $(T, \mathcal{X})$ is branching and has the minimum h-signature, subject to that, with the minimum signature and the maximum goodness. Let ν be a function mapping $V(G) \cup E(G)$ to positive real numbers such that $\nu(x)$ are rationally independent for $x \in V(G) \cup E(G)$. We pick an edge e with ends in the root bag and define λ to be the tie-breaker of separations of G defined by e, ν. We say that a θ'-cell of $(T, \mathcal{X})$ is *fat* if its volume is at least $3\theta' + 1$. For every fat θ'-cell L in $(T_i, \mathcal{X}_i)$, define $\mathcal{T}_L$ to be the tangle of order $\theta' + 1$ induced by L, and define $\mathcal{L}_L$ to be the location mentioned in Theorem 6.3.11 contained in $\mathcal{T}_L$. Note that the order of $\mathcal{L}_L$ is less than θ. By Theorem 7.1.6, $\mathcal{L}_L$ fits a frame in a well-behaved family.

Let $\mathcal{S}$ be the set of the tangles induced by some fat θ'-cells of $(T, \mathcal{X})$. The first objective is to revise $\mathcal{L}_L$ such that it θ'-isolates $\mathcal{T}_L$ from $\mathcal{S}$ for each fat θ'-cell L of $(T, \mathcal{X})$. For every fat θ'-cell L of $(T, \mathcal{X})$, define r_L to be the root of L, and for every node t in $T - \{r_L\}$, define $(A_{t,L}, B_{t,L})$ to be the separation of G such that the following hold.

- $V(A_{t,L}) \cap V(B_{t,L}) = X_t$.
- $V(B_{t,L}) = \bigcup_{t' \in T_{t,L}} X_{t'}$, where $T_{t,L}$ is the union of $\{t\}$ and the component of $T - \{t\}$ containing r_L, and
- If the root of T is in $T_{t,L}$, then $E(G[V(A) \cap V(B)]) \subseteq E(B)$; otherwise, $E(G[V(A) \cap V(B)]) \subseteq E(A)$.

Claim 1: For different fat θ'-cells L, L' of $(T, \mathcal{X})$, there exists a node t of T in the path in T from L to L' such that either $(A_{t,L}, B_{t,L})$ is the $(\mathcal{T}_L, \mathcal{T}_{L'})$-distinction, or $(A_{t,L'}, B_{t,L'})$ is the $(\mathcal{T}_{L'}, \mathcal{T}_L)$-distinction.

Proof of Claim 1: Let (A, B) be the $(\mathcal{T}_L, \mathcal{T}_{L'})$-distinction. So (B, A) is the $(\mathcal{T}_{L'}, \mathcal{T}_L)$-distinction. By Theorem 3.2.3, there exist $t^* \in V(T)$ in the path in T from L to L' such that $X_{t^*} = V(A) \cap V(B)$. Let t, t' be such a t^* that is the closest to L, L', respectively.

First, we assume that the root of one cell is an ancestor of the other. By symmetry, we may assume that the root of L' is an ancestor of the root of L. Since $(T, \mathcal{X})$ is branching, $B_{t,L} - V(A_{t,L})$ is connected. And by the definition of $(A_{t,L}, B_{t,L})$, we know that $E(G[X_t]) \subseteq E(A_{t,L})$. So $(A_{t,L}, B_{t,L})$ is the $(\mathcal{T}_L, \mathcal{T}_{L'})$-distinction and the $(\mathcal{T}_L, \mathcal{T}_{L'})$-moat.

So we may assume that none of the root of L, L' is an ancestor of the other. The same argument shows that $(A_{t,L}, B_{t,L})$ is the $(\mathcal{T}_L, \mathcal{T}_{L'})$-moat and $(A_{t,L'}, B_{t,L'})$ is the $(\mathcal{T}_{L'}, \mathcal{T}_L)$-moat. By Lemma 3.2.2, either $(A, B) = (A_{t,L}, B_{t,L})$ or $(B, A) = (A_{t,L'}, B_{t,L'})$. This completes the proof. $\square$

Let Z_L be the subset Z of $V(G_L) \cup E(G_L)$ with respect to $\mathcal{T}_L$ mentioned in Theorem 6.3.11. For every fat θ'-cell L of (T, X), we define $\mathcal{M}_L$ to be the collection of separations (A, B) of G satisfying the following properties.

- The order of (A, B) is less than θ.
- $(A, B) \in \mathcal{T}_L - \mathcal{T}'$ for some $\mathcal{T}' \in \mathcal{S}$.
- $(A, B) = (A_{t,L}, B_{t,L})$ for some $t \in V(T) - V(L)$.
- $Z_L \cup \{e\} \not\subseteq V(B) \cup E(B)$.
- If (C, D) satisfies the above conditions and let $t_{(C,D)}$ be a node in $t \in V(T) - V(L)$ such that $(C, D) = (A_{t_{(C,D)},L}, B_{t_{(C,D)},L})$, then either
  - $(C, D) \notin \mathcal{M}_L$ and there exists an internal node t' of the path in T from t to L such that $(A_{t',L}, B_{t',L}) \in \mathcal{M}_L$, or
  - $(C, D) \in \mathcal{M}_L$ and there is no internal node t' of the path in T from t to L such that $(A_{t',L}, B_{t',L}) \in \mathcal{M}_L$.

Clearly, $\mathcal{M}_L$ is a location for every fat θ'-cell of (T, X) by the third condition. Note that for each $(A, B) \in \mathcal{M}_L$, some element of $Z_L \cup \{e\}$ is in $(V(A) - V(B)) \cup E(A)$. Since $\mathcal{M}_L$ is location, $|\mathcal{M}_L| \leq |Z_L| + 1 \leq \theta$. Notice that if $(A, B) \in \mathcal{M}_L - \mathcal{T}'$ for some $\mathcal{T}' \in \mathcal{S}$, then $V(A) \cap V(B) = V(A^*) \cap V(B^*)$, where (A^*, B^*) is the $(\mathcal{T}, \mathcal{T}')$-moat by Theorem 3.2.3.

