

FUNDAMENTAL PROPERTIES OF CONVEX MIXED-INTEGER PROGRAMS

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FUNDAMENTAL PROPERTIES OF CONVEX MIXED-INTEGER PROGRAMS

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SUMMARY

In this Ph.D. dissertation research, we lay the mathematical foundations of various fundamental concepts in *convex* mixed-integer programs (MIPs), that is, optimization problems where all the decision variables belong to a given convex set and, in addition, a subset of them are required to be integer. In particular, we study properties of their feasible region and properties of cutting planes. The main contribution of this work is the extension of several fundamental results from the theory of *linear* MIPs to the case of *convex* MIPs.

In the first part, we study properties of general closed convex sets that determine the closedness and polyhedrality of their integer hulls. We first present necessary and sufficient conditions for the integer hull of a general convex set to be closed. This leads to useful results for special classes of convex sets such as pointed cones, strictly convex sets, and sets containing integer points in their interior. We then present a sufficient condition for the integer hulls of general convex sets to be polyhedra. This result generalizes the well-known result due to Meyer in the case of *linear* MIPs. Under a simple technical assumption, we show that these sufficient conditions are also necessary for the integer hull of general convex sets to be polyhedra.

In the second part, we apply the previous results to mixed-integer second order conic programs (MISOCPs), a special case of nonlinear *convex* MIPs. We show that there exists a polynomial time algorithm to check the closedness of the mixed-integer hulls of simple MISOCPs. Moreover, in the special case of pure integer problems, we present sufficient conditions for verifying the closedness of the integer hull of intersection of simple MISOCPs that can also be checked in polynomial

time.

In the third part, we present an extension of the duality theory for *linear* MIPs to the case of *conic* MIPs. In particular, we construct a subadditive dual to *conic* MIPs. Under a simple condition on the primal problem, we are able to prove strong duality.

In the fourth part, we study properties of maximal S -free convex sets, where S is a subset of integers contained in an arbitrary convex set. An S -free convex set is a convex set not containing any points of S in its interior. In this part, we show that maximal S -free convex sets are polyhedra and discuss some properties of these sets.

In the fifth part, we study some generalizations of the split closure in the case of *linear* MIPs. Split cuts form a well-known class of valid inequalities for *linear* MIPs. Cook et al. (1990) showed that the split closure of a rational polyhedron - that is, the set of points in the polyhedron satisfying all split cuts - is again a polyhedron. In this thesis, we extend this result from a single rational polyhedron to the union of a finite number of rational polyhedra. We also show how this result can be used to prove that some generalizations of split cuts, namely cross cuts, also yield closures that are rational polyhedra.

CHAPTER I

INTRODUCTION

Mixed-integer programs (MIPs) are optimization problems where some of the decision variables are constrained to take integer values. More precisely, let $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and let $K \subseteq \mathbb{R}^{p+q}$. Then a MIP is a problem of the form:

$$\inf\{c^T x + d^T y \mid (x, y) \in K \cap (\mathbb{Z}^p \times \mathbb{R}^q)\}. \quad (1)$$

MIPs have been extensively studied for more than fifty years because of their great range of applications and the beauty of their mathematical underlying structure. From the theoretical point of view, MIPs are a very complicated class of optimization problems, in fact they are NP-hard. Moreover, they are also extremely difficult to solve in practice, and especially in large-scale problems, often it is required to use sophisticated techniques in addition to state-of-the-art optimization software.

The set K in (1) is called the continuous relaxation of the mixed-integer program. MIPs are often classified according to the nature of their continuous relaxation in several classes of MIPs, including convex, linear, nonlinear, nonconvex, polynomial MIPs, among others. A particularly important class of mixed-integer programs is the one given by *convex* MIPs, that is, those MIPs for which their continuous relaxation is a convex set (K is a convex set).

1.1 Convex MIPs

A *convex* MIPs is a mixed-integer program of the form (1) for which K is a convex set. In the special case when K is a (rational) polyhedron, the *convex* MIP is called a *linear* MIP. *Linear* MIPs have been widely studied in the literature and they possess a great range of applications (see for instance [73]). However, despite the

usefulness of *linear* MIPs in solving real-world problems, there are many engineering and business applications that cannot be modeled by using linear constraints only. Therefore, there is a need to understand basic properties of MIPs that require more intricate constraints, such as (nonlinear) *convex* MIPs.

A noteworthy example of (nonlinear) *convex* MIPs are the so-called *conic* MIPs, which are obtained by taking $K = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \mid Ax + By - b \in \mathbf{C}\}$ in (1), where A, B and b are integral matrices of appropriate dimensions and $\mathbf{C} \subseteq \mathbb{R}^t$ is a pointed closed convex cone. Observe that by setting $\mathbf{C} = \mathbb{R}_+^t$ we retrieve the special case of *linear* MIPs and thus *conic* MIPs are a generalization of *linear* MIPs. Other notable example of *conic* MIPs is the case of second order *conic* MIPs, which are obtained by considering $\mathbf{C} = \{(w, z) \in \mathbb{R}^{t-1} \times \mathbb{R} \mid \|w\| \leq z\}$, where $\|\cdot\|$ denotes the Euclidean norm; that is, \mathbf{C} is the Lorentz cone in \mathbb{R}^t .

Nonlinear *convex* MIPs have a number of applications, including chemical engineering [42, 43], environmental impact mitigation [41], design of pulse coders for audio amplication [24], layout design [26], logistics [18, 45, 63], machine-job assignment [1], network design [22], portfolio optimization [19, 21, 55], trim-loss optimization [47, 48], and water distribution systems design [62].

In this thesis, we investigate basic properties of *convex* MIPs. The study of properties of optimization problems is not only interesting from a mathematical point of view, but, in fact, has proven crucial in the design of algorithms to find optimal solutions. We will illustrate this point by analyzing the case of *linear* MIPs next.

1.2 Linear MIPs

A *linear* MIPs is a mixed-integer program of the form (1) for which K is a rational polyhedron, that is, $K = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \mid Ax + By \geq b\}$, where A, B are integral matrices and b is an integral vectors of appropriate dimensions.

Linear MIPs are well-understood in several aspects, both from the theoretical and practical point of view: (1) Structure of their feasible region. (2) Algorithmic techniques (Ex: cutting planes, branch and bound). (3) Properties of cutting planes. (4) Duality theory (5) Complexity results, among others. All these properties are a consequence of the nice structure and properties of the continuous relaxation K in the case of *linear* MIPs. The rational polyhedron K satisfies a number of properties: finite representation result (Minkowski-Weyl Theorem), rational polyhedral recession cone, duality for linear programs and so forth.

We will describe next some important concepts and related properties of *linear* MIPs. For ease of exposition, we will describe these properties in the pure integer case ($q = 0$), but similar results are also valid in the mixed-integer case ($q \geq 1$).

1.2.1 Structural properties of *linear* MIPs

The feasible region of a *linear* MIP is given by the integer points inside rational polyhedron. The fundamental theorem of integer programming (Meyer [67]) states that the convex hull of solutions to a *linear* MIP is also a rational polyhedron. Therefore, the feasible region of a *linear* MIP has a very ‘nice’ structure: it can be finitely described in terms of a finite set of vertices and a finite set of directions. Figure 1 shows the integer hull of a rational polyhedron.

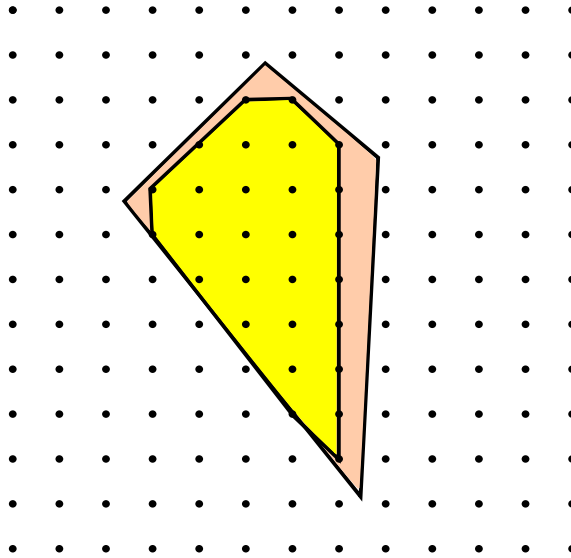


Figure 1: The integer hull of a rational polyhedron

An important consequence of the integer hull being a rational polyhedron is that a *linear* MIP can be solved as a linear program if the integer hull is known (of course, this set is not easy to obtain in general, as *linear* MIPs are NP-hard problems).

1.2.2 Cutting planes for *linear* MIPs

Cutting planes are one of the most successful algorithmic tools for *linear* MIPs. The main idea behind this algorithmic technique is to sequentially obtain tighter approximations of the convex hull of integer feasible solutions. This is achieved by the addition of cutting planes, that is, linear inequalities that separate fractional points from the convex hull of integer feasible solutions. Figure 2 illustrate the concept of cutting plane.

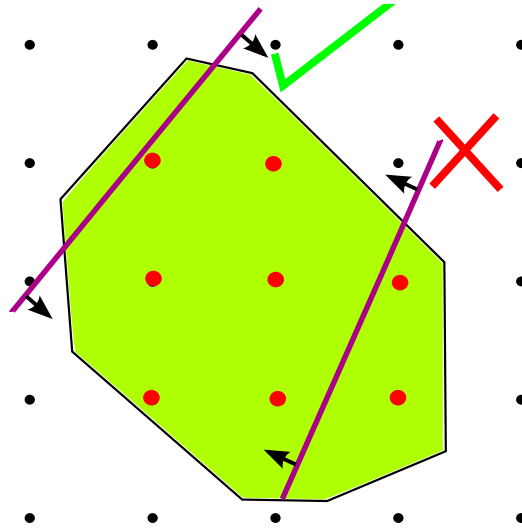


Figure 2: Definition of cutting plane

The main idea of the cutting plane algorithm for solving *linear* MIPs is to iteratively solve a simpler problem: We start by solving the linear program given by the continuous relaxation K . If the optimal solution is an integral vector, we stop, since it is feasible for the *linear* MIP. Otherwise, we add a cutting plane in order to cut off the fractional optimal solution, and then we iteratively repeat this process. Figure 3 shows a fractional optimal solution to the linear programming relaxation and the corresponding cutting plane used to cut off this fractional vector from the integer hull of the feasible region.

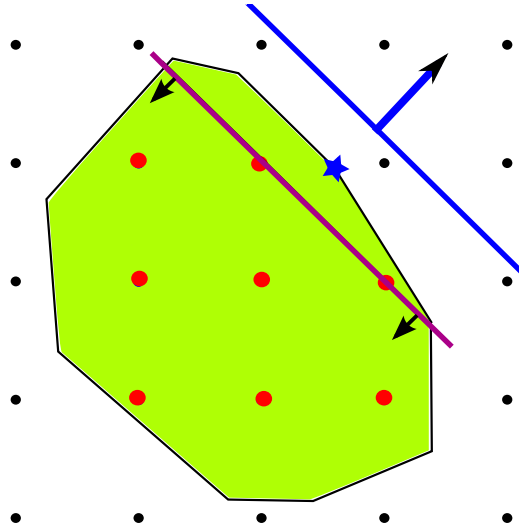


Figure 3: Application of cutting planes

We briefly describe next two important properties of cutting planes for *linear* MIPs.

1.2.2.1 Duality for linear MIPs

Let $\Omega \subseteq \mathbb{R}^m$, and $g : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$ be a function. Then we say that g is a subadditive function if for all $u, v \in \Omega$ s.t. $u + v \in \Omega$, we have $g(u + v) \leq g(u) + g(v)$. On the other hand, g is said to be nondecreasing if for $u, v \in \Omega$ such that $u \geq v$ we obtain $g(u) \geq g(v)$.

In the pure integer case ($q = 0$), it is known that a *linear* MIP of the form $\min\{c^T x \mid Ax \geq b, x \in \mathbb{Z}^p\}$ has a strong dual given by the optimization problem $\max\{g(b) \mid g \in \mathcal{F}\}$, where \mathcal{F} is a set of subadditive and nondecreasing functions (that depends on A and c). The following result is a consequence of strong duality: All cutting planes for a *linear* MIP are equal or dominated by a cutting plane of the form:

$$\sum_{i=1}^n g(A^i)x_i \geq g(b),$$

where A^i is the i th column of A and $g \in \mathcal{F}$ is a subadditive nondecreasing function.

This result implies that in order to solve *linear* MIPs, it suffices to consider cutting planes that are given by a subadditive functions. Moreover, such functions g that are independent of the data A, b can be use to generate cutting planes for arbitrary *linear* MIPs by using the formula above. A remarkable example is the case of the extremely useful Gomory Mixed-integer cuts (GMI cuts), that can be explicitly computed by using subadditive functions.

1.2.2.2 Cutting planes from lattice-free sets

In the pure integer case ($q = 0$) we have that $K = \{x \in \mathbb{R}^p \mid Ax \geq b\}$. In order to compute cutting planes for K we can consider a geometric approach that we will describe next. A convex set is called *lattice-free* if it contains no integer points in its interior. A *maximal lattice-free convex set* is a lattice-free convex set not strictly contained in any lattice-free convex set. Now, given a lattice-free set B , in order to compute cutting planes, we consider the following relaxation of the integer hull of K :

$$\text{conv}(K \setminus \text{int}(B)).$$

Observe that since B is a lattice-free set, this relaxation contains exactly the same integer points as the polyhedron K . Depending on the structure of the set B obtaining cutting planes by using this relaxation may be more useful than just considering the original continuous relaxation K . Notice that the largest the set B the better this relaxation approximates the integer hull of K , and thus, we obtain stronger cuts. Consequently, we are interested in studying maximal lattice-free sets. Lovász [65] and Basu et al. [13] proved that maximal lattice-free convex sets

are polyhedra. A remarkable example of maximal lattice-free convex sets are the split sets. A split set is a polyhedral lattice-free set defined by two parallel hyperplanes, as shown in Figure 4. In the figure we also observe a cutting plane obtained by using a split set.

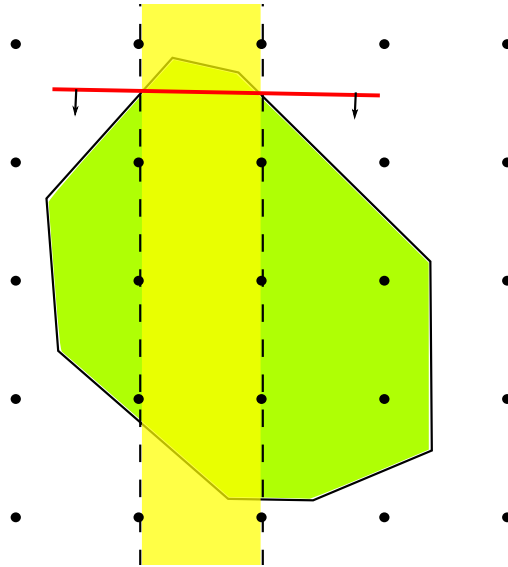


Figure 4: Example of Split cut

It can be proven that Splits cuts are in some sense equivalent to GMI cuts, and thus, split cuts are also a very important class of cutting planes from a practical point of view.

1.2.3 From *linear* MIPs to *convex* MIPs.

As illustrated in the previous sections, there are a number of properties of *linear* MIPs that have been studied in the literature. Hence, to some extent these optimization problems are well-understood. Moreover, we have a understanding of which properties have had a greater impact in the advance of *linear* MIPs, both from a theoretical and practical point of view. This knowledge may help us to determine the crucial properties of *convex* MIPs that need to be studied. The idea is not only to better understand the more complicated case of general *convex* MIPs,

but at the same time to push the limits of what we know in the case of *linear* MIPs, and thus gain a better understanding of this simpler case as well.

1.3 Contributions of this thesis

In this thesis we study *convex* MIPs, that is, K in (1) is a general convex set. As in the *linear* MIP case, these problems have a number of practical applications; however, general *convex* MIPs are not well understood. Motivated by this fact, we investigated various concepts that are relevant to the design of optimization algorithms for *nonlinear convex* MIPs, such as properties of their feasible region and properties of cutting planes. The main contribution of our work is the extension of several fundamental results from the theory of *linear* MIPs to the case of *convex* MIPs.

1.3.1 Structural properties of *Convex* MIPs

We start by studying the structure of the convex hull of feasible solutions to (1), that is, the integer hull of K . As discussed in the previous section, the integer hull of a *linear* MIP is a rational polyhedron. In the case of (nonlinear) *convex* MIPs this is unlikely: in some cases, the convex hull of solutions to a *convex* MIP is not even a closed set. This fact motivated us to study the structure of the feasible region of *convex* MIPs. Our research contribution to this topic is twofold:

- **General characterizations:** We found necessary and sufficient conditions for the integer hull of general closed convex sets to be ‘nice’, that is, to be either a closed set or a (rational) polyhedron.
- **Good characterizations:** A basic question is whether the conditions in these characterizations of ‘nice’ integer hulls are ‘easy’ to check. The representability results we obtained for general convex sets are quite general and the corresponding conditions are involved. This motivated us to study nonlinear

convex sets with a particular structure in order to obtain simpler conditions for the closedness of their integer hulls.

Next we will give a more elaborated description of our results.

1.3.1.1 General characterizations

This section is based on the following papers: “Some properties of convex hulls of integer points contained in general convex sets” [35] and “A Polynomial-Time Algorithm to Check Closedness of Simple Second Order Mixed-Integer Sets” [69]. In these papers we studied some structural properties of the convex hull of feasible solutions to (1) in the case K is a general closed convex set. In Figure 5 we show a convex set that posses a nonpolyhedral integer hull.

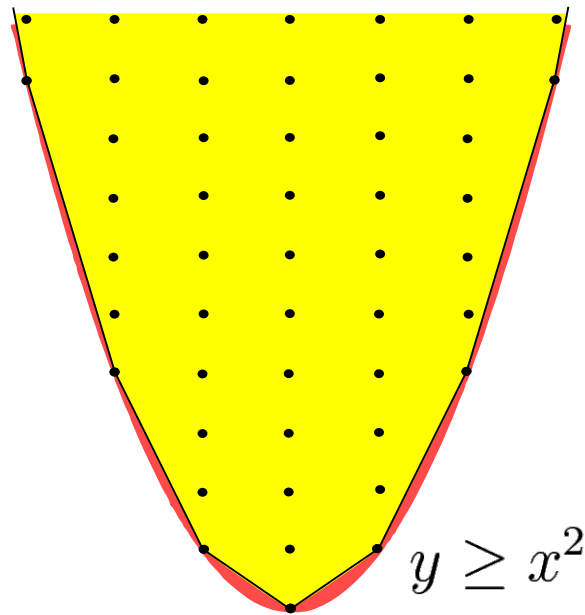


Figure 5: Integer hull of a quadratic nonlinear convex set

In [35] some properties of closedness of integer hulls are presented for the pure integer case ($q = 0$ in (1), that is, no continuous variables). In [69] we show the extension of some of these results to the case of a general mixed-integer lattice ($q \geq 1$ in (1)). We now summarize our main contributions:

- **Closedness:** We state some necessary and sufficient conditions for the mixed-integer hull of closed convex sets to be closed. The fundamental result is a characterization of the closedness of mixed-integer hulls in the case of closed convex sets with no lines (Section 2.3.1). We then use this result to study the closedness of the mixed-integer hull of some special class of convex sets: (1) Closed convex sets with no lines that contain a mixed-integer point in their interior (Section 2.3.2); (2) Strictly convex sets (Section 2.3.3); (3) Pointed closed convex cones (Section 2.3.4), and (4) Closed convex sets with lines (Section 2.3.5).
- **Polyhedrality:** In the special case of $\mathcal{L} = \mathbb{Z}^n$, we present some necessary and sufficient conditions for the integer hull of closed convex sets to be polyhedra. The main result of this section is a sufficient condition for the integer hull of a convex set to be a polyhedron. We also show that this condition is necessary in the case that the closed convex set contains an integer point in its interior (Section 2.4).
- **Operations that preserve closedness of mixed-integer hulls:** We study whether some operations (like intersection, union, Minkowski's sum, affine linear mappings, and preimages of affine linear mappings) applied to a closed convex set that possesses a closed mixed-integer hull preserve the closedness of the mixed-integer hull. Our main results are: (1) We verify that under some specific conditions, affine mappings maintain closedness of mixed-integer hulls (Section 3.3.5). (2) We show in the pure integer case ($\mathcal{L} = \mathbb{Z}^n$) that if the integer hulls of two closed convex sets are closed and the lineality space of their intersection is rational, then the integer hull of their intersection is also closed (Section 3.6).

1.3.1.2 Good characterizations

This section is based in the following paper: “A Polynomial-Time Algorithm to Check Closedness of Simple Second Order Mixed-Integer Sets” [69]. In this paper we studied a particular class of *conic* MIPs and proved that the closedness characterization can be simplified. Furthermore, the conditions in this characterization can be checked in polynomial time, an interesting fact that is somewhat unexpected to be true for integer hulls of nonlinear sets.

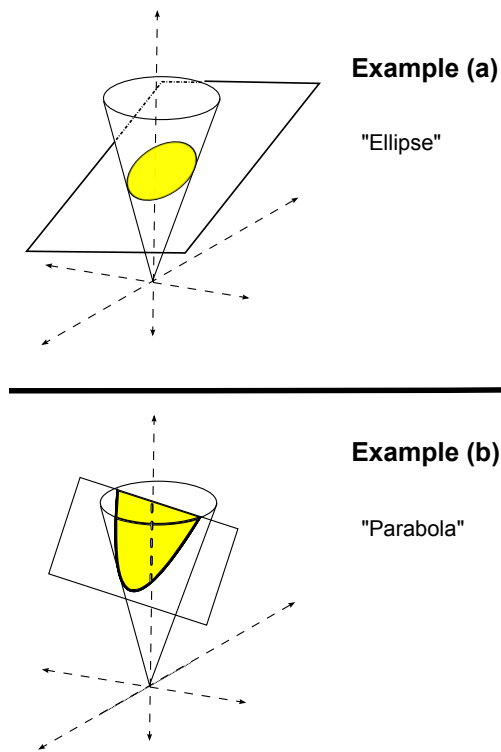


Figure 6: Examples of *Conic* MIPs

In Figure 6 some examples of *Conic* MIPS, that is, MIPS for which their continuous relaxation are sets of the form $K = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \mid Ax + By - b \in C\}$, are shown. (Note: The sets are drawn in \mathbb{R}^t , that is, after applying the mapping $(x, y) \mapsto Ax + By - b$.)

We now summarize our main contributions:

- **Closedness:** we applied some of the results given in [35] to the special case of a ‘simple’ *conic* MIP (that is, K is the feasible region of a conic program). In particular, we presented a simpler characterization of the closedness of the convex hull of feasible solutions to this simple mixed-integer nonlinear set (Section 3.3).
- **Checking closedness in polynomial time:** We showed that the conditions given in the characterization above can be checked in polynomial time with respect to the size of the data defining the problem if the conic program satisfies some additional conditions (Section 3.4).
- **Special case of second order cones:** We proved that the Lorentz cone that satisfies these additional conditions (Section 3.5). This result provides a polynomial time algorithm to check the closedness of integer hulls of simple second order conic programs (Section 3.5).

1.3.2 Properties of Cutting planes for *Convex* MIPs

As a second stream of research, we studied properties of cutting planes for *convex* MIPs. Since the success of this tool in the case of *linear* MIPs can be explained by the fact that properties of cutting planes are well-understood in this setting, it becomes crucial to extend this knowledge to the case of general *convex* MIPs. As part of this work we have the following contributions:

- **S -free convex sets:** In [68] we showed that all maximal S -free convex sets are polyhedra, where $S = \mathbf{P} \cap \mathbb{Z}^n$ for some general convex set \mathbf{P} . This result is an extension of an already known complete characterization of maximal S -free convex sets in the special case the set \mathbf{P} is a rational polyhedron.
- **Strong dual for *conic* MIPs:** In [70] we showed that in the special case that K is the feasible region of a general conic program, the problem (1) possesses

a strong dual problem.

- **Generalizations of Split cuts** In [32] we studied some properties of cross cuts, a class of cutting planes for *linear* MIPs that is a generalization of splits cuts (a class of cutting planes that is extremely useful in practice).

1.3.2.1 Cutting planes from S -free sets

This section is based in the following paper: “On maximal S -free convex sets” [68].

Consider in a general *convex* MIP in (1) where $K = \mathbf{P}$ is a general convex set (not necessarily closed). Let S be the set of integer points contained in \mathbf{P} , that is $S = \mathbf{P} \cap \mathbb{Z}^n$. A set K is called *S -free convex set* if $\text{int}(K) \cap S = \emptyset$. Hence the concept of S -free convex sets represents a generalization of the concept of lattice-free convex sets. *Maximal S -free convex sets* are defined analogously to maximal lattice-free convex sets.

In this setting, cutting planes for (1) can be obtained by considering relaxations of \mathbf{P} of the form

$$\text{conv}(\mathbf{P} \setminus (\text{int}(B) \times \mathbb{R}^q)),$$

where $B \subseteq \mathbb{R}^p$ is a S -free convex set with $S = \{x \in \mathbb{Z}^p \mid (x, y) \in \mathbf{P}\}$. Figure 7 show an illustration of this procedure in the pure integer case ($q = 0$).

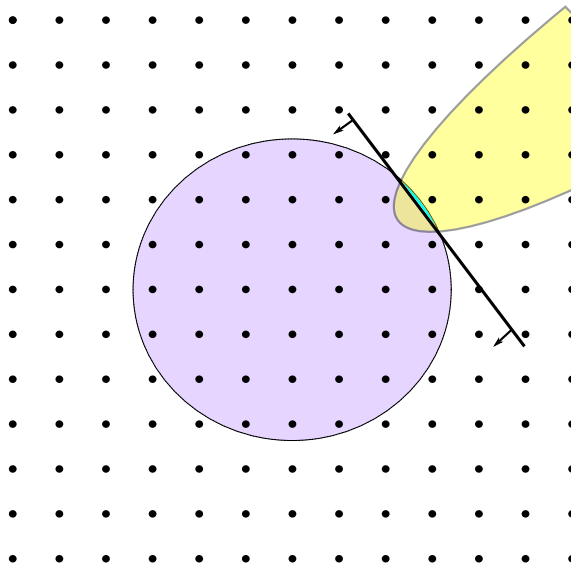


Figure 7: Cutting plane generated by S -free convex set

Observe that we do not require the set B to be a lattice-free set, since in order to obtain cutting planes for \mathbf{P} by using the set B it suffices that B does not contain any integer point of S in its interior. On the other hand, it is clear that the bigger the set, the better the cutting planes we can obtain. Thus, we are interested in studying maximal S -free convex sets. The fact that maximal S -free convex sets are polyhedra is of interest for the following reason: if the set $B \times \mathbb{R}^q$ is a polyhedron, then the set $\text{conv}(K \setminus (\text{int}(B) \times \mathbb{R}^q))$ is somewhat ‘simple’. Therefore, cutting planes are likely to be characterized in an ‘easy’ way.

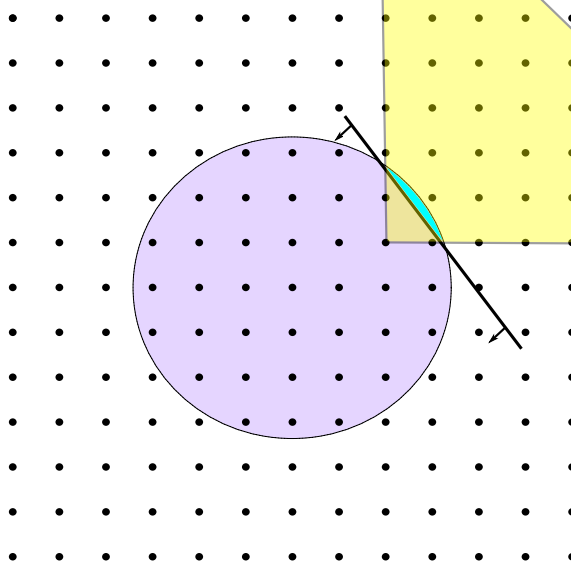


Figure 8: Cutting plane generated by a maximal S -free convex set

Figure 8 shows a cutting plane generated by a (polyhedral) maximal S -free convex set that contains the S -free convex set shown in Figure 7.

We now summarize our main contributions:

- **Polyhedrality of maximal S -free convex sets:** Let $S \subseteq \mathbb{Z}^n$ be the set of all integer points contained in a convex set \mathbf{P} . We prove that all maximal S -free convex sets are polyhedra (Section 4.3.1). Surprisingly, this result only use convexity of the original set (no need to be closed).
- **Structure of facets of maximal S -free convex sets:** Let $K \subseteq \mathbb{R}^n$ be a full-dimensional maximal S -free polyhedron, where S is defined as above. We show that for each facet of K either there is a point of S in its relative interior or there are points of S arbitrarily close to its relative interior. We also verify that the number of facets of K is at most 2^n (Section 4.3.2).
- **A property of the lineality space of maximal S -free convex sets:** In the case $\mathbf{P} = \mathbb{R}^n$ or when \mathbf{P} is a rational polyhedron, the lineality space of any maximal

S -free convex set K satisfies a nice property: if $K \cap \text{conv}(S)$ has nonempty interior, and r belongs to the recession cone of $\mathbf{P} \cap K$, then $-r$ belongs to the recession cone of K . There are some examples that show that this nice property is not necessarily true in the case of a general convex set \mathbf{P} . We state a sufficient condition for the convex set \mathbf{P} so that all maximal S -free convex sets have a lineality space satisfying this nice property (Section 4.4.1).

1.3.2.2 Duality for conic MIPs

This section is based on the paper “A Strong Dual for Conic Mixed-Integer Programs” [70]. In [70] we showed that in the special case that K is the feasible region of a general conic program, the problem (1) possesses a strong dual problem. This dual is a generalization of the well-known subadditive dual for *linear* MIPs. The motivation for this result is that in the *linear* MIP case there are some dual feasible solutions that can be used to generate very ‘good’ cutting planes. Our main result can be summarized as follows:

- **Subadditive dual for *conic* MIPs:** We extend the well-known subadditive dual for *linear* MIPs to the case of *conic* MIPs. In particular, under a simple condition we are able to prove strong duality. This condition plays a similar role as the assumption of rational data in the case of *linear* MIPs in the proof of the strong duality result (Chapter 5).

1.3.2.3 Cutting planes from cross cuts

In this section we study some properties of cutting planes for *linear* MIPs. This section is based on the paper “On some generalizations of split cuts.” [32]. In [32], we studied some generalizations of split cuts, namely, cross cuts. Cross cuts are obtained in a similar way as split cuts are obtained. More precisely, cross cuts are

cutting planes obtained by considering the following relaxation

$$\text{conv}(K \setminus (\text{int}(S_1) \cup \text{int}(S_2))),$$

where S_1, S_2 are split sets. Figure 9 illustrates a cross cut.

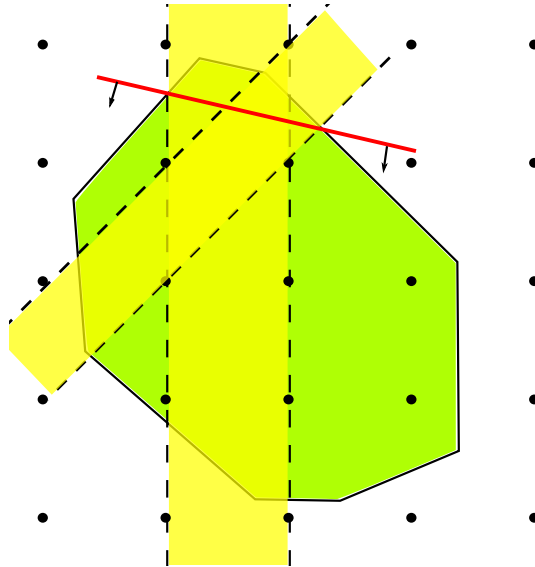


Figure 9: Example of Cross cut

Our main result can be summarized as follows:

- **Polyhedrality of the cross closure:** As a generalization of a property of split cuts, we showed that the cross closure of a rational polyhedron P - that is, the set of points in P satisfying all cross cuts for P - is again a rational polyhedron (Chapter 6).

1.4 Notation and some definitions

In this section we collect some of the notation and definitions that are used consistently throughout the chapters of this thesis. We note here that the chapters are self-contained in terms of notation, hence the purpose of this section is to serve as a quick reference.

1.4.1 Basic notation

- For $a, b \in \mathbb{R}^n$ we denote
 - $a^T b$: the standard scalar product of these two vectors.
 - $\|a\|$: the norm corresponding to this scalar product, that is, $\|a\| = \sqrt{a^T a}$ (Euclidean norm).
 - For $\epsilon > 0$, we use the notation $B(a, \epsilon)$ to denote the set $\{x \in \mathbb{R}^n \mid \|x - a\| \leq \epsilon\}$.
- Let $K \subseteq \mathbb{R}^n$. Then
 - \overline{K} : the closure of K .
 - $\text{int}(K)$: the interior of K .
 - $\text{rel.int}(K)$: the relative interior of K .
 - $\text{bd}(K)$: the boundary of K , that is, $\text{bd}(K) = \overline{K} \setminus \text{int}(K)$.
 - $\text{rel.bd}(X)$: the relative boundary of K , that is, $\text{rel.bd}(X) = \overline{X} \setminus \text{rel.int}(X)$.
 - $\text{dim}(K)$: the dimension of K .
 - $\text{conv}(K)$: the convex hull of K , given by $\text{conv}(K) = \{\sum_{i=1}^N \lambda_i x_i \mid \lambda_i \geq 0, x_i \in K, \forall i, \sum_i \lambda_i = 1, N \in \mathbb{N}\}$.
 - $\overline{\text{conv}}(K)$: the closure of $\text{conv}(K)$.
 - $\text{aff}(K)$: the affine hull of K , given by $\text{conv}(K) = \{\sum_{i=1}^N \lambda_i x_i \mid \sum_i \lambda_i = 1, N \in \mathbb{N}\}$.
 - $\text{cone}(K)$: the conic hull of K , given by $\text{cone}(K) = \{\sum_{i=1}^N \lambda_i x_i \mid \lambda_i \geq 0, x_i \in K, \forall i, N \in \mathbb{N}\}$.
 - $\text{rec}(K)$: the recession cone of K . We use the definition of recession cone for general convex sets given in Section 8 of [74]: $\text{rec}(K) = \{d \in \mathbb{R}^n \mid x + \lambda d \in K \forall x \in K, \forall \lambda \geq 0\}$.

– $\text{lin.space}(K)$: the lineality space of K , given by $\text{lin.space}(K) = \{d \in \mathbb{R}^n \mid x + \lambda d \in K, \forall x \in K, \forall \lambda \in \mathbb{R}\}$

• For a linear subspace $L \subseteq \mathbb{R}^n$ we denote:

– $L^\perp \subseteq \mathbb{R}^n$ the orthogonal subspace to L .

– $\text{Proj}_{L^\perp}(\cdot)$ the orthogonal projection onto L .

We also say that L is a rational (linear) subspace if there exists a basis of L formed by rational vectors.

1.4.2 Some definitions

Definition 1. A strictly convex set is a convex set S such that for all $x, y \in S$, $x \neq y$ and for all $\alpha \in (0, 1)$ we have $\alpha x + (1 - \alpha)y \in \text{rel.int}(S)$.

Definition 2 (Lattice-free convex sets). A set $K \subseteq \mathbb{R}^n$ is called lattice-free convex set if K is convex and $\text{rel.int}(K) \cap \mathbb{Z}^n = \emptyset$. A convex set K is called maximal lattice-free convex set if K is a lattice-free convex set and there does not exist a set $K' \subseteq \mathbb{R}^n$ such that $K' \neq K$, $K' \supseteq K$ and K' is a lattice-free convex set.

CHAPTER II

PROPERTIES OF INTEGER HULLS OF CLOSED CONVEX SETS

2.1 *Introduction*

An important goal in the study of mathematical programming is to analyze properties of the convex hull of feasible solutions. The Fundamental Theorem of Integer Programming (see Section 2.5 in [28]), due to Meyer [67], states that the convex hull of feasible points in a mixed integer linear set defined by rational data is a polyhedron. The proof of this result relies on (i) the Minkowski-Weyl Representation Theorem for polyhedra and (ii) the fact that the recession cone is a rational polyhedral cone and thus generated by a finite number of integer vectors. In the world of mixed integer linear programming (MILP) problems, these sufficient conditions for polyhedrality of the convex hull of feasible solutions are reasonable since we expect most instances to be described using rational data.

A convex integer program is an optimization problem where the feasible region is of the form $K \cap \mathbb{Z}^n$ where $K \subseteq \mathbb{R}^n$ is a closed convex set and \mathbb{Z}^n denotes the set of n -dimensional integral vectors. Let $\text{conv}(K \cap \mathbb{Z}^n)$ represent the convex hull of $K \cap \mathbb{Z}^n$. In this setting we do not have Minkowski-Weyl Representation Theorem for K or nice properties of recession cone of K . Therefore a natural question is to generalize Meyer's Theorem, in order to understand properties of the set K that lead to $\text{conv}(K \cap \mathbb{Z}^n)$ being a polyhedron. Note that [49] presents condition about the set $K \cap \mathbb{Z}^n$ (and more generally any subset of \mathbb{Z}^n) such that elements of $K \cap \mathbb{Z}^n$ have a finite integral generating set. In contrast, here we are interested in properties of the set K that allow us to deduce that $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron.

Observe that if $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron, then $\text{conv}(K \cap \mathbb{Z}^n)$ is a closed

set. To the best of our knowledge, even the basic question of conditions that lead to $\text{conv}(K \cap \mathbb{Z}^n)$ being closed is not well-understood. (See [71] for some sufficient conditions for closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ when K is a polyhedron that is not necessarily described by rational data). We therefore divide this chapter into two parts: (a) conditions for closedness and (b) conditions for polyhedrality. All the main results of these two parts are collected in Section 2.2.

In the first part of this chapter (Section 2.3), we present necessary and sufficient conditions for $\text{conv}(K \cap \mathbb{Z}^n)$ to be closed when K contains no lines (Theorem 1). This characterization also leads to useful results for special classes of convex sets such as sets containing integer points in their interior (Theorem 2), strictly convex sets (Theorem 3), and pointed cones (Theorem 4). The necessary and sufficient conditions we present in Theorem 2 generalize the result presented in [71]. The case where K contains lines is then dealt separately (Theorem 5).

In the second part of this chapter (Section 2.4), we present sufficient conditions for the convex hull of integer points contained in general convex sets to be polyhedra (Theorem 6). These sufficient conditions generalize the result presented in [67]. For a general convex set K containing at least one integer point in its interior, we show that these sufficient conditions are also necessary for $\text{conv}(K \cap \mathbb{Z}^n)$ to be a polyhedron (Theorem 6).

We conclude with some remarks in Section 2.5.

2.2 *Main results.*

2.2.1 Notation

We denote the standard scalar product on \mathbb{R}^n as $\cdot^T \cdot$ and the norm corresponding to this scalar product as $\|\cdot\|$. For $u \in \mathbb{R}^n$ and $\epsilon > 0$, we use the notation $B(u, \epsilon)$ to denote the set $\{x \in \mathbb{R}^n \mid \|x - u\| \leq \epsilon\}$. Let $K \subseteq \mathbb{R}^n$. In this chapter \overline{K} represents the closure of K , $\text{int}(K)$ represents the interior of K , $\text{bd}(K)$ denotes the boundary of K ,

$\text{rel.int}(K)$ denotes the relative interior of K , $\text{dim}(K)$ represents the dimension of K , $\text{rec}(K)$ represents the recession cone of K , $\text{lin.space}(K)$ represents the lineality space of K , $\text{conv}(K)$ denotes the convex hull of K , $\overline{\text{conv}}(K)$ denotes the closure of $\text{conv}(K)$ and $\text{aff}(K)$ represents the affine hull of K .

2.2.2 Results on closedness of $\text{conv}(K \cap \mathbb{Z}^n)$

Definition 3 ($u(K)$). *Given a convex set $K \subseteq \mathbb{R}^n$ and $u \in K \cap \mathbb{Z}^n$, we define $u(K) = \{d \in \mathbb{R}^n \mid u + \lambda d \in \text{conv}(K \cap \mathbb{Z}^n) \forall \lambda \geq 0\}$.*

The main result is the following characterization of closedness of $\text{conv}(K \cap \mathbb{Z}^n)$.

Theorem 1. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$.*

Furthermore, we present some refinements and consequences of this result when the closed convex set K contains an integer point in its interior (Theorem 2), K is a strictly closed convex set (Theorem 3) and K is a pointed closed cone (Theorem 4).

Theorem 2. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line and containing an integer point in its interior. Then the following are equivalent.*

1. $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.
2. $u(K) = \text{rec}(K) \forall u \in K \cap \mathbb{Z}^n$.
3. *The following property holds for every proper exposed face F of K : If $F \cap \mathbb{Z}^n \neq \emptyset$, then for all $u \in F \cap \mathbb{Z}^n$ and for all $r \in \text{rec.cone}(F)$, $\{u + \lambda r \mid \lambda \geq 0\} \subseteq \text{conv}(F \cap \mathbb{Z}^n)$.*

Theorem 3. *If $K \subseteq \mathbb{R}^n$ is a full-dimensional closed strictly convex set, then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.*

Theorem 4. *Let K be a full-dimensional pointed closed convex cone in \mathbb{R}^n . Then $\overline{\text{conv}}(K \cap \mathbb{Z}^n) = K$. In particular, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if every extreme ray of K is rational scalable (i.e. it can be scaled to be an integral vector).*

Finally, we present an extension of Theorem 1 for the case of the closed convex set K contains lines. Given $L \subseteq \mathbb{R}^n$ a linear subspace, we denote by L^\perp the linear subspace orthogonal to L and we denote by P_{L^\perp} the projection onto the set L^\perp . L is said to be a rational linear subspace if there exists a basis of L formed by rational vectors.

Theorem 5. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set such that the lineality space $L = \text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$ is not trivial. Then, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if the following two conditions hold:*

1. $\text{conv}(K \cap L^\perp \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n))$ is closed.
2. L is a rational linear subspace.

2.2.3 Results on polyhedrality of $\text{conv}(K \cap \mathbb{Z}^n)$

Definition 4 (Thin Convex set). *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. We say K is thin if the following holds for all $c \in \mathbb{R}^n$: $\min\{c^T x \mid x \in K\} = -\infty$ if and only if there exist $d \in \text{rec}(K)$ such that $d^T c < 0$.*

The main result is a sufficient and necessary condition for $\text{conv}(K \cap \mathbb{Z}^n)$ to be a polyhedron.

Theorem 6. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. If K is thin and recession cone of K is a rational polyhedral cone, then $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron. Moreover, if $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ and $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron, then K is thin and $\text{rec}(K)$ is a rational polyhedral cone.*

2.3 Closedness of $\text{conv}(K \cap \mathbb{Z}^n)$

Before presenting the results of this section, we develop some intuition by examining some examples.

Example 1. *If K is a bounded convex set, then $\text{conv}(K \cap \mathbb{Z}^n)$ is a polytope. Therefore properties of the recession cone play an important role in determining the closedness of $\text{conv}(K \cap \mathbb{Z}^n)$. Intuitively, it appears that irrational extreme recession directions of K may cause $\text{conv}(K \cap \mathbb{Z}^n)$ to be not closed. However this is not entirely true as illustrated in the next few examples.*

1. *First consider the set $K^1 = \{x \in \mathbb{R}^2 \mid x_2 - \sqrt{2}x_1 \leq 0, x_2 \geq 0\}$. It is easily verified that in this case $\text{conv}(K^1 \cap \mathbb{Z}^2)$ is not closed (see Theorem 2, Section 2.2; also see Figure 10). In particular, the half-line $\{x \in \mathbb{R}^2 \mid x_2 - \sqrt{2}x_1 = 0, x_2 > 0\}$ is contained in $\overline{\text{conv}(K^1 \cap \mathbb{Z}^2)}$ but not in $\text{conv}(K^1 \cap \mathbb{Z}^2)$. In this case it is clear that the irrational data describing the polyhedron causes $\text{conv}(K^1 \cap \mathbb{Z}^2)$ to be not closed.*

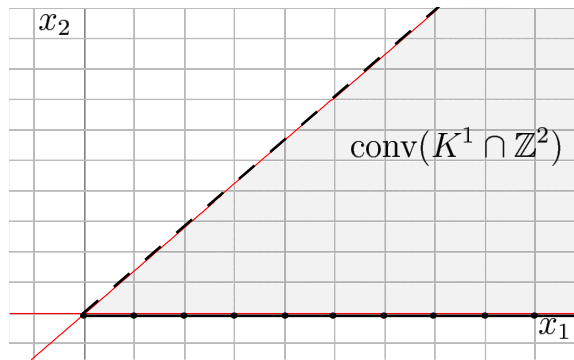


Figure 10: K^1 and $\text{conv}(K^1 \cap \mathbb{Z}^2)$.

2. *Now consider the set $K^2 = \{x \in \mathbb{R}^2 \mid x_2 - \sqrt{2}x_1 \leq 0, x_2 \geq 0, x_1 \geq 1\}$. Notice that the recession cone of K^1 and K^2 are the same. In fact $(K^1 \cap \mathbb{Z}^2) = (K^2 \cap \mathbb{Z}^2) \cup \{(0,0)\}$. However, we can verify (see Theorem 2, Section 2.2; also see Figure 11) that $\text{conv}(K^2 \cap \mathbb{Z}^2)$ is closed.*

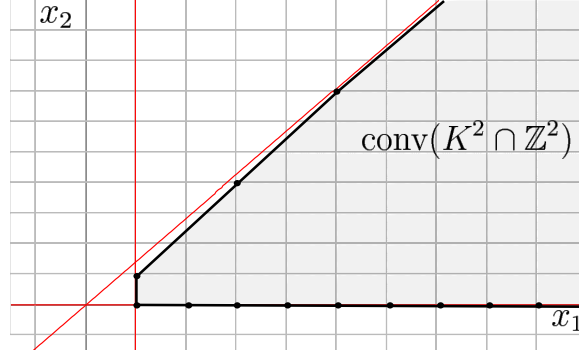


Figure 11: K^2 and $\text{conv}(K^2 \cap \mathbb{Z}^2)$.

We next illustrate a similar observation (i.e. recession cone of K has irrational extreme ray, but $\text{conv}(K \cap \mathbb{Z}^n)$ is closed) using non-polyhedral sets.

- Let $K^3 = \{x \in \mathbb{R}^2 \mid x_2 \geq x_1^2\}$. The recession cone of K^3 is $\{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\}$. It can be shown that $\text{conv}(K^3 \cap \mathbb{Z}^2)$ is closed (see Theorem 3, Section 2.2; also see Figure 12).

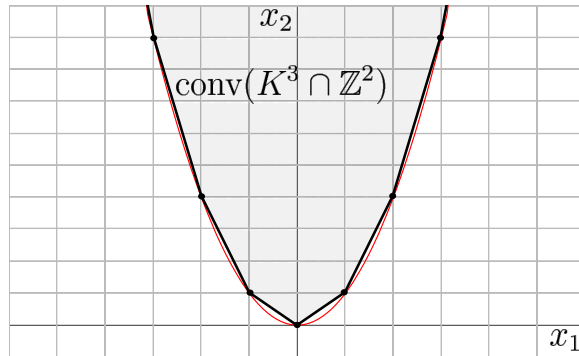


Figure 12: K^3 and $\text{conv}(K^3 \cap \mathbb{Z}^2)$.

- Now consider the set where we rotate the parabola K^3 such that the new recession cone is $\{x \in \mathbb{R}^2 \mid \sqrt{2}x_1 = x_2, x_2 \geq 0\}$, i.e., consider the set $K^4 = \{x \in \mathbb{R}^2 \mid \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} x \in K^3\}$. In this case, even though the recession cone is a non-rational polyhedral set, it can be verified that $\text{conv}(K^4 \cap \mathbb{Z}^2)$ is closed (see Theorem

3, Section 2.2).

Observe that all the sets discussed above have polyhedral recession cones. However, sets whose recession cone are non-polyhedral can also lead to $\text{conv}(K \cap \mathbb{Z}^n)$ being closed.

5. Consider the set $K^5 = \{(0, 0, 1)\} \cup \{(0, 1, 1)\} \cup \left\{ \left(\frac{1}{n}, \frac{1}{n^2}, 1 \right) \mid n \in \mathbb{Z}, n \geq 1 \right\}$. Then K^5 is closed, since it contains all its limit points. Therefore, K^5 is a compact set and thus $\text{conv}(K^5)$ is compact (Theorem 17.2 [74]). Therefore, we obtain that the set $K^6 = \text{conv}\left(\left\{ \sum_{u \in K^5} \lambda_u u \mid \lambda_u \geq 0 \forall u \in K^5 \right\}\right)$ is a closed convex cone. Finally, it can be verified that $\text{conv}(K^6 \cap \mathbb{Z}^3) = K^6$ is closed (see Theorem 4, Section 2.2).

2.3.1 Necessary and sufficient conditions for closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ for sets with no lines

In this section we will prove the following result (for the definition of $u(K)$ see Definition 3 in Section 2.2).

Theorem 1. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$.*

Note here that when $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$, Theorem 1 implies that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed and therefore we obtain $u(K) = \text{rec}(\text{conv}(K \cap \mathbb{Z}^n))$ is closed for every $u \in K \cap \mathbb{Z}^n$.

It is not difficult to verify that,

$$\text{conv}(K \cap \mathbb{Z}^n) = \text{conv}\left(\bigcup_{u \in K \cap \mathbb{Z}^n} (u + u(K))\right). \quad (2)$$

Hence Theorem 1 states that if the recession cone of each $u + u(K)$ is identical, then the convex hull of their union is closed. Therefore Theorem 1 is very similar in flavor to the following result.

Lemma 1 (Corollary 9.8.1 in [74]). *If K_1, \dots, K_m are non-empty closed convex sets in \mathbb{R}^n all having the same recession cone C , then $\text{conv}(K_1 \cup \dots \cup K_m)$ is closed and has C as its recession cone.*

Note however that Lemma 1 is not directly useful in verifying the ‘sufficient part’ of Theorem 1 since the number of integer points in a general convex set is not necessarily finite and thus the union in the right-hand-side of equation (2) is possibly over a countably infinite number of sets. Lemma 1 does not extend to infinite unions, in fact it does not hold even if the individual sets are polyhedra with the same recession cone. (Consider for example $\text{conv}(\bigcup_{i \in \mathbb{Z}, i \geq 1} K_i)$ where $K_i = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = \frac{1}{i}, x_2 \geq 0\}$.) However, we note here that the proof of Theorem 2 presented here will eventually use Lemma 1 in some cases, by suitably converting the set $\text{conv}(\bigcup_{u \in K \cap \mathbb{Z}^n} (u + u(K)))$ to the convex hull of the union of a finite number of appropriate sets.

We begin by presenting some results that are required for the proof of Theorem 1.

Lemma 2 (Corollary 8.3.1 in [74]). *Let $K \subseteq \mathbb{R}^n$ be a convex set. Then*

$$\text{rec}(\text{rel.int}(K)) = \text{rec}(\overline{K}) \supseteq \text{rec}(K).$$

The following crucial result is a direct consequence of Theorem 3.5 in [61].

Lemma 3 ([61]). *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed set. Then every extreme point of $\overline{\text{conv}}(K)$ belongs to K .*

Lemma 4. *Let U be a $n \times n$ unimodular matrix and let $K \subseteq \mathbb{R}^n$ be a closed convex set. Then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if $\text{conv}((UK) \cap \mathbb{Z}^n)$ is closed.*

Theorem 7 (Theorem 18.5 [74]). *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Let S be the set of extreme points of K and let D be the set of extreme rays of $\text{rec}(K)$. Then $K = \text{conv}(S) + \text{cone}(D)$. (Where $\text{cone}(D) = \{\sum_{i=1}^N \mu_i d_i \mid N \in \mathbb{Z}_+, d_i \in D, \mu_i \geq 0, \forall i = 1, \dots, N\}$.)*

A convex set $K \subseteq \mathbb{R}^n$ is called *lattice-free*, if $\text{int}(K) \cap \mathbb{Z}^n = \emptyset$. A lattice-free convex set $K \subseteq \mathbb{R}^n$ is called *maximal lattice-free convex set* if there does not exist a lattice-free convex set $K' \subseteq \mathbb{R}^n$ satisfying $K' \supsetneq K$.

We note here that every lattice-free convex set is contained in a maximal lattice-free convex set. The following characterization of maximal lattice-free convex set is from [65]. See also [13] for a related result.

Theorem 8 ([13], [65]). *A full-dimensional lattice-free convex set $Q \subseteq \mathbb{R}^n$ is a maximal lattice-free convex set if and only if Q is a polyhedron of the form $Q = P + L$, where P is a polytope and L is a rational linear subspace and every facet of Q contains a point of \mathbb{Z}^n in its relative interior.*

We now present the proof of the main result of this section.

