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About This Document

This resource contains curriculum for the distance education version of a course offered at the Georgia Institute of Technology, Math 1502, in Fall 2014. This distance education course explored linear algebra, infinite series, and differential equation concepts during lectures and recitations. Recitations are synchronous sessions that offer students an opportunity to apply and review course concepts, which they have been exposed to in lectures.

Contained in this curriculum are materials for 26 recitations, available in PDF and presentation slide formats. The slide format is offered for teaching assistants to import directly into web-conferencing software. Slides contain activities that students would solve during recitations. The associated notes contain solutions to corresponding activities and are available in PDF format.

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An similar version of this work, that corresponds to activities conducted in the Fall 2013 semester, is available through SMARTech at http://smartech.gatech.edu/handle/1853/51343

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For Further Information

Questions regarding this document can be directed to Greg Mayer (gsmayer@gmail.com), who would be happy to hear your suggestions on how to improve this document.

List of Topics

The following table presents a list of topics that were explored in the recitation activities. Numbers in brackets correspond to section numbers in the course textbook (Lay, D., Linear Algebra and its Applications, Fourth Edition).

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Welcome to Your Distance Calculus Recitation!

We’ll get started at 8:05.

While we are waiting, see if you can use the chat window (bottom right) to join the discussion.

Today:
- Introduction to Adobe Connect
- What are Recitations?
- Icebreaker
- Numerical integration (8.7)

If you can’t hear the TA, click the speaker icon.
Adobe Connect
Microphones, Webcams, Tablets

We can loan you a wacom bamboo tablet, if you’d like to borrow one please send me an email.

If you have a mic or a webcam, you are welcome to use them.
The Whiteboard

- You need to be a presenter to write on the board
- Only a host can change permission levels
- All writing on the board is anonymous
- Please respect other students taking this course (and your TA): you are responsible for your learning
Logging Into Adobe Connect for Recitations

Thursday’s recitation is at
https://georgiatech.adobeconnect.com/math1502-08-19-14
Adobe Connect Technical Problems?

You can:

• reload your browser
• log in/out
• use a different web browser
• reboot
• get help from another student and/or your TA

I strongly recommend that, if possible, you use a wired connection.
What are Recitations?

- Our goal: help students understand course material so that they can complete assignments and prepare for quizzes and exams.

- please bring questions about the homework or lectures
Our Section in a Nutshell

• students in Math 1502 are divided into many sections
• ours is the only section that
  • doesn’t have on-campus students
  • uses Connect for recitations
• Why Adobe Connect?
  • it’s cheaper
  • you can interact with students at other schools
Tablets

• Students in our section can borrow tablets.
• If you already have a tablet you want to use, that's ok
• Equipment need to be returned to your facilitator
• If you don’t have a tablet and want to borrow one, email me
• Tablets (should) come with a CD, use it to configure tablet settings
Course Websites

• Recordings of recitations and lectures: tegrity.gatech.edu
• Discussion forum: piazza.com
• Live lectures: gtcourses.gatech.edu
• Textbook and homework: www.mymathlab.com

• First homework due _______________
## Grading Weights

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Grades will be made available through T-Square
Your TA: Greg

- email: greg.mayer@gatech.edu
- Text me or call me at: __________
- Skype: greg.s.mayer

- Canadian, eh
- alors mon français est tres mauvais
- moved to the US ~2 years ago
- post-doctoral fellow at GT
Icebreaker

Everyone:
• type/say your name,
• one thing about yourself,
• place a dot on the map that approximates your current location.
Recitation 02

Recitations run from 8:05 – 8:55.

Today:
• Improper integrals
• Recitation logistics
• Numerical integration (8.7)

Questions about the course, homework?

While we are waiting to start, calculate:

a) the integral of $\frac{1}{x^2}$ from 1 to infinity
b) the integral of $e^{-1}$ from 1 to infinity
Improper Integrals

\[ \int_{1}^{\infty} \frac{1}{x^2} \, dx \]

\[ \int_{1}^{\infty} e^{-x} \, dx \]
Logistics

HW1
send me questions: greg.mayer@gatech.edu, or skype __________

Chat Pod
• You can send private messages to students
• If we have a Q&A pod, do we need a chat pod?
• I’d like group discussion in chat pod to be:
  i. positive
  ii. respectful of others

Are there any other conditions that you would like to add?
Example: Integrate $\frac{1}{x}$ from 1 to 2

a) What is the exact answer?

b) Set up but don’t evaluate an expression for the area using Simpson’s Rule.
Example: Integrate $1/x$ from 1 to 2

c) Find the number of subintervals required for four digit accuracy using Simpson’s rule.
Today:
1. Improper integrals: comparison test
2. A few announcements
3. Improper integrals: techniques of integration

Use the comparison test to determine whether the following integral converges.

\[ \int_{1}^{\infty} \frac{1}{\sqrt{1 + x^2}} \, dx \]
Announcements

HW2
• due tomorrow
• send me questions:

Invitation to Participate in a Study
• you’re receiving snail mail soon!
• please review forms with your parents and send them back

Online Survey
• anonymous!
• available at:
• please provide feedback on webcams, collaboration, the chat pod, etc

Chat Pod
• I’d like group discussion in chat pod to be:
  i. positive
  ii. respectful of others
Greg recommends:
1) one person volunteer to share whiteboard
2) discuss how to start question 4 & draw on board
3) solve question 4
4) proceed to question 5
Group Work

when you are in a breakout room:

Greg recommends:
1) one person volunteer to share whiteboard
2) discuss how to start question 4 & draw on board
3) solve question 4
4) proceed to question 5

you can change slides!
Group Work Questions

Evaluate the following integrals.

4) \( \int_{5}^{\infty} \frac{1}{x^2 + 25} \, dx \)

5) \( \int_{0}^{\frac{144}{4}} \frac{1}{\sqrt{144 - x}} \, dx \)

6) \( \int_{0}^{\frac{12}{4}} \frac{1}{\sqrt{144 - x^2}} \, dx \)

7) \( \int_{0}^{\infty} \frac{1}{x^2 + 7x + 6} \, dx \)

8) \( \int_{-\infty}^{\infty} \frac{Ax}{(x^2 + B)^{12}} \, dx \)

9) \( \int x^3 \ln x \, dx \)
Today:
1. 1\textsuperscript{st} order linear DE
2. A few announcements
3. Group Work: Separable & Linear DEs

Consider:

\[ xy' - y = 2x \ln x \]

a) Is the DE separable?
b) Is the DE 1\textsuperscript{st} order linear in \( y(t) \)?
c) What is the integrating factor?
Solve the differential equation:

\[ xy' - y = 2x \ln x \]
Group Work

when you are in a breakout room:

Greg recommends:
1) saying hello :D
2) decide who will activate whiteboard/document
3) discuss which question you want to start on
For each of the following DE's,

a) State whether the DE is separable.
b) State whether the DE is 1st order linear in y.
c) Solve the DE.

1) $xy' + 3y = 3 - \frac{4}{x}$

2) $xy' + y = (1 + x)e^x$

3) $\frac{1}{x}y' = ye^{x^2} + 2\sqrt{ye^{x^2}}$

4) $(\sin t)y' + (\cos t)y = \tan t$

5) $\cos(y) + (1 + e^{-x})(\sin y)y' = 0$
For each of the following DE’s,
a) State whether the DE is separable.
b) State whether the DE is 1\textsuperscript{st} order linear in y.
c) Solve the DE.

1) \( xy' + 3y = 3 - \frac{4}{x} \)
For each of the following DE’s,

a) State whether the DE is separable.
b) State whether the DE is 1\textsuperscript{st} order linear in y.
c) Solve the DE.

2) \( xy' + y = (1 + x)e^x \)
For each of the following DE’s,

a) State whether the DE is separable.
b) State whether the DE is 1st order linear in y.
c) Solve the DE.

3) \( \frac{1}{x} y' = ye^{x^2} + 2\sqrt{y}e^{x^2} \)
Group Work Questions

For each of the following DE’s,
a) State whether the DE is separable.
b) State whether the DE is 1st order linear in y.
c) Solve the DE.

4) \((\sin t)y' + (\cos t)y = \tan t\)
For each of the following DE’s, 

a) State whether the DE is separable.

b) State whether the DE is 1st order linear in y.

c) Solve the DE.

5) \( \cos(y) + (1 + e^{-x})(\sin y) y' = 0 \)
Today:
1. Geometric Series
2. Group Work: Infinite Series

The sum of a geometric series is: \[ \sum_{k=1}^{\infty} ar^{k-1} = \]

The geometric series is convergent iff \[ \text{___________} \].
Express 1.7979797979 ... as a rational number.
Express $1.7979797979 \ldots$ as a rational number.
Input interpretation:
1.79797979797979

Rational approximation:
\[
\frac{178}{99} = 1 + \frac{79}{99}
\]
Determine if the following series convergent or divergent. If it is convergent, find its sum.

\[ \sum_{k=1}^{\infty} \frac{4^{k+1}}{5^k} \]
Group Work

When you are in a breakout room:

Greg recommends:
1) saying hello :D
2) advance to the next slide with arrows located bottom left

Group Work Questions

1) Express 1.1232323 … as a ratio of integers.

2) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

A) \( \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} \)

B) \( \sum_{k=1}^{\infty} 2^{2k} 3^{1-k} \)

C) \( \sum_{k=1}^{\infty} \frac{(-5)^k}{4^{k+1}} \)

D) \( \sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)} \)
Group Work Questions

1) Express 0.301301301 …. as a ratio of integers.

2) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

A) \[ \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} \]

B) \[ \sum_{k=1}^{\infty} 2^{2k} \cdot 3^{1-k} \]

C) \[ \sum_{k=1}^{\infty} \frac{(-1)^k}{4^{k+1}} \]

D) \[ \sum_{k=1}^{\infty} \frac{1}{(3k - 2)(3k + 1)} \]

3) Express 1.1232323 …. as a ratio of integers.
Group Work Questions

1) Express 0.301301301 … as a ratio of integers.
Group Work Questions

Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ A) \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} \]
Group Work Questions

Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ B) \sum_{k=1}^{\infty} 2^{2k} 3^{1-k} \]
Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ C) \sum_{k=1}^{\infty} \frac{(-1)^k}{4^{k+1}} \]
Group Work Questions

Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ D) \sum_{k=1}^{\infty} \frac{1}{(3k - 2)(3k + 1)} \]
Group Work Questions

3) Express 1.1232323 .... as a ratio of integers.
Recitation 06

Today:
1. Test for Divergence
2. Group Work: Infinite Series, Differential Equations

The Test for Divergence (Theorem 7 from 10.2)

If \( \lim_{k \to \infty} a_k \) is not zero or does not exist, then \( \sum_{k=1}^{\infty} a_k \) diverges.

Note that:

If \( \lim_{k \to \infty} a_k \) is equal to zero, then ______________

Use the test for divergence to determine whether these series converge.

\[
A) \sum_{k=1}^{\infty} \frac{1}{k + 2}
\]

\[
B) \sum_{k=1}^{\infty} \frac{k}{k + 2}
\]
Group Work

When you are in a breakout room:

Greg recommends:
1) Saying hello :D
2) Each student write & chat in their favourite color
3) Advance to the next slide with arrows located bottom left

Group Work Questions

1) Express 1.1232323 ... as a ratio of integers.

2) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

   A) \[ \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} \]
   B) \[ \sum_{k=1}^{\infty} 2^{2k} \cdot 3^{-k} \]
   C) \[ \sum_{k=1}^{\infty} \frac{(-5)^k}{4^{k+1}} \]
   D) \[ \sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)} \]
Group Work Questions

1) Express 0.002100210021 … as a ratio of integers.

2) Solve the following DE for $x > 0$: $x^3 y' + (2 - 3x^2)y = x^3$

3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ A) \sum_{k=1}^{\infty} \frac{k^2}{5k^2 + 5} \]
\[ B) \sum_{k=1}^{\infty} \frac{1}{k(k+3)} \]
\[ C) \sum_{k=1}^{\infty} 3(0.4)^k - 2(-0.1)^{k+1} \]
\[ D) \sum_{k=1}^{\infty} \left( \sin \left( \frac{1}{k} \right) - \sin \left( \frac{1}{k+1} \right) \right) \]
\[ E) \sum_{k=1}^{\infty} \arctan(k) \]

4) Consider the DE $y' + y = y^2 e^x$
   a) Determine if the DE is separable, and/or 1st order linear in $y$.
   b) Solve the DE for $x > 0$. *Hint: let $z = 1/y$.***
Group Work Questions

1) Express 0.002100210021 …. as a ratio of integers.
2) Solve the following DE for $x > 0$: $x^3 y' + (2 - 3x^2) y = x^3$
3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ A) \sum_{k=1}^{\infty} \frac{k^2}{5k^2 + 5} \]
3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ B) \sum_{k=1}^{\infty} \frac{1}{k(k+3)} \]
3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ C) \sum_{k=1}^{\infty} 3(0.4)^k - 2(-0.1)^{k+1} \]
3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ D) \sum_{k=1}^{\infty} \left( \sin \left( \frac{1}{k} \right) - \sin \left( \frac{1}{k+1} \right) \right) \]
3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ E \sum_{k=1}^{\infty} \arctan(k) \]
4) Consider the DE $y' + y = y^2 e^x$
   a) Determine if the DE is separable, and/or 1\textsuperscript{st} order linear in $y$.
   b) Solve the DE for $x > 0$. \textit{Hint: let $z = 1/y$.}
Recitation 07

Today:
1. Announcements
2. Quiz Review

For what values of \( x \) does the following converge? Why?

\[ \sum_{k=1}^{\infty} x^{k-1} = 1 + x + x^2 + x^3 + \ldots \]

When the series converges, what is the series equal to?

Express the following as an infinite series.

\[ \frac{1}{1 - x^4} \]
How Quizzes Work

1. Facilitator gets a copy of quiz
2. You write quiz on Thursday, during recitation (8:05 – 8:55)
3. Quiz is proctored (perhaps by your facilitator)
4. Give your completed quiz to your proctor.

Format of Quiz

• Are calculators allowed?
• Are textbooks and notes allowed?
• Quiz length: probably three pages, 1 question per page, questions can have multiple parts
• Don’t write on back of pages
• Leave a 1 inch margin around edges of page
• If run out of space, you can use extra pages
How Quizzes Work

If You Have Questions During the Quiz

1. If your proctor agrees, you can use Adobe Connect to ask your TA questions (you’d need to be closely proctored)
2. Otherwise: call/text me at _____________

Before The Quiz

Find your facilitator, and discuss
• where you are writing the quiz, and
• whether you are connecting via Adobe Connect.

What Happens After The Quiz

1. Your facilitator scans/emails your quiz to GT,
2. they get graded in about a week or two,
3. your TA enters grades in T-Square,
4. your graded quiz is returned via email to you.
Group Work

When you are in a breakout room:

Greg recommends:
1. Saying hello :D
2. Each student write & chat in their favourite color
3. Advance to the next slide with arrows located bottom left

Group Work Questions

1) Express 1.1232323 ..., as a ratio of integers.

2) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

   A) \[ \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} \]

   B) \[ \sum_{k=1}^{\infty} 2^{2k} 3^{1-k} \]

   C) \[ \sum_{k=1}^{\infty} \frac{(-5)^k}{4^{k+1}} \]

   D) \[ \sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)} \]

   advance slides
1) Solve \( xy' + 2y = \cos(x) / x \), with \( y(\pi) = 1 \).
2) Find the sum of the following series.

\[ \sum_{k=3}^{\infty} \frac{(-2)^{k+2}}{5^{k-1}} \]
3) Find the number of intervals, \( N \), needed to ensure an accuracy of at least 0.01 using Simpson’s Rule for the following integral.

\[ \int_{1}^{e} \ln x \, dx \]
4) Determine whether the following converge. Do not evaluate these integrals.

\[
A) \int_{e}^{\infty} \frac{1}{x \ln x} \, dx
\]

\[
B) \int_{0}^{\infty} \cos(x)\sin(x)e^{-2x} \, dx
\]
5) Express as an infinite series. Assume $|x| < 1$. 

\[ \frac{x}{1 + x^2} \]
6) Find the sum of the following series.

\[
\sum_{k=0}^{\infty} \frac{5}{(k+1)(k+5)}
\]
7) Determine whether the following series converges.

\[
A) \sum_{k=1}^{\infty} \frac{k^2 + 3}{k3^k}
\]
Recitation 09

Today: 10.7 (Power Series, Radius/Interval of Convergence and Absolute Convergence)

A) The interval of convergence is _____________________

B) If an infinite series $\sum a_n$ is called absolutely convergent if ___________

C) If a series is absolutely convergent, then it is ______________

D) A series is called conditionally convergent if it is ______________________

Given the series below, determine
1) the radius and interval of convergence
2) values of $x$ where series is absolutely convergent
3) values of $x$ where the series is conditionally convergent

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k}$$

Quiz 1 will be graded on Tuesday afternoon, grades entered Wednesday.
How can recitations be improved to better prepare you for quizzes?
Given the series below, determine
1) the radius and interval of convergence
2) values of x where series is absolutely convergent
3) values of x where the series is conditionally convergent

\[ \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k} \]
Given the series below, determine
1) the radius and interval of convergence
2) values of x where series is absolutely convergent
3) values of x where the series is conditionally convergent

\[ \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k} \]
Given the series below, determine

1) the radius and interval of convergence
2) values of x where series is absolutely convergent
3) values of x where the series is conditionally convergent

\[ \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k} \]
1) Find the radius and interval of convergence.

\[ \sum_{k=1}^{\infty} \frac{x^k}{k} \]
2) Find the radius and interval of convergence.

\[ \sum_{k=1}^{\infty} (-k)^{4k} x^{4k} \]
3) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

\[ a) \sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2 + 1} \]

\[ b) \sum_{k=1}^{\infty} \frac{\cos(\pi k)}{k} \]
4) Find the radius and interval of convergence. Then find the values of $x$ for which series is

a) absolutely convergent  
b) conditionally convergent  
c) divergent.

\[
1 - \frac{x}{2} + \frac{2x^2}{4} - \frac{3x^3}{8} + \frac{4x^4}{16} - \ldots
\]
Complete the Following Formulas

The $N^{th}$ order Taylor Polynomial about $x = a$ is:

$$P_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The Taylor Series about $x = a$ is:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The phrase “about $x = a$” means that ________________________________

Remainder formula for expansion about $x = a$:

$$|R_n(x)| \leq \left( \max_{t \in [0,x]} |f^{(n+1)}(t)| \right) \frac{|x-a|^{n+1}}{(n+1)!}$$

Quiz 1 has been graded, grades are in T-Square, you’ll receive your graded quizzes this week via email, please make sure they were graded correctly and let me know if you have questions.
Example
a) Plot a rough sketch of \( f(x) = 1/(1 - x)^2 \) for \( x \) between -1 and + 1.

b) Use the Taylor expansion for \( 1/(1 - x) \) to find \( P_0(x) \), \( P_1(x) \), and \( P_2(x) \) of \( f(x) \) about \( x = 0 \).

Hint: the Taylor expansion of \( 1 / (1 - x) \) about \( x = 0 \) is \( \frac{1}{1 - x} = 1 + x + x^2 + x^3 + ... \)
Example

a) Plot a rough sketch of \( f(x) = \frac{1}{(1 - x)^2} \) for \( x \) between -1 and +1.

b) Use the Taylor expansion for \( \frac{1}{1 - x} \) to find \( P_0(x) \), \( P_1(x) \), and \( P_2(x) \) of \( f(x) \) about \( x = 0 \).

Hint: the Taylor expansion of \( \frac{1}{1 - x} \) about \( x = 0 \) is

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots
\]
Group Work Suggestions

• everyone pick a color to write in, and match it with their text
• use mics if you have them
• before moving to the next question:
  • ask if everyone agrees with answer
  • ask if everyone understands how to get the answer
• message me if your group gets stuck

Make sure you can solve Question #4 before the next quiz.
1) Find the Taylor expansion about \( x = 0 \) of \( f(x) = \frac{2x}{1 + x^2} \)

Hint: write the Taylor expansion for \( 1/(1-x) \) at \( x = 0 \), and then apply suitable modifications to the expansion.
2) Fill in the blank: the Maclaurin series is just the Taylor series about the point ________.

The Maclaurin series for $e^x$ is

$$e^x = \sum_{k=0}^{\infty} \frac{(e^x)^{(k)}(0)}{k!} x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Use this expansion to find the Maclaurin series for $f(x)$, and $P_2(x)$ about $x = 0$, where:

$$f(x) = e^{-x^2} = \exp(-x^2)$$
3) Estimate the following integral to within 0.01 using series. 
\textit{Hint: use the Alternating Series Remainder Theorem.}
\[ \int_{0}^{1} x^4 e^{-x^2} \, dx \]
4) Estimate $e^{3/2}$ to within $10^{-4}$.
Note: this is a question from a Math 1502 quiz from 2012.
Hint: use the Taylor Series Remainder Theorem, and approximate $e^{3/2}$ with 5.
5) Let \( f(x) = \frac{\exp(2x^2) - 1 - x^2}{x^4} \). Note: \( \exp(x) = e^x \)

a) Find \( P_2(x) \) about \( x = 0 \)

b) Use the result from part a) to evaluate the limit of \( f(x) \) as \( x \) goes to zero.
Recitation 11

Today: 4.5 (l’Hospital’s Rule)

Describe how you would evaluate the following limits.

\[ \lim_{x \to a} \frac{f(x)}{g(x)} \] is of the form 0/0 or ∞/∞, then we can:

\[ \lim_{x \to a} f(x) - g(x) \] is of the form ∞ − ∞, then we can:

\[ \lim_{x \to a} f(x)^{g(x)} \] is of the form 0^0, 1^∞, or ∞^0, then we can:

Quiz 1 has been graded, grades are in T-Square, you’ll receive your graded quizzes this week via email, please make sure they were graded correctly and let me know if you have questions.
POP QUIZ #1

- Start time: 8:10?
- Ends at: 8:25
- Pop quiz grading
  - 5 points: correct
  - 4 points: something correct
  - 3 points: name on the page
  - 0 points: did not take pop quiz

- To submit your work, choose any of the following:

  A. **work on whiteboard in breakout room**
     - type A in text chat so I know you want to work in breakout room
     - submit work by letting me know when done, and/or email me a screen capture of your work

  B. **work on paper and give work to facilitator**
     - type B in text chat so I know you’re doing this
     - leave 2 inch margin on paper
     - write your name and QH8 at the top
     - facilitator is receiving instructions today on how to submit your work

  C. **work on paper and email a photo of your work to me**
     - type C in text chat so I know you are emailing your work to me
     - email your photo to me before 8:40
Find the Taylor series, centered at $a = 0$, of $\sin(5x^2)$. 
Evaluate: \[ \lim_{x \to \infty} \left( \cos \frac{1}{x} \right)^x \]
Group Work Suggestions

• everyone pick a color to write in, and match it with their text
• use mics if you have them
• before moving to the next question:
  • ask if everyone agrees with answer
  • ask if everyone understands how to get the answer
• message me if your group gets stuck
1) Evaluate the following limit by using l'Hospital's rule, if possible.

Note: this limit is a well-known definition for an important number.

$$\lim_{x \to 1} x^{1/(x-1)}$$
2) Evaluate the following limit by using l'Hospital's rule, if possible.

\[
\lim_{x \to \infty} x \sin \left( \frac{\pi}{x} \right)
\]
3) Evaluate the following limit by using l'Hospital's rule, if possible.

$$\lim_{{x \to 0^+}} x(\ln x)^2$$
4) Evaluate the following limit by using l'Hospital's rule, if possible. *Hint: only one of these limits exists.*

a) \( \lim_{x \to \infty} \frac{1}{x} \int_{0}^{x} e^{t^{2}} \, dt \)

b) \( \lim_{x \to 0} \left( \frac{1}{\ln(1 + x)} - \frac{1}{x} \right) \)
5) a) \( \vec{v} \cdot \vec{v} = \|\vec{v}\|^2 \), is the ______________ of vector \( \vec{v} \).

b) A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.
Suppose we have the points P(1,2,3), Q(1,3,4), R(2,2,2).

a) Find a vector that is normal to the plane that contains the three points.
b) Find an equation of the plane.

