Effects of Insufficient Time-Scale Separation in Cascaded, Networked Systems

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Abstract— **In this paper, we investigate the effect of insufficient time-scale separation between inner and the outer loops in a cascaded, networked system under multiple clients. Inspired by the AQM (inner loop) and TCP (outer loop) at the Internet transport layer, a qualitative model is developed where the stability of the cascaded system is analyzed in terms of the gains acting at the outer and inner loops.**

I. Introduction

Networked control systems comprise of a number of nodes whose dynamic behavior depend on their interactions with adjacent nodes in the network. Such systems have been studied extensively during the last decade, with applications found in areas as diverse as multi-robot systems, mobile sensor networks, power grid control, data and communication networks, and manufacturing chains [1]. Distributed coordination and control strategies have been devised for, achieving and maintaining geometric formations [2], [3], coveraging areas [4], establishing connected communication links [5], or agreeing on global state values across the network [6], just to name a few. These strategies can be thought of as control laws that are internal to the network in the sense that they dictate how the node values should evolve over time as a function of the locally available information. For example, the AQM (Active Queue Management) protocol at the Transport Layer of the Internet [7], or the consensus protocol for reaching agreement among multiple agents [6], constitute such internal control strategies.

These internal controllers can be contrasted with external, or end-to-end controllers that adjust the input to the network at some peripheral nodes based on the performance of the network, as measured at other nodes in the network. For example, an operator could be driving leader nodes in a multi-robot network based on how the centroid or how specialized output nodes are behaving [8], or the rate at which data is injected into a network under the TCP protocol at the sender-side is based on the received data rate and congestion at the receiver-side in the network [9]. These types of constructions are examples of cascaded control design, and it is well-known that as long as the time-scales are sufficiently separated – the inner control loops are significantly faster than the outer ones – the addition of an outer loop does not harm the stability properties of the system, e.g., [10].

However, as has been observed in the networking community, as these time-scales approach each other, the performance of the system is affected in a negative way, e.g., when TCP and AQM start to act at similar time-scales [9], [11]. Similarly, when human operators are to interact with large collections of mobile robots, the performance deteriorates significantly if the input nodes and the output nodes are too far away, resulting in systems that are very hard for the human operator to interact with [12], [13]. Moreover, when multiple operators access the output nodes at once, the congestion occurs on relay servers and the performance deteriorates much more.

In this paper, we investigate these effects. In particular, we develop a simple model in which a number of these informal observations can be made concrete. This model is loosely inspired by TCP/AQM but two things should be noted already at this point: 1. We do not attempt to model TCP/AQM in any great level of detail. Instead, 2. We are simply interested in capturing qualitative aspects of such systems in a way that constitutes a first stepping stone towards understanding the general issue of insufficient time-scale separation.

The outline of the paper is as follows: First, Section II gives the problem formulation. Next, Section III analyzes the stability of the simple model of the cascaded system under no congestion. Section IV considers the networked system in the situation that congestion occurs. Then, Section V illustrates the validness of the analysis results through several numerical examples. Finally, Section VI concludes the discussion.

II. Problem Formulation

Notations: Let $\mathbb{R}$ and $\mathbb{C}$ be the sets of the real and complex numbers. The imaginary unit is denoted as j. For a complex number $\lambda \in \mathbb{C}$, its real and imaginary parts are denoted as $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$. For a matrix $A \in \mathbb{R}^{n\times n}$, $\sigma(A) \subset \mathbb{C}$ represents the set of all eigenvalues of A. Let $\lambda_{\mathrm{Rmax}}(A) \in \mathbb{R}$ be the maximum real part of the eigenvalues of A, that is,

$$\lambda_{\mathrm{Rmax}}(A) = \max_{\lambda\in\sigma(A)} \mathrm{Re}(\lambda).$$

It is said that a complex number $\lambda \in \mathbb{C}$ is stable if $\mathrm{Re}(\lambda) < 0$, and that a matrix $A \in \mathbb{R}^{n\times n}$ is stable if $\lambda_{\mathrm{Rmax}}(A) < 0$. Let ${}_n\mathrm{C}_i$ be the number of i-combination from n-elements. ■

The networked systems under consideration in this paper consist of subsystems that are themselves networks organized

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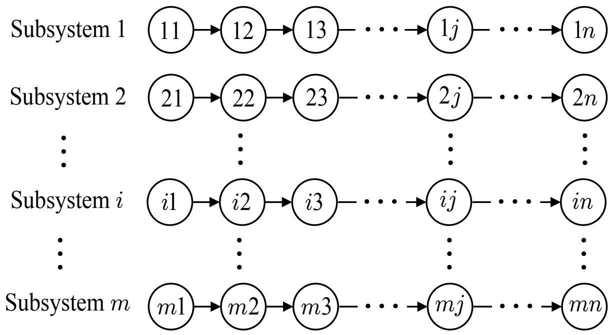