Proof. of Theorem 1 If $K \cap \mathbb{Z}^n = \emptyset$, then the result is trivial. Therefore, we will assume that $K \cap \mathbb{Z}^n \neq \emptyset$.

Let us prove “ \Rightarrow ”. If $\text{conv}(K \cap \mathbb{Z}^n)$ is closed, then $\forall u \in K \cap \mathbb{Z}^n$, $u(K) = \text{rec}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Thus $u(K)$ is identical for all $u \in K \cap \mathbb{Z}^n$.

Let us prove “ \Leftarrow ”. Observe that for all $u \in K \cap \mathbb{Z}^n$ we have that

$$\text{rec}(\text{conv}(K \cap \mathbb{Z}^n)) \subseteq u(K) \subseteq \text{rec}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)). \quad (3)$$

The first inclusion follows directly by definition of $\text{rec}(\text{conv}(K \cap \mathbb{Z}^n))$ and $u(K)$. The second inclusion is due to the fact that for a closed convex set, its recession cone gives the recession directions for every point in the set.

Assume now that $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$. We first claim that $u(K) = \text{rec}(\text{conv}(K \cap \mathbb{Z}^n)) \forall u \in K \cap \mathbb{Z}^n$. Let $r \in u(K)$ and $x \in \text{conv}(K \cap \mathbb{Z}^n)$. We can write $x = \sum_{i=1}^N \alpha_i z_i$, where $z_i \in K \cap \mathbb{Z}^n$, $\alpha_i \geq 0$ for all $i = 1, \dots, N$ and $\sum_{i=1}^N \alpha_i = 1$. Since $r \in z_i(K)$, $\forall i = 1, \dots, N$, we have $z_i + \lambda r \in \text{conv}(K \cap \mathbb{Z}^n)$ for all $\lambda \geq 0$. Since $x + \lambda r = \sum_{i=1}^N \alpha_i (z_i + \lambda r)$, we obtain that $x + \lambda r \in \text{conv}(K \cap \mathbb{Z}^n)$ for all $\lambda \geq 0$. Thus, $u(K) \subseteq \text{rec}(\text{conv}(K \cap \mathbb{Z}^n))$ and by (3) we obtain that

$$u(K) = \text{rec}(\text{conv}(K \cap \mathbb{Z}^n)) \quad \forall u \in K \cap \mathbb{Z}^n. \quad (4)$$

We will now show that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. There are two cases:

- **Case 1:** $\text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n \neq \emptyset$. We will verify that $\text{conv}(K \cap \mathbb{Z}^n) \supseteq \overline{\text{conv}}(K \cap \mathbb{Z}^n)$. Let $u \in \text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n$. Since $u \in \text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n$ we obtain that if $r \in \text{rec}(\text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)))$, then $u + \lambda r \in \text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)) \subseteq \text{conv}(K \cap \mathbb{Z}^n)$ for all $\lambda \geq 0$. Therefore, $\text{rec}(\text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n))) \subseteq u(K)$. Since $\text{rec}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = \text{rec}(\text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)))$ (by Lemma 2), by using (3) we conclude that $u(K) = \text{rec}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Therefore, by using (4) we obtain that $\text{rec}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Observe that, by Lemma 3, the extreme points of $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ belong to $K \cap \mathbb{Z}^n$. Since $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \subseteq K$, it does not contain any lines. Thus, by Theorem 7, $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ is given by the convex hull of its extreme points plus its recession cone. Since the extreme points of $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ belongs to $\text{conv}(K \cap \mathbb{Z}^n)$ and $\text{rec}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$, we obtain that $\text{conv}(K \cap \mathbb{Z}^n) \supseteq \overline{\text{conv}}(K \cap \mathbb{Z}^n)$. Therefore, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.
- **Case 2:** $\text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n = \emptyset$. We will use induction on the dimension of $\text{conv}(K \cap \mathbb{Z}^n)$. The base case, $\dim(\text{conv}(K \cap \mathbb{Z}^n)) = 1$ is straightforward to verify.

Suppose now the property is true for every closed convex set K' such that $\dim(\text{conv}(K' \cap \mathbb{Z}^n)) < \dim(\text{conv}(K \cap \mathbb{Z}^n))$ and $\text{rel.int}(\text{conv}(K' \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n = \emptyset$.

First for convenience, notice that we may assume $K = K \cap \text{aff}(K \cap \mathbb{Z}^n)$. Therefore $\dim(K \cap \mathbb{Z}^n) = \dim(K)$. Let $z \in K \cap \mathbb{Z}^n$. We now translate K as $K - \{z\}$ and note that it is sufficient to show that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed for this new set K . Observe that $\text{aff}(K \cap \mathbb{Z}^n)$ is a rational linear subspace, since it is generated by integer vectors. Now by selecting a suitable unimodular matrix (see [76]) and by the application of Lemma 4, we may assume that $\text{aff}(K \cap \mathbb{Z}^n)$ is of the form $\{x \mid x_i = 0 \ \forall i = k + 1, \dots, n\}$. Finally, we can project out the last $n - k$ components (every point in K has zero in these components) and note that

it is sufficient to show that $\text{conv}(K \cap \mathbb{Z}^k)$ is closed for this new set $K \subseteq \mathbb{R}^k$. In particular, without loss of generality, we may assume that $\text{conv}(K \cap \mathbb{Z}^n)$ is full-dimensional.

Note now that $\text{conv}(K \cap \mathbb{Z}^n)$ is lattice-free, and therefore there exists a full-dimensional maximal lattice-free polyhedron $Q \subseteq \mathbb{R}^n$ such that $\text{conv}(K \cap \mathbb{Z}^n) \subseteq Q$ and $Q = P + L$, where P is a polytope and L is a rational linear subspace.

Let F_i , $i = 1, \dots, N$ be the facets of Q such that $K \cap F_i \cap \mathbb{Z}^n \neq \emptyset$. We will verify that

$$\text{conv}(K \cap \mathbb{Z}^n) \cap F_i = \text{conv}(K \cap F_i \cap \mathbb{Z}^n). \quad (5)$$

Since $\text{conv}(K \cap \mathbb{Z}^n) \cap F_i$ is a convex set and contains $K \cap F_i \cap \mathbb{Z}^n$ we have $\text{conv}(K \cap \mathbb{Z}^n) \cap F_i \supseteq \text{conv}(K \cap F_i \cap \mathbb{Z}^n)$. On the other hand, let $x \in \text{conv}(K \cap \mathbb{Z}^n) \cap F_i$. Therefore $x = \sum_{j=1}^M \alpha_j z_j$, where $z_j \in K \cap \mathbb{Z}^n$, $\alpha_j \geq 0$ for all $j = 1, \dots, M$, and $\sum_{j=1}^M \alpha_j = 1$. Since $K \cap \mathbb{Z}^n \subseteq Q$ and $x \in F_i$, we must have $z_j \in F_i$, $\forall j = 1, \dots, M$, so $x \in \text{conv}(K \cap F_i \cap \mathbb{Z}^n)$.

Next, for all $i = 1, \dots, N$, we verify that

$$u(K \cap F_i) = u(K) \cap L \quad \forall u \in K \cap F_i \cap \mathbb{Z}^n. \quad (6)$$

Let $r \in u(K \cap F_i)$. Then, by definition we have that $u + \lambda r \in \text{conv}(K \cap F_i \cap \mathbb{Z}^n) \forall \lambda \geq 0$. By (5), this is equivalent to $u + \lambda r \in \text{conv}(K \cap \mathbb{Z}^n) \cap F_i \forall \lambda \geq 0$. This is also equivalent to $u + \lambda r \in \text{conv}(K \cap \mathbb{Z}^n) \forall \lambda \geq 0$ and $u + \lambda r \in F_i \forall \lambda \geq 0$. Thus equivalently we obtain that $r \in u(K)$ and $r \in \text{rec}(F_i)$. By the form of Q (see Theorem 8) we have, $\forall i = 1, \dots, N$, that $\text{rec}(F_i) = \text{rec}(Q) = L$. We obtain $r \in u(K) \cap L$. Therefore, we conclude $u(K \cap F_i) = u(K) \cap L$.

Since $u(K)$ is identical for all $u \in K \cap \mathbb{Z}^n$, (6) implies that $u(K \cap F_i)$ is identical for every $u \in K \cap F_i \cap \mathbb{Z}^n$ and $\forall i = 1, \dots, N$. Moreover, since $\text{conv}(K \cap F_i \cap \mathbb{Z}^n) \subseteq$

F_i , we obtain $\dim(\text{conv}(K \cap F_i \cap \mathbb{Z}^n)) < \dim(\text{conv}(K \cap \mathbb{Z}^n))$. So we can use either case 1 or the induction hypothesis to conclude that $\text{conv}(K \cap F_i \cap \mathbb{Z}^n)$ is a closed set.

We now have that $u(K \cap F_i) = \text{rec}(\text{conv}(K \cap F_i \cap \mathbb{Z}^n)) = u(K) \cap L$ for all $i = 1, \dots, N$. So the recession cone of $\text{conv}(K \cap F_i \cap \mathbb{Z}^n)$ is the same for all $i = 1, \dots, N$. Observe that,

$$\text{conv}(K \cap \mathbb{Z}^n) = \text{conv} \left[\bigcup_{i=1}^N \text{conv}(K \cap F_i \cap \mathbb{Z}^n) \right].$$

Since the convex hull of a finite union of closed convex sets with the same recession cone is closed (Lemma 1), we conclude that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. ■

We note here that the condition that K contains no line in the statement of Theorem 1 is not artificial. This is illustrated in the next Example.

Example 2. Consider the set $K^7 = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0.5 \leq x_2 - \sqrt{2}x_1 \leq 0.7\}$ which contains a line and let $u \in K^7 \cap \mathbb{Z}^2$. Since $\text{bd}(K^7) \cap \mathbb{Z}^2 = \emptyset$, we have that $u \in \text{int}(K^7)$. Since $u \in \text{int}(K^7)$, it can be verified that $u(K^7) = \text{rec}(K^7)$ (see Lemma 8 in Section 2.3.2). Thus $u(K^7)$ is identical for all $u \in K^7 \cap \mathbb{Z}^2$. However, $\text{conv}(K^7 \cap \mathbb{Z}^2)$ is not closed, since $\text{conv}(K^7 \cap \mathbb{Z}^2) = \text{int}(K^7)$. To see this, first observe that the lines defining the boundary of K^7 do not contain any integer point, so $\text{conv}(K^7 \cap \mathbb{Z}^2) \subseteq \text{int}(K^7)$. The other inclusion is a consequence of the Dirichlet Diophantine Approximation Theorem.

2.3.2 Closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ where $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$

In this section, we simplify the conditions of Theorem 1 for the case where $\text{int}(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n \neq \emptyset$. We will assume that K is full-dimensional throughout this section. In particular if K is not full-dimensional, then by application of Lemma 4 as in

the proof of Theorem 1, we can modify K and subsequently apply projection to achieve full-dimensionality of K and $K \cap \mathbb{Z}^n$.

In this section, we prove the following result.

Theorem 2. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line and containing an integer point in its interior. Then the following are equivalent.*

1. $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.
2. $u(K) = \text{rec}(K) \forall u \in K \cap \mathbb{Z}^n$.
3. *The following property holds for every proper exposed face F of K : If $F \cap \mathbb{Z}^n \neq \emptyset$, then for all $u \in F \cap \mathbb{Z}^n$ and for all $r \in \text{rec.cone}(F)$, $\{u + \lambda r \mid \lambda \geq 0\} \subseteq \text{conv}(F \cap \mathbb{Z}^n)$.*

Theorem 2 converts the question of verification of closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ to the verification of a somewhat simpler property of the faces of the set K . To see a simple application of Theorem 2 consider the cases (1.) and (2.) presented in Example 1. Note that both K^1 and K^2 contain integer points in their interior and $\text{conv}(K \cap \mathbb{Z}^n)$ is full-dimensional. In (1.), the facet $F := \{x \in \mathbb{R}^2 \mid x_2 = \sqrt{2}x_1, x_2 \geq 0\}$ contains only the point $(0,0)$ and thus does not satisfy the property presented in Theorem 2. Hence we deduce that $\text{conv}(K^1 \cap \mathbb{Z}^n)$ is not closed. On the other hand, since the facet $\{x \in \mathbb{R}^2 \mid x_2 = \sqrt{2}x_1, x_2 \geq 0, x_1 \geq 1\}$ contains no integer point and all other faces of K^2 also satisfy the property presented in Theorem 2, we can deduce that $\text{conv}(K^2 \cap \mathbb{Z}^n)$ is closed.

We note that Theorem 2 generalizes sufficient conditions for closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ presented in [71]. [71] shows that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if K is a polyhedron that contains no lines, $\text{rec}(K)$ is full-dimensional and every face of K satisfies the conditions described in the statement of Theorem 2. We note here that Theorem 2 is not true if the condition $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ is removed. The set K^{10} in Example 4 (Section 2.4) illustrates this.

Before we present the proof of Theorem 2, we first present a sequence of preliminary lemmas.

The following lemma is a direct consequence of the Dirichlet Diophantine Approximation Theorem (see Theorem 6.1 in [76]) and was proven in this form in [13].

Lemma 5 ([13]). *If $x \in \mathbb{Z}^n$ and $r \in \mathbb{R}^n$, then for all $\epsilon > 0$ and $\gamma \geq 0$, there exists a point of \mathbb{Z}^n at a distance less than ϵ from the half line $\{x + \lambda r \mid \lambda \geq \gamma\}$.*

Lemma 6. *Let $V \subseteq \mathbb{R}^n$ be a linear subspace of \mathbb{R}^n of dimension m . Let $\{a_1, \dots, a_m\}$ be a basis of V . Then there exists $\delta > 0$ such that if $b_i \in B(-a_i, \delta) \cap V$, for $i = 1, \dots, m$, then we have $0 \in \text{conv}(\{b_1, \dots, b_m, a_1, \dots, a_m\})$.*

Proof. Without loss of generality, we may assume $V = \mathbb{R}^n$ and that $\{a_1, \dots, a_n\}$ is the canonical basis of \mathbb{R}^n . Let $\delta > 0$, such that $\delta < \frac{1}{n}$. For $i = 1, \dots, n$, consider $b_i \in B(-a_i, \delta)$, that is, there exist $v_i \in \mathbb{R}^n$ with $\|v_i\| \leq 1$ such that $b_i = -a_i + \delta v_i$.

Let $\mu \in \mathbb{R}^n$ be the vector with all components equal to 1 and let $\lambda = \mu - \delta \sum_{i=1}^n v_i$. Since $\delta < \frac{1}{n}$ and $\|v_i\| \leq 1$, we have that $\lambda \geq 0$. We observe that

$$0 = \lambda - \mu + \delta \sum_{i=1}^n v_i = \sum_{i=1}^n \lambda_i a_i + \sum_{i=1}^n \mu_i b_i.$$

Thus, we conclude $0 \in \text{conv}(\{b_1, \dots, b_n, a_1, \dots, a_n\})$. ■

We will call a vector $r \in \mathbb{R}^n$ *rational scalable* if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\lambda r \in \mathbb{Z}^n$.

Lemma 7. *Let $r \in \mathbb{R}^n$ be a vector that is not rational scalable and $\gamma > 0$. Let \mathcal{P} be the projection of the set $\{x \in \mathbb{Z}^n \mid x^T r \geq \gamma\}$ on the subspace $r^\perp := \{x \in \mathbb{R}^n \mid x^T r = 0\}$. Then for all $\epsilon > 0$, $0 \in \text{conv}(B(0, \epsilon) \cap \mathcal{P})$.*

Proof. Let V be the linear subspace of r^\perp generated by $\text{int}(B(0, \epsilon)) \cap \mathcal{P}$ and let $m = \dim(V)$. Notice that since r is not rational scalable, by Lemma 5 we have that $m \geq 1$. Let $\{a_1, \dots, a_m\} \subseteq \text{int}(B(0, \epsilon)) \cap \mathcal{P}$ be a basis of V . For $i = 1, \dots, m$, let $x_i \in \mathbb{Z}^n$ such that a_i is the projection on r^\perp of x_i . Let $\delta > 0$ be such that if $b_i \in B(-a_i, \delta) \cap V$, for $i = 1, \dots, m$, then we have $0 \in \text{conv}(\{b_1, \dots, b_m, a_1, \dots, a_m\})$ (Lemma 6).

Since $\{-a_1, \dots, -a_m\} \subseteq \text{int}(B(0, \epsilon))$ there exists $r > 0$ such that $r \leq \delta$ and for all $i = 1, \dots, m$, $B(-a_i, r) \subseteq B(0, \epsilon)$. Since $-x_i \in \mathbb{Z}^n$, by Lemma 5 we obtain that for all $i = 1, \dots, m$ there exists $b_i \in \mathcal{P}$ at distance less or equal than r from $-a_i$. Thus, we have $b_i \in B(-a_i, \delta) \cap V$ and therefore, by the selection of δ , we obtain that

$$0 \in \text{conv}(\{b_1, \dots, b_m, a_1, \dots, a_m\}) \subseteq \text{conv}(B(0, \epsilon) \cap \mathcal{P}).$$

■

Lemma 8. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set, let $u \in K \cap \mathbb{Z}^n$ and let $d = \{u + \lambda r \mid \lambda > 0\} \subseteq \text{int}(K)$. Then $\{u\} \cup d \subseteq \text{conv}(K \cap \mathbb{Z}^n)$.*

Proof. If r is rational scalable, then the result is straightforward. Suppose therefore that r is not rational scalable. Also without loss of generality we may assume $\|r\| = 1$.

Observe that $u \in \text{conv}(K \cap \mathbb{Z}^n)$. Therefore it is sufficient to show that for $\gamma > 0$, $u + \gamma r \in \text{conv}(K \cap \mathbb{Z}^n)$. Note now that $d = \{u + \lambda r \mid \lambda > 0\} \subseteq \text{int}(K)$, is equivalent to $\exists \epsilon > 0$ such that $B(u + \lambda r, \epsilon) \subseteq K \forall \lambda \geq \gamma$. Without loss of generality we may assume that $u = 0$. We will show then that there exists $\mu \geq \gamma$ such that $\mu r \in \text{conv}(K \cap \mathbb{Z}^n)$. Let \mathcal{P} be the projection of $\{x \in \mathbb{Z}^n \mid r^T x \geq \gamma\}$ on the linear subspace $\{x \in \mathbb{R}^n \mid r^T x = 0\}$.

By Lemma 7, we have that $0 = \sum_{i=1}^p \lambda_i v^i$ where $v^i \in \mathcal{P} \cap B(0, \epsilon)$, $0 < \lambda_i \leq 1$ for all $i = 1, \dots, p$, $\sum_{i=1}^p \lambda_i = 1$. Let v^1, \dots, v^p be the projection of the integer points u^1, \dots, u^p where, for all $i = 1, \dots, p$, we have $u^i = v^i + \mu_i r \in \mathbb{Z}^n$ and $\mu_i \geq \gamma$. For all $i = 1, \dots, p$, since the distance between v^i and the half-line $\{\lambda r \mid \lambda \geq 0\}$ is less than ϵ and $\mu_i \geq \gamma$, we obtain that $u^i \in \{\lambda r \mid \lambda \geq \gamma\} + B(0, \epsilon) \subseteq K$. Therefore $u^i \in K \cap \mathbb{Z}^n$.

Now observe that

$$\sum_{i=1}^p \lambda_i (v^i + \mu_i r) = \sum_{i=1}^p \lambda_i v^i + r \sum_{i=1}^p \lambda_i \mu_i = r \sum_{i=1}^p \lambda_i \mu_i.$$

Since $\sum_{i=1}^p \lambda_i = 1$, we obtain that $\sum_{i=1}^p \lambda_i \mu_i \geq \gamma$. Thus, a point of the form μr where $\mu \geq \gamma$ belongs to $\text{conv}(K \cap \mathbb{Z}^n)$, completing the proof. \blacksquare

Now we have all the tools needed to verify Theorem 2.

Proof. of Theorem 2. Let $u \in \text{int}(K) \cap \mathbb{Z}^n$. Then we claim that $u(K) = \text{rec}(K)$. Observe first that since K is closed, we obtain that $u(K) \subseteq \text{rec}(K)$. Let $r \in \text{rec}(K)$. Now observe that since $u \in \text{int}(K) \cap \mathbb{Z}^n$, $\{u + \lambda r \mid \lambda > 0\} \subseteq \text{int}(K)$. Thus by Lemma 8, the half-line $\{u + \lambda r \mid \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Thus, $r \in u(K)$, completing the proof of the claim.

Now observe Theorem 1 implies (2.) \Rightarrow (1.) and the above claim together with Theorem 1 implies (1.) \Rightarrow (2.). We now verify (1.) \iff (3.).

Let us prove " \Leftarrow ". Assume that every exposed face of K satisfies the condition. We will verify that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. By Theorem 1, it is sufficient to show that $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$. Observe that we have verified that $u(K) = \text{rec}(K) \forall u \in \text{int}(K) \cap \mathbb{Z}^n$. Therefore, it remains to be shown that $u(K) = \text{rec}(K)$ for all $u \in \text{bd}(K)$. Consider any $u \in \text{bd}(K)$ and let $r \in \text{rec}(K)$. Then either $u + \lambda r \in \text{int}(K)$ for all $\lambda > 0$ or $r \in \text{rec}(F)$ for some proper exposed face F . In the first case by Lemma 8, the half-line $\{u + \lambda r \mid \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. In the second case, by the condition, we have that $\{u + \lambda r \mid \lambda \geq 0\} \subseteq \text{conv}(F \cap \mathbb{Z}^n) \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Thus $u(K) = \text{rec}(K)$, completing the proof.

Let us prove " \Rightarrow ". Let $\text{conv}(K \cap \mathbb{Z}^n)$ be closed. Then by Theorem 1, we know that $u(K)$ is closed and identical for all $u \in K \cap \mathbb{Z}^n$. Since we have verified that $u(K) = \text{rec}(K)$ for all $u \in \text{int}(K) \cap \mathbb{Z}^n$, we obtain that $u(K) = \text{rec}(K)$ for all $u \in K \cap \mathbb{Z}^n$.

Now examine any proper exposed face of F . If $F \cap \mathbb{Z}^n \neq \emptyset$, $u \in F \cap \mathbb{Z}^n$ and $r \in \text{rec}(F)$, then we have that $r \in \text{rec}(K)$ and thus $\{u + \lambda r \mid \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Therefore, it remains to verify that $\text{conv}(K \cap \mathbb{Z}^n) \cap F = \text{conv}(F \cap \mathbb{Z}^n)$ to complete the proof. Clearly $\text{conv}(F \cap \mathbb{Z}^n) \subseteq \text{conv}(K \cap \mathbb{Z}^n) \cap F$. If $x \in \text{conv}(K \cap \mathbb{Z}^n) \cap F$, then x is a convex combination of z^1, \dots, z^p where $z^i \in K \cap \mathbb{Z}^n$ for $i \in \{1, \dots, p\}$. However, since $x \in F$, $z^i \in F$ for all $i \in \{1, \dots, p\}$. Thus, $x \in \text{conv}(F \cap \mathbb{Z}^n)$, completing the proof. ■

2.3.3 Closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ where K is a strictly convex set

A set $K \subseteq \mathbb{R}^n$ is called a *strictly convex set*, if K is a convex set and for all $x, y \in K$, $\lambda x + (1 - \lambda)y \in \text{rel.int}(K)$ for $\lambda \in (0, 1)$.

Theorem 3. *If $K \subseteq \mathbb{R}^n$ is a full-dimensional closed strictly convex set, then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.*

Proof. First note that if K is bounded or if $K \cap \mathbb{Z}^n = \emptyset$, then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. Therefore we assume that K is unbounded and $K \cap \mathbb{Z}^n \neq \emptyset$.

We first verify that K does not contain a line. Assume by contradiction that K contains a line in the direction $r \neq 0$. Examine $x \in \text{bd}(K)$. Then points of the form $x + \lambda r$ and $x - \lambda r$ belong to K , where $\lambda > 0$. In particular, $x + \lambda r, x - \lambda r \in \text{bd}(K)$ since $x \in \text{bd}(K)$. However this contradicts the fact that K is strictly convex.

Consider a point $u \in K \cap \mathbb{Z}^n$. Let $r \in \text{rec}(K)$. Since K is strictly convex, we obtain that that set $\{u + \lambda r \mid \lambda > 0\}$ is contained in the interior of K . Therefore, by Lemma 8 we obtain that the set $\{u + \lambda r \mid \lambda \geq 0\}$ is contained in $\text{conv}(K \cap \mathbb{Z}^n)$. Thus, $u(K) = \text{rec}(K)$ for all $u \in K \cap \mathbb{Z}^n$. Therefore, by Theorem 1 we obtain that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. ■

Thus in the case of full-dimensional closed strictly convex set K , $\text{conv}(K \cap \mathbb{Z}^n)$ is closed independent of the recession cone. The sets K^3 and K^4 in Example 1 are examples of this fact.

It is easily verified that every face of K is zero-dimensional, i.e. a single point. Therefore in fact the statement of Theorem 3 follows straightforwardly from Theorem 2 in the case when K is not lattice-free. It turns out that if $K \subseteq \mathbb{R}^n$ is a full-dimensional unbounded closed strictly convex set and $K \cap \mathbb{Z}^n \neq \emptyset$, then K is not lattice-free. The proof would follow from Lemma 5.

2.3.4 Closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ where K is a full-dimensional pointed closed convex cone

In this section we prove the following result.

Theorem 4. *Let K be a full-dimensional pointed closed convex cone in \mathbb{R}^n . Then $\overline{\text{conv}}(K \cap \mathbb{Z}^n) = K$. In particular, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if every extreme ray of K is rational scalable.*

Proof. We first verify that $\overline{\text{conv}}(K \cap \mathbb{Z}^n) = K$. By convexity of K , we obtain that $\text{conv}(K \cap \mathbb{Z}^n) \subseteq K$. Since K is also closed, we obtain that $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \subseteq K$. Now, let $r \in \text{int}(K)$. Clearly, we have $d = \{0 + \lambda r \mid \lambda > 0\} \subseteq \text{int}(K)$. So, by Lemma 8, we obtain $\{0\} \cup d \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Hence, $\text{int}(K) \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Since K is a full-dimensional closed convex set, we have $K = \overline{\text{int}(K)}$. Thus, by taking the closure on both sides of the inclusion $\text{int}(K) \subseteq \text{conv}(K \cap \mathbb{Z}^n)$, we obtain $K \subseteq \overline{\text{conv}}(K \cap \mathbb{Z}^n)$.

We now verify that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if all the extreme rays of K are rational scalable rays. Suppose $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. Then $\text{conv}(K \cap \mathbb{Z}^n) = K$. If r is any extreme ray of K , then observe that $K \setminus \{\lambda r \mid \lambda > 0\}$ is a convex set. Since $\{\lambda r \mid \lambda > 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$, there must be an integer point in the set $\{\lambda r \mid \lambda > 0\}$. In other words, r is rational scalable.

Now assume that every extreme ray of K is rational scalable. Let R be the set of all extreme rays of K . Then observe that

$$K = \text{cone}(R) \subseteq \text{conv}(K \cap \mathbb{Z}^n) \subseteq K,$$

where the first equality follows from Theorem 7. Thus, $\text{conv}(K \cap \mathbb{Z}^n) = K$ or equivalently $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. \blacksquare

We note here that K^6 in Example 1 is an example for a non-polyhedral cone where each extreme ray is rational scalable. Therefore $\text{conv}(K^6 \cap \mathbb{Z}^3) = K^6$.

2.3.5 Closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ where K contains lines

Given a set K and a half-line $d = \{u + \lambda r \mid \lambda \geq 0\}$ we say K is *coterminal* with d if

$$\sup\{\mu \mid \mu > 0, u + \mu r \in K\} = \infty.$$

This definition is originally presented in [61]. Given a closed convex set K , a face F of K is called extreme facial ray of K if F is a closed half-line.

In this section, we will verify the following result.

Theorem 5. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set such that the lineality space $L = \text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$ is not trivial. Then, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if the following two conditions hold:*

1. $\text{conv}(K \cap L^\perp \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n))$ is closed.
2. L is a rational linear subspace.

Note that when L is a rational linear subspace (otherwise we already know that $\text{conv}(K \cap \mathbb{Z}^n)$ is not closed), we obtain that $\text{Proj}_{L^\perp}(\mathbb{Z}^n)$ is a lattice. Therefore, if the set $K \cap L^\perp$ does not contain any line, then we can characterize the closedness of $\text{conv}(K \cap L^\perp \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n))$ using the properties we have for convex sets not containing lines (these results can be easily extended for a general lattice).

Before presenting the proof of Theorem 5, we describe some useful corollaries.

Corollary 1. *Let K be a closed convex set and let $\text{rec}(K)$ be a rational polyhedral cone. Then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.*

Proof. Observe that if $K \cap \mathbb{Z}^n = \emptyset$, the result is trivial. We assume for the rest of the proof that $K \cap \mathbb{Z}^n \neq \emptyset$.

Let $L = \text{lin.space}(K)$. Since L is rational, $\text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = L$. Therefore, by Theorem 5, we only need to verify that $\text{conv}(K \cap L^\perp \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n))$ is closed.

Notice that the set $K \cap L^\perp$ does not contain any line. To simplify the proof, by using Lemma 4 we may assume without loss of generality that $L^\perp = \{x \in \mathbb{R}^n \mid x_i = 0 \forall i = k+1, \dots, n\}$. Thus, it is sufficient (after projecting out the last $n-k$ components) to show that $\text{conv}(K' \cap \mathbb{Z}^k)$ is closed, where $K' \subseteq \mathbb{R}^k$ is a closed convex set not containing any line and $\text{rec}(K')$ is a rational polyhedral cone. However, note now that $u(K') \supseteq \text{rec}(K') \supseteq \text{rec}(\overline{\text{conv}}(K' \cap \mathbb{Z}^k)) \supseteq u(K')$ for all $u \in K' \cap \mathbb{Z}^n$, where the first inclusion is due to the fact that $\text{rec}(K')$ is a rational polyhedral cone, the second inclusion is due to the fact that K' is closed and the last inclusion is the same as (3). Thus $u(K')$ is closed and identical for all $u \in K' \cap \mathbb{Z}^k$. Therefore, by Theorem 1 we conclude that K' is closed which completes the proof. ■

Corollary 2. *If $\text{lin.space}(K)$ is not a rational linear subspace and $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$, then $\text{conv}(K \cap \mathbb{Z}^n)$ is not closed.*

Proof. By Lemma 8, we conclude $\text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = \text{lin.space}(K)$, which completes the proof. ■

Next we present some results needed to verify Theorem 5. The crucial results needed are the following theorems from [52].

Theorem 9 ([52]). *If $A \subseteq \mathbb{R}^n$ is a closed set not containing a line, then $\text{conv}(A)$ is closed if and only if A is coterminal with all the extreme facial rays of $\overline{\text{conv}}(A)$.*

Theorem 10 ([52]). *Let $A \subseteq \mathbb{R}^n$ such that $L = \text{lin.space}(\overline{\text{conv}}(A))$ is not trivial. Then, $\text{conv}(A)$ is closed if and only if*

1. *The set $\text{Proj}_{L^\perp}(A)$ is coterminal with every extreme facial ray of $\overline{\text{conv}}(A) \cap L^\perp$.*

2. For every extreme point z of $\overline{\text{conv}}(A) \cap L^\perp$, $\text{conv}(A \cap (z + L)) = z + L$.

The following straightforward lemmas, that we present without any proofs, show some properties of the projection operation.

Lemma 9. *Let $A, B \subseteq \mathbb{R}^n$ and denote $L \subseteq \text{lin.space}(\overline{\text{conv}}(A))$. We have the following:*

1. $\text{Proj}_{L^\perp}(\overline{B}) \subseteq \overline{\text{Proj}_{L^\perp}(B)}$.
2. $\text{Proj}_{L^\perp}(\text{conv}(B)) = \text{conv}(\text{Proj}_{L^\perp}(B))$.
3. $\text{Proj}_{L^\perp}(\overline{\text{conv}}(A)) = \overline{\text{conv}}(A) \cap L^\perp$.
4. $\text{Proj}_{L^\perp}(\overline{\text{conv}}(A)) = \overline{\text{conv}}(\text{Proj}_{L^\perp}(A))$.

Lemma 10. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. Denote $L = \text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Then $\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n) = \text{Proj}_{L^\perp}(K) \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n)$. In particular, $\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n) = K \cap L^\perp \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n)$.*

We now have all the tools for proving Theorem 5.

Proof. of Theorem 5 Observe that if $K \cap \mathbb{Z}^n = \emptyset$, the result is trivial. We assume for the rest of the proof that $K \cap \mathbb{Z}^n \neq \emptyset$.

Claim 1: If L is a rational linear subspace, then (1.) of Theorem 5 is equivalent to (1.) of Theorem 10 with $A = K \cap \mathbb{Z}^n$. Since L is a rational linear subspace, $\text{Proj}_{L^\perp}(\mathbb{Z}^n)$ is a lattice and therefore $\text{Proj}_{L^\perp}(\mathbb{Z}^n)$ is a closed set. Hence $K \cap L^\perp \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n)$ is a closed set. Moreover, by Lemma 10, we have that

$$K \cap L^\perp \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n) = \text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n). \quad (7)$$

Thus, $\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n)$ is closed and $\text{conv}(K \cap L^\perp \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n))$ is closed if and only if $\text{conv}(\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n))$ is a closed set. Since $\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n)$ is closed, we may apply Theorem 9 to $\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n)$ to obtain that $\text{conv}(\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n))$ is closed if and only if the set $\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n)$ is coterminial with every extreme facial ray of

$\overline{\text{conv}}(\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n))$. By Lemma 9, $\overline{\text{conv}}(\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n)) = \text{Proj}_{L^\perp}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = \overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$. Therefore, $\text{conv}(K \cap L^\perp \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n))$ is closed if and only if $\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n)$ is coterminal with every extreme facial ray of $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$.

Observe that since the set $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$ does not contain any lines, it must have at least one extreme point.

Let us prove “ \Rightarrow ”. Suppose $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. Then, by Theorem 10 for every extreme point z of $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$ we have that $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) = z + L$. Thus $z + L$ is the convex hull of some nonempty subset of integer points and therefore L is a rational linear subspace. This proves (2.) of Theorem 5. Moreover, since L is a rational linear subspace, we have (1.) of Theorem 5 by Claim 1.

Let us prove “ \Leftarrow ”. Now suppose (1.) and (2.) of Theorem 5. Then by Claim 1, we have (1.) of Theorem 10. We will prove (2.) of Theorem 10, that is, for every extreme point z of $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$, we have $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) = z + L$. We first prove that $(z + L) \cap K \cap \mathbb{Z}^n \neq \emptyset$. Since $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp = \overline{\text{conv}}(\text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n))$, by Lemma 3 we have that $z \in \text{Proj}_{L^\perp}(K \cap \mathbb{Z}^n)$, and therefore there exists $l \in L$ such that $z + l \in K \cap \mathbb{Z}^n$. Hence, $(z + L) \cap K \cap \mathbb{Z}^n \neq \emptyset$. Now let $\{l_1, \dots, l_p\} \subseteq \mathbb{Z}^n$ be a basis of L and let $w \in (z + L) \cap K \cap \mathbb{Z}^n$. Since $L \subseteq \text{lin.space}(K)$ for all $\lambda_1, \dots, \lambda_p \in \mathbb{Z}$, the points $w, w + \lambda_1 l_1, \dots, w + \lambda_p l_p$ belong to $(z + L) \cap K \cap \mathbb{Z}^n$. Thus, by convexity of $\text{conv}((z + L) \cap K \cap \mathbb{Z}^n)$, for all $\lambda_1, \dots, \lambda_p \in \mathbb{R}$, the points $w, w + \lambda_1 l_1, \dots, w + \lambda_p l_p$ belong to $\text{conv}((z + L) \cap K \cap \mathbb{Z}^n)$. Hence, $\text{conv}((z + L) \cap K \cap \mathbb{Z}^n)$ contains an affine subspace whose dimension is the same as dimension of $z + L$. Since $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) \subseteq z + L$, we obtain that $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) = z + L$. Thus we obtain (2.) of Theorem 10. Therefore, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. \blacksquare

2.4 Polyhedrality of $\text{conv}(K \cap \mathbb{Z}^n)$

We use the following notation in this section. Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then $\sigma_K : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as $\sigma_K(a) = \sup\{a^T x \mid x \in K\}$ is the

support function of K . Given a cone T , we represent its polar by T^* . In particular, $(\text{rec}(K))^* = \{d \in \mathbb{R}^n \mid d^T u \leq 0 \forall u \in \text{rec}(K)\}$.

Let us develop some intuition regarding the question of polyhedrality of $\text{conv}(K \cap \mathbb{Z}^n)$. Suppose for simplicity that K contains no lines, K is full-dimensional and $\text{int}(K) \cap \mathbb{Z}^n$ is non-empty. Then by Theorem 2, we obtain that a necessary condition for $\text{conv}(K \cap \mathbb{Z}^n)$ to be closed is that $\text{rec}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec}(K)$. Therefore, in this setting, if we require $\text{conv}(K \cap \mathbb{Z}^n)$ to be a rational polyhedron, it is necessary that K has a rational polyhedral recession cone. However, this is not sufficient. Consider the case of the parabola K^3 presented in Example 1. It is easy to verify that $\text{conv}(K^3 \cap \mathbb{Z}^2)$ is not a polyhedron. To see what is ‘going wrong’, observe that $\min\{x_1 \mid x \in K^3\} = -\infty$, even though $(-1, 0)$ is orthogonal to the all vectors in the recession cone. Intuitively, this causes $\text{conv}(K^3 \cap \mathbb{Z}^2)$ to have an infinite number of extreme points. This motivates the definition of ‘thin set’ (see Definition 4 in Section 2.2). In terms of its support function and the polar of its recession cone, a closed convex set K is thin if and only if the following holds for all $c \in \mathbb{R}^n$: $\sigma_K(c) < +\infty$ if and only if $c \in (\text{rec}(K))^*$.

In this section we verify the following result.

Theorem 6. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. If K is thin and recession cone of K is a rational polyhedral cone, then $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron. Moreover, if $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ and $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron, then K is thin and $\text{rec}(K)$ is a rational polyhedral cone.*

Since every polyhedron is a thin set, Theorem 6 generalizes the result in [67]. We present a simple example illustrating Theorem 6 when K is not a polyhedral set.

Example 3. *Consider the set $K^8 = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\}$. It is straightforward to verify that K^8 is thin and $\text{rec}(K^8) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0, y_2 \geq 0\}$ is a rational polyhedron. Thus, $\text{conv}(K^8 \cap \mathbb{Z}^2) = \{(x_1, x_2) \mid x_1 \geq 1, x_2 \geq 1\}$ is a polyhedron. On the other*

hand observe that while each of the sets K^1, K^2, K^3, K^4, K^6 in Example 1 contains integer points in its interior, none of them are both thin and have rational polyhedral recession cone. Thus by Theorem 6, the convex hull of integer points in all these sets is non-polyhedral.

2.4.1 Sufficient conditions for $\text{conv}(K \cap \mathbb{Z}^n)$ to be polyhedral

The following variant of Gordan-Dickson's Lemma is from [50].

Lemma 11. *Let $m \in \mathbb{N}$ and $X \subseteq \mathbb{Z}^m$ and assume there exists $x_0 \in \mathbb{Z}^m$ such that $x \geq x_0$ for every $x \in X$. Then there exists a finite set $Y \subseteq X$ such that for every $x \in X$ there exists $y \in Y$ satisfying $y \leq x$.*

Proposition 1 (Sufficient Condition). *If K is thin and recession cone of K is a rational polyhedral cone, then $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron.*

Proof. Observe that if $K \cap \mathbb{Z}^n = \emptyset$, then $\text{conv}(K \cap \mathbb{Z}^n) = \emptyset$ is a polyhedron. Therefore, assume for the rest of the proof that $K \cap \mathbb{Z}^n \neq \emptyset$. Since $\text{rec}(K)$ is a rational polyhedron, we obtain $\text{rec}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec}(K)$. Choose an integer matrix $A \in \mathbb{Z}^{m \times n}$ such that

$$\text{rec}(K) = \{x \in \mathbb{R}^n \mid Ax \geq 0\}.$$

For $i = 1, \dots, m$ let $a_i \in \mathbb{R}^n$ denote the i th row of A .

Since K is thin, for all $i = 1, \dots, m$ we have $\inf\{a_i^T x \mid x \in K\} > -\infty$, so we obtain $\inf\{a_i^T x \mid x \in K \cap \mathbb{Z}^n\} > -\infty$. Thus, by defining the vector $x_0 \in \mathbb{Z}^m$ as follows

$$(x_0)_i = \inf\{a_i^T x \mid x \in K \cap \mathbb{Z}^n\}, \quad \forall i = 1, \dots, m$$

we conclude that $Ax \geq x_0$ for every $x \in K \cap \mathbb{Z}^n$. Therefore, by applying Lemma 11 to the set $\{Ax \mid x \in K \cap \mathbb{Z}^n\} \subseteq \mathbb{Z}^m$, we obtain that there exists a finite set $Y \subseteq K \cap \mathbb{Z}^n$

such that for every $x \in K \cap \mathbb{Z}^n$ there exists $y \in Y$ satisfying $Ay \leq Ax$. To prove that $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron, it is sufficient to show that

$$\text{conv}(K \cap \mathbb{Z}^n) = \text{conv}(Y) + \text{rec}(K).$$

Let $x \in K \cap \mathbb{Z}^n$ and $y \in Y$ as above, that is, such that $A(x - y) \geq 0$. We have $x - y \in \text{rec}(K)$. This shows $K \cap \mathbb{Z}^n \subseteq Y + \text{rec}(K)$ and yields the inclusion $\text{conv}(K \cap \mathbb{Z}^n) \subseteq \text{conv}(Y) + \text{rec}(K)$. The reverse inclusion $\text{conv}(Y) + \text{rec}(K) \subseteq \text{conv}(K \cap \mathbb{Z}^n)$ follows directly from the fact $Y \subseteq K \cap \mathbb{Z}^n$ and that $\text{rec}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec}(K)$. ■

We note here that the proof technique of Proposition 1 was suggested by an anonymous referee.

2.4.2 Necessary conditions for $\text{conv}(K \cap \mathbb{Z}^n)$ to be polyhedral

We begin with a few lemmas before presenting the ‘necessary direction’ of Theorem 6.

Lemma 12. *Let $Q \subseteq \mathbb{R}^n$ be a full-dimensional maximal lattice free convex set and let $c^T x \leq d$ be a valid inequality for Q . Then there exists $\delta > 0$ such that $c^T x \geq d - \delta$ is a valid inequality for Q .*

Proof. Assume by contradiction that $\inf\{c^T x \mid x \in Q\} = -\infty$. Since Q is a polyhedron (by Theorem 8), we obtain that there exists a recession direction r of Q such that $r^T c < 0$. However, because $\text{rec}(Q) = \text{lin.space}(Q)$, we have that $-r$ is a recession direction of Q . Then $\sup\{c^T x \mid x \in Q\} = +\infty$, contradicting the assumption that $c^T x \leq d$ is a valid inequality for Q . ■

Lemma 13. *If $K \subseteq \mathbb{R}^n$ is thin and $T \subseteq \mathbb{R}^n$ is a closed subset of K such that $\text{rec}(T) = \text{rec}(K)$, then T is thin.*

Proof. Suppose $\inf\{c^T x | x \in T\}$ is unbounded. Then, $\inf\{c^T x | x \in K\}$ is unbounded. Since K is thin, there exists $d \in \text{rec}(K) = \text{rec}(T)$ such that $d^T c < 0$. If $\inf\{c^T x | x \in T\}$ is bounded, then $d^T c \geq 0$ for all $d \in \text{rec}(T)$. ■

Proposition 2 (Necessary Condition). *Let $K \subseteq \mathbb{R}^n$ be a closed convex set such that $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$. If $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron, then K is thin and $\text{rec}(K)$ is a rational polyhedral cone.*

Proof. Let $P = \{x \in \mathbb{R}^n | a_i^T x \leq b_i, i = 1, \dots, m\}$ be a description of $\text{conv}(K \cap \mathbb{Z}^n)$. Note that P is a rational polyhedron. We will show first that for all $i = 1, \dots, m$, we have $\sup\{a_i^T x | x \in K\} < \infty$. Let $i \in \{1, \dots, m\}$ and assume by contradiction that $\sup\{a_i^T x | x \in K\} = \infty$. Consider the set $K_i = K \cap \{x \in \mathbb{R}^n | a_i^T x \geq b_i\}$. Notice that $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$, so K must be a full-dimensional set. Also, by assumption, we have $K \not\subseteq \{x \in \mathbb{R}^n | a_i^T x \leq b_i\}$. Therefore it can be verified that $\text{int}(K) \cap \{x \in \mathbb{R}^n | a_i^T x > b_i\} \neq \emptyset$. This implies $\text{int}(K_i) = \text{int}(K) \cap \{x \in \mathbb{R}^n | a_i^T x > b_i\} \neq \emptyset$ and thus K_i is of full dimension.

Moreover, we have $\text{int}(K_i) \cap \mathbb{Z}^n = (\text{int}(K_i) \cap K) \cap \mathbb{Z}^n \subseteq \text{int}(K_i) \cap P = \emptyset$, so K_i is a lattice-free set. Hence, there exists a full-dimensional maximal lattice-free polyhedron $Q = \{x \in \mathbb{R}^n | c_j^T x \leq d_j, j = 1, \dots, q\}$ such that $K_i \subseteq Q$.

Since K is not lattice-free we obtain that $K \not\subseteq Q$. Therefore there exists $x_0 \in K \setminus Q$, that is, $x_0 \in K$, $a_i^T x_0 < b_i$, and there exists $j \in \{1, \dots, q\}$ such that $c_j^T x_0 > d_j$. By Lemma 12, there exists $\delta > 0$ such that $x \in Q$ implies $c_j^T x \geq d_j - \delta$.

Let $\{x_n\}_{n \geq 1} \subseteq K_i$ such that $\lim_{n \rightarrow \infty} a_i^T x_n = \infty$ and $\lambda_n \in (0, 1)$ such that the point $y_n = (1 - \lambda_n)x_0 + \lambda_n x_n$ satisfies $a_i^T y_n = b_i$. Since $x_0, x_n \in K$, by convexity of K , we have $y_n \in K$. Therefore we obtain that $y_n \in K_i$.

On the other hand,

$$\begin{aligned}
c_j^T y_n - d_j &= (1 - \lambda_n)c_j^T x_0 + \lambda_n c_j^T x_n - d_j \\
&\geq (1 - \lambda_n)(c_j^T x_0 - d_j) - \lambda_n \delta \\
&= (c_j^T x_0 - d_j) - \lambda_n[(c_j^T x_0 - d_j) + \delta].
\end{aligned} \tag{8}$$

where the inequality follows from the fact that $\{x_n\}_{n \geq 1} \subseteq K_i \subseteq Q \subseteq \{x \in \mathbb{R}^n : c_j^T x \geq d_j - \delta\}$.

Notice that, by definition, $\lambda_n = \frac{b_i - a_i^T x_0}{a_i^T x_n - a_i^T x_0}$ and thus $\lim_{n \rightarrow \infty} \lambda_n = 0$. Hence, by 52, for sufficiently large n , we have $c_j^T y_n > d_j$, a contradiction with the fact $y_n \in K_i \subseteq Q$. So, we must have $\sup\{a_i^T x \mid x \in K\} < \infty$, for all $i \in \{1, \dots, m\}$.

We conclude that there exist numbers \bar{b}_i , for all $i = 1, \dots, m$, with $b_i \leq \bar{b}_i < \infty$ such that $K \subseteq P' := \{x \mid a_i^T x \leq \bar{b}_i, i = 1, \dots, m\}$. Hence, since $P \subseteq K \subseteq P'$, we have $\text{rec}(K) = \{x \mid a_i^T x \leq 0, i = 1, \dots, m\}$, so $\text{rec}(K)$ is a rational polyhedral cone. Moreover, every polyhedron is thin, so by Lemma 13, we conclude K is also thin, as desired.

■

We note here that the additional technical condition that $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ in Proposition 2 is not artificial. We illustrate this with examples next.

Example 4. 1. *Here is an example that shows that $\text{conv}(K \cap \mathbb{Z}^n)$ can be a polyhedron and yet it is not thin, since it is lattice-free. Consider the set*

$$\begin{aligned}
K^9 &:= \text{conv}(\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0, x_1 = 0, x_2 \geq 0\} \\
&\cup \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0.5, x_2 \geq x_1^2\} \\
&\cup \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 1, x_1 = 0, x_2 \geq 0\}).
\end{aligned}$$

Observe that $\text{conv}(K^9 \cap \mathbb{Z}^3) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, x_2 \geq 0, 0 \leq x_3 \leq 1\}$ is a polyhedron. However note that K^9 is not thin since $\text{rec}(K^9) = \{\lambda(0, 1, 0) \mid \lambda \geq 0\}$ and $\inf\{(-1, 0, 0)^T x \mid x \in K^9\} = -\infty$ but $(0, 1, 0)^T(-1, 0, 0) = 0$. Finally note that K^9 is lattice-free.

2. Here is an example that shows that $\text{conv}(K \cap \mathbb{Z}^n)$ can be a polyhedron and yet $\text{rec}(K)$ is not a rational polyhedral cone, since it is lattice-free. Consider the set

$$\begin{aligned} K^{10} &:= \text{conv}(\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 - \sqrt{2}x_1 = 0, x_3 = 0\} \\ &\cup \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 - \sqrt{2}x_1 = 1, x_3 = 0.5\} \\ &\cup \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 - \sqrt{2}x_1 = -1, x_3 = 0.5\} \\ &\cup \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 - \sqrt{2}x_1 = 0, x_3 = 1\}). \end{aligned}$$

Then $K^{10} \cap \mathbb{Z}^3 = \{(0, 0, 0), (0, 0, 1)\}$ and thus $\text{conv}(K^{10} \cap \mathbb{Z}^3)$ is a polyhedron. However, note that $\text{rec}(K^{10})$ is not a rational polyhedral cone. Also observe that K^{10} is lattice-free.

2.5 Remarks

We first remark that all the key results in this chapter (Theorem 1, Theorem 2, Theorem 3, Theorem 4, Theorem 5, Theorem 6) hold if we replace \mathbb{Z}^n by any general lattice $\Gamma \subseteq \mathbb{R}^n$ and investigate the closedness and polyhedrality of $\text{conv}(K \cap \Gamma)$.

It is possible to relax the requirement of $\text{conv}(K \cap \mathbb{Z}^n)$ being a polyhedron and ask the question when $\text{conv}(K \cap \mathbb{Z}^n)$ is *locally polyhedron*, i.e., the intersection of $\text{conv}(K \cap \mathbb{Z}^n)$ with any polytope is also a polytope. To the best of our knowledge the most general sufficient conditions known for $\text{conv}(K \cap \mathbb{Z}^n)$ to be locally polyhedral are presented in [71] for the case where K is general polyhedron (not necessary rational). Coming up with necessary and sufficient conditions for $\text{conv}(K \cap \mathbb{Z}^n)$ to be locally polyhedron in the case where K is a general convex set is an interesting open question.

Another important question is determining necessary and sufficient conditions for the following optimization problem

$$\begin{aligned} z^* &= \min d^T x \\ \text{s.t. } & x \in K \cap \mathbb{Z}^n, \end{aligned} \tag{9}$$

to be solvable, i.e., if z^* is bounded and $K \cap \mathbb{Z}^n \neq \emptyset$ implies there exists $x^* \in K \cap \mathbb{Z}^n$ such that $d^T x^* \leq d^T x \forall x \in K \cap \mathbb{Z}^n$. Clearly if $\text{conv}(K \cap \mathbb{Z}^n)$ is a rational polyhedron, then the optimization problem is solvable for any d . Another sufficient condition that can be easily verified is that d is a rational vector. However, finding general necessary and sufficient conditions for (9) to be solvable is a challenging question.

CHAPTER III

PROPERTIES OF MIXED-INTEGER HULLS OF CONIC PROGRAMS

3.1 Introduction

A mixed-integer convex program is the optimization problem of minimizing a linear function over the mixed-integer points contained in a closed convex set, that is, a problem of the form

$$\min \{c^T x \mid x \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\}, \quad (10)$$

where $c \in \mathbb{R}^{n_1+n_2}$ and K is a closed convex set. The integer hull of K is defined as the convex hull of the mixed-integer points contained in K . Since the nonconvex optimization problem (10) can be solved by minimizing the same linear objective over the integer hull of K (a convex optimization problem), it is of interest to study integer hulls of closed convex sets. Understanding the structure of integer hulls has proven to be critical in the design of various algorithms for solving mixed-integer programs.

