Dynamical systems
Dynamical systems
Dynamical systems: a statistical approach
Dynamical systems: a statistical approach

Linear

Non-linear
Dynamical systems: a statistical approach

Completely integrable

Chaotic
In the chaotic case positions and directions get uniformly distributed:

Typical questions:
How long do we have to wait to have uniform distribution?
Are there periodic orbits and what information do they contain?
In the chaotic case positions and directions get uniformly distributed:

Recent work on the rate of decay for billiards by 
Baladi–Demers–Liverani ’15
A dynamical analogue of the Riemann zeta function:

\[ \zeta(s) = \prod_p (1 - p^{-s})^{-1} \]
A dynamical analogue of the Riemann zeta function: Ruelle zeta function
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Replace primes with prime closed orbits in $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$
A dynamical analogue of the Riemann zeta function: Ruelle zeta function

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Replace $p$ by $\log l_\gamma$ where $l_\gamma$ is the length of a prime closed orbit.
A dynamical analogue of the Riemann zeta function: Ruelle zeta function

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\[
\zeta_D(s) = \prod_{\gamma}(1 - e^{-s\ell_{\gamma}})
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Replace \( p \) by \( e^{\ell_{\gamma}} \) where \( \ell_{\gamma} \) is the length of a prime closed orbit.

It turns out that the zeros and poles of \( \zeta_D \) contain information about statistical properties of the chaotic dynamical system. That includes the time at which we achieve uniform distribution.
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Dynamical zeta functions have been studied by many authors:

\[ Y(\ell_1, \ell_2, \phi) \text{ for } \ell_1 = \ell_2 = 7, \phi = \frac{\pi}{2} \]
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Smale '67 conjectured that for Anosov flows $\zeta_D$ is meromorphic in $\mathbb{C}$: “I must admit a positive answer would be a little shocking!”
What is an Anosov flow?
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\[ T_\rho X = E_0(\rho) \oplus E_s(\rho) \oplus E_u(\rho), \quad \rho \mapsto E_\bullet(\rho) \text{ continuous}, \]
\[
\begin{align*}
  d\varphi_t(\rho)E_\bullet(\rho) &= E_\bullet(\varphi_t(\rho)), \\
  |d\varphi_t(\rho)v|_{\varphi_t(\rho)} &\leq Ce^{-\theta|t|}|v|_\rho, \quad v \in E_u(\rho), \quad t < 0, \\
  |d\varphi_t(\rho)v|_{\varphi_t(\rho)} &\leq Ce^{-\theta|t|}|v|_\rho, \quad v \in E_s(\rho), \quad t > 0.
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Example: \( X = S^*M := \{(x, \xi) \in T^*M; |\xi|^2_g = 1\}, \) where \((M, g)\) is a compact Riemannian manifold of negative curvature.
Theorem (Giulietti–Liverani–Pollicott ’12, Dyatlov–Z ’13)

For an Anosov flow on a compact manifold $\zeta_D(s)$ has a meromorphic continuation to $\mathbb{C}$. 
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Our proof uses the microlocal approach to Anosov flows due to Faure–Sjöstrand ’11 and radial point propagation of singularities estimates due to Melrose ’94 and developed further by Vasy ’13.
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The noncompact case (essentially the full Smale conjecture) recently completed by Dyatlov–Guillarmou.
Scattering theory

\[ \xi^2 + V(x) = E \]

Faure–Sjöstrand
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Scattering theory

Faulre–Sjöstrand

Dyatlov–Z

Dyatlov–Guillarmou
Special case:
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Let \((M, g)\) be a smooth oriented compact Riemannian surface of 

\[ \zeta_D(s) = \prod \gamma (1 - e^{-s \ell}) \]

\(\Re s \gg 1\).

Theorem (Dyatlov–Z ’16)
Let \(g\) be the genus of \(M\). Then 

\[ s^2 - 2g \zeta_D(s) \]

is holomorphic near 

\(s = 0\)

and 

\[ s^2 - 2g \zeta_D(s)|_{s=0} \neq 0 \].

In particular, the set of lengths of closed orbits (the length spectrum) determines the genus.