Define $\mathcal{L}'_L = \mathcal{M}_L \cup \{(C \cap (\bigcap_{(A,B) \in \mathcal{M}_L} B), D \cup (\bigcup_{(A,B) \in \mathcal{M}_L} A)) : (C, D) \in \mathcal{L}_L\}$.

Claim 2: $\mathcal{L}'_L$ is a location for every fat θ'-cell L in $(T, \mathcal{X})$.

Proof of Claim 2: Let $(A, B), (A', B')$ be different separations in $\mathcal{L}'_L$. We shall prove that $A \subseteq B'$. If $(A, B), (A', B') \in \mathcal{M}_L$, then we are done. If $(A, B) \in \mathcal{M}_L$, then $(A', B') = (C \cap (\bigcap_{(U,U') \in \mathcal{M}_L} U'), D \cup (\bigcup_{(U,U') \in \mathcal{M}_L} U))$ for some $(C, D) \in \mathcal{L}_L$, so $A \subseteq B'$. So we may assume that $(A, B) \notin \mathcal{M}_L$, and hence there exists $(C, D) \in \mathcal{L}_L$

such that $(A,B) = (C \cap (\bigcap_{(U,U') \in \mathcal{M}_L} U'), D \cup (\bigcup_{(U,U') \in \mathcal{M}_L} U))$. Hence, it is clear that if $(A',B') \in \mathcal{M}_L$, then $A \subseteq B'$. Therefore, we may assume that there exists $(C',D') \in \mathcal{L}_L$ such that $(A',B') = (C' \cap (\bigcap_{(U,U') \in \mathcal{M}_L} U'), D' \cup (\bigcup_{(U,U') \in \mathcal{M}_L} U))$. However, $\mathcal{L}_L$ is a location, so $C \subseteq D'$ and hence $A \subseteq B'$. This proves the claim. □

Note that the order of $\mathcal{L}'_L$ is less than $\theta + |\mathcal{M}_L|(\theta - 1) \leq \theta^2 = \theta'$. Furthermore, $\mathcal{L}'_L \subseteq \mathcal{T}_L$, since $C \cap (\bigcap_{(A,B) \in \mathcal{M}_L} B) \subseteq C$ for each $(C,D) \in \mathcal{T}_L$. Note that $\mathcal{L}'_L$ fits a frame in a well-behaved family by Lemma 7.2.3.

Claim 3: $\mathcal{L}'_L$ θ'-isolates $\mathcal{T}_L$ from $\mathcal{S}$ for every fat θ'-cell L of $(T,\mathcal{X})$.

Proof of Claim 3: Suppose that $\mathcal{L}'_L$ does not θ^2-isolate $\mathcal{T}_L$ from $\mathcal{S}_i$. Then there exist $(C',D') \in \mathcal{L}'_L$ and $\mathcal{T}' \in \mathcal{S}$ such that $(D',C') \in \mathcal{T}'$, but either $A \not\subseteq C'$ or $D' \not\subseteq B$, where (A,B) is the $(\mathcal{T}_L,\mathcal{T}')$-distinction. Observe that if $\mathcal{M}_L \cup \mathcal{L}_L \subseteq \mathcal{T}'$, then $\mathcal{L}'_L \subseteq \mathcal{T}'$. So there exists a separation in $\mathcal{M}_L \cup \mathcal{L}_L$ distinguishing $\mathcal{T}_L$ from $\mathcal{T}'$. Therefore, the order of (A,B) is less than the order of $\mathcal{M}_L \cup \mathcal{L}_L$, which is θ. Let L' be a fat θ'-cell such that $\mathcal{T}' = \mathcal{T}_{L'}$. If $Z_L \cup \{e\} \not\subseteq V(B) \cup E(B)$, then by Claim 1, there exists a node t in the path in T from L and L' such that $(A,B) = (A_{t,L}, B_{t,L})$ or $(A,B) = (B_{t,L'}, A_{t,L'})$. For the former, by definition of $\mathcal{M}_L$, there exists $(A^*,B^*) \in \mathcal{M}_L$ such that $A \subseteq A^*$ and $B^* \subseteq B$; for the latter, $B_{t,L'} \subseteq A_{t,L}$ and $B_{t,L} \subseteq A_{t,L'}$, so $Z_L \cup \{e\} \not\subseteq V(B_{t,L}) \cup E(B_{t,L})$, and hence there exists $(A^*,B^*) \in \mathcal{M}_L$ such that $A \subseteq A^*$ and $B^* \subseteq B$. But $\mathcal{L}'_L$ is a location and $(A^*,B^*), (C',D') \in \mathcal{L}'_L$, so $A^* \subseteq D'$ and $C' \subseteq B^*$. Hence $A \subseteq A^* \subseteq D'$ and $C' \subseteq B^* \subseteq B$. However, $(B,A) \in \mathcal{T}', (D',C') \in \mathcal{T}'$ and $B \cup D' \supseteq B \cup A = G$, a contradiction. Therefore, $Z_L \cup \{e\} \subseteq V(B) \cup E(B)$. That is, $C' \cap D' \cap Z_L = C' \cap D' \cap B \cap Z_L$.