In the case of mixed-integer linear programs, a particularly important structural result, due to Meyer [67], states that the integer hull of a rational polyhedron is again a rational polyhedron. In [71] some sufficient conditions are given in order for integer hulls of general polyhedra to be locally polyhedral (a set whose intersection with any polytope is again a polytope). Closedness and other properties of integer hulls of closed convex sets arising in the context of nonconvex mixed-integer quadratic programming are studied in [25]. The structure of integer hulls of general closed convex sets is studied in [35] and [72]. In [35] the authors give

necessary and sufficient conditions for the integer hull to be a rational polyhedron, and more generally, to be a closed set. In [72] the author studied the facial structure of integer hulls of closed convex sets and provided a necessary and sufficient condition for these integer hulls to be semidefinite representable. It is worth noting that the results presented in [35], [71] and [72] only consider the pure integer case ($n_2 = 0$).

A basic question is whether these structural properties are ‘good’, in the sense that the corresponding conditions are ‘easy’ to check. Observe that the conditions given in the polyhedrality result in [67] (rationality of data) and in the closedness result in [25] (being a particular nonlinear convex set) are straightforward to check. On the other hand, the closedness and polyhedrality results in [35] and the semidefinite representability result in [72] are quite general and, not surprisingly, the corresponding conditions are much more involved. In fact, these conditions require to check some nontrivial properties of faces and recession cones of general convex sets. Moreover, it is very unlike that we are able to simplify these conditions as much as they are in [67] and [25], unless we study nonlinear convex sets with a particular structure.

In this chapter, we first extend some of the results from [35] to the mixed-integer case ($n_2 > 0$) and then show that these results eventually lead to ‘good’ characterizations of integer hulls by studying a special class of convex sets. Specifically, we study properties of integer hulls of simple nonlinear convex sets of the form:

$$\mathbf{P} := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid Ax + By - b \in \mathbf{C}\},$$

where A and B are rational matrices of suitable dimensions, b is a rational vector and $\mathbf{C} \subseteq \mathbb{R}^m$ is a full-dimensional closed convex cone obtained by taking the conic hull of a strictly convex set. In contrast to the case of mixed-integer linear programs, the integer hull of \mathbf{P} is unlikely a rational polyhedron. We therefore

explore the more basic question of whether this integer hull is closed or not.

While the integer hull of \mathbf{P} is not always closed, we are able to provide a characterization of when it is closed. Furthermore, if the cone \mathbf{C} satisfies some additional properties, this characterization yields a polynomial-time algorithm to verify whether the integer hull of \mathbf{P} is closed or not. We find it interesting that it is possible to construct an algorithm (let alone one that runs in polynomial-time) to test the closedness of integer hulls of unbounded nonlinear sets. We also apply our results to the case when \mathbf{C} is the Lorentz cone and obtain that for an important class of mixed-integer second order conic programs, conditions for the closedness of the corresponding integer hulls can be checked in polynomial time.

3.2 *Main Results*

We begin with some definitions. Given a matrix B , we use $\text{Kernel}(B)$ to denote the kernel (null space) of the matrix B and $\langle B \rangle$ to denote the linear subspace generated by the columns of the matrix B . For a linear subspace $L \subseteq \mathbb{R}^n$ we denote by $L^\perp \subseteq \mathbb{R}^n$ the orthogonal subspace to L and we denote by $\text{Proj}_{L^\perp}(\cdot)$ the orthogonal projection onto L . We say that L is a rational (linear) subspace if there exists a basis of L formed by rational vectors. The Euclidian norm is denoted by $\|\cdot\|$. For a set X we denote its dimension by $\dim(X)$, its relative interior by $\text{rel.int}(X)$, its interior by $\text{int}(X)$, its closure by \overline{X} , its boundary by $\text{bd}(X) := \overline{X} \setminus \text{int}(X)$, its relative boundary by $\text{rel.bd}(X) := \overline{X} \setminus \text{rel.int}(X)$, its recession cone by $\text{rec}(X)$ and its lineality space by $\text{lin.space}(X)$. The convex hull of X is the set $\text{conv}(X)$. The conic hull of X is the set $\text{cone}(X)$.

We say that a convex cone is pointed if it does not contain lines. In this chapter, we will only consider convex cones that are pointed and closed.

Definition 5. *A generator for a pointed closed convex cone $\mathbf{C} \subseteq \mathbb{R}^m$ is a bounded closed convex set $\mathbf{S} \subseteq \mathbb{R}^m$ of dimension $\dim(\mathbf{C}) - 1$ such that $\mathbf{C} = \text{cone}(\mathbf{S})$. We say that \mathbf{C} is*

generated by \mathbf{S} .

Notice that the previous definition implies that: (1) For all $x \in \mathbf{C} \setminus \{0\}$, there exists a unique $\lambda > 0$ such that $\lambda x \in \mathbf{S}$; and (2) The extreme rays of \mathbf{C} are in one to one correspondence with the extreme points of \mathbf{S} , in the sense that $r \in \mathbf{C}$ is an extreme ray of \mathbf{C} if and only if r can be scaled by a positive scalar to be an extreme point of \mathbf{S} .

3.2.1 Characterization of closedness

Our first result is a characterization of the closedness of integer hulls of simple conic sets.

Theorem 11. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a strictly closed convex set. Let $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$ and $b \in \mathbb{Q}^m$. Let*

$$\mathbf{P} := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid Ax + By - b \in \mathbf{C}\}, \quad (11)$$

$V := \{Ax + By - b \mid (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\}$ and $\mathcal{L} := \{Ax + By \mid (x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}\}$. Then $\text{conv}(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed if and only if one of the following holds:

1. $b \notin \mathcal{L}$.
2. $b \in \mathcal{L}$, and $\dim(\mathbf{C} \cap V) \leq 1$.
3. $b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) = 2$, $\dim(\langle B \rangle) \leq 0$ and the two extreme rays of the cone $\mathbf{C} \cap V$ can be scaled by a non-zero scalar so that they belong to the lattice $\{Ax \mid x \in \mathbb{Z}^{n_1}\}$.
4. $b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) \geq 2$ and $\dim(\langle B \rangle) \geq \dim(V) - 1$.

The proof of Theorem 11 relies on three sets of results: (1) Understanding when affine rational maps preserve closedness. (2) In a recent paper [35] we presented some properties on the closedness of integer hulls of closed convex sets in the pure integer case, with applications to strictly convex sets and cones. We generalize

these results from the pure integer case to the mixed-integer case. (3) Geometric properties of cones generated by strictly convex sets. We present a proof of Theorem 11 in Section 3.3.

3.2.2 Complexity of checking closedness

For \mathbf{P} given by (11) we denote by $\text{size}(\mathbf{P})$ the sum of the size of the (usual) binary representation of the matrices A, B and b . (That is for a matrix $M \in \mathbb{R}^{m \times n}$, we have $\text{size}(M) = mn + \sum_{i=1}^m \sum_{j=1}^n \log(\lceil m_{ij} \rceil + 1)$.)

Definition 6. *A pointed closed convex cone $\mathbf{C} \subseteq \mathbb{R}^m$ is said to be poly-checkable if for all $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$ the following can be done in polynomial time with respect to $\text{size}([A \ B])$: (I) To decide whether $\dim(\mathbf{C} \cap \langle [A \ B] \rangle) \leq 1$ or not, and, whenever $\dim(\mathbf{C} \cap \langle [A \ B] \rangle) \geq 2$, to compute $\dim(\mathbf{C} \cap \langle [A \ B] \rangle)$; and (II) Checking condition (3.) of Theorem 11. (Notice that checking condition (3.) of Theorem 11 only depends on $\text{size}(A)$).*

Theorem 11 yields a polynomial-time algorithm to check the closedness of integer hulls of simple conic sets whenever the cone \mathbf{C} is poly-checkable. Formally, we have the following result.

Theorem 12. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a strictly closed convex set. Assume that \mathbf{C} is poly-checkable. Let $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$, $b \in \mathbb{Q}^m$ and let \mathbf{P} be as defined in (11). Then there exists an algorithm that runs in polynomial-time with respect to $\text{size}(\mathbf{P})$ to check whether $\text{conv}(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed or not.*

The algorithm in Theorem 12 is constructed by showing that each of the cases in Theorem 11 can be verified in polynomial-time. We present a proof of Theorem 12 in Section 3.4.

3.2.3 The Lorentz cone is poly-checkable

The Lorentz cone $L^m \subseteq \mathbb{R}^m$ is defined as $L^m := \{(w, z) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|w\| \leq z\}$. The following result shows that the class of poly-checkable cones contains the Lorentz cone.

Theorem 13. *The Lorentz cone is poly-checkable.*

Among the two conditions that we need to verify in order to prove that the Lorentz cone is poly-checkable (see Definition 6 in Section 3.2.2), the most ‘interesting’ is (II): to check condition (3.) in Theorem 11 in polynomial-time, the key idea is to reduce the verification of this condition to whether a suitable number is a perfect square, via the use of the Hermite normal form algorithm and properties of the Lorentz cone. We present a proof of Theorem 13 in Section 3.5.

Notice that as a consequence of Theorem 12 and Theorem 13 we obtain that there exist a polynomial-time algorithm to check the closedness of integer hulls of simple second order conic sets.

3.2.4 A property of integer hulls

In the case of pure integer programs, we prove the following result.

Theorem 14. *Let $K_i \subseteq \mathbb{R}^n$, $i = 1, 2$, be closed convex sets. Assume $\text{conv}(K_i \cap \mathbb{Z}^n)$ is closed, for $i = 1, 2$. If $L = \text{lin.space}(K_1 \cap K_2)$ is generated by integer points, then $\text{conv}[(K_1 \cap K_2) \cap \mathbb{Z}^n]$ is closed.*

The proof of Theorem 14 uses as its building block a characterization of closedness of integer hulls of general closed convex sets from [35]. A proof of this result is presented in Section 3.6. We obtain the following straightforward corollary to Theorem 14.

Corollary 3. *Consider the sets $P_i := \{x \in \mathbb{R}^n \mid A_i x - b_i \in C^{m_i}\}$, where for all $i = 1, \dots, q$, we have $A_i \in \mathbb{Q}^{m_i \times n}$, $b_i \in \mathbb{Q}^{m_i}$, and $C^{m_i} \subseteq \mathbb{R}^{m_i}$ is a poly-checkable pointed closed convex*

cone in \mathbb{R}^{m_i} that is generated by a strictly convex set. If the integer hull of \mathbf{P}_i is closed for all $i = 1, \dots, q$, then $\text{conv}(\bigcap_{i=1}^q \mathbf{P}_i \cap \mathbb{Z}^n)$ is closed.

Notice that by the application of Theorem 12 for each \mathbf{P}_i , the sufficient condition of Corollary 3 can be verified in time that is polynomial in the size of the input data. We finally note that Theorem 14 does not hold for the mixed-integer case as illustrated in the next example.

Example 5. Let $K_1 = \{(x, y) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \mid y \geq x_2 - \sqrt{2}x_1\}$ and $K_2 = \{(x, y) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \mid y \geq \sqrt{2}x_1 - x_2\}$. It is straightforward to check that $\text{conv}(K_1 \cap (\mathbb{Z}^2 \times \mathbb{R})) = K_1$ and that $\text{conv}(K_2 \cap (\mathbb{Z}^2 \times \mathbb{R})) = K_2$. Thus, the integer hulls of K_1 and K_2 are closed. However, we will verify next that $\text{conv}((K_1 \cap K_2) \cap (\mathbb{Z}^2 \times \mathbb{R}))$ is not closed. Denote $X = \{(x, y) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \mid y = 0\}$. Let $r := \{\lambda(1, \sqrt{2}, 0) \mid \lambda \geq 0\} = K_1 \cap K_2 \cap X$. Thus, r is a ray with irrational slope contained in X . By the application of Dirichlet Approximation Theorem, we can verify that there are mixed-integer points $(x, y) \in \mathbb{Z}_+^2 \times \mathbb{R}_+$ in $K_1 \cap K_2$ that are arbitrarily close to the ray r . This implies that r belongs to the closure of $\text{conv}((K_1 \cap K_2) \cap (\mathbb{Z}^2 \times \mathbb{R}))$. On the other hand, since r is a face of $K_1 \cap K_2$ and $(0, 0, 0)$ is the only mixed-integer point in this face, we obtain that $r \cap \text{conv}((K_1 \cap K_2) \cap (\mathbb{Z}^2 \times \mathbb{R})) = \{(0, 0, 0)\}$. Therefore, we conclude that $\text{conv}((K_1 \cap K_2) \cap (\mathbb{Z}^2 \times \mathbb{R}))$ is not a closed set.

We note here that Example 5 does not exclude the possibility of a result of the form of Corollary 3 for the mixed-integer case when each of the simple second order conic sets are defined using rational data. We have not been able to resolve this question.

We note here that an extended abstract containing some of the results of this chapter appeared in [36].

3.3 Proof of Theorem 11

We first present in Section 3.3.1 a sketch of the proof of Theorem 11. In particular, we specify the crucial results needed in the proof. Then we present some

basic results needed to prove Theorem 11 in Section 3.3.2 (Properties of convex sets), Section 3.3.3 (Properties of cones generated by strictly convex sets) and Section 3.3.4 (Properties of mixed-integer lattices). Next, we present the proofs of the crucial results mentioned in Section 3.3.1: Proposition 3 is proved in Section 3.3.5, Proposition 4 and Proposition 6 are proved in Section 3.3.6 and the proofs of Proposition 5 and Proposition 7 can be found in Section 3.3.7. The final step of the proof of Theorem 11 is presented in Section 3.3.8.

3.3.1 Sketch of Proof of Theorem 11

We present a definition of mixed-integer lattices before presenting the sketch of our proof.

Definition 7 (Mixed-integer lattice [20]). *Let $A = [a_1 | \dots | a_{n_1}] \in \mathbb{R}^{m \times n_1}$ and $B = [b_1 | \dots | b_{n_2}] \in \mathbb{R}^{m \times n_2}$, where $\{a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2}\}$ is a linearly independent set of \mathbb{R}^m . Then*

$$\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$$

is said to be the mixed-integer lattice generated by A and B .

We note here that in the case $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$ it can be proved that a set \mathcal{L} defined as above is a mixed-integer lattice, even in the case the set $\{a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2}\}$ is not linearly independent (see Proposition 8 in Section 3.3.4).

An affine subspace W is said to be generated by a mixed-integer lattice \mathcal{L} if $W = \text{aff}(W \cap \mathcal{L})$. In the special case $\mathcal{L} = \mathbb{Z}^n$ we also say that the affine subspace is rational.

Proof Outline:

1. **Simplifying the set $\mathbf{P} := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid Ax + By - b \in \mathbf{C}\}$.** To simplify the analysis, we apply the affine map $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^m$ defined as $T(x, y) = Ax + By - b$ to the set $(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$. The image of $(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ under the

map T is the set $((\mathbf{C} \cap V) \cap (\mathcal{L} - b))$ where $V := \{Ax + By - b \mid (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\}$ is an affine subspace and $\mathcal{L} := \{Ax + By \mid (x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}\}$ is a mixed-integer lattice (since A and B are rational matrices). Thus, we obtain the ‘simple’ set $\mathbf{C} \cap V$ in place of \mathbf{P} , at the cost of a ‘complicated’ translated mixed-integer lattice $\mathcal{L} - b$ in place of the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$. The closedness of a set is usually not invariant under affine transformations. However, in Section 3.3.5 we verify the following result:

Proposition 3. *Let $K \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be a closed convex set. Let $G : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^m$ defined as $G(x, y) := Ex + Fy - g$ be an affine map, where $E \in \mathbb{R}^{m \times n_1}$, $F \in \mathbb{R}^{m \times n_2}$ and $g \in \mathbb{R}^m$. Assume that E, F satisfy the following:*

- (a) $\text{Kernel}([E \ F]) \subseteq \text{lin.space}(K)$,
- (b) $\text{Kernel}([E \ F])$ is generated by points in the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.

Then

$$\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \text{ is closed} \Leftrightarrow \text{conv}[G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})] \text{ is closed.}$$

As a consequence of Proposition 3 applied to the affine mapping $T(x, y) = Ax + By - b$ we obtain that

$$\text{conv}(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \text{ is closed} \Leftrightarrow \text{conv}[(\mathbf{C} \cap V) \cap (\mathcal{L} - b)] \text{ is closed.} \quad (12)$$

2. **Case Analysis.** Next we analyze the set $\mathbf{C} \cap V$. Observe that since \mathbf{C} is a cone generated by a closed strictly convex set and V is an affine set, there are two natural cases (see Figure 13):

- (a) **Case 1: $\mathbf{C} \cap V$ is strictly convex set.** If $0 \notin V$, then $\mathbf{C} \cap V$ is a strictly convex set. We verify the following result in Section 3.3.6.

Proposition 4. *Let $K \subseteq \mathbb{R}^n$ be a closed strictly convex set, $t \in \mathbb{R}^n$ and \mathcal{L} a mixed-integer lattice. Then $\text{conv}(K \cap [\mathcal{L} + t])$ is closed.*

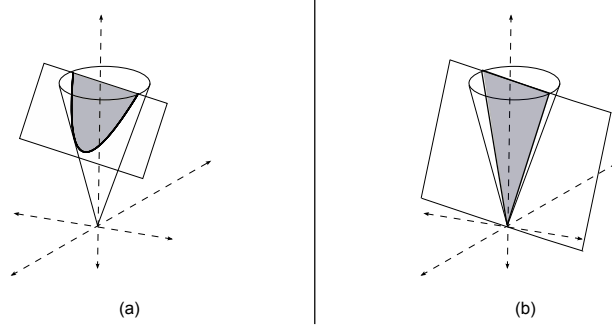


Figure 13: Different cases for $\mathbf{C} \cap V$: (a) Strictly convex set (b) Pointed closed convex cone.

Proposition 4 is a generalization of a result about integer hulls of strictly convex sets from [35]. As a consequence of Proposition 4 and (12) we obtain that $\text{conv}(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is always closed in this case. Observe that this is case (1.) in Theorem 11 when $0 \notin V$.

(b) **Case 2: $\mathbf{C} \cap V$ is a cone.** If $0 \in V$, then $V = \langle [A \ B] \rangle = \text{aff}(\mathcal{L})$ and $\mathbf{C} \cap V$ is a closed pointed convex cone. We have two subcases.

Subcase 1: $b \notin \mathcal{L}$. In this case, $\mathcal{L} - b \neq \mathcal{L}$. Moreover, $\mathcal{L} - b$ is not a mixed-integer lattice. We need the following property.

Proposition 5. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set. Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}$ be a mixed-integer lattice, where A, B are rational matrices. Denote $V = \text{aff}(\mathcal{L})$, and let $b \in (V \cap \mathbb{Q}^m) \setminus \mathcal{L}$. Then $\text{conv}((\mathbf{C} \cap V) \cap (\mathcal{L} - b))$ is closed.*

This result is a consequence of some properties of the closedness of integer hulls of general closed convex sets from [35] and is proven in Section 3.3.7. We can apply Proposition 5 to verify that $\text{conv}[(\mathbf{C} \cap V) \cap (\mathcal{L} - b)]$ is a closed set in this case. Notice this is case (1.) in Theorem 11 when $0 \in V$. In particular, this completes the examination of (1.) in Theorem 11.

Subcase 2: $b \in \mathcal{L}$. We begin the analysis of this case by verifying the following result.

Proposition 6. *Let C be a closed pointed convex cone in \mathbb{R}^n and let \mathcal{L} be a mixed-integer lattice. Then $\overline{\text{conv}}(C \cap \mathcal{L}) = C \cap W$, where $W = \text{aff}(C \cap \mathcal{L})$. In particular, $\text{conv}(C \cap \mathcal{L})$ is closed if and only if every extreme ray of $C \cap W$ can be scaled by a non-zero scalar to belong to \mathcal{L} .*

Proposition 6 is a generalization of a result about integer hulls of cones from [35] (for a proof see Section 3.3.6). As a consequence of Proposition 6, verifying closedness is equivalent to verifying whether the extreme rays of $C \cap V$ can be scaled by a non-zero number to belong to \mathcal{L} .

When $\dim(C \cap V) \leq 1$, it is straightforward to verify that this is always the case. This is case (2.) in Theorem 11.

For analyzing the case where $\dim(C \cap V) > 1$ we need the following additional result.

Proposition 7. *Let $C \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by the closed strictly convex set. Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ be a mixed-integer lattice, where A, B are rational matrices. Denote $V = \text{aff}(\mathcal{L})$. Then*

- i. Assume $\dim(C \cap V) = 2$. If $\dim(\langle B \rangle) \geq \dim(V) - 1$, then every extreme ray of $C \cap V$ can be scaled by a non-zero scalar to belong to \mathcal{L} .*
- ii. Assume $\dim(C \cap V) \geq 3$. Then $\dim(\langle B \rangle) \geq \dim(V) - 1$ if and only if every extreme ray of $C \cap V$ can be scaled by a non-zero scalar to belong to \mathcal{L} .*

The proof of Proposition 7 (presented in Section 3.3.7) is based on the geometric properties of $C \cap V$ and on the cardinality of its set of extreme rays, that can be countable or not depending on the dimension of the

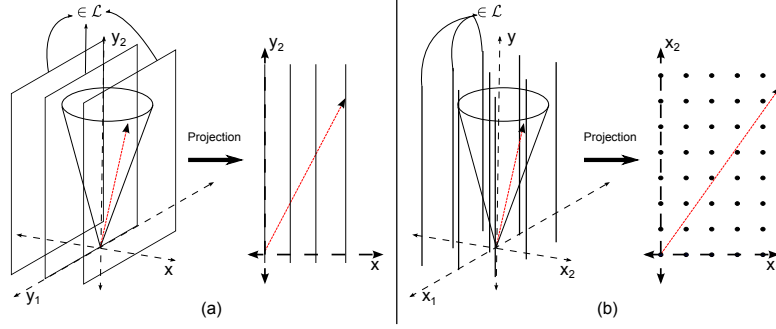


Figure 14: $V = \mathbb{R}^3$. Extreme rays of $\mathbf{C} \cap V$: (a) All scalable to belong to $\mathcal{L} = \mathbb{Z} \times \mathbb{R}^2$ (b) Not all scalable to belong to $\mathcal{L} = \mathbb{Z}^2 \times \mathbb{R}$.

cone.

Proposition 7 is essentially stating that when $\dim(\mathbf{C} \cap V) \geq 3$, in order for every extreme ray to be scalable to belong to the mixed-integer lattice \mathcal{L} , there should be “sufficient” number of continuous components in the mixed-integer lattice \mathcal{L} . See Figure 14 for an illustration. Therefore we obtain that if $\dim(\mathbf{C} \cap V) \geq 3$, then $\text{conv}(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed if and only if $\dim(\langle B \rangle) \geq \dim(V) - 1$. Moreover if $\dim(\mathbf{C} \cap V) = 2$ and $\dim(\langle B \rangle) \geq 1$, then $\text{conv}(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is also closed. Together, this constitutes case (4.) in Theorem 11.

The only case that remains is where $\dim(\mathbf{C} \cap V) = 2$ and $\dim(\langle B \rangle) \leq 0$. In this case, we need to explicitly check whether the two extreme rays of $\mathbf{C} \cap V$ can be scaled by a non-zero scalar to belong to the lattice \mathcal{L} . This is case (3.) in Theorem 11.

3.3.2 Properties of convex sets

The proofs of the following lemmas are standard and hence omitted.

Lemma 14. *Let $K \subseteq \mathbb{R}^n$ be a convex set and let $W \subseteq \mathbb{R}^n$ be an affine subspace. If $\text{rel.int}(K) \cap W \neq \emptyset$, then*

1. $\text{aff}(K \cap W) = \text{aff}(K) \cap W$.

$$2. \text{rel.int}(K \cap W) = \text{rel.int}(K) \cap W.$$

$$3. \text{rel.bd}(K \cap W) = \text{rel.bd}(K) \cap W.$$

Lemma 15. *Let $X \subseteq \mathbb{R}^n$ be a closed strictly convex set and $W \subseteq \mathbb{R}^n$ be an affine subspace. Then $X \cap W$ is a closed strictly convex set.*

3.3.3 Properties of cones generated by strictly convex sets

We will use the following lemmas.

Lemma 16. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set, and let $W \subseteq \mathbb{R}^m$ be an affine subspace. Assume that $\dim(\mathbf{C} \cap W) \geq 2$. Then*

1. $\text{int}(\mathbf{C}) \cap W \neq \emptyset$. Consequently, we obtain that $\dim(\mathbf{C} \cap W) = \dim(W)$ and $\text{rel.bd}(\mathbf{C} \cap W) = \text{bd}(\mathbf{C}) \cap W$.
2. $(\text{bd}(\mathbf{C}) \setminus \{0\}) \cap W \neq \emptyset$.

Proof.

1. Since $\dim(\mathbf{C} \cap W) \geq 2$, there exist three affinely independent points $x, y, z \in \mathbf{C} \cap W$. If $\{x, y, z\} \cap \text{int}(\mathbf{C}) \cap W \neq \emptyset$, we are done. Now, assume that $x, y, z \in \text{bd}(\mathbf{C}) \cap W$. Since $\{x, y, z\}$ is an affinely independent set, we may assume without loss of generality that $x, y \neq 0$ and $x \notin \{\lambda y \mid \lambda \geq 0\}$. Let $\hat{x}, \hat{y} \in \mathbf{S}$ such that $x = \alpha \hat{x}$ and $y = \beta \hat{y}$ for some $\alpha, \beta > 0$. Notice that $\hat{x} \neq \hat{y}$. Since \mathbf{S} is a strictly convex set, we obtain that $\text{conv}(\{\hat{x}, \hat{y}\}) \cap \text{rel.int}(\mathbf{S}) \neq \emptyset$. Let $\lambda \hat{x} + (1 - \lambda) \hat{y} \in \text{rel.int}(\mathbf{S})$, where $\lambda \geq 0$. Then we have

$$w := \frac{1}{\left(\frac{\lambda}{\alpha} + \frac{(1-\lambda)}{\beta}\right)} (\lambda \hat{x} + (1 - \lambda) \hat{y}) = \frac{\frac{\lambda}{\alpha} x + \frac{(1-\lambda)}{\beta} y}{\frac{\lambda}{\alpha} + \frac{(1-\lambda)}{\beta}}. \quad (13)$$

Since $\lambda\hat{x} + (1 - \lambda)\hat{y} \in \text{rel.int}(\mathbf{S})$, we obtain $w \in \text{int}(\mathbf{C})$. Therefore, by (13) we obtain $w \in \text{conv}(\{x, y\}) \cap \text{int}(\mathbf{C})$. Since $\text{conv}(\{x, y\}) \subseteq W$, we conclude that $\text{int}(\mathbf{C}) \cap W \neq \emptyset$.

Finally, the facts that $\dim(\mathbf{C} \cap W) = \dim(W)$ and $\text{rel.bd}(\mathbf{C} \cap W) = \text{bd}(\mathbf{C}) \cap W$ are straightforward consequences of Lemma 14.

2. By (1.) we obtain $\dim(\text{int}(\mathbf{C}) \cap W) = \dim(\text{rel.int}(\mathbf{C} \cap W)) \geq 2$. Thus, there exist vectors $w_0, w_1, w_2 \in \text{int}(\mathbf{C}) \cap W$ such that $w_1 - w_0$ and $w_2 - w_0$ are linearly independent vectors. For $i = 1, 2$ consider the line generated by w_0 and w_i , that is $L_i = \{\alpha_i w_0 + (1 - \alpha_i)(w_i - w_0) \mid \alpha_i \in \mathbb{R}\}$. Since \mathbf{C} is a pointed cone we have that $L_i \not\subseteq \text{int}(\mathbf{C})$. Thus, since $L_i \cap \text{int}(\mathbf{C}) \neq \emptyset$, we obtain $L_i \cap \text{bd}(\mathbf{C}) \neq \emptyset$. For $i = 1, 2$ consider $x_i \in L_i \cap \text{bd}(\mathbf{C})$. Since $w_1 - w_0$ and $w_2 - w_0$ are linearly independent vectors, we have that $L_1 \neq L_2$. Hence, since $w_0 \in L_1 \cap L_2$, we obtain that $x_1 \neq x_2$. In particular, without loss of generality we may assume that $x_1 \neq 0$. Since W is an affine subspace, we have that $x_1 \in L_1 \subseteq W$. Therefore, we conclude that $(\text{bd}(\mathbf{C}) \setminus \{0\}) \cap W \neq \emptyset$.

■

The next lemma states the two possible cases for the structure of the set $\mathbf{C} \cap W$, where \mathbf{C} is a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set and W is an affine subspace.

Lemma 17. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set $\mathbf{S} \subseteq \mathbb{R}^m$, and let $W \subseteq \mathbb{R}^m$ be an affine subspace. Then*

1. *If $0 \notin W$, then $\mathbf{C} \cap W$ is a strictly convex set.*
2. *If $0 \in W$ and $\dim(\mathbf{C} \cap W) \geq 2$, then*

- (a) $\mathbf{C} \cap W$ is a pointed closed convex cone of dimension $\dim(W)$.
- (b) $\mathbf{S} \cap W$ is a strictly convex set and a generator for $\mathbf{C} \cap W$.
- (c) If $\dim(W) \geq 3$, then $\mathbf{C} \cap W$ has an uncountable number of extreme rays. Equivalently, $\text{rel.bd}(\mathbf{S} \cap W)$ is uncountable.

Proof.

1. The case $\dim(\mathbf{C} \cap W) \leq 1$ is straightforward. Assume $\dim(\mathbf{C} \cap W) \geq 2$. Let F be a proper face of $\mathbf{C} \cap W$, and let $x, y \in F$. We will show that $F = \{x\}$. By (1.) of Lemma 16 we have that $\text{rel.bd}(\mathbf{C} \cap W) = \text{bd}(\mathbf{C}) \cap W$. Therefore, since F is a proper face of $\mathbf{C} \cap W$, we obtain that x and y belong to a face of \mathbf{C} . Since \mathbf{C} is generated by a strictly convex set, then all of its faces have dimension one. Thus, we obtain that there exists $\lambda \geq 0$, such that $y = \lambda x$. Since W is an affine subspace, we have that the set $L := \{x + \alpha(y - x) \mid \alpha \in \mathbb{R}\} = \{x + \alpha(\lambda - 1)x \mid \alpha \in \mathbb{R}\}$ is a subset of W . This implies we must have $\lambda = 1$, for otherwise, we obtain that $0 \in L \subseteq W$. Thus, $F = \{x\}$. Therefore, we conclude that $\mathbf{C} \cap W$ is a strictly convex set.
2. (a) Since \mathbf{C} and W are cones, we obtain that $\mathbf{C} \cap W$ is also a cone. The fact that $\dim(\mathbf{C} \cap W) = \dim(W)$ follows directly from (1.) of Lemma 16.
 (b) Since \mathbf{S} is a strictly convex set, by Lemma 15 we conclude that $\mathbf{S} \cap W$ is also a strictly convex set.

In order to prove that $\mathbf{S} \cap W$ is a generator for $\mathbf{C} \cap W$ we need to show that $\mathbf{C} \cap W = \text{cone}(\mathbf{S} \cap W)$ and that $\dim(\mathbf{S} \cap W) = \dim(\mathbf{C} \cap W) - 1$.

- We prove next that $\mathbf{C} \cap W = \text{cone}(\mathbf{S} \cap W)$. Clearly, $\text{cone}(\mathbf{S} \cap W) \subseteq \mathbf{C} \cap W$. We now prove the inclusion $\mathbf{C} \cap W \subseteq \text{cone}(\mathbf{S} \cap W)$. Let $r \in \mathbf{C} \cap W$ with $r \neq 0$. Then, by definition of \mathbf{C} , there exists $\hat{r} \in \mathbf{S}$ such that $r = \alpha \hat{r}$ for some $\alpha > 0$. Notice that since $r \in \mathbf{C} \cap W$, the ray

$\{\lambda r \mid \lambda \geq 0\} \subseteq \mathbf{C} \cap W$. Thus, we obtain that $\hat{r} \in W$. Since $r = \alpha \hat{r}$ and $\hat{r} \in \mathbf{S} \cap W$, we conclude that $r \in \text{cone}(\mathbf{S} \cap W)$.

- We now prove that $\dim(\mathbf{S} \cap W) = \dim(\mathbf{C} \cap W) - 1$. Observe first that since $\dim(\mathbf{C} \cap W) \geq 2$, we have that $\dim(\mathbf{C} \cap W) = \dim(W)$. We therefore need to verify that $\dim(\mathbf{S} \cap W) = \dim(W) - 1$.

Next we claim $\text{rel.int}(\mathbf{S}) \cap W \neq \emptyset$: Since $\dim(\mathbf{C} \cap W) \geq 2$, we obtain by (1.) of Lemma 16 that $\text{int}(\mathbf{C}) \cap W \neq \emptyset$. Let $r \in \text{int}(\mathbf{C}) \cap W$. By definition of \mathbf{C} , there exists $\hat{r} \in \mathbf{S}$ such that $r = \alpha \hat{r}$ for some $\alpha > 0$. Since W is a subspace, we obtain $\hat{r} \in W$. Moreover, since $r \in \text{int}(\mathbf{C})$, we conclude that $\hat{r} \in \text{rel.int}(\mathbf{S}) \cap W$. Thus, $\text{rel.int}(\mathbf{S}) \cap W \neq \emptyset$.

Observe that $0 \notin \text{aff}(\mathbf{S})$, for otherwise we would have $\mathbf{C} \subseteq \text{aff}(\mathbf{S})$, a contradiction with the fact $\dim(\mathbf{S}) = \dim(\mathbf{C}) - 1$. Since $\text{rel.int}(\mathbf{S}) \cap W \neq \emptyset$, by Lemma 14 we obtain that $\text{aff}(\mathbf{S} \cap W) = \text{aff}(\mathbf{S}) \cap W$. Therefore, since $0 \in W$ and $0 \notin \text{aff}(\mathbf{S})$, we obtain that $\text{aff}(\mathbf{S} \cap W)$ is strictly contained in W . Thus, $\dim(\mathbf{S} \cap W) \leq \dim(W) - 1$.

On the other hand, we have

$$\mathbf{C} \cap W = \text{cone}(\mathbf{S} \cap W) \subseteq \text{aff}((\mathbf{S} \cap W) \cup \{0\}).$$

Thus, since $0 \notin \mathbf{S} \cap W$, we obtain that $\dim(\mathbf{C} \cap W) \leq \dim(\mathbf{S} \cap W) + 1$. Hence, since $\dim(\mathbf{C} \cap W) = \dim(W)$, we conclude that $\dim(\mathbf{S} \cap W) \geq \dim(W) - 1$.

- (c) By (2.), we obtain that $\dim(\mathbf{S} \cap W) = \dim(W) - 1$. Thus, $\dim(\mathbf{S} \cap W) \geq 2$. Therefore, since $\mathbf{S} \cap W$ is a bounded and closed convex set of dimension at least 2, we must have that $\text{rel.bd}(\mathbf{S} \cap W)$ is uncountable.

■

3.3.4 Properties of mixed-integer lattices

The next proposition shows that if the data defining the set $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ is rational, then \mathcal{L} is a mixed-integer lattice, even in the case the matrix $[A \ B]$ does not have linearly independent columns.

Proposition 8. *Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ and denote $W = \langle B \rangle$. Assume that the data defining \mathcal{L} is rational, that is, $A \in \mathbb{Q}^{m \times n_1}$ and $B \in \mathbb{Q}^{m \times n_2}$. Then there exists $p_1 \leq n_1$, a matrix $A' \in \mathbb{Q}^{m \times p_1}$, whose columns are linearly independent and contained in W^\perp and there exists $p_2 \leq n_2$, a matrix $B' \in \mathbb{Q}^{m \times p_2}$, whose columns are linearly independent and $\langle B' \rangle = \langle B \rangle$, such that*

$$\mathcal{L} = \{x \in \mathbb{R}^m \mid x = A'z + B'y, z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}.$$

Proof.

Since the columns of A are rational, the set $\mathcal{M} = \{x \in \mathbb{R}^m \mid x = Az, z \in \mathbb{Z}^{n_1}\}$ is a lattice (even in the case the columns of A are not linearly independent). Moreover, since W is a rational subspace, we have that $\text{Proj}_{W^\perp}(\mathcal{M})$ is a lattice generated by rational vectors. Let $A' \in \mathbb{Q}^{m \times p_1}$ be matrix whose columns form a basis of $\text{Proj}_{W^\perp}(\mathcal{M})$, that is, the columns of A' are linearly independent and

$$\text{Proj}_{W^\perp}(\mathcal{M}) = \{x \in \mathbb{R}^m \mid x = A'z, z \in \mathbb{Z}^{p_1}\}.$$

Let B' be the matrix whose columns are a basis of W (for instance these columns could be any maximal linearly independent subset of columns of B).

We obtain that

$$\begin{aligned} \mathcal{L} &= \mathcal{M} + W \\ &= \text{Proj}_{W^\perp}(\mathcal{M}) + W \\ &= \{x \in \mathbb{R}^m \mid x = A'z, z \in \mathbb{Z}^{p_1}\} + W \\ &= \{x \in \mathbb{R}^m \mid x = A'z + B'y, z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}. \end{aligned}$$

Now, observe that the columns of A' are linearly independent. Since we have $\text{Proj}_{W^\perp}(\mathcal{M}) \subseteq W^\perp$ and since the columns of B' are linearly independent and are contained in W , we have that the columns of $[A' \mid B']$ are linearly independent. Thus, \mathcal{L} is a mixed-integer lattice. \blacksquare

For the rest of this section we will assume that the columns of $[A \ B]$ are linearly independent. Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ be a mixed-integer lattice and let $\Psi_{\mathcal{L}} : \text{aff}(\mathcal{L}) \rightarrow \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be defined as $\Psi_{\mathcal{L}}(Az + By) = (z, y)$, that is, for a vector $x \in \text{aff}(\mathcal{L})$, $\Psi_{\mathcal{L}}(x)$ are the coordinates of x in the basis of the linear subspace $\text{aff}(\mathcal{L})$ formed by the columns of $[A \ B]$. We obtain the following straightforward lemma.

Lemma 18. *Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ be a mixed-integer lattice. Then $\Psi_{\mathcal{L}} : \text{aff}(\mathcal{L}) \rightarrow \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is an invertible mapping and $\Psi_{\mathcal{L}}(\mathcal{L}) = \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.*

Let $\mathcal{L} \subseteq \mathbb{R}^m$ be a mixed-integer lattice and $W \subseteq \mathbb{R}^m$ be a linear subspace. The following technical lemma states that $W \cap \mathcal{L}$ is a mixed-integer lattice.

Lemma 19. *Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ be a mixed-integer lattice. Let $W \subseteq \mathbb{R}^m$ be a linear subspace. Then $W \cap \mathcal{L}$ is a mixed-integer lattice.*

Proof. By Lemma 18 and replacing W by $W \cap \text{aff}(\mathcal{L})$, we may assume that $\mathcal{L} = \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ and that $W \subseteq \mathbb{R}^{n_1+n_2}$ by applying the linear mapping $\Psi_{\mathcal{L}}$ to \mathcal{L} and $W \cap \text{aff}(\mathcal{L})$, respectively. Let $\mathcal{L}' = W \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. We want to prove that \mathcal{L}' is a mixed-integer lattice. Denote $M = W \cap (\{0\}^{n_1} \times \mathbb{R}^{n_2})$ and $\Lambda = (W \cap M^\perp) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. We will prove that (a) Λ is a lattice and that (b) $\mathcal{L}' = \Lambda + M$. Since $\Lambda \subseteq M^\perp$, (a) and (b) together imply that \mathcal{L}' is a mixed-integer lattice.

First we prove (a). It suffices to prove that Λ is a discrete additive subgroup (Theorem 1.4 [11]). Since $W \cap M^\perp$ and $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ are additive subgroups, we conclude Λ is an additive subgroup. On the other hand, let $(z, y_1), (z, y_2) \in \Lambda$. Then

$(z, y_1) - (z, y_2) = (0, y_1 - y_2) \in W \cap M^\perp \subseteq M^\perp$ and also, by definition of M , we have $(0, y_1 - y_2) \in M$. Thus, we must have $y_1 = y_2$. So, we obtain that if $(z_1, y_1), (z_2, y_2) \in \Lambda$ are distinct, then $z_1 \neq z_2$. Therefore, we have $\|(z_1, y_1) - (z_2, y_2)\|^2 \geq (z_1 - z_2)^2 \geq 1$. We conclude Λ is a discrete set. Therefore, Λ is a lattice.

We next verify (b). The inclusion $\mathcal{L}' \supseteq \Lambda + M$ is easy, since $\mathcal{L}' = W \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$, $\Lambda \subseteq \mathcal{L}'$, $M \subseteq \mathcal{L}'$, and $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, W are additive subgroups. For the inclusion $\mathcal{L}' \subseteq \Lambda + M$, let $(z, y) \in \mathcal{L}'$. We can write

$$(z, y) = (z, u) + (0, v),$$

where (z, u) is the projection on M^\perp of (z, y) and $(0, v)$ is the projection on M of (z, y) . Since $(z, y) - (0, v) \in W \cap \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, we conclude $(z, u) \in \Lambda$. Thus, $(z, y) \in \Lambda + M$, as desired. ■

Proposition 9. *Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ be a mixed-integer lattice. Let $K \subseteq \mathbb{R}^m$ be a closed convex set such that $K \cap \mathcal{L} \neq \emptyset$. Then there exist $p_1, p_2 \in \mathbb{Z}_+$ with $p_1 \leq n_1, p_2 \leq n_2, p_1 + p_2 = \dim(K \cap \mathcal{L})$, and a full-dimensional closed convex set $K' \subseteq \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ such that*

$$\text{conv}(K \cap \mathcal{L}) \text{ is closed if and only if } \text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2})) \text{ is closed.}$$

Moreover, K' can be taken as $K' = \Phi(K \cap \text{aff}(K \cap \mathcal{L}))$, where $\Phi : \text{aff}(K \cap \mathcal{L}) \rightarrow \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ is an invertible affine mapping such that $\Phi(\mathcal{L} \cap \text{aff}(K \cap \mathcal{L})) = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$. When $0 \in K$, we can chose Φ to be a linear mapping.

Proof. We first show that by translating K by a vector in $k \in K \cap \mathcal{L}$ we may assume $0 \in K$. Let $k \in K \cap \mathcal{L}$. Observe that since $k \in \mathcal{L}$, we have $\mathcal{L} - k = \mathcal{L}$. Hence, we obtain

$$\begin{aligned} \text{conv}(K \cap \mathcal{L}) &= \text{conv}(K \cap \mathcal{L}) + (k - k) \\ &= \text{conv}((K - k) \cap (\mathcal{L} - k)) + k. \end{aligned} \tag{14}$$

Thus, by (14) we conclude that $\text{conv}(K \cap \mathcal{L})$ is closed if and only if $\text{conv}((K - k) \cap \mathcal{L})$ is closed. Therefore, we may assume $0 \in K$.

Denote $W = \text{aff}(K \cap \mathcal{L})$. By Lemma 19, since W is a linear subspace, we obtain that $\mathcal{L}' = W \cap \mathcal{L}$ is a mixed-integer lattice, that is, there exists $p_1 \leq n_1, p_2 \leq n_2$, $A' \in \mathbb{R}^{m \times p_1}$, and $B' \in \mathbb{R}^{m \times p_2}$ such that $[A' \ B']$ has linearly independent columns, and

$$\mathcal{L}' = \{x \in \mathbb{R}^m \mid x = A'z + B'y, z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}.$$

Let $\Psi_{\mathcal{L}'} : \text{aff}(\mathcal{L}') \rightarrow \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ be the invertible linear mapping in Lemma 18 for the particular case of the mixed-integer lattice \mathcal{L}' , and let $K' = \Psi_{\mathcal{L}'}(K \cap W)$. We have

$$\begin{aligned} \text{conv}(K \cap \mathcal{L}) &= \text{conv}((K \cap W) \cap (\mathcal{L} \cap W)) \\ &= \text{conv}((K \cap W) \cap \mathcal{L}') \\ &= \Psi_{\mathcal{L}'}^{-1}[\text{conv}(\Psi_{\mathcal{L}'}(K \cap W) \cap \Psi_{\mathcal{L}'}(\mathcal{L}'))] \\ &= \Psi_{\mathcal{L}'}^{-1}[\text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))] \end{aligned}$$

The first equality uses the fact that $K \cap \mathcal{L} \subseteq W$, the third equality uses the fact that $\Psi_{\mathcal{L}'}$ is an invertible mapping and the last equality uses $\Psi_{\mathcal{L}'}(\mathcal{L}') = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$. Therefore, since $\Psi_{\mathcal{L}'}$ is a homeomorphism, we conclude $\text{conv}(K \cap \mathcal{L})$ is closed if and only if $\text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))$ is closed.

Finally, we show that K' can be taken as $K' = \Phi(K \cap \text{aff}(K \cap \mathcal{L}))$, where $\Phi : \text{aff}(K \cap \mathcal{L}) \rightarrow \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ is an invertible affine mapping such that $\Phi(\mathcal{L} \cap \text{aff}(K \cap \mathcal{L})) = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$. If $0 \in K$, it suffices to take $\Phi = \Psi_{\mathcal{L}'}$ (hence Φ is a linear mapping). If $0 \notin K$, then for an arbitrary $k \in K \cap \mathcal{L}$, we can define the following invertible affine mapping $\Phi : \text{aff}(K \cap \mathcal{L}) \rightarrow \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, where for all $x \in \text{aff}(K \cap \mathcal{L})$, $\Phi(x) = \Psi_{\mathcal{L}'}(x - k)$. Then, by the previous arguments in the proof, it is clear we can take $K' = \Phi(K \cap \text{aff}(K \cap \mathcal{L}))$. Since $k \in K \cap \mathcal{L}$, we have $\mathcal{L} \cap \text{aff}(K \cap \mathcal{L}) = \mathcal{L} \cap \text{aff}(K \cap \mathcal{L}) + k$. Thus, $\Phi(\mathcal{L} \cap \text{aff}(K \cap \mathcal{L})) = \Psi_{\mathcal{L}'}(\mathcal{L}') = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$.

■

3.3.5 Affine Maps Preserving Closedness and Proof of Proposition 3

Lemma 20. *Let \mathcal{L}' be a mixed-integer lattice, let $K \subseteq \mathbb{R}^n$ be a closed convex set, and let $L \subseteq \text{lin.space}(K)$ be a linear subspace. Then we have*

$$\text{Proj}_{L^\perp}(K \cap \mathcal{L}') = \text{Proj}_{L^\perp}(K) \cap \text{Proj}_{L^\perp}(\mathcal{L}').$$

Proof. The inclusion $\text{Proj}_{L^\perp}(K \cap \mathcal{L}') \subseteq \text{Proj}_{L^\perp}(K) \cap \text{Proj}_{L^\perp}(\mathcal{L}')$ is always true. Now, to prove the other inclusion, let $x \in \text{Proj}_{L^\perp}(K) \cap \text{Proj}_{L^\perp}(\mathcal{L}')$. Then there exist $l_1, l_2 \in L$ such that $x + l_1 \in K$ and $x + l_2 \in \mathcal{L}'$. Since $x + l_1 \in K$, $(l_2 - l_1) \in L$ and $L \subseteq \text{lin.space}(K)$, we obtain that $x + l_1 + (l_2 - l_1) \in K$. Thus, $x + l_2 \in K$. Therefore, we conclude that $x \in \text{Proj}_{L^\perp}(K \cap \mathcal{L}')$. ■

Proposition 10. *Let \mathcal{L}' be a mixed-integer lattice, let $K \subseteq \mathbb{R}^n$ be a closed convex set, and let $L \subseteq \text{lin.space}(K)$ be a linear subspace. If L is generated by points in \mathcal{L}' , then*

$$\text{conv}(K \cap \mathcal{L}') = \text{conv}((K \cap L^\perp) \cap \text{Proj}_{L^\perp}(\mathcal{L}')) + L.$$

In particular, $\text{conv}(K \cap \mathcal{L}')$ is closed $\Leftrightarrow \text{conv}((K \cap L^\perp) \cap \text{Proj}_{L^\perp}(\mathcal{L}'))$ is closed.

Proof. Let $\{l_1, \dots, l_q\} \subseteq \mathcal{L}'$ be a basis of L . Denote $\mathcal{M} = \{x \in \mathbb{R}^p \mid x = \sum_{i=1}^q z_i l_i, z_i \in \mathbb{Z}, \forall i = 1, \dots, q\}$. Since $\mathcal{M} \subseteq (\text{lin.space}(K) \cap \mathcal{L}')$, we obtain that

$$K \cap \mathcal{L}' = (K \cap \mathcal{L}') + \mathcal{M}. \tag{15}$$

Recall that for all $X, Y \subseteq \mathbb{R}^p$, we have $\text{conv}(X + Y) = \text{conv}(X) + \text{conv}(Y)$. Thus, by (15) we have

$$\text{conv}(K \cap \mathcal{L}') = \text{conv}(K \cap \mathcal{L}') + L. \tag{16}$$

Therefore, as a consequence of (16) we can write

$$\begin{aligned}
\text{conv}(K \cap \mathcal{L}') &= \text{Proj}_{L^\perp}(\text{conv}(K \cap \mathcal{L}')) + L \\
&= \text{conv}(\text{Proj}_{L^\perp}(K \cap \mathcal{L}')) + L \\
&= \text{conv}(\text{Proj}_{L^\perp}(K) \cap \text{Proj}_{L^\perp}(\mathcal{L}')) + L \\
&= \text{conv}((K \cap L^\perp) \cap \text{Proj}_{L^\perp}(\mathcal{L}')) + L,
\end{aligned}$$

where the second equality is by lineality of $\text{Proj}_{L^\perp}(\cdot)$, the third equality uses Lemma 20, and the last equality uses the fact that $L \subseteq \text{lin.space}(K)$.

Finally, the fact that $\text{conv}(K \cap \mathcal{L}')$ is closed $\Leftrightarrow \text{conv}((K \cap L^\perp) \cap \text{Proj}_{L^\perp}(\mathcal{L}'))$ is closed is a straightforward consequence of the identity $\text{conv}(K \cap \mathcal{L}') = \text{conv}((K \cap L^\perp) \cap \text{Proj}_{L^\perp}(\mathcal{L}')) + L$ and the fact that $\text{conv}((K \cap L^\perp) \cap \text{Proj}_{L^\perp}(\mathcal{L}')) \subseteq L^\perp$. \blacksquare

For a function $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$ we denote by G_W its restriction to the linear subspace $W \subseteq \mathbb{R}^n$, that is, $G_W : W \rightarrow \mathbb{R}^m$ and $G_W(x) = G(x)$ for all $x \in W$. The following lemma is straightforward to verify.

Lemma 21. *Let $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined as $G(x) = Mx - g$, where $M \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$. Let $W = \text{Kernel}(M)$. Let $G_{W^\perp} : W^\perp \rightarrow \mathbb{R}^m$ the restriction of G to W^\perp . Then*

1. G_{W^\perp} is injective.
2. For all $X, Y \subseteq W^\perp$, we have $G_{W^\perp}(X \cap Y) = G_{W^\perp}(X) \cap G_{W^\perp}(Y)$.
3. For all $X \subseteq \mathbb{R}^n$, we have $G_{W^\perp}(\text{Proj}_{W^\perp}(X)) = G(X)$.

Proposition 3. *Let $K \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be a closed convex set. Let $G : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^m$ defined as $G(x, y) := Ex + Fy - g$ be an affine map, where $E \in \mathbb{R}^{m \times n_1}$, $F \in \mathbb{R}^{m \times n_2}$ and $g \in \mathbb{R}^m$. Assume that E, F satisfy the following:*

1. $\text{Kernel}([E \ F]) \subseteq \text{lin.space}(K)$,
2. $\text{Kernel}([E \ F])$ is generated by points in the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.

Then

$\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed $\Leftrightarrow \text{conv}[G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]$ is closed.

Proof. Let us denote $W = \text{Kernel}([E \ F])$, and let G_{W^\perp} be the restriction of G to W^\perp .