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Pollicott-Ruelle Resonances

\begin{align*}
\rho_{f,g}(t) &= \int_X e^{-itP} f(x) g(x) \, dx \\
\hat{\rho}_{f,g}(\lambda) &= \int_0^\infty \rho_{f,g}(t) e^{i\lambda t} \, dt
\end{align*}
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Resonances: poles of $\hat{\rho}_{f, g}(\lambda)$.
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m(\lambda) := \dim \{ u \in D'(X) : (\frac{1}{i} V - \lambda)^r u = 0, \ WF(u) \subset E_u^* \} \\
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Blank–Keller–Liverani’02 ... Faure–Sjöstrand ’11...
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\( \nu_1 > 0 \) for contact flows (and in particular for geodesic Anosov flows): Dolgopyat’98, Liverani ’04, Tsujii’12, Nonnenmacher–Z’15.
A “real” life investigation of the gap
A “real” life investigation of the gap

Spectral RP-gap observed through the Niño 3 index ($\delta_s = 0.1$)

Spectral RP-gap observed through the Niño 3 index ($\delta_s = 0.95$)
A “real” life investigation of the gap

Microlocal analysis (semiclassical version)

- **Phase space:** \((x, \xi) \in T^*X\)
- **Semiclassical parameter:** \(h \to 0\), the effective wavelength
- **Classical observables:** \(a(x, \xi) \in C^\infty(T^*X)\)
- **Quantization:** \(\text{Op}_h(a) = a(x, \frac{h}{i} \partial_x) : C^\infty(X) \to C^\infty(X)\), semiclassical pseudodifferential operator
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Basic examples

- \(a(x, \xi) = x_j \implies \text{Op}_h(a) = x_j\) multiplication operator
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Classical-quantum correspondence

- \([\text{Op}_h(a), \text{Op}_h(b)] = \frac{h}{i} \text{Op}_h(\{a, b\}) + O(h^2)\)
- \(\{a, b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b = H_a b, \quad e^{tH_a}\) Hamiltonian flow
- Example: \([\text{Op}_h(\xi_k), \text{Op}_h(x_j)] = \frac{h}{i}\delta_{jk}\)
Standard semiclassical estimates

General question

\[ P = \text{Op}_h(p), \quad Pu = f \quad \implies \quad \|u\| \preccurlyeq \|f\| ? \]
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\[ P = \text{Op}_h(p), \quad P u = f \quad \implies \quad \| u \| \lesssim \| f \| ? \]

Control \( u \) microlocally:

\[ \| \text{Op}_h(a) u \| \lesssim \| f \| + O(h^\infty) \| u \| \]
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\[ \{p = 0\} \begin{array}{c} \text{supp } a \end{array} \]
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Propagation of singularities

\[ \begin{array}{c} \{b \neq 0\} \quad e^{tH_p} \quad \text{supp } a \end{array} \]
Phase space description of singularities
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\[ \exists \, a \in \mathcal{C}_c^\infty(T^*X), \, a(x, \xi) \neq 0, \, \|a(x, hD)u\|_{L^2} = \mathcal{O}(h^\infty) \]
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- \( \text{WF}(\delta(ax - y)) = \{(x, ax; -a\eta, \eta) : y \in \mathbb{R}, \xi \in \mathbb{R} \setminus 0\} \)
Return to the Ruelle zeta function (with a slight change of convention):

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$$\zeta_k(\lambda) := \left(- \sum_{m=1}^{\infty} \sum_{\gamma} \frac{e^{im\lambda \ell_\gamma} \operatorname{tr} \wedge^k P^m_\gamma}{m|\det(I - P^m_\gamma)|}\right), \quad \operatorname{Im} \lambda \gg 1.$$
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\( P_{\gamma} \) is the linearized Poincaré map:
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**Theorem (Dyatlov–Z '16)**