Note that $\mathcal{L}_L - Z_L$ is linked, so the order of $((C' \cup A) - Z_L, D' \cap B - Z_L)$ is at least the order of $(C' - Z_L, D' - Z_L)$. Therefore, $|(C' \cup A) \cap D' \cap B| = |(C' \cup A) \cap D' \cap B \cap Z_L| + |(C' \cup A) \cap D' \cap B - Z_L| \geq |C' \cap D' \cap B \cap Z_L| + |C' \cap D' - Z_L| = |C' \cap D' \cap Z_L| + |C' \cap D' - Z_L| = |C' \cap D'|$. It implies that the order of $(C' \cap A, D' \cup B)$ is at most the order of (A,B). As (A,B) is the $(\mathcal{T}_L,\mathcal{T}')$-distinction and $(C' \cup A, D' \cup B) \in \mathcal{T}_L - \mathcal{T}'$, we know that the order of $(C' \cap A, D' \cup B)$ is equal to the order of (A, B). Since $e \in B$, $\lambda(C' \cap A, D' \cup B) \leq \lambda(A, B)$. So $C' \cap A = A$ and $D' \cup B = B$. In other words, $A \subseteq C'$ and $D' \subseteq B$, a contradiction. This proves that $\mathcal{L}'_L$ θ'-isolates $\mathcal{T}_L$ from $\mathcal{S}$. $\Box$

By Theorem 7.3.2, there exist a set $\mathcal{O} \subseteq \{\mathcal{L}'_L : L$ is a fat θ'-cell of $(T, \mathcal{X})\}$ and a mapping ι from the set of fat θ'-cells of $(T, \mathcal{X})$ to $\mathcal{O}$ such that $\iota(L) \subseteq \mathcal{T}_L$ and $\mathcal{L} \not\subseteq \mathcal{T}_L$, for every fat θ'-cell L and $\mathcal{L} \in \mathcal{O} - \{\iota(L)\}$, and there is no path in T passing through nodes of three different fat θ'-cells L_1, L_2, L_3 in order such that $\iota(L_1) = \iota(L_3) \neq \iota(L_2)$.

We say that two fat θ'-cell L_1, L_2 of $(T, \mathcal{X})$ are *near* if there is no path in T from L_1 to L_2 containing a node of a fat θ'-cell of $(T, \mathcal{X})$ other than L_1, L_2.

Claim 4: Let L_1, L_2 are two near fat θ'-cells of $(T, \mathcal{X})$ such that $\iota(L_1) \neq \iota(L_2)$. If t is a node in the path in T from L_1 to L_2 such that X_t realizes the $(\mathcal{T}_{L'_1}, \mathcal{T}_{L'_2})$-moat (or distinction, respectively) for some fat θ'-cells L'_1, L'_2 of $(T, \mathcal{X})$ with $\iota(L'_1) = \iota(L_1) \neq \iota(L_2) = \iota(L'_2)$, then X_t realizes the $(\mathcal{T}_{L_1}, \mathcal{T}_{L_2})$-moat (or distinction, respectively).

Proof of Claim 4: Let (A', B') be the $(\mathcal{T}_{L'_1}, \mathcal{T}_{L'_2})$-moat (or distinction, respectively). By Theorem 3.2.3, there exists t^* in the path in T from L_1 to L_2 such that X_{t^*} realizes the $(\mathcal{T}_{L_1}, \mathcal{T}_{L_2})$-moat (or distinction, respectively), denoted by (A, B). We are done unless $X_t \neq X_{t^*}$, so we suppose that $X_t \neq X_{t^*}$. Since (A', B') distinguishes $\mathcal{T}_{L_1}$ from $\mathcal{T}_{L_2}$, the order of (A', B') is at least the order of (A, B). Similarly, (A, B) distinguishes $\mathcal{T}_{L'_1}$ from $\mathcal{T}_{L'_2}$. So the order of (A, B) equals the order of (A', B').

We first assume that (A, B) and (A', B') are moats. If t^* is closer to L_1 than t, then $A' \subseteq A$, but (A', B') is the $(\mathcal{T}_{L'_1}, \mathcal{T}_{L'_2})$-moat, so $(A, B) = (A', B')$. Similarly, $(A, B) = (A', B')$ if t is closer to L_1 than t^*. Hence $t = t^*$.

Then we assume that (A, B) and (A', B') are distinctions. Since (A', B') distinguishes $\mathcal{T}_{L_1}$ from $\mathcal{T}_{L_2}$, $\lambda(A', B') \geq \lambda(A, B)$. Similarly, (A, B) distinguishes $\mathcal{T}_{L'_1}$ from $\mathcal{T}_{L'_2}$, so $\lambda(A, B) \geq \lambda(A', B')$. So $\lambda(A, B) = \lambda(A', B')$. That is, $(A, B) = (A', B')$. $\Box$

Let $\mathcal{R}$ be a set of fat θ'-cells of $(T, \mathcal{X})$ such that $\iota(R) \neq \iota(R')$ for different members R, R' of $\mathcal{R}$, and for every fat θ'-cell L of $(T, \mathcal{X})$, there exists $R \in \mathcal{R}$ such that

$\iota(L) = \iota(R)$. For every $R \in \mathcal{R}$, define ∂R to be the set of nodes of T such that each member of ∂R realizes a $(\mathcal{T}_L, \mathcal{T}_{L'})$-distinction for some near fat θ'-cells L, L' with $\iota(L) = \iota(R) \neq \iota(L')$, and subject to that, the component of $T - \partial R$ containing R is minimal, and subject to that, ∂R is minimal.

Claim 5: For every $R \in \mathcal{R}$ and for every $(\mathcal{T}_L, \mathcal{T}_{\mathcal{L}'})$-distinction (A, B) for some fat θ'-cells L, L' of (T, X) with $\iota(L) = \iota(R) \neq \iota(L')$, there uniquely exists $(C, D) \in \iota(R)$ such that $A \subseteq C$ and $D \subseteq B$.