2) The equation $Ax + By + Cz = D$ is a plane. A vector perpendicular to the plane is:
Find a parametrization of the line that is the intersection of the planes

P: \( x - 2y + z = 3 \)
Q: \( 2x + y + z = 1 \)

Note: this is a question from a 2012 Math 1502 quiz.
Group Work Suggestions

• everyone pick a color to write in, and match it with their text

• use mics if you have them

• before moving to the next question:
  • ask if everyone agrees with answer
  • ask if everyone understands how to get the answer

• message me if your group gets stuck
1) Find the equation for the line that is perpendicular to the yz-plane, and also passes through P(1,4,3).
2) Do these lines intersect each other? Why/why not?

\begin{align*}
  x_1 &= 1 + t & x_2 &= 1 - u \\
  y_1 &= -1 - t & y_2 &= 1 + 3u \\
  z_1 &= -4 + 2t & z_2 &= -2u
\end{align*}
3) a) If $\mathbf{a} \times \mathbf{b} = 0$ and $\mathbf{a} \cdot \mathbf{b} = 0$, what can we conclude about vectors $\mathbf{a}$ and $\mathbf{b}$? Explain your reasoning.

b) Which of the following make sense? Explain why/why not.

i. $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$

ii. $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$

iii. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

iv. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
4) Vectors that are co-planar are in the same plane. Determine whether the vectors are co-planar:

\[ P: \quad j - k \]
\[ Q: \quad 3i - j + 2k \]
\[ R: \quad 3i - 2j + 3k \]
5) a) \( \vec{v} \cdot \vec{v} = \left\| \vec{v} \right\|^2 \), is the ______________ of vector \( \vec{v} \).

b) A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.
Recitation 13

Today: Quiz 2 Review

Estimate to within 0.0001 with a Taylor Polynomial.

\[ \int_{0}^{1/2} \frac{\ln(1 + x)}{x} \, dx \]
Group Work Suggestions

• before moving to the next question:
  • ask if everyone agrees with answer
  • ask if everyone understands how to get the answer

• everyone pick a color to write in, and match it with their text

• use mics if you have them

• message me if your group gets stuck
1) Evaluate by a) using power series, and b) l’Hôpital’s rule.

\[
\lim_{x \to 0} \frac{\cos(x) - 1}{x \sin x}
\]
2) Find the interval of convergence.

\[ \sum_{k=1}^{\infty} \frac{\ln k}{k} (x + 1)^k \]
3) Find the distance between planes P1 and P2.

P1:   \( x + 2y + z = 3 \)
P2:   \( x + 2y + z = 9 \)
4) The line $L$ is determined from $P_1$ and $P_2$. The plane $Q$ is determined by $Q_1$, $Q_2$, $Q_3$. Does $L$ intersect $Q$? If so, where?

$P_1(1,-1,2)$
$P_2(-2,3,1)$
$Q_1(2,0,-4)$
$Q_2(1,2,3)$
$Q_3(-1,2,1)$
5) From 2012 Quiz 1

Tyler Hamilton would like you to find

a) A series for $\int_0^x \sin(\pi^2/2) \, dt$
Greg Lemond wants you to use series and error bounds to estimate \( e^{7/2} \) to within \( 10^{-3} \). You must use (an) error bound. You do not have to actually sum, just say how many terms, (or the highest power of \( 7/2 \)). (You may use \( e^{(7/2)} \leq 35 \)).
7) Vectors that are co-planar are in the same plane. Determine whether the vectors are co-planar:

P: \( \mathbf{i} + \mathbf{j} - \mathbf{k} \)

Q: \( 2\mathbf{i} - \mathbf{j} \)

R: \( 3\mathbf{i} - \mathbf{j} - \mathbf{k} \)
8) Find a parametrization of the line that is the intersection of the planes

\[
\begin{align*}
P: & \quad x + y + z + 1 = 0 \\
Q: & \quad x - y + z + 2 = 0
\end{align*}
\]

Note: this is a modified version of a question from a 2012 Math 1502 quiz.
9) a) \( \vec{v} \cdot \vec{v} = \| \vec{v} \|^2 \), is the ________________ of vector \( \mathbf{v} \).

b) A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.
Recitation 15

Today: Solving linear systems of equations

The following was a 2013 pop quiz question. For what values of $a$ does the following system have a solution?

\[
\begin{align*}
7x + 2y - 3z &= 25 \\
y + 3z &= 5 \\
3y + az &= 3
\end{align*}
\]

Please take a few minutes to fill out the technical issues survey. There are 30 recitations in the semester: we’re ~50% through Math 1502!
A Few Definitions

a) A system of linear equations is **consistent** if it has ________________________ .

b) A system of linear equations is **inconsistent** if it has ________________________ .

c) A system of linear equations that is overdetermined has ________________________ .

d) Can a system of linear equations be overdetermined and consistent? If yes, provide an example with at least 3 equations.
3) Find \( h \) and \( k \) such that the system has
   a) no sol’n
   b) a unique sol’n
   c) infinitely many solutions

\[
\begin{align*}
  x_1 + hx_2 &= 2 \\
  4x_1 + 8x_2 &= k
\end{align*}
\]
4) A 3 x 4 coefficient matrix has three pivot columns. Is the system consistent? Why/why not?

5) Find the general solution to the system whose augmented matrix is given below.
6) True or false.

a) The reduced echelon form of a matrix is unique.

b) If a system has free variables, the solution set has many solutions.

c) If a row in an echelon form of an augmented matrix is \[ \begin{bmatrix} 0 & 0 & 0 & 3 & 0 \end{bmatrix} \], then the linear system is inconsistent.
Recitation 16

Today: Span, Linear Dependence

Definitions

Assume that $v_1$ and $v_2$ are arbitrary vectors.

a) The sum $c_1v_1 + c_2v_2$ is a ______________ combination of vectors $v_1$ and $v_2$.

b) The set of all possible ________ combinations of $v_1$ and $v_2$ is the ________ of $v_1$ and $v_2$.

c) Any vector in the __________ of $v_1$ and $v_2$ can be written as a ______________________

of vectors $v_1$ and $v_2$.

Question 1

If you haven’t already, please take a few minutes to fill out the technical issues survey.
No lectures/recitations next Monday/Tuesday.
1) This is similar to the first question on your next HW.
Linear Dependence

Vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3, ..., \vec{v}_N \) are linearly dependent (LD) if \( \exists \ c_1, c_2, c_3, ..., c_N \) not all \( \underline{\text{__________}} \), such that

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + ... + c_N \vec{v}_N = \vec{0}
\]

If the vectors are not LD, they are \( \underline{\text{________________________}} \).

Example:

To determine whether a set of vectors are \( \underline{\text{____}} \), we solve:

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + ... + c_N \vec{v}_N = \vec{0}
\]

which has the same solution as the linear system whose augmented matrix is \( \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & ... & \vec{v}_N & \vec{0} \end{bmatrix} \).
2) Determine whether the following vectors are LI.

\[
\begin{bmatrix}
5 \\
1
\end{bmatrix},
\begin{bmatrix}
2 \\
8
\end{bmatrix},
\begin{bmatrix}
1 \\
3
\end{bmatrix},
\begin{bmatrix}
-1 \\
7
\end{bmatrix}
\]
Group Work Suggestions

• before moving to the next question:
  • ask if everyone agrees with answer
  • ask if everyone understands how to get the answer

• everyone pick a color to write in, and match it with their text

• use mics if you have them

• message me if your group gets stuck
3) This is similar to the second question on your next HW.

Determine whether vector $b$ is in the set spanned by the columns of matrix $A$.

\[
A = \begin{pmatrix}
1 & 0 & 5 \\
-2 & 1 & -6 \\
0 & 2 & 8
\end{pmatrix},
\quad
b = \begin{pmatrix}
2 \\
-1 \\
6
\end{pmatrix}
\]

Hint: let the columns of $A$ be $a_1, a_2, a_3$. If $b$ is in the set spanned by these 3 vectors, there is a linear combination of $a_1, a_2, a_3$ that equals $b$. 
4) This is similar to the third question on your next HW.

Determine whether vector $b$ can be written as a linear combination of vectors $a_1$ and $a_2$. In other words, determine whether $x_1$ and $x_2$ exist such that $x_1a_1 + x_2a_2 = b$. If possible, find $x_1$ and $x_2$.

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$
5) Determine whether the following vectors are LI.

\[
\begin{bmatrix}
5 \\
-3 \\
-1
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
-7 \\
2 \\
4
\end{bmatrix}
\]
6) Find values of \( h \) so that the following vectors are LD.

\[
\begin{bmatrix}
2 \\
-2 \\
4
\end{bmatrix}, \quad \begin{bmatrix}
4 \\
-6 \\
7
\end{bmatrix}, \quad \begin{bmatrix}
-2 \\
2 \\
h
\end{bmatrix}
\]
Recitation 18

Today: Span, Linear Dependence, Linear Transforms

Quiz 3 Next Thursday
Make sure you can solve these questions from old quizzes:
2012: Quiz 2 #2 and #3
2012: Quiz 3 #1 and #2
2013: Quiz 3 #1 and #3
We’ll solve some of these in Tuesday’s recitation.

QH8 Office Hours Next Week
Tuesday and Wednesday 7:30 to 8:30 pm
At the same place as last time:

Online Drop-in Tutoring
Wednesdays, 5:30 to 7:00 pm
For all ~450 distance calculus students
Facilitated by Greg, who will answer questions and review problems from QH8 recitations

If you haven’t already, please take a few minutes to fill out the technical issues survey.
1) This is similar to the second and third questions on the transforms HW.

Let \( \vec{u}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \) and \( T \) be a linear transformation that maps \( \vec{u}_1 \) onto \( \vec{v}_1 \), and \( \vec{u}_2 \) onto \( \vec{v}_2 \). Find \( T \) and the image of \( \begin{bmatrix} 3 \\ 0 \end{bmatrix} \) under \( T \).
2) Plot \( u \) and \( v \), their images under \( T \), and provide a geometric interpretation of what \( T \) does to vectors in \( \mathbb{R}^2 \).

\[
\begin{align*}
\mathbf{u} &= \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \\
\end{align*}
\]

\( a) \quad T(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \)

\( b) \quad T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \)
3) From 2012 Quiz 2

Let \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( B = \begin{pmatrix} \cos\left(\frac{-\pi}{3}\right) & -\sin\left(\frac{-\pi}{3}\right) \\ \sin\left(\frac{-\pi}{3}\right) & \cos\left(\frac{-\pi}{3}\right) \end{pmatrix} \), \( C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \). Compute the image of the house under the transformation \( ABC \). Show the intermediate steps.
4) Fill in the elements of the 3x3 matrix.
*Hint: the elements can be identified by inspection.*

\[
\begin{bmatrix}
  x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
  2x_1 - 6x_3 \\
x_3 - 7x_1 \\
-x_2 - 5x_3
\end{bmatrix}
\]
5) Find values of $h$ so that the vectors are LD, and values of $h$ so that the vectors are LI.

\[
\begin{bmatrix}
3 \\
-6 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
-6 \\
4 \\
-3
\end{bmatrix}, \quad \begin{bmatrix}
9 \\
h \\
3
\end{bmatrix}
\]

If __________ , then the vectors are LD.

If __________ , then the vectors are LI.
Recitation 19

Today: Quiz 3 Review

Quiz 3 Next Thursday
Make sure you can solve these questions from old quizzes:
- 2012: Quiz 2 #2 and #3
- 2012: Quiz 3 #2
- 2013: Quiz 3 #1 and #3

Review recent HWs on Span, Lin Transforms, Gauss Jordan
Review sections 1.1, 1.2, 1.3, 1.7, 1.8, 1.9

QH8 Office Hours Next Week
Tuesday and Wednesday 7:30 to 8:30 pm
We'll solve practice quiz problems & go over specific areas you’d like to review.
At the same place as last time:

Online Drop-in Tutoring
Wednesdays, 5:30 to 7:00 pm
For all ~450 distance calculus students
Facilitated by Greg, who will answer questions and review problems from QH8 recitations

Adobe 9.3: What’s New?
- new Adobe Connect Add-in (not needed, I hope)
- fewer technical issues?
- new drawing tools
- participants can be given drawing powers
- writing on board is still anonymous
1) From 2013, Quiz 3, #3

If $T$ is a linear transformation consisting of rotating counterclockwise by $\frac{\pi}{3}$ radians followed by a reflection about the line $x = y$, find the matrix such that $T(x) = Ax$. 
2) From 2012, Quiz 2, #2

For what values of $b$ is $y$ a linear combination of $u$ and $v$?

$\begin{bmatrix}
-1 \\
1 \\
3 \\
\end{bmatrix} = 
\begin{bmatrix}
4 \\
2 \\
3 \\
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
b \\
\end{bmatrix}$

If __________ , then $y$ is a linear combination of $u$ and $v$.

If __________ , then $y$ is not a linear combination of $u$ and $v$. 
3) From 2013, Quiz 3, #1

If $T$ is a linear transformation, and:

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

find the matrix, $A$, such that $T(x) = Ax$. 

4) From 2012, Quiz 3, #2

For what values of $a$ are the following vectors LI?

\[
\begin{bmatrix}
1 \\
0 \\
2
\end{bmatrix}, \begin{bmatrix}
2 \\
4 \\
3
\end{bmatrix}, \begin{bmatrix}
2 \\
2 \\
a
\end{bmatrix}
\]

If ___________ , then the vectors are LD.

If ___________ , then the vectors are LI.
5) State whether the following statements are true or false and explain your reasoning.

a) The columns of matrix A are LI if the equation $Ax = 0$ has the trivial solution.

   The statement is ____________ because:

b) If $S$ is a set of LI vectors, then each vector in $S$ is a linear combination of the other vectors in $S$.

   The statement is ____________ because:

c) If a set of vectors contains fewer vectors than there are entries in the vectors, the set is LI.

   The statement is ____________ because:
Recitation 21

Today: Matrix Inverses, LU Decomposition

1a) State the formula for the inverse of the matrix

\[ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad P \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad P \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \]
2) Solve the equation $Ax = b$, using the LU decomposition of $A$, where

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}, b = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$
3) Find the LU decomposition of $A$.

\[
A = \begin{bmatrix}
-5 & 0 & 4 \\
10 & 2 & -5 \\
10 & 10 & 16
\end{bmatrix}
\]
4) State whether the following statements are true or false and explain your reasoning.

a) A can be row reduced to the identity matrix iff A is invertible.
   
   The statement is ___________ because:

b) If matrix B is the inverse of matrix A, the equations AB = I and BA = I must both be true.
   
   The statement is ___________ because:

c) If A and B are both N×N and invertible, then the product A⁻¹B⁻¹ is the inverse of AB.
   
   The statement is ___________ because:

d) If an N×N matrix A is invertible, then the columns of Aᵀ are LI.
   
   The statement is ___________ because:
b) (3 points) The two vectors
\[
\begin{bmatrix}
1 \\
3 \\
1
\end{bmatrix}, \begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}
\]
span a plane passing through the origin. Find a vector that is normal to this plane.
POP QUIZ #2

• Start time: 8:10?
• Ends at: start time + 15 minutes.
• Pop quiz grading
  • 5 points: correct
  • 4 points: something correct
  • 3 points: name on the page
  • 0 points: did not take pop quiz

• To submit your work, choose any of the following:

A. work on whiteboard in breakout room
  • type A in text chat so I know you want to work in breakout room
  • submit work by letting me know when done, and/or email me a screen capture of your work

B. work on paper and give work to facilitator
  • type B in text chat so I know you’re doing this
  • leave 2 inch margin on paper
  • write your name and QH8 at the top
  • facilitator is receiving instructions today on how to submit your work

C. work on paper and email a photo of your work to me
  • type C in text chat so I know you are emailing your work to me
  • email your photo to me before: end time
Find a basis for the column space of:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 3 & 6 & 9
\end{pmatrix}
\]
2) Fill in the blanks:

a) Coll A is the set of all linear combinations of the _________ of matrix A.

b) Nul A is the set of all solutions to ______________ .

c) The _________ columns of matrix A form a basis for the column space of A.

d) The rank of matrix A is the ______________________ .

e) The nullity of matrix A is the ______________________ .
Find i) a basis for Col A, and ii) a basis for Nul A.

\[ A = \begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -3 & 0 & 6 \\ 0 & 2 & 6 & 0 & -4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
Recitation 23

Today: Column and Null Space

From 2012 Quiz 3: Find a basis for the nullspace of $A$. Also find its rank and nullity.

$$A = \begin{bmatrix} 3 & -1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2 & 4 \\ 0 & 1 & 3 & 2 & 4 \end{bmatrix}$$
**Starting This Week:**

Evidence of inappropriate behavior will be forwarded to the course instructors, and possibly also to the chair of the School of Mathematics and High school facilitators. Evidence will be reviewed to determine if further action is required. Such action could either result in

1) the Georgia Tech's Office of Undergraduate Admissions being made aware of student behavior, and/or
2) all students from a particular school moved to another section where interactions between students from different schools is not possible.

Behavior is inappropriate if it can interpreted as hurtful or disrespectful.

Students can request to be moved to another section at any time.

Questions can be directed to the students teaching assistant and/or the course instructors at any time.
2) A 2013 pop quiz question: find the coordinates of $\mathbf{b}$ with respect to $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{b} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
3) Is $\lambda = 2$ an eigenvalue of matrix $A$? Why/why not?

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$$
4) Find a basis for the eigenspace of $A$, for the eigenvalue $\lambda = -5$.

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}$$
5) Find the characteristic polynomial and eigenvalues of:

a) \( X = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} \)

b) \( Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix} \)
6) Find the LU decomposition of the following matrix (likely not enough time for this).

Hints: start by finding U with row operations, eliminating one element at a time.

\[
B = \begin{bmatrix}
3 & 1 & 2 \\
-9 & 0 & -4 \\
9 & 9 & 14
\end{bmatrix}
\]
Suppose $A$ and $B$ are square matrices.

1) If a multiple of one row of $A$ is added to another row to produce $B$, then $\det(B) = \underline{_______}$.

2) If two rows of $A$ are interchanged to produce $B$, then $\det(B) = \underline{________}$.

3) If one row of $A$ is multiplied by $K$ to produce $B$, then $\det(B) = \underline{________}$.

4) If $A$ is a triangular matrix, then $\det(A) = \underline{____________}$.

Compute $\det(A)$

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$$
Theorems from Section 3.2
Suppose A and B are square matrices.
1) A is not invertible iff det(A) = __________ .
2) det(AB) = ______________ .
3) det(A + B) = ______________ .
4) det(A^T) = ______________ .

Determine whether the following vectors are LI.
\[
\begin{bmatrix}
5 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
-3 \\
5
\end{bmatrix},
\begin{bmatrix}
-1 \\
-2 \\
3
\end{bmatrix}
\]
Section 5.3: Diagonalization

A matrix $A$ is diagonalizable if it can be written in the form:

$$A = \begin{pmatrix} \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\ \end{pmatrix}$$

where

$P$ is $\begin{pmatrix} \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\ \end{pmatrix}$

$D$ is $\begin{pmatrix} \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\ \end{pmatrix}$

Suppose $A$ is $N \times N$. To diagonalize $A$:

1. find all $\begin{pmatrix} \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\ \end{pmatrix}$ of $A$ to construct $D$

2. find $N$ ______ eigenvectors of $A$ to construct $P$

3. find $P^{-1}$ (we don’t yet have a method for finding inverse of $3 \times 3$ matrix)

4. write $A = \begin{pmatrix} \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\ \end{pmatrix}$. 
Diagonalize the following matrices, if possible.

\[ A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}; \quad B = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}; \quad C = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \]

where \( \lambda \)'s of \( C \) are 2, 2, 5
Recitation 25

Today: Quiz Review

Quiz 4 on Thursday
Review recent HWs on span, determinants, and eigenvalues
Review sections 2.8, 2.9, 3.1, 3.2, 5.1, 5.2

QH8 Office Hours
Tuesday and Wednesday 7:30 to 8:30 pm
We’ll solve practice quiz problems & go over specific areas you’d like to review.
At the same place

Online Drop-in Tutoring
Wednesdays, 5:30 to 7:00 pm
For all ~450 distance calculus students
Facilitated by Greg, who will answer questions and review problems from QH8 recitations

Question 1 (from 2013 Quiz 4)
a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

\[
A = \begin{bmatrix}
10 & 3 & -1 \\
2 & 9 & 1 \\
-2 & 3 & 11 \\
\end{bmatrix}
\]

b) Find as many LI eigenvectors for this eigenvalue as possible.
Group Work
Writing on board disappears when I enter room (sometimes). So lets try this:
• None of the breakout rooms have the questions.
• Every breakout room has a whiteboard.
• Write Question on the whiteboard when you get into the breakout room, solve it, then move to Question 2.
• You’ve got about 15 minutes.

Question 1 (from 2013 Quiz 4)
a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

\[ A = \begin{bmatrix} 10 & 3 & -1 \\ 2 & 9 & 1 \\ -2 & 3 & 11 \end{bmatrix} \]

b) Find as many LI eigenvectors for this eigenvalue as possible.

Question 2
Compute \( \det(B) \) by using row reduction.

\[ B = \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix} \]
Question 1 (from 2013 Quiz 4)

a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

b) Find as many LI eigenvectors for this eigenvalue as possible.

\[
A = \begin{bmatrix}
10 & 3 & -1 \\
2 & 9 & 1 \\
-2 & 3 & 11
\end{bmatrix}
\]
Question 2
Compute \( \text{det}(B) \) by using row reduction.

\[
B = \begin{bmatrix}
1 & 5 & -3 \\
3 & -3 & 3 \\
2 & 13 & -7
\end{bmatrix}
\]
3) State whether the following statements are true or false and explain your reasoning.

a) The dimension of Col A is the number of pivot columns of A.

*The statement is* ____________ *because:*

b) The dimension of Nul A is the number of variables in equation Ax = 0.

*The statement is* ____________ *because:*

c) If A is a square matrix, and det(A⁴) = 0, then A is not invertible.

*The statement is* ____________ *because:*
4) Find all values of $h$ so that the eigenspace for $D$, for $\lambda = 4$, is two dimensional.

$$D = \begin{bmatrix}
4 & 2 & 3 & 3 \\
0 & 2 & h & 3 \\
0 & 0 & 4 & 14 \\
0 & 0 & 0 & 2
\end{bmatrix}$$
Recitation 27

Today: Diagonalization (5.3), Orthogonality (6.1, 6.2)

Let \( v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \ v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \).

Show that these are (pairwise) orthogonal. If

\[
\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = a_1 v_1 + a_2 v_2 + a_3 v_3,
\]

where the \( v_i \)'s are as above and the \( a_i \)'s are scalars, FIND \( a_2 \)
Question 2: Group Work
Writing on board disappears when I enter room (sometimes). So lets try this:
- None of the breakout rooms have the questions.
- Every breakout room has a whiteboard.
- Write Question on the whiteboard when you get into the breakout room
- You’ve got about 10 minutes.

Question 2 (parts a and b are from 2014 Quiz 4)

a) Find all eigenvalues
b) Find a eigenbasis for each eigenvalue.
c) Is it possible to diagonalize A? Why/why not?

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & 3 \end{bmatrix} \]
Question I (parts a and b are from 2014 Quiz 4, Question 3)

a) Find all eigenvalues

b) Find a eigenbasis for each eigenvalue.

c) Is it possible to diagonalize $A$? Why/why not?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$
Recitation 28

Today: Orthogonality (6.1, 6.2, 6.3)

True or False:
a) Eigenvalues must be nonzero scalars.

This is _______, because

b) Eigenvectors must be nonzero vectors.

This is _______, because
Quiz 4, Question 1b
Solutions were emailed to students yesterday.

Question

\[
b) \text{ Find a basis for the null space of } A. \quad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & 6 \end{pmatrix}
\]

An Answer

A basis for \( \text{Nul}(A) \) is the set:

\[
\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \end{bmatrix} \right\}
\]

Is this correct? How can we check to see if this answer is correct?

The basis vectors must ______________ , and ______________
Orthogonality (6.2)

a) Compute the orthogonal projection (OP) of \( y \) onto the line, \( L \), that passes through the origin and is parallel to \( u \).

b) Sketch \( y \), \( u \), \( L \), and the OP.

\[
y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \quad u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}
\]
Orthogonality (6.2)

c) Calculate the distance between $y$ and its OP.

d) Write $y$ as a sum of a vector in Span($u$) and a vector orthogonal to $u$.

$y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$
Orthogonality and Linear Independence

a) Are the columns of $A$ LI?
b) Do the columns of $A$ form a basis for $\mathbb{R}^4$?
c) Are the columns of $A$ mutually orthogonal?

\[
A = \begin{bmatrix}
3 & -5 & 1 \\
1 & 1 & 1 \\
-1 & 1 & -2 \\
3 & -7 & 8
\end{bmatrix}
\]
Orthogonality and Linear Independence

Find an orthogonal basis for the column space of $A$.