Observe that since G_{W^\perp} is an affine map we obtain that

$$G_{W^\perp} \left[\text{conv}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})] \right] = \text{conv} \left[G_{W^\perp}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})] \right]. \quad (17)$$

Now, by using (2.) and (3.) of Lemma 21 and the fact that $W \subseteq \text{lin.space}(K)$ we have

$$\begin{aligned} \text{conv} \left[G_{W^\perp}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})] \right] &= \text{conv} \left[G_{W^\perp}(K \cap W^\perp) \cap G_{W^\perp}(\text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \right] \\ &= \text{conv} \left[G_{W^\perp}(\text{Proj}_{W^\perp}(K)) \cap G_{W^\perp}(\text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \right] \\ &= \text{conv} [G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]. \end{aligned} \quad (18)$$

By combining (17) and (18), we obtain

$$G_{W^\perp} \left[\text{conv}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})] \right] = \text{conv} [G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]. \quad (19)$$

Moreover, by (1.) of Lemma 21, the mapping G_{W^\perp} is an homeomorphism from W^\perp to $G(\mathbb{R}^n)$. Hence, as a consequence of (19) we obtain that

$$\text{conv}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})] \text{ is closed} \Leftrightarrow \text{conv} [G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})] \text{ is closed.} \quad (20)$$

On the other hand, since $W \subseteq \text{lin.space}(K)$ and W is generated by points in the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, by Proposition 10 we have that

$$\text{conv}[(K \cap W^\perp) \cap \text{Proj}_{W^\perp}(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})] \text{ is closed} \Leftrightarrow \text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \text{ is closed.} \quad (21)$$

Therefore, by putting together identities (20) and (21) we conclude that

$$\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \text{ is closed} \Leftrightarrow \text{conv}[G(K) \cap G(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})] \text{ is closed.}$$

■

3.3.6 Proofs of Proposition 4 and Proposition 6

In [35] some properties of closedness of mixed-integer hulls are presented for the case $\mathcal{L} = \mathbb{Z}^n$. In this section we show the extension of some of these results to the case of a general mixed-integer lattice \mathcal{L} . Since the proof techniques of the results of this section are a simple generalization of those in [35], we only present the proofs of Proposition 4 and Proposition 6. We begin with some definitions and preliminary results.

We recall the following theorem we have already used in Chapter 2.

Theorem 7 (Theorem 18.5 [74]). *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Let S be the set of extreme points of K and let D be the set of extreme rays of $\text{rec}(K)$. Then $K = \text{conv}(S) + \text{cone}(D)$.*

The following definition is a generalization of Definition 3 in Section 2.2

Definition 8 ($u(K, \mathcal{L})$). *Given a convex set $K \subseteq \mathbb{R}^n$ and $u \in K \cap \mathcal{L}$, we define $u(K, \mathcal{L}) = \{d \in \mathbb{R}^n \mid u + \lambda d \in \text{conv}(K \cap \mathcal{L}) \forall \lambda \geq 0\}$.*

The following result, modified from [35], is a characterization of closedness of mixed-integer hulls for general closed convex sets.

Theorem 15. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. If $\text{conv}(K \cap \mathcal{L})$ is closed, then $u(K, \mathcal{L})$ is identical for all $u \in K \cap \mathcal{L}$. Conversely, if $u(K, \mathcal{L})$ is identical for all $u \in K \cap \mathcal{L}$ and K contains no lines, then $\text{conv}(K \cap \mathcal{L})$ is closed.*

For a proof of Theorem 15 see Section 3.7.1 in the Appendix. The following lemma, a generalization of a result from [35], is also crucial.

Lemma 22. *Let $K \subseteq \mathbb{R}^n$ be a full-dimensional closed convex set and let $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. If $\{u + \lambda d \mid \lambda > 0\} \subseteq \text{int}(K)$, then $\{u + \lambda d \mid \lambda \geq 0\} \subseteq \text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$.*

Now we present the proofs of Proposition 4 and Proposition 6.

Proposition 4. *Let $K \subseteq \mathbb{R}^n$ be a closed strictly convex set that is generated by a closed strictly convex set, $t \in \mathbb{R}^n$ and \mathcal{L} a mixed-integer lattice. Then $\text{conv}(K \cap [\mathcal{L} + t])$ is closed.*

Proof. Case 1: $t = 0$. Since $\text{conv}(K \cap \mathcal{L}) = \text{conv}([K \cap \text{aff}(K \cap \mathcal{L})] \cap \mathcal{L})$ and since by Lemma 15 we have that $K \cap \text{aff}(K \cap \mathcal{L})$ is a strictly convex set, we may assume that $K = K \cap \text{aff}(K \cap \mathcal{L})$. Moreover, since invertible affine functions map closed strictly convex sets to closed strictly convex sets, by Proposition 9, we may assume that K is a full-dimensional strictly convex set and that $\mathcal{L} = \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.

First note that if K is bounded or if $K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$, then $\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed. Therefore, we assume that K is unbounded and $K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$.

We first verify that K does not contain a line. Assume by contradiction that K contains a line in the direction $r \neq 0$. Examine $x \in \text{bd}(K)$. Then points of the form $x + \lambda r$ and $x - \lambda r$ belong to K , where $\lambda > 0$. In particular, $x + \lambda r, x - \lambda r \in \text{bd}(K)$ since $x \in \text{bd}(K)$. However, this contradicts the fact that K is strictly convex.

For ease of notation, for $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ let us denote $u(K) := u(K, \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. We show next that $u(K) = \text{rec}(K)$ for all $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Consider a point $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Clearly, as K is a closed set and $\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \subseteq K$, we obtain $u(K) \subseteq \text{rec}(K)$. We now show the other inclusion. Let $r \in \text{rec}(K)$. Since K is strictly convex, we obtain that that set $\{u + \lambda r \mid \lambda > 0\}$ is contained in the interior of K . Therefore, by Lemma 22 we obtain that the set $\{u + \lambda r \mid \lambda \geq 0\}$ is contained in $\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$, so $r \in u(K)$. Thus, $u(K) = \text{rec}(K)$ for all $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Therefore, by Theorem 15 we obtain that $\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed.

Case 2: $t \neq 0$. Observe that

$$\begin{aligned}
\text{conv}(K \cap (\mathcal{L} + t)) &= \text{conv}([(K - t) \cap \mathcal{L}] + t) \\
&= \text{conv}([(K - t) \cap \mathcal{L}]) + t.
\end{aligned} \tag{22}$$

On the other hand, since K is a strictly convex set, we obtain that $K - t$ is also a strictly convex set. Thus, by **Case 1**, we have that $\text{conv}([(K - t) \cap \mathcal{L}])$ is closed. Therefore, by equation (22), we conclude that $\text{conv}(K \cap (\mathcal{L} + t))$ is a closed set. ■

Proposition 6. *Let C be a closed pointed convex cone in \mathbb{R}^n and let \mathcal{L} be a mixed-integer lattice. Then $\overline{\text{conv}}(C \cap \mathcal{L}) = C \cap W$, where $W = \text{aff}(C \cap \mathcal{L})$. In particular, $\text{conv}(C \cap \mathcal{L})$ is closed if and only if every extreme ray of $C \cap W$ can be scaled by a non-zero scalar to belong to \mathcal{L} .*

Proof. Since $\overline{\text{conv}}(C \cap \mathcal{L}) = \overline{\text{conv}}((C \cap W) \cap \mathcal{L})$ and $C \cap W$ is a pointed closed convex cone, we may assume that $C = C \cap W$. Furthermore, by Proposition 9 we can map C to a full-dimensional closed pointed convex cone $C' = \Phi(C) \subseteq \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, where Φ is an affine map that also satisfies $\Phi(\mathcal{L} \cap W) = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$. Since Φ is an invertible affine mapping, we have that $\overline{\text{conv}}(C \cap \mathcal{L}) = C$ is equivalent to $\overline{\text{conv}}(C' \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) = C'$, and that the closedness of $\text{conv}(C \cap \mathcal{L})$ is equivalent to the closedness of $\text{conv}(C' \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$. Therefore, in order to prove the result, we may assume that C is a full-dimensional closed pointed convex cone and that $\mathcal{L} = \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.

We need to show that $\overline{\text{conv}}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) = C$ and that $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed if and only if every extreme ray of C can be scaled by a non-zero scalar to belong to $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.

We first verify that $\overline{\text{conv}}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) = C$. By convexity of C , we obtain that $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \subseteq C$. Since C is also closed, we obtain that $\overline{\text{conv}}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \subseteq C$. This shows one inclusion. To show the other inclusion, let $r \in \text{int}(C)$. Clearly, we have $\{0 + \lambda r \mid \lambda > 0\} \subseteq \text{int}(C)$. So, by Lemma 22 we obtain $\{0 + \lambda r \mid \lambda \geq$

$0\} \subseteq \text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$. Hence, $\text{int}(C) \subseteq \text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$. Since C is a full-dimensional closed convex set, we have $C = \overline{\text{int}(C)}$. Thus, by taking the closure on both sides of the inclusion $\text{int}(C) \subseteq \text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$, we obtain $C \subseteq \overline{\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))}$.

We now verify that $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed if and only if every extreme ray of C can be scaled by a non-zero scalar to belong to $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$. Suppose $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed. Then $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) = C$. If r is any extreme ray of C , then observe that $C \setminus \{\lambda r \mid \lambda > 0\}$ is a convex set. Since $\{\lambda r \mid \lambda > 0\} \subseteq \text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$, there must be a point $x \in (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ in the set $\{\lambda r \mid \lambda > 0\}$. In other words, r can be scaled by a non-zero scalar to belong to $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.

Now assume that every extreme ray of C can be scaled by a non-zero scalar to belong to $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$. Let R be the set of all extreme rays of C . Then observe that

$$C = \text{cone}(R) \subseteq \text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \subseteq C,$$

where the first equality follows from Theorem 7. Thus, $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) = C$ or equivalently $\text{conv}(C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed. \blacksquare

3.3.7 Proofs of Proposition 5 and Proposition 7

We start with some preliminary results.

Lemma 23. *Let $C \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone. Let $t \in \mathbb{R}^m$, $u \in \text{bd}(C+t) \setminus \{t\}$ and let $r \in C$. Then*

$$u + r \in \text{bd}(C+t) \Leftrightarrow r = \lambda(u - t), \text{ for some } \lambda \geq 0.$$

Proof.

Without loss of generality, we may assume that $t = 0$. We need to prove that $u + r \in \text{bd}(C) \Leftrightarrow r = \lambda u$, for some $\lambda \geq 0$. Assume that C is generated by the $(m-1)$ -dimensional bounded closed strictly convex set $S \subseteq \mathbb{R}^m$.

(\Leftarrow) If $r = \lambda u$, for some $\lambda \geq 0$, then $u + r = (1 + \lambda)u$. Thus, since $u \in \text{bd}(\mathbf{C})$, we obtain that $u + r \in \text{bd}(\mathbf{C})$

(\Rightarrow) The case $r = 0$ is straightforward. Let us assume that $r \neq 0$. Observe that, since $u, r \in \mathbf{C} \setminus \{0\}$, we have that there exists $\alpha, \beta > 0$, $\hat{u}, \hat{r} \in \mathbf{S}$ such that $\hat{u} = \alpha u$ and $\hat{r} = \beta r$. We obtain that

$$u + r = \left(\frac{\alpha + \beta}{\alpha\beta} \right) \frac{\beta\hat{u} + \alpha\hat{r}}{\alpha + \beta}. \quad (23)$$

Let us assume for contradiction that $r \neq \lambda u$, for all $\lambda \geq 0$. Hence, the definition of generator ($\mathbf{C} = \text{cone}(\mathbf{S})$ and \mathbf{S} is a $(m - 1)$ -dimensional bounded closed strictly convex set) implies that $\hat{u} \neq \hat{r}$.

Since $\hat{u}, \hat{r} \in \mathbf{S}$, $\hat{u} \neq \hat{r}$ and \mathbf{S} is a strictly convex set, we have that $\frac{\beta\hat{u} + \alpha\hat{r}}{\alpha + \beta} \in \text{rel.int}(\mathbf{S})$. Therefore, by equation (23) we conclude that $u + r \in \text{int}(\mathbf{C})$, a contradiction. ■

We need the following corollary to Lemma 22.

Corollary 4. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set and let \mathcal{L} be a mixed-integer lattice. Let $u \in K \cap \mathcal{L}$. Assume that $\text{aff}(K) = \text{aff}(K \cap \mathcal{L})$. If $\{u + \lambda d \mid \lambda > 0\} \subseteq \text{rel.int}(K)$, then $\{u + \lambda d \mid \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathcal{L})$.*

Proof. Let $\Phi : \text{aff}(K \cap \mathcal{L}) \rightarrow \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ be an invertible affine mapping as in Proposition 9 such that $K' = \Phi(K \cap \text{aff}(K \cap \mathcal{L}))$ is a full-dimensional set and $\Phi(\mathcal{L} \cap \text{aff}(K \cap \mathcal{L})) = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$. The properties of K' and Φ imply that $\{u + \lambda d \mid \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathcal{L})$ is equivalent to $\{\Phi(u) + \lambda\Phi(d) \mid \lambda \geq 0\} \subseteq \text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))$. We will show next the inclusion $\{\Phi(u) + \lambda\Phi(d) \mid \lambda \geq 0\} \subseteq \text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))$.

Since by assumption $\text{aff}(K) = \text{aff}(K \cap \mathcal{L})$, we obtain that $K' = \Phi(K)$. This implies that $\text{int}(K') = \Phi(\text{rel.int}(K))$. Therefore, since $\{u + \lambda d \mid \lambda > 0\} \subseteq \text{rel.int}(K)$, we obtain

$\{\Phi(u) + \lambda\Phi(d) \mid \lambda > 0\} \subseteq \text{int}(K')$. Moreover, since $\Phi(u) \in \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$, by Lemma 22, we conclude $\{\Phi(u) + \lambda\Phi(d) \mid \lambda \geq 0\} \subseteq \text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))$. ■

We need the following technical lemma.

Lemma 24. *Let \mathcal{L} be a mixed-integer lattice and let $V = \text{aff}(\mathcal{L})$. Let $K \subseteq V$ be a non-empty closed convex set such that $\text{aff}(\text{rec}(K)) = V$. Then $\text{aff}(\text{conv}(K \cap \mathcal{L})) = V$.*

Proof. First notice that since $K \neq \emptyset$ and $\text{aff}(\text{rec}(K)) = V$, we obtain that $\text{aff}(K) = V$. Let $\Psi_{\mathcal{L}} : V \rightarrow \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ be an invertible mapping as in Lemma 18 such that $\Psi_{\mathcal{L}}(\mathcal{L}) = \mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}$ and $K' = \Psi_{\mathcal{L}}(K) \subseteq \mathbb{R}^{p_1+p_2}$ is a full-dimensional closed convex set. Also by construction we have that $\text{aff}(\text{conv}(K \cap \mathcal{L})) = V$ if and only if $\text{aff}(\text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))) = \mathbb{R}^{p_1+p_2}$. Moreover, $\text{aff}(\text{rec}(K')) = \mathbb{R}^{p_1+p_2}$, that is $\text{rec}(K')$ is full-dimensional. Let $T \subseteq \text{rec}(K')$ be a rational polyhedral full-dimensional cone. Let $v \in K' \cap \mathbb{Q}^{p_1+p_2}$, that is v be a rational point in K' . Then $v+T \subseteq K'$ and $v+T$ is a rational polyhedron with a full-dimensional recession cone. In particular, $(v+T) \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}) \neq \emptyset$. Therefore $\text{rec}(\text{conv}((v+T) \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))) = T$. Since $K' \supseteq v+T$, we obtain that $\text{rec}(\text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))) \supseteq T$ or equivalently $\text{aff}(\text{rec}(\text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2})))) = \mathbb{R}^{p_1+p_2}$. Thus, $\text{aff}(\text{conv}(K' \cap (\mathbb{Z}^{p_1} \times \mathbb{R}^{p_2}))) = \mathbb{R}^{p_1+p_2}$, as required. ■

We also recall a lemma we used in Chapter 2.

Lemma 2 (Corollary 8.3.1 in [74]). *Let $K \subseteq \mathbb{R}^n$ be a convex set. Then*

$$\text{rec}(\text{rel.int}(K)) = \text{rec}(\overline{K}) \supseteq \text{rec}(K).$$

We now present the proofs of Proposition 5 and Proposition 7.

Proposition 5. *Let $C \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set. Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}$ be a mixed-integer lattice, where A, B are rational matrices. Denote $V = \text{aff}(\mathcal{L})$, and let $b \in (V \cap \mathbb{Q}^m) \setminus \mathcal{L}$. Then $\text{conv}((C \cap V) \cap (\mathcal{L} - b))$ is closed.*

Proof. Observe that

$$\begin{aligned}
\text{conv}((\mathbf{C} \cap V) \cap (\mathcal{L} - b)) &= \text{conv}((\mathbf{C} \cap V) \cap (\mathcal{L} - b)) + (b - b) \\
&= \text{conv}([(\mathbf{C} \cap V) + b] \cap [(\mathcal{L} - b) + b]) - b \quad (24) \\
&= \text{conv}([(\mathbf{C} \cap V) + b] \cap \mathcal{L}) - b.
\end{aligned}$$

Thus, we conclude that $\text{conv}((\mathbf{C} \cap V) \cap (\mathcal{L} - b))$ is closed if and only if $\text{conv}([(\mathbf{C} \cap V) + b] \cap \mathcal{L})$ is closed.

Denote $\mathbf{C}_V := \mathbf{C} \cap V$.

If $\dim(\mathbf{C}_V) \leq 1$, then it is straightforward to verify that $\text{conv}(\mathbf{C}_V \cap (\mathcal{L} - b))$ is closed. Therefore, we assume $\dim(\mathbf{C}_V) \geq 2$. We claim that $\text{aff}(\mathbf{C}_V + b) = \text{aff}((\mathbf{C}_V + b) \cap \mathcal{L})$: By the application of (1.) of Lemma 16 and Lemma 14 we obtain that $\text{aff}(\mathbf{C}_V) = V$. Let $K := (\mathbf{C} + b) \cap V = \mathbf{C}_V + b$. Then $\text{rec}(K) = \mathbf{C}_V$. Thus $\text{aff}(\text{rec}(K)) = V$. Therefore by applying Lemma 24 with the above defined K , we obtain that $\text{aff}((\mathbf{C}_V + b) \cap \mathcal{L}) = \text{aff}(\text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})) = V$. Thus, since $\text{aff}((\mathbf{C}_V + b) \cap \mathcal{L}) \subseteq \text{aff}(\mathbf{C}_V + b) \subseteq V$, we obtain that $\text{aff}(\mathbf{C}_V + b) = V = \text{aff}((\mathbf{C}_V + b) \cap \mathcal{L})$.

We will prove that $\text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})$ is closed by using Theorem 15 and showing that for all $u \in (\mathbf{C}_V + b) \cap \mathcal{L}$ we have that $u(\mathbf{C}_V + b, \mathcal{L}) := \{d \in \mathbb{R}^m \mid u + \lambda d \in \text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})\}$ is the same cone. In particular, we will show that $u(\mathbf{C}_V + b, \mathcal{L}) = \mathbf{C}_V$ for all $u \in (\mathbf{C}_V + b) \cap \mathcal{L}$. To simplify the notation, we will write $u(\mathbf{C}_V + b)$ instead of $u(\mathbf{C}_V + b, \mathcal{L})$ for the rest of the proof.

Notice that since $\text{conv}((\mathbf{C}_V + b) \cap \mathcal{L}) \subseteq \mathbf{C}_V + b$, $\mathbf{C}_V + b$ is a closed set and $\text{rec}(\mathbf{C}_V + b) = \mathbf{C}_V$, the inclusion $u(\mathbf{C}_V + b) \subseteq \mathbf{C}_V$ is straightforward to obtain.

To prove the other inclusion we consider two cases.

Case 1: $u \in \text{rel.int}(\mathbf{C}_V + b) \cap \mathcal{L}$. To show that $\mathbf{C}_V \subseteq u(\mathbf{C}_V + b)$ it suffices to verify that $u + \mathbf{C}_V \subseteq \text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})$. By using Lemma 2 we obtain that $\mathbf{C}_V = \text{rec}(\mathbf{C}_V + b) = \text{rec}(\text{rel.int}(\mathbf{C}_V + b))$. Thus, we have $u + \mathbf{C}_V \subseteq \text{rel.int}(\mathbf{C}_V + b)$. Also we have verified that $\text{aff}(\mathbf{C}_V + b) = \text{aff}((\mathbf{C}_V + b) \cap \mathcal{L})$. Therefore, we obtain by Corollary 4 that $u + \mathbf{C}_V \subseteq \text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})$.

Case 2: $u \in \text{rel.bd}(\mathbf{C}_V + b) \cap \mathcal{L}$. Observe first that since $b \in (V \cap \mathbb{Q}^m) \setminus \mathcal{L}$, we obtain that $u \neq b$. Let $d \in \mathbf{C}_V$. We want to show that $d \in u(\mathbf{C}_V + b)$. We have two subcases.

- Let $d \in \mathbf{C}_V \setminus \{\alpha(u-b) \mid \alpha \geq 0\}$. Since $u \neq b$ and $d \neq \alpha(u-b)$ for all $\alpha \geq 0$, Lemma 23 implies that $u + \lambda d \in \text{int}(\mathbf{C} + b)$, for all $\lambda \geq 0$. Since $\text{rel.int}(\mathbf{C}_V + b) = \text{int}(\mathbf{C} + b) \cap V$ (a consequence of Lemma 17 and $\dim(\mathbf{C}_V) \geq 2$), and $u, d \in V$ we obtain that $\{u + \lambda d \mid \lambda > 0\} \subseteq \text{rel.int}(\mathbf{C}_V + b)$. Thus, since $\text{aff}(\mathbf{C}_V + b) = \text{aff}((\mathbf{C}_V + b) \cap \mathcal{L})$, by Corollary 4 we have that $d \in u(\mathbf{C}_V + b)$.
- Now, let $d \in \{\alpha(u-b) \mid \alpha \geq 0\}$. The case $d = 0$ is straightforward to verify. Let us assume then that $d = \alpha(u-b)$, where $\alpha > 0$. Since $\{u + \lambda \alpha(u-b) \mid \lambda \geq 0\} = \{u + \lambda(u-b) \mid \lambda \geq 0\}$, to show that $d \in u(\mathbf{C}_V + b)$ it suffices to show that $u-b \in u(\mathbf{C}_V + b)$. As a consequence of $b \in (V \cap \mathbb{Q}^m) \setminus \mathcal{L}$ and $u \in \mathcal{L}$, we obtain that $u-b \in \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Q}^{p_1} \setminus \mathbb{Z}^{p_1}, y \in \mathbb{Q}^{p_2}\}$. Hence, since $u \in \mathcal{L}$ we obtain that $\{u + \lambda(u-b) \mid \lambda \geq 0\} \subseteq \text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})$. Thus, we conclude that $u-b \in u(\mathbf{C}_V + b)$, and therefore, $d \in u(\mathbf{C}_V + b)$.

This finishes the proof of $\mathbf{C}_V \subseteq u(\mathbf{C}_V + b)$. Therefore, we obtain $u(\mathbf{C}_V + b) = \mathbf{C}_V$ for all $u \in (\mathbf{C}_V + b) \cap \mathcal{L}$. We conclude, by Theorem 15 that $\text{conv}((\mathbf{C}_V + b) \cap \mathcal{L})$ is closed. ■

Proposition 7. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set. Let $\mathcal{L} = \{x \in \mathbb{R}^m \mid x = Az + By, z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$ be a mixed-integer lattice, where A, B are rational matrices. Denote $V = \text{aff}(\mathcal{L})$. Then*

1. *Assume $\dim(\mathbf{C} \cap V) = 2$. If $\dim(\langle B \rangle) \geq \dim(V) - 1$, then every extreme ray of $\mathbf{C} \cap V$ can be scaled by a non-zero scalar to belong to \mathcal{L} .*
2. *Assume $\dim(\mathbf{C} \cap V) \geq 3$. Then $\dim(\langle B \rangle) \geq \dim(V) - 1$ if and only if every extreme ray of $\mathbf{C} \cap V$ can be scaled by a non-zero scalar to belong to \mathcal{L} .*

Proof.

Since $[A \ B] \in \mathbb{Q}^{m \times n}$, by Proposition 8 we obtain that \mathcal{L} is a mixed integer lattice, that is, for some $A' \in \mathbb{R}^{m \times p_1}$ and $B' \in \mathbb{R}^{m \times p_2}$, such that $[A' \ B']$ has linearly independent columns, $V = \langle [A \ B] \rangle = \langle [A' \ B'] \rangle$, $\langle B' \rangle = \langle B \rangle$, and we have

$$\mathcal{L} = \{x \in \mathbb{R}^m \mid x = A'z + B'y, z \in \mathbb{Z}^{p_1}, y \in \mathbb{R}^{p_2}\}.$$

1. We will show that whenever $\dim(\mathbf{C} \cap V) \geq 2$ the following implication is true: If $\dim(\langle B \rangle) \geq \dim(V) - 1$, then every extreme ray of $\mathbf{C} \cap V$ can be scaled by a non-zero scalar to belong to \mathcal{L} .

The proof for the case $\dim(\langle B' \rangle) = \dim(V)$ is straightforward and so it is omitted. Assume for the rest of the proof that $\dim(\langle B' \rangle) = \dim(V) - 1$. Since $\dim(\langle B' \rangle) = \dim(V) - 1$, we have that $A' = a$, where $a \in \mathbb{Q}^m$. Since $[a \ B']$ is a linearly independent set generating V , we have

$$V = \langle a \rangle + \langle B' \rangle. \tag{25}$$

Now, let r be an extreme ray of $\mathbf{C} \cap V$. We next show that there exists $r' \in \mathcal{L}$, such that r and r' generate the same extreme ray of $\mathbf{C} \cap V$, that is, there exists $\beta > 0$ such that $r' = \beta r$. Using (25) we can write

$$r = \lambda a + b, \tag{26}$$

where $\lambda \in \mathbb{R}$ and $b \in \langle B' \rangle$. If $\lambda = 0$, then we can take $r' = r$. If $\lambda \neq 0$, consider $r' = \frac{1}{|\lambda|}r$. By equation (26) we have $r' = \frac{\lambda}{|\lambda|}a + \frac{1}{|\lambda|}b$. Thus, we conclude that $r' \in \mathcal{L}$, as desired.

2. By (1.), we only need to prove that the following implication is true: if every extreme ray of $\mathbf{C} \cap V$ can be scaled by non-zero to belong to \mathcal{L} and $\dim(\mathbf{C} \cap V) \geq 3$, then $\dim(\langle B' \rangle) \geq \dim(V) - 1$.

Note that \mathbf{C} is generated by an $(m - 1)$ -dimensional bounded closed strictly convex set, call this set $\mathbf{S} \subseteq \mathbb{R}^m$.

We show first that $\dim(\langle B' \rangle) \geq 2$. Let $r \in \text{rel.bd}(\mathbf{S} \cap V)$ be an extreme ray of $\mathbf{C} \cap V$ (Lemma 17). By hypothesis, there exists $\lambda_r \geq 0$, $z_r \in \mathbb{Z}^{p_1}$ such that $r \in \lambda_r[(A'z_r + \langle B' \rangle) \cap \mathbf{C}]$. Since $C_{z_r} := \text{cone}[(A'z_r + \langle B' \rangle) \cap \mathbf{C}] \subseteq \mathbf{C} \cap V$, we obtain that r must define an extreme ray of C_{z_r} . Now, assume for a contradiction that $\dim(\langle B' \rangle) \leq 1$. Then, since $\dim((A'z_r + \langle B' \rangle) \cap \mathbf{C}) \leq 1$, we obtain that $\dim(C_{z_r}) \leq 2$. Thus, we have that every cone C_{z_r} has at most two extreme rays. In particular, no more than two distinct elements of $\text{rel.bd}(\mathbf{S} \cap V)$ can define an extreme ray of a cone of the form C_{z_r} , for some $r \in \text{rel.bd}(\mathbf{S} \cap V)$. Therefore, since \mathbb{Z}^{p_1} is countable and every element in $\text{rel.bd}(\mathbf{S} \cap V)$ defines an extreme ray of a cone of the form C_{z_r} , for some $r \in \text{rel.bd}(\mathbf{S} \cap V)$, we obtain that the set $\text{rel.bd}(\mathbf{S} \cap V)$ is countable. On the other hand, since $\dim(\mathbf{C} \cap V) \geq 3$, by (2.(c)) of Lemma 17 we obtain $\text{rel.bd}(\mathbf{S} \cap V)$ is uncountable, a contradiction.

We show next that

$$\mathbf{C} \cap V \subseteq \bigcup_{z \in \mathbb{Z}^{p_1}} \text{cone}[A'z + \langle B' \rangle]. \quad (27)$$

Let $x \in \mathbf{C} \cap V$. We have two cases depending if $x \in \text{rel.bd}(\mathbf{C}) \cap V$ or not. Observe that by Lemma 16, we have $\text{rel.bd}(\mathbf{C} \cap V) = \text{rel.bd}(\mathbf{C}) \cap V$.

Case 1: $x \in \text{rel.bd}(\mathbf{C} \cap V)$. The case $x = 0$ is straightforward. Let us assume $x \neq 0$. Then, there exists $r \in \text{rel.bd}(\mathbf{S} \cap V)$ and $\lambda > 0$ such that $x = \lambda r$. Since, by hypothesis, there exists $\lambda_r \geq 0$, $z_r \in \mathbb{Z}^{p_1}$ such that $r \in \lambda_r[(A'z_r + \langle B' \rangle) \cap \mathbf{C}]$, we conclude that $x \in \bigcup_{z \in \mathbb{Z}^{p_1}} \text{cone}[A'z + \langle B' \rangle]$.

Case 2: $x \notin \text{rel.bd}(\mathbf{C} \cap V)$. Consider the affine subspace $x + \langle B' \rangle$. Since $x \in \text{rel.int}(\mathbf{C} \cap V)$ and $\dim(\mathbf{C} \cap V) \geq 2$, we obtain that $x \in \text{int}(\mathbf{C})$ (by Lemma 16 and Lemma 14). This implies that $\dim(\mathbf{C} \cap (x + \langle B' \rangle)) = \dim(\langle B' \rangle) \geq 2$. Thus, by (2.) of Lemma 16 applied to the cone \mathbf{C} and the affine subspace $x + \langle B' \rangle$,

we obtain that $[\text{bd}(\mathbf{C}) \setminus \{0\}] \cap [(x + \langle B' \rangle)] \neq \emptyset$. Therefore, since $x + \langle B' \rangle \subseteq V$ we have that there exists $r \in \text{rel.bd}(\mathbf{S} \cap V)$ and $\lambda > 0$ such that $\lambda r \in x + \langle B' \rangle$. We have

$$\begin{aligned} \lambda r \in x + \langle B' \rangle &\Leftrightarrow \lambda r - x \in \langle B' \rangle \\ &\Leftrightarrow x - \lambda r \in \langle B' \rangle \\ &\Leftrightarrow x \in \lambda r + \langle B' \rangle \\ &\Leftrightarrow \frac{x}{\lambda} \in r + \langle B' \rangle. \end{aligned}$$

By hypothesis, let $\lambda_r > 0$, $z_r \in \mathbb{Z}^{p_1}$ such that $r \in \lambda_r[(A'z_r + \langle B' \rangle) \cap \mathbf{C}]$. Since $r \in \lambda_r[A'z_r + \langle B' \rangle]$ and $\lambda_r[A'z_r + \langle B' \rangle] + \langle B' \rangle = \lambda_r[A'z_r + \langle B' \rangle]$ we obtain $r + \langle B' \rangle \subseteq \lambda_r[A'z_r + \langle B' \rangle]$. Thus, we have

$$\frac{x}{\lambda} \in \lambda_r[A'z_r + \langle B' \rangle].$$

Therefore, $x \in \text{cone}[A'z_r + \langle B' \rangle]$. This proves (27).

Since $\dim(\mathbf{C} \cap V) = \dim(V)$ and \mathbb{Z}^{p_1} is a countable set, by equation (27) we obtain that there exists $z \in \mathbb{Z}^{p_1} \setminus \{0\}$ such that $\dim(\text{cone}(A'z + \langle B' \rangle)) = \dim(V)$. Therefore, since $\dim(\text{cone}(A'z + \langle B' \rangle)) \leq \dim(\langle B' \rangle) + 1$, we conclude $\dim(\langle B' \rangle) = \dim(\langle B' \rangle) \geq \dim(V) - 1$, as desired. ■

3.3.8 Final step of the proof of Theorem 11

Recall the set $\mathbf{P} := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid Ax + By - b \in \mathbf{C}\}$, the affine map $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^m$ defined as $T(x, y) = Ax + By - b$, and the affine subspace $V = T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. We obtain the following corollary to Proposition 3.

Corollary 5. *If $\text{lin.space}(\mathbf{P}) = \text{Kernel}([A \ B])$ is a rational subspace, then*

$\text{conv}(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed $\Leftrightarrow \text{conv}[(\mathbf{C} \cap V) \cap T((\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))]$ is closed.

In the rest of this section, we will present the proof of Theorem 11.

Theorem 11. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set. Let $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$ and $b \in \mathbb{Q}^m$. Let*

$$\mathbf{P} := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid Ax + By - b \in \mathbf{C}\},$$

$V := \{Ax + By - b \mid (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\}$ and $\mathcal{L} := \{Ax + By \mid (x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}\}$. Then $\text{conv}(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed if and only if one of the following holds:

1. $b \notin \mathcal{L}$.
2. $b \in \mathcal{L}$, and $\dim(\mathbf{C} \cap V) \leq 1$.
3. $b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) = 2$, $\dim(\langle B \rangle) \leq 0$ and the two extreme rays of the cone $\mathbf{C} \cap V$ can be scaled by a non-zero scalar so that they belong to the lattice $\{Ax \mid x \in \mathbb{Z}^{n_1}\}$.
4. $b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) \geq 2$ and $\dim(\langle B \rangle) \geq \dim(V) - 1$.

Proof. First, since A and B are rational by Corollary 5, the closedness of $\text{conv}[\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]$ is equivalent to the closedness of $\text{conv}[\mathbf{C} \cap V \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]$. Therefore, it suffices to show that conditions (1.), (2.), (3.), and (4.) of Theorem 11 are equivalent to the closedness of $\text{conv}[(\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]$.

Notice that \mathcal{L} is a mixed-integer lattice, $T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \mathcal{L} - b$ and that if $b \in \mathcal{L}$, then $\mathcal{L} = T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$.

(\Rightarrow) By contrapositive.

Case 1: If $b \in \mathcal{L}$ and $\dim(\mathbf{C} \cap V) = 2$, $\dim(\langle B \rangle) \leq 0$ and not all the extreme rays of the cone $\mathbf{C} \cap V$ belong to the lattice $\{Ax \mid x \in \mathbb{Z}^{n_1}\}$, then by Proposition 6 we obtain that $\text{conv}[(\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1})]$ is not closed.

Case 2: Now assume that $b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) \geq 3$ and $\dim(\langle B \rangle) < \dim(V) - 1$. Then, by (2.) of Proposition 7, we obtain that not all of the extreme rays of the cone $\mathbf{C} \cap V$ are generated by points of the mixed-integer lattice $\mathcal{L} = T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Therefore, by Proposition 6 we conclude that $\text{conv}((\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is not closed.

(\Leftarrow)

Case 1: If $b \notin \mathcal{L}$ we consider two subcases.

- Assume $0 \notin V$. Then by (1.) of Lemma 17, we obtain that $\mathbf{C} \cap V$ is a strictly convex set. Notice that $T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \mathcal{L} - b$. Therefore, by Proposition 4, we conclude that $\text{conv}((\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed.
- Assume $0 \in V$. Then since $b \in \langle [A \ B] \rangle$, and A, B, b are rational we conclude that $b \in \{Ax + By \mid (x, y) \in (\mathbb{Q}^{n_1} \setminus \mathbb{Z}^{n_1}) \times \mathbb{Q}^{n_2}\}$. Therefore, since $T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \mathcal{L} - b$ and $V = \text{aff}(\mathcal{L})$, by Proposition 5 we conclude $\text{conv}((\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is a closed set.

Case 2: If $b \in \mathcal{L}$, then $\mathbf{C} \cap V$ is a cone. We consider three subcases.

- Assume $b \in \mathcal{L}$ and $\dim(\mathbf{C} \cap V) \leq 1$. In this case $\mathbf{C} \cap V$ is either a point or a ray, thus $\text{conv}((\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is a closed set.
- Assume $b \in \mathcal{L}$ and $\dim(\mathbf{C} \cap V) = 2$, $\dim(\langle B \rangle) \leq 0$ and that all the extreme rays of the cone $\mathbf{C} \cap V$ belong to the lattice $\{Ax \mid x \in \mathbb{Z}^n\}$. Then, by Proposition 6, we obtain that $\text{conv}[(\mathbf{C} \cap V) \cap T(\mathbb{Z}^n)]$ is closed.
- Assume $b \in \mathcal{L}$, $\dim(\mathbf{C} \cap V) \geq 2$ and $\dim(\langle B \rangle) \geq \dim(V) - 1$. Then, by Proposition 7, we obtain that all of the extreme rays of $\mathbf{C} \cap V$ are generated by points of $\mathcal{L} = T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Therefore, by Proposition 6 we conclude that $\text{conv}((\mathbf{C} \cap V) \cap T(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed.

■

3.4 Proof of Theorem 12

In this section we prove the following result.

Theorem 12. *Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional pointed closed convex cone that is generated by a closed strictly convex set. Assume that \mathbf{C} is poly-checkable. Let $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$, $b \in \mathbb{Q}^m$ and let \mathbf{P} be as defined in (11). Then there exists an algorithm that runs in polynomial-time with respect to $\text{size}(\mathbf{P})$ to check whether $\text{conv}(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed or not.*

That is, this result states that the closedness of $\text{conv}(\mathbf{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ can be checked in polynomial-time, whenever the cone \mathbf{C} is poly-checkable. To prove Theorem 12 we need to verify that all the conditions of Theorem 11 can be checked in polynomial time with respect to the size of the data. The definition of a poly-checkable cone already implies that some statements of these conditions can be verified in polynomial time. Specifically, in the case $V := \langle [A \ B] \rangle - b$ is a linear subspace, we can decide whether $\dim(\mathbf{C} \cap V) \leq 1$ or not, and in the case $\dim(\mathbf{C} \cap V) \geq 2$ we can compute $\dim(\mathbf{C} \cap V)$ and also we can check condition (3.) of Theorem 11 in polynomial time. In order to check the rest of conditions, we use the fact that the dimension of linear subspaces generated by rational matrices can be computed in polynomial time, and the following well-known result.

Lemma 25. *Let $b \in \mathbb{Q}^m$ and $\mathcal{L} = \{Ax + By \mid (x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}\}$ be a mixed-integer lattice, where $A \in \mathbb{Q}^{m \times n_1}$ and $B \in \mathbb{Q}^{m \times n_2}$. Then the condition $b \in \mathcal{L}$ can be checked in polynomial-time with respect to the size of A, B, b .*

Proof of Theorem 12. First observe that by Lemma 25, we can check if $b \in \mathcal{L}$ or not in polynomial time with respect to $\text{size}(\mathbf{P})$. This implies that we can distinguish whether we are in the case defined by condition (1.) of Theorem 11 or not. In particular, condition (1.) of Theorem 11 can be checked in polynomial time.

Now assume that $b \in \mathcal{L}$. Then $b \in \langle [A \ B] \rangle$, and so $V := \langle [A \ B] \rangle - b$ is a linear subspace. Thus, we can compute the dimension of $\langle B \rangle$ and V in polynomial time (by the Gaussian algorithm of Edmonds [40]). Since \mathbf{C} is poly-checkable, we obtain that deciding whether $\dim(\mathbf{C} \cap V) \leq 1$ or not, and in the case $\dim(\mathbf{C} \cap V) \geq 2$, computing $\dim(\mathbf{C} \cap V)$ can be done in polynomial time with respect to $\text{size}(\mathbf{P})$. This implies that we can identify which case among the ones defined by conditions (2.), (3.) or (4.) we need to analyze. Moreover, since we have already computed $\dim(\langle B' \rangle)$ and $\dim(V)$, we conclude that we can check conditions (2.) and (4.) of Theorem 11 in polynomial time. Finally, in the case given by condition (3.) of Theorem 11, since \mathbf{C} is poly-checkable, we conclude that checking this condition can be done in polynomial time with respect to $\text{size}(\mathbf{P})$. ■

3.5 *The Lorentz cone is poly-checkable*

Recall that the Lorentz cone $\mathbf{L}^m \subseteq \mathbb{R}^m$ is defined as $\mathbf{L}^m := \{(w, z) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|w\| \leq z\}$. A generator for \mathbf{L}^m is given by $\mathbf{S}^m := \{(w, 1) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|w\| \leq 1\}$. In this section, we prove the following result.

Theorem 13. *The Lorentz cone is poly-checkable.*

In order to prove Theorem 13, we need to verify that for all $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$ the following conditions are satisfied:

- (I) To decide whether $\dim(\mathbf{L}^m \cap \langle [A \ B] \rangle) \leq 1$ or not, and, whenever $\dim(\mathbf{L}^m \cap \langle [A \ B] \rangle) \geq 2$, to compute $\dim(\mathbf{L}^m \cap \langle [A \ B] \rangle)$ can be done in polynomial-time with respect to $\text{size}([A \ B])$.
- (II) To check condition (3.) of Theorem 11 in polynomial time can be done in polynomial-time with respect to $\text{size}([A \ B])$.

In Section 3.5.1 we present all the required results to prove that the Lorentz cone satisfies Condition (I). Checking that the Lorentz cone satisfies Condition (II) is more involved, and is presented in Section 3.5.2.

3.5.1 Verifying the validity of Condition (I)

In order to verify Condition (I) we will show that for an arbitrary rational matrix $[A \ B] \in \mathbb{Q}^{m \times n}$, we can decide whether $\dim(\mathbf{L}^m \cap \langle [A \ B] \rangle) \leq 1$ or not in polynomial time with respect to the size of A, B and that whenever $\dim(\mathbf{L}^m \cap \langle [A \ B] \rangle) \geq 2$ we can compute $\dim(\mathbf{L}^m \cap \langle [A \ B] \rangle)$, also in polynomial time with respect to the size of A, B .

For a linear subspace $W \subseteq \mathbb{R}^n$, recall that Proj_W denotes the orthogonal projection onto the linear subspace W .

Lemma 26. *Let $[A \ B] \in \mathbb{Q}^{m \times n}$ be a rational matrix and let $a \in \mathbb{Q}^m$ be a vector of polynomial size with respect to the size of A, B . Denote $W = \langle [A \ B] \rangle$. Then $\text{Proj}_W(a)$ can be computed in polynomial-time with respect to the size of A, B .*

The following lemma can be used to decide whether $\dim(\mathbf{L}^m \cap W) \leq 1$ or not, for an arbitrary linear subspace W ¹.

Lemma 27. *Let $W \subseteq \mathbb{R}^m$ be a linear subspace. Then*

1. $\dim(\mathbf{L}^m \cap W) \leq 1$ if and only if $(\text{int}(\mathbf{L}^m) \cap W = \emptyset \text{ or } \dim(W) \leq 1)$.
2. Let us denote $a := (0, 1) \in \mathbb{R}^{m-1} \times \mathbb{R}$. Assume $\dim(W) \geq 2$. Then

$$\dim(\mathbf{L}^m \cap W) \geq 2 \text{ if and only if } \text{Proj}_W(a) \in \text{int}(\mathbf{L}^m).$$

Proof.

¹We are grateful to Arkadi Nemirovski for a preliminary version of this idea.

1. (\Rightarrow) If $\text{int}(\mathbf{L}^m) \cap W \neq \emptyset$ and $\dim(W) \geq 2$, then we obtain by a standard result in convex analysis (see Lemma 14) that $\text{aff}(\mathbf{L}^m \cap W) = W$. Therefore, $\dim(\mathbf{L}^m \cap W) \geq 2$.

(\Leftarrow) If $\dim(W) \leq 1$, then clearly $\dim(\mathbf{L}^m \cap W) \leq 1$. On the other hand, if $\text{int}(\mathbf{L}^m) \cap W = \emptyset$, then by (1.) of Lemma 16, we obtain that $\dim(\mathbf{L}^m \cap W) \leq 1$.

2.

(\Leftarrow) If $\text{Proj}_W(a) \in \text{int}(\mathbf{L}^m)$, then $\text{int}(\mathbf{L}^m) \cap W \neq \emptyset$. Therefore, since $\dim(W) \geq 2$, by (1.) we conclude that $\dim(\mathbf{L}^m \cap W) \geq 2$.

(\Rightarrow) Now assume $\dim(\mathbf{L}^m \cap W) \geq 2$. Therefore, by (1.) of Lemma 16, we obtain that $\text{int}(\mathbf{L}^m) \cap W \neq \emptyset$. Let $(l, l_m) \in \text{int}(\mathbf{L}^m) \cap W$, where $l \in \mathbb{R}^{m-1}$ and $l_m \in \mathbb{R}$. Since W is a linear subspace, without loss of generality we may assume that $l_m = 1$. Hence $(l, l_m) \in \mathbf{S}^m$. By (2.) of Lemma 17, we have that $\mathbf{S}^m \cap W$ is a generator of the cone $\mathbf{L}^m \cap W$. Thus, since $(l, l_m) \in \text{int}(\mathbf{L}^m)$, we must have $r := \|l\| < 1$. Let $K' = \text{cone}(\{(x, 1) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|x\| = r\})$. Observe that since $r < 1$, we have $K' \cap \mathbf{S}^m \subseteq \text{rel.int}(\mathbf{S}^m)$. Thus we obtain that $K' \setminus \{0\} \subseteq \text{int}(\mathbf{L}^m)$. Let d be the distance between a and the ray $R := \{\lambda(l, l_m) \mid \lambda \geq 0\}$. Since $R \subseteq W$, we obtain that $\text{Proj}_W(a) \in B(a, d) := \{x \in \mathbb{R}^m \mid \|x - a\| \leq d\}$. Using the symmetry of K' , a simple two dimensional argument shows that $B(a, d) \subseteq K' \setminus \{0\}$. Therefore, we conclude that $\text{Proj}_W(a) \in (K' \setminus \{0\}) \subseteq \text{int}(\mathbf{L}^m)$. ■

In the following proposition we verify that Condition (I) is satisfied in the case of the Lorentz cone.

Proposition 11. *Let $[A \ B] \in \mathbb{Q}^{m \times n}$ be a rational matrix, and denote $W = \langle [A \ B] \rangle$.*

1. *The condition $\dim(\mathbf{L}^m \cap W) \leq 1$ can be checked in polynomial-time with respect to the size of A, B .*

2. If $\dim(\mathbf{L}^m \cap W) \geq 2$, then $\dim(\mathbf{L}^m \cap W)$ can be computed in polynomial-time with respect to the size of A, B .

Proof. Observe that $\dim(W) = \dim(\langle [A \ B] \rangle)$ and thus, since A, B are rational matrices, it can be computed in polynomial-time with respect to the size of A, B, b by the Gaussian algorithm of Edmonds [40].

1. Since $\dim(W)$ can be computed in polynomial-time, we can check whether $\dim(W) \leq 1$ or not in polynomial-time with respect to the size of A, B . Now, to verify whether $\dim(\mathbf{L}^m \cap W) \leq 1$ or not we use Lemma 27 as follows. If $\dim(W) \leq 1$, by (1.) of Lemma 27 we conclude that $\dim(\mathbf{L}^m \cap W) \leq 1$. If $\dim(W) \geq 2$, then by (2.) of Lemma 27 to verify whether $\dim(\mathbf{L}^m \cap W) \geq 2$ or not, we need to check if $\text{Proj}_W(a) = (u, u_m) \notin \text{int}(\mathbf{L}^m)$ or not. Thus, we only need to compute $\|u\|^2$, and compare it with u_m^2 . By Lemma 26 the size of (u, u_m) is polynomial in the size of A, B , therefore we obtain that this comparison also can be done in polynomial-time with respect to the size of A, B .
2. Since $\dim(\mathbf{L}^m \cap W) \geq 2$, then by (1.) of Lemma 16 we obtain $\dim(\mathbf{L}^m \cap W) = \dim(W)$. By previous claim, $\dim(W)$ can be computed in polynomial-time.

■

3.5.2 Verifying the validity of Condition (II)

In this section we assume $V := \{Ax \mid x \in \mathbb{R}^{n_1}\}$, $\mathcal{L} := \{Ax \mid x \in \mathbb{Z}^{n_1}\}$, $b \in \mathcal{L}$ and $\dim(\mathbf{L}^m \cap V) = 2$. Since A is a rational matrix, a basis for \mathcal{L} can be found in polynomial-time with respect to $\text{size}(A)$ by computing the Hermite normal form of A (and the size of these basis vectors are bounded by a polynomial function of $\text{size}(A)$). Let the vectors $(A_1, a_1), (A_2, a_2) \in \mathbb{Q}^{m-1} \times \mathbb{Q}$ form a basis of \mathcal{L} . We denote $S_V := \mathbf{S}^m \cap V$.

The following lemma characterizes the extreme rays of $\mathbf{L}^m \cap V$ in terms of the basis of the lattice \mathcal{L} .

Lemma 28. Let $V := \{Ax \mid x \in \mathbb{R}^n\}$. Assume that $\dim(\mathbf{L}^m \cap V) = 2$. Then

1. The numbers a_1 and a_2 cannot be both zero.
2. The relative boundary of S_V is given by the solutions of the following system of two equations:

$$\begin{aligned} \|\alpha_1 A_1 + \alpha_2 A_2\|^2 &= 1 \\ \alpha_1 a_1 + \alpha_2 a_2 &= 1. \end{aligned} \tag{28}$$

3. Let (α_1, α_2) and (α'_1, α'_2) be the solutions of the system of equations (28). Then the two extreme rays of $\mathbf{L}^m \cap V$ can be written as

$$\alpha_1 \begin{pmatrix} A_1 \\ a_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} A_2 \\ a_2 \end{pmatrix} \quad \text{and} \quad \alpha'_1 \begin{pmatrix} A_1 \\ a_1 \end{pmatrix} + \alpha'_2 \begin{pmatrix} A_2 \\ a_2 \end{pmatrix}.$$

Proof. Observe that

$$\{(A_1, a_1), (A_2, a_2)\} \text{ is a basis of } V. \tag{29}$$

1. Since $\dim(\mathbf{L}^m \cap V) = 2$, by (1.) of Lemma 16 we obtain that $\text{int}(\mathbf{L}^m) \cap V \neq \emptyset$. Thus, since $\text{int}(\mathbf{L}^m) \cap \mathbb{R}^{m-1} \times \{0\} = \emptyset$, we have that $\mathbf{L}^m \cap V \not\subseteq \mathbb{R}^{m-1} \times \{0\}$. In particular, $V \not\subseteq \mathbb{R}^{m-1} \times \{0\}$. Therefore, by (29), we conclude that a_1 and a_2 cannot be both zero.
2. Since $S_V = \mathbf{S}^m \cap V \subseteq V$ and by (29), we obtain that

$$S_V = \left\{ \alpha_1 \begin{pmatrix} A_1 \\ a_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} A_2 \\ a_2 \end{pmatrix} \mid \|\alpha_1 A_1 + \alpha_2 A_2\|^2 \leq 1; \alpha_1 a_1 + \alpha_2 a_2 = 1 \right\}.$$

3. By (2.) of Lemma 17 we have $\mathbf{L}^m \cap V = \text{cone}(S_V)$ and that r is an extreme ray of $\mathbf{L}^m \cap V$ if and only if r can be scaled to belong to the relative boundary of S_V . Therefore, the extreme rays of $\mathbf{L}^m \cap V$ can be found using equation (28). ■

Notice that by (1.) of Lemma 28 we have that either $a_1 \neq 0$ or $a_2 \neq 0$. Thus, we may assume without loss of generality throughout the rest of this section that $a_2 \neq 0$.

Lemma 29. *Let (α_1, α_2) and (α'_1, α'_2) be the solutions of the system of equations (28). Then the extreme rays of $\mathbf{L}^m \cap V$ can be scaled by a non-zero scalar to belong to \mathcal{L} if and only if $\alpha_1, \alpha'_1 \in \mathbb{Q}$.*

Proof.

(\Rightarrow) We use (3.) of Lemma 28 to characterize the extreme rays of $\mathbf{L}^m \cap V$ in terms of (α_1, α_2) and (α'_1, α'_2) . First we consider the extreme ray associated to (α_1, α_2) . There exists $\lambda > 0$ and $\gamma \in \mathbb{Q}^m$ such that

$$\lambda \left[\alpha_1 \begin{pmatrix} A_1 \\ a_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} A_2 \\ a_2 \end{pmatrix} \right] = \gamma. \quad (30)$$

Since $\alpha_1 a_1 + \alpha_2 a_2 = 1$, by considering the last constraint in (30) we obtain that $\lambda \in \mathbb{Q} \setminus \{0\}$. Thus, we obtain that (α_1, α_2) is the unique solution to a system of linear equations with rational data and thus α_1, α_2 are rational. Similarly α'_1, α'_2 are also rational.

(\Leftarrow) Observe first that since (α_1, α_2) and (α'_1, α'_2) are the solutions to (28), we obtain that

$$\alpha_1 a_1 + \alpha_2 a_2 = 1 \quad \text{and} \quad \alpha'_1 a_1 + \alpha'_2 a_2 = 1.$$

If $\alpha_1 = 0$, then α_2 is rational. If $\alpha_1 \neq 0$, then α_1 is rational if and only if α_2 is rational, since a_1 and a_2 are rational. Therefore in general α_1 is rational if and only if α_2 is rational. Thus by hypothesis we obtain that $(\alpha_1, \alpha_2), (\alpha'_1, \alpha'_2) \in \mathbb{Q}^2$. Hence, there exists $\lambda, \lambda' > 0$ such that $\lambda(\alpha_1, \alpha_2), \lambda'(\alpha'_1, \alpha'_2) \in \mathbb{Z}^2$. Therefore, by (3.) of Lemma 28 we obtain that the extreme rays of $\mathbf{L}^m \cap V$ can be scaled to belong to \mathcal{L} . ■

The following proposition verifies the validity of Condition (II) for the Lorentz cone.