Proof of Claim 5: Since $\iota(L) \not\in \mathcal{T}_{L'}$, there exists $(C, D) \in \iota(L) - \mathcal{T}_{L'}$. As $\iota(L)$ θ'-isolates $\mathcal{T}_L$ from $\mathcal{S}$, $A \subseteq C$ and $D \subseteq B$. Suppose that there exists $(C', D') \in \iota(R) - \{(C, D)\}$ such that $A \subseteq C'$ and $D' \subseteq B$. Then $A \subseteq C' \subseteq D \subseteq B$. Since A is disjoint from B, A is empty. So $(G, \emptyset) = (B, A) \in \mathcal{T}_{L'}$, a contradiction. $\Box$

For every $R \in \mathcal{R}$, define N_R to be the component of $T - \partial R$ containing R. Note that either N_R contains the root of T, or the parent, denoted by p_R, of the root of N_R is in ∂R. In the latter case, we define (C_R, D_R) to be the separation in $\iota(R)$ such that $A \subseteq C_R$ and $D_R \subseteq B$, where (A, B) is the $(\mathcal{T}_L, \mathcal{T}_{L'})$-distinction realized by p_R for some fat θ'-cells L, L' with $\iota(L) = \iota(R) \neq \iota(L')$. Note that the existence of (C_R, D_R) follows from Claim 5. For every $R \in \mathcal{R}$, define N'_R to be the subtree of T induced by $N_R \cup \partial R$.

The next objective is to construct a new tree-decomposition such that for each $R \in \mathcal{R}$, there exists a node of the new tree such that $\iota(R)$ is "realized" by this node.

For every $R \in \mathcal{R}$, define a rooted tree T_R as follows.

- $V(T_R) = \{t^*_R, t_{(C,D)} : (C, D) \in \iota(R)\} \cup \bigcup_{(C,D)\in\iota(R)} V(T_{(C,D)})$, where $T_{(C,D)}$ is a copy of N'_R for each $(C, D) \in \iota(R)$.
- $E(T_R) = \{t^*_R t_{(C,D)}, t_{(C,D)} r_{(C,D)} : (C, D) \in \iota(R)\} \cup \bigcup_{(C,D)\in\iota(R)} E(T_{(C,D)})$, where $r_{(C,D)}$ is the copy of r_R in $T_{(C,D)}$.
- The edges of T_R is oriented as follows.

– If (C_R, D_R) is defined, then the tail of $t^*_R t_{(C_R,D_R)}$ is $t_{(C_R,D_R)}$, and the tail of $t_{(C_R,D_R)} r_{(C_R,D_R)}$ is $r_{(C_R,D_R)}$, and the root of $T_{(C,D)}$ is the copy of the root of N'_R in $T_{(C,D)}$.

– For every other $(C, D) \in \iota(R)$, the tail of $t^*_R t_{(C,D)}$ is t^*_R, and the tail of $t_{(C,D)} r_{(C,D)}$ is $t_{(C,D)}$, and the root of $T_{(C,D)}$ is $r_{(C,D)}$.

Hence the root of T_R is t^*_R if (C_R, D_R) is undefined, and is the copy of the root of N'_R in $T_{(C_R,D_R)}$ otherwise.

For every node t of T, we define (A_t, B_t) to be the separation of G such that $V(A_t)$ is the union of the bags of t and the descendants of t, and $V(B_t) = (V(G)-V(A_t)) \cup X_t$, and subject to that, B_t is maximal. Define $\mathcal{X}_R = (X_{R,t} : t \in V(T_R))$ as the follows.

- $X_{R,t^*_R} = \bigcap_{(C,D)\in\iota(R)} D$.
- $X_{R,t_{(C,D)}} = V(C) \cap V(D)$ for every $(C, D) \in \iota(R)$.
- $X_{R,r_{(C,D)}} = (X_{r_R} \cap V(C)) \cup (V(C) \cap V(D))$ for every $(C, D) \in \iota(R)$.
- For every $(C, D) \in \iota(R)$ and for every $t \in V(T_{(C,D)}) - \{r_{(C,D)}\}$, $X_{R,t} = V(C \cap A_{t,R}) \cap V(D \cup B_{t,R})$.

By Claim 5, for every $t \in \partial R$, there uniquely exists $(C, D) \in \iota(R)$ such that $A_{t,R} \subseteq C$ and $D \subseteq B_{t,R}$. And in fact, $X_{R,t_{(C,D)}} = X_t$.

Define T' be the rooted tree that is obtained from the union of the components M of $T - \bigcup_{R\in\mathcal{R}} N_R$ and $\bigcup_{R\in\mathcal{R}} T_R$ by identifying the pairs of nodes t, t', where

- t is in a component M of $T - \bigcup_{R\in\mathcal{R}} N_R$, and t is in ∂R_t for some $R_t \in \mathcal{R}$, and
- t' is in T_{R_t}, and
- $X_{R_{t'},t'} = X_{t'}$.

We remark that R_t may not be unique, and t is identified with more than one node in that case. Also, we only identify nodes that have the same bags. Define $\mathcal{X}' = (X'_t : t \in V(T'))$, where $X'_t = X_{R,t}$ if $t \in T_R$ for some $R \in \mathcal{R}$, and $X'_t = X_t$ otherwise. Actually, $(T', \mathcal{X}')$ can be obtained from $(T, \mathcal{X})$ by repeatedly applying the operation mentioned right before Lemma 7.3.1. So $(T', \mathcal{X}')$ is a tree-decomposition of G.

The next step is to prove that $(T', \mathcal{X}')$ realizes many nice edge-cuts. Let $f(k, h)$ be the bound of the (W, s)-depth for every W, s mentioned in Theorem 4.2.5. We recall that every node in $\bigcup_{R \in \mathcal{R}} \partial R$ is also in $(T', \mathcal{X}')$.

Claim 6: For every $W \subseteq V(G)$ and $0 \leq s \leq h$, if there exists a directed path P' in T' passing through at least $4h^3k(f(k, h) + 3) + 1$ nodes of T' that are in $\bigcup_{R \in \mathcal{R}} \partial R$ such that each bag of these nodes contains W, and the bags of these nodes are pairwise disjoint and of size s after deleting W, and no other node of P' has bag size less than $s + |W|$ in (T', X'), then there exists a node t^* in P' such that $|X'_{t^*}| = s + |W|$ and X'_{t^*} corresponds to an edge-cut modulo W.