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$
If time permits *(from 2014 Quiz 4)*:

**Extra Credit: (5 points)**

Find a 2 x 2 matrix such that the column space is the line $x_1 + 2x_2 = 0$, and the null space is the line $x_1 = x_2$. 
If time permits *(from 2012 Quiz 4)*:
Find the eigenvalues and eigenvectors of $A$ and use them to find a formula for $A^k$.

$$A = \begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}$$
Recitation 29

Today: Orthogonality (6.1 to 6.5)

Orthogonality and Linear Independence

a) Are the columns of \( A \) LI?

b) Do the columns of \( A \) form a basis for \( \mathbb{R}^4 \)?

c) Are the columns of \( A \) mutually orthogonal?

\[
A = \begin{bmatrix}
3 & -5 & 1 \\
1 & 1 & 1 \\
-1 & 1 & -2 \\
3 & -7 & 8 \\
\end{bmatrix}
\]
Orthogonality and Linear Independence

Find an orthogonal basis for the column space of $A$. 

\[ A = \begin{bmatrix} 
3 & -5 & 1 \\
1 & 1 & 1 \\
-1 & 1 & -2 \\
3 & -7 & 8 
\end{bmatrix} \]
Orthogonality and Linear Independence

Find an orthogonal basis for the column space of $A$. 

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$
2) Least Squares (slide 1/3)
Consider the system $Ax = b$, where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

i. Does $A$ have LI columns?

ii. Do the columns of $A$ form a basis for $\mathbb{R}^3$?

iii. Is $b$ in $\text{Col}(A)$?

iv. Is there a solution to $Ax = b$?
2) Least Squares (slide 2/3)
Consider the system \( Ax = b \), where

\[
A = \begin{bmatrix}
4 & 0 \\
0 & 2 \\
1 & 1
\end{bmatrix}, \quad b = \begin{bmatrix}
2 \\
0 \\
11
\end{bmatrix}
\]

Our best solution to this system: the vector \( \hat{u} \), such that \( ||b - A\hat{u}|| \leq ||b - A__|| \). What does this mean?

Is \( b \) in \( \text{Col}(A) \)?

Is \( A\hat{u} \) in \( \text{Col}(A) \)?

Is \( (b - A\hat{u}) \) perpendicular to all vectors in \( \text{Col}(A) \)?

Thus, \( A^T \cdot (b - A\hat{u}) = \)

Solving for \( \hat{u} \) gives us:
2) Least Squares (slide 3/3)

Consider the system $Ax = b$, where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Find the least squares solution to this system.

$$A^TA =$$

$$A^Tb =$$

$$(A^TA)^{-1} =$$

$$\hat{u} =$$

The vector we found, $\hat{u}$, has the property that:
Least Squares (slide 3/3)
Consider the system $Ax = b$, where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Find the least squares solution to this system.

$$A^T A =$$

$$A^T b =$$

$$(A^T A)^{-1} =$$

$$x =$$

The vector we found, $x$, has the property that:
Least Squares: A Special Case
Find a least squares sol’n to Ax=b, where:

\[ A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix} \]
Recitation 31

Today: QR Decomposition, Orthogonality Review (6.1 to 6.5)

**QR Factorization**
A = QR, and R is an upper triangular matrix.

\[
A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -6 \\ 1 & 5 \end{bmatrix},
Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}
\]

a) If we weren’t given Q, we could find it by using _______________ .

b) The columns of Q are _______________ .

c) The columns of Q form a _______________ basis for ____________
QR Factorization
A = QR, and R is an upper triangular matrix.

\[ A = \begin{bmatrix}
  5 & 9 \\
  1 & 7 \\
 -3 & -6 \\
  1 & 5 \\
\end{bmatrix}, 
\]
\[ Q = \begin{bmatrix}
  5/6 & -1/6 \\
  1/6 & 5/6 \\
 -3/6 & 1/6 \\
  1/6 & 3/6 \\
\end{bmatrix}. 
\]

d) The dimensions of R have to be __________, because ____________________.

e) Find R.

f) Show that your answer for part (e) is correct.
Least Squares (LS) Formulas
For each case, state a formula for the LS solution of $Ax = b$.

1) $A$ is a square matrix and $A^T A$ is invertible.

2) $A$ is square and has **orthogonal** columns.

3) $A$ is square and has **orthonormal** columns.
Least Squares Solutions

If $A$ is square, and $A^T A$ is not invertible, can we find a LS solution to $Ax = b$? Why/why not?
Least Squares Solutions

Describe all LS solutions to the system:

\[ x_1 + x_2 = 2 \]
\[ x_1 + x_2 = 4 \]
Recitation 32

Today: Final Exam Review

Quiz Grades
• Quiz grades “locked” today.
• Please check your graded quizzes to see if they were graded correctly.

If You are Writing Math 1502 Final Exam
• Part I: Mon, Dec 8
• Part II: Tue, Dec 9
• Work with your facilitator to find a time/place to write.

QH8 Office Hours
Sat Dec 6, Sun Dec 7
Please use text chat when you are free with “I’m free Sat ___ and Sun ___” so that we can try to find times that work, for most of you.

Today
Group work: 3 groups, we may return to main room if/when groups are getting stuck
1) Section 4.6: Row and Col Space of $A^T$ (Slide 1 of 2)

Row(A) is the set of all possible linear combinations of the rows of A.

Theorem (from Section 4.6)

If two matrices A and B are row equivalent, then their row spaces are _________.

If B is in echelon form, the nonzero rows of B form a basis for ________________, as well as ________________.

A proof of this theorem uses the fact: if B is obtained from row operations on A, the rows of B are ______________________ of the rows of A.
1) Section 4.6: Row and Col Space of $A^T$ (Slide 2 of 2)

Matrix $A$ and its row echelon form are given. Find a basis for

1) $\text{Col}(A)$
2) $\text{Row}(A)$
3) $\text{Row}(A^T)$
4) $\text{Col}(A^T)$

*Hint: the answers for all of the above do not require any calculation.*

$$A = \begin{bmatrix}
1 & -4 & 9 & -7 \\
-1 & 2 & -4 & 1 \\
5 & -6 & 10 & 7 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & -1 & 5 \\
0 & -2 & 5 & -6 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$$
2) From 2012 Quiz 4:
Find the eigenvalues and eigenvectors of $A$ and use them to find a formula for $A^k$.

*Hint: one of the eigenvalues is 3.*

$$A = \begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}$$
3) QR Factorization
A = QR, and R is an upper triangular matrix.

\[ A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}, \quad Q = \begin{bmatrix} -2/7 & 5/7 \\ 5/7 & 2/7 \\ 2/7 & -4/7 \\ 4/7 & 2/7 \end{bmatrix} \]

a) The dimensions of R have to be __________.
b) Calculate matrix R. *Hint: save time by factoring 1/7 out of matrix Q first.*
c) Compute a few elements of the product QR to check your answer for part (b)
4) Fill in the Blanks

Matrix A is invertible and has dimensions N\times N.

a) The columns of A form a basis for _____

b) rank(A) = _____

c) Nul(A) = _____

d) dim(Nul(A)) = _____

e) dim(Nul(A^T)) = _____

f) dim(Row(A^T)) = _____

g) dim(Col(A^T)) = _____

h) dim(Col(A)) + dim(Nul(A)) = _____ + _____ = _______

i) dim(Row(A)) + dim(Nul(A)) = _____ + _____ = _______
5) A modified version of a 2014 Quiz 2 Question

Compute the integral, using a Taylor polynomial, to an accuracy of at least 0.01.

*Hint: N is bigger than 30.*

$$\int_{0}^{1} e^{-x^4} \, dx$$
6) True or False

A) The LS solution of $Ax = b$ is the point in $\text{Col}(A)$ closest to $b$.

This statement is ______ because

B) If $x$ is in subspace $W$, then $x - \text{proj}_W x \neq 0$

This statement is ______ because
7) True or False

A) If \( \{v_1, v_2, v_3\} \) is an orthogonal basis and \( c \) is a constant, then \( \{v_1, v_2, cv_3\} \) is another, different orthogonal basis.

This statement is ______ because

B) If \( \hat{u} \) is a LS solution to \( Au = b \), then \( \hat{u} = (A^TA)^{-1}A^Tb \).

This statement is ______ because
8) Eigenvalues and Orthogonality

1) What can we say about the eigenvalues of an orthonormal matrix? 
   *Hint: look at $||Av||^2$, where $A$ is orthonormal, and $v$ is an e-vector of $A$."

2) What can we say about the eigenvalues of a matrix that has LD columns?

3) What can we say about the eigenvalues of a matrix that is upper triangular?
Welcome to Your Distance Calculus Recitation!

We’ll get started at 8:05.

While we are waiting, see if you can use the chat window (bottom right) to join the discussion.

Today:
- Introduction to Adobe Connect
- What are Recitations?
- Icebreaker
- Numerical integration (8.7)

If you can’t hear the TA, click the speaker icon
Adobe Connect
Microphones, Webcams, Tablets

We can loan you a wacom bamboo tablet, if you’d like to borrow one please send me an email.

If you have a mic or a webcam, you are welcome to use them.
The Whiteboard

• You need to be a presenter to write on the board
• Only a host can change permission levels
• All writing on the board is anonymous
• Please respect other students taking this course (and your TA): you are responsible for your learning
Logging Into Adobe Connect for Recitations

Thursday’s recitation is at https://georgiatech.adobeconnect.com/math1502-08-19-14
Adobe Connect Technical Problems?

You can:
• reload your browser
• log in/out
• use a different web browser
• reboot
• get help from another student and/or your TA

I strongly recommend that, if possible, you use a wired connection.
What are Recitations?

- Our goal: help students understand course material so that they can complete assignments and prepare for quizzes and exams.

- please bring questions about the homework or lectures
Our Section in a Nutshell

- students in Math 1502 are divided into many sections
- ours is the only section that
  - doesn’t have on-campus students
  - uses Connect for recitations
- Why Adobe Connect?
  - it’s cheaper
  - you can interact with students at other schools
Tablets

• Students in our section can borrow tablets.
• If you already have a tablet you want to use, that's ok.
• Equipment need to be returned to your facilitator.
• If you don’t have a tablet and want to borrow one, email me.
• Tablets (should) come with a CD, use it to configure tablet settings.
Course Websites

- Recordings of recitations and lectures: tegrity.gatech.edu
- Discussion forum: piazza.com
- Live lectures: gtcourses.gatech.edu
- Textbook and homework: www.mymathlab.com

- First homework due ______________
# Grading Weights

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<tr>
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<th>Weight (%)</th>
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<tr>
<td>Homework</td>
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<td>Final</td>
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<td>Quizzes</td>
<td>60</td>
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<tr>
<td>Pop Quizzes</td>
<td>5</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>100</strong></td>
</tr>
</tbody>
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Grades will be made available through T-Square
Recitation 02

Recitations run from 8:05 – 8:55.

Today:
- Improper integrals
- Recitation logistics
- Numerical integration (8.7)
- Questions about the course, homework?

While we are waiting to start, calculate:

a) the integral of \(\frac{1}{x^2}\) from 1 to infinity
b) the integral of \(e^{-1}\) from 1 to infinity
**Improper Integrals**

\[
\int_{1}^{\infty} \frac{1}{x^2} \, dx = \lim_{b \to \infty} \left[ x^{-1} \right]_{1}^{b} = \lim_{b \to \infty} \left( \frac{-1}{b} - \frac{-1}{1} \right) = 1
\]

\[
\int_{1}^{\infty} e^{-x} \, dx = \lim_{b \to \infty} \left[ -e^{-x} \right]_{1}^{b} = \lim_{b \to \infty} \left( -e^{-b} \right) - (-e^{-1}) = 0 + \frac{1}{e} = \frac{1}{e}
\]
Example: Integrate $1/x$ from 1 to 2

a) What is the exact answer?

\[
\int_1^2 \frac{1}{x} \, dx = \ln x \bigg|_1^2 = \ln 2 \approx 0.69314718 \ldots
\]

b) Set up but don’t evaluate an expression for the area using Simpson’s Rule.

Let \( n = 4 \):

\[
S = \frac{1}{3} \left( \frac{1}{4} + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{8} \right) \quad \text{(Not necessary)}
\]

\[
S = \frac{1}{3} \left( \frac{1}{4} + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{8} \right) = \frac{1747}{2520}
\]

\[
\approx 0.693253968 \ldots
\]

Note: \( n = 4 \) implies 5 function evaluations
Example: Integrate $1/x$ from 1 to 2

c) Find the number of subintervals required for four digit accuracy using Simpson’s rule.

\[
E_S \leq \frac{b-a}{180} \max |f'''| (\Delta x)^4,
\]

\[
= \frac{(b-a)^5}{180N^4} \cdot 24
\]

\[
= \frac{1}{180N^4} \cdot 24
\]

For 4 digit accuracy, we need

\[
E_S < 0.00005
\]

\[
\Rightarrow \frac{24}{180N^4} < 0.00005
\]

\[
\Rightarrow N^4 > \frac{20000}{24} \Rightarrow N = 8
\]

**NOTE:** $E_T: \frac{24}{12N^2} < 0.00005, N > 14.2 \Rightarrow N = 15$
Recitation 03

Today:
1. Improper integrals: comparison test
2. A few announcements
3. Improper integrals: techniques of integration

Use the comparison test to determine whether the following integral converges.

\[ \int_1^\infty \frac{1}{\sqrt{1 + x^2}} \, dx \]

Note that: \( \frac{1}{\sqrt{1 + x^2}} < \frac{1}{1 + x} \) on \( x \in [1, \infty) \)

So:

\[ \frac{1}{\sqrt{1 + x^2}} > \frac{1}{1 + x} \] on \( x \in [1, \infty) \)

\[ \Rightarrow \int_1^\infty \frac{1}{\sqrt{1 + x^2}} \, dx > \int_1^\infty \frac{1}{1 + x} \, dx = \ln|1 + x| \bigg|_1^\infty \]

\[ \Rightarrow \text{by comparison test, integral diverges} \]
Group Work

when you are in a breakout room:

message TA

Greg recommends:
1) one person volunteer to share whiteboard
2) discuss how to start question 4 & draw on board
3) solve question 4
4) proceed to question 5
Group Work

when you are in a breakout room:

change slides

Greg recommends:
1) one person volunteer to share whiteboard
2) discuss how to start question 4 & draw on board
3) solve question 4
4) proceed to question 5
Group Work Questions

Evaluate the following integrals.

4) \[ \int_{5}^{\infty} \frac{1}{x^2 + 25} \, dx \]

7) \[ \int_{0}^{\infty} \frac{1}{x^2 + 7x + 6} \, dx \]

5) \[ \int_{0}^{144} \frac{1}{\sqrt{144 - x}} \, dx \]

8) \[ \int_{-\infty}^{\infty} \frac{Ax}{(x^2 + B)^{12}} \, dx \]

6) \[ \int_{0}^{144} \frac{1}{\sqrt{144 - x^2}} \, dx \]

9) \[ \int x^3 \ln x \, dx \]
4) \[ \int_{5}^{\infty} \frac{1}{x^2 + 25} \, dx = \frac{1}{25} \int_{5}^{\infty} \frac{1}{(\frac{x}{5})^2 + 1} \, dx, \quad u = \frac{x}{5}, \quad dx = 5 \, du \]
\[ = \frac{1}{25} \int_{1}^{\infty} \frac{1}{u^2 + 1} \, (5 \, du) \]
\[ = \frac{1}{5} \left[ \arctan u \right]_{1}^{\infty} \]
\[ = \frac{1}{5} \left( \frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi}{20} \]

5) \[ \int_{0}^{144} \frac{1}{\sqrt{144-x^2}} \, dx = \int_{144}^{0} \frac{1}{\sqrt{u}} \, (-du), \quad u = 144 - x \]
\[ = \int_{144}^{0} \frac{1}{\sqrt{u}} \, du \]
\[ = 2 \cdot u^{\frac{1}{2}} \bigg|_{0}^{144} \]
\[ = 24 \]

6) \[ \int_{0}^{12} \frac{1}{\sqrt{144-x^2}} \, dx = \int_{0}^{12} \frac{1}{12 \sqrt{1 - \frac{x^2}{144}}} \, (12 \, dx) \]
\[ = \theta \bigg|_{0}^{\frac{12}{12}} = \frac{\pi}{2} \]
\[ \text{NOTE: } \arcsin 1 = \frac{\pi}{2} \quad \text{arcsin 0 = 0} \]
7) \( \int_0^\infty \frac{1}{x^2+7x+6} \, dx = \int_0^\infty \frac{1}{(x+6)(x+1)} \, dx \), \( \frac{1}{x+6} + \frac{1}{x+1} = \frac{A}{x+6} + \frac{B}{x+1} \)

\[ \begin{bmatrix} 1 & 6 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

or 'just': \( A+6B = 1 \), \( A+B = 0 \)

\( \Rightarrow 5B = 1, \) so \( B = \frac{1}{5}, \) \( A = -\frac{1}{5} \)

\[ \frac{1}{5} \ln \frac{x+1}{x+6} \bigg|_0^\infty \]

\( = \frac{1}{5} \left( \ln 1 - \ln \frac{1}{6} \right) \)

\( = -\frac{1}{5} \ln \left( \frac{1}{6} \right) \)

\( = \frac{1}{5} \ln (6) \)

\( \int_0^\infty \frac{Ax}{(x^2+B)^2} \, dx \)

8) \( \int_{-\infty}^{+\infty} \frac{Ax}{(x^2+B)^2} \, dx = 0 \), \( A = 2 \)

ODD FUNCTION OVER SYMMETRIC INTERVAL

9) \( \int x^3 \ln x \, dx \)

let: \( u = \ln x \)

\( du = \frac{1}{x} \, dx \)

\( v = x^4/4 \)

\( \int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \cdot \frac{1}{x} \, dx + C \)

\( = \frac{x^4}{4} \ln x - \frac{1}{16} x^4 + C \)
Recitation 04

Today:
1. 1\textsuperscript{st} order linear DE
2. A few announcements
3. Group Work: Seperable & Linear DEs

Consider: 

\[ xy' - y = 2x \ln x \]

\textit{Standard Form}: \[ y' - \frac{1}{x} y = 2x \ln x \]

a) Is the DE separable?  \textit{No}

b) Is the DE 1\textsuperscript{st} order linear in \( y(t) \)?  \textit{Yes}

c) What is the integrating factor? 

I.F. is 

\[ \int \frac{1}{x} \, dx = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x} \]
\[ xy' - y = 2x \ln x \]

**STEP 1:** Standard form:
\[ y' - \frac{1}{x}y = 2 \ln x \]

**STEP 2:** I.F.: \( \frac{1}{x} \)

**STEP 3:** Multiply by I.F.
\[ x y' - x^{-1} y = 2 \frac{\ln x}{x} \]

\[ x y' = 2 \frac{\ln x}{x} + c \]

**STEP 4**
\[ \frac{d}{dx} \left( x^{-1} y \right) = 2 \frac{\ln x}{x} \text{ by \ - \ rule} \]

**STEP 5**
\[ x^{-1} y = 2 \int \frac{\ln x}{x} \, dx \]
\[ = 2 (\ln x)^2 + c \]
\[ y = 2x (\ln x)^2 + C x \]
Announcements

Invitation to Participate in a Study
- you’re receiving snail mail soon!
- please review forms with your parents and send them back

Online Survey
- anonymous!
- available at:
- please provide feedback on webcams, collaboration, the chat pod, etc

Quiz 1
- Sep 11 (two weeks from today)
Group Work

when you are in a breakout room:

message TA

Greg recommends:
1) saying hello
2) decide who will activate whiteboard
3) discuss which question you want to start on
Group Work Questions

For each of the following DE’s,

a) State whether the DE is separable.

b) State whether the DE is 1st order linear in y.

c) Solve the DE.

1) \( xy' + 3y = 3 - \frac{4}{x} \)

2) \( \frac{1}{x} y' = ye^{x^2} + 2\sqrt{ye^{x^2}} \)

3) \( xy' + y = (1 + x)e^x \)

4) \( (\sin t)y' + (\cos t)y = \tan t \)

5) \( \cos(y) + (1 + e^{-x})(\sin y)y' = 0 \)
1) \( xy' + 3y = 3 - \frac{4}{x} \) is linear. **STANDARD FORM:** \( y' + \frac{3}{x}y = \frac{3 - 4/x^2}{x} \)

**IF:** \( e^{\int \frac{3}{x} \, dx} = e^{\ln x^3} = x^3 \)

**MULTIPLY BY:** \( IF: \)

\[ x^3 y' + 3x^2 y = 3x^2 - 4x \]

\[ x^3 y = \int 3x^2 - 4x \, dx + C \]

\[ = x^3 - 2x^2 + C \]

\[ y = 1 - \frac{2x}{x^3} + \frac{C}{x^3} \]
$2) \quad xy' + y = (1 + x)e^x$

S.F.: \( y' + \frac{1}{x}y = \frac{1 + x}{x}e^x \)

I.F. is \( e^{\int \frac{1}{x} dx} = x \)

\( \Rightarrow \quad xy' + y = (1 + x)e^x \)

\[ xy = \int (1 + x)e^x \, dx \]

\[ = e^x + \int xe^x \, dx + C \]

\[ = e^x + xe^x - \int e^x \, dx + C \]

\[ = xe^x + C \]

\[ y = e^x + c/x \]
3) \( \frac{1}{x} y' = ye^{x^2} + 2\sqrt{y} e^{x^2} \)

not linear, but separable:

\[ \frac{1}{x} y' = e^{x^2} (y + 2\sqrt{y}) \]

\[ \frac{1}{y + 2\sqrt{y}} dy = xe^{x^2} \, dx \]

Let \( \sqrt{y} = u \), \( du = \frac{1}{2\sqrt{y}} dy \)

\[ \frac{1}{u^2 + 2u} 2u \, du = \int xe^{x^2} \, dx \]

\[ u = x^2, \quad du = 2x \, dx \]

\[ 2 \frac{1}{u+2} \, du = \frac{1}{2} \int e^u \, du \]

\[ 2 \ln(\sqrt{y} + 2) = \frac{1}{2} e^{x^2} + C \]

\[ \ln(\sqrt{y} + 2) = \frac{e^{x^2}}{4} + C \]

\[ y = (e^{c} + e^{x^2/4} - 2)^2 \]
4) \( \sin t \ y' + \cos t \ y = \tan t \)

RF: \( y' + \frac{e^{-t}}{y} = \frac{t}{e} \)

IF: \( \int e^{-t} dt \), let \( u = \sin t \), \( du = \cos t \ dt \)

\[
\frac{1}{u} du = e^{-t} = w = s
\]

\[
\Rightarrow s y' + c y = \tan t
\]

\[
s y = \int \tan t \ dt = -\ln \cos t + c \quad e + c
\]

5) \( \cos y + (1+e^{-x})(\sin y) \ y' = 0 \)

\[
\frac{1}{e^{x}} dy = -\frac{1}{1+e^{-x}} \quad u = 1+e^{-x}
\]

\[
\ln |\sec y| = -\int \frac{e^{x}}{e^{x}+1} \ dx \quad u = e^{x}, \ du = u \ dx
\]

\[
\ln |\sec y| = -\int \frac{u}{u+1} \ du
\]

\[
\ln |\sec y| = -(1+e^{-x}) + c
\]
Recitation 05

Today:
1. Geometric Series
2. Group Work: Infinite Series

The sum of a geometric series is: \[ \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \]

The geometric series is convergent iff \(|r| < 1\).
Express 1.7979797979 ... as a rational number.

\[
\begin{align*}
1.79 &= 1 + 79 \cdot 10^{-2} + 79 \cdot 10^{-4} + \ldots \\
&= 1 + 79 \left( 10^{-2} + 10^{-4} + 10^{-6} + \ldots \right) \\
&= 1 + 79 \sum_{k=1}^{\infty} 10^{-k \cdot 2} = 1 + 79 \sum_{k=1}^{\infty} \left( \frac{1}{100} \right)^k \\
&= 1 + \frac{79}{100} \left( \frac{1}{1-\frac{1}{100}} \right) \\
&= 1 + \frac{79}{99} \\
&= 1.797979797979...
\end{align*}
\]

Please complete Start-of-Term Survey at:
https://www.surveymonkey.com/s/Math1501StartOfTerm2014
Input Interpretation:
1.79797979797979

Rational approximation:
\[
\frac{178}{99} = 1 + \frac{79}{99}
\]
Determine if the following series convergent or divergent. If it is convergent, find its sum.

\[
\sum_{k=1}^{\infty} \frac{4^{k+1}}{5^k} = \sum \frac{4 \cdot 4^k}{5 \cdot 5^k} = \frac{16}{5} \sum \left(\frac{4}{5}\right)^{k-1}
\]

\[
= \frac{16}{5} \cdot \frac{1}{1 - \frac{4}{5}}
\]

\[
= 16
\]

CONVERGENT BECAUSE |ABS VAL| COMMON RATIO

IS \(\left|\frac{4}{5}\right| < 1\)
Group Work

when you are in a breakout room:

Greg recommends:
1) saying hello :D
2) click the arrow that points right, located bottom left of screen
Group Work Questions

1) Express $1.1232323\ldots$ as a ratio of integers.

2) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

$$A) \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k}$$

$$B) \sum_{k=1}^{8} 2^{2k}3^{1-k}$$

$$C) \sum_{k=1}^{8} \frac{(-5)^k}{4^{k+1}}$$

$$D) \sum_{k=1}^{8} \frac{1}{(3k-2)(3k+1)}$$

\[0.301301\ldots\]
1. \( 1.123 = 1.1 + 2.3 \left( 10^{-3} + 10^{-5} + \ldots \right) \)

\[
= 1.1 + \frac{2.3}{10} \left( 10^{-2} + 10^{-4} + \ldots \right)
\]

\[
= 1.1 + \frac{2.3}{10} \sum_{k=1}^{\infty} 10^{-2k}
\]

\[
= \frac{11}{10} + \frac{2.3}{10} \sum_{k=1}^{\infty} \left( \frac{1}{100} \right)^k
\]

\[
= \frac{11}{10} + \frac{2.3}{1000} \frac{1}{1 - \frac{1}{100}}
\]

\[
= \frac{11}{10} + \frac{2.3}{1000} \frac{100}{99}
\]

\[
= \frac{11}{10} + \frac{2.3}{990}
\]

\[
= \frac{556}{495} = 1 \frac{61}{495}
\]
\[
0.301 = 301 \left(10^{-3} + 10^{-6} + \ldots\right) \\
= 301 \left(\sum_{k=1}^{\infty} 10^{-3k}\right) \\
= 301 \left(\sum_{k=1}^{\infty} 10^{-3k}\right) \\
= 301 \frac{1}{1000} \sum_{k=1}^{\infty} \left(\frac{1}{1000}\right)^k \\
= \frac{301}{1000} \frac{1}{1 - \frac{1}{1000}} \\
= \frac{301}{999} \checkmark
\]

\[
2A) \quad \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k + \left(\frac{4}{3}\right)^k \\
= \frac{1}{2} \sum_{k=1}^{\infty} 5^{k-1} + \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k-1} \\
= \frac{1}{2} \frac{\frac{1}{1-5}}{1} + \frac{1}{3} \frac{\frac{1}{1-\frac{1}{3}}}{1} \\
= 1 + \frac{1}{2}
\]
2B) \[ \sum_{k=1}^{\infty} 2^k 3^{1-k} = \sum 4^k \cdot 3 \cdot 3^{-k} = 3 \sum \left(\frac{4}{3}\right)^k \Rightarrow \text{diverges} \]

2C) \[ \sum_{k=1}^{8} \frac{(-1)^k}{4^{k+1}} = \sum \frac{(-1)(-1)^{k-1}}{4 \cdot 4 \cdot 4^{k-1}} = \frac{-1}{16} \sum_{k=1}^{8} \frac{(-1)^{k-1}}{4^{k-1}} \]

\[ = \frac{-1}{16} \cdot \frac{1}{1 - (-\frac{1}{4})} = \frac{-1}{16} \cdot \frac{1}{\frac{5}{4}} = \frac{-1}{20} \]

\( \checkmark \)

we did a slightly different question (we should have used: \( \sum (-\frac{1}{4})^k \) diverges)

2D) \[ \sum_{k=1}^{8} \frac{1}{(3k-2)(3k+1)} = \sum \frac{1}{3} \frac{1}{3k-2} - \frac{1}{3} \frac{1}{3k+1} \]

\[ = \left(\frac{1}{3} - \frac{1}{12}\right) + \left(\frac{1}{12} - \frac{1}{21}\right) + \left(\frac{1}{21} - \frac{1}{30}\right) + \left(\frac{1}{30} - \cdots \right) \]

\[ = 1/3 \]
Recitation 06

Today:
1. Test for Divergence
2. Group Work: Infinite Series, Differential Equations

The Test for Divergence (Theorem 7 from 10.2)

If \( \lim_{k \to \infty} a_k \) is not zero or does not exist, then \( \sum_{k=1}^{\infty} a_k \) diverges.

Note that:

If \( \lim_{k \to \infty} a_k \) is equal to zero, then the divergence test is inconclusive.

Use the test for divergence to determine whether these series converge.

A) \( \sum_{k=1}^{\infty} \frac{1}{k+2} \)

The test for divergence is inconclusive, because \( \lim_{k \to \infty} \frac{1}{k+2} = 0 \)

B) \( \sum_{k=1}^{\infty} \frac{k}{k+2} \)

Diverges, because \( \lim_{k \to \infty} \frac{k}{k+2} = \lim_{k \to \infty} \frac{1}{1 + \frac{2}{k}} = 1 \neq 0 \)
Group Work

When you are in a breakout room:

Group Work Questions

1) Express $1.1232323 \ldots$ as a ratio of integers.

2) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

$$A) \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k}$$

$$B) \sum_{k=1}^{\infty} 2^{2^k}3^{1-k}$$

$$C) \sum_{k=1}^{\infty} \frac{(-5)^k}{4^{k+1}}$$

$$D) \sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}$$

Greg recommends:
1) Saying hello :D
2) Each student write & chat in their favourite color
3) Advance to the next slide with arrows located bottom left
Group Work Questions

1) Express 0.002100210021 …. as a ratio of integers.

2) Solve the following DE for $x > 0$: $x^3 y' + (2 - 3x^2) y = x^3$

3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

   A) $\sum_{k=1}^{\infty} \frac{k^2}{5k^2 + 5}$

   B) $\sum_{k=1}^{\infty} \frac{1}{k(k+3)}$

   C) $\sum_{k=1}^{\infty} 3(0.4)^k - 2(-0.1)^{k+1}$

   D) $\sum_{k=1}^{\infty} \left( \sin \left( \frac{1}{k} \right) - \sin \left( \frac{1}{k+1} \right) \right)$

   E) $\sum_{k=1}^{\infty} \arctan(k)$

4) Consider the DE $y' + y = y^2 e^x$
   a) Determine if the DE is separable, and/or 1st order linear in $y$.
   b) Solve the DE for $x > 0$. Hint: let $z = 1/y$. 
1) Express 0.002100210021 \ldots as a ratio of integers.

\[
\frac{21}{10^4 + 10^{-8} + 10^{-12} + \ldots}
\]

\[
= 21 \left( \sum_{k=1}^{\infty} 10^{-4k} \right)
\]

\[
= 21 \left( \sum_{k=1}^{\infty} \left( \frac{1}{10000} \right)^k \right)
\]

\[
= 21 \left( \frac{\frac{1}{10000}}{1 - \frac{1}{10000}} \right)
\]

\[
= \frac{21 \times 10000}{10000 - 1}
\]

\[
= \frac{210000}{9999}
\]

\[
= \frac{7}{3333}
\]
2) Solve the following DE for $x > 0$: $x^3 y' + (2 - 3x^2) y = x^3$

**Standard Form:**

$$ y' + \frac{2 - 3x^2}{x^3} y = 1 $$

**I.F. is** $e^{\int \frac{2 - 3x^2}{x^3} \, dx} = e^{\int (x^{-2} - \ln x^3) \, dx} = F(x)$

**DE Becomes:**

$$ (F)y' + (F)\frac{2 - 3x^2}{x^3} y = F $$

$$ F y = \int F \, dx = \int \frac{e^{x^{-2}}}{x^3} \, dx = \frac{1}{2} e^{-1/x^2} + C $$

$$ y = \frac{x^3}{2} + C x^3 e^{1/x^2} $$
3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ A) \sum_{k=1}^{\infty} \frac{k^2}{5k^2 + 5} \]

\[
\lim_{k \to \infty} \frac{k^2}{5k^2 + 2} = \lim_{k \to \infty} \frac{1}{5 + \frac{2}{k^2}} = \frac{1}{3}
\]

\( \Rightarrow \) diverges by the divergence test
Group Work Questions

3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[
B) \sum_{k=1}^{\infty} \frac{1}{k(k+3)} = \sum_{k=1}^{\infty} \left( \frac{1/3}{k} - \frac{1/3}{k+3} \right) = 1 = A(k+3) + B(k)
\]

\[A + B = 0\]
\[3A = 1\]

\[= \frac{1}{3} \left[ (1 - \frac{1}{4}) + (\frac{1}{2} - \frac{1}{5}) + (\frac{1}{3} - \frac{1}{6}) + (\frac{1}{4} - \frac{1}{7}) + \ldots \right]\]

\[= \frac{1}{3} \left( 1 + \frac{1}{2} + \frac{1}{3} \right)\]

\[= \frac{1}{3} \left( \frac{6}{6} + \frac{3}{6} + \frac{2}{6} \right)\]

\[= \frac{11}{18}\]

\[\approx 0.61111\ldots\]
Group Work Questions

3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ C) \sum_{k=1}^{\infty} 3(0.4)^k - 2(-0.1)^{k+1} \]

\[ = 3 \left( \frac{0.4}{1-0.4} \right) - 2 \sum_{k=1}^{\infty} (-0.1)^{k+1} \]

\[ = 2 - 2 \sum_{k=1}^{\infty} (0.1) (0.1)^{k-1} \]

\[ = 2 - 0.02 \frac{1}{1-(-0.1)} \]

\[ = 2 - \frac{0.02}{1.1} \]

(\text{Note: } \frac{0.02}{1.1} = 0.0181818\ldots)

\[ = 2 - \frac{2}{110} = \frac{718}{110} = 0.65272727\ldots \]
3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ D) \sum_{k=1}^{\infty} \left( \sin \left( \frac{1}{k} \right) - \sin \left( \frac{1}{k+1} \right) \right) \]

\[ = \left( \sin 1 - \sin \frac{1}{2} \right) + \left( \sin \frac{1}{2} - \sin \frac{1}{3} \right) + \ldots \]

\[ = \sin 1 \]
3) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

\[ E) \sum_{k=1}^{\infty} \arctan(k) \]

\[
\lim_{k \to \infty} \arctan k \to \frac{\pi}{2} \text{ as } k \to \infty
\]

So \[ \sum_{k=1}^{\infty} \arctan k \text{ D.N.E., by the divergence test.} \]
\[ y' + y = y^2 e^x \]

Not separable. Not linear in y.

Let \( z = \frac{1}{y} \). Then

The DE becomes:

\[ \frac{d}{dx} \left( y \right) + y = y^2 e^x \]

\[ -y^2 \frac{dz}{dx} + \frac{1}{z} = \frac{1}{z^2} e^x \]

\[ -\frac{1}{z^2} \frac{dz}{dx} + \frac{1}{z} = \frac{1}{z^2} e^x \]

Using the chain rule:

\[ \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \]

\[ \frac{dy}{dx} = -y^2 \]

So

\[ \frac{dz}{dx} = -y^2 \frac{dz}{dx} \]

and

\[ z = \frac{1}{y} \]

This is a linear DE in \( z \). If is \( e^{-x} \)

\[ e^{-x} \frac{dz}{dx} - e^{-x} z = 1 \]

\[ e^{-x} z = x + C \Rightarrow z = \frac{x}{y} = xe^x + ce^x \]

or: \( y = \frac{1}{x e^x + ce^x} \)
Recitation 07

Today:
1. Announcements
2. Quiz Review

For what values of $x$ does the following converge? Why?

\[ \sum_{k=1}^{\infty} x^{k-1} = 1 + x + x^2 + x^3 + \ldots \quad |x| < 1, \text{ geometric series} \]

When the series converges, what is the series equal to?

\[ = \frac{1}{1-x} \]

Express the following as an infinite series.

\[ \frac{1}{1-x^4} = \frac{1}{1-u} \quad u = x^4 \]

\[ = \sum_{k=1}^{\infty} u^{k-1} = \sum_{k=1}^{\infty} (x^4)^{k-1} = \sum_{k=1}^{\infty} x^{4k-4} \]
How Quizzes Work

1. Facilitator gets a copy of quiz
2. You write quiz on Thursday, during recitation (8:05 – 8:55)
3. Quiz is proctored (perhaps by your facilitator)
4. Give your completed quiz to your proctor.

Format of Quiz

- Are calculators allowed?
- Are textbooks and notes allowed?
- Quiz length: probably three pages, 1 question per page, questions can have multiple parts
- Don’t write on back of pages
- Leave a 1 inch margin around edges of page
- If run out of space, you can use extra pages
How Quizzes Work

If You Have Questions During the Quiz
1. If your proctor agrees, you can use to Adobe Connect to ask your TA questions (you’d need to be closely proctored)
2. Otherwise: call/text me at 212-521-6488

Before The Quiz
Find your facilitator, and discuss
- where you are writing the quiz, and
- whether you are connecting via Adobe Connect.

What Happens After The Quiz
1. Your facilitator scans-emails your quiz to GT,
2. they get graded in about a week or two,
3. your TA enters grades in T-Square,
4. your graded quiz is returned via email to you.
Group Work

When you are in a breakout room:

message TA

Group Work Questions

1) Express $1.1232323\ldots$ as a ratio of integers.

2) Determine if the following are convergent or divergent. If it is divergent, explain why. If it is convergent, find its sum.

   \[
   \begin{align*}
   A) & \quad \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} \\
   B) & \quad \sum_{k=1}^{\infty} 2^{2k}3^{1-k} \\
   C) & \quad \sum_{k=1}^{\infty} \frac{(-5)^k}{4^{k+1}} \\
   D) & \quad \sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}
   \end{align*}
   \]

advance slides

Greg recommends:

1) Saying hello :D
2) Each student write & chat in their favourite color
3) Advance to the next slide with arrows located bottom left
1) Solve $xy' + 2y = \frac{\cos(x)}{x}$, with $y(\pi) = 1$.

**Standard Form:**

\[
y' + \frac{2}{x}y = \frac{\cos x}{x^2}
\]

**I.F.:**

\[
e^{\int \frac{2}{x} \, dx} = e^{2 \ln x} = x^2
\]

**D.E. Becomes:**

\[
x^2y' + 2xy = \cos x
\]

\[
x^2y = \int \cos x \, dx + C
\]

\[
= \sin x + C
\]

\[
y = \frac{\sin x}{x^2} + \frac{C}{x^2}
\]

\[
C = \pi^2
\]

\[
\Rightarrow y = \frac{\sin x}{x^2} + \frac{\pi^2}{x^2}
\]
2) Find the sum of the following series.

\[ \sum_{k=3}^{\infty} \frac{(-2)^{k+2}}{5^{k-1}} = \sum_{k=1}^{8} \frac{(-2)^{k+4}}{5^{k+1}} \]

\[ = \sum_{k=1}^{8} \frac{(-2)^{k}}{5^{2}} \cdot \frac{(-2)^{k-1}}{5^{k-1}} \]

\[ = \frac{-32}{25} \sum_{k=1}^{8} \left( \frac{-2}{5} \right)^{k-1} \]

\[ = \frac{-32}{25} \frac{1}{\frac{-2}{5}} \]

\[ = \frac{-32}{25} \cdot \frac{5}{7} \]

\[ = -\frac{32}{35} \]
3) Find the number of intervals, \( N \), needed to ensure an accuracy of at least 0.01 using Simpson’s Rule for the following integral.

\[
\int_{1}^{e} \ln x \, dx
\]

\[
E_{S} \leq \frac{(6-a)^{5}}{180 \, N^{4}} \cdot M
\]

\[
M = \max_{x \in [1, e]} \left| \frac{d^{4}}{dx^{4}} \ln x \right|
\]

\[
\frac{d^{4}}{dx^{4}} \ln x = \frac{-6}{x^{4}}
\]

\[
\Rightarrow \text{use } M = 6
\]

\[
\frac{(e-1)^{5}}{180 \, N^{4}} \cdot 6 \leq 0.01
\]

\[
\frac{(e-1)^{5} \cdot 6}{180 \cdot 0.01} \leq N^{4}
\]

\[
N \geq 2.658 \quad \ldots
\]

\[
\Rightarrow N \text{ must be at least } 3
\]
4) Determine whether the following converge. Do not evaluate these integrals.

\[ A) \int_{e}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \frac{1}{b} \int_{e}^{b} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \frac{1}{2} \left( \ln(\ln b) \right) \Bigg|_{b}^{e} \quad \text{(D.N.E.)} \]

\[ \text{(can use } u \text{ substitution with } u = \ln x) \]

\[ \text{(D.N.E. = Does Not Exist)} \]

\[ B) \int_{0}^{\infty} \cos(x) \sin(x) e^{-2x} \, dx < \]

\[ | \cos x \sin x e^{-2x} | \leq e^{-2x} \text{ for } x \in [0, \infty] \]

and \( \int_{0}^{\infty} e^{-2x} \, dx \) exists, so the given integral converges.
Express as an infinite series.

\[
\frac{x}{1 + x^2} = \frac{x}{1 - (-x^2)} = \frac{x}{1 - u}, \quad u = -x^2
\]

\[
= \sum_{k=1}^{\infty} u^k
\]

\[
= \sum_{k=1}^{\infty} (-x^2)^k
\]

\[
= \sum_{k=1}^{\infty} (-1)^k x^{2k+1}
\]
6) Find the sum of the following series.

\[
\sum_{k=0}^{\infty} \frac{5}{(k+1)(k+5)} = 5 \sum_{k=0}^{\infty} \frac{1/4}{k+1} - \frac{1/4}{k+5} \quad \text{(partial fractions)}
\]

= \frac{5}{4} \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots \right)

(TELESCOPIC SERIES)

= \frac{125}{48}

---

WORK FOR PARTIAL FRACTION DECOMPOSITION

\[
\frac{1}{(k+1)(k+5)} = \frac{A}{k+1} + \frac{B}{k+5}
\]

1 = A(k+5) + B(k+1)

1 = k(A+B) + (5A+B)

0 \cdot k + 1 = k(A+B) + (5A+B)

\Rightarrow A + B = 0 \quad \Rightarrow A = -B, 5A + (-A) = 1

5A + B = 1 \quad \Rightarrow A = \frac{1}{4}, B = -\frac{1}{4}
7) Determine whether the following series converges.

A) \[ \sum_{k=1}^{\infty} \frac{k^2 + 3}{k3^k} \]

**Use Ratio Test**

\[ \lim_{k \to \infty} \frac{(k+1)^2 + 3}{k^2 + 3} \cdot \frac{k}{(k+1)(3^{k+1})} \]

\[ = \lim_{k \to \infty} \frac{k^2 + 2k + 4}{k^2 + 3} \cdot \frac{k}{(k+1)(3^{k+1})} \]

\[ = \lim_{k \to \infty} \frac{\frac{k^2 + 2k + 4}{k^2 + 3}}{\frac{(k+1)}{k} \cdot \frac{1}{3}} \]

\[ = \frac{1}{3} \]

\[ \Rightarrow \text{converges by ratio test.} \]
Recitation 09

Today: 10.7 (Power Series, Radius/Interval of Convergence and Absolute Convergence)

A) The interval of convergence is set of all values of $x$ st. the series converges

B) If an infinite series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ converges

C) If a series is absolutely convergent, then it is convergent

D) A series is called conditionally convergent if it is convergent but not abs. convergent.

Given the series below, determine
1) the radius and interval of convergence
2) values of $x$ where series is absolutely convergent
3) values of $x$ where the series is conditionally convergent

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k}$$
Given the series below, determine
1) the radius and interval of convergence
2) values of $x$ where series is absolutely convergent
3) values of $x$ where the series is conditionally convergent

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k}$$

1) Try ratio test. Let $a_k = \frac{(-1)^k x^k}{k2^k}$

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{x^{k+1} \cdot \frac{k}{k+1} \cdot \frac{2^k}{2^{k+1}}} {x^k \cdot \frac{k}{k+1} \cdot \frac{2^k}{2^{k+1}}} \right|$$

$$= \lim_{k \to \infty} \left| \frac{x}{2} \right|$$

$$= \left| \frac{x}{2} \right|$$

We need $\left| \frac{x}{2} \right| < 1 \Rightarrow \text{radius is 2}$

At $x = \pm 2$, ratio test is inconclusive

At $x = -2$, $\sum \frac{(-1)^k (-2)^k}{k2^k}$ is divergent.

At $x = 2$, $\sum \frac{(-1)^k 2^k}{k2^k} = \sum \frac{1}{k}$ is convergent.

Interval is $x \in (-2, 2)$ or $-2 < x < 2$
Given the series below, determine
1) the radius and interval of convergence
2) values of $x$ where series is absolutely convergent
3) values of $x$ where the series is conditionally convergent

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k}$$

2) series is abs. convergent where $\sum \frac{x^k}{k2^k}$ converges
   - by ratio test: abs. convergent for $(-2, +2)$
   - at $x=\pm2$: divergent
   $$\Rightarrow \text{A.C. for } x \in (-2, +2)$$

3) C.C. at $x = +2$
   (because at $x = 2$, series is convergent but not A.convergent
1) Find the radius and interval of convergence.

\[ \sum_{k=1}^{\infty} \frac{x^k}{k} \]

**TRY: Ratio test** with \( a_k = \frac{x^k}{k} \)

\[ \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{x^{k+1}}{x^k} \cdot \frac{k}{k+1} \right| = \left| \frac{x}{1} \right| = |x| \Rightarrow \text{RADIUS } = 1 \]

At \( x = +1 \), divergent p-series, \( p = 1 \)

At \( x = -1 \), convergent by alternating series test

Interval is \(-1 < x < +1\)
2) Find the radius and interval of convergence.

\[ \sum_{k=1}^{\infty} (-k)^4 k^{4k} x^{4k} \]

TRY RATIO TEST:

\[
\lim_{k \to \infty} \left| \frac{(-k+1)^4 (k+1)^{4k}}{(-k)^4 k^{4k}} \cdot \frac{x^{4k}}{x^{4k}} \right| = \lim_{k \to \infty} \left| \frac{(k+1)^4 (k+1)^{4k}}{k^{4k}} \cdot x^4 \right|
\]

The limit ONE, unless \( x = 0 \).

\[ \Rightarrow \text{ RADIUS IS ZERO} \]

\[ \text{INTERVAL IS THE POINT } x = 0 \]
3) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

\[ a) \sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2 + 1} \]

Not abs. convergent: \( \frac{k}{k^2 + 1} > \frac{k}{2k^2} = \frac{1}{2k} \) (p-series)

But \( \lim_{k \to \infty} \frac{k}{k^2 + 1} = 0 \), so the series is conditionally convergent by the alt. series test.

\[ b) \sum_{k=1}^{\infty} \frac{\cos(\pi k)}{k} = \sum_{k=1}^{\infty} (-1)^k k \]

Not abs. convergent, because \( \cos(\pi k) = (-1)^k \)

But is conditionally convergent by the alternating series test.
4) Find the radius and interval of convergence. Then find the values of \( x \) for which series is

a) absolutely convergent

b) conditionally convergent

c) divergent.

\[
1 - \frac{x}{2} + \frac{2x^2}{4} - \frac{3x^3}{8} + \frac{4x^4}{16} - \ldots
\]

\[
= \left( \sum_{k=1}^{\infty} \frac{kx^k \cdot (-1)^k}{2^k} \right) + 1
\]

Try ratio test: \( \left| \frac{\frac{k+1}{k} \cdot \frac{2^{k+1}}{2^{k+1}} \cdot \frac{x^{k+1}}{x^k}}{\frac{2^{k}}{2^{k}} \cdot \frac{x^k}{x^k}} \right| = \left| 2x \right| < 1 \Rightarrow \text{radius is } \frac{1}{2}
\]

At \( x = \pm 2 \), the series is divergent by the test for divergence.

\( \Rightarrow \) interval is just \( x \in (-0.5, +0.5) \).