Proposition 12. *If $\dim(\mathbf{L}^m \cap V) = 2$, then whether the two extreme rays of the cone $\mathbf{L}^m \cap V$ can be scaled by a non-zero scalar to belong to \mathcal{L} can be checked in polynomial-time.*

Proof. Let (α_1, α_2) and (α'_1, α'_2) be the solutions of the system of equations (28). Since $a_2 \neq 0$, we can write $\alpha_2 = \frac{1-\alpha_1 a_1}{a_2}$ and $\alpha'_2 = \frac{1-\alpha'_1 a_1}{a_2}$. Therefore, by Lemma 29, in order to check whether the extreme rays of the cone $\mathbf{L}^m \cap V$ can be scaled to belong to \mathcal{L} , we only need to verify if the solutions α_1, α'_1 to the quadratic equation

$$\left\| \alpha A_1 + \frac{1-\alpha a_1}{a_2} A_2 \right\|^2 = 1 \quad (31)$$

belong to \mathbb{Q} . We will show that this can be done in polynomial-time with respect to the data $A_1, A_2 \in \mathbb{Q}^{m-1}$, $a_1, a_2 \in \mathbb{Q}$. Since the size of the product of all the denominators of the components of the vectors and the scalars appearing in (31) is polynomial with respect to the size of the original data, without loss of generality we obtain the following equivalent equation

$$\|\alpha p + q\|^2 = r, \quad (32)$$

where $p, q \in \mathbb{Z}^{m-1}$, $r \in \mathbb{Z}$ and $\text{size}(p)$, $\text{size}(q)$ and $\text{size}(r)$ are polynomial with respect to the size of the original data. Notice that equation (32) can be written as

$$\left(\sum_{i=1}^{m-1} p_i^2 \right) \alpha^2 + \left(\sum_{i=1}^{m-1} 2p_i q_i \right) \alpha + \sum_{i=1}^{m-1} q_i^2 - r = 0. \quad (33)$$

Let $c_1 = \sum_{i=1}^{m-1} p_i^2$, $c_2 = \sum_{i=1}^{m-1} 2p_i q_i$ and $c_3 = \sum_{i=1}^{m-1} q_i^2 - r$. Observe that $\text{size}(c_1)$, $\text{size}(c_2)$ and $\text{size}(c_3)$ are polynomial with respect to the size of the original data. Using this notation, and by solving the quadratic equation (33) we obtain

$$\alpha_1 = \frac{-c_2 + \sqrt{c_2^2 - 4c_1 c_3}}{2c_1} \quad \text{and} \quad \alpha'_1 = \frac{-c_2 - \sqrt{c_2^2 - 4c_1 c_3}}{2c_1}.$$

Therefore, $\alpha, \alpha' \in \mathbb{Q}$ if and only if $c_2^2 - 4c_1c_3$ is a perfect square. Since the latter can be checked in polynomial-time with respect to the size of c_1, c_2, c_3 (see, for example, Section 1.7 of [27]), we conclude that we can determine if $\alpha, \alpha' \in \mathbb{Q}$ in polynomial-time with respect to size of the original data. \blacksquare

3.6 Invariance of Closedness of Integer Hulls Under Finite Intersection in the Pure Integer Case

The proof of Theorem 14 relies on a characterization of closedness of integer hulls that we proved in a recent paper [35] (see Section 3.3.6).

We need the following straightforward corollary to Lemma 22 (see also Corollary 4).

Corollary 6. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set such that $\text{aff}(K)$ is a rational subspace. Let $u \in K \cap \mathbb{Z}^n$. If $\{u + \lambda d \mid \lambda > 0\} \subseteq \text{rel.int}(K)$, then $\{u + \lambda d \mid \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$.*

We now prove the main result in this section.

Theorem 14. *Let $K_i \subseteq \mathbb{R}^n$, $i = 1, 2$, be closed convex sets. Assume $\text{conv}(K_i \cap \mathbb{Z}^n)$ is closed, for $i = 1, 2$. If $L = \text{lin.space}(K_1 \cap K_2)$ is generated by integer points, then $\text{conv}[(K_1 \cap K_2) \cap \mathbb{Z}^n]$ is closed.*

Proof. If $(K_1 \cap K_2) \cap \mathbb{Z}^n = \emptyset$, then we are done. Assume $(K_1 \cap K_2) \cap \mathbb{Z}^n \neq \emptyset$.

We may assume that $K_1 = \text{conv}(K_1 \cap \mathbb{Z}^n)$ and $K_2 = \text{conv}(K_2 \cap \mathbb{Z}^n)$. By Theorem 15 we know that $u(K_i) = U_i$ for all $u \in K_i \cap \mathbb{Z}^n$, $i = 1, 2$.

We have two cases:

Case 1: $L = \{0\}$, that is, $(K_1 \cap K_2)$ **does not contain lines.**

By Theorem 15, to prove that $\text{conv}((K_1 \cap K_2) \cap \mathbb{Z}^n)$ is closed it is sufficient to show that for all $u \in (K_1 \cap K_2) \cap \mathbb{Z}^n$ we have $u(K_1 \cap K_2) = U_1 \cap U_2$.

We first verify $u(K_1 \cap K_2) \subseteq U_1 \cap U_2$. Since $\text{conv}[(K_1 \cap K_2) \cap \mathbb{Z}^n] \subseteq \text{conv}(K_1 \cap \mathbb{Z}^n) \cap \text{conv}(K_2 \cap \mathbb{Z}^n)$, we have $u(K_1 \cap K_2) \subseteq u(K_1) \cap u(K_2) = U_1 \cap U_2$.

Now we verify that $u(K_1 \cap K_2) \supseteq U_1 \cap U_2$. Let $u \in (K_1 \cap K_2) \cap \mathbb{Z}^n$ and let $d \in U_1 \cap U_2$. Since K_1 is a closed convex set, there exists a face F_1 of K_1 (F_1 may be K_1) such that $u \in F_1$ and $\{u + \lambda d \mid \lambda > 0\} \subseteq \text{rel.int}(F_1)$. Similarly, let F_2 be the face of K_2 such that $u \in F_2$ and $\{u + \lambda d \mid \lambda > 0\} \subseteq \text{rel.int}(F_2)$. Let $Q = F_1 \cap F_2$. Observe that $\{u + \lambda d \mid \lambda > 0\} \subseteq \text{rel.int}(F_1) \cap \text{rel.int}(F_2)$, thus we have $\text{rel.int}(Q) = \text{rel.int}(F_1) \cap \text{rel.int}(F_2)$. Hence, by a standard result in convex analysis, we obtain that $\text{aff}(Q) = \text{aff}(F_1) \cap \text{aff}(F_2)$. Thus, since $\text{aff}(F_1)$ and $\text{aff}(F_2)$ are rational affine subspaces, we obtain that $\text{aff}(Q)$ is a rational affine subspace. Therefore, by Corollary 6, $\{u + \lambda d \mid \lambda \geq 0\} \subseteq \text{conv}(Q \cap \mathbb{Z}^n) \subseteq \text{conv}[(K_1 \cap K_2) \cap \mathbb{Z}^n]$ and so, $d \in u(K_1 \cap K_2)$.

Therefore, for all $u \in (K_1 \cap K_2) \cap \mathbb{Z}^n$, $u(K_1 \cap K_2) = u(K_1) \cap u(K_2) = U_1 \cap U_2$.

Case 2: $L \neq \{0\}$, that is, $(K_1 \cap K_2)$ contains lines.

Since L is generated by integer points, by the Hermite normal form algorithm, there exists an unimodular matrix U such that $UL = \mathbb{R}^p \times \{0\}^{n-p}$. Thus, since $U\mathbb{Z}^n = \mathbb{Z}^n$ and the invertible linear mapping defined by U preserves closedness, we may assume that $L = \mathbb{R}^p \times \{0\}^{n-p}$. For $i = 1, 2$ let $K'_i \subseteq \mathbb{R}^{n-p}$ be the convex set such that $K_i \cap L^\perp = \{0\}^p \times K'_i$. Notice that by Proposition 10 we only need to show that $\text{conv}((K_1 \cap K_2 \cap L^\perp) \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n)) = \text{conv}(\{0\}^p \times (K'_1 \cap K'_2 \cap \mathbb{Z}^{n-p}))$ is closed. This is equivalent to show that $\text{conv}(K'_1 \cap K'_2 \cap \mathbb{Z}^{n-p})$ is closed. Observe that for $i = 1, 2$ we have that $\text{conv}(K_i \cap \mathbb{Z}^n)$ is closed. Hence, by Proposition 10 we obtain that $\text{conv}((K_i \cap L^\perp) \cap \text{Proj}_{L^\perp}(\mathbb{Z}^n)) = \text{conv}(\{0\}^p \times (K'_i \cap \mathbb{Z}^{n-p}))$ is closed, $i = 1, 2$. Equivalently, $\text{conv}(K'_i \cap \mathbb{Z}^{n-p})$ is closed, $i = 1, 2$. Now, notice that the set $(K'_1 \cap K'_2)$ does not contain lines. Thus, by Case 1 applied to the sets K'_1 and K'_2 , we obtain that $\text{conv}(K'_1 \cap K'_2 \cap \mathbb{Z}^{n-p})$ is closed, as desired. \blacksquare

As observed in Example 5 in Section 3.2.4, Theorem 14 is not necessarily true when we replace \mathbb{Z}^n by an arbitrary mixed-integer lattice \mathcal{L} . In order to see what is going wrong in the general case observe that in the proof of Theorem 14 we use the following implication: “If $\text{aff}(F_1)$ and $\text{aff}(F_2)$ are rational affine subspaces,

then $\text{aff}(F_1) \cap \text{aff}(F_2)$ is also a rational affine subspace". We show next that this implication is not true when we consider affine subspaces that are generated by a general mixed-integer lattice \mathcal{L} . Recall the sets $K_1 = \{(x, y) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \mid y \geq x_2 - \sqrt{2}x_1\}$ and $K_2 = \{(x, y) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \mid y \geq \sqrt{2}x_1 - x_2\}$ from Example 5. Let $F_1 = \{(x, y) \in K_1 \mid y = x_2 - \sqrt{2}x_1\}$ and let $F_2 = \{(x, y) \in K_2 \mid y = \sqrt{2}x_1 - x_2\}$. Then $\text{aff}(F_1)$ and $\text{aff}(F_2)$ are affine subspaces that are generated by the mixed-integer lattice $\mathbb{Z}^2 \times \mathbb{R}$, but the affine subspace $\text{aff}(F_1) \cap \text{aff}(F_2)$ is not generated by $\mathbb{Z}^2 \times \mathbb{R}$.

3.7 Appendix

3.7.1 Proof of Theorem 15

In the paper [35] some properties of integer hulls of closed convex sets are presented for the case $\mathcal{L} = \mathbb{Z}^n$. In this section we extend the main result of [35] to the case of general mixed-integer lattices.

The proof techniques we will use in this section are a simple generalization of those used in [35]. Moreover, the results from Section 3.3.4 allow us to reduce the analysis to the case $\mathcal{L} = \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$. We begin with some preliminary results that are used to present the proof of Theorem 15.

A convex set $B \subseteq \mathbb{R}^n$ is called lattice-free, if $\text{int}(B) \cap \mathbb{Z}^n = \emptyset$. A lattice-free convex set $B \subseteq \mathbb{R}^n$ is called maximal lattice-free convex set if does not exist a lattice-free convex set $B' \subseteq \mathbb{R}^n$ satisfying $B \subsetneq B'$.

We require a Corollary of Theorem 8 (See Chapter 2). Denote $\text{Proj}_{\mathbb{R}^{n_1}} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_1}$ the projection on the space of integer variables, that is, on the subspace $\mathbb{R}^{n_1} \times \{0\}$.

Corollary 7. *Let $B \subseteq \mathbb{R}^n$ be a convex set such that $\dim(\text{Proj}_{n_1}(B) \cap \mathbb{Z}^{n_1}) = n_1$. If $\text{rel.int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$, then there exists a polytope $P \subseteq \mathbb{R}^n$ and a rational subspace $L \subseteq \mathbb{R}^n$ such that $Q = P + L$ satisfies $\text{int}(Q) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$ and $B \subseteq Q$.*

Proof. Let $\text{int}_{\mathbb{R}^{n_1}}(\text{Proj}_{n_1}(B))$ denote the interior of $\text{Proj}_{n_1}(B)$ with respect to \mathbb{R}^{n_1} . By Theorem 6.6 of [74] and since $\text{rel.int}(B) = \text{int}(B)$, we obtain that $\text{rel.int}(\text{Proj}_{n_1}(B)) =$

$\text{Proj}_{n_1}(\text{rel.int}(B)) = \text{Proj}_{n_1}(\text{int}(B))$. Thus, $\dim(\text{Proj}_{n_1}(B)) = n_1$. Therefore, $\text{int}_{\mathbb{R}^{n_1}}(\text{Proj}_{n_1}(B)) = \text{rel.int}(\text{Proj}_{n_1}(B)) = \text{Proj}_{n_1}(\text{int}(B))$.

We show next that $\text{int}_{\mathbb{R}^{n_1}}(\text{Proj}_{n_1}(B)) \cap \mathbb{Z}^{n_1} = \emptyset$. Since $\text{int}_{\mathbb{R}^{n_1}}(\text{Proj}_{n_1}(B)) = \text{Proj}_{n_1}(\text{rel.int}(B))$, if $x \in \text{int}_{\mathbb{R}^{n_1}}(\text{Proj}_{n_1}(B))$, then there exists $y \in \mathbb{R}^{n_2}$ such that $(x, y) \in \text{rel.int}(B)$. Hence, since $\text{rel.int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$, we obtain that $x \notin \mathbb{Z}^{n_1}$. Thus, $\text{Proj}_{n_1}(B)$ is a full-dimensional lattice-free convex set of \mathbb{R}^{n_1} . Therefore, by Theorem 8, there exists a polytope $P_1 \subseteq \mathbb{R}^{n_1}$ and a rational subspace $L_1 \subseteq \mathbb{R}^{n_1}$ such that $Q_1 = P_1 + L_1$ satisfies $\text{int}_{\mathbb{R}^{n_1}}(Q_1) \cap \mathbb{Z}^{n_1} = \emptyset$ and $\text{Proj}_{n_1}(B) \subseteq Q_1$.

Now,

$$\begin{aligned} B &\subseteq \text{Proj}_{n_1}(B) \times \mathbb{R}^{n_2} \\ &\subseteq Q_1 \times \mathbb{R}^{n_2} \\ &= (P_1 + L_1) \times \mathbb{R}^{n_2} \\ &= [P_1 \times \{0\}] + [L_1 \times \mathbb{R}^{n_2}]. \end{aligned}$$

So, by taking, $P = P_1 \times \{0\}$, $L = L_1 \times \mathbb{R}^{n_2}$ and $Q = P + L$, and observing that $\text{int}(Q) = \text{int}_{\mathbb{R}^{n_1}}(Q_1) \times \mathbb{R}^{n_2}$, we arrive at the desired conclusion. \square

■

For ease of exposition, we recall some lemmas that we have already used in Chapter 2 to prove the result in the pure integer case (Theorem 1).

Lemma 1 (Corollary 9.8.1 in [74]). *If K_1, \dots, K_m are non-empty closed convex sets in \mathbb{R}^n all having the same recession cone C , then $\text{conv}(K_1 \cup \dots \cup K_m)$ is closed and has C as its recession cone.*

Lemma 3 ([52]). *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed set. Then every extreme point of $\overline{\text{conv}}(K)$ belongs to K .*

Now we are ready to present the proof of Theorem 15.

Theorem 15. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Then $\text{conv}(K \cap \mathcal{L})$ is closed if and only if $u(K, \mathcal{L})$ is identical for every $u \in K \cap \mathcal{L}$.*

Proof. If $K \cap \mathcal{L} = \emptyset$, then the result is trivial. Therefore, we will assume that $K \cap \mathcal{L} \neq \emptyset$. By Lemma 9, we may assume that $K = K \cap \text{aff}(K \cap \mathcal{L})$, $K \cap \mathcal{L}$ is full-dimensional and that $\mathcal{L} = \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$. For the rest of the proof we denote $u(K) = u(K, \mathcal{L})$.

Let us prove “ \Rightarrow ”. If $\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed, then for all $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$, we have that $u(K) = \text{rec}(\overline{\text{conv}}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})))$. Therefore, $u(K)$ is identical for all $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$.

Let us prove “ \Leftarrow ”. Observe that for all $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ we have that

$$\text{rec}(\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))) \subseteq u(K) \subseteq \text{rec}(\overline{\text{conv}}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))). \quad (34)$$

The first inclusion follows directly by definition of $\text{rec}(\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})))$ and $u(K)$. The second inclusion is due to the fact that for a closed convex set, its recession cone gives the recession directions for every point in the set.

Let us denote $K_I = \text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ (hence $\overline{K_I} = \overline{\text{conv}}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$). Assume that $u(K)$ is identical for every $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. We want to show that K_I is a closed set.

We first claim that $u(K) = \text{rec}(K_I) \forall u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. We start by proving $u(K) \subseteq \text{rec}(K_I)$. Let $r \in u(K)$ and $x \in K_I$. We can write $x = \sum_{i=1}^N \alpha_i z_i$, where $z_i \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$, $\alpha_i \geq 0$ for all $i = 1, \dots, N$ and $\sum_{i=1}^N \alpha_i = 1$. Since for all $i = 1, \dots, N$ we have that $u(K) = z_i(K)$, we obtain that $r \in z_i(K)$. Therefore, for all $i = 1, \dots, N$ we have $z_i + \lambda r \in K_I$ for all $\lambda \geq 0$. Since $x + \lambda r = \sum_{i=1}^N \alpha_i (z_i + \lambda r)$, we obtain that $x + \lambda r \in K_I$ for all $\lambda \geq 0$. Thus, $u(K) \subseteq \text{rec}(K_I)$ and by (34) we obtain that

$$u(K) = \text{rec}(K_I) \quad \forall u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}). \quad (35)$$

We will now show that K_I is closed. There are two cases:

- **Case 1:** $\text{rel.int}(K_I) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$. We will verify that $K_I \supseteq \overline{K_I}$. Let $u \in \text{rel.int}(K_I) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Since $u \in \text{rel.int}(K_I) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ we obtain that

for all $r \in \text{rec}(\text{rel.int}(K_I))$, $u + \lambda r \in \text{rel.int}(K_I) \subseteq K_I$ for all $\lambda \geq 0$. Therefore, $\text{rec}(\text{rel.int}(K_I)) \subseteq u(K)$. Since $\text{rec}(\overline{K_I}) = \text{rec}(\text{rel.int}(K_I))$ (by Lemma 2), by using (34) we conclude that $u(K) = \text{rec}(\overline{K_I})$. Therefore, by using (35) we obtain that $\text{rec}(K_I) = \text{rec}(\overline{K_I})$. Observe that, by Lemma 3, the extreme points of $\overline{K_I}$ belong to $K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$. Since $\overline{K_I} \subseteq K$, it does not contain any lines. Thus, by Theorem 7, $\overline{K_I}$ is given by the convex hull of its extreme points plus its recession cone. Since the extreme points of $\overline{K_I}$ belongs to $K_I = \text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ and $\text{rec}(K_I) = \text{rec}(\overline{K_I})$, we obtain that $K_I \supseteq \overline{K_I}$. Therefore, K_I is closed.

- **Case 2:** $\text{rel.int}(K_I) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$. We will use induction on the dimension of K_I . The base case, $\dim(K_I) = 1$ is straightforward to verify.

Suppose now the property is true for every closed convex set K' such that $\dim(\text{conv}(K' \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))) < \dim(K_I)$ and $\text{rel.int}(\text{conv}(K' \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$.

Recall that by Lemma 9 we may assume that $\text{aff}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is of full dimension. In particular, we obtain that $\dim(\text{Proj}_{n_1}(K) \cap \mathbb{Z}^{n_1}) = n_1$. Thus, K_I satisfies the hypothesis of Corollary 7. Therefore, there exists a full-dimensional maximal lattice-free polyhedron $Q \subseteq \mathbb{R}^n$ such that $K_I \subseteq Q$ and $Q = P + L$, where P is a polytope and L is a rational linear subspace.

Let F_i , $i = 1, \dots, N$ be the facets of Q such that $K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$. We will verify that

$$K_I \cap F_i = \text{conv}(K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})). \quad (36)$$

Since $K_I \cap F_i$ is a convex set and contains $K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ we have $K_I \cap F_i \supseteq \text{conv}(K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$. On the other hand, let $x \in K_I \cap F_i$. Therefore $x = \sum_{j=1}^M \alpha_j z_j$, where $z_j \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$, $\alpha_j \geq 0$ for all $j = 1, \dots, M$, and $\sum_{j=1}^M \alpha_j = 1$. Since $K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \subseteq Q$ and $x \in F_i$, we must have $z_j \in F_i$, $\forall j = 1, \dots, M$, so $x \in \text{conv}(K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$.

Next, for all $i = 1, \dots, N$, we verify that

$$u(K \cap F_i) = u(K) \cap L \quad \forall u \in K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}). \quad (37)$$

We have $r \in u(K \cap F_i)$ if and only if $u + \lambda r \in \text{conv}(K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \forall \lambda \geq 0$. By (36), this is equivalent to $u + \lambda r \in K_I \cap F_i \forall \lambda \geq 0$. This is also equivalent to $u + \lambda r \in K_I \forall \lambda \geq 0$ and $u + \lambda r \in F_i \forall \lambda \geq 0$. Thus equivalently we obtain that $r \in u(K)$ and $r \in \text{rec}(F_i)$. By the form of Q (see Theorem 8) we have that $\text{rec}(F_i) = \text{rec}(Q) = L$ for all $i = 1, \dots, N$. We obtain $r \in u(K) \cap L$. Therefore, we conclude $u(K \cap F_i) = u(K) \cap L$.

Since $u(K)$ is identical for all $u \in K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$, (37) implies that $u(K \cap F_i)$ is identical for every $u \in K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ and for all $i = 1, \dots, N$. Moreover, since $\text{conv}(K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \subseteq F_i$, we obtain $\dim(\text{conv}(K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))) < \dim(K_I)$. Thus, we can use either **Case 1** or the induction hypothesis to conclude that $\text{conv}(K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is a closed set.

Since for all $i = 1, \dots, N$ we have that $\text{conv}(K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is a closed set, we obtain that $u(K \cap F_i) = \text{rec}(\text{conv}(K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})))$. Therefore, since $u(K \cap F_i) = u(K) \cap L$, we obtain that the recession cone of $\text{conv}(K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is the same for all $i = 1, \dots, N$. Observe that,

$$K_I = \text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) = \text{conv} \left[\bigcup_{i \in \{1, \dots, N\}} \text{conv}(K \cap F_i \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \right].$$

Since the convex hull of a finite union of closed convex sets with the same recession cone is closed (Lemma 1), we conclude that $\text{conv}(K \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed.

■

CHAPTER IV

PROPERTIES OF MAXIMAL S -FREE CONVEX SETS

4.1 Introduction

A convex set is called *lattice-free* if it contains no integer points in its interior. A *maximal lattice-free convex set* is a lattice-free convex set not strictly contained in any lattice-free convex set. Lovász [65] and Basu et al. [13] proved that maximal lattice-free convex sets are polyhedra. Given a convex set $\mathbf{P} \subseteq \mathbb{R}^n$, let S be the set of integer points contained in \mathbf{P} , that is $S = \mathbf{P} \cap \mathbb{Z}^n$. A set K is called *S -free convex set* if $\text{int}(K) \cap S = \emptyset$. Hence the concept of S -free convex sets represents a generalization of the concept of lattice-free convex sets. *Maximal S -free convex sets* are defined analogously to maximal lattice-free convex sets. Dey and Wolsey [37] show that maximal S -free convex sets are polyhedra under restrictive conditions. Fukasawa and Günlük [44] show that maximal S -free convex sets are polyhedra where \mathbf{P} is a rational polyhedron in \mathbb{R}^2 . Basu et al. [14] prove that if \mathbf{P} is any general rational polyhedron, then maximal S -free convex sets are polyhedra. A natural question then is to ask whether the polyhedrality of maximal S -free convex sets carries through to the case where \mathbf{P} is an arbitrary convex set, that is the case where $S \subseteq \mathbb{Z}^n$ is assumed only to satisfy the property that $\text{conv}(S) \cap \mathbb{Z}^n = S$.

In this chapter we verify that maximal S -free convex sets are polyhedra, where S is the set of integer points in any convex set $\mathbf{P} \subseteq \mathbb{R}^n$. Since the proof in Basu et al. [14] explicitly uses that fact that $\text{conv}(S)$ is a rational polyhedron when S is the set of integer points contained in a rational polyhedron, the result in this chapter also provides an alternative proof to the result in [14]. Finally, we establish the polyhedrality of maximal S -free convex sets in some cases where S does not satisfy

$$\text{conv}(S) \cap \mathbb{Z}^n = S.$$

We now briefly describe one motivation for the study of maximal S -free convex sets. A key algorithmic technique in solving mixed integer optimization problems is to sequentially obtain tighter approximations of the convex hull of integer feasible solutions. This is achieved by the addition of cutting planes, that is inequalities that separate fractional point from the convex hull of integer feasible points. See for example Marchand et al. [66] and Johnson et al. [60] for description of cutting plane methods. A connection between S -free convex sets and cutting planes for mixed integer linear programming was first discovered by Balas [9]. The main idea is the following. Consider the mixed integer set

$$T := \{(x, y) \in \mathbb{Z}^p \times \mathbb{R}^q \mid (x, y) \in \mathcal{C}\}, \quad (38)$$

where \mathcal{C} is a convex set. Now suppose that we are able to identify a set $S \subseteq \mathbb{Z}^n$ such that $S \supseteq \text{Proj}_x(T)$. If $B \subseteq \mathbb{R}^p$ is an S -free convex set, then by letting $\hat{B} = B \times \mathbb{R}^q$ we can construct a valid relaxation of $\text{conv}(T)$ as $\text{conv}(\mathcal{C} \setminus \text{int}(\hat{B}))$. Often by a good choice of B , $\text{conv}(\mathcal{C} \setminus \text{int}(\hat{B}))$ is a much better approximation of the convex hull of T than \mathcal{C} . A classical example of this procedure is that of split disjunctions; see Cook et al. [29]. Notice that if B^1 and B^2 are S -free convex sets such that $B^1 \supseteq B^2$, then $\text{conv}(\mathcal{C} \setminus \text{int}(\hat{B}^1)) \subseteq \text{conv}(\mathcal{C} \setminus \text{int}(\hat{B}^2))$. This motivates the search for maximal S -free convex sets. Various families of cutting planes based on maximal S -free convex sets (for different S) have been proposed. See for example Andersen et al. [2], Andersen et al. [3], Basu et al. [14], Borozan and Cornuéjols [23], Cornuéjols and Margot [30], Dey and Wolsey [37], Fukasawa and Günlük [44], Johnson [59], Zambelli [79]. Since we verify that maximal S -free convex sets are polyhedra, one possible way to compute $\text{conv}(\mathcal{C} \setminus \text{int}(\hat{B}))$ is by computing the convex hull of the following disjunction

$$\bigcup_{i=1}^t (\mathcal{C} \cap \{(x, y) \mid a_i^T x \geq b_i\}), \quad (39)$$

where $B = \{x \in \mathbb{R}^p \mid a_i^T x \leq b_i, i \in \{1, \dots, t\}\}$. For many classes of convex sets \mathcal{C} , the convex hull of the union of sets in (39) can be described ‘conveniently’; see for example the case of second order conic representable sets in Ben-Tal and Nemirovski [17].

The rest of the chapter is organized as follows. In Section 5.4, we review some standard results from convex analysis and present a key result about maximal lattice-free convex sets in affine subspaces due to Basu et al. [13]. In Section 4.3, we present the characterization of maximal S -free convex sets. In Section 4.4, we discuss some differences between the properties of maximal S -free convex sets in the general case to the case where S is the set of integer points contained in a rational polyhedron and point out some generalizations of the results presented in Section 4.3.

4.2 Preliminaries

Let W be an affine subspace of \mathbb{R}^n . We use $\text{int}_W(A)$ to denote the interior of the set $A \subseteq W$ with respect to the topology induced by \mathbb{R}^n on W . Therefore $\text{rel.int}(A) = \text{int}_{\text{aff.hull}(A)}(A)$. We will call a set H a half-space (resp. hyperplane) of W if H is the intersection of W with a half-space (resp. hyperplane) of \mathbb{R}^n and $W \not\subseteq H$. For $u \in \mathbb{R}^n$ and $\varepsilon > 0$, $B(u, \varepsilon)$ is the open ball around u of radius $\varepsilon > 0$, that is $B(u, \varepsilon) := \{x \in \mathbb{R}^n \mid \|x - u\| < \varepsilon\}$.

We will frequently use the following basic result from convex analysis that we prove for completeness.

Lemma 30. *Let $U \subseteq V \subseteq \mathbb{R}^n$ be affine subspaces and let $A \subseteq V$ be a convex set such that $\text{int}_V(A) \cap U \neq \emptyset$. Then $\text{int}_V(A) \cap U = \text{int}_U(A \cap U)$.*

Proof. The inclusion $\text{int}_V(A) \cap U \subseteq \text{int}_U(A \cap U)$ is straightforward. For the other inclusion, assume that $y \in \text{int}_U(A \cap U)$. Then there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \cap U \subseteq A \cap U$. Since $\text{int}_V(A) \cap U \neq \emptyset$, there exists $x \in \text{int}_V(A) \cap U$. If $x = y$, then the proof is complete. Otherwise, since U is a affine subspace, we obtain that

$z = y + \frac{\varepsilon}{2\|x-y\|}(y-x) \in B(y, \varepsilon) \cap U$. It follows that y is a strict convex combination of $z \in A$ and $x \in \text{int}_V(A)$, so $y \in \text{int}_V(A)$, and then $y \in \text{int}_V(A) \cap U$. ■

Using Lemma 30, we restate a version of separation theorem for convex sets that we use later.

Theorem 16 (Separation Theorem). *Let $W \subseteq \mathbb{R}^n$ be an affine subspace. If $A, B \subseteq W$, A and B are convex sets, and $\text{rel.int}(A) \cap \text{rel.int}(B) = \emptyset$, then there exists half-space H of W such that $A \subseteq H$ and $\text{int}_W(H) \cap B = \emptyset$.*

Proof. By Theorem 4.14 in Hiriart-Urrut and Lemaréchal [51], there exists $s \in \mathbb{R}^n$ such that

$$\sup_{y \in A} s^T y \leq \inf_{y \in B} s^T y \quad (40)$$

$$\inf_{y \in A} s^T y < \sup_{y \in B} s^T y. \quad (41)$$

Let $\tilde{H} = \{x \in \mathbb{R}^n \mid s^T x \leq \sup_{y \in A} s^T y\}$ and $H = \tilde{H} \cap W$. By (41), s is not orthogonal to W . Therefore H is a halfspace of W . Finally, by Lemma 30 we obtain $\text{int}_{\mathbb{R}^n}(\tilde{H}) \cap W = \text{int}_W(H)$, which completes the proof. ■

Listed below are some properties of interiors and relative interiors of convex sets that are also used frequently. See Hiriart-Urrut and Lemaréchal [51] and Rockafellar [74] for proofs.

Proposition 13. *Let W be an affine subspace and let $A, B \subseteq W$ be convex sets. Then,*

1. $\text{int}_W(A) \cap \text{int}_W(B) = \text{int}_W(A \cap B)$.
2. $\text{rel.int}(A) \cap \text{rel.int}(B) \subseteq \text{rel.int}(A \cap B)$.
3. *If $\text{rel.int}(A) \cap \text{rel.int}(B) \neq \emptyset$, then $\text{rel.int}(A) \cap \text{rel.int}(B) = \text{rel.int}(A \cap B)$.*
4. $A \subseteq \overline{\text{rel.int}(A)}$, where \overline{C} represent the closure of C .
5. $\text{rel.int}(A) + \text{rel.int}(B) = \text{rel.int}(A + B)$.

Definition 9 (*S-free and Lattice-free Convex sets*). Let $W \subseteq \mathbb{R}^n$ be an affine subspace of dimension $\dim(W) \geq 1$, $\mathbf{P} \subseteq W$ be a convex set and $S = \mathbf{P} \cap \mathbb{Z}^n$. A set K is called *S-free* (resp. *lattice-free*) convex set of W if $K \subseteq W$, K is convex and $\text{int}_W(K) \cap S = \emptyset$ (resp. $\text{int}_W(K) \cap \mathbb{Z}^n = \emptyset$). A convex set $K \subseteq W$ is called *maximal S-free* (resp. *lattice-free*) convex set of W if K is *S-free* (resp. *lattice-free*) convex set and there does not exist a set $K' \subseteq W$ such that $K' \neq K$, $K' \supseteq K$ and K' is an *S-free* (resp. *lattice-free*) convex set of W .

Basu et al. [13] proved that maximal lattice-free convex sets of affine subspaces are polyhedra. This is a crucial ingredient in the proof presented in this chapter. We present this result next.

Theorem 17 (Basu et al. [13]). Let $W \subseteq \mathbb{R}^n$ be an affine subspace containing an integral point and V be the affine hull of $W \cap \mathbb{Z}^n$. A set $K \subseteq W$ is a maximal lattice-free convex set of W if and only if one of the following holds:

1. K is a polyhedron in W whose dimension equals $\dim(W)$, $K \cap V$ is a maximal lattice-free convex set of V whose dimension equals $\dim(V)$, and for every facet F of K , $F \cap V$ is a facet of $K \cap V$,
2. K is an affine hyperplane of W such that $K \cap V$ is an irrational hyperplane of V ,
3. K is a half-space of W that contains V on its boundary.

The proof of Theorem 17 in Basu et al. [13] involves showing that if K is a lattice-free convex set of W , then K must be contained in maximal lattice-free convex set of W . We will therefore use the following simplified version of Theorem 17.

Theorem 18 (Basu et al. [13]). Let $W \subseteq \mathbb{R}^n$ be an affine subspace containing an integral point. If $K \subseteq W$ is a lattice-free convex set of W , then it is contained in a maximal lattice-free convex set B of W , where B is a polyhedron.

4.3 Maximal S -free Convex Sets

As observed in Basu et al. [13], the existence of maximal S -free convex sets is a consequence of Zorn's Lemma.

We will verify the following results in this section.

Theorem 19. *Let $W \subseteq \mathbb{R}^n$ be an affine subspace of dimension $\dim(W) \geq 1$, V the affine hull of $W \cap \mathbb{Z}^n$, $\mathbf{P} \subseteq W$ a convex set, $S = \mathbf{P} \cap \mathbb{Z}^n$ and $\mathbf{P}_I = \text{conv}(S)$. If $K \subseteq W$ is a maximal S -free convex set, then one of the following holds:*

1. $\dim(K) = \dim(W)$ and $\text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I) \neq \emptyset$: Then K is a polyhedron in W , $K \cap V$ is a maximal S -free convex set of V whose dimension equals $\dim(V)$, and for every facet F of K , $F \cap V$ is a facet of $K \cap V$,
2. $\dim(K) < \dim(W)$: Then K is an hyperplane of W ,
3. $\dim(K) = \dim(W)$ and $\text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I) = \emptyset$: Then K is a halfspace of W ,
4. $S = \emptyset$ and $K = W$.

If S is the set of integer points contained in an rational polyhedron, then every facet of maximal S -free polyhedron contains a point of S in its relative interior. A slightly weaker result holds in the general case. Before we present this result, we require some additional notation. Let K be a polyhedron of W , that is let $K = \{x \in W : a_i^T x \leq b_i \ \forall i \in \{1, \dots, N\}\}$. Then we denote the i^{th} facet of K as $F_i(K)$. Also for all $\varepsilon > 0$, let $F_i^\varepsilon(K) = \{x \in W : a_j^T x < b_j \ \forall j \neq i \text{ and } b_i < a_i^T x < b_i + \varepsilon\}$.

Theorem 20. *Let $W \subseteq \mathbb{R}^n$ be an affine subspace of dimension $\dim(W) \geq 1$, $\mathbf{P} \subseteq W$ a convex set and $S = \mathbf{P} \cap \mathbb{Z}^n$. Let K be a S -free convex set such that $\dim(K) = \dim(W)$. If K is a maximal S -free convex set, then K is a polyhedron with N facets such that*

1. $(\text{rel.int}(F_i) \cup F_i^\varepsilon(K)) \cap S \neq \emptyset \quad \forall \varepsilon > 0 \ \forall i \in \{1, \dots, N\}$ and
2. $N \leq 2^{\dim(\text{conv}(S))}$.

The rest of the section is organized as follows. In Section 4.3.1, we prove that maximal S -free convex sets are polyhedra. In Section 4.3.2, we verify properties of facets of maximal S -free convex sets. In particular, if $\dim(K) = \dim(W)$ and $\text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I) \neq \emptyset$, then we show that $K \cap V$ is a maximal S -free convex set of V whose dimension equals $\dim(V)$, and for every facet F of K , $F \cap V$ is a facet of $K \cap V$ (where $V = \text{aff.hull}(W \cap \mathbb{Z}^n)$). We also verify part (1.) of Theorem 20 in this section. Finally in Section 4.3.3, we obtain an upper bound on the number of facets of maximal S -free convex sets, completing the proof of Theorem 20. This upper bound is obtained using the upper bound result for maximal lattice-free sets (Doignon [39], Bell [16], Scarf [75]) but involves a little more work as facets of maximal S -free convex sets do not in general contain points of S in their relative interior.

4.3.1 Polyhedrality of Maximal S -free Convex Sets

To show that a maximal S -free convex set is a polyhedron, it is sufficient to show that every S -free convex set is contained in a polyhedral S -free convex set. This is verified next.

Proposition 14. *Let $W \subseteq \mathbb{R}^n$ be an affine subspace of dimension $\dim(W) \geq 1$, $\mathbf{P} \subseteq W$ be a convex set, $S = \mathbf{P} \cap \mathbb{Z}^n$ and $\mathbf{P}_I = \text{conv}(S)$. Let $K \subseteq W$ be an S -free convex set of W . Then one of the following holds:*

1. $\dim(K) = \dim(W)$ and $\text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I) \neq \emptyset$: K is contained in an S -free convex set $B \subseteq W$ such that B is a polyhedron,
2. $\dim(K) < \dim(W)$: Then K is contained in an S -free hyperplane of W ,
3. $\text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I) = \emptyset$: Then K is contained in an S -free halfspace of W ,
4. $S = \emptyset$ and W is an S -free convex set (K is contained in W).

Since the proof of Proposition 14 is technical, we summarize the main steps here. The proof is by induction on the dimension of W . We first prove cases (2.), (3.) and (4.). The proof of these three cases does not require the use of the induction hypothesis. To prove case (1.), we carefully separate K from S , i.e., determine half-spaces of W such that the polyhedron defined by the intersection of these half-spaces contains K and does not contain any point of S in its interior (wrt W). To achieve this goal, we first observe that $K \cap \mathbf{P}_I$ is a lattice-free convex set and thus there exists a maximal lattice-free polyhedron $Q \subseteq W$ containing $K \cap \mathbf{P}_I$. Then we examine each facet of Q . Corresponding to each facet F of Q , we separate K from the subset of S that the inequality defining F separates from $K \cap \mathbf{P}_I$, using a finite number of separating half-spaces. In most cases, this is achieved via a straightforward application of the Separation Theorem. However for facets of Q where the defining hyperplane, say H^\pm , has a non-empty intersection with S and also with the interior of K , these separating half-spaces can be found by first determining a maximal $(S \cap H^\pm)$ -free polyhedron contained in $(W \cap H^\pm)$. The existence of this polyhedron is by the induction hypothesis. In the last step of the proof, it is verified that the polyhedron constructed as the intersection of the above separating half-spaces (corresponding to all the facets of Q) does not contain any point of S in its interior (wrt W). Since Q has a finite number of facets, the intersection of all these separating half-spaces is a S -free polyhedron containing K .

Proof. The proof of (1.), (2.), (3.) and (4.) is by induction on the dimension of W . Consider first the case where $\dim(W) = 1$. Then since $K \subseteq W$ is a convex set and $\dim(K) \leq 1$, we conclude that \bar{K} is an S -free polyhedron. Therefore the cases (1.), (2.), (3.) and (4.) are easily verifiable when $\dim(W) = 1$.

Assume now that we have (1.), (2.), (3.) and (4.) for every affine subspace of dimension less than $\dim(W)$. We first prove cases (2.), (3.) and (4.) and then we prove the more complicated case (1.)

If $S = \emptyset$, then W is an S -free convex set and this completes the proof. Hence, we can assume $S \neq \emptyset$. Observe that since \mathbf{P} is convex, $S = \mathbf{P}_I \cap \mathbb{Z}^n$.

If $\dim(K) < \dim(W)$, then K is contained in a hyperplane H of W . Notice that we must have $\text{int}_W(H) = \emptyset$, for otherwise $\dim(H) = \dim(W)$ and therefore $H = W$, a contradiction with $W \subsetneq H$. Since $\text{int}_W(H) \cap S = \emptyset$, the proof is complete.

If $\text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I) = \emptyset$, then by separation theorem there exists a half-space H^\leq of W such that $K \subseteq H^\leq$ and $\mathbf{P}_I \cap \text{int}_W(H^\leq) = \emptyset$. Thus, K is contained in an S -free half-space. Therefore we can assume $\dim(K) = \dim(W)$ and $\text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I) \neq \emptyset$.

Since K is a S -free convex set, we obtain that $\text{int}_W(K \cap \mathbf{P}_I) \cap \mathbb{Z}^n = \text{int}_W(K) \cap \text{int}_W(\mathbf{P}_I) \cap \mathbb{Z}^n \subseteq \text{int}_W(K) \cap (\mathbf{P}_I \cap \mathbb{Z}^n) = \text{int}_W(K) \cap S = \emptyset$. So we conclude $K \cap \mathbf{P}_I$ is a lattice-free convex set of W . Also since $S \neq \emptyset$, we obtain that $W \cap \mathbb{Z}^n \neq \emptyset$. Hence, by Theorem 18, there exists a maximal lattice-free convex set of W , $Q \subseteq W$ such that $K \cap \mathbf{P}_I \subseteq Q$ and Q is a polyhedron. Observe also that since $S \neq \emptyset$, $Q \subsetneq W$. We therefore obtain that

$$Q = \bigcap_{i=1}^m H_i^\leq,$$

where each H_i^\leq , $i = 1, \dots, m$ is a half-space of W defining a facet of Q . For $i \in I = \{1, \dots, m\}$, denote $H_i^\geq = W \setminus (\text{int}_W(H_i^\leq))$ and $H_i^\dagger = H_i^\leq \cap H_i^\geq$.

We partition the set I into four disjoint subsets.

1. $I_1 := \{i \in I : \text{rel.int}(\mathbf{P}_I) \cap \text{rel.int}(H_i^\geq) \neq \emptyset\}$. If $i \in I_1$, then $\text{rel.int}(\mathbf{P}_I \cap H_i^\geq) = \text{rel.int}(\mathbf{P}_I) \cap \text{rel.int}(H_i^\geq)$. Therefore

$$\begin{aligned} \text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I \cap H_i^\geq) &= \text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I) \cap \text{rel.int}(H_i^\geq) \\ &= \text{rel.int}(K \cap \mathbf{P}_I) \cap \text{rel.int}(H_i^\geq) \\ &\subseteq Q \cap \text{rel.int}(H_i^\geq) = \emptyset. \end{aligned}$$

The second equality holds because $\text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I) \neq \emptyset$. Since $\text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I \cap H_i^{\geq}) = \emptyset$, we obtain that there exists G_i , a half-space of W , such that

$$K \subseteq G_i, \quad (42)$$

$$\text{int}_W(G_i) \cap \mathbf{P}_I \cap H_i^{\geq} = \emptyset. \quad (43)$$

2. Consider $I_2 := \{i \in I \setminus I_1 : \text{int}_W(K) \cap H_i^{\leq} = \emptyset, S \cap H_i^{\leq} \neq \emptyset\}$.

In this case, we verify that $K \subseteq H_i^{\leq}$. Since $K \subseteq \overline{\text{rel.int}(K)} = \overline{\text{int}_W(K)}$ and H_i^{\leq} is a closed convex set, it is sufficient to verify that $\text{int}_W(K) \subseteq H_i^{\leq}$. Assume by contradiction, that there exists $x \in \text{int}_W(K) \cap \text{int}_W(H_i^{\geq})$. Since $\emptyset \neq \text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I) \subseteq K \cap \mathbf{P}_I \subseteq H_i^{\leq}$, we obtain that $\text{rel.int}(K) \cap H_i^{\leq} \neq \emptyset$. However, note that since the affine hull of $\text{rel.int}(K)$ and H_i^{\leq} is W , $\text{rel.int}(K)$ is an open set (wrt W) and H_i^{\leq} is a half space of W , we obtain that $\text{rel.int}(K) \cap \text{rel.int}(H_i^{\leq}) \neq \emptyset$. (Choose $\tilde{y} \in \text{rel.int}(K) \cap H_i^{\leq}$. If $\tilde{y} \in \text{rel.int}(H_i^{\leq})$, then we are done. Otherwise, since $\tilde{y} \in \text{rel.int}(K)$ and affine hull of $\text{rel.int}(K)$ is W , it is possible to choose a neighborhood $B(\tilde{y}, \varepsilon) \cap W$ of \tilde{y} contained in $\text{rel.int}(K)$. However, since $\tilde{y} \notin \text{rel.int}(H_i^{\leq})$, we obtain that $\tilde{y} \in H_i^{\leq}$. Therefore, $B(\tilde{y}, \varepsilon) \cap W \cap \text{rel.int}(H_i^{\leq}) \neq \emptyset$ and thus there exists $y \in \text{rel.int}(K) \cap \text{rel.int}(H_i^{\leq})$. Let $y \in \text{rel.int}(K) \cap \text{rel.int}(H_i^{\leq})$. Hence every convex combination of x and y belongs to $\text{int}_W(K)$ since $x, y \in \text{int}_W(K)$. Moreover, since $x \in \text{int}_W(H_i^{\geq})$ and $y \in \text{int}_W(H_i^{\leq})$, there exists a convex combination of x and y that belongs to $\text{int}_W(K) \cap H_i^{\leq}$. Therefore $\text{int}_W(K) \cap H_i^{\leq} \neq \emptyset$, which contradicts the fact that $i \in I_2$. If $i \in I_2$, then define $G_i = H_i^{\leq}$. Therefore from the above, we obtain

$$K \subseteq G_i, \quad (44)$$

$$H_i^{\leq} \cap \text{int}_W(G_i) = \emptyset. \quad (45)$$

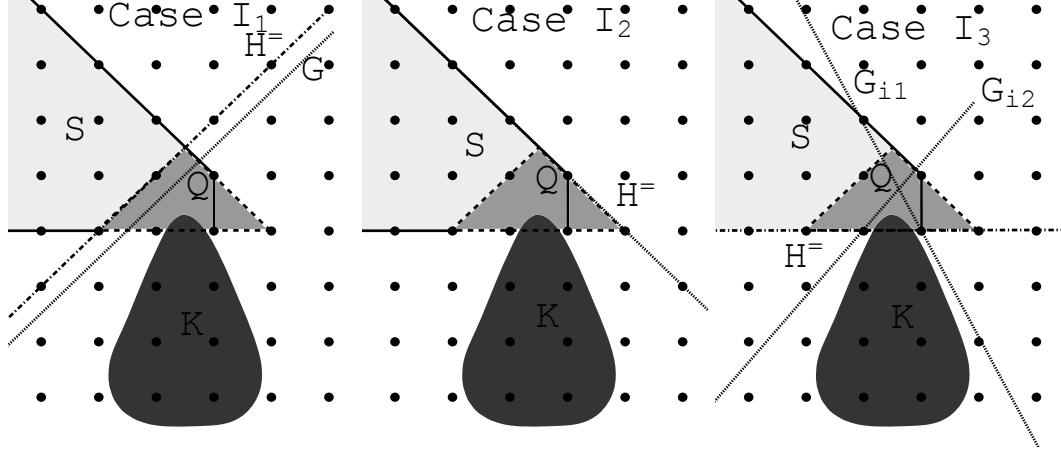


Figure 15: Illustration of case I_1 , I_2 , and I_3 .

3. Consider $I_3 := \{i \in I \setminus I_1 : \text{int}_W(K) \cap H_i^- \neq \emptyset, S \cap H_i^- \neq \emptyset\}$. If $i \in I_3$, then by Lemma 30 we obtain that $\text{int}_{H_i^-}(K \cap H_i^-) = \text{int}_W(K) \cap H_i^-$.

Since K is an S -free convex set and $\text{int}_{H_i^-}(K \cap H_i^-) = \text{int}_W(K) \cap H_i^-$ we obtain that $\text{int}_{H_i^-}(K \cap H_i^-) \cap S = \emptyset$. Hence, $K \cap H_i^-$ is a $(S \cap H_i^-)$ -free convex set of H_i^- . Note that since $W \subsetneq H_i^\leq$, we obtain that $\dim(H_i^-) = \dim(W) - 1$, i.e., $1 \leq \dim(H_i^-) < \dim(W)$. By the induction hypothesis, there exists a polyhedron $T_i \subseteq H_i^-$ such that $K \cap H_i^- \subseteq T_i$, T_i is a $(S \cap H_i^-)$ -free convex set of H_i^- . Note that since $(S \cap H_i^-) \neq \emptyset$, we obtain that $T_i \neq H_i^-$. Therefore $T = \bigcap_{j=1}^{m_i} F_{ij}^\leq$, where $F_{ij}^\leq \subseteq H_i^-$, $j = 1, \dots, m_i$ are the half-spaces of H_i^- defining the facets of T_i . Denote $F_{ij}^\geq = H_i^- \setminus (\text{int}_{H_i^-}(F_{ij}^\leq))$. We have:

$$\begin{aligned}
 \text{rel.int}(F_{ij}^\geq) \cap \text{rel.int}(K) &= \text{rel.int}(F_{ij}^\geq) \cap H_i^- \cap \text{int}_W(K) \\
 &\subseteq F_{ij}^\geq \cap \text{int}_{H_i^-}(K \cap H_i^-) \\
 &\subseteq F_{ij}^\geq \cap \text{int}_{H_i^-}(T_i) = \emptyset.
 \end{aligned}$$

Therefore there exists G_{ij} , half-spaces of W , such that:

$$K \subseteq G_{ij}, \quad (46)$$

$$F_{ij}^{\geq} \cap \text{int}_W(G_{ij}) = \emptyset. \quad (47)$$

In this case, define $G_i = \bigcap_{j=1}^{m_i} G_{ij}$.

We verify a property that we require later. Note that since $\text{int}_W(K) \subseteq \text{int}_W(G_{ij})$ and $\text{int}_W(K) \cap H_i^- \neq \emptyset$, we obtain that $\text{int}_W(G_{ij}) \cap H_i^- \neq \emptyset$. Therefore using Lemma 30 we obtain the equality $\text{int}_{H_i^-}(G_{ij} \cap H_i^-) = \text{int}_W(G_{ij}) \cap H_i^-$. Therefore, (47) is equivalent to

$$F_{ij}^{\geq} \cap \text{int}_{H_i^-}(G_{ij} \cap H_i^-) = \emptyset. \quad (48)$$

4. Finally define $I_4 := \{i \in I \setminus I_1 : S \cap H_i^- = \emptyset\}$. Note that $I = I_1 \cup I_2 \cup I_3 \cup I_4$.

Before the final step in the proof, we verify that if $\text{rel.int}(\mathbf{P}_I) \cap \text{rel.int}(H_i^{\geq}) = \emptyset$, then

$$\mathbf{P}_I \cap \text{rel.int}(H_i^{\geq}) = \emptyset. \quad (49)$$

As $\text{rel.int}(\mathbf{P}_I) \cap \text{rel.int}(H_i^{\geq}) = \emptyset$, we obtain $\text{rel.int}(\mathbf{P}_I) \subseteq H_i^{\leq}$. Since H_i^{\leq} is a closed set, we obtain $\overline{\text{rel.int}(\mathbf{P}_I)} \subseteq H_i^{\leq}$. Finally, note that since \mathbf{P}_I is a convex set, we obtain $\mathbf{P}_I \subseteq \overline{\text{rel.int}(\mathbf{P}_I)} \subseteq H_i^{\leq}$ (see Proposition 13). Thus $\mathbf{P}_I \cap \text{rel.int}(H_i^{\geq}) = \emptyset$.