Proof of Claim 6: We fix $W \subseteq V(G)$ and $0 \leq s \leq h$ in the proof of this claim. For each node t of T, let (A_t, B_t) be a separation such that $V(A_t) \cap V(B_t) = X_t$ and $V(B_t) = X_t \cup \bigcup_{t' \in C} X_{t'}$, where C is the component of $T - \{t\}$ containing the root of T.

Since every node in $\bigcup_{R \in \mathcal{R}} \partial R$ is also in $(T', \mathcal{X}')$, there exists a directed path P in T passing though these $4h^3k(f(k, h) + 3) + 1$ mentioned nodes. Since the (W, s)-depth of $(T, \mathcal{X})$ is at most $f(k, h)$, there exist different fat θ'-cells $L_0, L_1, ..., L_{4h^3k+1}$ of $(T, \mathcal{X})$ such that for each $1 \leq j \leq 4h^3k$, there exist a pair of distinct nodes $u_{j,1}, u_{j,2} \in \bigcup_{R \in \mathcal{R}} \partial R$ and a node c_j in P such that the following hold.

- P passes through $u_{1,2}, c_1, u_{1,1}, u_{2,2}, c_2, u_{2,1}, ..., u_{j,2}, c_j, u_{j,1}, ..., u_{4h^3k,2}, c_{4h^3k}, u_{4h^3k,1}$ in order.
- Each X_{c_j} corresponding an edge-cut modulo W of size $s + |W|$ in $(T, \mathcal{X})$.

- $u_{j,1} \neq u_{j+1,2}$.

- $(A_{u_{j,1}}, B_{u_{j,1}}) = (A_{u_{j,1},L_{j-1}}, B_{u_{j,1},L_{j-1}})$ and $(A_{u_{j,2}}, B_{u_{j,2}}) = (A_{u_{j,2},L_j}, B_{u_{j,2},L_j})$.

Note that we may assume that $c_j \neq u_{2,j}$, otherwise, we are done since X'_t corresponds to an edge-cut of G under modulo W, where t is the copy of c_j in $T_{(C,D)}$ for some $R \in \mathcal{R}$ and $(C, D) \in \iota(R)$. We further choose c_j to be as close to $u_{j,1}$ as possible for each j.

As each $u_{i,j}$ corresponds to an edge-cut under modulo W, for every vertex in $V(A_{u_{i,j}}) \cap V(B_{u_{i,j}}) - W$, there exists exactly one edge between it and $V(B_{u_{i,j}}) - V(A_{u_{i,j}})$. Let $P_1, P_2, ..., P_s$ the s disjoint paths in $G[A_{u_{1,2}} \cap B_{u_{4h^3k,1}}] - W$, such that these s paths defines the order of the vertices in the bags of the nodes in P of size $|W| + s$ but not in W. For each node $t \in V(P)$ with $|X_t| = s + |W|$ and for every $j, \ell \in \{1, 2, ..., s\}$, we say that the j-th vertex v of X_t *jump right* (and *left*, respectively) *to* ℓ if the following hold.

- $\ell \neq j$.

- $v \in V(P_j) \cap X_t$.

- There exist two edge-disjoint paths in $B_t - E(G[X_t])$ (and $A_t - E(G[X_t])$, respectively) with a common end v such that the other end of one path is on P_ℓ and otherwise disjoint from P_ℓ, and the other end of the other path is in P_j.

- None of these two paths has the both ends in X_t.

For each $1 \leq q \leq 4h^3k$, we say that q *jumps out* if there exist $x, y \in V(B_{u_{q,1}}) \cap V(A_{u_{q,2}})$ (possibly $x = y$), integers j, ℓ_x, ℓ_y with $1 \leq j, \ell_x, \ell_y \leq s$, and nodes t_x, t_y on P with $|X_{t_x}| = |X_{t_y}| = |W| + s$ such that the following hold.

- x, y are the j-th vertex of X_{t_x}, X_{t_y}, respectively.

- P passes $u_{q,2}, t_y, t_x, u_{q,1}$ in order. (Possibly $t_x = t_y$.)

- $j \neq \ell_x$ and $j \neq \ell_y$.

- x jumps left to ℓ_x, and y jumps right to ℓ_y.

- Either $x = y$, or there exist two edge-disjoint path from x to y in $G[V(B_{u_{q,1}}) \cap V(A_{u_{q,2}})]$ such that these two paths does not intersect in any internal vertex of the paths in those jumps.

Suppose that each q with $1 \leq q \leq 4h^3k$ and $q \equiv 2 \pmod 4$ jumps out. Then there exist at least k such q's whose corresponding j, ℓ_x, ℓ_y in the definition of jumping out are the same. But it implies that G_i contains a topological minor isomorphic to the Robertson chain of length k, a contradiction. Therefore, there exists q^* with $1 \leq q^* \leq 4h^3k$ and $q \equiv 2 \pmod 4$ that does not jump out. Our objective is to prove that $(A_{c_{q^*}, R_{c_{q^*}}} \cap C', B_{c_{q^*}, R_{c_{q^*}}} \cup D')$ is an edge-cut under modulo W, where (C', D') is the separation in $\iota(R_{c^*})$ such that $A_{u_{q^*,1}} \subseteq C'$ and $D' \subseteq B_{u_{q^*,2}}$. Note that it implies that the bag of some node in P' is corresponding to an edge-cut under modulo W and leads to a contradiction.