9) ABS. CONVERGENT ONLY ON \( -\frac{1}{2} < x < \frac{1}{2} \)

b) C. CONVERGENT NO WHERE

c) DIVERGENT FOR \( |x| > \frac{1}{2} \)
Recitation 10

Today: 10.8, 10.9 (Taylor Polynomials and Series)

Complete the Following Formulas

The N\textsuperscript{th} order Taylor Polynomial about \( x = a \) is:

\[
P_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(a)}{k!} (x-a)^k
\]

The Taylor Series about \( x = a \) is:

\[
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = P_N(x) - R_N(x)
\]

The phrase "about \( x = a \)" means that the expansion is "centered" at \( x = a \)

Remainder formula for expansion about \( x = a \):

\[
|R_n(x)| \leq \left( \max_{t \in [a, x]} |f^{(n+1)}(t)| \right) \frac{|x-a|^{n+1}}{(n+1)!}
\]

Quiz 1 has been graded, grades are in T-Square, you'll receive your graded quizzes this week via email, please make sure they were graded correctly and let me know if you have questions.
Example

a) Plot a rough sketch of \( f(x) = \frac{1}{1 - x^2} \) for \( x \) between -1 and +1.

b) Use the Taylor expansion for \( \frac{1}{1 - x} \) to find \( P_0(x), P_1(x), \) and \( P_2(x) \) of \( f(x) \) about \( x = 0 \).

c) Sketch \( P_0, P_1, P_2 \)

Hint: the Taylor expansion of \( \frac{1}{1 - x} \) about \( x = 0 \) is \( \frac{1}{1 - x} = 1 + x + x^2 + x^3 + ... \)

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]

\[
\frac{d}{dx} \left( \frac{1}{1-x} \right) \cdot (-1) = \frac{d}{dx} \left( 1 + x + x^2 + x^3 + ... \right) = 1 + 2x + 3x^2 + ...
\]

\[
\Rightarrow P_0 = 1, \quad P_1 = 1 + 2x, \quad P_2 = 1 + 2x + 3x^2
\]

Alternate approaches:

1) Expand & collect

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]

2) Use formula:

\[
P_0 = \frac{f^{(0)}(a)}{0!} (x-a)^0 = 1, \quad P_1 = \frac{f^{(1)}(a)}{1!} (x-a) + P_0 = 1 + 2x, \quad e^c
\]
Group Work Suggestions

- everyone pick a color to write in, and match it with their text
- use mics if you have them
- before moving to the next question:
  - ask if everyone agrees with answer
  - ask if everyone understands how to get the answer
- message me if your group gets stuck

Make sure you can solve Question #4 before the next quiz.
1) Find the Taylor expansion about $x = 0$ of $f(x) = \frac{2x}{1+x^2}$

Hint: write the Taylor expansion for $1/(1-x)$ at $x = 0$, and then apply suitable modifications to the expansion.

\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots
\]

\[
\frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots
\]

\[
\frac{2x}{1+x^2} = 2x \left(1-x^2 + x^4 - x^6 + \cdots\right) = f(x)
\]

DONE.

IF YOU PREFER: $f(x) = \sum_{k=0}^{\infty} 2x (-1)^k (x^2)^k$
2) Fill in the blank: the Maclaurin series is just the Taylor series about the point $x=0$.

The Maclaurin series for $e^x$ is

$$e^x = \sum_{k=0}^{\infty} \frac{(e^x)^{(k)}(0)}{k!} x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Use this expansion to find the Maclaurin series for $f(x)$, where:

$$f(x) = e^{-x^2} = \exp(-x^2)$$

Also find $P_2(x)$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

(because $(e^x)^{(k)}(0) = 1$ for all $k$)

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}$$

$$P_2(x) = 1 + (-x^2) + \frac{(-x^2)^2}{2!} = 1 - x^2 + \frac{x^4}{2}$$
3) Estimate the following integral to within 0.01 using series.
*Hint: use the Alternating Series Remainder Theorem.*

\[
\int_0^1 x^4 e^{-x^2} \, dx
\]

\[
e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = (-x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots)
\]

\[
x^4 e^{-x^2} = x^4 - x^6 + \frac{x^8}{2!} - \frac{x^{10}}{3!} + \ldots
\]

\[
\int_0^1 x^4 e^{-x^2} \, dx = \left(\frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9 \cdot 2} - \frac{x^{11}}{11 \cdot 3 \cdot 2} + \ldots\right) \bigg|_0^1
\]

\[
= \frac{1}{5} - \frac{1}{7} + \frac{1}{18} - \frac{1}{288} + \ldots
\]

The Alt. Series Remainder Theorem is \(|R_n| < |a_{n+1}|\)

\[a_4 = \frac{1}{56} > 0.01, \quad a_5 = \frac{1}{288} < 0.01. \text{ Thus } \int_0^1 x^4 e^{-x^2} \, dx \approx \frac{1}{5} - \frac{1}{7} + \frac{1}{18} - \frac{1}{288} \ldots\]

ie: First four terms needed.
4) Estimate \( e^{3/2} \) to within \( 10^{-4} \).

Note: this is a question from a Math 1502 quiz from 2012.
Hint: use the Taylor Series Remainder Theorem, and approximate \( e^{3/2} \) with 5.

\[
\left| R_N(x) \right| = \left| f(x) - P_N(x) \right| = \max_c f^{(N+1)}(c) \left| x \right|^{N+1} / (N+1)!, \quad c \in [0, x].
\]

**WHAT DO WE USE FOR \( \max_c f^{(N+1)}(c) \)? Use 5, because \( e^{3/2} \approx 4.48. \)**

<table>
<thead>
<tr>
<th>( N )</th>
<th>( 5 \cdot (3/2)^{N+1} / (N+1)! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.625</td>
</tr>
<tr>
<td>2</td>
<td>2.8125</td>
</tr>
<tr>
<td>8</td>
<td>0.0005296...</td>
</tr>
<tr>
<td>9</td>
<td>0.00008 &lt; 10^{-4} = 0.0001</td>
</tr>
</tbody>
</table>

Thus \( e^{3/2} \approx P_q \left( \frac{3}{2} \right) = \left( 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^q}{q!} \right) \bigg|_{x=3/2} \)

**DONE. (Note if you evaluate this, you get 4.48167....)**
5) Let \( f(x) = \frac{\exp(2x^2) - 1 - x^2}{x^4} \). Note: \( \exp(x) = e^x \)

a) Find \( P_2(x) \) about \( x = 0 \)

b) Use the result from part a) to evaluate the limit of \( f(x) \) as \( x \) goes to zero.

\[
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\]

Substitute \( x \) with \( 2x^2 \).

\[
e^{2x^2} = 1 + \frac{(2x^2)^1}{1!} + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \ldots
\]

Therefore:

\[
f(x) = \frac{\left(e^{2x^2} - 1 - x^2\right)}{x^4} = \frac{\left[\left(1 + \frac{2x^2}{1!} + \frac{4x^4}{2!} + \frac{8x^6}{3!} + \ldots\right) - 1 - x^2\right]}{x^4}
\]

\[
= \frac{[0 + x^2 + 2x^4 + \frac{8}{3!}x^6 + \ldots]}{x^4}
\]

\[
= 0 + x^{-2} + 2 + \frac{8}{3!}x^2 + \ldots
\]

\[
\Rightarrow P_2(x) = 2 + \frac{8}{3!}x^2
\]

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(0 + \frac{1}{x^2} + 2 + \frac{8}{3!}x^2 + \ldots\right) \text{ which does not exist.}
\]

\[
\lim_{x \to 0} f(x) \Rightarrow f(x) \to \infty \text{ as } x \to 0.
\]

(THANK YOU ANDREW DURBIN FOR CORRECTING MY ERROR ON THIS QUESTION!)
Recitation 11

Today: 4.5 (l'Hospital's Rule)

Describe how you would evaluate the following limits.

\[
\lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{is of the form } 0/0 \text{ or } \infty/\infty, \text{ then we can:}
\]

\[
\text{we evaluate } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

\[
\lim_{x \to a} f(x) - g(x) \quad \text{is of the form } \infty - \infty, \text{ then we can:}
\]

\[
\text{find common denominator, use } \ast
\]

\[
\lim_{x \to a} f(x)^{g(x)} \quad \text{is of the form } 0^0, 1^\infty, \text{ or } \infty^0, \text{ then we can:}
\]

\[
\text{use } \lim_{x \to a} e^{\ln f^g} = \lim_{x \to a} e^{g \ln f}, \text{ use } \ast
\]

Quiz 1 has been graded, grades are in T-Square, you'll receive your graded quizzes this week via email, please make sure they were graded correctly and let me know if you have questions.
POP QUIZ #1

- Start time: 8:10?
- Ends at: 8:25
- Pop quiz grading
  - 5 points: correct
  - 4 points: something correct
  - 3 points: name on the page
  - 0 points: did not take pop quiz

- To submit your work, choose any of the following:

  A. **work on whiteboard in breakout room**
     - type A in text chat so I know you want to work in breakout room
     - submit work by letting me know when done, and/or email me a screen capture of your work

  B. **work on paper and give work to facilitator**
     - type B in text chat so I know you’re doing this
     - leave 2 inch margin on paper
     - write your name and QH8 at the top
     - facilitator is receiving instructions today on how to submit your work

  C. **work on paper and email a photo of your work to me**
     - type C in text chat so I know you are emailing your work to me
     - email your photo to me before 8:40
Find the Taylor series, centered at $a = 0$, of $\sin(5x^2)$.

Taylor series at $a=0$ of $\sin x$ is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(1+2k)!}$$

$$\Rightarrow \sin(5x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (5x^2)^{2k+1}}{(1+2k)!}$$

DONE.
Evaluate: \( \lim_{x \to \infty} \left( \cos \frac{1}{x} \right)^x = 1^{\infty} \)

\[ = \lim_{x \to \infty} e^{\ln \cos (1/x) x} \]

\[ = \lim_{x \to \infty} e^{x \ln \cos 1/x} \]

\[ = \lim_{x \to \infty} e^{(\ln \cos 1/x)'(1/x/1/x)} \]

\[ = \lim_{x \to \infty} e^{-\cos(1/x) \cdot \sin(1/x)(-1/x^2) / (1/x)} \]

\[ = \lim_{x \to \infty} e^{-\sin(1/x)/x^2} \]

\[ = e^0 = 1 \]
1) Evaluate the following limit by using l'Hospital's rule, if possible.

*Note: this limit is a well-known definition for an important number.*

\[
\lim_{x \to 1} x^{1/(x-1)}
\]

\[
= \ln x^{1/(x-1)}
\]

\[
= \lim_{x \to 1} \ln x
\]

\[
= \lim_{x \to 1} \frac{1}{x-1} \ln x
\]

\[
= \lim_{x \to 1} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} (x-1)}
\]

\[
= \lim_{x \to 1} e^{1/x} = e
\]
2) Evaluate the following limit by using l'Hospital's rule, if possible.

\[
\lim_{x \to \infty} x \sin \left( \frac{\pi}{x} \right) , \quad \text{i.F. type } 0 \cdot \infty \quad (\text{i.F. = indeterminate form})
\]

\[
= \lim_{x \to \infty} \frac{\sin \left( \frac{\pi}{x} \right)}{\frac{1}{x}}, \quad \text{i.F. type } 0
\]

\[
\overset{H}{=} \lim_{x \to \infty} \cos \left( \frac{\pi}{x} \right) \cdot \frac{1}{x^2}
\]

\[
= \lim_{x \to \infty} \cos \left( \frac{\pi}{x} \right) \cdot \pi \cdot (-x^{-2})
\]

\[
= \lim_{x \to \infty} \frac{\cos \left( \frac{\pi}{x} \right) \cdot \pi \cdot (-x^{-2})}{-x^{-2}}
\]

\[
= \pi
\]
3) Evaluate the following limit by using l'Hospital's rule, if possible.

\[ \lim_{x \to 0^+} x (\ln|x|)^2 \]

\[ = \lim_{x \to 0} \frac{(\ln x)^2}{1/x} , \quad \text{I.F. type } \frac{\infty}{\infty} \]

\[ = \lim_{x \to 0} \frac{2 \ln(x) - \frac{1}{x}}{-\frac{1}{x^2}} \]

\[ = 2 \lim_{x \to 0} \frac{\ln(x)}{1/x} , \quad \text{I.F. type } \frac{\infty}{\infty} \]

\[ = 2 \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \]

\[ = 2 \lim_{x \to 0} (-x) = 0 \]
4) Evaluate the following limit by using l'Hospital's rule, if possible.

a) \[ \lim_{x \to \infty} \frac{1}{x} \int_0^x e^t \, dt \]
\[ \overset{\text{H}}{=} \lim_{x \to 0} \frac{e^{x^2}}{1} = \infty \]

b) \[ \lim_{x \to 0} \left( \frac{1}{\ln(1+x)} - \frac{1}{x} \right) \]
\[ = \lim_{x \to 0} \frac{x - \ln(1+x)}{x \ln(1+x)} \]
\[ \overset{\text{H}}{=} \lim_{x \to 0} \frac{1 - (1+x)^{-1}}{\ln(1+x) + \frac{x}{1+x}} \]
\[ = \lim_{x \to 0} \frac{(1+x) - 1}{(1+x) \ln(1+x) + x} \]
\[ \overset{\text{H}}{=} \lim_{x \to 0} \frac{1}{\ln(1+x) + 1 + 1} \]
\[ = \frac{1}{2} \]
5) \( \vec{v} \cdot \vec{v} = \|\vec{v}\|^2 \), is the \underline{length} of vector \( \vec{v} \).

b) A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.

\textit{Solution next recitation}
Suppose we have the points P(1,2,3), Q(1,3,4), R(2,2,2).

a) Find a vector that is normal to the plane that contains the three points.

b) Find an equation of the plane.

Thus, \( \mathbf{N} = \mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix} i & j & k \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} \\ 1 & 0 & -1 \end{vmatrix} = \hat{i} + \hat{j} - \hat{k} \) is \( \perp \) to the plane. \( \text{DONE} \)

Let \( \mathbf{r} = (x, y, z) \) be a point in the plane. So \( \mathbf{F} - \mathbf{P} \) is a vector in the plane.

Thus: \( \mathbf{O} = \mathbf{N} \cdot \mathbf{r} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 2 \\ z - 3 \end{pmatrix} = -x + y - z + 1 - 2 + 3 \Rightarrow -x + y - z = -2 \)

2) The equation \( Ax + By + Cz = D \) is a plane. A vector perpendicular to the plane is:

\[
\begin{pmatrix} A \\ B \\ C \end{pmatrix} \text{ or } A \hat{i} + B \hat{j} + C \hat{k}
\]
Find a parametrization of the line that is the intersection of the planes

\[ P: \quad x - 2y + z = 3 \]
\[ Q: \quad 2x + y + z = 1 \]

we need a line of form \( \vec{r} = \vec{r}_0 + \vec{v}t \)

Note: this is a question from a 2012 Math 1502 quiz.

- Vectors \( \perp \) to P & Q are \( \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \)

- A vector \( \perp \) to normals is \( \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \)

- A point in intersection can be found by finding any point that satisfies both equations:
  - Let \( z = 0 \), then \( x - 2y = 3 \) \( \Rightarrow \) \( x = -1 \)
  - \( 2x + y = 1 \) \( \Rightarrow \) \( y = 1 \)

\[ \Rightarrow ( -1 , 1 , 0 ) \text{ is a point in intersection.} \]

\[ \Rightarrow \text{line is} \quad \vec{r} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}t \]
1) Find the equation for the line that is perpendicular to the yz-plane, and also passes through P(1,4,3).

A vector \( \perp \) to yz-plane is \((\hat{i} + \hat{0j} + \hat{0k})\).

\[ \Rightarrow \text{line is } \vec{r} = (4) + (0) t, \quad \text{DONE.} \]

To see why the vector \((\hat{i})\) is \( \perp \) to the yz-plane, consider that:

* The yz-plane has the equation \( x = 0 \),

  or \( 1x + 0y + 0z = 0 \).

* The vector, or a vector \( \perp \) to the plane is

  the coefficients of \( x, y, z \). So the coefficients are \( 1, 0, 0 \), so a vector \( \perp \) is \((\hat{i})\).
2) Do these lines intersect each other? Why/why not?

\[ x_1 = 1 + t \quad \quad x_2 = 1 - u \]
\[ y_1 = -1 - t \quad \quad y_2 = 1 + 3u \]
\[ z_1 = -4 + 2t \quad \quad z_2 = -2u \]

If they intersect, \( \exists \ t, u \) st.

1. \( 1 + t = 1 - u \Rightarrow t = -u \)
2. \( -1 - t = 1 + 3u \Rightarrow t + 3u = 2 \), or \( t + 3(-t) = 2 \)
3. \( -4 + 2t = -2u \)

Equation 3 must be satisfied by \( t = 1 \) and \( u = -1 \). Is it?

\[-4 + 2(1) = -2u\]
\[-4 + 2(1) = -2(-1)\]
\[-6 = -2 \quad \text{NOT SATISFIED} \]

\[ \Rightarrow \text{LINES DON'T INTERSECT} \text{ (i.e., they are "skew")} \]
3) a) If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{b} = 0$, what can we conclude about vectors $\mathbf{a}$ and $\mathbf{b}$? Explain your reasoning.

$\mathbf{a} \times \mathbf{b} = \mathbf{0}$ means $\mathbf{a} \parallel \mathbf{b}$, or at least one of $\mathbf{a}, \mathbf{b}$ is $\mathbf{0}$.

$\mathbf{a} \cdot \mathbf{b} = 0$ means $\mathbf{a} \perp \mathbf{b}$, or at least one of $\mathbf{a}, \mathbf{b}$ is $\mathbf{0}$.

$\Rightarrow$ at least one of $\mathbf{a}, \mathbf{b}$ is zero vector.

b) Which of the following make sense? Explain why/why not.

i. $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$  **NONSENSE**: $\mathbf{a} \cdot \mathbf{b}$ is a scalar, $\mathbf{a} \times (\text{scalar})$ is illegal

ii. $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$  **NONSENSE**: $\mathbf{a} \cdot (\text{scalar})$ is illegal

iii. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$  **OK**: $\mathbf{a} \times \text{vector}$ is legal

iv. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$  **OK**: $\mathbf{a} \cdot \text{vector}$ is legal
4) Vectors that are co-planar are in the same plane. Determine whether the vectors are co-planar:

\[ \mathbf{P} : j - k \]

\[ \mathbf{Q} : 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} \]

\[ \mathbf{R} : 3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \]

\[ (\mathbf{P} \times \mathbf{Q}) \cdot \mathbf{R} = \mathbf{0} \Rightarrow \text{co-planar} \]

\[ (\mathbf{P} \times \mathbf{Q}) \cdot \mathbf{R} = \mathbf{0} \Rightarrow \text{not co-planar} \]

\[ \mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 3 & -1 & 2 \end{vmatrix} = \mathbf{i} + 3\mathbf{j} - 3\mathbf{k} \]

\[ \mathbf{P} \times \mathbf{Q} \cdot \mathbf{R} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} = 3 - 6 - 9 \neq 0 \]

\[ \Rightarrow \text{not co-planar} \]
Recitation 13

Today: Quiz 2 Review

Estimate to within 0.0001 by using series.

\[ I = \int_0^{1/2} \ln(1 + x) \frac{dx}{x} \]

\[ = \int_0^1 \frac{1}{x} \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \right) dx \]

\[ = \int_0^1 \left( 1 - \frac{x}{2} + \frac{x^2}{3} - \cdots \right) dx \]

\[ = \left[ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \right] \mid_0^{1/2} \]

\[ = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2 2^k} \]

By the alternating series remainder theorem, the first term that is less than 0.0000 is \( k = 8 \).

\[ \Rightarrow \quad \frac{\varepsilon}{2} < I < \sum_{k=1}^{8} \frac{(-1)^{k-1}}{k^2 2^k} \]

\[ I \approx \sum_{k=1}^{8} \frac{(-1)^{k-1}}{k^2 2^k} \]

Office hours: Tuesday and Wednesday 7:30 to 8:30 pm
https://georgiatech.adobeconnect.com/distancecalculusofficehours
Group Work Suggestions

- before moving to the next question:
  - ask if everyone agrees with answer
  - ask if everyone understands how to get the answer

- everyone pick a color to write in, and match it with their text

- use mics if you have them

- message me if your group gets stuck
1) Evaluate by a) using power series, and b) l'Hopital's rule.

\[
\lim_{x \to 0} \frac{\cos(x) - 1}{x \sin x}
\]

a) \[
\lim_{x \to 0} \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - 1}{x \left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)} = \lim_{x \to 0} \frac{-\frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}{x^2 \left(1 - \frac{x^2}{3!} + \cdots\right)} = -\frac{1}{2}
\]

b) \[
\lim_{x \to 0} \frac{-\sin x}{\sin x + x \cos x} = \lim_{x \to 0} \frac{-\cos x}{\cos x + \cos x - x \sin x}
\]

\[
= \frac{-1}{2} = \frac{-1}{1 + 1 - 0}
\]

\[(\text{PART b) USES 'HOP. TWICE'})\]
2) Find the interval of convergence.

\[ \sum_{k=1}^{\infty} \frac{\ln k}{k} (x+1)^k \]

**Ratio Test:**

\[
\lim_{k \to \infty} \left| \frac{\ln (k+1)}{\ln k} \cdot \frac{k}{k+1} \cdot \frac{(x+1)^{k+1}}{(x+1)^k} \right| = |x+1| < 1
\]

\[ (-1 < x+1 < 1) \]

\[ (-2 < x < 0) \]

\[ \Rightarrow \text{converges for } x \in (-2, 0), \text{ and at endpoints:} \]

\[ x = -2, \text{ the series is } \sum_{k=1}^{\infty} \frac{\ln k}{k} \text{ diverges by comparison with harmonic series,} \]

\[ x = 0, \text{ the series is } \sum_{k=1}^{\infty} \frac{\ln k}{k} (1)^k \text{ converges by A.S.T,} \]

\[ \Rightarrow \text{interval is } [-2, 0) \]
3) Find the distance between planes P1 and P2.

P1: \( x + 2y + z = 3 \)

P2: \( x + 2y + z = 9 \)

A point on \( P_1 \) is \((0,0,3)\).

The normal to \( P_1 \) is \([1, 2, 1]\).

A line connecting \( P_1 \) to \( P_2 \) is \( L = [0, 0, 3] + t [1, 2, 1] \).

The point on \( P_2 \) closest to \((9,9,3)\) is the point where:

\[
x + 2y + z = 9 \\
(0 + t) + 2(0 + 2t) + (3 + t) = 9
\]

Solving: \( 6t = 6 \), \( t = 1 \). Thus, line intersects \( P_2 \) when \( t = 1 \).

\( \Rightarrow \) this point is \((0+1, 0+2, 3+1)\) or \((1, 2, 4)\).

Distance between \((1, 2, 4)\) and \((9,9,3)\) is \( \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6} \).
4) The line $L$ is determined from $P_1$ and $P_2$. The plane $Q$ is determined by $Q_1$, $Q_2$, $Q_3$.

Does $L$ intersect $Q$? If so, where?

$L$ is $\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$

$L$ is $\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$

$Q$ has normal $\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$, or $\begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}$

$\Rightarrow Q$ has equation $\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \cdot \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} \right) = 0$

$\Rightarrow x + 4y - 2 = 6$

If $L$ intersects $Q$, $\exists t$ s.t.

$(1 - 3t) + 4(-1 + 4t) - (2 - t) = 6 \Rightarrow t = \frac{11}{14}$

$\Rightarrow x = 1 - 3 \cdot \frac{11}{14} = \frac{-19}{14}$

$y = -1 + 4 \cdot \frac{11}{14} = \frac{15}{7}$

$z = 2 - \frac{11}{14} = \frac{17}{14}$
5) From 2012 Quiz 1

Tyler Hamilton would like you to find

a) A series for $\int_{0}^{x} \sin(\pi t^2/2) \, dt$

\[
= \int_{0}^{x} \left(1 - \frac{(\pi t^2)^3}{3!} + \frac{(\pi t^2)^5}{5!} - \ldots\right) \, dt
\]

\[
= x - \frac{\pi^3 t^7}{2^3 \cdot 3! \cdot 7} + \frac{\pi^5 t^{11}}{2^5 \cdot 5! \cdot 11} - \ldots
\]
6) From 2012 Quiz 1

Greg Lemond wants you to use series and error bounds to estimate $e^{7/2}$ to within $10^{-3}$. You must use (an) error bound. You do not have to actually sum, just say how many terms, (or the highest power of $7/2$). (You may use $e^{7/2} \leq 35$).

**LAGRANGE FORM OF ERROR:** 

$$35 \cdot \frac{(7/2)^{n+1}}{(n+1)!}$$

FIND A VALUE OF $N$ S.T. THIS IS LESS THAN $10^{-3}$.

15 works,

$\Rightarrow$ 15 terms.
7) Vectors that are co-planar are in the same plane. Determine whether the vectors are co-planar:

\[ P: \quad \vec{i} + \vec{j} - \vec{k} \]
\[ Q: \quad 2\vec{i} - \vec{j} \]
\[ R: \quad 3\vec{i} - \vec{j} - \vec{k} \]

**METHOD 1**
\[
\vec{P} \times \vec{Q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & 1 & 0 \end{vmatrix} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \cdot \vec{R} = 3 + 2 + 1 = 6
\]

If co-planar, \( \vec{P} \times \vec{Q} \cdot \vec{R} = 0 \), \( \Rightarrow \) not co-planar.

**ALTERNATE METHOD:**
- Co-planar iff \( \vec{P} \vec{Q} \parallel \vec{P} \times \vec{R} \), \( \text{Notation: } \parallel = \text{parallel} \)

\[
\vec{P} \times \vec{Q} = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}
\]
\[
\vec{P} \times \vec{R} = \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix}
\]

These vectors are not scalar multiples of each other, and so are not parallel.
8) Find a parametrization of the line that is the intersection of the planes

\[ P: \quad -x + 2y + z = 2 \]
\[ Q: \quad x + y + 2z = 11 \]
\[ x + y + z + 1 = 0 \]
\[ x - y + z + 2 = 0 \]

Note: this is a modified version of a question from a 2012 Math 1502 quiz.