\(\zeta_k(\lambda)\) extends to an entire function and the multiplicities of its zeros are given by

\[
\dim \left\{ u \in \mathcal{D}'(X, \Omega_0^k) : (\frac{1}{i} \mathcal{L}_V - \lambda)^r u = 0, \ WF(u) \subset E_u^* \right\}
\]

where \(\Omega_0^k\) are k-forms satisfying \(\iota_V u = 0\).
The starting point for relating the zeta function to the
distributional spectrum of \( \frac{1}{i} \mathcal{L}_V \) is the Atiyah–Bott–Guillemin
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The starting point for relating the zeta function to the distributional spectrum of $\frac{1}{i} L_V$ is the Atiyah–Bott–Guillemin trace formula:

$$\text{tr} \ e^{-itP/h} = \sum_{m=1}^{\infty} \sum_{\gamma} \frac{l_\gamma \delta(t - m\ell_\gamma)}{|I - P^m_\gamma|}, \quad P = hV/i.$$
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This is related to the first building block of the zeta function:

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\zeta_0(\lambda) := \exp \left( - \sum_{m=1}^{\infty} \sum_{\gamma} \frac{e^{im\lambda \ell_{\gamma}}}{m|\det(I - P_m^n)|} \right)
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Since

$$(P - z)^{-1} = \frac{i}{\hbar} \int_{0}^{\infty} e^{-itP/h} dt$$
The starting point for relating the zeta function to the distributional spectrum of $\frac{1}{i}\mathcal{L}_V$ is the Atiyah–Bott–Guillemin trace formula:

$$\text{tr } e^{-itP/h} = \sum_{m=1}^{\infty} \sum_{\gamma} l_{\gamma} \delta(t - m\ell_{\gamma}) \frac{\ell_{\gamma} \delta(t - m\ell_{\gamma})}{|I - P_{\gamma}|}, \quad P = hV/i.$$  

This is related to the first building block of the zeta function:

$$\zeta_0(\lambda) := \exp \left( - \sum_{m=1}^{\infty} \sum_{\gamma} \frac{e^{im\lambda\ell_{\gamma}}}{m|\det(I - P_{\gamma})|} \right)$$

Since

$$(P - z)^{-1} = \frac{i}{h} \int_{0}^{\infty} e^{-itP/h} dt$$

$$e^{it_0 \lambda} h \text{tr } \varphi_{-t_0}(P - h\lambda)^{-1} = \frac{\partial}{\partial \lambda} \log \zeta_0(\lambda).$$
\[ e^{it_0 \lambda} \hbar \text{tr} \varphi_{-t_0}^\ast (P - h\lambda)^{-1} = \frac{\partial}{\partial \lambda} \log \zeta_0(\lambda). \]
\[ e^{i t_0 \lambda} h \text{tr} \varphi^*_{-t_0} (P - h\lambda)^{-1} = \frac{\partial}{\partial \lambda} \log \zeta_0(\lambda). \]

Here the trace is a **formal** trace obtained by integration over the diagonal:
\[ e^{it_0 \lambda} h \text{tr} \phi_0^*(P - h\lambda)^{-1} = \frac{\partial}{\partial \lambda} \log \zeta_0(\lambda). \]

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\[ Au(x) = \int_X K_A(x, y) u(y) dy \]

\[ \text{tr} A := \int_X K_A(x, x) dx \]
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Allowed under a wave front set condition

\[ \text{WF}(K_A) \cap \{(x, x; \xi, -\xi) : x \in X, \xi \in T^*_x X \setminus 0\} \]
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Example: \( Au(x) := u(ax), x \in \mathbb{R}, \)
$$e^{it_0\lambda} h \, \text{tr} \, \varphi_{-t_0}^*(P - h\lambda)^{-1} = \frac{\partial}{\partial \lambda} \log \zeta_0(\lambda).$$

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\[ \operatorname{WF}(K_A) = \{(x, ax; -a\eta, \eta) : \eta \neq 0\} \]
\[ e^{i t_0 \lambda} h \text{tr} \varphi_{-t_0}^* (P - h\lambda)^{-1} = \frac{\partial}{\partial \lambda} \log \zeta_0(\lambda). \]

Need:

\[ \text{WF}(K_{\varphi_{-t_0}^* (P-z)^{-1}}) \cap \{(x, x; \xi, -\xi) : x \in X, \xi \in T_x X \setminus 0\} = \emptyset. \]
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Propagation through radial sinks \((E^*_u)\) and sources \((E^*_s)\) based on earlier work of Melrose '94 and Vasy '13, gives this.
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Propagation through radial sinks \((E_u^*)\) and sources \((E_s^*)\) based on earlier work of Melrose ’94 and Vasy ’13, gives this.
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Propagation through radial sinks \((E_u^*)\) and sources \((E_s^*)\) based on earlier work of Melrose '94 and Vasy '13, gives this.