Note that $(A_{c_{q^*}}, B_{c_{q^*}}) \in \mathcal{T}_{L_{q^*-1}}$, and $(A_{u_{q^*,1}}, B_{u_{q^*,1}})$ is $\mathcal{T}_{L_{q^*}} - \mathcal{T}_{L_{q^*-1}}$, and $V(A_{u_{q^*,1}}) \cap V(B_{u_{q^*,1}}) = V(K) \cap V(K')$, where (K, K') is the $(\mathcal{T}_{L_{q^*}}, \mathcal{T}_{L_{q^*-1}})$-distinction. Next, we show that every vertex in $X_{u_{q^*,1}}$ jumps left. Suppose to the contrary. Let v_j be a vertex in $X_{u_{q^*,1}}$ that does not jump left, where v_j is the j-th vertex in $X_{u_{q^*,1}}$. Delete $\bigcup_{\ell \neq j} P_\ell$ from G, and look at the path of blocks from the block containing v_j to the block containing the vertex in the intersection of P_j and $X_{u_{q^*+1,2}}$. Since v_j does not jump left, there exists a separation $(A, B) \in \mathcal{T}_{L_{q^*}}$ of order two such that $v_j \in V(A) \cap V(B) \subseteq V(P_j)$, and A contains some mentioned blocks. Then $(A_{u_{q^*,1}, L_{q^*}} \cap B, B_{u_{q^*,1}, L_{q^*}} \cup A)$ is in $\mathcal{T}_{L_{q^*}} - \mathcal{T}_{L_{q^*+1}}$, but it has smaller λ-order than the $(\mathcal{T}_{L_{q^*}}, \mathcal{T}_{L_{q^*+1}})$-distinction, a contradiction. This proves that every vertex in $X_{u_{q^*,1}}$ jumps left.

Since q^* does not jump out, for every $1 \leq j \leq s$, there exists an edge e_j in $G[N[A_{u_{q^*,2},L_{q^*-1}}] \cap V(B_{u_{q^*,1},L_{q^*}})] \cap P_j$ such that either e_j is the only edge between $v_{q^*,j}$ and $V(B_{u_{q^*,j},L_{q^*-1}}) - V(A_{u_{q^*,j},L_{q^*-1}})$, or the vertex in $X_{u_{q^*,1}} \cap P_j$ is disconnected with the vertex in $X_{u_{q^*,2}} \cap P_j$ in $G - (\{e_j\} \cup \bigcup_{i \neq j} V(P_i))$, where $v_{q^*,j}$ is the vertex in $P_j \cap X_{u_{q^*,1}}$. We pick such e_j as close to $v_{q^*,j}$ as possible for each $1 \leq j \leq s$. So there exists a separation (A, B) of order $|W| + s$ such that $W \subseteq V(A) \cap V(B)$, $A_{u_{q^*,1},L_{q^*}} \subseteq A$, and every path in B passing through a vertex in $V(B) - V(A)$ with both ends in $V(A) \cap V(B) - W$ contains one of $e_1, e_2, ..., e_s$.

For each $1 \leq j \leq s$, let P_j' be the subpath of P_j not containing e_j and whose ends are the vertex in $P_j \cap X_{u_{q^*,1}}$ and an end of e_j. Let t^* be a node of a path in T from c_{q^*} and $u_{q^*,1}$ such that $A_{t^*,L_{q^*}} \subseteq A$ and $B \subseteq B_{t^*,L_{q^*}}$ and $|X_{t^*}| = |W| + s$. Note that such a t^* exists as $u_{q^*,1}$ is a candidate. Furthermore, X_{t^*} contains exactly one vertex in P_j' for each $1 \leq j \leq s$. Let x_j be the vertex in $X_{t^*} \cap V(P_j')$, and let d_j be the distance between x_j and the vertex in $V(P_j) \cap X_{u_{q^*,1}}$ in P_j', for each $1 \leq j \leq s$. We choose t^* such that $(d_1, d_2, ..., d_s)$ is lexicographically minimal. We shall prove that $(A_{t^*,L_{q^*}}, B_{t^*,L_{q^*}}) = (A, B)$.

Suppose to the contrary. So there exists $1 \leq p \leq s$ such that the vertex in $V(P_p) \cap X_{t^*}$ is an end of a path in $B_{t^*,L_{q^*}}$ passing through a vertex in $V(B_{t^*,L_{q^*}}) - V(A_{t^*,L_{q^*}})$ with ends in $V(A_{t^*,L_{q^*}}) \cap V(B_{t^*,L_{q^*}})$ containing an edge other than e_p. We assume that p is the minimum number satisfying the above property. So the goodness of t^* is $p - 1$. Since q^* does not jump out, there exists no right jump from the vertex in $X_{t^*} \cap V(P_p)$. However, we can repeatedly swap subtrees of $(T, \mathcal{X})$ to increase the goodness of $(T, \mathcal{X})$ without changing the signature and loss of the branching property, a contradiction. Hence, $(A_{t^*,L_{q^*}}, B_{t^*,L_{q^*}}) = (A, B)$ corresponds to an edge-cut modulo W.

Finally, we shall prove that $(A_{t^*,L_{q^*}} \cap C', B_{t^*,L_{q^*}} \cup D')$ corresponds to an edge-cut under modulo W for some $(C', D') \in \iota(R)$, where $R \in \mathcal{R}$ such that $\iota(L_{q^*}) = \iota(R)$.

Let (C', D') be the separation in $\iota(R)$ such that $A_{u_{q^*,1},L_{q^*}} \subseteq C'$ and $D' \subseteq B_{u_{q^*,1},L_{q^*}}$.

Recall that $\iota(R) = \mathcal{L}'_{L_{q^*}} = \mathcal{M}_{L_{q^*}} \cup \{(C \cap (\bigcap_{(A,B)\in\mathcal{M}_{L_{q^*}}} B), D \cup (\bigcup_{(A,B)\in\mathcal{M}_{L_{q^*}}} A)) : (C, D) \in \mathcal{L}_{L_{q^*}}\}$. We first assume that $(C', D') \in \mathcal{M}_{L_{q^*}}$. By the definition of $\mathcal{M}_{L_{q^*}}$, there is a node t^{**} in the path in T from t^* to L_{q^*}, such that $(A_{t^{**},L_{q^*}}, B_{t^{**},L_{q^*}}) = (C', D')$. So $A_{t^*,L_{q^*}} \subseteq C'$, $D' \subseteq B_{t^*,L_{q^*}}$, and $(A_{t^*,L_{q^*}} \cap C', B_{t^*,L_{q^*}} \cup D') = (A_{t^*,L_{q^*}}, B_{t^*,L_{q^*}})$ corresponds to an edge-cut modulo W. Hence, we may assume that $(C', D') = (C \cap (\bigcap_{(A,B)\in\mathcal{M}_{L_{q^*}}} B), D \cup (\bigcup_{(A,B)\in\mathcal{M}_{L_{q^*}}} A))$ for some $(C, D) \in \mathcal{L}_{L_{q^*}}$, and $Z_{L_{q^*}} \cup \{e\} \subseteq V(B_{t^*,L_{q^*}}) \cup E(B_{t^*,L_{q^*}})$.