Normals to \( P \) & \( Q \) are \([1, 1, 1]\) and \([-1, -1, 1]\).

A vector \( \vec{v} \) to desired line is
\[ \frac{1}{2}\vec{v} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \]

A point in intersection is \( x = 0, \quad \begin{cases} y + z = -1 \\ y + z = -2 \end{cases} \) \( \implies y = \frac{1}{2}, \quad z = -\frac{3}{2} \)

\[ \vec{e} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \]
Recitation 15

Today: Solving linear systems of equations

The following was a 2013 pop quiz question. For what values of \( a \) does the following system have a solution?

\[
\begin{align*}
7x + 2y - 3z &= 25 \\
y + 3z &= 5 \\
3y + az &= 3
\end{align*}
\]

If you prefer, we can write out augmented matrices:

\[
\begin{align*}
\mathbf{R}_3 - 3\mathbf{R}_2 & \rightarrow \begin{pmatrix} 7 & 0 & -9 & 15 \\ 0 & 1 & 3 & 5 \\ 0 & 3 & a-9 & 3 \end{pmatrix} \\
\mathbf{R}_1 - 2\mathbf{R}_2 & \rightarrow \begin{pmatrix} 7 & 0 & -9 & 15 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & a-9 & -12 \end{pmatrix} \\
\mathbf{R}_2 - 3\mathbf{R}_3 & \rightarrow \begin{pmatrix} 7 & 0 & -9 & 15 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -12/(a-9) \end{pmatrix}
\end{align*}
\]

\[ x_3 = \frac{-12}{a-9} \quad \Rightarrow \quad as \ a \to 9, \ then \ x_3 \to +\infty. \quad \text{Thus } a \neq 9. \]

But if \( a \neq 9, \ x_3 \) has a solution, as does the system:

\[
\begin{align*}
x_1 &= \frac{1}{9} \left( 15 + 9x_3 \right) \\
x_2 &= 5 - 3x_3 \\
x_3 &= \frac{12}{9-a}
\end{align*}
\]

There are 30 recitations in the semester (including quizzes). We're \( \sim 50\% \) through the course.
A Few Definitions

a) A system of linear equations is consistent if it has at least one solution.

b) A system of linear equations is inconsistent if it has no solutions.

c) A system of linear equations that is overdetermined has more equations than unknowns.

d) Can a system of linear equations be overdetermined and consistent? If yes, provide an example with at least 3 equations.

\[
\text{YES: } \begin{pmatrix}
1 & 2 & 5 \\
1 & 3 & 7 \\
2 & 4 & 10
\end{pmatrix}, \text{ i.e } x_1=1, x_2=2
\]
3) Find $h$ and $k$ such that the system has
   a) no sol'n
   b) a unique sol'n
   c) infinitely many solutions

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

$\begin{pmatrix} 1 & h & 2 \\ 4 & 8 & k \end{pmatrix} \sim \begin{pmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{pmatrix}$

**PART a)**

a) $\Rightarrow$ if $h=2$ and $k \neq 8$ then (2) is inconsistent, so the
   system is inconsistent (because LHS=0, RHS $\neq 0$)

**PART c)**

If $k=8, h=2$ we have $0x_1 + 0x_2 = 0$ and $x_1 = 2 - hx_2$
   which has $\infty$ many solutions. (because $x_2$ is free)

b) If $k$ = anything and $h \neq 2$, unique sol'n.
4) A $3 \times 4$ coefficient matrix has three pivot columns. Is the system consistent? Why/why not?

Consider: 
\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
is a coeff. matrix, which yields an infinite number of solutions so yes.

(ii.e. the augmented matrix can't have row of form \([0\ 0\ 0\ c]\)

5) Find the general solution to the system whose augmented matrix is given below.

\[
\begin{bmatrix}
3 & -2 & 4 & 0 \\
9 & -6 & 12 & 0 \\
6 & -4 & 8 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
3 & -2 & 4 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\Rightarrow x_1 = \frac{1}{3}(2x_2 - 4x_3)
\]

\(x_2, x_3\) are free
6) True or false.
   a) The reduced echelon form of a matrix is unique.
   b) If a system has free variables, the solution set has many solutions.
   c) If a row in an echelon form of an augmented matrix is [0 0 0 3 0], then the linear system is inconsistent.

a) **TRUE**

b) **FALSE!** eg Aug MTX $\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ INCONSISTENT 3 F.V.

c) **FALSE!** row just says that $3x_4 = 0$. By itself does not mean system is inconsistent.
Recitation 16

Today: Span, Linear Dependence

Definitions

Assume that \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are arbitrary vectors.

a) The sum \( c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 \) is a ______ combination of vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).

b) The set of all possible ______ combinations of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) is the ______ of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).

c) Any vector in the ______ of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) can be written as a ______ of vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).

Question 1

Let \( \mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \), \( \mathbf{a}_2 = \begin{bmatrix} -6 \\ -17 \\ 2 \end{bmatrix} \), and \( \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ h \end{bmatrix} \). For what value(s) of \( h \) is \( \mathbf{b} \) in the plane spanned by \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \)?

If you haven’t already, please take a few minutes to fill out the technical issues survey.
1) This is similar to the **first** question on your next HW.

Let \( \mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \), \( \mathbf{a}_2 = \begin{bmatrix} -6 \\ -17 \\ 2 \end{bmatrix} \), and \( \mathbf{b} = \begin{bmatrix} 4 \\ h \end{bmatrix} \). For what value(s) of \( h \) is \( \mathbf{b} \) in the plane spanned by \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \)?

**If \( \mathbf{b} \) is in the plane, \( \exists \ c_1 \& \ c_2 \) s.t.

\[
\mathbf{c}_1 \mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_2 = \mathbf{b}
\]

(Or \( A \mathbf{v} = \mathbf{b} \), \( A \) is the m×x formed from \( \mathbf{a}_1 \).

Aug. matrix:

\[
\begin{bmatrix}
1 & -6 & 4 \\
4 & -17 & 2 \\
-1 & 2 & h
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -6 & 4 \\
0 & 7 & -14 \\
0 & -4 & h+4
\end{bmatrix}
\]

\[
0c_1 - 4c_2 = h + 4, \quad -4c_2 = h + 4 \implies h = 4
\]

if \( h = 4 \), \( \mathbf{b} \) is co-planar with \( \overrightarrow{a}_1, \overrightarrow{a}_2 \)
Linear Dependence

Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_N$ are linearly dependent (LD) if $\exists c_1, c_2, c_3, \ldots, c_N$ not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \ldots + c_N\vec{v}_N = \vec{0}$$

If the vectors are not LD, they are \textit{linearly independent}.


To determine whether a set of vectors are LD, we solve:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \ldots + c_N\vec{v}_N = \vec{0}$$

which has the same solution as the linear system whose augmented matrix is $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \ldots \ \vec{v}_N \ 0]$. If $c_i$ not all zero, LD. If $c_i = 0$ for all $i$, LI.
2) Determine whether the following vectors are LI.

\[
\begin{bmatrix}
5 \\
1 \\
\end{bmatrix}, \begin{bmatrix}
2 \\
8 \\
\end{bmatrix}, \begin{bmatrix}
1 \\
3 \\
\end{bmatrix}, \begin{bmatrix}
-1 \\
7 \\
\end{bmatrix}
\]

If LI, \( \exists \: c_1, c_2, c_3, c_4 \) s.t.

\[
c_1 \vec{v_1} + c_2 \vec{v_2} + c_3 \vec{v_3} + c_4 \vec{v_4} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

2 equations, 4 unknowns.

\( \Rightarrow \) at least 2 f.v.

\( \Rightarrow \) not all \( c_1, c_2, c_3, c_4 \) are zero.

\( \Rightarrow \) LI.
3) This is similar to the second question on your next HW.

Determine whether vector \( \mathbf{b} \) is in the set spanned by the columns of matrix \( \mathbf{A} \).

\[
\mathbf{A} = \begin{pmatrix}
1 & 0 & 5 \\
-2 & 1 & -6 \\
0 & 2 & 8
\end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix}
2 \\
-1 \\
6
\end{pmatrix}
\]

If \( \mathbf{b} \) spanned by \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \), then

\[
\mathbf{c}_1 \mathbf{a}_1 + \mathbf{c}_2 \mathbf{a}_2 + \mathbf{c}_3 \mathbf{a}_3 = \mathbf{b}
\]

\[
\begin{pmatrix}
1 & 0 & 5 \\
-2 & 1 & -6 \\
0 & 2 & 8
\end{pmatrix}
\begin{pmatrix}
\mathbf{c}_1 \\
\mathbf{c}_2 \\
\mathbf{c}_3
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 5 & 2 \\
0 & 1 & 4 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\Rightarrow \mathbf{c}_1 = 3 - 4 \mathbf{c}_3, \quad \mathbf{c}_2 = 2 - 5 \mathbf{c}_3, \quad \mathbf{c}_3 \text{ free} \Rightarrow \mathbf{b} \text{ is in the set because } \mathbf{c}_3 \text{ is not all zero}
\]

(Also: \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b} \) are LD, because \( \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \) exist and are not all zero)
4) This is similar to the third question on your next HW.

Determine whether vector $b$ can be written as a linear combination of vectors $a_1$ and $a_2$. In other words, determine whether $x_1$ and $x_2$ exist such that $x_1a_1 + x_2a_2 = b$. If possible, find $x_1$ and $x_2$.

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

If $b$ is a lin. comb. of $a_1$ & $a_2$, then $\exists x_1 \& x_2 s.t. \ x_1a_1 + x_2a_2 = b$.

Form aug. $m \times x$:

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 14 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 6 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_1 + 0x_2 = 3$

$0x_1 + x_2 = 2$

$0x_1 + 0x_2 = 0$

$\Rightarrow x_1 = +3, \ \ x_2 = +2$

$\Rightarrow \quad b$ is a lin. comb. of $a_1, a_2$

and $x_1 = +3, \ x_2 = +2$
5) Determine whether the following vectors are L.I.

\[
\begin{bmatrix}
5 \\
-3 \\
-1
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-7 \\
2 \\
4
\end{bmatrix}
\]

If L.I., then \( \exists \ c_1, c_2, c_3 \) not all zero s.t.

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}
\]

Make augmented matrix:

\[
\begin{bmatrix}
5 & 0 & -7 & 0 \\
-3 & 2 & 0 & 0 \\
-1 & 0 & 4 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & -4 & 0 \\
0 & 0 & -10 & 0 \\
0 & 0 & -13 & 0
\end{bmatrix}
\]

\( c_1 = 4 \ c_3 \)

\( c_1, c_3 \) free

\( \Rightarrow \) \( c_i \)'s exist, not all zero

s.t. \( c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0} \)

\( \Rightarrow \) \( \{ \} \), not L.I.
6) Find values of \( h \) so that the following vectors are LD.

\[
\begin{bmatrix}
2 \\
-2 \\
4
\end{bmatrix}, \quad \begin{bmatrix}
4 \\
-6 \\
7
\end{bmatrix}, \quad \begin{bmatrix}
-2 \\
2 \\
h
\end{bmatrix}
\]

\[
\begin{pmatrix}
2 & 4 & -2 & 0 \\
2 & -6 & 2 & 0 \\
4 & 7 & h & 0
\end{pmatrix}^{R_1/2} \begin{pmatrix}
1 & 2 & -1 & 0 \\
0 & -2 & 0 & 0 \\
0 & 0 & 1 & -1 & 0
\end{pmatrix}
\]

If \( h = -4 \), \( c_3 = 0 \). But \( c_2 = 0 \) and \( c_1 = c_3 \). So if \( h = -4 \), \( c_1 = c_2 = c_3 = 0 \), vectors are LI.

If \( h \neq -4 \), vectors are LD.
Recitation 19

Today: Span, Linear Dependence, Linear Transforms

Quiz 3 Next Thursday

Make sure you can solve these questions from old quizzes:

2012: Quiz 2 #2 and #3
2012: Quiz 3 #1 and #2
2013: Quiz 3 #1 and #3

We'll solve some of these in Tuesday's recitation.

QH8 Office Hours Next Week

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If you haven't already, please take a few minutes to fill out the technical issues survey.
1) This is similar to the second and third questions on the transforms HW.

Let \( \vec{u}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \), and \( T \) be a linear transformation that maps \( \vec{u}_1 \) onto \( \vec{v}_1 \), and \( \vec{u}_2 \) onto \( \vec{v}_2 \). Find \( T \) and the image of \( \begin{bmatrix} 3 \\ 0 \end{bmatrix} \) under \( T \).

Let \( T(x^1) = A \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \).

\[
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix},
\]
so

\[
3a_{11} + 4a_{12} = 4 \quad (1)
\]

\[
3a_{21} + 4a_{22} = 1 \quad (2)
\]

Similarly:

\[
3a_{11} + 3a_{12} = -1 \quad (3)
\]

\[
3a_{21} + 3a_{22} = 3 \quad (4)
\]

Combining (1) and (3) gives

\[
a_{12} = 5, \quad a_{11} = -16/3
\]

Equations (2) and (4) gives

\[
a_{21} = -2, \quad a_{22} = 3
\]

Therefore,

\[
T(x^1) = \begin{bmatrix} -16/3 & 5 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}
\]

\[
T(\begin{bmatrix} 3 \\ 0 \end{bmatrix}) = \begin{bmatrix} -16 \\ -6 \end{bmatrix}
\]
2) Plot $u$ and $v$, their images under $T$, and provide a geometric interpretation of what $T$ does to vectors in $\mathbb{R}^2$.

$u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, v = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$a) \quad T(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\overrightarrow{T(u)} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \overrightarrow{T(v)} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

$\overrightarrow{T(u)}$, $\overrightarrow{T(v)}$

$b) \quad T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\overrightarrow{T(u)} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \overrightarrow{T(v)} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$\overrightarrow{T(u)}$, $\overrightarrow{T(v)}$

Reflection through origin

Reflection through line $y = x$
3) From 2012 Quiz 2

Let \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( B = \begin{pmatrix} \cos \left( \frac{-\pi}{3} \right) & -\sin \left( \frac{-\pi}{3} \right) \\ \sin \left( \frac{-\pi}{3} \right) & \cos \left( \frac{-\pi}{3} \right) \end{pmatrix} \), 
\( C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \).

Compute the image of the house under the transformation \( ABC \). Show the intermediate steps.

- Image of (1) under \( C \): \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)
- Image of (1) under \( C \): \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

- Image under \( BC \) is a shearing then clockwise rotation by \( \frac{\pi}{3} \):
  \( BC(1) = B(C(1)) = \begin{pmatrix} 1/2 \\ -0.9 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.9 \end{pmatrix} \)
  \( BC(0) = B(C(0)) = \begin{pmatrix} 0.9 \\ -0.5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.9 \\ -0.5 \end{pmatrix} \)

- Image under \( ABC \):
  \( ABC(1) = A(BC(1)) = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \begin{pmatrix} 1/2 \\ -0.9 \end{pmatrix} = \begin{pmatrix} -0.45 \\ 0.5 \end{pmatrix} \)
  \( ABC(0) = A(BC(0)) = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \begin{pmatrix} 0.9 \\ -0.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.9 \end{pmatrix} \)

notice how the yellow dot never moved.
4) Fill in the elements of the 3x3 matrix.
*Hint: the elements can be identified by inspection.*

\[
\begin{bmatrix}
2 & 0 & -6 \\
-7 & 0 & 1 \\
0 & -1 & -5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
2x_1 - 6x_3 \\
x_3 - 7x_1 \\
-x_2 - 5x_3
\end{bmatrix}
\]
5) Find values of $h$ so that the vectors are LD, and values of $h$ so that the vectors are LI.

$$\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}$$

If $h = -18$, then the vectors are LD.
If $h \neq -18$, then the vectors are LI.

$$\begin{pmatrix} 2 & -6 & 9 & 0 \\ -6 & 4 & h & 0 \\ 1 & -3 & +3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & -8 & h+18 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_2 = 0$

1. Second row gives:

$0x_1 - 8x_2 + (h+18)x_3 = 0$

But $x_2 = 0$, so $(h+18)x_3 = 0$
If $h \neq -18$, $x_3 = 0$, and
vectors are LI
because $x_1 = x_2 = x_3 = 0$
Recitation 20

Today: Quiz 3 Review

Quiz 3 Next Thursday
Make sure you can solve these questions from old quizzes:
2012: Quiz 2 #2 and #3
2012: Quiz 3 #2
2013: Quiz 3 #1 and #3
Review recent HWs on Span, Lin Transforms, Gauss Jordan
Review sections 1.1, 1.2, 1.3, 1.7, 1.8, 1.9

QH8 Office Hours Next Week
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If you haven’t already, please take a few minutes to fill out the technical issues survey.
1) From 2013, Quiz 3, #3

If \( T \) is a linear transformation consisting of rotating counterclockwise by \( \pi/3 \) radians followed by a reflection about the line \( x = y \), find the matrix such that \( T(x) = Ax \).

Let \( T(\mathbf{x}) = A\mathbf{x} \). Then matrix \( A \) is a 2x2 matrix, and \( \mathbf{x} \) is a column vector with two elements.

\[
T(\mathbf{x}) = A\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

\[
= \begin{pmatrix} \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \\ \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]

If you prefer:

\[
T(\mathbf{x}) = A\mathbf{x} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]

A common mistake would be to write \( T = \begin{pmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \)

but that is wrong: the rotation should be applied first.
2) From 2012, Quiz 2, #2

For what values of \( b \) is \( y \) a linear combination of \( u \) and \( v \)?

\[
\begin{bmatrix}
-1 \\
1 \\
3
\end{bmatrix}, \quad \begin{bmatrix}
4 \\
2 \\
3
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
1 \\
b
\end{bmatrix}
\]

If \( y \) is a L.C. of \( u \) and \( v \), then \( \exists c_1, c_2 \) such that \( c_1u + c_2v = y \). Form augmented matrix:

\[
\begin{bmatrix}
4 & 1 & -1 \\
2 & 1 & 3 \\
1 & b/3 & 1
\end{bmatrix} \sim \begin{bmatrix}
0 & -1 & -3 \\
2 & 0 & -2 \\
1 & b/3 & 1
\end{bmatrix} \sim \begin{bmatrix}
0 & 1 & 3 \\
0 & 0 & 1 \\
1 & b/3 & 1
\end{bmatrix}
\]

First and 2\(^{nd} \) rows give us \( c_1 = -1, c_2 = -3 \).

The 3\(^{rd} \) row is:

\[
\begin{align*}
-1 + b/3 (3) &= 1 \\
b &= 2.
\end{align*}
\]

If \( b \neq 2 \), then the system is not consistent, i.e. \( y \) is not a L.C. of \( u \) and \( v \).

If \( b = 2 \), then \( y \) is a linear combination of \( u \) and \( v \).

If \( b \neq 2 \), then \( y \) is not a linear combination of \( u \) and \( v \).
3) From 2013, Quiz 3, #1

If \( T \) is a linear transformation, and:

\[
T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\]

find the matrix, \( A \), such that \( T(x) = Ax \).

Let \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \). Then:

\[
T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \end{pmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \end{bmatrix} \quad \text{(1)}
\]

\[
T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a_{11} + 2a_{12} \end{bmatrix} \quad \text{(2)}
\]

First rows of equations (1) and (2):

\[
\begin{align*}
\begin{cases}
a_{11} + a_{12} = 1 \\
a_{11} = 0, \quad a_{12} = 1
\end{cases}
\end{align*}
\]

(by inspection)

Second rows of equations (1) and (2):

\[
\begin{align*}
\begin{cases}
a_{21} + a_{22} = 0 \\
a_{21} + 2a_{22} = 3
\end{cases}
\end{align*}
\]

(by inspection)

\[
T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]
4) From 2012, Quiz 3, #2

For what values of \( a \) are the following vectors LI?

\[
\begin{bmatrix}
1 \\
0 \\
2
\end{bmatrix}, \quad
\begin{bmatrix}
2 \\
4 \\
3
\end{bmatrix}, \quad
\begin{bmatrix}
2 \\
a \\
a
\end{bmatrix}
\]

If LI, then the equation \( c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0} \) can only be satisfied when \( c_1 = c_2 = c_3 = 0 \).

Form augmented matrix:

\[
\begin{pmatrix}
1 & 2 & 2 & 0 \\
3 & 2 & 1 & 0 \\
2 & 4 & -1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 2 & 0 \\
0 & 2 & -4 & 0 \\
0 & 4 & a-7 & 0 \\
\end{pmatrix}
\]

The second row is \( 0c_1 + 0c_2 + (2a-7)c_3 = 0 \), or \( (2a-7)c_3 = 0 \).

If \( 2a-7 \neq 0 \), then \( c_3 \) must be zero. But if \( c_3 = 0 \), then from the other rows, \( c_2 = c_1 = 0 \). Thus, if \( 2a-7 \neq 0 \), \( c_1 = c_2 = c_3 = 0 \) and vectors must be LI.

If vectors are not LI they are LD, and vice versa.

If \( a = \frac{7}{2} \), then the vectors are LD.

If \( a \neq \frac{7}{2} \), then the vectors are LI.
Recitation 21

Today: Matrix Inverses, LU Decomposition

1a) State the formula for the inverse of the matrix

\[ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad M^{-1} = \frac{1}{\det M} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]

1b) Use the formula in (1a) to find a 2×2 matrix P such that:

\[ P \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad P \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

By construction:

\[ P \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \text{The inverse of } B \text{ is } \frac{1}{3 \cdot 3 - 3 \cdot 4} \begin{bmatrix} 3 & -3 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix} \]

\[ PB^{-1} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} B^{-1} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -16/3 & 5 \\ 3 & -2 \end{bmatrix} \]

\[ \Rightarrow P = \begin{bmatrix} -16/3 & 5 \\ 3 & -2 \end{bmatrix} \]

(check: \[ P \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -16/3 & 5 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \checkmark \])
2) Solve the equation $Ax = b$, using the LU decomposition of $A$, where

$$
A = \begin{bmatrix}
3 & -7 & -2 \\
-3 & 5 & 1 \\
6 & -4 & 0
\end{bmatrix}, \quad L = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
3 & -7 & -2 \\
-2 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad b = \begin{bmatrix}
-7 \\
5 \\
2
\end{bmatrix}
$$

$L = \text{lower triangular matrix}$

$U = \text{upper triangular matrix}$

Let $y = Ux$. Then $Ly = b$:

$$
\begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
2 & -5 & 1
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix} =
\begin{bmatrix}
-7 \\
5 \\
2
\end{bmatrix}

\Rightarrow
\begin{align*}
y_1 &= -7 \\
- y_1 + y_2 + 0y_3 &= 5, \quad y_2 &= -2 \\
2y_1 - 5y_2 + y_3 &= 2, \quad y_3 &= 6
\end{align*}
$$

But $Ux = y$:

$$
\begin{bmatrix}
3 & -7 & -2 \\
0 & -2 & -1 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
-7 \\
-2 \\
6
\end{bmatrix}

\Rightarrow
\begin{align*}
x_3 &= -6 \\
-2x_2 - x_3 &= -2, \quad x_2 = 4 \\
3x_1 - 7x_2 - 2x_3 &= -7, \quad x_1 = 3
\end{align*}
$$

$\Rightarrow X = \begin{bmatrix}
3 \\
4 \\
-6
\end{bmatrix}$
3) Find the LU decomposition of \( A \).

\[
A = \begin{bmatrix}
-5 & 0 & 4 \\
10 & 2 & -5 \\
10 & 10 & 16
\end{bmatrix}
\]

\( E_{21} \) = "elimination matrix" that eliminates \( a_{21} \)

\[
E_{21} = \begin{pmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

\[
E_{21} = \begin{pmatrix}
5 & 0 & 4 \\
0 & 2 & 3 \\
10 & 10 & 16
\end{pmatrix}
\]

\[
E_{21} A = \begin{pmatrix}
5 & 0 & 4 \\
0 & 2 & 3 \\
10 & 10 & 16
\end{pmatrix}
\]

\( E_{31} \) =

\[
E_{31} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -2 & 1
\end{pmatrix}
\]

\[
E_{31} E_{21} A = \begin{pmatrix}
5 & 0 & 4 \\
0 & 2 & 3 \\
0 & 10 & 24
\end{pmatrix}
\]

\( E_{32} \) =

\[
E_{32} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
E_{32} E_{31} E_{21} A = \begin{pmatrix}
5 & 0 & 4 \\
0 & 2 & 3 \\
0 & 0 & 9
\end{pmatrix}
\]

\( L \) = \( E_{31}^{-1} E_{32}^{-1} \) = \[
\begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
2 & 5 & 1
\end{pmatrix}
\]

\( L \) = \( E_{21}^{-1} \) \( E_{31}^{-1} \) \( E_{32}^{-1} \)

\( U \) = \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\( A = \begin{pmatrix} E_{32} E_{31} E_{21} \end{pmatrix}^{-1} U \)
3) Find the LU decomposition of \( A \).

\[
A = \begin{bmatrix}
-5 & 0 & 4 \\
10 & 2 & -5 \\
10 & 10 & 16
\end{bmatrix}
\]

**Find U:** (sequence of row ops to obtain REF)

\[
\begin{align*}
\text{R}_2 + 2\text{R}_1 & \rightarrow \begin{bmatrix}
-5 & 0 & 4 \\
0 & 2 & 13 \\
10 & 10 & 16
\end{bmatrix} \\
\text{R}_3 + 2\text{R}_1 & \rightarrow \begin{bmatrix}
-5 & 0 & 4 \\
0 & 2 & 13 \\
0 & 0 & 24
\end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix}
-5 & 0 & 4 \\
0 & 2 & 3 \\
0 & 0 & 9
\end{bmatrix} = U
\]

**Find L**

\[
\begin{bmatrix}
-5 \\
10 \\
10
\end{bmatrix}
\begin{bmatrix}
2 \\
10 \\
-2
\end{bmatrix} = \begin{bmatrix}
9 \\
2 \\
5
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 0 & 0 \\
-2 & 1 & 0 \\
2 & 5 & 1
\end{bmatrix} = L
\]

**Actually:**

This method is the same as the other, we just wrote out fewer steps.
4 b) State whether the following statements are true or false and explain your reasoning.

a) A can be row reduced to the identity matrix iff A is invertible.