To complete the proof we will verify that the set $B = \bigcap_{i \in I \setminus I_4} G_i \subseteq W$ is an S -free convex set of W containing K . By (42), (44) and (46), we obtain that $K \subseteq B$. We need to prove that B is an S -free convex set to complete the proof. Assume by contradiction that there exists $y \in S \cap \text{int}_W(B)$, that is $y \in S$ and $y \in \text{int}_W(G_i) \forall i \in I \setminus I_4$.

We have

$$\begin{aligned}
\mathbf{P}_I &= W \cap \mathbf{P}_I \\
&= \left[\bigcup_{i \in I} H_i^{\geq} \cup \text{int}_W(Q) \right] \cap \mathbf{P}_I \\
&\subseteq \left[\bigcup_{i \in I} (H_i^{\geq} \cap \mathbf{P}_I) \right] \cup \text{int}_W(Q) \\
&= \left[\bigcup_{i \in I_1} (H_i^{\geq} \cap \mathbf{P}_I) \cup \bigcup_{i \in I_2} (H_i^{\bar{=}} \cap \mathbf{P}_I) \cup \bigcup_{i \in I_3} (H_i^{\bar{=}} \cap \mathbf{P}_I) \cup \bigcup_{i \in I_4} (H_i^{\bar{=}} \cap \mathbf{P}_I) \right] \\
&\quad \cup \text{int}_W(Q). \tag{50}
\end{aligned}$$

The last equality is a consequence of (49), since for $i \in I_2 \cup I_3 \cup I_4$ we have that $(H_i^{\geq} \cap \mathbf{P}_I) = (\text{rel.int}(H_i^{\geq}) \cap \mathbf{P}_I) \cup (H_i^{\bar{=}} \cap \mathbf{P}_I) = (H_i^{\bar{=}} \cap \mathbf{P}_I)$.

Since $y \in S = \mathbf{P}_I \cap \mathbb{Z}^n$ and $S \cap H_i^{\bar{=}} = \emptyset$ for $i \in I_4$, we obtain that $y \notin H_i^{\bar{=}} \cap \mathbf{P}_I$ for $i \in I_4$. Moreover Q is lattice-free. Therefore, using (50) there are three cases:

1. For $i \in I_1$, if $y \in H_i^{\geq} \cap \mathbf{P}_I$ and $y \in \text{int}_W(G_i)$, then we obtain a contradiction to (43).
2. For $i \in I_2$, if $y \in H_i^{\bar{=}}$ and $y \in \text{int}_W(G_i)$, then we obtain a contradiction to (45).
3. For $i \in I_3$, if $y \in H_i^{\bar{=}}$, then since $y \in \bigcap_{j=1}^{m_i} \text{int}_W(G_{ij}) \cap H_i^{\bar{=}} = \bigcap_{j=1}^{m_i} \text{int}_{H_i^{\bar{=}}}(G_{ij} \cap H_i^{\bar{=}}) \subseteq \bigcap_{j=1}^{m_i} (H_i^{\bar{=}} \setminus F_{ij}^{\geq}) = \bigcap_{j=1}^{m_i} \text{int}_{H_i^{\bar{=}}}(F_{ij}^{\leq})$ (the last inclusion is a consequence of (48)), we obtain that $y \in \text{int}_{H_i^{\bar{=}}}(T_i)$ which is in contradiction with the fact that T_i is an $S \cap H_i^{\bar{=}}$ -free convex set of $H_i^{\bar{=}}$.

Therefore B is S -free polyhedron containing K . ■

Cases (2.), (3.) and (4.) of Theorem 19 follow from maximality of K and cases (2.), (3.) and (4.) of Proposition 14 respectively. Case (1.) of Proposition 14 shows that maximal S -free convex sets are polyhedra when $\dim(K) = \dim(W)$ and $\text{rel.int}(K) \cap \text{rel.int}(\mathbf{P}_I) \neq \emptyset$. To complete the proof of Theorem 19, we need to show

that $K \cap V$ is a maximal S -free convex set of V whose dimension equals $\dim(V)$, and for every facet F of K , $F \cap V$ is a facet of $K \cap V$ (where $V = \text{aff.hull}(W \cap \mathbb{Z}^n)$). This is verified in the next section.

4.3.2 Structure of Facets of Maximal S -free Convex Sets

For convenience we repeat the definition of some notation. Let K be a polyhedron of W , that is let $K = \{x \in W : a_i^T x \leq b_i \forall i \in \{1, \dots, N\}\}$. Then we denote the i^{th} facet of K as $F_i(K)$. Also for all $\varepsilon > 0$, let $F_i^\varepsilon(K) = \{x \in W : a_j^T x < b_j \forall j \neq i \text{ and } b_i < a_i^T x < b_i + \varepsilon\}$.

Before completing the proof of Theorem 19, we prove part (1.) of Theorem 20.

Proposition 15. *Let $K = \{x \in W : a_i^T x \leq b_i, \forall i \in \{1, \dots, N\}\} \subseteq \mathbb{R}^n$ be a maximal S -free convex set, such that $\dim(K) = \dim(W)$. Then*

$$\left(\text{rel.int}(F_i(K)) \cup F_i^\varepsilon(K) \right) \cap S \neq \emptyset \quad \forall \varepsilon > 0 \forall i \in \{1, \dots, N\}.$$

Proof. For $\varepsilon > 0$ and $i \in \{1, \dots, N\}$ consider the set

$$K_i^\varepsilon = \{x \in W : a_j^T x \leq b_j \forall j \neq i \text{ and } a_i^T x \leq b_i + \varepsilon\}.$$

Since $K \subsetneq K_i^\varepsilon$ we obtain that K_i^ε is not an S -free convex set, that is, $\text{int}_W(K_i^\varepsilon) \cap S \neq \emptyset$.

Hence, the set $\text{int}_W(K_i^\varepsilon) \setminus \text{int}_W(K)$ must contain a point of S . Observe that:

$$\begin{aligned} \text{int}_W(K_i^\varepsilon) \setminus \text{int}_W(K) &= \{x \in W : a_j^T x < b_j \forall j \neq i \text{ and } b_i \leq a_i^T x < b_i + \varepsilon\} \\ &= F_i^\varepsilon(K) \cup \text{rel.int}(F_i(K)). \end{aligned}$$

Therefore $\emptyset \neq \text{int}_W(K_i^\varepsilon) \setminus \text{int}_W(K) \cap S = (F_i^\varepsilon(K) \cup \text{rel.int}(F_i(K))) \cap S$. ■

Note that Proposition 15 highlights an important difference between maximal S -free convex sets for general S and for the case where S is the set of integer points contained in a rational polyhedron. In the case of general S , there is no guarantee

that every facet of a full-dimensional (wrt W) maximal S -free convex set contains points belonging to S in its relative interior. This is illustrated in the next example.

Example 6. Let $S = \{x \in \mathbb{Z}^2 \mid \sqrt{2}x_1 + x_2 \leq 0, x_1 \geq 1\}$. Then the set $K = \{x \in \mathbb{R}^2 \mid \sqrt{2}x_1 + x_2 \geq 0\}$ is a maximal S -free convex set, but the facet of K defined by $\{x \in \mathbb{R}^2 \mid \sqrt{2}x_1 + x_2 = 0\}$ contains no point belonging to S .

Now we complete the proof of Theorem 19. The proof of the first part of the next proposition is similar to the proof of a similar property for maximal lattice-free convex sets, appearing in Basu et al. [13].

Proposition 16. Let $K = \{x \in W : a_i^T x \leq b_i, \forall i \in \{1, \dots, N\}\} \subseteq W$ be a maximal S -free convex set such that $\dim(K) = \dim(W)$ and $\text{int}_W(K) \cap \text{rel.int}(\mathbf{P}_I) \neq \emptyset$. Let V be the affine hull of $W \cap \mathbb{Z}^n$. Then $K \cap V$ is a maximal S -free convex set of V such that $\dim(K \cap V) = \dim(V)$ and $F \subseteq V$ is a facet of $K \cap V$ if and only if $F = F_i(K) \cap V$ for some $i \in \{1, \dots, N\}$.

Proof. Since $\text{int}_W(K) \cap \text{rel.int}(\mathbf{P}_I) \neq \emptyset$, we obtain that $\text{int}_W(K) \cap V \neq \emptyset$. Therefore Lemma 30 implies that $\text{int}_W(K) \cap V = \text{int}_V(K \cap V)$.

We first verify that $K \cap V$ is a maximal S -free convex set of V . Since K is an S -free convex set and $\text{int}_W(K) \cap S = \text{int}_W(K) \cap S \cap V = \text{int}_V(K \cap V) \cap S$, we obtain that $K \cap V$ is an S -free convex set of V . If $K \cap V$ is not maximal, then there exist $B \subseteq V$, an S -free convex set, such that $B \supsetneq K \cap V$. Consider $K' = \text{conv}(K \cup B)$. Then $K' \cap V = B$. We have $\text{int}_W(K') \cap S = \text{int}_W(K') \cap S \cap V = \text{int}_V(B) \cap S = \emptyset$. Therefore K' is an S -free convex set of W . Since $K' \supsetneq K$ and $B \supsetneq K \cap V$, we obtain $K' \supsetneq K$ which is in contradiction with the fact that K is maximal S -free convex set. Therefore, $K \cap V$ is a maximal S -free convex set of V .

Since $\text{int}_V(K \cap V) \neq \emptyset$, we obtain $\dim(K \cap V) = \dim(V)$. Finally, we verify that $F \subseteq V$ is a facet of $K \cap V$ if and only if $F = F_i(K) \cap V$ for some $i \in \{1, \dots, N\}$.

(\Rightarrow) If F is a facet of $K \cap V = \{x \in V : a_i^T x \leq b_i, \forall i \in \{1, \dots, N\}\}$, then $F = F_i(K) \cap V$ for some $i \in \{1, \dots, N\}$.

(\Leftarrow) Given $\varepsilon > 0$, by the use of Proposition 15 we obtain that for all $i = 1, \dots, N$, $\emptyset \neq (\text{relint}(F_i(K)) \cup F_i^\varepsilon(K)) \cap S = (\text{rel.int}(F_i(K)) \cup F_i^\varepsilon(K)) \cap V \cap S \subseteq \text{int}_W(\{x \in W : a_k^T x \leq b_k, \forall k \in \{1, \dots, N\} \setminus \{i\}\}) \cap V \cap S = \text{int}_V(\{x \in V : a_k^T x \leq b_k, \forall k \in \{1, \dots, N\} \setminus \{i\}\}) \cap S$. Note that the last equality is obtained as a consequence of Lemma 30, $\{x \in W : a_k^T x \leq b_k, \forall k \in \{1, \dots, N\} \setminus \{i\}\} \supseteq K$ and $\text{int}_W(K) \cap V \neq \emptyset$. As $K \cap V$ is an S -free convex set of V and $\dim(K \cap V) = \dim(V)$, we conclude that $a_i^T x \leq b_i$ must define a facet of $K \cap V$, that is $F_i(K) \cap V$ is a facet of $K \cap V$ for all $i \in \{1, \dots, N\}$. \blacksquare

4.3.3 Upper Bound on the Number of Facets of Maximal S -free Convex Sets

When S is the set of integer points contained in a general convex set, full-dimensional (wrt W) maximal S -free convex sets do not need to have points of S in the relative interior of each facet. Therefore, proving upper bound on the number of facets is slightly more involved than the case of maximal lattice-free convex sets.

We first begin with a Lemma that states what is known about maximal S -free convex sets in the easy case when the set S is defined by a polytope.

Lemma 31. *Let $K' \subseteq W$, \mathbf{P}' a polytope, $S' = \mathbf{P}' \cap \mathbb{Z}^n$, $S' \neq \emptyset$ and $\dim(K') = \dim(W)$. Then K' is a maximal S' -free convex set of W if and only if K' is a S -free polyhedron with a point of S' in the relative interior of each of its facets.*

The next Lemma is a standard result. See Schrijver [76] for a proof.

Lemma 32. *If $L \subseteq \mathbb{R}^n$ is a rational affine subspace (i.e. $\text{aff.hull}(L \cap \mathbb{Z}^n) = L$), then there exists an affine transformation $T : \mathbb{R}^{\dim(L)} \mapsto L$ such that T is invertible and $T(\mathbb{Z}^{\dim(L)}) = L \cap \mathbb{Z}^n$.*

Using Lemma 31 and Lemma 32 and a proof similar to that in Doignon [39], Bell [16], Scarf [75] we obtain the following result.

Lemma 33. *Let $K' \subseteq W$, \mathbf{P}' a polytope, $S' = \mathbf{P}' \cap \mathbb{Z}^n$, $S' \neq \emptyset$ and $\dim(K') = \dim(W)$. If K' is a maximal S' -free convex set of W , then K' is a polyhedron with at most $2^{\dim(\mathbf{P}')}$ facets.*

Proof. By Lemma 31, K' is a polyhedron with a point of S' in the relative interior of each of its facets. Let $L = \text{aff.hull}(\mathbf{P}')$ (in particular $\dim(L) = \dim(\mathbf{P}')$) and $T : \mathbb{R}^{\dim(\mathbf{P}')} \mapsto L$ the affine transformation given by the Lemma 32. Assume by contradiction that K' has strictly more than $2^{\dim(\mathbf{P}')}$ facets. Since $S' \subseteq V \cap \mathbb{Z}^n$ and by the Pigeon hole principle, there exists $a, b \in \mathbb{Z}^{\dim(\mathbf{P}')}$ with the same parity, such that $T(a), T(b)$ are points in the relative interior of two different facets of K' . We have that $\frac{a+b}{2} \in \mathbb{Z}^{\dim(\mathbf{P}')}$, so by the properties of T , we have $\frac{T(a)+T(b)}{2} = T(\frac{a+b}{2}) \in L \cap \mathbb{Z}^n$. Since \mathbf{P}' is convex, $\frac{T(a)+T(b)}{2} \in S'$ and therefore $\text{int}_W(K') \cap S' \neq \emptyset$, a contradiction with the fact K' is S' -free. Therefore, K' has at most $2^{\dim(\mathbf{P}')}$ facets. \blacksquare

We now have all the tools needed to verify the upper bound on the number of facets of maximal S -free convex sets.

Proposition 17. *If $K = \{x \in W : a_i^T x \leq b_i, \forall i \in \{1, \dots, N\}\} \subseteq W$ is a maximal S -free convex set such that $\dim(K) = \dim(W)$, then $N \leq 2^{\dim(\text{conv}(S))}$.*

Proof. Let $\varepsilon > 0$. Consider the sets I_1 and I_2 defined as:

1. $i \in I_1$, if $S \cap \text{rel.int}(F_i(K)) \neq \emptyset$.
2. $i \in I_2$, if $S \cap \text{rel.int}(F_i(K)) = \emptyset$ and, $F_i^\varepsilon(K) \cap S \neq \emptyset$.

By Proposition 15, these sets are well defined and $I_1 \cup I_2 = \{1, \dots, N\}$. If $I_1 \neq \emptyset$, then for each $i \in I_1$ take a point $x_i \in \text{rel.int}(F_i(K)) \cap S$ and if $I_2 \neq \emptyset$, then for each $i \in I_2$, take a point $x_i \in F_i^\varepsilon(K) \cap S$.

Define $\mathbf{P}' = \text{conv}(\{x_1, \dots, x_N\})$ and $S' = \mathbf{P}' \cap \mathbb{Z}^n$. Since $\mathbf{P}' \subseteq \mathbf{P}$, we have that $S' \subseteq S$. This implies that $\text{int}_W(K) \cap S' \subseteq \text{int}_W(K) \cap S = \emptyset$, so K is an S' -free convex set.

Note that \mathbf{P}' is a rational polytope, so by Corollary 31 and Lemma 33 every maximal S' -free convex set K' is a polyhedron that has at most $2^{\dim(\mathbf{P}')}$ facets and contains an integer point of S' in the relative interior of each of its facets. Observe that $\dim(\mathbf{P}') \leq \dim(\text{conv}(S))$.

If $I_2 = \emptyset$, then K is a maximal S' -free convex set. Therefore $N \leq 2^{\dim(\mathbf{P}')} \leq 2^{\dim(\text{conv}(S))}$.

If $I_2 \neq \emptyset$, then consider $i \in I_2$. We will construct a polyhedron K_1 with N facets such that:

1. K_1 is an S' -free convex set.
2. $F_j^\varepsilon(K) \subseteq F_j^\varepsilon(K_1), \forall j \in I_2 \setminus \{i\}$.
3. K_1 has N facets that are in one-to-one correspondence with the facets of K .
4. K_1 has at least $|I_1| + 1$ facets with a point of S' in the relative interior.

We construct K_1 from K by changing the right hand side of the i^{th} inequality. Since \mathbf{P}' is bounded and $S' \subseteq \mathbb{Z}^n$, we obtain that $|S'|$ is finite. So $S' \cap F_i^\varepsilon(K)$ is also finite. Moreover, since $x_i \in S' \cap F_i^\varepsilon(K)$, we have $S' \cap F_i^\varepsilon(K) \neq \emptyset$. Hence, there exists $z_i \in S' \cap F_i^\varepsilon(K)$ such that

$$\min\{a_i^T x : x \in S' \cap F_i^\varepsilon(K)\} = a_i^T z_i.$$

Denote $b'_i = a_i^T z_i$. Consider the polyhedron

$$K_1 = \{x \in W : a_j^T x \leq b_j, \forall j \in I_1 \cup I_2 \setminus \{i\}, a_i^T x \leq b'_i\}$$

We verify that K_1 satisfies (1.), (2.), (3.), and (4.):

1. Since K is S' -free, we only need to show that $\text{int}(K_1) \setminus \text{int}(K)$ does not contain

points of S' . Since $b'_i \leq b_i + \varepsilon$ we have:

$$\begin{aligned} \text{int}(K_1) \setminus \text{int}(K) &= \{x \in W : a_j^T x < b_j, \forall j \in I_1 \cup I_2 \setminus \{i\}, b_i \leq a_i^T x < b'_i\} \\ &\subseteq \{x \in W : a_j^T x < b_j \forall j \neq i \text{ and } b_i \leq a_i^T x < b_i + \varepsilon\} \\ &= F_i^\varepsilon(K) \cup \text{rel.int}(F_i(K)) \end{aligned}$$

Since $i \in I_2$, $S \cap \text{rel.int}(F_i(K)) = \emptyset$ so $(\text{int}(K_1) \setminus \text{int}(K)) \cap S' \subseteq F_i^\varepsilon(K) \cap S'$. On the other hand, since $\forall x \in S' \cap F_i^\varepsilon(K)$, $a_i^T x \geq b'_i$, we conclude $(\text{int}(K_1) \setminus \text{int}(K)) \cap S' = \emptyset$. Therefore, K_1 is a S' -free convex set.

2. Using the fact that $b_i \leq b'_i$ and $j \in I_2 \setminus \{i\}$ we have,

$$\begin{aligned} F_j^\varepsilon(K) &= \{x \in W : a_k^T x < b_k \forall k \neq j \text{ and } b_j < a_j^T x < b_j + \varepsilon\} \\ &\subseteq \{x \in W : a_k^T x < b_k \forall k \neq i, j, a_i^T x < b'_i \text{ and } b_j < a_j^T x < b_j + \varepsilon\} \\ &= F_j^\varepsilon(K_1). \end{aligned}$$

3. By definition, the point z_i satisfies $a_i^T z_i = b'_i$ and $\forall j \neq i$, $a_j^T z_i < b_j$. Therefore the inequality $a_i^T x \leq b'_i$ is a facet defining inequality for K_1 . Observe also that the inequalities $a_j^T x \leq b_j$ for $j \neq i$ are facet-defining inequality for K_1 . Therefore K_1 has N facets.

4. As verified earlier, z_i belongs to the relative interior of the facet of K_1 defined by $\{x \in K_1 : a_i^T x = b'_i\}$. Also for $j \in I_1$, since $b_i \leq b'_i$, we have:

$$\begin{aligned} \text{rel.int}(F_j(K)) &= \{x \in W : a_k^T x < b_k \forall k \neq j \text{ and } a_j^T x = b_j\} \\ &\subseteq \{x \in W : a_k^T x < b_k \forall k \neq i, j, a_i^T x < b'_i \text{ and } a_j^T x = b_j\}. \end{aligned}$$

Therefore, since $x_j \in \text{rel.int}(F_j) \cap S'$, the facet of K_1 defined by $\{x \in K_1 : a_j^T x = b_j\}$ has a point of S' in its relative interior. In conclusion, K_1 has at least $|I_1| + 1$ facets with a point of S' in its relative interior.

So we have a procedure to construct, from K , an S' -free polyhedron K_1 with N facets that has at least one more facet that contains a point of S' in its relative interior than K has. If K_1 is not a maximal S' -free convex set, then we can choose a facet of K_1 without a point of S' in its relative interior and by properties (1.) and (2.), repeat the above procedure. We can repeat this a finite number of times, obtaining a sequence K_1, K_2, \dots, K_T of polyhedra with the same number of facets as K and such that K_T does not have any facet without a point of S' in its relative interior. So K_T is a maximal S' -free convex set. Thus, $N \leq 2^{\dim(\mathbf{P}')} \leq 2^{\dim(\text{conv}(S))}$.

■

4.4 Notes

4.4.1 Differences Between General Maximal S -free Convex Sets and the Case Where S is the Set of Integer Points Contained in a Rational Polyhedron

As discussed in Section 4.3.2, in the case of general S , full-dimensional (wrt W) maximal S -free convex sets do not necessarily have integer points in the relative interior of each facet.

There is another difference between maximal S -free convex sets in the general case and the case when S is the set of integer points contained in a rational polyhedron. The result of Lovász [65] states that maximal lattice-free convex sets are in the form of a polytope plus a cylinder, that is, if r is a recession direction of maximal lattice-free convex set, then so is $-r$. Similarly, Basu et al. [14] prove the following result.

Proposition 18 (Basu et al. [14]). *If S is a nonempty set of integer points contained in a rational polyhedron \mathbf{P} , K is a maximal S -free convex set such that $K \cap \text{conv}(S)$ has nonempty interior, and r belongs to the recession cone of \mathbf{P} and K , then $-r$ belongs to the recession cone of K .*

Proposition 18 is an important property and is useful in verifying many results.

See for example Basu et al. [12]. The following example shows that this result is not true in general.

Example 7. Let $\mathbf{P} = \{u \in \mathbb{R}^3 \mid u_3 \geq u_2 - \sqrt{2}u_1, u_3 \geq \sqrt{2}u_1 - u_2\}$. Then $K = \{v \in \mathbb{R}^3 \mid v_3 \leq 1, v_2 \geq 0\}$ is a maximal S -free convex set. Also $r = (1, \sqrt{2}, 0) \in \text{rec}(K \cap \mathbf{P})$. However, $-r \notin \text{rec}(K)$.

We present some sufficient conditions on \mathbf{P} for a property like the one presented in Proposition 18 to hold. For simplicity we restrict the discussion to the case where $W = \mathbb{R}^n$.

Given a closed convex set $\mathbf{P} \subseteq \mathbb{Z}^n$, and a non-zero vector $r \in \text{rec}(\mathbf{P})$, let $\mathcal{Z}(\mathbf{P}, r) = \{x \in \mathbb{Z}^n \mid \exists \lambda \geq 0, \text{ s.t. } x + \lambda r \in \mathbf{P}\}$.

Definition 10 (Convex Set with Dirichlet Property). We say a closed convex set $\mathbf{P} \subseteq \mathbb{R}^n$ satisfies Dirichlet Property if \mathbf{P} satisfies the following conditions: For all $r \in \text{rec}(\mathbf{P}) \setminus \{0\}$ and for all $x \in \mathcal{Z}(\mathbf{P}, r)$, given any $\epsilon > 0$ and $\gamma \geq 0$ there exists $\bar{x} \in \mathbf{P} \cap \mathbb{Z}^n$ such that the distance between the integer point \bar{x} and the half line $\{x + \lambda r \mid \lambda \geq \gamma\}$ is less than ϵ .

We first show that the Dirichlet property indeed implies a property similar to that presented in Proposition 18.

Proposition 19. Let $\mathbf{P} \subseteq \mathbb{R}^n$ be a closed convex set satisfying Dirichlet Property, let $S = \mathbf{P} \cap \mathbb{Z}^n$, and let K be a full-dimensional maximal S -free convex set. If r belongs to the recession cone of \mathbf{P} and K , then $-r$ belongs to the recession cone of K .

Proof. By part (5.) of Proposition 13, we obtain $\text{int}(K + \{\lambda r \mid \lambda \in \mathbb{R}\}) = \text{int}(K) + \text{rel.int}(\{\lambda r \mid \lambda \in \mathbb{R}\}) = \text{int}(K) + \{\lambda r \mid \lambda \in \mathbb{R}\}$.

To prove the result of this proposition we need to verify that $\text{int}(K + \{\lambda r \mid \lambda \in \mathbb{R}\}) \cap S = \emptyset$. Assume by contradiction that $x \in \text{int}(K + \{\lambda r \mid \lambda \in \mathbb{R}\}) \cap S$. By the previous claim, there exists $\bar{\lambda} \in \mathbb{R}_+$ such that $x + \bar{\lambda}r \in \text{int}(K)$. Therefore $B(0, \delta) + \{x + \lambda r \mid \lambda \geq \bar{\lambda}\}$ is contained in the interior of K for some suitable small and positive δ .

Since r is a recession direction of \mathbf{P} , we obtain that $x \in \mathcal{Z}(\mathbf{P}, r)$. As \mathbf{P} satisfies Dirichlet property, there exists an integer point z belonging to \mathbf{P} in the interior of the set $B(0, \delta) + \{x + \lambda r \mid \lambda \geq \bar{\lambda}\}$. However, this set lies in the interior of K . Therefore $z \in \mathbf{P}$ and lies in interior of K , a contradiction. ■

We note here that if $\text{conv}(S)$ satisfies Dirichlet Property, then the statement of Proposition 19 holds with \mathbf{P} replaced by $\text{conv}(S)$, i.e., if r belongs to the recession cone of $\text{conv}(S)$ and K , then $-r$ belongs to the recession cone of K . This observation is useful in conjunction with the result of Proposition 21 that is presented in the next Section.

Our motivation for the name ‘Dirichlet property’ is due to the fact that we often use the following consequence of Dirichlet Theorem to prove the ‘Dirichlet property’.

Lemma 34 (Basu et al. [13]). *If $x \in \mathbb{Z}^n$ and $r \in \mathbb{R}^n$, then for all $\epsilon > 0$ and $\gamma \geq 0$, there exists a point of \mathbb{Z}^n at a distance less than ϵ from the half line $\{x + \lambda r \mid \lambda \geq \gamma\}$.*

4.4.1.1 Some Convex Sets Satisfying Dirichlet Property

Next we give examples of convex sets with Dirichlet Property.

Proposition 20. *Every full dimensional closed convex set in \mathbb{R}^2 is a convex set satisfying Dirichlet Property.*

Proof. Let \mathbf{P} be a full-dimensional, closed convex set in \mathbb{R}^2 . Consider any $\epsilon > 0$ and $\gamma \geq 0$. Let r be a non zero vector in the recession cone of \mathbf{P} and $x \in \mathcal{Z}(\mathbf{P}, r)$. Denote $\eta = \min\{\lambda \in \mathbb{R}_+ \mid x + \lambda r \in \mathbf{P}\}$.

1. Since \mathbf{P} is a closed set, we obtain that if r is a rational vector, then there exists $\bar{x} \in \mathbf{P} \cap \mathbb{Z}^2$ such that the distance between \bar{x} and the half line $\{x + \lambda r \mid \lambda \geq \gamma\}$ is 0.

2. Suppose now that r is not a rational vector (i.e. $\lambda r \notin \mathbb{Z}^2 \forall \lambda > 0$). Assume by contradiction that there exists no $\bar{x} \in \mathbf{P} \cap \mathbb{Z}^2$ such that the distance between \bar{x} and the half line $\{x + \lambda r \mid \lambda \geq \gamma\}$ is less than ϵ . Since \mathbf{P} is full dimensional and $L := \{x + \lambda r \mid \lambda \geq \eta\} \subseteq \mathbf{P}$, the set $M := \{y \in \mathbf{P} \mid \text{distance}(y, L) \leq \epsilon\}$ is a full-dimensional convex subset of \mathbf{P} . Note now that M is lattice-free and r belongs to the recession cone of M . Therefore M is contained in a maximal lattice-free convex set. Since the only class of unbounded full-dimensional maximal lattice-free convex set in \mathbb{R}^2 is the split set (a set of form $\{y \in \mathbb{R}^2 \mid b \leq a^T y \leq b + 1\}$ where $a \in \mathbb{Z}^2$ and $b \in \mathbb{Z}$), we obtain that r is a rational vector which is a contradiction.

■

We next verify that all the examples which do not satisfy the property similar to that presented in Proposition 18, also satisfy the condition that $\text{conv}(S)$ is not closed. Note that Proposition 20 can be used to construct examples where $\text{conv}(S)$ is not closed and yet the property similar to that presented in Proposition 18 is satisfied.

Proposition 21. *If $S \subseteq \mathbb{Z}^n$ and $\text{conv}(S)$ is full-dimensional and closed, then $\text{conv}(S)$ satisfies the Dirichlet property.*

Proof. Let $\mathbf{P} := \text{conv}(S)$. If $n = 1$, then the result is straightforward to verify.

The proof is now by induction on n . Consider any $\epsilon > 0$ and $\gamma \geq 0$. Let r be a non zero vector in the recession cone of \mathbf{P} and $x \in \mathcal{Z}(\mathbf{P}, r)$. Denote $\eta = \min\{\lambda \in \mathbb{R}_+ \mid x + \lambda r \in \mathbf{P}\}$.

1. Since \mathbf{P} is a closed set, we obtain that if r is a rational vector, then there exists $\bar{x} \in \mathbf{P} \cap \mathbb{Z}^n$ such that the distance between \bar{x} and the half line $\{x + \lambda r \mid \lambda \geq \gamma\}$ is 0.

2. Suppose now that r is not a rational vector. There are two cases:

(a) Suppose $\exists \delta > 0$ and $\zeta > 0$ such that $B(x + \zeta r, \delta) \subseteq \mathbf{P}$. Let $\epsilon' = \min\{\delta, \epsilon\}$.

Now as a consequence of the Dirichlet Theorem there exists an integer point \bar{x} at a distance less than ϵ' from the half line $\{x + \lambda r \mid \lambda \geq \max\{\zeta, \gamma\}\}$. Then \bar{x} belongs to \mathbf{P} , completing the proof.

(b) The half-line $\{x + \lambda r \mid \lambda \geq \eta\}$ lies on the boundary of $\text{conv}(S)$. Let F be a proper face of $\text{conv}(S)$ containing this half-line. (Recall that a face F of a closed convex set \mathbf{P} is a closed convex subset such that if $x \in F$ and x can be written as convex combination of $x^1, x^2 \in \mathbf{P}$, then $x^1, x^2 \in F$). Then we claim:

i. $F = \text{conv}(S \cap F)$: Since $S \cap F \subseteq F$ and F is a convex set, we obtain that $\text{conv}(S \cap F) \subseteq F$. For the other inclusion, observe that if $x \in F$, then x can be written as a convex combination of a finite number of vectors in S (since $F \subseteq \text{conv}(S)$). However by definition of a face, each of these vectors belong to F . Thus $x \in \text{conv}(S \cap F)$, or equivalently $F \subseteq \text{conv}(S \cap F)$.

ii. $\text{aff.hull}(F)$ is a rational affine half-space containing an integer point: By the above claim $S \cap F \neq \emptyset$ and $\text{aff.hull}(F)$ can be generated by vectors in S , which are integral vectors. Thus, $\text{aff.hull}(F)$ is a rational affine half-space.

Now, we apply the invertible affine transformation $A : \text{aff.hull}(F) \mapsto \mathbb{R}^{\dim(\text{aff.hull}(F))}$ to F , where A is the function described in Lemma 32. Observe now that $A(F)$ is a closed convex set, $A(F) = \text{conv}(A(S \cap F))$, Ar is a recession direction of $A(F)$, $Ax \in \mathcal{Z}(A(F), Ar)$ and $1 \leq \dim(\text{aff.hull}(F)) < n$. Therefore by the induction hypothesis, for all $\delta > 0$, there exists an integer point \hat{x} belonging to $A(\text{conv}(S \cap F))$ that lies within a distance

of δ from the half-line $\{u \mid u = Ax + \lambda Ar, \lambda \geq \max\{\eta, \gamma\}\}$. However, this implies that there exists a point $\bar{x} \in S$ such that \bar{x} lies at a distance less than ϵ to the half line $\{u \mid u = x + \lambda r, \lambda \geq \max\{\eta, \gamma\}\}$.

■

We remark here that by the use of Lemma 32, the result of Proposition 21 can be extended to the case where $\text{aff.hull}(\text{conv}(S))$ is not full-dimensional since $\text{aff.hull}(\text{conv}(S))$ is a rational affine subspace.

Next we show an interesting class of non-polyhedral convex sets that satisfy Dirichlet Property.

Proposition 22. *Every full dimensional, closed, strictly convex set satisfies Dirichlet Property.*

Proof. Let \mathbf{P} be a full dimensional, closed, strictly convex set. Consider any $\epsilon > 0$ and $\gamma \geq 0$. Let r be a non zero vector in the recession cone of \mathbf{P} and $x \in \mathcal{Z}(\mathbf{P}, r)$. Denote $\eta = \min\{\lambda \in \mathbb{R}_+ \mid x + \lambda r \in \mathbf{P}\}$.

1. Since \mathbf{P} is a closed set, we obtain that if r is a rational vector, then there exists $\bar{x} \in \mathbf{P} \cap \mathbb{Z}^n$ such that the distance between \bar{x} and the half line $\{x + \lambda r \mid \lambda \geq \gamma\}$ is 0.
2. Suppose now that r is not a rational vector. Let $\bar{\gamma} := \max\{\gamma, \eta + 1\}$. The point $x + \bar{\gamma}r$ belongs to \mathbf{P} and is not an extreme point of \mathbf{P} . Since \mathbf{P} is strictly convex, the point $x + \bar{\gamma}r$ lies in the interior of \mathbf{P} . Therefore, there exists a ball of radius $\delta > 0$ around the point $x + \bar{\gamma}r$ that lies in \mathbf{P} . Set $\epsilon' := \min\{\epsilon, \frac{\delta}{2}\}$. Now as a consequence of the Dirichlet Theorem there exists an integer point \bar{x} at a distance less than ϵ' from the half line $\{x + \lambda r \mid \lambda \geq \bar{\gamma}\}$. Since \bar{x} lies at a distance less than δ from the half line $\{x + \lambda r \mid \lambda \geq \bar{\gamma}\}$ and the set $B(0, \delta) + \{x + \lambda r \mid \lambda \geq \bar{\gamma}\}$ belongs to \mathbf{P} , we obtain that \bar{x} belongs to \mathbf{P} . Moreover, the point \bar{x} lies at a distance less than ϵ from the half line $\{x + \lambda r \mid \lambda \geq \gamma\}$.

■

4.4.2 Other Extensions

1. Instead of defining $S \subseteq \mathbb{Z}^n$, we can define S as a subset of points belonging to a general lattice. All the results in Section 4.3 carry through.
2. The condition that $S \subseteq \mathbb{Z}^n$ such that $\text{conv}(S) \cap \mathbb{Z}^n = S$ is not necessary for the polyhedrality of maximal S -free convex sets. For example, the following corollary of Theorem 19 can be proven.

Corollary 8. *Let $S_i \subseteq \mathbb{Z}^n \cap W$, $i = 1, \dots, N$ such that $S_i = \text{conv}(S_i) \cap \mathbb{Z}^n$. Denote $S = \bigcup_{i=1}^N S_i$. Let $K \subseteq W$ be an S -free convex set of W . Then there exists an S -free polyhedron $B \subseteq W$ such that $K \subseteq B$.*

Proof. For all $i = 1, \dots, N$, $\text{int}_W(K) \cap S_i \subseteq \text{int}_W(K) \cap S = \emptyset$. Therefore K is an S_i -free convex set of W . By Theorem 19, there exists an S_i -free polyhedron $B_i \subseteq W$ such that $K \subseteq B_i$. The polyhedron $B = \bigcap_{i=1}^N B_i$ satisfies $K \subseteq B$ and $\text{int}_W(B) \cap S = \bigcap_{i=1}^N \text{int}_W(B_i) \cap \bigcup_{j=1}^N S_j = \bigcup_{j=1}^N [S_j \cap \bigcap_{i=1}^N \text{int}_W(B_i)] \subseteq \bigcup_{j=1}^N [S_j \cap \text{int}_W(B_j)] = \emptyset$. Thus B is an S -free polyhedron containing K . ■

An analysis similar to that in Section 4.3.1 and 4.3.2 can be carried out for this more general class of S . The upper bound on the number of facets of maximal S -free convex sets presented in Section 4.3.3 is not valid for this class of more general S . For example, if $S = \{(0, 0), (1, 0), (-1, 1), (2, 1), (0, 2), (1, 2)\}$, then the hexagon with vertices $\{(0.5, -0.25), (2, 0.5), (2, 1.5), (0.5, 2.25), (-1, 1.5), (-1, 0.5)\}$ is a maximal S -free convex set.

CHAPTER V

A STRONG DUAL FOR CONIC MIXED-INTEGER PROGRAMS

5.1 *Introduction*

One of the fundamental goals of optimization theory is the study of structured techniques to obtain bounds on the optimal objective function value for a given class of optimization problems. For a minimization (resp. maximization) problem, upper (resp. lower) bounds on the optimal objective function value are provided by points belonging to the feasible region. Dual bounds, that is, lower (resp. upper) bounds on the optimal objective function value for a minimization (resp. maximization) problem are typically obtained by constructing various types of dual optimization problems whose feasible solutions provide these bounds. We will say that a minimization (resp. maximization) problem is finite if its feasible region is nonempty and the objective function is bounded from below (resp. above). A strong dual is typically characterized by two properties:

1. The primal program is finite if and only if the dual program is finite.
2. If the primal and the dual are finite, then the optimal objective function values of the primal and dual are equal.

In the case of linear programming problems and more generally for conic (convex) optimization problems the dual optimization problem is well understood and plays a key role in various algorithmic devices [17]. The subadditive dual for mixed-integer linear programs is also well understood [53, 54, 56, 58, 78]. In this chapter, we evaluate the possibility of extending the subadditive dual to the case of mixed-integer conic programs.

The rest of the chapter is organized as follows. In Section 5.2, we present the necessary notation, definitions and the statement of our main result. In Section 5.3, we verify the basic weak duality result, that is, the fact that the dual feasible solutions produce valid bounds. Apart from weak duality, the proof of strong duality relies on the following additional three results: (i) The finiteness of the primal being equivalent to the finiteness of its continuous relaxation. (ii) Strong duality for conic programs. (iii) The possibility of constructing a subadditive function defined over \mathbb{R}^m such that it is dual feasible and matches the value function of the primal on a relevant subset of \mathbb{R}^m . In Section 5.4, we develop and present (in the case of conic duality) these preliminary results. In Section 5.5, we present the proof of the strong duality result. In particular, in Section 5.5.1, we present a sufficient condition for the finiteness of the primal program being equivalent to the finiteness of the dual program. In Section 5.5.2, we prove that under this sufficient condition, if the primal and dual are finite, then their optimal values must be equal. In Section 5.6, we discuss valid subadditive inequalities for conic mixed-integer programs. Finally, in Section 5.7 we present the dual for an alternative form of the primal conic mixed-integer program.

5.2 Notation, definitions and main result

Let $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional, closed and pointed cone. A conic vector inequality is defined as follows ([17]):

Definition 11 (Conic vector inequality). *For $a, b \in \mathbb{R}^m$, $a \succeq_{\mathbf{C}} b$ if and only if $a - b \in \mathbf{C}$. In addition, we write $a \succ_{\mathbf{C}} b$ whenever $a - b \in \text{int}(\mathbf{C})$.*

A mixed-integer conic programming problem (the primal optimization problem) is an optimization problem of the following form:

$$(\mathcal{P}) \left\{ \begin{array}{l} z^* = \inf \quad c^T x \\ \text{s.t.} \quad Ax \succeq_{\mathbf{C}} b \\ x_i \in \mathbb{Z}, \forall i \in \mathcal{I}, \end{array} \right.$$

where $\mathcal{I} = \{1, \dots, n_1\} \subseteq \{1, \dots, n\}$ is the set of indices of integer variables.

Notice that problems of the form of (\mathcal{P}) are a generalization of mixed-integer linear programming problems, which are recovered by setting $\mathbf{C} = \mathbb{R}_+^m$. Hence, a natural way of defining a dual optimization problem for mixed-integer conic programming is to generalize the well-known subadditive dual for mixed-integer linear programming (see, for example, [46] and [73]). Consequently, to define the dual of (\mathcal{P}) , we first present some notation and definitions that are slight variations of those necessary to define the subadditive dual for mixed-integer linear programming problems.

Definition 12 (Subadditive). *Let $\mathcal{S} \subseteq \mathbb{R}^m$. A function $g : \mathcal{S} \mapsto \mathbb{R} \cup \{-\infty\}$ is said to be subadditive if for all $u, v \in \mathcal{S}$ such that $u + v \in \mathcal{S}$, the inequality $g(u + v) \leq g(u) + g(v)$ holds.*

Definition 13 (Nondecreasing w.r.t. \mathbf{C}). *Let $\mathcal{S} \subseteq \mathbb{R}^m$. A function $g : \mathcal{S} \mapsto \mathbb{R} \cup \{-\infty\}$ is said to be nondecreasing w.r.t. \mathbf{C} if for all $u, v \in \mathcal{S}$ such that $u \succeq_{\mathbf{C}} v$, the inequality $g(u) \geq g(v)$ holds.*

We define the subadditive dual problem for (\mathcal{P}) as follows:

$$(\mathcal{D}) \left\{ \begin{array}{l} \rho^* = \sup \quad g(b) \\ \text{s.t.} \quad g(A^i) = -g(-A^i) = c_i, \quad \forall i \in \mathcal{I} \\ \bar{g}(A^i) = -\bar{g}(-A^i) = c_i, \quad \forall i \in \mathcal{C} \\ g(0) = 0 \\ g \in \mathcal{F}, \end{array} \right.$$

where $\mathcal{C} = \{n_1 + 1, \dots, n\}$ is the set of indices of continuous variables, A^i denotes the i th column of A , for a function $g : \mathbb{R}^m \mapsto \mathbb{R}$ we write $\bar{g}(d) = \limsup_{\delta \rightarrow 0^+} \frac{g(\delta d)}{\delta}$ and $\mathcal{F} = \{g : \mathbb{R}^n \mapsto \mathbb{R} \mid g \text{ is subadditive and nondecreasing w.r.t. } \mathbf{C}\}$.

Notice that when $\mathbf{C} = \mathbb{R}_+^m$, we retrieve the subadditive dual for a mixed-integer linear programming problem. In this chapter, we generalize the subadditive dual for mixed-integer linear programming as presented in Section II.3.3 of [73]. In [46] a different form of primal is used, and as a consequence, the form of the dual presented in that paper is slightly different from the dual shown in [73]. In the mixed-integer linear case both approaches are equivalent.

In the subadditive duality theory for mixed-integer linear programming ($\mathbf{C} = \mathbb{R}_+^m$), a sufficient condition to have strong duality is the rationality of the data defining the problem, that is, $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$. The main result of this chapter is to show that strong duality for mixed-integer conic programming holds under the following technical condition

$$\text{there exists } \hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \text{ such that } A\hat{x} \succ_{\mathbf{C}} b. \quad (*),$$

where $n_2 = n - n_1$. We state the main result formally next.

Theorem 21 (Strong duality). *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_{\mathbf{C}} b$, then*

1. *(\mathcal{P}) is finite if and only if (\mathcal{D}) is finite.*
2. *If (\mathcal{P}) is finite, then there exists a function g^* feasible for (\mathcal{D}) such that $g^*(b) = z^*$ and consequently $z^* = \rho^*$.*

Condition (*) in the case of mixed-integer conic programs plays the same role as the assumption of rational data in the case of mixed-integer programs in the proof of the strong duality result. In particular, we will see in Section 5.4.1 that both are sufficient conditions for the finiteness of the corresponding convex mixed-integer problem being equivalent to the finiteness of its continuous relaxation.

5.3 Weak duality

As in the case of mixed-integer linear programming, weak duality is a straightforward consequence of the definition of the subadditive dual. We first require a well-known result relating g and \bar{g} when g is a subadditive function.

Theorem 22 ([53], [57], and [73]). *If $g : \mathbb{R}^m \mapsto \mathbb{R}$ is a subadditive function such that $g(0) = 0$, then $\forall d \in \mathbb{R}^m$ with $\bar{g}(d) < \infty$ and $\forall \lambda \geq 0$ we have that $g(\lambda d) \leq \lambda \bar{g}(d)$.*

Proposition 23 (Weak duality). *For all $x \in \mathbb{R}^n$ feasible for (\mathcal{P}) and for all $g : \mathbb{R}^m \mapsto \mathbb{R}$ feasible for (\mathcal{D}) , we have that $g(b) \leq c^T x$.*

Proof. Let $u, v \geq 0$ such that $x = u - v$. We have

$$\begin{aligned}
g(b) &\leq g(Ax) \\
&= g(Au - Av) \\
&= g\left(\sum_{i=1}^n A^i u_i + \sum_{i=1}^n (-A^i) v_i\right) \\
&= g\left(\sum_{i \in \mathcal{I}} A^i u_i + \sum_{i \in \mathcal{I}} (-A^i) v_i + \sum_{i \in \mathcal{C}} A^i u_i + \sum_{i \in \mathcal{C}} (-A^i) v_i\right) \\
&\leq \sum_{i \in \mathcal{I}} g(A^i u_i) + \sum_{i \in \mathcal{I}} g(-A^i v_i) + \sum_{i \in \mathcal{C}} g(A^i u_i) + \sum_{i \in \mathcal{C}} g(-A^i v_i) \\
&\leq \sum_{i \in \mathcal{I}} g(A^i) u_i + \sum_{i \in \mathcal{I}} g(-A^i) v_i + \sum_{i \in \mathcal{C}} \bar{g}(A^i) u_i + \sum_{i \in \mathcal{C}} \bar{g}(-A^i) v_i \\
&= \sum_{i=1}^n c_i u_i + \sum_{i=1}^n (-c_i) v_i \\
&= c^T x.
\end{aligned}$$

The first inequality relies on the fact that x satisfies $Ax \succeq_{\mathcal{C}} b$ and g is nondecreasing w.r.t. \mathcal{C} and the second inequality relies on the fact that g is subadditive. The third inequality is based on the subadditivity of g , the fact that $g(0) = 0$ and Theorem 22. ■

We obtain the following corollary of Proposition 23.

Corollary 9.

1. *If the primal problem (\mathcal{P}) is unbounded, then the dual problem (\mathcal{D}) is infeasible.*
2. *If the dual problem (\mathcal{D}) is unbounded, then the primal problem (\mathcal{P}) is infeasible.*

5.4 Preliminary results for proving strong duality

5.4.1 Finiteness of a convex mixed-integer problem being equivalent to the finiteness of its continuous relaxation

In this section, we study a sufficient condition for the finiteness of the primal (\mathcal{P}) being equivalent to the finiteness of its continuous relaxation. This condition is required to show that the primal program is finite if and only if the dual program is finite.

The main result of this section is a sufficient condition for property

$$\inf\{c^T x \mid x \in B\} = -\infty \Leftrightarrow \inf\{c^T x \mid x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\} = -\infty \quad (51)$$

to hold in the context of general convex mixed-integer optimization, that is, when the feasible region of the primal is of the form $B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$, where $B \subseteq \mathbb{R}^n$ is a closed convex set and $n = n_1 + n_2$.

The next example shows that property (51) is not always satisfied, not even when the feasible set is a polyhedron.

Example 8. *Consider the polyhedral set $B_1 = \{x \in \mathbb{R}^2 \mid x_2 - \sqrt{2}x_1 = 0\}$ and let the objective function be given by $c = (1, \sqrt{2})$. In this case $B_1 \cap \mathbb{Z}^2 = \{0, 0\}$, so $\inf\{c^T x \mid x \in B_1 \cap \mathbb{Z}^n\} = 0$. On the other hand, $\inf\{c^T x \mid x \in B_1\} = -\infty$. Therefore, the integer programming problem has finite optimal objective function value, but its relaxation has unbounded objective function value.*

When the feasible set is a polyhedron, a well-known sufficient condition for property (51) to be true is that the polyhedron is defined by rational data. However, the following example shows that (51) is not necessarily true when the convex set B is full-dimensional, $\text{conv}(B \cap \mathbb{Z}^n)$ is a polyhedron and B is conic quadratic representable using rational data.

Example 9. Consider the set

$$\begin{aligned} B_2 = & \text{conv}(\{x \in \mathbb{R}^3 \mid x_3 = 0, x_1 = 0, x_2 \geq 0\} \\ & \cup \{x \in \mathbb{R}^3 \mid x_3 = 0.5, x_2 \geq x_1^2\} \\ & \cup \{x \in \mathbb{R}^3 \mid x_3 = 1, x_1 = 0, x_2 \geq 0\}). \end{aligned}$$

Notice B_2 is full-dimensional, closed (the sets defining B_2 have the same recession cone) and is defined by rational data. Observe that $\text{conv}(B_2 \cap \mathbb{Z}^3) = \{x \in \mathbb{R}^3 \mid x_1 = 0, x_2 \geq 0, 0 \leq x_3 \leq 1\}$ is a polyhedron. Since we have $\inf\{x_1 \mid x \in B_2\} = -\infty$ and $\inf\{x_1 \mid x \in B_2 \cap \mathbb{Z}^3\} = 0 > -\infty$, the set B_2 does not satisfy property (51).

Finally, it can be shown that the set B_2 is conic quadratic representable using rational data, that is, there exists a rational matrix A and a rational vector b such that

$$B_2 = \left\{ x \in \mathbb{R}^3 \mid \exists u \ A \begin{pmatrix} x \\ u \end{pmatrix} \succeq_{\mathbf{C}} b \right\},$$

where \mathbf{C} is a direct product of Lorentz cones (see [17]).

Before we state the sufficient condition for property (51) to hold, we give some definitions and preliminary results that will be needed to prove the validity of this condition.

A linear subspace $L \subseteq \mathbb{R}^n$ is said to be a rational linear subspace if there exists a basis of L formed by rational vectors. A convex set $B \subseteq \mathbb{R}^n$ is called lattice-free, if $\text{int}(B) \cap \mathbb{Z}^n = \emptyset$. A lattice-free convex set $B \subseteq \mathbb{R}^n$ is called maximal lattice-free convex set if does not exist a lattice-free convex set $B' \subseteq \mathbb{R}^n$ satisfying $B \subsetneq B'$.

For ease of exposition, we recall a corollary to Theorem 8 we have already used in Section 3.7.1 of Chapter 3.

Corollary 7. *Let $B \subseteq \mathbb{R}^n$ be a full-dimensional convex set. Let $n_1 + n_2 = n$. If $\text{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$, then there exists a polytope $P \subseteq \mathbb{R}^n$ and a rational subspace $L \subseteq \mathbb{R}^n$ such that $Q = P + L$ satisfies $\text{int}(Q) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$ and $B \subseteq Q$.*

The sufficient condition for (51) to hold is stated in the following result. The proof of this result is modified from a result in [35].

Proposition 24. *Let $n_1 + n_2 = n$ and let $B \subseteq \mathbb{R}^n$ be a convex set such that $\text{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$. Then*

$$\inf\{c^T x \mid x \in B\} = -\infty \Leftrightarrow \inf\{c^T x \mid x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\} = -\infty.$$

Proof.

(\Leftarrow) Clearly, if $\inf\{c^T x \mid x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\} = -\infty$, then we must have that $\inf\{c^T x \mid x \in B\} = -\infty$.

(\Rightarrow) Suppose $\inf\{c^T x \mid x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\} = d > -\infty$. We will show that $\inf\{c^T x : x \in B\} > -\infty$. Assume for a contradiction that $\inf\{c^T x : x \in B\} = -\infty$. Consider the set $B^\leq = B \cap \{x \in \mathbb{R}^n : c^T x \leq d\}$. Notice that since $\text{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$, we obtain that B is a full-dimensional set. Also, by assumption, we have that $B \not\subseteq \{x \in \mathbb{R}^n : c^T x \geq d\}$. Therefore, $\text{int}(B) \cap \{x \in \mathbb{R}^n : c^T x < d\} \neq \emptyset$. This implies that $\text{int}(B^\leq) = \text{int}(B) \cap \{x \in \mathbb{R}^n : c^T x < d\} \neq \emptyset$ and thus B^\leq is a full-dimensional set.