The meromorphy of \(z \mapsto (P - z)^{-1} : C^\infty(X) \to \mathcal{D}'(X)\) shows that the poles are simple and residues are positive integers.
Now suppose that $X = S^* M$, $M$ an orientable Riemannian surface of negative curvature.
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In general,

\[ \zeta_D(\lambda) = \frac{\zeta_1(\lambda)}{\zeta_0(\lambda)\zeta_2(\lambda)} \]

\[ m = m_1(0) - m_0(0) - m_2(0) \]

\[ m_j(0) = \dim \left\{ u \in \mathcal{D}'(X, \Omega^j_0) : \mathcal{L}_V u = 0, \text{WF}(u) \subset E^*_u \right\} \]

where $\Omega^j_0(X)$ are $j$-forms satisfying $\iota_V u = 0$. 
Claim: \( m_0(0) = m_2(0) = 1, \quad m_1(0) = \dim H^1(X, \mathbb{C}). \)
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Regularity result:

\[ \text{Re} \langle Vu, u \rangle_{L^2} \geq 0, \Vu \in C^\infty, \WF(u) \subset E_u^* \implies u \in C^\infty. \]
Claim: \( m_0(0) = m_2(0) = 1, \quad m_1(0) = \dim H^1(X, \mathbb{C}) \).

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m_0(0) = \dim \{ u \in \mathcal{D}'(X) : Vr u = 0, \text{WF}(u) \subset E_u^* \} = 1.
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Regularity result:

\[
\text{Re} \langle Vu, u \rangle_{L^2} \geq 0, \quad Vu \in C^\infty, \quad \text{WF}(u) \subset E_u^* \implies u \in C^\infty.
\]

Here \( L^2(X) \) is defined using the smooth invariant measure on \( S^*M \):

\[
d\text{vol} := \alpha \wedge d\alpha
\]

where \( \alpha \) is the contact form,

\[
\alpha = zd\zeta|_{S^*M}, \quad S^*M := \{(z, \zeta) \in T^*M : |\zeta|^2_{g(z)} = 1\},
\]

\[
\alpha(V) = 1, \quad \iota_V d\alpha = 0, \quad \ker \alpha(x) = E_u(x) \oplus E_s(x)
\]
\[ m_0(0) := \dim \{ u \in \mathcal{D}'(X) : V^r u = 0, \WF(u) \subset E_u^* \} = 1. \]
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**Proof:**

If \( V^2 u = 0 \), \( \text{WF}(u) \subset E^*_u \) then, by the regularity result, \( u \in C_\infty(X) \). Also, \( u(x) = u(x) \). Hence

\[
\langle du|_E, v \rangle \rightarrow 0 \quad \{ t \rightarrow +\infty \quad v \in E_s(x), \quad t \rightarrow -\infty \quad v \in E_u(x) \}.
\]

It follows that \( du|_{E_u \oplus E_s} = 0 \) and that means that \( du = \varphi_\alpha = \alpha \wedge d(du) = \varphi_\alpha \wedge d\alpha = \Rightarrow \varphi = 0 = \Rightarrow du = 0 = \Rightarrow u = \text{const} \). If \( V^2 u = 0 \), \( \text{WF}(u) \subset E^*_u \) then \( Vu = \text{const} \). But

\[
\int_X V u \, \text{vol} = 0 \quad \Rightarrow \text{const} = 0.
\]
\[ m_0(0) := \dim \{ u \in \mathcal{D}'(X) : V^r u = 0, \WF(u) \subset E_u^* \} = 1. \]

**Proof:** If \( Vu = 0, \WF(u) \subset E_u^* \) then, by the regularity result, \( u \in C^\infty(X) \).
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**Proof:** If \( Vu = 0, \text{WF}(u) \subset E_u^* \) then, by the regularity result, \( u \in \mathcal{C}^\infty(X) \). Also, \( u(e^{tV}x) = u(x) \).
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Hence

\[
\langle du(x), \nu \rangle = \langle du(e^{tV}x), de^{tV}(x)\nu \rangle \to 0 \quad \left\{ \begin{array}{ll}
t \to +\infty & \nu \in E_s(x), \\
t \to -\infty & \nu \in E_u(x). \\
\end{array} \right.
\]
\[ m_0(0) := \dim \{ u \in D'(X) : V^r u = 0, \WF(u) \subset E_u^* \} = 1. \]