The statement is _____ because: "this is how we find inverse" (Theorem 2.2)

b) If matrix B is the inverse of matrix A, the equations AB = I and BA = I must both be true.

The statement is _____ because: by def'n of invertible

c) If A and B are both N x N and invertible, then the product A⁻¹B⁻¹ is the inverse of AB.

The statement is _____ because: \((AB)^{-1} = B^{-1}A^{-1}\)

d) If an N x N matrix A is invertible, then the columns of \(A^T\) are LI.

The statement is _____ because: if A is invertible, then \(A^T\)

is also invertible and

any invertible matrix (square) has LI cols.

See Invertible matrix theorem, section 2.3
Recitation 22

Today: Column Space and Null Space, LU Decomposition

From quiz 3:

b) (3 points) The two vectors

\[
\begin{bmatrix}
1 \\
3 \\
1
\end{bmatrix}, \begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}
\]

span a plane passing through the origin. Find a vector that is normal to this plane.

The given vectors are both in the plane.

A vector perpendicular to them is given by:

\[
\begin{bmatrix}
1 \\
3 \\
1
\end{bmatrix} \times \begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix} = \begin{bmatrix}
1 \\
3 \\
1
\end{bmatrix} \begin{bmatrix}
1 & 3 & 1
\end{bmatrix} = \begin{bmatrix}
3-1 \\
-1-6 \\
1-1
\end{bmatrix} = \begin{bmatrix}
2 \\
-5 \\
0
\end{bmatrix}
\]

\[\text{DONE.}\]

Why does that work?

\[\begin{bmatrix}
2 \\
1 \\
-5
\end{bmatrix}\text{ is normal to all vectors in this plane,}
\]

\[\begin{bmatrix}
2 \\
1 \\
-5
\end{bmatrix} \cdot (c_1 \begin{bmatrix}
1 \\
3 \\
1
\end{bmatrix} + c_2 \begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}) = c_1 \cdot 0 + c_2 \cdot 0 = 0\]
POP QUIZ #2

- Start time: 8:10?
- Ends at: 8:25
- Pop quiz grading
  - 5 points: correct
  - 4 points: something correct
  - 3 points: name on the page
  - 0 points: did not take pop quiz

- To submit your work, choose any of the following:

  A. **work on whiteboard in breakout room**
     - type A in text chat so I know you want to work in breakout room
     - submit work by letting me know when done, and/or email me a screen capture of your work
  
  B. **work on paper and give work to facilitator**
     - type B in text chat so I know you’re doing this
     - leave 2 inch margin on paper
     - write your name and QH8 at the top
     - facilitator is receiving instructions today on how to submit your work
  
  C. **work on paper and email a photo of your work to me**
     - type C in text chat so I know you are emailing your work to me
     - email your photo to me before 8:40
Find a basis for the column space of:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 3 & 6 & 9
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Pivot columns are in columns 1 and 4, so (1) and (4) form a basis for the column space of the given matrix.

Check: column 6 is:

\[
\begin{bmatrix}
5 \\
3 \\
9
\end{bmatrix} = 3 \begin{bmatrix}
4 \\
1 \\
3
\end{bmatrix} - 6 \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} \checkmark \text{ (similar calculations for other columns)}
\]
2) Fill in the blanks:

a) Coll A is the set of all linear combinations of the **columns** of matrix A.
   (or pivot columns)

b) Nul A is the set of all solutions to $Ax = \vec{0}$.

c) The **LI** columns of matrix A form a basis for the column space of A.
   (or the pivot columns)

d) The rank of matrix A is the **number of LI cols**.

e) The nullity of matrix A is the $(\text{number of cols}) - \text{rank}$.

$\Rightarrow \text{num of cols} = \text{rank} + \text{nullity}$
3) Find i) a basis for \text{Col} A, and ii) a basis for \text{Nul} A.

\[ A = \begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -3 & 0 & 6 \\ 0 & 2 & 6 & 0 & -4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

i) A basis for \text{Col} A is just \begin{bmatrix} \frac{2}{3} \\ \frac{3}{3} \\ \frac{-1}{3} \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}. I.e. the pivot columns.

ii) \( x_3 \) & \( x_5 \) are free.

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_5 - x_3 \\ 2x_5 - 3x_3 \\ x_3 \\ 2x_5 \\ -x_5 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

(first row gives: \( 3x_1 - x_2 - 3x_3 + 0x_4 + 6x_5 = 0 \)
\( 3x_1 - (2x_5 - 3x_3) - 3x_2 + 6x_5 = 0 \) \( \Rightarrow 3x_1 = -4x_5 + 0x_3 + 3x_5 \) \( \Rightarrow x_1 = \frac{-4}{3}x_5 + \frac{3}{3}x_5 \))
Recitation 23

Today: Column and Null Space, Eigenvalues, LU Decomposition

From 2012 Quiz 3: Find a basis for the nullspace of A. Also find its rank and nullity.

\[ A = \begin{bmatrix} 3 & -1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2 & 4 \\ 0 & 1 & 3 & 2 & 4 \end{bmatrix} \]

\[
\begin{align*}
R_1 + R_2 & \Rightarrow \begin{pmatrix} 3 & 0 & 4 & 16 & 10 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
R_3 - R_2 & \Rightarrow \begin{pmatrix} 3 & 0 & 4 & 16 & 10 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]

\[ \Rightarrow 2 \text{ pivot cols} \Rightarrow \text{rank} = 2 \]

And nullity = (\# of cols) - (rank) = \(5 - 2 = 3\)

Find basis for nullspace. Three free variables: \(x_3, x_4, x_5\).

Row 1: \[3x_1 + 0x_2 + 4x_3 + x_4 + 6x_5 = 0 \Rightarrow x_1 = \frac{-4}{3}x_3 - \frac{1}{3}x_4 - 2x_5\]

Row 2: \[0x_1 + x_2 + 2x_3 + 3x_4 + 4x_5 = 0 \Rightarrow x_2 = -2x_3 - 3x_4 - 4x_5\]

\[
\begin{bmatrix}
\text{Row } 1 \\
\text{Row } 2 \\
\text{Row } 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= \begin{bmatrix}
x_3 \\
x_4 \\
x_5
\end{bmatrix}
\begin{bmatrix}
-\frac{4}{3} \\
-\frac{1}{3} \\
0
\end{bmatrix}
+ \begin{bmatrix}
x_3 \\
x_4 \\
x_5
\end{bmatrix}
\begin{bmatrix}
-2 \\
-4 \\
0
\end{bmatrix}
+ \begin{bmatrix}
x_3 \\
x_4 \\
x_5
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

Vectors \(\vec{v}_1, \vec{v}_2, \vec{v}_3\) are a basis for nullspace of \(A\).
2) A 2013 pop quiz question: find the coordinates of $\mathbf{b}$ with respect to $\mathbf{v}_1$ and $\mathbf{v}_2$.

$\mathbf{b} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

If $\mathbf{b}$ is in the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\exists c_1, c_2$ s.t.

$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{b}$

or:

$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

If $c_1$ & $c_2$ exist, then $\mathbf{b}$ is in basis, & $c_1$ & $c_2$ are its "coordinates".

Solving system (with augmented matrix) yields

$c_1 = \frac{5}{3}$

$c_2 = \frac{2}{3}$

thus, coordinates are $(\frac{5}{3}, \frac{2}{3})$
3) Is \( \lambda = 2 \) an eigenvalue of matrix \( A \)? Why/why not? (from Section 5.1)

\[
A = \begin{bmatrix}
3 & 2 \\
3 & 8 \\
\end{bmatrix}
\]

If \( \lambda = 2 \) is an e-vector, then it must satisfy:

\[
A \vec{v} = \lambda \vec{v}, \quad \text{or} \quad (A - \lambda I) \vec{v} = \vec{0}. \quad \text{(by def'n)}
\]

Substitute in known quantities:

\[
\vec{v} = (A - \lambda I) \vec{v}
\]

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} = \begin{bmatrix}
3 & 2 \\
3 & 8
\end{bmatrix} \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 & 2 \\
3 & 6
\end{bmatrix} \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

Are columns of \( A - \lambda I \) LI? NO.

\[
\Rightarrow \exists \text{many sol'ns (infinitely many)}
\]

\[
\Rightarrow \lambda = 2 \text{ is an eigenvalue.}
\]

Comprehension Check:

Q: IF the columns of \( A - \lambda I \) were LI, what would \( \vec{v} \) have to be?

\( \vec{v} \) would have to be \( \vec{0} \).

\( \lambda \) would not be an e-value, because e-vectors must be non-zero.
4) Find a basis for the eigenspace of $A$, for the eigenvalue $\lambda = -5$. 

$$
A = \begin{bmatrix}
-4 & 1 & 1 \\
2 & -3 & 2 \\
3 & 3 & -2
\end{bmatrix}
$$

The corresponding e-vectors to $\lambda = -5$ are solutions to:

$$(A - \lambda I) \mathbf{v} = \mathbf{0},$$

or

$$
\begin{bmatrix}
-4 -(-5) & 1 & 1 \\
2 & -3 -(-5) & 2 \\
3 & 3 & -2+5
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} = \mathbf{0}
$$

or:

$$
\begin{pmatrix}
1 & 1 & 0 \\
2 & 2 & 0 \\
3 & 3 & 0
\end{pmatrix}
\begin{bmatrix}
\text{aug. matrix} \\
v_1 \\
v_2 \\
v_3
\end{bmatrix} = \mathbf{0}
$$

$\Rightarrow x_2$ & $x_3$ free!

And $x_1 = -x_2 - x_3$.

If $x_2 = 1, x_3 = 0$, then $x_1 = -1$

$\Rightarrow \mathbf{v}_1 = \begin{bmatrix}
-1 \\
0
\end{bmatrix}$

If $x_2 = 0, x_3 = 1$, then $x_1 = -1$

$\mathbf{v}_2 = \begin{bmatrix}
0 \\
1
\end{bmatrix}$

$\mathbf{v}_1, \mathbf{v}_2$ are LI e-vectors that form basis for e-space of $A$ for $\lambda = -5$. 
5) Find the characteristic polynomial and eigenvalues of:

a) \( X = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} \)

\[
0 = \det (A - \lambda I) \\
= \begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix} \\
= (2-\lambda)(2-\lambda) - 49 \\
= \lambda^2 - 4\lambda - 45 \\
= (\lambda - 9)(\lambda + 5) \\
\Rightarrow \lambda = 9, -5
\]

b) \( Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix} \)

\[
0 = \det (A - \lambda I) \\
= (\lambda - 3)(\lambda + 2)^2 \\
\Rightarrow \text{e-values are } 0, 3, 2
\]
Find the LU decomposition of the following matrix (if its before 8:40 am, in group work).

\[ B = \begin{bmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{bmatrix} \]

\[ \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 9 & 9 & 14 \end{bmatrix} = \Xi_{21} A \]

\[ \Xi_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \Xi_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix} \]

\[ \Xi_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \Xi_{32} \Xi_{31} \Xi_{21} A = U \]

\[ L = \Xi_{21}^{-1} \Xi_{31}^{-1} \Xi_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ +3 & 4 & 1 \end{pmatrix} \]
Find the LU decomposition of the following matrix (if it's before 8:40 am, in group work).

\[
B = \begin{bmatrix}
3 & 1 & 2 \\
-9 & 0 & -4 \\
9 & 9 & 14
\end{bmatrix}
\]

**ALTERNATE METHOD**

**FIND U:**

\[
\begin{align*}
R_2 &+ 3R_1 \rightarrow \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 6 & 8 \end{bmatrix} \\
R_3 &- 3R_1 \rightarrow \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix} = U
\end{align*}
\]

**FIND L:**

COLUMN 1

\[
\begin{bmatrix} 9 \\ \frac{9}{3} \\ \frac{9}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ \frac{1}{3} \end{bmatrix}
\]

COLUMN 2

\[
\begin{bmatrix} 0 \\ \frac{3}{3} \\ \frac{6}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}
\]

COLUMN 3

\[
\begin{bmatrix} 0 \\ \frac{4}{4} \\ \frac{4}{4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
3 & 2 & 1
\end{bmatrix}
\]
Recitation 24

Today: Determinants (3.1, 3.2), Diagonalization (5.3)

Suppose $A$ and $B$ are square matrices.
1) If a multiple of one row of $A$ is added to another row to produce $B$, then $\det(B) = \det(A)$.
2) If two rows of $A$ are interchanged to produce $B$, then $\det(B) = -\det(A)$.
3) If one row of $A$ is multiplied by $K$ to produce $B$, then $\det(B) = K\det(A)$.
4) If $A$ is a triangular matrix, then $\det(A) =$ product of diagonal elements.

Compute $\det(A)$.

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{bmatrix}$$

$$\Rightarrow \quad \det(A) = -(1)(3)(-5) = +15.$$
Theorems from Section 3.2

Suppose A and B are square matrices.

1) If A is not invertible, \( \det(A) = \boxed{0} \).

2) \( \det(AB) = \det(A) \cdot \det(B) \).

3) \( \det(A + B) \neq \det(A) + \det(B) \). \( \text{ie - } \det(A+B) \neq \det(A) + \det(B) \).

4) \( \det(A^T) = \det(A) \).

Determine whether the following vectors are LI.

\[
\begin{bmatrix}
5 \\
1 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
-3 \\
5
\end{bmatrix}
\begin{bmatrix}
-1 \\
-2 \\
3
\end{bmatrix}
\]

If LI, then \( \det \begin{bmatrix}
5 & 0 & -1 \\
1 & 3 & 2 \\
0 & 5 & 3
\end{bmatrix} \) would be non-zero.

Cofactor Method:

\[
\det \begin{bmatrix}
5 & 0 & -1 \\
1 & 3 & 2 \\
0 & 5 & 3
\end{bmatrix} = 5 \begin{vmatrix}
3 & 2 \\
5 & 3
\end{vmatrix} - 0 \begin{vmatrix}
1 & 2 \\
0 & 3
\end{vmatrix} + (-1) \begin{vmatrix}
1 & 3 \\
0 & 5
\end{vmatrix}
\]

\[
= 5((-9 + 10) - 5) = 5(-1) = -5
\]

\( \Rightarrow \) vectors are LD.
Section 5.3: Diagonalization

A matrix $A$ is diagonalizable if it can be written in the form:

$$ A = P D P^{-1} $$

where

$P$ is a matrix of **e-vectors** (columns are e-vectors)

$D$ is a matrix of **e-values** (diagonal elements are e-values, off diagonal are 0)

Suppose $A$ is $N \times N$. To diagonalize $A$:

1. find all **e-values** of $A$

2. find $N$ **eigenvectors** of $A$

3. construct $P$ from eigenvectors

4. construct $D$ from eigenvalues

5. find $P^{-1}$.
2) Diagonalize the following matrices, if possible.

\[ A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}; \quad B = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}; \quad C = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \]

where \( \lambda \)'s of \( C \) are 2, 2, 5

A) \( 0 = \det(A - \lambda I) \)
\[ = \det \begin{bmatrix} 1 - \lambda & 3 \\ 4 & 2 - \lambda \end{bmatrix} \]
\[ = (1 - \lambda)(2 - \lambda) - 12 \]
\[ = \lambda^2 - 3\lambda - 10 \]
\[ \Rightarrow \lambda_1, \lambda_2 = -2, 5 \]

if \( \lambda = -2 \), \( \vec{v} = [-1] \)

if \( \lambda = 5 \), \( \vec{v} = \left[ \begin{array}{c} 3/4 \\ 1 \end{array} \right] \)

\[ p = \begin{bmatrix} 1 & 3/4 \\ -1 & 1 \end{bmatrix}, \quad p^{-1} = \begin{bmatrix} 1 & -3/4 \\ 1 & 1 \end{bmatrix} \]
\[ \Rightarrow A = \begin{bmatrix} -2 & 0 & 0 \\ 3/4 & -3/4 & 0 \\ 1 & 0 & 1/4 \end{bmatrix} \]

B) \( 0 = (3 - \lambda)(3 - \lambda) - 0 \)
\[ \Rightarrow \lambda_1, \lambda_2 = +3 \]

When \( \lambda_1 = 3 \):
\[ (A - 3I) \vec{v} = \begin{bmatrix} 0 & 2 \\ 0 & -6 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
\[ \Rightarrow \text{one eigenvector is} \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], \text{but we can't find another LI eigenvector. So P does not exist, and B is not diagonalizable.} \]

C) if \( \lambda = 2 \):
\[ A - 2I = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} \]
\[ \Rightarrow \vec{v}_1 = \left[ \begin{array}{c} 1 \\ 1 \end{array} \right], \quad \vec{v}_2 = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]. \]

if \( \lambda = 5 \):
\[ A - 5I = \begin{bmatrix} -4 & -3 \\ 0 & -2 \end{bmatrix} \]
\[ \approx \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \]
\[ \vec{v}_3 = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]. \]

To find \( P^{-1} \): (Use Section 2.2)
\[ \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \]
\[ \approx \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ \Rightarrow c = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \]

Finding inverse of 3x3 hasn't been covered in Math 302.
Recitation 25

Today: Quiz Review

Quiz 4 Next Thursday
- Review recent HWs on span, determinants, and eigenvalues
- Review sections 2.8, 2.9, 3.1, 3.2, 5.1, 5.2

QH8 Office Hours
- Tuesday and Wednesday 7:30 to 8:30 pm
- We’ll solve practice quiz problems & go over specific areas you’d like to review.
- At the same place as last time:
  https://georgiatech.adobeconnect.com/distancecalculusofficehours

Online Drop-in Tutoring
- Wednesdays, 5:30 to 7:00 pm
- For all ~450 distance calculus students
- Facilitated by Greg, who will answer questions and review problems from QH8 recitations

Question 1 (from 2013 Quiz 4)

a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

\[ A = \begin{bmatrix} 10 & 3 & -1 \\ 2 & 9 & 1 \\ -2 & 3 & 11 \end{bmatrix} \]

b) Find as many LI eigenvectors for this eigenvalue as possible.
Group Work
Writing on board disappears when I enter room (sometimes). So let's try this:
• None of the breakout rooms have the questions.
• Every breakout room has a whiteboard.
• Write Question on the whiteboard when you get into the breakout room, solve it, then move to Question 2.
• You’ve got about 15 minutes.

Question 1 (from 2013 Quiz 4)

a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

\[ A = \begin{bmatrix} 10 & 3 & -1 \\ 2 & 9 & 1 \\ -2 & 3 & 11 \end{bmatrix} \]

b) Find as many LI eigenvectors for this eigenvalue as possible.

Question 2
Compute \( \det(B) \) by using row reduction.

\[ B = \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix} \]
Question 1 (from 2013 Quiz 4)

a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

\[
A = \begin{bmatrix} 10 & 3 & -1 \\ 2 & 9 & 1 \\ -2 & 3 & 11 \end{bmatrix}
\]

b) Find as many LI eigenvectors for this eigenvalue as possible.

\[a) \text{ EASY WAY: IF } 12 \text{ IS AN E-VALUE, THERE ARE NON-TRIVIAL SOLNS TO } (A-12I)x = 0.\]

Since \( A-12I = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -3 & 1 \\ -2 & 3 & -1 \end{bmatrix} \) has LD columns, non-trivial solutions exist, so \( \lambda = 12 \) must be an E-value.

\[b) \text{ ANOTHER WAY: } \lambda = 12 \text{ IS AN E-VALUE IF ITS ROOT OF THE CHAR. POLY:} \]

\[
det(A-12I) = \begin{vmatrix} -2 & 3 & -1 \\ 2 & -3 & 1 \\ -2 & 3 & -1 \end{vmatrix} = -2(3-3) - 3(-2+2) - 1(6-6) = 0
\]

\[\Rightarrow \text{ 12 IS AN E-VALUE.} \]

Yet another approach:

**FIND CHARACTERISTIC POLYNOMIAL:** Find its roots, and see if 12 is one of them.

\[
\begin{pmatrix} -2 & 3 & -1 & 0 \\ 2 & -3 & 1 & 0 \\ -2 & 3 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[2x_1 = 3x_2 - x_3 \]

\[x_2, x_3 \text{ free.} \]

\( \Rightarrow \text{ basis is } \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \right\} \)

\( \Rightarrow \text{ only 2 LI e-vectors.} \)
**Question 2**

Compute \( \text{det}(B) \) by using row reduction.

\[
\begin{vmatrix}
1 & 5 & -3 \\
3 & -3 & 3 \\
2 & 13 & -7
\end{vmatrix}
= \begin{vmatrix}
1 & 5 & -3 \\
0 & -18 & 12 \\
0 & -3 & -1
\end{vmatrix}
= \begin{vmatrix}
1 & 5 & -3 \\
0 & -3 & 2 \\
0 & 0 & 1
\end{vmatrix}
= 6 \cdot 1 = 6
\]

\[
\text{CHECK ANSWER:}
|B| = 1(21 - 39) - 5(-21 - 6) - 3(39 + 6)
= -18 + 5 \cdot 27 - 3 \cdot 45
= -18 + 135 - 135
= -18
\]
3) State whether the following statements are true or false and explain your reasoning:

\[ \text{If } A \text{ is a square matrix, and } \det(A^T) = 0, \text{ then } A \text{ is not invertible.} \]

The statement is **false** because:

\[ \text{det}(A^T) = \text{det}(A) \text{ for all } A \text{ in } M_{n \times n}, \]

The dimension of Nul A is not equal to the number of variables in equation \( Ax = 0 \).

The dimension of Col A is not equal to the number of pivot columns of A.
4) Find all values of $h$ so that the eigenspace for $D$, for $\lambda = 4$, is two dimensional.

$$
D = \begin{bmatrix}
4 & 2 & 3 & 3 \\
0 & 2 & h & 3 \\
0 & 0 & 4 & 14 \\
0 & 0 & 0 & 2
\end{bmatrix}
$$

The basis for $E$-space spans solutions to:

$$(D - 4I)\vec{v} = \vec{0}.$$ 

$$D - 4I = \begin{pmatrix}
0 & 2 & 3 & 3 \\
0 & 2 & h & 3 \\
0 & 0 & 0 & 14 \\
0 & 0 & 0 & 2
\end{pmatrix} \sim \begin{pmatrix}
0 & 2 & 3 & 3 \\
0 & 2 & h & 3 \\
0 & 0 & 0 & 14 \\
0 & 0 & 0 & 2
\end{pmatrix}
$$

If $h \neq 3$, there are 2 pivot columns, 7 variables, so $\dim (eigenspace) = 2$.

$$\Rightarrow \text{there are 2 free variables}$$

If $h = -3$:

There are 2 pivot columns, 7 variables, so $\dim (eigenspace) = 1$.

$\Rightarrow h$ must be $-3$ for there to be a 2D eigenspace for $\lambda = 4$.

Still confused?

If $h = -3$, then our aug. matrix is $\begin{pmatrix}
0 & 2 & 3 & 3 \\
0 & 0 & 0 & 6
\end{pmatrix}$, so $x_1$ & $x_3$ free, $x_2 = -3x_3$, $x_4 = 6$, and $\vec{v} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$.