Moreover, we have that $\text{int}(B^\leq) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$, since $\text{int}(B^\leq) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = (\text{int}(B^\leq) \cap B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \subseteq \text{int}(B^\leq) \cap (B \cap \{x \in \mathbb{R}^n : c^T x \geq d\}) \subseteq \{x \in \mathbb{R}^n : c^T x < d\} \cap \{x \in \mathbb{R}^n : c^T x \geq d\} = \emptyset$. Hence, by Corollary 7 there exists a full-dimensional polyhedron $Q = \{x \in \mathbb{R}^n \mid a_k^T x \leq b_k, k \in \{1, \dots, q\}\}$ such that $Q = P + L$, where P is a polytope and L a rational linear subspace, $\text{int}(Q) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$ and $B^\leq \subseteq Q$.

Since $\text{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$, we obtain that $B \not\subseteq Q$. Therefore, there exists

$x_0 \in B \setminus Q$, that is, $x_0 \in B$ and $a_j^T x_0 > b_j$, for some $j \in \{1, \dots, q\}$. Notice that, since $B^\leq \subseteq Q$, we have that $x_0 \notin B^\leq$. Thus, we obtain that $c^T x_0 > d$.

On the other hand, since $Q \subseteq \{x \in \mathbb{R}^n \mid a_j^T x \leq b_j\}$, we have that $\sup\{a_j^T x \mid x \in Q\} < \infty$. Therefore, since $\text{rec}(Q) = L$, we must have $a_j^T r = 0$, for all $r \in \text{rec}(Q)$. Hence, $\inf\{a_j^T x \mid x \in Q\} > -\infty$, implying that there exists $M > 0$ such that $Q \subseteq \{x \in \mathbb{R}^n \mid a_j^T x \geq b_j - M\}$.

Let $\{x_n\}_{n=1}^\infty \subseteq B^\leq$ such that $\lim_{n \rightarrow \infty} c^T x_n = -\infty$ and let $\lambda_n \in (0, 1]$ such that $y_n = (1 - \lambda_n)x_0 + \lambda_n x_n$ satisfies $c^T y_n = d$.

Notice that

$$\begin{aligned} a_j^T y_n - b_j &= (1 - \lambda_n)a_j^T x_0 + \lambda_n a_j^T x_n - b_j \\ &\geq (1 - \lambda_n)(a_j^T x_0 - b_j) - \lambda_n M \\ &= (a_j^T x_0 - b_j) - \lambda_n[(a_j^T x_0 - b_j) + M], \end{aligned} \tag{52}$$

where the inequality follows from the fact that $\{x_n\}_{n=1}^\infty \subseteq B^\leq \subseteq Q \subseteq \{x \in \mathbb{R}^n \mid a_j^T x \geq b_j - M\}$.

On the other hand, by definition of λ_n , we have that $\lambda_n = \frac{d - c^T x_0}{c^T x_n - c^T x_0}$ and thus $\lim_{n \rightarrow \infty} \lambda_n = 0$. Hence, by (52), for sufficiently large N , we obtain that $a_j^T y_N > b_j$. Also, since B is a convex set and y_N is a convex combination of $x_0, x_N \in B$ we obtain that $y_N \in B$. Thus, $y_N \in B^\leq \subseteq Q$, a contradiction with $a_j^T y_N > b_j$. Therefore, we must have $\inf\{c^T x : x \in B\} > -\infty$. \blacksquare

The condition that there exists a mixed-integer feasible solution in the interior of the continuous relaxation is crucial for Proposition 24. This is illustrated in Example 8 and Example 9, where B_1 and B_2 , respectively, do not satisfy property (51). Finally, observe that the converse of Proposition 24 is not true; consider any lattice-free rational unbounded polyhedron.

5.4.2 Strong duality for conic programming

In mixed-integer linear programming, the proof of strong duality for the corresponding subadditive dual relies on the existence of a strong duality result for linear programming. Unfortunately, unlike the case of linear programming, strong duality for conic programming requires some additional assumptions. Therefore, it is not surprising that we require the extra condition (*) to prove strong duality for mixed-integer conic programming. We recall the duality theorem for conic programming ([17]).

Theorem 23 (Duality for conic programming). *Let $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Let $\mathbf{C} \subseteq \mathbb{R}^m$ be a full-dimensional, closed and pointed cone. Denote $\mathbf{C}_* = \{y \in \mathbb{R}^m \mid y^T x \geq 0, \forall x \in \mathbf{C}\}$, the dual cone of \mathbf{C} . Then:*

1. (Weak duality) *For all $x \in \{x \in \mathbb{R}^n \mid Ax \geq_{\mathbf{C}} b\}$ and $y \in \{y \in \mathbb{R}^m \mid A^T y = c, y \geq_{\mathbf{C}_*} 0\}$ we have that $b^T y \leq c^T x$.*
2. (Strong duality) *If there exists $\hat{x} \in \mathbb{R}^n$ such that $A\hat{x} >_{\mathbf{C}} b$ and $\inf\{c^T x : Ax \geq_{\mathbf{C}} b\} > -\infty$, then*

$$\inf\{c^T x : Ax \geq_{\mathbf{C}} b\} = \max\{b^T y : A^T y = c, y \geq_{\mathbf{C}_*} 0\}.$$

5.4.3 Value function of (\mathcal{P})

In this section we study some properties of the value function of (\mathcal{P}) . The motivation is the following: we will verify that the value function satisfies all the properties of the feasible functions of the dual (\mathcal{D}) , except that it might not be defined over all of \mathbb{R}^m .

We begin with some notation. For $u \in \mathbb{R}^m$, let $S(u) = \{x \in \mathbb{R}^n \mid Ax \geq_{\mathbf{C}} u, x_i \in \mathbb{Z}, \forall i \in \mathcal{I}\}$ be the feasible region with right-hand-side u and let $P(u) = \{x \in \mathbb{R}^n \mid Ax \geq_{\mathbf{C}} u\}$ be its continuous relaxation. Let $\Omega = \{u \in \mathbb{R}^m \mid S(u) \neq \emptyset\}$. Notice that since $0 \in S(0)$, we have that $\Omega \neq \emptyset$.

Definition 14 (Value function of (\mathcal{P})). *The value function of a mixed-integer conic program is the function $f : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$, defined as*

$$f(u) = \inf\{c^T x \mid x \in S(u)\}.$$

We show next some basic properties of the value function.

Proposition 25. *Let $f : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$ be the value function of (\mathcal{P}) , then*

1. f is subadditive on Ω .
2. f is nondecreasing w.r.t. \mathbf{C} on Ω .
3. If $f(0) = 0$, then $f(A^i) = -f(-A^i) = c_i, \forall i \in \mathcal{I}$.
4. If $f(0) = 0$, then $\bar{f}(A^i) = -\bar{f}(-A^i) = c_i, \forall i \in \mathcal{C}$.
5. Let $u \in \Omega$. If $f(u) > -\infty$, then $f(0) = 0$.

Proof.

1. Let $u_1, u_2 \in \Omega$, $x_1 \in S(u_1)$ and $x_2 \in S(u_2)$. By additivity of $\succeq_{\mathbf{C}}$, we have that $x_1 + x_2 \in S(u_1 + u_2)$. This implies that $c^T x_1 + c^T x_2 \geq f(u_1 + u_2)$. By taking the infimum over $x_i \in S(u_i)$, $i = 1, 2$ we conclude that $f(u_1) + f(u_2) \geq f(u_1 + u_2)$.
2. Let $u_1, u_2 \in \Omega$, $u_1 \succeq_{\mathbf{C}} u_2$. Let $x \in S(u_1)$. By transitivity of $\succeq_{\mathbf{C}}$, we have that $x \in S(u_2)$. Therefore, $S(u_1) \subseteq S(u_2)$, and thus we obtain that $f(u_1) \geq f(u_2)$.
3. For $i \in \mathcal{I}$ and $\alpha \in \{-1, 1\}$, since the vector $x_i = \alpha$, $x_j = 0, \forall j \neq i$ is feasible for (\mathcal{P}) with right-hand-side $b = \alpha A^i$, we have that $f(\alpha A^i) \leq \alpha c_i$. Since f is subadditive, we obtain $0 = f(0) \leq f(\alpha A^i) + f(-\alpha A^i)$. Therefore, $\alpha c_i \leq -f(-\alpha A^i) \leq f(\alpha A^i) \leq \alpha c_i$, and thus we obtain that $f(\alpha A^i) = \alpha c_i$. Equivalently, for $i \in \mathcal{I}$, we have that $f(A^i) = c_i$ and $f(-A^i) = -c_i$.

4. For $i \in \mathcal{C}$ and $\delta \in \mathbb{R}$, since the vector $x_i = \delta$, $x_j = 0, \forall j \neq i$ is feasible for (\mathcal{P}) with right-hand-side $b = \delta A^i$, we have that $f(\delta A^i) \leq \delta c_i$. Since f is subadditive, we obtain $0 = f(0) \leq f(\delta A^i) + f(-\delta A^i)$. Therefore, $\delta c_i \leq -f(-\delta A^i) \leq f(\delta A^i) \leq \delta c_i$, so, $f(\delta A^i) = \delta c_i$. This implies that $\bar{f}(A^i) = \limsup_{\delta \rightarrow 0^+} \frac{f(\delta A^i)}{\delta} = c_i$ and $\bar{f}(-A^i) = \limsup_{\delta \rightarrow 0^+} \frac{f(\delta(-A^i))}{\delta} = -c_i$. Therefore, for $i \in \mathcal{C}$, $\bar{f}(A^i) = c_i$ and $\bar{f}(-A^i) = -c_i$.
5. We verify the contrapositive of this statement. Assume $f(0) < 0$. Then there exists $\bar{x} \in S(0)$ such that $c^T \bar{x} < 0$. For all $\lambda \in \mathbb{Z}_+$, we have that $\lambda \bar{x} \in S(0)$ and $c^T(\lambda \bar{x}) = \lambda c^T \bar{x} < 0$. Let $x \in S(u)$. By additivity of $\succeq_{\mathcal{C}}$, we have that $x + \lambda \bar{x} \in S(u)$ for all $\lambda \in \mathbb{Z}_+$. Since $c^T(x + \lambda \bar{x}) = c^T x + \lambda c^T \bar{x}$, we obtain that $f(u) = -\infty$. ■

Since the value function f might not be defined over \mathbb{R}^m , it is not necessarily a feasible solution to the dual. In the next section, we will construct a new function that is equal to f on b , is finite-valued over \mathbb{R}^m , and continues to satisfy all the conditions of the dual (\mathcal{D}) . The following corollary of Proposition 25 and the subsequent two propositions are crucial in this construction.

Corollary 10. *Let $\mathbf{C}_1 \subseteq \mathbb{R}^p$, $\mathbf{C}_2 \subseteq \mathbb{R}^q$ be full-dimensional closed and pointed convex cones. Let $S(u, v) = \{(x, y) \in \mathbb{R}^{(p+q)} \mid A_1 x + A_2 y \succeq_{\mathbf{C}_1} u, Fy \succeq_{\mathbf{C}_2} v, x_i \in \mathbb{Z}, \forall i \in \mathcal{I}_p, y_i \in \mathbb{Z}, \forall i \in \mathcal{I}_q\}$, where $\mathcal{I}_p \subseteq \{1, \dots, p\}$, $\mathcal{I}_q \subseteq \{1, \dots, q\}$, and let $\Omega_p = \{u \in \mathbb{R}^p \mid S(u, 0) \neq \emptyset\}$. Let*

$$G(u, v) = \inf\{c^T x + d^T y \mid (x, y) \in S(u, v)\}.$$

Consider $g : \Omega_p \mapsto \mathbb{R}$ defined as $g(u) = G(u, 0)$. Then

1. *g is subadditive on Ω_p .*
2. *g is nondecreasing w.r.t. \mathbf{C}_1 on Ω_p .*
3. *If $G(0, 0) = 0$, then $g(A^i) = -g(-A^i) = c_i, \forall i \in \mathcal{I}_p$ and $\bar{g}(A^i) = -\bar{g}(-A^i) = c_i, \forall i \in \{1, \dots, p\} \setminus \mathcal{I}_p$.*

Proof.

1. Observe that, by (1.) of Proposition 25, G is subadditive on its domain. Let $u_1, u_2 \in \Omega_p$. Then

$$g(u_1 + u_2) = G[(u_1, 0) + (u_2, 0)] \leq G(u_1, 0) + G(u_2, 0) = g(u_1) + g(u_2),$$

where the inequality is a consequence of G being subadditive.

2. Observe that, by (2.) of Proposition 25, G is nondecreasing w.r.t. $\mathbf{C}_1 \times \mathbf{C}_2$ on its domain. Also, $u_1 \succeq_{\mathbf{C}_1} u_2$ if and only if $(u_1, 0) \succeq_{\mathbf{C}_1 \times \mathbf{C}_2} (u_2, 0)$. Therefore, if $u_1 \succeq_{\mathbf{C}_1} u_2$, then $g(u_1) = G(u_1, 0) \geq G(u_2, 0) = g(u_2)$, as desired.
3. If $G(0, 0) = 0$, then, by (3.) and (4.) of Proposition 25, for $\alpha \in \{-1, 1\}$ we obtain that $g(\alpha A^i) = G(\alpha A^i, 0) = \alpha c_i$, $\forall i \in \mathcal{I}_p$ and $\bar{g}(\alpha A^i) = \bar{G}(\alpha A^i, 0) = \alpha c_i$, $\forall i \in \{1, \dots, p\} \setminus \mathcal{I}_p$.

■

The next proposition states a sufficient condition for $\Omega = \mathbb{R}^m$ to hold, that is, $S(b) \neq \emptyset$ for all $b \in \mathbb{R}^m$.

Proposition 26. *If there exists $\hat{x} \in \mathbb{R}^n$ such that $A\hat{x} \succ_{\mathbf{C}} 0$, then $\forall b \in \mathbb{R}^m$, there exists $x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $Ax \succ_{\mathbf{C}} b$.*

Proof. Let $b \in \mathbb{R}^m$. It is sufficient to prove that there exists $x \in \mathbb{Z}^n$ such that $Ax \succ_{\mathbf{C}} b$. We will show this next. Since $A\hat{x} \succ_{\mathbf{C}} 0$, there exists $\varepsilon > 0$ such that $B(A\hat{x}, \varepsilon) \subseteq \mathbf{C}$, where $B(A\hat{x}, \varepsilon)$ is the open ball of radius ε around $A\hat{x}$. Therefore, by continuity of Ax and density of \mathbb{Q}^n in \mathbb{R}^n , there exists $q \in \mathbb{Q}^n$ such that $Aq \in B(A\hat{x}, \varepsilon)$. This implies, by a suitable positive scaling of q , that there exists $z \in \mathbb{Z}^n$ such that $Az \in \text{int}(\mathbf{C})$. Hence, there exists $\delta > 0$ such that $B(Az, \delta) \subseteq \mathbf{C}$. For $M \in \mathbb{N}$ sufficiently large, we obtain that $Az - \frac{b}{M} \in B(Az, \delta) \subseteq \mathbf{C}$. Thus, scaling by $M > 0$, we obtain that $A(Mz) - b \in \text{int}(\mathbf{C})$, that is, $A(Mz) \succ_{\mathbf{C}} b$, as desired. ■

The following result gives a condition such that if the primal is finite for some right-hand-side b , then it is also finite for every right-hand-side $u \in \Omega$.

Proposition 27. *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_{\mathbf{C}} b$ and $f(b) > -\infty$, then $\forall u \in \Omega$ we have that $\inf\{c^T x \mid x \in P(u)\} > -\infty$. In particular, $\forall u \in \Omega$, we have that $f(u) > -\infty$.*

Proof. Since $\{x \in \mathbb{R}^n \mid Ax \succ_{\mathbf{C}} b\} = \text{int}(P(b))$, we have that $\text{int}(P(b)) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$. Therefore, since $f(b) > -\infty$, by Proposition 24, we obtain that $\inf\{c^T x \mid x \in P(b)\} > -\infty$. This implies, by (2.) of Theorem 23, that the set $\{y \mid A^T y = c, y \succeq_{\mathbf{C}_*} 0\}$ is nonempty.

Let $u \in \Omega$ and let $\bar{y} \in \{y \mid A^T y = c, y \succeq_{\mathbf{C}_*} 0\}$. By (1.) of Theorem 23 we obtain that $\inf\{c^T x \mid x \in P(u)\} \geq u^T \bar{y}$, as required. \blacksquare

5.5 Strong duality

For ease of exposition, we recall next the main result of this chapter.

Theorem 21 (Strong duality). *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_{\mathbf{C}} b$, then*

1. *(\mathcal{P}) is finite if and only if (\mathcal{D}) is finite.*
2. *If (\mathcal{P}) is finite, then there exists a function g^* feasible for (\mathcal{D}) such that $g^*(b) = z^*$ and, consequently, $z^* = \rho^*$.*

We will prove Theorem 21 in the following two subsections.

5.5.1 Finiteness of the primal being equivalent to the finiteness of the dual

In this section we present a sufficient condition for the following to hold: (\mathcal{P}) is finite if and only if (\mathcal{D}) is finite. Observe that essentially we need conditions under which the converse of Corollary 9 holds.

We begin by showing that ‘a part’ of the converse of Corollary 9 holds generally. The proof is modified from a result in [46].

Proposition 28. *If the primal problem is infeasible, then the dual is unbounded or infeasible.*

Proof. If the dual problem is feasible, then we need to verify that it is unbounded. Define $G : \mathbb{R}^m \mapsto \mathbb{R}$ as $G(d) = \min\{x_0 \mid Ax + x_0 d \succeq_{\mathbf{C}} d, x_i \in \mathbb{Z}, i \in \mathcal{I}, x_0 \in \mathbb{Z}_+\}$. Notice $G(d) \in \{0, 1\}$ for all $d \in \mathbb{R}^m$, because $x = 0$ and $x_0 = 1$ is always a feasible solution. We have that $G(d) = 0$ if and only if $\{x \mid Ax \succeq_{\mathbf{C}} d, x_i \in \mathbb{Z}, i \in \mathcal{I}\} \neq \emptyset$. Hence, for $d_1, d_2 \in \mathbb{R}^m$, $G(d_1) = G(d_2) = 0$ implies $G(d_1 + d_2) = 0$. Therefore, we obtain that G is subadditive. Also, for $d_1, d_2 \in \mathbb{R}^m$ such that $d_1 \succeq_{\mathbf{C}} d_2$, we have that $G(d_1) = 0$ implies $G(d_2) = 0$. Hence, we obtain that G is nondecreasing w.r.t. \mathbf{C} . For $i \in \mathcal{I}$, $\alpha \in \{-1, 1\}$, we obtain that $G(\alpha A^i) = 0$, because $x_i = \alpha$, $x_j = 0, \forall j \neq i$ is a feasible solution when $d = \alpha A^i$. Similarly, for $i \in \mathcal{C}$, we have that $\overline{G}(A^i) = \overline{G}(-A^i) = 0$. Moreover, $G(0) = 0$ and since the primal is infeasible, we obtain that $G(b) = 1$.

Let g be a feasible solution for the dual. Then $g + \lambda G$ is also a feasible solution for the dual for all $\lambda \geq 0$. Since $[g + \lambda G](b) = g(b) + \lambda$ for all $\lambda \geq 0$, we conclude that the dual is unbounded. ■

Proposition 29. *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_{\mathbf{C}} b$, then (\mathcal{P}) is finite if and only if (\mathcal{D}) is finite.*

Proof.

(\Leftarrow) Assume that the dual is finite. Then, by Proposition 28, we obtain that the primal is feasible. Thus, Corollary 9 implies that the primal is finite.

(\Rightarrow) Assume that the primal is finite. Corollary 9 implies that if the dual is feasible then the dual cannot be unbounded. We next verify that dual is indeed feasible.

First observe that since $\text{int}(P(b)) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$ and $f(b) > -\infty$ hold, by the application of Proposition 24, we obtain that $\inf\{c^T x \mid Ax \succeq_{\mathbf{C}} b\} > -\infty$.

Since $\inf\{c^T x \mid Ax \succeq_{\mathbf{C}} b\}$ is finite and there exists \hat{x} such that $A\hat{x} \succ_{\mathbf{C}} b$, by the application of (2.) of Theorem 23, we obtain that the set $\{y \in \mathbb{R}^m \mid A^T y = c, y \succeq_{\mathbf{C}^*} 0\}$

is nonempty. Let $\hat{y} \in \{y \in \mathbb{R}^m \mid A^T y = c, y \succeq_{C^*} 0\}$. Then the function $g(u) = \hat{y}^T u$ is a feasible solution of (\mathcal{D}) , so the dual problem is feasible. \blacksquare

Notice that Proposition 29 gives a proof for (1.) of Theorem 21. In the next section, we will refine the second half of the proof of Proposition 29, to show that when there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_C b$ and the primal is finite, not only is the dual finite, but also its optimal objective function value is equal to that of the primal.

5.5.2 Feasible optimal solution for (\mathcal{D})

In this section, we construct a feasible optimal solution for the dual problem (\mathcal{D}) . The next proposition shows how this can be done.

Proposition 30. *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_C b$ and (\mathcal{P}) is finite, then there exists a function $g^* : \mathbb{R}^m \mapsto \mathbb{R}$, feasible for (\mathcal{D}) such that $g^*(b) = z^*$ and consequently $z^* = \rho^*$.*

Proof. Let f be the value function of (\mathcal{P}) . By Proposition 27, we obtain that $f(u) > -\infty$ for all $u \in \Omega$. Therefore, $f : \Omega \mapsto \mathbb{R}$ is a well defined function.

If $\Omega = \mathbb{R}^n$, then by Proposition 25, $f : \mathbb{R}^n \mapsto \mathbb{R}$ is feasible for the dual (\mathcal{D}) . Moreover, by definition of f , $f(b) = z^*$. Thus, by considering $g^* = f$, we obtain that $g^*(b) = z^*$, as desired.

If $\Omega \subsetneq \mathbb{R}^n$, then we will use f to construct $g^* : \mathbb{R}^m \mapsto \mathbb{R}$ feasible for the dual (\mathcal{D}) such that $g^*(b) = f(b)$.

For $\lambda \geq 0$ denote $f_R(\lambda b) = \inf\{c^T x \mid x \in P(\lambda b)\}$. By Proposition 24, we have that $f_R(b) > -\infty$. Therefore, since $f_R(\lambda b) = \lambda f_R(b)$ we obtain that $f_R(\lambda b) = \lambda f_R(b) > -\infty$, for all $\lambda \geq 0$. Notice also that for all $y \in \mathbb{Z}$ we have that $yb \in \Omega$, implying that $f(yb) > -\infty$ for all $y \in \mathbb{Z}$.

Denote $X(u) = \{(x, y) \in \mathbb{R}^{n+1} \mid Ax - yb \succeq_C u, y \geq 0, x_i \in \mathbb{Z}, \forall i \in \mathcal{I}, y \in \mathbb{Z}\}$. Now we will show how to construct g^* . We have to consider three cases. For each of

these three cases we give a different construction of g^* and show that $g^*(0) = 0$ and $g^*(b) = f(b)$.

Case 1: $f(b) \geq 0$ and $f_R(b) \geq 0$. Define

$$g^*(u) = \inf\{c^T x + [f(b) - f_R(b)]y \mid (x, y) \in X(u)\}.$$

First we prove that $g^*(0) = 0$. Let $(x, y) \in X(0)$. Then we have

$$\begin{aligned} c^T x + [f(b) - f_R(b)]y &\geq f(yb) + f(b)y - f_R(b)y \\ &= [f(yb) - f_R(yb)] + f(b)y \\ &\geq 0. \end{aligned}$$

By considering the feasible solution $(0, 0)$, with objective value 0, we conclude that $g^*(0) = 0$.

Now we prove that $g^*(b) = f(b)$. Let $(x, y) \in X(b)$ with $y \geq 1$. Then we have

$$\begin{aligned} c^T x + [f(b) - f_R(b)]y &\geq f((y+1)b) + f(b)y - f_R(b)y \\ &= [f((y+1)b) - f_R((y+1)b)] + f_R(b) + f(b)y \\ &\geq f(b)y \\ &\geq f(b). \end{aligned}$$

On the other hand, notice that $(x, 0) \in X(b)$ if and only if $x \in S(b)$. For $(x, 0) \in X(b)$ we have $c^T x + [f(b) - f_R(b)]0 = c^T x$. Therefore, by taking the infimum over $(x, 0) \in X(b)$, we conclude that $g^*(b) = f(b)$.

Case 2: $f(b) \leq 0$ and $f_R(b) \leq 0$. In this case, define

$$g^*(u) = \inf\{c^T x - 2f_R(b)y \mid (x, y) \in X(u)\}.$$

First we prove that $g^*(0) = 0$. Let $(x, y) \in X(0)$. Then we have

$$\begin{aligned} c^T x - 2f_R(b)y &\geq f(yb) - f_R(b)y - f_R(b)y \\ &= [f(yb) - f_R(yb)] - f_R(b)y \\ &\geq 0. \end{aligned}$$

By considering the feasible solution $(0, 0)$, with objective value 0, we conclude that $g^*(0) = 0$.

Now we prove that $g^*(b) = f(b)$. Let $(x, y) \in X(b)$ with $y \geq 1$. Then we have

$$\begin{aligned} c^T x - 2f_R(b)y &\geq f((y+1)b) - f_R(b)y - f_R(b)y \\ &= [f((y+1)b) - f_R((y+1)b)] - f_R(b)(y-1) \\ &\geq 0 \\ &\geq f(b). \end{aligned}$$

For $(x, 0) \in X(b)$ we have $c^T x - 2f_R(b)0 = c^T x$. Therefore, by taking the infimum over $(x, 0) \in X(b)$, we conclude that $g^*(b) = f(b)$.

Case 3: $f(b) \geq 0$ and $f_R(b) \leq 0$. In this case, define

$$g^*(u) = \inf\{c^T x + [f(b) - 2f_R(b)]y \mid (x, y) \in X(u)\}.$$

First we prove that $g^*(0) = 0$. Let $(x, y) \in X(0)$. Then we have

$$\begin{aligned} c^T x + [f(b) - 2f_R(b)]y &\geq f(yb) - f_R(b)y + [f(b) - f_R(b)]y \\ &= [f(yb) - f_R(yb)] + [f(b) - f_R(b)]y \\ &\geq 0. \end{aligned}$$

By considering the feasible solution $(0, 0)$, with objective value 0, we conclude that $g^*(0) = 0$.

Now we prove that $g^*(b) = f(b)$. Let $(x, y) \in X(b)$ with $y \geq 1$. Then we have

$$\begin{aligned} c^T x + [f(b) - 2f_R(b)]y &\geq f((y+1)b) - f_R(b)y + [f(b) - f_R(b)]y \\ &= [f((y+1)b) - f_R((y+1)b)] + f_R(b) + [f(b) - f_R(b)]y \\ &\geq f_R(b)(1-y) + yf(b) \\ &\geq yf(b) \\ &\geq f(b). \end{aligned}$$

For $(x, 0) \in X(b)$ we have $c^T x + [f(b) - 2f_{\mathbb{R}}(b)]0 = c^T x$. Therefore, by taking the infimum over $(x, 0) \in X(b)$, we conclude that $g^*(b) = f(b)$.

We show next that in all the three cases described above, we have that g^* is feasible for the dual (\mathcal{D}) . Observe that since $A\hat{x} - b \succ_{\mathbf{C}} 0$; $1 > 0$, by the application of Proposition 26, we obtain that $X(u) \neq \emptyset$ for all $u \in \mathbb{R}^m$. Moreover, since $g^*(0) = 0$, we have that $g^*(u) > -\infty$ for all $u \in \mathbb{R}^m$ (Proposition 27). Thus, we have defined a function $g^* : \mathbb{R}^m \mapsto \mathbb{R}$. Finally, by Corollary 10, observe that g^* satisfies all the constraints of the dual (\mathcal{D}) . In conclusion, g^* is feasible for the dual (\mathcal{D}) and $g^*(b) = f(b) = z^*$, thus completing the proof. \blacksquare

Notice that Proposition 30 gives a proof for (2.) of Theorem 21.

5.6 Valid inequalities

A valid inequality for the feasible region of the primal (\mathcal{P}) (that is, $S(b)$) is a linear inequality $\pi^T x \geq \pi_0$ such that for all $x \in S(b)$, we have $\pi^T x \geq \pi_0$. In the case of mixed-integer linear programs defined by rational data ($\mathbf{C} = \mathbb{R}_+^m$, $A \in \mathbb{Q}^{m \times n}$, and $b \in \mathbb{Q}^m$) it can be shown that all interesting valid inequalities $\pi^T x \geq \pi_0$ for $S(b)$ are of the form

$$\sum_{i \in \mathcal{I}} g(A^i)x_i + \sum_{i \in \mathcal{C}} \bar{g}(A^i)x_i \geq g(b),$$

where g is feasible for the dual (\mathcal{D}) with $c = \pi$ and $g(b) \geq \pi_0$ (see [46], [53] and [56]).

Similarly, in the case of mixed-integer conic programming we can use subadditive functions that are nondecreasing with respect to \mathbf{C} to generate valid inequalities. In particular, Proposition 23 (Weak duality) and Theorem 21 (Strong duality) yield the following corollary.

Corollary 11. *1. Assume that the problems (\mathcal{P}) and (\mathcal{D}) are both feasible. If g is feasible for the dual (\mathcal{D}) , then the inequality $\sum_{i \in \mathcal{I}} g(A^i)x_i + \sum_{i \in \mathcal{C}} \bar{g}(A^i)x_i \geq g(b)$ is a valid inequality for $S(b)$.*

2. Assume there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_{\mathcal{C}} b$. Given a valid inequality $\pi^T x \geq \pi_0$ for $S(b)$, then there exists $g \in \mathcal{F}$ satisfying $g(0) = 0$, $g(A^i) = -g(-A^i) = \pi_i \forall i \in \mathcal{I}$, $\bar{g}(A^i) = -\bar{g}(-A^i) = \pi_i \forall i \in \mathcal{C}$, and $g(b) \geq \pi_0$. Then, $\sum_{i \in \mathcal{I}} g(A^i)x_i + \sum_{i \in \mathcal{C}} \bar{g}(A^i)x_i \geq g(b)$ is a valid inequality for $S(b)$ that is equivalent to or dominates $\pi^T x \geq \pi_0$.

In [78], it is shown that in the case of pure integer linear programs, given a rational left-hand-side matrix A , there exists a finite set of subadditive functions that yields the subadditive dual for any choice of the right-hand-side b . Such a result is unlikely in the integer conic setting since, in general, the convex hull of feasible points is not necessarily a polyhedron. The following example illustrates this behavior.

Example 10. Let $S \subseteq \mathbb{R}^2$ be the epigraph of the parabola $x_2 = x_1^2$, that is, $S = \{x \in \mathbb{R}^2 \mid x_1^2 \leq x_2\}$. It is well-known that S is a conic quadratic representable set ([17]).

Indeed, we have

$$S = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \succeq_{\mathbf{L}^3} \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right\},$$

where $\mathbf{L}^3 = \{x \in \mathbb{R}^3 \mid \sqrt{x_1^2 + x_2^2} \leq x_3\}$ is the Lorentz cone in \mathbb{R}^3 .

On the other hand, we have that $\text{conv}(S \cap \mathbb{Z}^2)$ is a nonpolyhedral closed convex set; also see [25] and [35]. In fact, we have

$$\text{conv}(S \cap \mathbb{Z}^2) = \bigcap_{n \in \mathbb{Z}} \{x \in \mathbb{R}^2 \mid x_2 - (2n+1)x_1 \geq -(n^2 + n)\},$$

where all these inequalities define facets of $\text{conv}(S \cap \mathbb{Z}^2)$.

By (2.) of Corollary 11, we have that for all $n \in \mathbb{Z}$, there exists a subadditive function

$g_n : \mathbb{R}^3 \mapsto \mathbb{R}$, such that g_n is nondecreasing w.r.t. \mathbf{L}^3 , $g_n(0) = 0$, for $\alpha \in \{-1, 1\}$

$$g_n \left[\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] = -\alpha(2n+1), \quad g_n \left[\alpha \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right] = \alpha, \quad \text{and } g_n \left[\begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right] \geq -(n^2+n).$$

Moreover, for this particular case, we explicitly present these functions. For all $n \in \mathbb{Z}$, the function g_n can be defined as $g_n(y) = [\mu_n^T y]$, where $\mu_n \in \mathbf{L}^3$ is given by $\mu_n = (-(2n+1), 1 - (n^2+n), 1 + (n^2+n))^T$.

Therefore, we conclude that we can write the convex hull of the integer points in S in terms of an infinite number of linear inequalities given by subadditive functions that are nondecreasing w.r.t. \mathbf{L}^3 , that is

$$\text{conv}(S \cap \mathbb{Z}^2) = \bigcap_{n \in \mathbb{Z}} \left\{ x \in \mathbb{R}^2 \mid g_n \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] x_1 + g_n \left[\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right] x_2 \geq g_n \left[\begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right] \right\}.$$

Finally, notice that since $\text{conv}(S \cap \mathbb{Z}^2)$ is a nonpolyhedral set, it cannot be described in terms of a finite number of linear inequalities.

5.7 Primal problems with particular structure

It is sometimes convenient to write the dual for specially structured problems like the ones with some nonnegative variables, with some equality constraints, etc. Finding the appropriate version of the dual and showing that it satisfies strong duality, using the results of this chapter, is a relatively simple exercise. We illustrate this for problems with some nonnegative variables. This problem is given by

$$(\mathcal{P}') \left\{ \begin{array}{l} z' = \inf \quad c^T x \\ \text{s.t.} \quad Ax \geq_c b \\ \quad \quad x_i \in \mathbb{R}_+, \forall i \in \mathcal{J} \\ \quad \quad x_i \in \mathbb{Z}, \forall i \in \mathcal{I}. \end{array} \right.$$

A subadditive dual for (\mathcal{P}') is given by

$$(\mathcal{D}') \left\{ \begin{array}{l} \rho' = \sup \quad g(b) \\ \text{s.t.} \quad g(A^i) \leq c_i, \quad \forall i \in \mathcal{I} \cap \mathcal{J} \\ \bar{g}(A^i) \leq c_i, \quad \forall i \in \mathcal{C} \cap \mathcal{J} \\ g(A^i) = -g(-A^i) = c_i, \quad \forall i \in \mathcal{I} \setminus \mathcal{J} \\ \bar{g}(A^i) = -\bar{g}(-A^i) = c_i, \quad \forall i \in \mathcal{C} \setminus \mathcal{J} \\ g(0) = 0 \\ g \in \mathcal{F}. \end{array} \right.$$

We formally state this result as a corollary of Theorem 21.

Corollary 12. 1. (Weak duality) For all $x \in \mathbb{R}^n$ feasible for (\mathcal{P}') and for all $g : \mathbb{R}^m \mapsto \mathbb{R}$ feasible for (\mathcal{D}') , we have $g(b) \leq c^T x$.

2. (Strong duality) If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_{\mathcal{C}} b$ and $\hat{x}_i > 0$ for all $i \in \mathcal{J}$, then

(a) (\mathcal{P}') is finite if and only if (\mathcal{D}') is finite.

(b) If (\mathcal{P}') is finite, then there exists a function g^* feasible for (\mathcal{D}') such that $g^*(b) = z'$ and consequently $z' = \rho'$.

Proof. The proof of weak duality is just a slight modification of the proof of weak duality for (\mathcal{P}) and (\mathcal{D}) (Proposition 23). We now verify the strong duality result.

We want to show that a strong dual of (\mathcal{P}') is given by (\mathcal{D}') . Let $q = |\mathcal{J}|$ and let $l : \mathcal{J} \mapsto \{1, \dots, q\}$ be a bijection. Notice that we can write (\mathcal{P}') in the form of (\mathcal{P}) as follows

$$(\mathcal{P}'') \left\{ \begin{array}{l} z^* = \inf \quad c^T x \\ \text{s.t.} \quad \begin{bmatrix} A \\ E \end{bmatrix} x \succeq_{\mathcal{C} \times \mathbb{R}_+^q} \begin{pmatrix} b \\ 0 \end{pmatrix} \\ x_i \in \mathbb{Z}, \forall i \in \mathcal{I}, \end{array} \right.$$

where $E \in \mathbb{R}^{q \times n}$ is the matrix whose columns are defined next: for $i \notin \mathcal{J}$ the column is the 0 vector and for $i \in \mathcal{J}$, the column is $e_{l(i)}$ (the $l(i)$ th vector of the canonical basis of \mathbb{R}^q).

By Theorem 2.4, the problem

$$\left. \begin{array}{l}
 \rho^* = \sup \quad G(b, 0) \\
 \text{s.t.} \quad G(A^i, e_{l(i)}) = -G(-A^i, -e_{l(i)}) = c_i, \quad \forall i \in \mathcal{I} \cap \mathcal{J} \\
 \quad \quad \bar{G}(A^i, e_{l(i)}) = -\bar{G}(-A^i, -e_{l(i)}) = c_i, \quad \forall i \in \mathcal{C} \cap \mathcal{J} \\
 \quad \quad G(A^i, 0) = -G(-A^i, 0) = c_i, \quad \forall i \in \mathcal{I} \setminus \mathcal{J} \\
 \quad \quad \bar{G}(A^i, 0) = -\bar{G}(-A^i, 0) = c_i, \quad \forall i \in \mathcal{C} \setminus \mathcal{J} \\
 \quad \quad G(0) = 0 \\
 \quad \quad G : \mathbb{R}^{m+q} \mapsto \mathbb{R} \text{ s.t. } G \text{ is subadditive} \\
 \quad \quad \text{and nondecreasing w.r.t } \mathbf{C} \times \mathbb{R}_+^q,
 \end{array} \right\} (\mathcal{D}'')$$

is a strong dual for (\mathcal{P}'') .

Let $G^* : \mathbb{R}^{m+q} \mapsto \mathbb{R}$ be the function given by Theorem 2.4, that is, G^* is feasible for (\mathcal{D}'') and $G^*(b, 0) = z^*$. For $x \in \mathbb{R}^m$, define $g^*(x) = G^*(x, 0)$. Notice that $g^*(0) = G^*(0, 0) = 0$. Also, since G^* is subadditive, we obtain that g^* is subadditive. Since $x \succeq_{\mathbf{C}} y$ implies $(x, 0) \succeq_{\mathbf{C} \times \mathbb{R}_+^q} (y, 0)$, we have that g is nondecreasing w.r.t \mathbf{C} . For $i \in \mathcal{I} \cap \mathcal{J}$, since G^* is nondecreasing w.r.t $\mathbf{C} \times \mathbb{R}_+^q$, we have that $g^*(A^i) = G^*(A^i, 0) \leq G^*(A^i, e_{l(i)}) = c_i$. Similarly, for $i \in \mathcal{C} \cap \mathcal{J}$ and $\delta \geq 0$, we have that $g^*(\delta A^i) \leq G^*(\delta A^i, \delta e_{l(i)})$. Hence, by definition of \bar{g}^* and \bar{G}^* , we obtain that $\bar{g}^*(A^i) \leq \bar{G}^*(A^i, e_{l(i)}) = c_i$. Therefore, we have that g^* is feasible for (\mathcal{D}') .

Finally, since $g^*(b) = G^*(b, 0) = z^*$, we conclude that (\mathcal{D}') is a strong dual of (\mathcal{P}') , as desired. \blacksquare

We note here that Corollary 12 allows us to consider a somewhat simpler form of dual for the problem (\mathcal{P}') than the one given directly by Theorem 21. In particular, the feasible functions of (\mathcal{D}') have a domain of smaller dimension than the

feasible functions of (\mathcal{D}) and some of the constraints in (\mathcal{D}') are less restrictive than the corresponding constraints in (\mathcal{D}) .

CHAPTER VI

ON SOME GENERALIZATIONS OF THE SPLIT CLOSURE

6.1 *Introduction*

Cutting planes (or *cuts*, for short) are crucial for solving mixed-integer programs (MIPs), and currently the most effective cuts for general MIPs are the *split cuts*, which can also be seen as a class of disjunctive cuts that generalize GMI cuts. In their seminal paper Cook, Kannan and Schrijver [29] studied split cuts and showed that the split closure of a rational polyhedron P – that is, the set of points in P satisfying all split cuts for P – is again a polyhedron. This is not a trivial result as one has to consider infinitely many split cuts associated with P .

Recently there has been substantial work on generalizing split cuts in different ways to obtain new and more effective classes of cutting planes, and analogues of the polyhedrality of the split closure result have been obtained for some of these classes. Andersen et. al. [6] studied cuts obtained from two dimensional convex lattice-free sets, and Andersen, Louveaux and Weismantel [5] showed that the set of points in a rational polyhedron satisfying all cuts from lattice-free sets with bounded max-facet-width is a polyhedron. Averkov [7] gives a short proof of this latter result. Averkov, Wagner and Weismantel [8] show that the closure with respect to integral lattice-free sets is a polyhedron. In another recent paper, Basu et. al. [15] show that the triangle closure (points satisfying cuts obtained from maximal lattice-free triangles) of a polyhedron in a specific family (the two-row continuous group relaxation) is a polyhedron.

As a generalization of split cuts, recently Dash, Dey and Günlük [33] studied cuts which are obtained by considering two split sets simultaneously. These cuts

are called *cross cuts* and are equivalent to the 2-branch split cuts of Li and Richard [64]. In this chapter, we generalize Cook et al's result from a single rational polyhedron to the union of a finite number of rational polyhedra and use this result to show that the cross cut closure of a rational polyhedron is a polyhedron.

6.1.1 Summary of proof techniques and results

We next formally define split sets, split cuts for a given polyhedron (all polyhedra in this chapter are assumed to be rational) and the split closure of a polyhedron. Let $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, then the split set associated with (π, π_0) is defined to be

$$S(\pi, \pi_0) = \{x \in \mathbb{R}^n \mid \pi_0 < \pi^T x < \pi_0 + 1\}.$$

Clearly, $S(\pi, \pi_0) \cap \mathbb{Z}^n = \emptyset$ and consequently the integer points contained in a polyhedron $P \subset \mathbb{R}^n$ are the same as the ones contained in $\text{conv}(P \setminus S(\pi, \pi_0)) \subset \mathbb{R}^n$. Linear inequalities that are valid for $\text{conv}(P \setminus S(\pi, \pi_0))$ are called split cuts generated by the split set $S(\pi, \pi_0)$.

Let $\mathcal{S}^* = \{S(\pi, \pi_0) \mid (\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}\}$ denote the collection of all split sets and let $\mathcal{S} \subseteq \mathcal{S}^*$ be given. The split closure of a set $A \subseteq \mathbb{R}^n$, with respect to \mathcal{S} is defined as

$$\text{SC}(A, \mathcal{S}) = \bigcap_{S \in \mathcal{S}} \text{conv}(A \setminus S),$$

where for a given set $X \subseteq \mathbb{R}^n$, we denote its convex hull by $\text{conv}(X)$. We refer to $\text{SC}(A, \mathcal{S}^*)$ as the split closure of A and denote it as $\text{SC}(A)$.

Cook, Kannan and Schrijver [29] showed that if P is a rational polyhedron, then $\text{SC}(P)$ is also a rational polyhedron, by proving the following statement.

Given a polyhedron P , there is a finite set of split disjunctions such that any split cut is implied by a nonnegative linear combination of split cuts derived from these disjunctions.

Each split disjunction yields a finite set of nonredundant split cuts, and therefore the above statement implies that $SC(P)$ is a polyhedron. Andersen, Cornuéjols and Li [4] proved the following stronger version of this result.

Given a polyhedron P , there is a finite set of split disjunctions and associated splits such that any split cut is implied by a nonnegative linear combination of split cuts derived from linearly independent subsets of constraints of P using disjunctions from the finite set.

Furthermore, they introduce proof ideas substantially different from Cook, Kannan and Schrijver [29], in particular, an analysis of the possible points of intersection of edges of a rational, pointed polyhedron with the hyperplanes bounding split sets. Other proofs of the polyhedrality of $SC(P)$ can be found in [77], [34] and [5]. The first two of the above three papers give explicit bounds on the sizes of coefficients defining ‘nonredundant’ split disjunctions, thereby implying that only finitely many split disjunctions can be nonredundant. The last paper builds on the proof technique in [4] to show that the closure of a polyhedron with respect to cuts from lattice-free sets having bounded max-facet-width (split sets have max-facet-width 1) is polyhedral. In [7], Averkov builds on proof techniques in [4] and [5]; his results imply the following strong generalization of Cook, Kannan and Schrijver’s result.

Given a polyhedron P , there is a finite set of split disjunctions and associated set \mathcal{S} of split sets such that any other split disjunction and associated split set S' is dominated by a disjunction from \mathcal{S} in the sense that $\text{conv}(P \setminus S') \supseteq \text{conv}(P \setminus S)$ for some $S \in \mathcal{S}$.

In other words, Averkov’s result implies that any split cut is implied by a nonnegative linear combination of split cuts obtained from a single split disjunction from a finite list of disjunctions.

One can view the above results as proving that there exists a finite set $\widehat{\mathcal{S}} \subseteq \mathcal{S}^*$ such that $\text{SC}(P, \mathcal{S}) = \text{SC}(P, \widehat{\mathcal{S}})$. When such $\widehat{\mathcal{S}}$ exists, we say that the split closure is *finitely generated*. For a non-polyhedral set the split closure is not necessarily polyhedral. Even then, in some cases it may be finitely generated (see for example [31]).

As a generalization of split cuts, recently Dash, Dey and Günlük [33] studied *cross cuts*. A cross disjunction is a pair (S_1, S_2) , where $S_1 \in \mathcal{S}^*, S_2 \in \mathcal{S}^*$. Let $\mathcal{L} \subseteq \mathcal{S}^* \times \mathcal{S}^*$ be a collection of cross disjunctions. The cross closure of a set $A \subseteq \mathbb{R}^n$, with respect to \mathcal{L} , is defined as

$$\mathcal{C}(A, \mathcal{L}) = \bigcap_{(S_1, S_2) \in \mathcal{L}} \text{conv}(A \setminus (S_1 \cup S_2)),$$

and the cross closure of P is $\mathcal{C}(P, \mathcal{S}^* \times \mathcal{S}^*)$, denoted simply by $\mathcal{C}(P)$. In Section 6.5, we give our main result, which is a generalization of Cook, Kannan and Schrijver's result to cross cuts.

Theorem 24. *Let P be a rational pointed polyhedron and let $\mathcal{L} \subset \mathcal{S}^* \times \mathcal{S}^*$ be a collection of cross disjunctions. Then $\mathcal{C}(P, \mathcal{L})$ is a polyhedron. More precisely,*

$$\mathcal{C}(P, \mathcal{L}) = \bigcap_{(S_1, S_2) \in \hat{\mathcal{L}}} \text{conv}(P \setminus (S_1 \cup S_2))$$

where $\hat{\mathcal{L}} \subset \mathcal{L}$ is a finite set.

Our proof draws on techniques from the proofs of the highlighted results above, namely from [29], [4] and [7]. An important (for our overall proof) intermediate result we prove is the following generalization of Cook, Kannan and Schrijver's result to a finite union of rational polyhedra.

Theorem 25. *Let \mathcal{K} be a finite set. Let P_k ($k \in \mathcal{K}$) be a collection of rational polyhedra and let $P = \bigcup_{k \in \mathcal{K}} P_k$. Then $\text{SC}(P, \mathcal{S})$ is finitely generated for any $\mathcal{S} \subseteq \mathcal{S}^*$. More precisely,*

$$\text{SC}(P, \mathcal{S}) = \bigcap_{S \in \hat{\mathcal{S}}} \text{conv}(P \setminus S)$$

where $\hat{S} \subset S$ is a finite set.

Note that Theorem 25 does not always implies that $\text{SC}(P, \mathcal{S})$ is polyhedral as it is easy to see that for $P_1 = \{(0, 0)\}$ and $P_2 = \{x \in \mathbb{R}^2 : x_2 = 1\}$ we have $\text{SC}(P_1 \cup P_2, \mathcal{S}^*) = \text{conv}(P_1 \cup P_2)$ which is not a polyhedron.

6.2 Preliminaries

For a convex set $K \subseteq \mathbb{R}^n$ its recession cone is $\text{rec}(K) := \{d \in \mathbb{R}^n \mid x + \lambda d \in K \ \forall \lambda \geq 0, x \in K\}$ and its lineality space is $\text{lin.space}(K) := \{d \in \mathbb{R}^n \mid x + \lambda d \in K \ \forall \lambda \in \mathbb{R}, x \in K\}$. For a linear subspace $W \subseteq \mathbb{R}^n$ we denote the orthogonal linear subspace to W as W^\perp . The orthogonal projection of a set K onto W is denoted as $\text{Proj}_W(K)$.

For a rational polyhedron P , we denote by $V(P) \subseteq \mathbb{Q}^n$ its set of vertices and by $E(P) \subseteq \mathbb{Q}^n$ its set of extreme rays. When $V(P) \neq \emptyset$ and $E(P) = \emptyset$, we say that the polyhedron is pointed (equivalently $\text{lin.space}(Q) = \{0\}$). Recall that every rational polyhedron $P \subseteq \mathbb{R}^n$ can be written in the form

$$P = Q + L,$$

where $L := \text{lin.space}(P)$ is a rational linear subspace and $Q \subseteq L^\perp$ is a pointed rational polyhedron.

6.2.1 Intersection points and Gordan-Dickson Lemma

The two main ingredients that we use in this chapter are the so-called Gordan-Dickson Lemma, and, the analysis of intersection points of (closed) split sets and half-lines that have their end point contained in the split set. In [4], Anderson, Cornuejols and Li give an alternate proof of the polyhedrality of the split closure of polyhedra using a new proof technique. We next summarize the relevant results from [4], and state the Gordan-Dickson Lemma.

6.2.1.1 Split sets and intersection points

We start with defining the point where a rational half-line $H = \{v + \lambda r \mid \lambda \geq 0\}$, where $v, r \in \mathbb{Q}^n$, intersects for the first time the complement of a split set $S \in \mathcal{S}^*$ that contains the end point v of H .

Definition 15 (Intersection point step size). *Let $v, r \in \mathbb{Q}^n$ and $S \in \mathcal{S}^*$ such that $v \in S$, then*

$$\lambda_{vr}(S) = \sup\{\lambda \mid v + \lambda r \in S\}.$$

Given a split set $S = S(\pi, \pi_0)$, the step size can be explicitly computed as follows:

$$\lambda_{vr}(S) = \begin{cases} \frac{\pi^T v - \pi_0}{-\pi^T r} & \pi^T r < 0 \\ \frac{\pi_0 + 1 - \pi^T v}{\pi^T r} & \pi^T r > 0 \\ +\infty & \pi^T r = 0 \end{cases}$$

Furthermore, notice that if $\pi^T r > 0$, then the point $p = v + \lambda_{vr}(S)r$ is the point where the half-line $H = \{v + \lambda r \mid \lambda \geq 0\}$ intersects the hyperplane $\{x \in \mathbb{R}^n \mid \pi^T x = \pi_0\}$. If, on the other hand, $\pi^T r < 0$ then p is the intersection point with the hyperplane $\{x \in \mathbb{R}^n \mid \pi^T x = \pi_0 + 1\}$. We next review some properties of $\lambda_{vr}(S)$ presented in [4] and [5].

Lemma 35 (Lemma 5 in [4]). *Let $S \in \mathcal{S}^*$ and $H = \{v + \lambda r \mid \lambda \geq 0\}$ where $v, r \in \mathbb{Q}^n$ and $v \in S$. If $\lambda_{v,r}(S) < +\infty$, then $\lambda_{vr}(S) < \min\{z \in \mathbb{Z}_+ \mid zr \in \mathbb{Z}^n\}$.*

Lemma 36 (Lemma 6 in [4]). *Let $\lambda^* > 0$ and $H = \{v + \lambda r \mid \lambda \geq 0\}$ where $v, r \in \mathbb{Q}^n$. Then there exists a finite set $\Lambda \in \mathbb{R}$ such that for all $S \in \mathcal{S}^*$, $\lambda_{vr}(S) \in \Lambda$ provided that $\infty > \lambda_{vr}(S) > \lambda^*$ and $v \in S$.*

Definition 16 (Relevant directions). *Let P be a pointed polyhedron. For a vertex*

$v \in V(P)$, we define

$$D_v(P) = \left\{ r \in E(P) \mid \{v + \lambda r \mid \lambda \in \mathbb{R}_+\} \text{ is a 1-dimensional face of } P \right\} \\ \cup \left\{ v' - v \mid v' \in V(P), \text{ and } \text{conv}(v, v') \text{ is a 1-dimensional face of } P \right\}$$

to denote the set of relevant directions for the vertex v .

Observe that for $v \in V(P)$, the relevant directions are the extreme rays of the radial cone at the vertex v in the polyhedron P .

The following result is originally presented in [4] for conic polyhedra and later generalized by Andersen, Louveaux, and Weismantel [5] to general polyhedra.