In this case, by the definition of $\mathcal{M}_{L_{q^*}}$, for every $(A, B) \in \mathcal{M}_{L^*}$, $A_{t^*,L_{q^*}} \subseteq B$ and $A \subseteq B_{t^*,L_{q^*}}$. Therefore, $A_{t^*,L_{q^*}} \cap C' = A_{t^*,L_{q^*}} \cap C$ and $B_{t^*,L_{q^*}} \cup D' = B_{t^*,L_{q^*}} \cup D$. Similarly, if $A_{t^*,L_{q^*}} \subseteq C$ and $D \subseteq B_{t^*,L_{q^*}}$, then $A_{t^*,L_{q^*}} \subseteq C'$ and $D' \subseteq B_{t^*,L_{q^*}}$, and hence we are done. So we may assume that either $A_{t^*,L_{q^*}} \not\subseteq C$, or $D \not\subseteq B_{t^*,L_{q^*}}$.

As $(C - W, D - W)$ is linked in $\mathcal{T}_{L_{q^*}} - W$, and $Z_{L_{q^*}} \subseteq V(B_{t^*,L_{q^*}}) \cup E(B_{t^*,L_{q^*}})$, we know that the order of $(A_{t^*,L_{q^*}} \cup C, B_{t^*,L_{q^*}} \cap D)$ is at least the order of (C, D), so the order of $(A_{t^*,L_{q^*}} \cap C, B_{t^*,L_{q^*}} \cup D)$ is at most the order of $(A_{t^*,L_{q^*}}, B_{t^*,L_{q^*}})$. On the other hand, the order of $(A_{t^*,L_{q^*}}, B_{t^*,L_{q^*}})$ is the same as the $(\mathcal{T}_{L^*}, \mathcal{T}')$-distinction for some tangle $\mathcal{T}'$ for which $(A_{t^*,L_{q^*}}, B_{t^*,L_{q^*}}) \not\in \mathcal{T}'$. So the order of $(A_{t^*,L_{q^*}} \cup C, B_{t^*,L_{q^*}} \cap D)$ equals the order of (C, D). Since $(A_{t^*,L_{q^*}}, B_{t^*,L_{q^*}})$ corresponds to an edge-cut modulo W, every vertex in $((A_{t^*,L_{q^*}} \cup C) \cap (B_{t^*,L_{q^*}} \cap D)) - (C \cap D)$ is incident with at most one edge whose the other end in $V(B_{t^*,L_{q^*}} \cap D) - V(A_{t^*,L_{q^*}} \cup C)$. It implies that every vertex in $(C \cap D \cap V(B_{t^*,L_{q^*}})) - V(A_{t^*,L_{q^*}})$ is incident with at most one edge whose the other end in $V(D) - V(C)$, by Theorem 6.3.11. Consequently, $(A_{t^*,L_{q^*}} \cap C, B_{t^*,L_{q^*}} \cup D)$ corresponds to an edge-cut under modulo W, unless some vertex in $V(A_{t^*,L_{q^*}}) \cap V(C) \cap V(B_{t^*,L_{q^*}}) \cap V(D)$ is adjacent to some vertex in $V(A_{t^*,L_{q^*}}) - V(C)$.

So we may assume that some vertex x in $V(A_{t^*,L_{q^*}}) \cap V(C) \cap V(B_{t^*,L_{q^*}}) \cap V(D)$ is adjacent to some vertex in $V(A_{t^*,L_{q^*}}) - V(C)$. Note that it implies that $x \in Z_{L_{q^*}}$ by Theorem 6.3.11. Let y be the vertex in $V(B_{t^*,L_{q^*}}) - V(A_{t^*,L_{q^*}})$ adjacent to x. But

$(A_{t^*,L_{q^*}} \cup \{y\}, B_{t^*,L_{q^*}} - \{x\}) \in \mathcal{T}_{L_{q^*}} - \mathcal{T}'$ having the order the same as the $(\mathcal{T}_{L_{q^*}}, \mathcal{T}')$-moat. Therefore, $x \in Z_{L_{q^*}} \cap V(A^*) - V(B^*)$, where (A^*, B^*) is the $(\mathcal{T}_{L_{q^*}}, \mathcal{T}')$-moat. It is a contradiction since $Z_{L_{q^*}} \subseteq V(B_{t^*,L_{q^*}}) \cup E(B_{t^*,L_{q^*}})$ and $(C, D) \notin \mathcal{M}_{L_{q^*}}$. This proves the claim. $\Box$

Define $g(k, h)$ to be $4h^3k(f(k, h) + 3) + 1$. Now, we define $\mathcal{Y}$ to be a subset of $V(T')$ satisfying the following properties.

- For each $W \subseteq V(G)$ and for each $0 \leq s \leq h$, if a directed path in T' passes through at least $g(k, h)$ nodes t in $\bigcup_{R \in \mathcal{R}} \partial R$ with $|X'_t| = |W| + s$ and $W \subseteq X'_t$ and $X'_t - W$ are pairwise disjoint such that every node t' in this path satisfies that $|X'_{t'}| \geq |W| + s$, then it passes through a node t'' in $\mathcal{Y}$ with $|X'_{t''}| \leq |W| + s$, and $X'_{t''}$ corresponds to an edge-cut modulo W.