Basis is $\{ \begin{bmatrix} 0 \\ 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} \}$, which has two linearly independent vectors.
Recitation 27

Today: Diagonalization (5.3), Orthogonality (6.1, 6.2, 6.3, 6.4)

Let \( v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \)

Show that these are (pairwise) orthogonal. If

\[
\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = a_1 v_1 + a_2 v_2 + a_3 v_3,
\]

where the \( v_i \)'s are as above and the \( a_i \)'s are scalars, FIND \( a_2 \)

\[
\begin{align*}
\overrightarrow{v_1} \cdot \overrightarrow{v_2} &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = -1 + 0 + 1 = 0 \\
\overrightarrow{v_1} \cdot \overrightarrow{v_3} &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -1 + 0 + 1 = 0 \\
\overrightarrow{v_2} \cdot \overrightarrow{v_3} &= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - 2 + 1 = 0
\end{align*}
\]

\( \{ \text{vectors must be pairwise orthogonal} \} \)

SECOND PART: FIND \( a_2 \)

THE TEDIOUS METHOD: solve

\[
\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} a_1 + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} a_2 + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} a_3
\]

THE FAST METHOD: \( v_2 \cdot (\frac{2}{3}) = v_2 \cdot (a_1 v_1 + a_2 v_2 + a_3 v_3) = a_1 v_1 \cdot v_2 + a_2 v_2 \cdot v_2 + a_3 v_3 \cdot v_2 = \)

\[
\begin{align*}
\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} &= 2 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 2 - 6 + 5 = 1, \\
\begin{pmatrix} -1 \\ 1 \end{pmatrix} &= 2 - 6 + 5 = 1, \text{ and } a_2 v_2 \cdot v_2 = 6,
\end{align*}
\]

SO \( a_2 \cdot 6, \) SO \( a_2 = \frac{1}{6}. \)
Question I (parts a and b are from 2014 Quiz 4, Question 3)

a) Find all eigenvalues
b) Find a eigenbasis for each eigenvalue.

c) Is it possible to diagonalize A? Why/why not?

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & 3 \end{bmatrix} \]

a) 
\[ 0 = \det(A - \lambda I) = (1-\lambda)(\lambda-2)^2 \Rightarrow \lambda_1 = 1, \quad \lambda_{2,3} = 2. \]

b) \( \lambda_1 = 1: \) 
\[ A - (1)I = \begin{bmatrix} 6 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \] 
first row: 0+2x_2+0 = 0 \Rightarrow x_2 = 0 
second row: 0+0+x_3 = 0 \Rightarrow x_3 = 0 

\Rightarrow \lambda_1 \text{ is free, choose } x_1 = 1, \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \]

\( \lambda_{2,3} = 2: \) 
\[ A - 2I = \begin{bmatrix} -1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \] 
\[ \Rightarrow x_3 \text{ free, } \quad x_2 = x_3, \quad x_1 = 5x_3 \]
\[ \Rightarrow \mathbf{v}_{2,3} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}. \]

\( \Rightarrow \text{Basis for } \mathbb{P}_1 \text{ is } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \]
\( \Rightarrow \text{Basis for } \mathbb{P}_{2,3} \text{ is } \left\{ \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \right\}. \]

\( \text{C) Not possible to diagonalize, because we don't have three LI eigenvectors} \)
Recitation 28

Today: Orthogonality (6.1, 6.2, 6.3)

True or False:

a) Eigenvalues must be nonzero scalars.

This is **FALSE**, because eigenvalues can be zero. When they are zero, we know that the given matrix has LD columns (because $A \vec{v} = \vec{0} \Rightarrow \vec{v} = 0$ will have non-trivial solutions).

b) Eigenvectors must be nonzero vectors.

This is **TRUE**, because we want to find non-trivial solutions to the equation $A \vec{v} = \lambda \vec{v}$.

The zero vector $\vec{v} = \vec{0}$ will always solve the equation $A \vec{v} = \lambda \vec{v}$, so we're only interested in the non-zero solutions.
**Quiz 4, Question 1b**
Solutions were emailed to students yesterday.

**Question**

b) Find a basis for the null space of $A$. 

$$ A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & 6 \end{pmatrix} $$

**An Answer**

A basis for $\text{Nul}(A)$ is the set: 

$$ \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \end{bmatrix} \right\} $$

Is this correct? How can we check to see if this answer is correct?

The basis vectors must be \textit{linearly independent}, and be in the null space of $A$.

$A\begin{bmatrix} -2 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, but $A\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}$ is not in $\text{Nul}(A)$, and so cannot be a basis vector.
Orthogonality

a) Compute the orthogonal projection of $\vec{y}$ onto the line, $L$, that passes through the origin and is parallel to $\vec{u}$.

b) Sketch $\vec{y}$, $\vec{u}$, and the orthogonal projection, and $L$.

c) Calculate the distance between $\vec{y}$ and $L$.

d) Write $\vec{y}$ as a sum of a vector in $\text{Span}(\vec{u})$ and a vector orthogonal to $u$.

\[
\vec{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}
\]

(a) orthogonal projection $= \frac{\vec{u} \cdot \vec{y}}{\vec{u} \cdot \vec{u}} \vec{u}$ (definition)

\[
= \frac{14 + 6}{49 + 1} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \hat{\vec{y}}
\]

(b) $L = \text{Span} \vec{u}$

$|\vec{y} - \hat{\vec{y}}| = \sqrt{(2 - 14/5)^2 + (6 - 2/5)^2} = \sqrt{0.64 + 28.16} \approx 5.46$

c) component of $\vec{y} \perp \vec{u}$

d) $\vec{y} = \hat{\vec{y}} + (\vec{y} - \hat{\vec{y}})$ (DONE)
**Question 3**

a) Are the columns of A LI?

b) Do the columns of A form a basis for $\mathbb{R}^4$?

c) Are the columns of A mutually orthogonal?

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

\[ R_3 + R_2 \sim \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & -2 & 7 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{YES, COLUMNS ARE LI.} \]

6) NO. Basis for a subspace of $\mathbb{R}^4$.

\[ \hat{a}_1 \cdot \hat{a}_2 = -15 + 1 - 1 - 21 \neq 0 \]

\[ \hat{a}_1 \cdot \hat{a}_3 = \text{doesn't matter} \]

\[ \hat{a}_2 \cdot \hat{a}_3 = \text{doesn't matter} \]

\{ not mutually orthogonal. \}
Recitation 29

Today: Orthogonality (6.1 to 6.5)

Orthogonality and Linear Independence

a) Are the columns of \( A \) LI?

b) Do the columns of \( A \) form a basis for \( \mathbb{R}^4 \)?

c) Are the columns of \( A \) mutually orthogonal?

\[
A = \begin{bmatrix}
3 & -5 & 1 \\
1 & 1 & 1 \\
-1 & 1 & -2 \\
3 & -7 & 8
\end{bmatrix}
\]

a) **YES** (we did this last recitation)

b) **NO**, we would need 4 columns.

The columns of \( A \) are a basis for a subspace of \( \mathbb{R}^4 \).

c) **NO**: \[
\begin{bmatrix} 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -36 \neq 0
\]

(No need to check the other 2 dot products)
Find an orthogonal basis for the column space of $A$. 

Let columns of $A$ be $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$. 

Let basis for $\text{Col}(A)$ be $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.

Set $\vec{v}_1 = \frac{1}{2} \vec{a}_1 = \left[ \begin{array}{c} \frac{3}{2} \\ -1 \\ -1 \end{array} \right]$, projection of $\vec{a}_2$ onto $\vec{v}_1$ = $\text{proj}_{\vec{v}_1} \vec{a}_2$

\[
\vec{v}_2 = \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \left[ \begin{array}{c} -5 \\ 1 \\ 1 \end{array} \right] - \frac{-40}{20} \left[ \begin{array}{c} 3 \\ 1 \\ 1 \end{array} \right] = \left[ \begin{array}{c} -3 \\ 3 \\ 3 \end{array} \right]
\]

\[
\vec{v}_3 = \vec{a}_3 - \frac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \left[ \begin{array}{c} 1 \\ -1 \\ -1 \end{array} \right] - \frac{1}{2} \vec{v}_1 - \left( \frac{1}{2} \right) \vec{v}_2 
\]

orthogonal basis is $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.

A) What would have happened if $\text{dim}(\text{Col}(A)) = 2$?

A) $\vec{v}_3 = 0$ and $\vec{v}_2 = 0$. 

Schematic: 

\[ \vec{a}_2 - \text{proj}_{\vec{v}_1} \vec{a}_2 = \vec{v}_2 \]
Least Squares (slide 1/3)
Consider the system \( Ax = b \), where
\[
A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}
\]

1) Are the columns of matrix \( A \) linearly independent?

\[ \text{YES! (They are not co-linear.)} \]

2) Do the columns of \( A \) form a basis for \( \mathbb{R}^3 \)?

\[ \text{NO! (we would need 3 columns.)} \]

3) Is \( b \) in \( \text{Col}(A) \)?

\[ \text{NOPE. If it were, then there would exist } x_1, x_2, \]
\[
A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b : \begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} \Rightarrow \text{INCONSISTENT.}
\]

4) Is there a solution to \( Ax = b \)?

\[ \text{NOPE.} \]

5) Therefore, we will:

\[ \text{FIND } \hat{x}, \text{ that is the "closest" solution to } Ax = b. \]
Least Squares (slide 2/3)
Consider the system $Ax = b$, where
\[ A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}. \]

Our best solution to this system: the vector $\hat{x}$, such that $||b - \hat{A}x|| \leq ||b - A\hat{x}||$.

What does this mean?

1) Is $b$ in Col $A$? **NO.**

2) Is $A\hat{x}$ in Col $A$? **YES.** $A\hat{x}$ is co-planar with $a_1$ and $a_2$.

3) Is $(b - A\hat{x})$ perpendicular to all vectors in Col($A$)? **YES!**

\[ A^T(b - A\hat{x}) = 0 = [0] \]

\[ \Rightarrow (\hat{x} = (A^TA)^{-1}A^Tb) \]
Least Squares (slide 3/3)

Consider the system $Ax = b$, where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Find the least squares solution to this system.

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \\ 11 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$\hat{x} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Note: $A\hat{x} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$ is the "closest" vector in $	ext{col}(A)$ to $b$. 

DONE
**Recitation 2.8: Least-Squares**

Example: Find A: L.S. soln of Ax = b: A = \([2\ 5\ 7] \quad \begin{bmatrix} 1 \\ 2 \\ 4 \\ 9 \\ 6 \\ 5 \end{bmatrix}\)

What is \(A\)? Because \(A^2\) is invertible, we would need \(A\) to be square.

If there were not at least \(m \geq n\), we would need \(A\) to be square.

What is \(x^T\)?

If \(b\) is the orthogonal projection of \(b\) on \(\text{col} A\), then...
Recitation 31

Today: QR Decomposition, Orthogonality Review (6.1 to 6.5)

QR Factorization

\[ A = QR, \text{ and } R \text{ is an upper triangular matrix.} \]

\[
A = \begin{bmatrix}
  a_1 & a_2 \\
  5 & 9 \\
  1 & 7 \\
 -3 & -6 \\
  1 & 5 \\
\end{bmatrix},
Q = \begin{bmatrix}
  q_1 & q_2 \\
  5/6 & -1/6 \\
  1/6 & 5/6 \\
 -3/6 & 1/6 \\
  1/6 & 3/6 \\
\end{bmatrix}
\]

a) If we weren’t given \( Q \), we could find it by using the \textbf{Gram Schmidt}.

\textit{The Gram Schmidt algorithm produces an orthogonal basis for \( \text{Col}(A) \).}

b) The columns of \( Q \) are \textbf{orthonormal}: so \( q_1 \cdot q_1 = q_2 \cdot q_2 = 1 \), \( q_1 \cdot q_2 = 0 \).

c) The columns of \( Q \) form an \textbf{orthonormal} basis for \( \text{col}(A) \)

\textbf{Note:} \( q_1 \) is a multiple of \( a_1 \).

\( q_2 \) is a multiple of \( a_1 \& a_2 \):

\[ q_2 = a_2 - \frac{a_2 \cdot q_1}{q_1 \cdot q_1} q_1 \]

\( q_1 \cdot q_1 = q_2 \cdot q_2 = 1 \), \( q_1 \cdot q_2 = 0 \).

\textit{Has the weightings for final exam exemption policy been announced?}
QR Factorization
A = QR, and R is an upper triangular matrix.

\[
A = \begin{bmatrix}
5 & 9 \\
1 & 7 \\
-3 & -6 \\
1 & 5 \\
\end{bmatrix}, \quad Q = \begin{bmatrix}
5/6 & -1/6 \\
1/6 & 5/6 \\
-3/6 & 1/6 \\
1/6 & 3/6 \\
\end{bmatrix}
\]

d) The dimensions of R have to be ___2x2___, because ___A = QR___.

e) Find R.

f) Show that your answer for part (e) is correct.

\[
\begin{align*}
\text{e)} & \quad A = QR \text{ (given)} \\
& \quad Q^T A = Q^T Q R \text{ (multiply by } Q^T) \\
& \quad Q^T A = I R, \text{ because } Q \text{ is an orthonormal matrix} \\
\Rightarrow & \quad R = Q^T A = \frac{1}{6} \begin{bmatrix}
5 & 1 & -3 & 4 \\
-1 & 5 & 1 & 3 \\
-3 & -6 & 1 & 5 \\
\end{bmatrix} = \frac{1}{6} \begin{bmatrix}
96 & 72 \\
0 & 36 \\
0 & 36 \\
\end{bmatrix} = \begin{bmatrix}
5 & 12 \\
0 & 6 \\
0 & 6 \\
\end{bmatrix}
\end{align*}
\]

\[
\text{f)} \quad QR \text{ should equal } A. \text{ Check: } \frac{1}{6} \begin{bmatrix}
5 & 1 & -3 & 4 \\
-1 & 5 & 1 & 3 \\
-3 & -6 & 1 & 5 \\
\end{bmatrix} \begin{bmatrix}
6 & 12 \\
0 & 6 \\
\end{bmatrix} = \frac{1}{6} \begin{bmatrix}
30 & 54 \\
0 & 36 \\
0 & 36 \\
\end{bmatrix} = \begin{bmatrix}
5 & 9 \\
0 & 6 \\
0 & 6 \\
\end{bmatrix}
\]

STOP CALCULATING HERE...
Least Squares (LS) Formulas
For each case, state a formula for the LS solution of $Ax = b$.

1) A is a square matrix and $A^TA$ is invertible.
   $$\hat{x} = (A^TA)^{-1}A^Tb$$

2) A is square, $A^TA$ is invertible, and A has **orthogonal** columns.
   Orthogonal projection of $b$ onto $\text{Col}(A)$ is
   $$\hat{b} = \frac{b \cdot a_1}{a_1 \cdot a_1} a_1 + \frac{b \cdot a_2}{a_2 \cdot a_2} a_2 + \ldots$$
   $$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{bmatrix}, \text{ and in general, } \hat{x}_i = \frac{b \cdot a_i}{a_i \cdot a_i}$$

3) A is square, $A^TA$ is invertible, and A has **orthonormal** columns.
   
   **METHOD 1**
   Since $A$ has orthonormal columns, $A^TA = I$, so
   $$\hat{x} = (I)^{-1}A^Tb = A^Tb$$

   **METHOD 2**
   Can also use: $\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{bmatrix}$, where $\hat{x}_i = \frac{b \cdot a_i}{a_i \cdot a_i} = \frac{b \cdot a_i}{1} = b \cdot a_i$
Least Squares Solutions

If A is square, and $A^T A$ is not invertible, can we find a LS solution to $Ax = b$? Why/why not?

The LS solution to $A\hat{x} = \hat{b}$ is the sol'n to $A^T A \hat{x} = A^T b$.

- $A^T A$ is some matrix that has LD columns
- $A^T b$ is some vector

$\Rightarrow$ we are solving a system with an infinite number of solutions

$\Rightarrow$ YES, we can find a LS solution, and there are infinitely many of them.
Describe all LS solutions to the system:

\[ x_1 + x_2 = 2 \]
\[ x_1 + x_2 = 4 \]

If \( A\hat{x} = b \), then \( A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \), \( \hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \).

The LS solutions of \( A\hat{x} = b \) solve:

\[ A^T A \hat{x} = A^T b \]

But \( A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \) \[ \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} \]

\( A^T b = \begin{bmatrix} 6 \\ 6 \end{bmatrix} \)

is a system with an infinite \# of solutions:

\[ 2x_1 + 2x_2 = 6 \]
\[ 1x_1 + x_2 = 3 \]

THE LS SOLUTION, \( \hat{x} \) IS THE VECTOR SUCH THAT:

\[ \|A\hat{x} - b\| \leq \|A\hat{x} - \hat{b}\| \]
Recitation 32

Today: Final Exam Review

Quiz Grades
- Quiz grades "locked" today.
- Please check your graded quizzes to see if they were graded correctly.

If You are Writing Math 1502 Final Exam
- Part I: Mon, Dec 8
- Part II: Tue, Dec 9
- Work with your facilitator to find a time/place to write.

QH8 Office Hours
Sat Dec 6, Sun Dec 7
Please use text chat when you are free with "I am free on Sat ___ and Sun ___" so that we can try to find times that work for most of you.

Today
Group work: 3 groups, we may return to main room if/when groups are getting stuck
1) Section 4.6: Row and Col Space of $A^T$ (Slide 1 of 2)

Row(A) is the set of all possible linear combinations of the rows of A.

**Theorem (from Section 4.6)**

If two matrices $A$ and $B$ are row equivalent, then their row spaces are equal.

If $B$ is in echelon form, the nonzero rows of $B$ form a basis for $\text{Row}(A)$, as well as $\text{Row}(B)$.

A proof of this theorem uses the fact: if $B$ is obtained from row operations on $A$, the rows of $B$ are a linear combination of the rows of $A$. 
1) Section 4.6: Row and Col Space of $A^T$ (Slide 2 of 2)

Matrix $A$ and its row echelon form are given. Find a basis for

1) $\text{Col}(A)$
2) $\text{Row}(A)$
3) $\text{Row}(A^T)$
4) $\text{Col}(A^T)$

*Hint: the answers for all of the above do not require any calculation.*

$$A = \begin{bmatrix}
    a_1 & a_2 & a_3 & a_4 \\
    1 & -4 & 9 & -7 \\
    -1 & 2 & -4 & 1 \\
    5 & -6 & 10 & 7 \\
\end{bmatrix} \sim \begin{bmatrix}
    1 & 0 & -1 & 5 \\
    0 & -2 & 5 & -6 \\
    0 & 0 & 0 & 0 \\
\end{bmatrix}$$

1) $\{ \vec{e_1}, \vec{e_2} \}$
2) $\{ \vec{r_1}, \vec{r_2} \}$

3) $A^T = \begin{bmatrix}
    1 & -1 & 5 \\
    -4 & 2 & -6 \\
    -9 & -4 & 10 \\
    -7 & 1 & 7 \\
\end{bmatrix}$, and we already know the third column of $A^T$ is a linear combination of the other two, so basis is $\{ \vec{a_1}, \vec{a_2} \}$.

4) Similar to previous question, basis is $\{ \vec{r_1^T}, \vec{r_2^T} \}$.
If time permits (from 2012 Quiz 4):

Find the eigenvalues and eigenvectors of $A$ and use them to find a formula for $A^k$.

$$A = \begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}$$

$$\det(A - \lambda I) = (5 - \lambda)(7 - \lambda) - 8 = \lambda^2 - 12 \lambda + 27 = (\lambda - 3)(\lambda - 9), \quad \lambda_1 = 3, \quad \lambda_2 = 9$$

$$\lambda_1 = 3: \quad A - 3I = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 9: \quad A - 9I = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$A^k = P D^k P^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 9^k \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix}$$
3) QR Factorization
A=QR, and R is an upper triangular matrix.

\[
A = \begin{bmatrix}
-2 & 3 \\
5 & 7 \\
2 & -2 \\
4 & 6
\end{bmatrix}, \quad Q = \begin{bmatrix}
-2/7 & 5/7 \\
5/7 & 2/7 \\
2/7 & -4/7 \\
4/7 & 2/7
\end{bmatrix}
\]

a) The dimensions of R have to be \(2 \times 2\).

b) Calculate matrix R. *Hint: save time by factoring 1/7 out of matrix Q first.*

c) Compute a few elements of the product QR to check your answer for part (b)

\[
R = Q^T A = \frac{1}{7} \begin{bmatrix}
-2 & 5 \\
5 & 2 \\
2 & 4 \\
4 & 6
\end{bmatrix} \begin{bmatrix}
-2 & 3 \\
5 & 7 \\
2 & -2 \\
4 & 6
\end{bmatrix} = \begin{bmatrix}
7 & 0 \\
0 & 7
\end{bmatrix}
\]

\[
QR = \frac{1}{7} \begin{bmatrix}
-2 & 5 \\
5 & 2 \\
2 & 4 \\
4 & 6
\end{bmatrix} \begin{bmatrix}
7 & 0 \\
0 & 7
\end{bmatrix} = \begin{bmatrix}
-2 & +3 \\
5 + 3 & -2 + 1
\end{bmatrix}
\]

sufficient to stop after two elements
4) Fill in the Blanks
Matrix A has dimensions N\times N and is invertible.

a) The columns of A form a basis for \( \mathbb{R}^N \).

b) \( \text{rank}(A) = N \), because A is invertible, it has N LI columns.

c) \( \text{Nul}(A) = \{ 0 \} \).

d) \( \text{dim}(\text{Nul}(A)) = 0 \), because A is invertible, it has N LI rows, so \( A^T \) has N LI columns.

e) \( \text{dim}(\text{Nul}(A^T)) = 0 \), because A is invertible, it has N LI rows, so \( A^T \) has N LI columns.

f) \( \text{dim}(\text{Row}(A^T)) = \text{dim}(\text{Col}(A)) = N \).

g) \( \text{dim}(\text{Col}(A^T)) = \text{dim}(\text{Row}(A)) = N \).

h) \( \text{dim}(\text{Col}(A)) + \text{dim}(\text{Nul}(A)) = N + 0 = N \).

i) \( \text{dim}(\text{Row}(A)) + \text{dim}(\text{Nul}(A)) = N + 0 = N \).
5) A modified version of a 2014 Quiz 2 Question

Compute the integral, using a Taylor polynomial, to an accuracy of at least 0.01.

\[
\int_0^2 e^{-x^4} \, dx
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^2 x^k \, dx
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[ \frac{x^{k+1}}{k+1} \right]_0^2
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{2^{k+1}}{k+1} - 0 \right)
\]

\[
< \frac{2^{4N+1}}{N!(4N+1)}
\]

\[
\text{need } N = 39 \quad \text{... AHH!}
\]
True or False

A) If \( \{v_1, v_2, v_3\} \) is an orthogonal basis and \( c \) is a constant, then \( \{v_1, v_2, cv_3\} \) is another, different orthogonal basis.

This statement is \[\boxed{F}\] because a basis is a set of all possible linear combinations of vectors.

B) The LS solution of \( Ax = b \) is the point in \( \text{Col}(A) \) closest to \( b \).

This statement is \[\boxed{F}\] because \( Ax \) is the closest point to \( b \), not \( \hat{x} \).

C) If \( x \) is in subspace \( W \), then \( x - \text{proj}_W x \neq 0 \).

This statement is \[\boxed{F}\] because if \( x \) is in \( W \), then \( x = \text{proj}_W x \), i.e. the projection of a vector onto that space is itself.
True or False

D) If \( \{v_1, v_2, v_3\} \) is an orthogonal basis and \( c \) is a constant, then \( \{v_1, v_2, cv_3\} \) is another, different orthogonal basis.

This statement is _____ because

E) If \( \hat{u} \) is a LS solution to \( Au = b \), then \( \hat{u} = (A^TA)^{-1}A^Tb \).

This statement is F because \( A^TA \) may not be invertible.
Eigenvalues and Orthogonality

What can we say about the eigenvalues of an orthogonal matrix?

Let \( A \bar{v} = \lambda \bar{v} \), \( A \) is orthogonal.

Then: \( \| A \bar{v} \| = \| \lambda \bar{v} \| \), but \( \| A \bar{v} \| = \| \bar{v} \| \), because \( A \)

is orthonormal.

\[ \Rightarrow \lambda = \pm 1 \]

\( \text{Because:} \)
\[ \| A x \|^2 = (A x)^T A x = x^T A^T A x = x^T x \]
\[ \Rightarrow \| A x \|^2 = \| x \|^2 \]
\[ \text{so} \| A x \| = \| x \| \]

(\text{note: norm is always non-negative})

1. If \( A \) has \( CD \) columns, at least one e-val is zero.

2. If \( A \) is upper triangular, e-vals are the diagonal elements of \( A \).