Lemma 37 (Lemmas 2.3, 2.4, 4.2 in [4, 5]). *Let P be a pointed rational polyhedron and let $S \in \mathcal{S}$. If $P \setminus S \neq \emptyset$, then (1) $\text{conv}(P \setminus S)$ is a rational polyhedron. (2) The extreme rays of $\text{conv}(P \setminus S)$ are the same as the extreme rays of P . (3) If u is a vertex of $\text{conv}(P \setminus S)$, then either $u \in V(P) \setminus S$, or, $u = v + \lambda_{vr}(S)r$, where $v \in V(P) \cap S$ and $r \in D_v(P)$ satisfies one of the following: (i) $r \in E(P)$ and $\lambda_{vr}(S) < +\infty$, or, (ii) $r = v' - v$ for some $v' \in V(P) \setminus S$ such that $\text{conv}(v, v')$ is an edge of P and $\lambda_{vr}(S) < 1$.*

6.2.1.2 Gordan-Dickson Lemma

Finally we state a very simple and useful lemma that shows that for any positive integer p , every set of p -tuples of natural numbers has finitely many minimal elements.

Lemma 38 (Gordan-Dickson Lemma). *Let $X \subseteq \mathbb{Z}_+^p$. Then there exists a finite set $Y \subseteq X$ such that for every $x \in X$ there exists $y \in Y$ satisfying $x \geq y$.*

6.2.2 Computing $\text{conv}(P \setminus (S_1 \cup S_2))$ in terms of Q and L

6.2.2.1 On unimodular matrices and split sets of a linear subspace

We next review some basic properties of *unimodular* matrices, i.e., square matrices with determinant ± 1 . If U is an $n \times n$ unimodular matrix, and $v \in \mathbb{R}^n$, the affine

transformation $\sigma(x) = Ux + v$ is a one-to-one, invertible, mapping of \mathbb{R}^n to \mathbb{R}^n with $\sigma^{-1}(x) = U^{-1}(x - v)$ and this transformation preserves volumes. If U is an integral unimodular matrix, then so is U^{-1} ; if in addition $v \in \mathbb{Z}^n$, then the function $\sigma(x)$ is a one-to-one, invertible, mapping of \mathbb{Z}^n to \mathbb{Z}^n . Further, if $a \in \mathbb{Z}^n, b \in \mathbb{Z}$, the set $\{x \in \mathbb{R}^n : a^T x = b\}$ is mapped to the set

$$\{x' \in \mathbb{R}^n : a^T U^{-1}(x' - v) = b\} \equiv \{x' \in \mathbb{R}^n : a^T U^{-1} x' = b + a^T U^{-1} v\}$$

and $a^T U^{-1} \in \mathbb{Z}^n$. Therefore, given a split set $S(a, b)$, $\sigma(S(a, b))$ and $\sigma^{-1}(S(a, b))$ are both split sets. If $a \in \mathbb{Z}^n$ and the g.c.d. of the coefficients of a is one, then there is a unimodular matrix U such that $a^T U = (0, \dots, 0, 1)$ and $a^T x = b$ has an integral solution for any integer b , say v^b (see [?, Corollary 4.1c]). Note that the previous statement implies that a^T is the last row of U^{-1} . Then, under the linear transformation $x \rightarrow Ux$ (with inverse transformation $x \rightarrow U^{-1}x + v^b$), there is a one-to-one mapping of $\{x \in \mathbb{R}^n : a^T x = b\}$ to the set $\{x \in \mathbb{R}^n : x_n = b\}$, and of the integer points in the respective sets. Further, any linear subspace L of $\{x : a^T x = 0\}$ is mapped to a subspace of $\{x \in \mathbb{R}^n : x_n = 0\}$, and therefore the intersection of a split set and L^\perp is a split set (as a subset of L^\perp). We summarize this result in the next lemma.

Lemma 39. *Let L be a rational subspace. Then*

1. *If $L \subseteq \text{lin.space}(S)$, then $\text{Proj}_{L^\perp}(S)$ is a split set of L^\perp .*
2. *If $L \not\subseteq \text{lin.space}(S)$, then $\text{Proj}_{L^\perp}(S) = L^\perp$.*

The set of all split sets of a linear subspace $W \subseteq \mathbb{R}^n$ will be denoted as \mathcal{S}^W . By Lemma 39, we have

$$\mathcal{S}^W = \{\text{Proj}_W(S) \mid S \in \mathcal{S} \text{ with } W^\perp \subseteq \text{lin.space}(S)\}.$$

When $W = \mathbb{R}^n$, we just write $\mathcal{S} := \mathcal{S}^{\mathbb{R}^n}$.

6.2.2.2 Formula for $\text{conv}(P \setminus (S_1 \cup S_2))$

We start with some basic properties of convex sets. We will omit the proof of the following two lemmas.

Lemma 40. *Let $A, B \subseteq \mathbb{R}^n$, then $\text{conv}(A + B) = \text{conv}(A) + \text{conv}(B)$.*

Lemma 41. *Let $L \subseteq \mathbb{R}^n$ be a linear subspace and let $A, B \subseteq L^\perp$. Then*

1. $(A + L)^c := \mathbb{R}^n \setminus (A + L) = (L^\perp \setminus A) + L$.
2. $(A + L) \cap (B + L) = (A \cap B) + L$.

The next proposition allows to compute $\text{conv}(P \setminus (S_1 \cup S_2))$ in terms of Q, L and the split sets S_1, S_2 .

Proposition 31. *Let $S_1, S_2, \dots, S_m \in \mathcal{S}^*$, and let $P = Q + L$ be a rational polyhedron, where L is a rational linear subspace and $Q \subseteq L^\perp$ is a pointed rational polyhedron. Let $I = \{1, \dots, m\}$ and let $J = \{i \in I \mid L \subseteq \text{lin.space}(S_i)\}$. Then*

1. $\text{conv}(P \setminus (\bigcup_{i \in I} S_i)) = \text{conv}(P \setminus (\bigcup_{i \in J} S_i))$.
2. *Assume that $J \neq \emptyset$. Then $\text{conv}(P \setminus (\bigcup_{i \in J} S_i)) = \text{conv}(Q \setminus (\bigcup_{i \in J} \widehat{S}_i)) + L$, where $\widehat{S}_i := \text{Proj}_{L^\perp}(S_i)$, $i \in J$, are splits sets of L^\perp .*

Proof. For $i \in I$ let $S_i = \{x \in \mathbb{R}^n \mid \pi_0^i < (\pi^i)^T x < \pi_0^i + 1\}$.

1. The inclusion $\text{conv}(P \setminus (\bigcup_{i \in I} S_i)) \subseteq \text{conv}(P \setminus \bigcup_{i \in J} S_i)$ is straightforward. We next prove that $P \setminus (\bigcup_{i \in J} S_i) \subseteq \text{conv}(P \setminus (\bigcup_{i \in I} S_i))$. Let $x_0 \in P \setminus (\bigcup_{i \in J} S_i)$. Consider an index $i \in J$. Since $L \subseteq \text{lin.space}(S_i)$, we obtain that $(\pi^i)^T(x_0 + \alpha v) = (\pi^i)^T x_0$ for all $\alpha \in \mathbb{R}$ and $v \in L$. Therefore, for all $i \in J$, $\alpha \in \mathbb{R}$ and $v \in L$

$$x_0 + \alpha v \in P \setminus S_i. \quad (53)$$

Assume, without loss of generality, that $I \setminus J = \{1, \dots, k\}$ for some $k > 0$. For all $i \in I \setminus J$ we have $L \not\subseteq \text{lin.space}(S_i)$. Then, since $\text{lin.space}(S_i) = \{x \in \mathbb{R}^n \mid (\pi^i)^T x =$

0}, we know that for all $i = 1, \dots, k$ there is a vector $v^i \in L$ such that $(\pi^i)^T v^i \neq 0$. By scaling, we can assume that there is a positive number Δ such that $\Delta - 1 \geq |(\pi^i)^T v^j| \geq 1$ for any $i, j \in \{1, \dots, k\}$ where $(\pi^i)^T v^j \neq 0$. Then $v = \sum_{j=1}^k \Delta^{j-1} v^j$ has the property that $(\pi^i)^T v \neq 0$ for $i = 1, \dots, k$: for any fixed $i \in \{1, \dots, k\}$, if l is the largest positive integer such that $(\pi^i)^T v^l \neq 0$ (such an integer exists by definition), then

$$\Delta^{l-1} |(\pi^i)^T v^l| \geq \Delta^{l-1} > (\Delta - 1) \sum_{j=1}^{l-1} \Delta^{j-1} \geq \sum_{j=1}^{l-1} \Delta^{j-1} |(\pi^i)^T v^j|.$$

Therefore, for large enough $\alpha > 0$, we obtain that for all $i \in I \setminus J$

$$x_0 + \alpha v, x_0 - \alpha v \in P \setminus S_i. \quad (54)$$

Now, by using (53) and (54) we obtain

$$x_0 + \alpha v, x_0 - \alpha v \in P \setminus \left(\bigcup_{i \in I} S_i \right). \quad (55)$$

Since $x_0 \in \text{conv}(\{x_0 + \alpha v, x_0 - \alpha v\})$, (55) implies that $x_0 \in \text{conv}(P \setminus (\bigcup_{i \in I} S_i))$.

Therefore, we conclude that $\text{conv}(P \setminus \bigcup_{i \in J} S_i) \subseteq \text{conv}(P \setminus \bigcup_{i \in I} S_i)$.

2. Since $L \subseteq \text{lin.space}(S_i)$ for $i \in J$, we can write

$$S_i = \widehat{S}_i + L.$$

Using this equality and the properties in Lemma 41, we obtain

$$\begin{aligned} P \setminus \left(\bigcup_{i \in J} S_i \right) &= (Q + L) \setminus \left[\bigcup_{i \in J} (\widehat{S}_i + L) \right] \\ &= (Q + L) \cap \left[\bigcap_{i \in J} (\widehat{S}_i + L)^c \right] \\ &= (Q + L) \cap \left[\bigcap_{i \in J} ((L^\perp \setminus \widehat{S}_i) + L) \right] \\ &= [Q \cap \bigcap_{i \in J} (L^\perp \setminus \widehat{S}_i)] + L \\ &= [Q \setminus \left(\bigcup_{i \in J} \widehat{S}_i \right)] + L. \end{aligned}$$

Therefore, by Lemma 40, we obtain that $\text{conv}(P \setminus (\bigcup_{i \in J} S_i)) = \text{conv}(Q \setminus (\bigcup_{i \in J} \widehat{S}_i)) + L$, as desired. \blacksquare

6.3 Split Closure of a Finite Collection of Polyhedral Sets

In this section, we show that given a finite collection of rational polyhedra, there exists a finite set of splits that define the split closure.

We start with some notation. Let $W \subseteq \mathbb{R}^n$ be a linear subspace and let $Q \subseteq W$ be a pointed polyhedron. For $V' \subseteq V(Q)$, we denote $\mathcal{S}_Q^W(V')$ the set of all splits sets of W that contain V' and does not contain $V(Q) \setminus V'$, that is $\mathcal{S}_Q^W(V') = \{S \in \mathcal{S}^W \mid V(Q) \cap S = V'\}$. When $W = \mathbb{R}^n$, we just write $\mathcal{S}_Q(V')$ and when $Q = \{v\}$ we write \mathcal{S}_v^W .

The following is a simple observation based on Lemma 37.

Corollary 13. *Let $W \subseteq \mathbb{R}^n$ be a linear subspace and let $Q \subseteq W$ be a pointed rational polyhedron. Let $V' \subseteq V(Q)$ and $S_1, S_2 \in \mathcal{S}_Q^W(V')$. Then*

$$\lambda_{vr}(S_1) \geq \lambda_{vr}(S_2), \text{ for all } v \in V' \text{ and } r \in D_v(Q) \Leftrightarrow \text{conv}(Q \setminus S_1) \subseteq \text{conv}(Q \setminus S_2). \quad (56)$$

Using Lemmas 35 and 36 we obtain the following result.

Lemma 42. *Let $W \subseteq \mathbb{R}^n$ be a rational linear subspace. Let $v, r \in \mathbb{Q}^n \cap W$. There exists a function $x_{vr} : \mathcal{S}_v^W \rightarrow \mathbb{Z}_+$ such that whenever $S_1, S_2 \in \mathcal{S}_v^W$, we have*

$$x_{vr}(S_1) \leq x_{vr}(S_2) \Leftrightarrow \lambda_{vr}(S_1) \geq \lambda_{vr}(S_2). \quad (57)$$

Proof. By applying an appropriate unimodular mapping, we may assume that $W = \mathbb{R}^n$. Let $\Lambda = \{\lambda_{vr}(S) \mid S \in \mathcal{S}_v \text{ and } \lambda_{vr}(S) < +\infty\}$ and $M_r = \min\{z \in \mathbb{Z}_+ \mid zr \in \mathbb{Z}^n\}$. Define

$$x_{vr}(S) = \begin{cases} 0, & \lambda_{vr}(S) = +\infty \\ |\Lambda \cap [\lambda_{vr}(S), M_r]|, & \lambda_{vr}(S) < +\infty. \end{cases}$$

Notice that $x_{vr}(S)$ is well defined for all $S \in \mathcal{S}_v$ as $\lambda_{vr}(S) < M_r$ by Lemma 35 and $|\Lambda \cap [\lambda_{vr}(S), M_r]| < +\infty$ by Lemma 36.

When $\lambda_{vr}(S_1) = +\infty$ we have $x_{vr}(S_1) = 0$ and the equivalence (57) clearly holds. If, on the other hand, $\lambda_{vr}(S_1) < +\infty$, we obtain that $\lambda_{vr}(S_1) \geq \lambda_{vr}(S_2)$ is equivalent to $x_{vr}(S_1) \leq x_{vr}(S_2)$, since it is easy to see that the latter occurs if and only if $|\Lambda \cap [\lambda_{vr}(S_1), M_r]| \leq |\Lambda \cap [\lambda_{vr}(S_2), M_r]|$. \blacksquare

Lemma 43. *Let $W \subseteq \mathbb{R}^n$ be a linear subspace and let $Q \subseteq W$ be a pointed rational polyhedron. Let $V' \subseteq V(Q)$. Let $p = \sum_{v \in V'} |D_v(Q)|$. Then there exists a function $x_{V'}^Q : \mathcal{S}_Q^W(V') \rightarrow \mathbb{Z}_+^p$ such that for all $S_1, S_2 \in \mathcal{S}_Q^W(V')$ we have*

$$x_{V'}^Q(S_2) \geq x_{V'}^Q(S_1) \Leftrightarrow \text{conv}(Q \setminus S_2) \supseteq \text{conv}(Q \setminus S_1).$$

Proof. For $S \in \mathcal{S}_Q^W(V')$ let's define a p -tuple $x_{V'}^Q(S)$, where for each $v \in V'$ and $r \in D_v(Q)$, the tuple has a unique entry that equals $x_{vr}(S)$ (from Lemma 42). Now, observe that by Lemma 42, we obtain that

$$x_{V'}^Q(S_2) \geq x_{V'}^Q(S_1) \Leftrightarrow \lambda_{vr}(S_1) \geq \lambda_{vr}(S_2), \text{ for all } v \in V' \text{ and } r \in D_v(Q).$$

Therefore, by Corollary 13 we obtain that

$$x_{V'}^Q(S_2) \geq x_{V'}^Q(S_1) \Leftrightarrow \text{conv}(Q \setminus S_2) \supseteq \text{conv}(Q \setminus S_1).$$

This observation together with the Gordan-Dickson Lemma (Lemma 38) can be used to show that the split closure of a polyhedron is again a polyhedron. In [7], Averkov uses a similar argument to show the polyhedrality of more general closures that include the split closure. We next use Lemma 43 for split closures of unions of polyhedra.

Let \mathcal{K} be a finite set. Consider a collection of rational polyhedra $P_k, k \in \mathcal{K}$ defined as

$$P_k = Q_k + L_k, \tag{58}$$

where for $k \in \mathcal{K}$, L_k is a rational linear subspace and $Q_k \subseteq L_k^\perp$ is a pointed rational polyhedron.

For $\mathcal{K}' \subseteq \mathcal{K}$, and $V' \subseteq \bigcup_{k \in \mathcal{K}'} V(Q_k)$ we denote

$$\mathcal{S}(\mathcal{K}', V') = \{S \in \mathcal{S} \mid \text{Proj}_{L_k^\perp}(S) \in \mathcal{S}_{Q_k}^{L_k^\perp}(V') \forall k \in \mathcal{K}' \text{ and } \text{Proj}_{L_k^\perp}(S) \notin \mathcal{S}_{Q_k}^{L_k^\perp}(V') \forall k \notin \mathcal{K}'\}.$$

$\mathcal{S}(\mathcal{K}', V')$ is the set of splits such that for all $k \in \mathcal{K}'$ we have that $\text{Proj}_{L_k^\perp}(S)$ is a split set of L_k that contains all the points in $V(Q_k) \cap V'$ but does not contains the points in $V(Q_k) \setminus V'$, and such that for all $k \notin \mathcal{K}'$ we have that $\text{Proj}_{L_k^\perp}(S)$ is not a split set of L_k .

Proposition 32. *Let $\mathcal{S} \subseteq \mathcal{S}^*$ and $\{P_k\}_{k \in \mathcal{K}}$ be a finite collection of pointed rational polyhedra. Then, there exists a finite set $\mathcal{S}_Y \subseteq \mathcal{S}$ such that for all $S_1 \in \mathcal{S}$ there exists $S_2 \in \mathcal{S}_Y$ such that*

$$\text{conv}(P_k \setminus S_2) \subseteq \text{conv}(P_k \setminus S_1) \quad \text{for all } k \in \mathcal{K}.$$

Proof. Notice that sets $\mathcal{S}(\mathcal{K}', V')$ for $\mathcal{K}' \subseteq \mathcal{K}$, and $V' \subseteq \bigcup_{k \in \mathcal{K}'} V(P_k)$ form a finite partition of \mathcal{S} , that is, $\mathcal{S}(\mathcal{K}', V') \cap \mathcal{S}(\mathcal{K}'', V'') = \emptyset$ if $V' \neq V''$ or $\mathcal{K}' \neq \mathcal{K}''$, and

$$\mathcal{S} = \bigcup_{V' \subseteq \bigcup_{k \in \mathcal{K}} V(P_k); \mathcal{K}' \subseteq \mathcal{K}} \mathcal{S}(\mathcal{K}', V').$$

Consequently, it suffices to show the existence of finite sets $\mathcal{S}_Y(\mathcal{K}', V') \subseteq \mathcal{S}(\mathcal{K}', V')$ for each $\mathcal{K}' \subseteq \mathcal{K}$, and $V' \subseteq \bigcup_{k \in \mathcal{K}'} V(P_k)$ that satisfy the claim when $S_1 \in \mathcal{S}(\mathcal{K}', V')$ and $S_2 \in \mathcal{S}_Y(\mathcal{K}', V')$.

We now consider a arbitrary sets $\mathcal{K}' \subseteq \mathcal{K}$, and $V' \subseteq \bigcup_{k \in \mathcal{K}'} V(P_k)$. We have two cases:

Case 1: If $V' = \emptyset$ or $\mathcal{K}' = \emptyset$, then by Proposition 31 and Corollary 13 it is easy to see that for all $S \in \mathcal{S}(\mathcal{K}', V')$ and $k \in \mathcal{K}$ we have $\text{conv}(P_k \setminus S) = P_k$. Thus, it is sufficient to take $\mathcal{S}_Y(\mathcal{K}', V') = \{S\}$, where $S \in \mathcal{S}(\mathcal{K}', V')$ can be chosen arbitrarily.

Case 2: If $V' \neq \emptyset$ and $\mathcal{K}' \neq \emptyset$. By Lemma 43 for all $k \in \mathcal{K}'$ we have that there exists a function $x_{V'}^{Q_k} : \mathcal{S}_{Q_k}^{L_k^\perp}(V') \rightarrow \mathbb{Z}_+^{P_k}$ such that for all $S_1, S_2 \in \mathcal{S}(\mathcal{K}', V')$ we have

$$x_{V'}^{Q_k}(\text{Proj}_{L_k^\perp}(S_2)) \geq x_{V'}^{Q_k}(\text{Proj}_{L_k^\perp}(S_1)) \Leftrightarrow \text{conv}(Q_k \setminus \text{Proj}_{L_k^\perp}(S_2)) \supseteq \text{conv}(Q_k \setminus \text{Proj}_{L_k^\perp}(S_1)). \quad (59)$$

By (59) we have that Proposition 31 implies that for all $k \in \mathcal{K}'$

$$x_{V'}^{Q_k}(\text{Proj}_{L_k^\perp}(S_2)) \geq x_{V'}^{Q_k}(\text{Proj}_{L_k^\perp}(S_1)) \Leftrightarrow \text{conv}(P_k \setminus S_2) \supseteq \text{conv}(P_k \setminus S_1). \quad (60)$$

Let $p = \sum_{k \in \mathcal{K}'} p_k$. For each $S \in \mathcal{S}(\mathcal{K}', V')$ we now define a p -tuple $x(S)$, where for each $k \in \mathcal{K}'$, the tuple has a unique entry that equals $x_{V'}^{Q_k}(\text{Proj}_{L_k^\perp}(S))$ (Lemma 43).

Collection of these p -tuples gives the following set contained in \mathbb{Z}_+^p :

$$X = \left\{ x(S) \mid S \in \mathcal{S}(\mathcal{K}', V') \right\}.$$

By Lemma 38, the set X possesses a finite set of minimal elements, that is, there exists a finite set $Y \subseteq X$ such that for every $x \in X$ there exists $y \in Y$ satisfying $x \geq y$. As $Y \subseteq X$ we can construct a finite set of splits $\mathcal{S}_Y(\mathcal{K}', V') \subseteq \mathcal{S}(\mathcal{K}', V')$ such that for any $S_1 \in \mathcal{S}(\mathcal{K}', V')$ there exists $S_2 \in \mathcal{S}_Y(\mathcal{K}', V')$ satisfying

$$x_{V'}^{Q_k}(\text{Proj}_{L_k^\perp}(S_2)) \geq x_{V'}^{Q_k}(\text{Proj}_{L_k^\perp}(S_1)) \text{ for all } k \in \mathcal{K}'. \quad (61)$$

By (60) we obtain that (61) is equivalent to

$$\text{conv}(P_k \setminus S_2) \supseteq \text{conv}(P_k \setminus S_1) \text{ for all } k \in \mathcal{K}'.$$

By Proposition 31 the last statement is equivalent to

$$\text{conv}(P_k \setminus S_2) \supseteq \text{conv}(P_k \setminus S_1) \text{ for all } k \in \mathcal{K}.$$

This finish the proof in Case 2.

To conclude the proof it suffices to let

$$\mathcal{S}_Y = \bigcup_{V' \subseteq \bigcup_{k \in \mathcal{K}} V(P_k); \mathcal{K}' \subseteq \mathcal{K}} \mathcal{S}_Y(\mathcal{K}', V') \quad \blacksquare$$

Theorem 25. Let \mathcal{K} be a finite set. Let P_k be a collection of rational polyhedra and let $P = \bigcup_{k \in \mathcal{K}} P_k$. Then $\text{SC}(P, \mathcal{S})$ is finitely generated for any $\mathcal{S} \subseteq \mathcal{S}^*$. More precisely,

$$\text{SC}(P, \mathcal{S}) = \bigcap_{S \in \hat{\mathcal{S}}} \text{conv}(P \setminus S)$$

where $\hat{\mathcal{S}} \subset \mathcal{S}$ is a finite set.

Proof. Note that for any $S_2 \in \mathcal{S}$: $\text{conv}\left(\left(\bigcup_{k \in \mathcal{K}} P_k\right) \setminus S_2\right) = \text{conv}\left(\bigcup_{k \in \mathcal{K}} (P_k \setminus S_2)\right) = \text{conv}\left(\bigcup_{k \in \mathcal{K}} \text{conv}(P_k \setminus S_2)\right)$. Furthermore, by Proposition 32, there is a finite set $\mathcal{S}_Y \subset \mathcal{S}$ such that for each $S_1 \in \mathcal{S}$ there exists $S_2 \in \mathcal{S}_Y$ that satisfies

$$\text{conv}\left(\bigcup_{k \in \mathcal{K}} \text{conv}(P_k \setminus S_2)\right) \subseteq \text{conv}\left(\bigcup_{k \in \mathcal{K}} \text{conv}(P_k \setminus S_1)\right) = \text{conv}\left(\bigcup_{k \in \mathcal{K}} P_k \setminus S_1\right).$$

As \mathcal{S}_Y is finite, to complete the proof, it suffices to observe that

$$\text{SC}(P, \mathcal{S}) = \bigcap_{S \in \mathcal{S}} \text{conv}\left(\bigcup_{k \in \mathcal{K}} P_k \setminus S\right) = \bigcap_{S \in \mathcal{S}_Y} \text{conv}\left(\bigcup_{k \in \mathcal{K}} P_k \setminus S\right).$$

■

6.4 Split Closure of a Union of Mixed-integer Sets

Consider a mixed-integer set defined by a polyhedron $P^{LP} \in \mathbb{R}^{n+l}$ and the mixed-integer lattice $\mathbb{Z}^n \times \mathbb{R}^l$ where n and l are positive integers:

$$P^I = P^{LP} \cap (\mathbb{Z}^n \times \mathbb{R}^l) \tag{62}$$

An inequality is called a split cut for P^{LP} with respect to the lattice $\mathbb{Z}^n \times \mathbb{R}^l$ if it is valid for $\text{conv}(P^{LP} \setminus S)$ for some $S \in \mathcal{S}_{n,l}^*$ where

$$\mathcal{S}_{n,l}^* = \{S(\pi, \pi_0) \in \mathcal{S}^* : \pi \in \mathbb{Z}^n \times \{0\}^l\}.$$

The split closure is then defined in the usual way as the intersection of all such split cuts. A straightforward extension of Theorem 25 is the following:

Corollary 14. Let $P_k \in \mathbb{R}^{n+l}$ be a rational polyhedron for $k \in \mathcal{K}$ where \mathcal{K} is a finite set and let $P = \bigcup_{k \in \mathcal{K}} P_k$. Then $\text{SC}(P, \mathcal{S})$ is finitely generated for any $\mathcal{S} \subseteq \mathcal{S}_{n,l}^*$.

6.5 Cross Closure of a Polyhedral Set

6.5.1 Reduction to the pointed polyhedral case

Recall that every rational polyhedra $P \subseteq \mathbb{R}^n$ is of the form $P = Q + L$, where $L := \text{lin.space}(P)$ is a rational linear subspace and $Q \subseteq L^\perp$ is a pointed rational polyhedron.

Observe that by Proposition 31 the cross closure of P can be written as

$$\mathcal{C}(P) = \bigcap_{\substack{S_1 \in \mathcal{S}^*, S_2 \in \mathcal{S}^* \\ L \not\subseteq \text{lin.space}(S_i), \\ i=1,2}} P \cap \bigcap_{\substack{S_1 \in \mathcal{S}^*, S_2 \in \mathcal{S}^* \\ L \subseteq \text{lin.space}(S_1), \\ L \not\subseteq \text{lin.space}(S_2)}} \text{conv}(P \setminus S_1) \cap \bigcap_{\substack{S_1 \in \mathcal{S}^*, S_2 \in \mathcal{S}^* \\ L \subseteq \text{lin.space}(S_i), \\ i=1,2}} [\text{conv}(Q \setminus (\widehat{S}_1 \cup \widehat{S}_2)) + L], \quad (63)$$

where for $i = 1, 2$ $\widehat{S}_i = \text{Proj}_{L^\perp}(S_i)$. Since in the right hand side the first term is equal to P and the second term is equal to $\text{SC}(P)$, we only need to prove the polyhedrality of

$$\bigcap_{\substack{S_1 \in \mathcal{S}^*, S_2 \in \mathcal{S}^* \\ L \subseteq \text{lin.space}(S_i), i=1,2}} [\text{conv}(Q \setminus (\widehat{S}_1 \cup \widehat{S}_2)) + L].$$

Since L is a rational subspace and $Q \subseteq L^\perp$ and $\widehat{S}_1, \widehat{S}_2$ are splits sets of L^\perp , by applying a suitable unimodular mapping we may assume that $L^\perp = \mathbb{R}^n$. Therefore, we conclude that to prove the polyhedrality of the cross closure for any rational polyhedron, it is sufficient to study the special case when the polyhedron P is pointed.

6.5.2 Proof in the pointed polyhedral case

In this section, we show that the cross closure of a rational polyhedron is again a polyhedron. We combine the proof technique of Cook, Kannan and Schrijver [29] for showing that the split closure of a polyhedron is polyhedral along with the results we derived in earlier sections based on proof techniques of Anderson, Cornuéjols, Li [4], and Averkov [7]. We need some definitions to discuss the overall techniques used. Lets denote by $\|\cdot\|$ the usual euclidean norm. Define the width

of a split set $S(\pi, \pi_0)$ as $w(S(\pi, \pi_0)) = 1/\|\pi\|$ (this is the geometric distance between the parallel hyperplanes bounding the split set). Then $w(S(\pi, \pi_0)) > \eta$ for some $\eta > 0$ implies that $\|\pi\| < 1/\eta$. Therefore, for any fixed $\eta > 0$ and $\pi_0 \in \mathbb{Z}^n$, there are only a finite number of $\pi \in \mathbb{Z}^n$ such that $w(S(\pi, \pi_0)) > \eta$.

Cook, Kannan, Schrijver (roughly) prove their polyhedrality result using the following idea. Assume P is a polyhedron, \mathcal{L} is a finite list of split sets and let $\text{SC}(P, \mathcal{L}) = \bigcap_{S \in \mathcal{L}} \text{conv}(P \setminus S)$. Suppose that for every face F of P , $\text{SC}(P, \mathcal{L}) \cap F \subseteq \text{SC}(F)$. Then (i) there are only finitely many split sets beyond the ones contained in \mathcal{L} which yield split cuts cutting off points of $\text{SC}(P, \mathcal{L})$ (they show that if $S(\pi, \pi_0)$ is such a split set, then π must have bounded norm). Therefore, (ii) if one assumes (by induction on dimension) that the number of split sets needed to define the split closure of each face of a polyhedron is finite, then so is the number of split sets needed to define the split closure of the polyhedron.

Santanu Dey [38] observed that idea (i) in the Cook, Kannan, Schrijver proof technique can also be used in the case of some disjunctive cuts which generalize split cuts. We apply a modification of idea (i) to cross cuts; namely we show in Lemma 48 that if \mathcal{L} is a finite list of cross disjunctions (represented as a pair of split sets) such that

$$\mathcal{C}(P, \mathcal{L}) = \bigcap_{(S_1, S_2) \in \mathcal{L}} \text{conv}(P \setminus (S_1 \cup S_2)) \quad (64)$$

intersected with each face of P is equal to the cross closure of each face, then cross disjunctions (S_1, S_2) where both $w(S_1)$ and $w(S_2)$ are at most some $\eta > 0$ can only yield cross cuts valid for $\mathcal{C}(P, \mathcal{L})$, and are therefore not needed to define the cross closure of P . We then only need to consider cross disjunctions (S_1, S_2) where one of $w(S_1), w(S_2)$ is greater than η (such cross disjunctions are still infinitely many in number).

We first need a generalization of Lemma 37, property (2.). Let $\text{rec}(P)$ denote the recession cone of P , $\text{aff}(P)$ denote the affine hull of P , and P^I denote the integer

hull of P .

Lemma 44. *Let P be a polyhedron and let $S_i \in \mathcal{S}^*$ be split sets for $i \in \{1, \dots, m\}$. Then, $\text{conv}(P \setminus (\cup_i S_i))$ is a polyhedron with the same recession cone as P .*

Proof. Remember that $\bar{S}_i = D_i^0 \cup D_i^1$ where D_i^0 and D_i^1 are two disjoint half-spaces corresponding to the two sides of the disjunction associated with S_i . We start with writing $P \setminus (\cup_i S_i)$ as a union of polyhedral sets. To this end let $B = \{0, 1\}^m$ and consider sets $P^b = P \cap D^b$ for $b \in B$ where

$$D^b = D_1^{b_1} \cap D_2^{b_2} \cap \dots \cap D_m^{b_m}.$$

We can then write

$$P \setminus (\cup_{i=1}^m S_i) = \cup_{b \in B} P^b. \quad (65)$$

To prove the claim, we will show that the convex hull of the right-hand-side of equation (65) is polyhedral. Let P_∞ denote the recession cone of P and let $P_\infty^b = P^b + P_\infty$. Note that, by definition, $P_\infty^b = \emptyset$ if $P^b = \emptyset$.

Now consider an arbitrary point $x \in P_\infty^b$ for some $b \in B$. We will next show that $x \in \text{conv}(\cup_{b \in B} P^b)$. Clearly, $x = x^b + d^b$ where $x^b \in P^b$ and $d^b \in P_\infty$. If $x \in P^b$, the claim holds. If, on the other hand, $x \notin P^b$, then d^b does not belong to the recession cone of D^b . As recession cone of D^b is the intersection of the recession cones of $D_i^{b_i}$ for $i \in I = \{1, \dots, m\}$, we have $d^b \in \text{rec}(D_i^{b_i})$ for $i \in \bar{I}$ for some $\bar{I} \subseteq I$ and $d^b \notin \text{rec}(D_i^{b_i})$ for $i \in I \setminus \bar{I}$. Now consider $\bar{b} \in B$ where $\bar{b}_i = b_i$ for $i \in \bar{I}$ and $\bar{b}_i = 1 - b_i$ for $i \in I \setminus \bar{I}$. Clearly d^b belongs to the recession cone of $D^{\bar{b}}$. Let $x(t)$ stand for $x^b + td^b$ and note that $x(t) \in P$ for all $t \geq 0$.

As $x^b \in P^b \subseteq \cap_{i \in \bar{I}} D_i^{b_i}$ and $d^b \in \text{rec}(D_i^{b_i})$ for $i \in \bar{I}$, we have $x(t) \in \cap_{i \in \bar{I}} D_i^{b_i}$ for all $t \geq 0$. Now consider $i \in I \setminus \bar{I}$ and let $D_i^{b_i} = \{x \in \mathbb{R}^n : \pi x \leq \pi_0\}$. As $d^b \notin \text{rec}(D_i^{b_i})$ we have $d^b \pi > 0$ and therefore $\pi(x^b + td^b) \geq \pi_0 + 1$ for all $t \geq \bar{t}$ for some $\bar{t} > 0$, implying $x(t) \in D_i^{1-b_i}$ for all $t \geq \bar{t}$. As $i \in I \setminus \bar{I}$ is chosen arbitrarily, we conclude that there exists a large enough $t^* > 0$ such that $x(t^*) \in D_i^{1-b_i}$ for all $i \in I \setminus \bar{I}$.

Consequently, $x(t^*) \in P^{\bar{b}}$ and we have

$$x \in \text{conv}(x^b, x^b + t^*d^b) \subseteq \text{conv}(P^b, P^{\bar{b}}) \subseteq \text{conv}(\cup_{b \in B} P^b).$$

As x is an arbitrary point in P_{∞}^b , we have that

$$\cup_{b \in B} P_{\infty}^b \subseteq \text{conv}(\cup_{b \in B} P^b)$$

which establishes that

$$\text{conv}(\cup_{b \in B} P_{\infty}^b) = \text{conv}(\cup_{b \in B} P^b)$$

as $P^b \subseteq P_{\infty}^b$ for all $b \in B$.

Finally, notice that $\text{conv}(\cup_{b \in B} P_{\infty}^b)$ is a polyhedral set since the sets P_{∞}^b , $b \in B$ are polyhedra with the same recession cone. ■

In the proof above, we also showed that the following corollary holds:

Corollary 15. *Let P be a polyhedron and let $S_i \in \mathcal{S}^*$ be split sets for $i \in \{1, \dots, m\}$. Then, $\text{conv}(P \setminus (\cup_i S_i))$ has the same recession cone as P .*

Observe that Lemma 44 implies that $\mathcal{C}(P, \mathcal{L})$ is a polyhedron with $\text{rec}(\mathcal{C}(P, \mathcal{L})) = \text{rec}(P)$ if $\mathcal{C}(P, \mathcal{L})$ is defined as in (64).

The proof of the following lemma is ommited.

Lemma 45. *Let P be a polyhedron and let F be a face of P . For any set B , $\text{conv}(P \setminus B) \cap F = \text{conv}(F \setminus B)$.*

The next result is essentially contained in Cook, Kannan and Schrijver [29], though our statement and proof are slightly different as we need to handle polyhedra which are not full-dimensional.

Lemma 46. *Let P and Q be pointed polyhedra in \mathbb{R}^n where $Q \subset P$ and P and Q have the same affine hull. Then there exists a number $r > 0$ such that for any $c \in \mathbb{R}^n$ satisfying*

(i) $\max\{c^T x : x \in Q\} < \max\{c^T x : x \in P\} < \infty$ and (ii) the first maximum is attained at a vertex of Q contained in the relative interior of P , there exists a ball of radius r in the relative interior of P with each point x in the ball satisfying $c^T x > \max\{c^T x : x \in Q\}$.

Proof. Let $P = \{x : a_i^T x \leq b_i \text{ for } i = 1, \dots, m, a_i^T x = b_i \text{ for } i = m+1, \dots, l\}$ where $a_i \in \mathbb{R}^n$ and $0 < m \leq l$. We can assume, without loss of generality, that for $i = 1, \dots, m$ the inequalities $a_i^T x \leq b_i$ are non-redundant and a_i belongs to the linear space L that is parallel to $\text{aff}(P)$. Further, by scaling we can assume $\max_{i=1}^m \{\|a_i\|\} \leq 1$. Let $V = \{v_1, \dots, v_k\}$ be the set of vertices of Q contained in the relative interior of P (if V is empty, then no c satisfying the conditions of the Lemma exists and therefore any $r > 0$ yields the result), and let

$$\epsilon = \min_{i=1, \dots, m, j=1, \dots, k} \{b_i - a_i^T v_j\} > 0.$$

Let c satisfy the conditions of the lemma. Further, let $c = \hat{c} + \bar{c}$ where \hat{c} is orthogonal to L and \bar{c} lies in L . As $\hat{c}^T x$ is a constant, say δ , for all $x \in \text{aff}(P)$, then for any point $x \in P$, $c^T x = \bar{c}^T x + \delta$ which implies that

$$\max\{c^T x : x \in Q\} = \max\{\bar{c}^T x : x \in Q\} + \delta <$$

$$\max\{\bar{c}^T x : x \in P\} + \delta = \max\{c^T x : x \in P\}.$$

Let the second maximum equal d and be obtained at a vertex $v \in V$, and let the third maximum equal d' and be obtained at a vertex $v' \in P$.

By LP duality, there exists multipliers $0 \leq \lambda \in \mathbb{R}^m$ such that $\bar{c} = \sum_{i=1}^m \lambda_i a_i$ and $d' = \sum_{i=1}^m \lambda_i b_i$ and $\tau = \sum_{i=1}^m \lambda_i > 0$; if $\tau = 0$, then $c^T v = c^T v'$, a contradiction. By scaling, we can assume that $\tau = 1$. Then $\|\bar{c}\| \leq 1$. Furthermore

$$d' - \bar{c}^T v = \sum_{i=1}^m \lambda_i (b_i - a_i^T v) \geq \epsilon.$$

By definition, the distance between the hyperplanes $\bar{c}^T x = d$ and $\bar{c}^T x = d'$ is at least $\epsilon/\|\bar{c}\| > \epsilon/2$. Therefore, any point z in the ball $B(v', \epsilon/2)$ satisfies $\bar{c}^T z > \bar{c}^T v$ (and also

$c^T z > c^T v$). We can find an $r > 0$ such that for each vertex u of P , $P \cap B(u, \epsilon/2)$ contains a ball of radius r . ■

In the proof above, we can assume that we construct a fixed set \mathcal{B} of balls, one per each vertex of P , such that one of these balls satisfies the desired property in Lemma 46 (for $Q = \mathcal{C}(P, \mathcal{L})$).

A (open) *strip* in \mathbb{R}^n is the set of points (strictly) between a pair of parallel hyperplanes and the width of a strip is the distance between its bounding hyperplanes.

Thus the topological closure of a split set is a strip. The *minimum width* of a closed, compact, convex set A is defined as the minimum width of a strip containing A and is denoted by $w(A)$. It is known (Bang [10]) that the sum of widths of a collection of strips containing A must exceed its minimum width. The following statement is a trivial consequence of Bang's result.

Lemma 47. *Let B be a ball of radius $r > 0$, and let S_1, S_2 be split sets such that $B \subseteq (S_1 \cup S_2)$. Then,*

$$w(S_1) + w(S_2) \geq 2r.$$

The next result generalizes a result in Cook, Kannan and Schrijver [29] on the action of split disjunctions on full-dimensional polyhedra to the action of cross disjunctions on polyhedra which may not be full-dimensional. We need some definitions. Given a list of cross disjunctions $\mathcal{L} \subseteq \mathcal{S}^* \times \mathcal{S}^*$, by abusing notation, we define

$$\text{Proj}_1(\mathcal{L}) = \{S : (S, S') \in \mathcal{L} \text{ for some split set } S'\}, \text{Proj}_2(\mathcal{L}) = \{S : (S', S) \in \mathcal{L} \text{ for some split set } S'\}.$$

Lemma 48. *Let P be a pointed polyhedron in \mathbb{R}^n . Let $Q \subset P$ be a polyhedron, and \mathcal{L} be a list of cross disjunctions such that $Q \cap F \subseteq \mathcal{C}(F, \mathcal{L})$ for each proper face F of P . Then there exist finite sets $\mathcal{S}_1 \subseteq \text{Proj}_1(\mathcal{L})$ and $\mathcal{S}_2 \subseteq \text{Proj}_2(\mathcal{L})$ such that for any cross*

disjunction $(S_1, S_2) \in \mathcal{L}$ either (i) every cross cut derived from the disjunction is valid for Q or (ii) the disjunction equals or is dominated by a cross disjunction (S', S'') where $S' \in \mathcal{S}_1$ or $S'' \in \mathcal{S}_2$.

Proof. Let $c^T x \leq \gamma$ be a nonredundant cross cut obtained from a cross disjunction $(S_1, S_2) \in \mathcal{L}$.

Suppose $c^T x \leq \gamma$ is not valid for Q . Then

$$\gamma = \max\{c^T x : x \in \text{conv}(P \setminus (S_1 \cup S_2))\} < \max\{c^T x : x \in Q\} < \infty.$$

The second inequality follows from Lemma 44 which implies that $\text{rec}(P) = \text{rec}(\text{conv}(P \setminus (S_1 \cup S_2)))$ and from the fact that $\text{rec}(Q) \subseteq \text{rec}(P)$. The second maximum above must be obtained at a vertex v of Q contained in the relative interior of P as for each F proper face of P we have

$$\max\{c^T x : x \in Q \cap F\} \leq \max\{c^T x : x \in \mathcal{C}(P, \mathcal{L}) \cap F\} \leq \max\{c^T x : x \in \text{conv}(P \setminus (S_1 \cup S_2)) \cap F\}.$$

As Q is contained in P and is a pointed polyhedron, Lemma 46 implies that there exists a ball B of radius r (for some r) in the relative interior of P with all points in the ball satisfying $c^T x > \gamma$.

Case 1: P is full-dimensional. Clearly $B \subseteq S_1 \cup S_2$, otherwise there exists an $x \in B \setminus (S_1 \cup S_2) \subseteq \text{conv}(P \setminus (S_1 \cup S_2))$; such an x satisfies $c^T x > \gamma$, a contradiction. Lemma 47 implies that $w(S_1) + w(S_2) \geq 2r$ which implies that for i either 1 or 2, we have (i) $S_i \cap B \neq \emptyset$ and (ii) $w(S_i) \geq r \Leftrightarrow \|\pi^i\| \leq 1/r$ where $S_i = S(\pi^i, \pi_0^i)$. There are only finitely many split sets in \mathcal{S}^* which satisfy properties (i) and (ii). Let \mathcal{S}_1 and \mathcal{S}_2 stand for, respectively, the set of such split sets in $\text{Proj}_1(\mathcal{L})$ and $\text{Proj}_2(\mathcal{L})$. Thus either $S_1 \in \mathcal{S}_1$ or $S_2 \in \mathcal{S}_2$.

Case 2: Let $k = \dim(P) < n$. By applying a unimodular transformation to P , we can assume that $\text{aff}(P) = \{x \in \mathbb{R}^n : x_{k+1} = \alpha_{k+1}, \dots, x_n = \alpha_n\}$, where $\alpha_{k+1}, \dots, \alpha_n$ are rational numbers, and $\alpha_i = p_i/q_i$ where p_i and q_i are coprime integers, for $i =$

$k+1, \dots, n$. Recall that any unimodular transformation maps split sets to split sets. Let Δ stand for the least common multiple of $q_i (i = k+1, \dots, n)$, the denominators of α_i . Define the vector $a \in \mathbb{R}^n$ such that $a_1 = \dots = a_k = 0$ and $a_{k+1} = \alpha_{k+1}, \dots, a_n = \alpha_n$. Assume $S_1 = S(\pi, \pi_0)$ and $S_2 = S(\pi', \pi'_0)$. Then

$$S_1 \cap \text{aff}(P) = \bar{S}_1 \times \mathbb{R}^{n-k} \text{ where } \bar{S}_1 = \{x \in \mathbb{R}^k : \pi_0 - a^T \pi < (\text{Proj}_{\mathbb{R}^k} \pi)^T x < \pi_0 + 1 - a^T \pi\}.$$

Therefore \bar{S}_1 is a strip in \mathbb{R}^k with $w(\bar{S}_1) = 1/\|\text{Proj}_{\mathbb{R}^k} \pi\|$. Let \bar{S}_2 be defined similarly in terms of S_2 .

Since B is contained in the relative interior of P , we can write $B = \bar{B} \times \mathbb{R}^{n-k}$, where $\bar{B} \subseteq \mathbb{R}^k$ is a ball of radius r . Now any point x in $\bar{B} \setminus (\bar{S}_1 \cup \bar{S}_2)$ satisfies $c^T x > \gamma$, and therefore $\bar{B} \subseteq \bar{S}_1 \cup \bar{S}_2$, which implies that either (i) $w(\bar{S}_1) \geq r$ and $\bar{S}_1 \cap \bar{B} \neq \emptyset$ or (ii) $w(\bar{S}_2) \geq r$ and $\bar{S}_2 \cap \bar{B} \neq \emptyset$. In other words, either $\|\text{Proj}_{\mathbb{R}^k} \pi\| \leq 1/r$ or $\|\text{Proj}_{\mathbb{R}^k} \pi'\| \leq 1/r$. Assume the first inequality holds. Then the first k coefficients of π come from a bounded set, but the last $n-k$ coefficients can be arbitrary integers. However, as $\alpha_{k+1}, \dots, \alpha_n$ are rational, $a^T \pi = \sum_{i=k+1}^n \alpha_i \pi_i$ is an integral multiple of Δ . In other words, the set \bar{S}_1 has the form

$$\hat{S} = \{x \in \mathbb{R}^k : \hat{\pi}_0 - \delta < \hat{\pi}^T x < \hat{\pi}_0 + 1 - \delta\}. \quad (66)$$

where δ is an integral multiple of Δ and $\hat{\pi}$ and $\hat{\pi}_0$ are integral. Furthermore, there are only finitely many choices of tuples $(\hat{\pi}, \hat{\pi}_0, \delta)$ (call this set Φ) such that $\hat{S} \cap B \neq \emptyset$ and $w(\hat{S}) \geq r$; for each tuple $\phi \in \Phi$, let $S(\pi^\phi, \pi_0)$ be a split set in $\text{Proj}_1(\mathcal{L})$ (if one exists) such that $a^T \pi^\phi = \delta, \text{Proj}_{\mathbb{R}^k} \pi^\phi = \hat{\pi}$ and $\pi_0 = \hat{\pi}_0$, and let \mathcal{S}_1 be the list of such split sets. Define \mathcal{S}_2 similarly in terms of split sets from $\text{Proj}_2(\mathcal{L})$. Then it is clear that \bar{S}_1 or \bar{S}_2 must equal a strip defined by some $\phi \in \Phi$. Then (S_1, S_2) is either dominated by $(S(\pi^\phi, \pi_0), S_2)$ with $S(\pi^\phi, \pi_0) \in \mathcal{S}_1$ or by $(S_1, S(\pi^\phi, \pi_0))$ with $S(\pi^\phi, \pi_0) \in \mathcal{S}_2$. ■

Theorem 24. *Let P be a rational pointed polyhedron and let $\mathcal{L} \subseteq \mathcal{S}^* \times \mathcal{S}^*$ be a list of cross*

disjunctions. Then $\mathcal{C}(P, \mathcal{L})$ is a polyhedron. More precisely for some finite set $\hat{\mathcal{L}} \subseteq \mathcal{L}$,

$$\mathcal{C}(P) = \bigcap_{(S_1, S_2) \in \hat{\mathcal{L}}} \text{conv}(P \setminus (S_1 \cup S_2)).$$

Proof. The proof is by induction on $\dim(P)$. For the base case, let $\dim(P) = 0$. Then P is a single point. If P belongs to $S_1 \cup S_2$ for any cross disjunction in \mathcal{L} , then $\mathcal{C}(P, \mathcal{L}) = \mathcal{C}(P, \{(S_1, S_2)\}) = \emptyset$. Otherwise $\mathcal{C}(P, \mathcal{L}) = P$.

We next assume that for all polyhedra Q of dimension strictly less than $\dim(P)$, $\mathcal{C}(Q, \mathcal{L})$ is defined by a finite set of cross disjunctions. Let F be a proper face of P . Since $\dim(F) < \dim(P)$, by the induction hypothesis we infer that there exists a finite set of cross disjunctions $\mathcal{L}(F) \subseteq \mathcal{L}$ in \mathbb{R}^n such that $\mathcal{C}(F, \mathcal{L}) = \mathcal{C}(F, \mathcal{L}(F))$. Therefore, if we define

$$\mathcal{L}^* = \bigcup_{F \text{ is proper face of } P} \mathcal{L}(F),$$

then \mathcal{L}^* is a finite list of cross disjunctions. Then for any proper face F of P , we have

$$\mathcal{C}(P, \mathcal{L}^*) \cap F = \mathcal{C}(F, \mathcal{L}^*) = \mathcal{C}(F, \mathcal{L}(F)) = \mathcal{C}(F, \mathcal{L})$$

where the first equality follows from Lemma 45 and the second and third inequality follow from $\mathcal{C}(F, \mathcal{L}(F)) = \mathcal{C}(F, \mathcal{L})$ and $\mathcal{L}(F) \subseteq \mathcal{L}^* \subseteq \mathcal{L}$. Note that $\mathcal{C}(P, \mathcal{L}^*) \subset P$ is a pointed polyhedron. Letting Q stand for $\mathcal{C}(P, \mathcal{L}^*)$ in Lemma 48, we can conclude the existence of finite sets of split sets $\mathcal{S}_1 \subseteq \text{Proj}_1(\mathcal{L})$ and $\mathcal{S}_2 \subseteq \text{Proj}_2(\mathcal{L})$ such that all cross cuts obtained from cross disjunctions (S_1, S_2) with $S_1 \notin \mathcal{S}_1$ and $S_2 \notin \mathcal{S}_2$ are valid for $\mathcal{C}(P, \mathcal{L}^*)$. Therefore, letting $(S_1, *) = \{S_2 : (S_1, S_2) \in \mathcal{L}\}$ and $(*, S_2) = \{S_1 : (S_1, S_2) \in \mathcal{L}\}$ for any split sets S_1 and S_2 , we have

$$\begin{aligned} \mathcal{C}(P, \mathcal{L}) &= \bigcap_{(S_1, S_2) \in \mathcal{L}} \text{conv}(P \setminus (S_1 \cup S_2)) = \\ \mathcal{C}(P, \mathcal{L}^*) \cap \bigcap_{S_1 \in \mathcal{S}_1, S_2 \in (S_1, *)} \text{conv}(P \setminus (S_1 \cup S_2)) &\cap \bigcap_{S_2 \in \mathcal{S}_2, S_1 \in (*, S_2)} \text{conv}(P \setminus (S_1 \cup S_2)). \end{aligned}$$

For any split set $S_1 \in \text{Proj}_1 \mathcal{L}$, the set $P \setminus S_1$ is a union of two pointed rational polyhedra (possibly empty), and therefore Theorem 25 implies that $\text{SC}(P \setminus S_1, (S_1, *))$ is finitely generated. Furthermore,

$$\bigcap_{S_1 \in \mathcal{S}_1, S_2 \in (S_1, *)} \text{conv}(P \setminus (S_1 \cup S_2)) = \bigcap_{S_1 \in \mathcal{S}_1} \bigcap_{S_2 \in (S_1, *)} \text{conv}((P \setminus S_1) \setminus S_2) = \bigcap_{S_1 \in \mathcal{S}_1} \text{SC}(P \setminus S_1, (S_1, *)),$$

and is therefore finitely generated. Therefore, we conclude that $\mathcal{C}(P, \mathcal{L})$ is finitely generated and, by Lemma 44, is a polyhedron. ■

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