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\[
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It follows that \( du|_{E_u \oplus E_s} = 0 \) and that means that \( du = \varphi \alpha \).
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\[ 0 = \alpha \wedge d(du) = \varphi \alpha \wedge d\alpha \implies \varphi = 0 \]

\[ \implies du = 0 \]

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\[ m_0(0) := \dim \{ u \in \mathcal{D}'(X) : V^r u = 0, WF(u) \subset E_u^* \} = 1. \]

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\langle du(x), \nu \rangle = \langle du(e^{tV}x), de^{tV}(x)\nu \rangle \longrightarrow 0 \quad \begin{cases} t \to +\infty & \nu \in E_s(x), \\
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If \( V^2u = 0, \WF(u) \subset E_u^* \) then \( Vu = \text{const} \).

But \( \int_X Vud\text{vol} = 0 \) so \( \text{const} = 0 \).
\[ m_1(0) = \dim Y_1 = \dim H^1(X, \mathbb{C}), \]

\[ Y_1 := \{ u \in \mathcal{D}'(\Omega^1(X)) : \mathcal{L}_V^r u = 0, \iota_V u = 0, \WF(u) \subset E_u^* \} \]
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1. \( \mathcal{L}_V u = 0 \) is equivalent to (since \( i_V u = 0 \)) to \( i_V d u = 0 \). Hence \( d u \) is a resonant state for 2-forms and \( d u = c d \alpha \). Since \( u \wedge d \alpha = i_V u \alpha \wedge d \alpha = 0 \) we have

\[
\text{cvol}(X) = \int d u \wedge \alpha = \int u \wedge d \alpha = 0.
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\[ m_1(0) = \dim Y_1 = \dim H^1(X, \mathbb{C}), \]
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1. \( L^r_V u = 0 \) is equivalent to (since \( \iota_V u = 0 \)) to \( \iota_V d u = 0 \). Hence \( d u \) is a resonant state for 2-forms and \( d u = c d \alpha \). Since \( u \wedge d \alpha = \iota_V u \alpha \wedge d \alpha = 0 \) we have

\[ c \text{vol}(X) = \int d u \wedge \alpha = \int u \wedge d \alpha = 0. \]

We conclude that \( d u = 0 \).
\[ m_1(0) = \dim Y_1 = \dim H^1(X, \mathbb{C}), \]

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We conclude that \( du = 0 \).

2. Hodge theory: \( \exists \varphi \in \mathcal{D}', \WF(\varphi) \subset E_u^*, u - d\varphi \in C^\infty(\Omega^1(X)) \).
\[ m_1(0) = \dim Y_1 = \dim H^1(X, \mathbb{C}), \]

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\[ Y_1 \ni u \mapsto [u - d\varphi] \in H^1(X, \mathbb{C}) \]

is an isomorphism
\[ m_1(0) = \dim Y_1 = \dim H^1(X, \mathbb{C}), \]
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2. Hodge theory: \( \exists \varphi \in D', \WF(\varphi) \subset E^*_u, u - d \varphi \in C^\infty(\Omega^1(X)) \).

\[ Y_1 \ni u \mapsto [u - d \varphi] \in H^1(X, \mathbb{C}) \]

is an isomorphism

Showing **semisimplity** is a little bit tricky...
\[ m_0(0) = m_2(0) = 1, \quad m_1(0) = \dim H^1(X, \mathbb{C}). \]

Since for surfaces of genus \( g \geq 2 \),

\[ H^1(S^* M, \mathbb{C}) \cong H^1(M, \mathbb{C}) \]

it follows that the order of vanishing of \( \zeta_D \) at 0 is

\[ m = -m_0(0) + m_1(0) - m_2(0) = -\text{Euler characteristic of } M = 2g - 2 \]
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Thanks for your attention!