- For every $R \in \mathcal{R}$, every directed path in T_R contains at most two nodes in T_R.

Note that the existence of $\mathcal{Y}$ follows from Claim 6 and the fact that X'_t corresponds to an edge-cut module W implies that X'_t corresponds to an edge-cut module W' for every $W' \supseteq W$.

We define T^* to be the rooted tree obtained from T' by contracting each component of $T' - (\bigcup_{R \in \mathcal{R}} \partial R \cup \mathcal{Y})$, and we define $X^*_t = X'_t$ if $t \in \bigcup_{R \in \mathcal{R}} \partial R \cup \mathcal{Y}$, and define X^*_t to be the union of the bags of nodes that is contracted into t for other nodes t of T^*. Let $\mathcal{X}^* = \{X^*_t : t \in V(T^*)\}$. Then $(T^*, \mathcal{X}^*)$ is a rooted tree-decomposition of G.

Recall that there exist foundation paths in G passing through the bags of the nodes in $(T, \mathcal{X})$, and $\bigcup_{R \in \mathcal{R}} \partial R \subseteq V(T^*) \cap V(T)$. In addition, if $y \in \mathcal{Y} \cap V(T_R)$ for some $R \in \mathcal{R}$, then $\min_{t \in \partial R} X'_t \leq |X'_y| \leq \max_{t \in \partial R} X'_t$. So there exist foundation paths in G passing through the bags of the nodes in $(T^*, \mathcal{X}^*)$. Therefore, $(T^*, \mathcal{X}^*)$ is an unimpeded rooted tree-decomposition of adhesion at most h. On the other hand, for every $W \subseteq V(G)$ and integer s with $0 \leq s \leq h$, the (W, s)-depth of $(T^*, \mathcal{X}^*)$ is at most $4g(k, h)$, by the choice of $\mathcal{Y}$.

Now we shall apply Lemma 5.1.8 to complete the proof of this theorem. Note that the node of T^* is either in T or is contracted by some nodes of T, so the rooted location that is made by a node and its children fits a center-essential frame. It is sufficient to show that those frames are in a well-behaved family. Let $t \in V(T^*)$. If $t \in V(T)$, then $|X_t^*| \leq h$, so we are done by Lemma 5.1.5. So we may assume that t is obtained by contracting nodes from T^*.

Recall that the rooted location made by t and its children has a tree-refinement given by the tree-decomposition of $(T', \mathcal{X}')$. We denote the tree-refinement by $(T(t), \mathcal{X}(t))$. If $T(t)$ does not contain any vertex in T_R for every $R \in \mathcal{R}$, then the tree-refinement is in T and we are done by Lemma 5.1.8. So we may assume that $T(t)$ contains a node in T_R for some $R \in \mathcal{R}$.

By Lemma 7.2.3, the rooted location made by t_R and its children fits a frame in a well-behaved set. So it is sufficient to prove that $(T_{(C,D)}, \mathcal{X}_{R,t_{(C,D)}})$ satisfies the condition of Lemma 5.1.8.

Note that for every $R \in \mathcal{R}$ and $(C, D) \in \iota(R)$, $|X_{R,t_{(C,D)}}| \leq \theta'$. Furthermore, $|X_{R,r_{(C,D)}}| \leq |X_{r_R} \cap V(C)| + |V(C) \cap V(D)| \leq 2\theta'$ by Lemma 3.2.1. Similarly, for every fat θ'-cell L with $\iota(L) = \iota(R)$, if t is a copy of a node of L in $T_{(C,D)}$, then $|X_{R,t}| \leq |V(A_{t,R}) \cap V(B_{t,R}) \cap V(C)| + |V(A_{t,R}) \cap V(C) \cap V(D)| \leq 2\theta'$. For $t \in V(T_{(C,D)}) - \{r_{(C,D)}\}$ that is not the copy of a node in any fat θ'-cell, $|X_{R,t}| \leq |V(A_{t,R}) \cap V(B_{t,R}) \cap V(C)| + |V(A_{t,R}) \cap V(C) \cap V(D)| \leq (3\theta' + 1) + \theta'$. Therefore, $|X_{R,t}| \leq 4\theta' + 1$ for every $t \in V(T_R) - \{t_R^*\}$.

For each subset Y of $V(C) \cap V(D)$, let T_Y be the forest of $T_{(C,D)}$ consisting of the nodes whose bags in $(T_{(C,D)}, \mathcal{X}_{R,t_{(C,D)}})$ contains Y but not contain any Y' with $Y \subset Y' \subseteq V(C) \cap V(D)$. As $T_{(C,D)}$ is a copy of a subtree of T, the (W, s)-depth of the subtree in $(T_{(C,D)}, \mathcal{X}_{R,t_{(C,D)}})$ is at most $f(k, h)$, for all $W \subseteq V(G)$ and $0 \leq s \leq h$. So for each $Y \subseteq V(C) \cap V(D)$, the (W, s)-depth of each component of T_Y in $(T_{(C,D)}, \mathcal{X}_{R,t_{(C,D)}})$ is at most $f(k, h)$ for all $W \subseteq V(G_i)$ and $0 \leq s \leq h$. Furthermore, there are at most

$2^{|V(C)\cap V(D)|} \leq 2^{\theta'}$ such sets Y. Therefore, the rooted location made by t and its children is in a well-behaved family by Lemmas 5.1.8 and 5.1.6. This completes the proof. ■

Proof of Theorem 1.4.5: By Theorem 7.3.3, Theorem 1.4.5 holds for graphs whose every vertex is incident with a loop. Since if e is a loop such that it is contained in the image of π_E, say $e = \pi_E(f)$, then f must be a loop. Therefore, this theorem holds for graphs without loops. Now, we prove the theorem for general graphs. Since $(\mathbb{N} \cup \{0\}, \leq)$ is a well-quasi-ordered set, we can label vertices by the number of loops incident with them and then remove all loops. Hence the theorem follows. □

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