Fig. 1. Cascaded, networked systems consisting of m subsystems of n length.

along a directed line topology as shown in Fig. 1. Each node in this system, say node ij – with index $i \in \{1, 2, \dots, m\}$ corresponding to subsystem i, and $j \in \{1, 2, \dots, n\}$ denoting the j:th position along the line in subsystem i – is tasked with balancing its state value to the state value to node $i(j-1)$, i.e., the node incident to node ij along the directed line graph, while at the same time regulating the sum across all subsystems to some reference value at each position. The interpretation here could for example be a network model, where multiple clients send data under congestion on relay servers. Each subsystem is a cascaded system which represents the path of packets sent by a client toward a destination via several servers. On each server, network congestion occurs which limits the amount of the data rates under the server's capacity.

Based on the previous discussion, subsystem $i \in \{1, 2, \dots, m\}$ is given by the n-dimensional system as

$$\left\{ \begin{array}{rcl} \dot{x}_{i1}(t) & = & \varepsilon(r_i - x_{in}(t)) + \eta(x_{in}(t) - x_{i1}(t)) \\ \dot{x}_{ij}(t) & = & x_{i(j-1)}(t) - x_{ij}(t) + \kappa\left(\bar{r}_j - \displaystyle\sum_{k=1}^{m} x_{kj}(t)\right), \\ & & \qquad j \in \{2, 3, \dots, n\} \end{array} \right. \tag{1}$$

where $x_{ij}(t) \in \mathbb{R},\ j \in \{1, 2, \dots, n\}$ are the state variables, $r_i \in \mathbb{R}$ is the reference of $x_{in}(t)$, $\bar{r}_j \in \mathbb{R}$ is the reference of $\sum_{k=1}^{m} x_{kj}(t)$, and non-negative numbers ε, η and κ are gains. The block diagram of this system is depicted in Fig. 2. From the viewpoint of the network model, $x_{i1}(t)$ represents the packet rate sent by client i, $x_{ij}(t),\ j \in \{2, 3, \dots, n-1\}$ denotes his packet rate transfered through relay server j and $x_{in}(t)$ means the packet rate arriving at the destination. The basic control objective is to send packets with the desired rate r_i throughout any servers, that is, $x_{ij}(t) = r_i$ for all j. The first equation in (1) describes a behavior model (or protocol) of client i. The two terms with the gains ε and η represent the outer and inner loops, respectively, as shown in Fig. 2. The first term denotes the rate control for adjusting his sending packet rate $x_{i1}(t)$ to the desired one r_i. The second term represents the congestion control for reducing $x_{i1}(t)$ if the packet rate $x_{in}(t)$ at the destination is small because of congestion. The second equation in (1) represents behavior of server j such that it attempts to relay packets with the same rate as $x_{i(j-1)}(t)$ of the previous server, but the actual rate is $x_{ij}(t)$ because of network congestion. The congestion is described by the term with the gain κ such that the amount of the all clients' packet rates, i.e. $\sum_{k=1}^{m} x_{kj}(t)$, is attracted to the capacity $\bar{r}_j$.

Note that if $\kappa = 0$, the subsystems (1) are not connected with each other. Then, each subsystem is described as the n-dimensional system

$$\left\{ \begin{array}{rcl} \dot{x}_{i1}(t) & = & \varepsilon(r_i - x_{in}(t)) + \eta(x_{in}(t) - x_{i1}(t)) \\ \dot{x}_{ij}(t) & = & x_{i(j-1)}(t) - x_{ij}(t),\ j \in \{2, 3, \dots, n\} \end{array} \right. . \tag{2}$$

This equation represents the network model under no congestion, where ε and η represent the feedback gains of the outer and inner loops. Thus, it is expected that if the outer loop gain ε is small enough, the system (2) is stable. Then, what happens on the connected systems (1) under the influence of the network congestion?

The main problem in this paper is as follows.

Problem 1: Derive (necessary/sufficient) conditions for the connected systems (1) and the disconnected systems (2) to be stable.

First, we expect to know what gains ε and η guarantee the stability of the disconnected systems (2) depending on the dimension n of the subsystems in order to understand the issue of insufficient time-scale separation. Next, we investigate the relationship between the stability of the connected systems (1) and that of the disconnected ones (2) in order to understand the effect of the network congestion.

III. Stability of the Disconnected Subsystem

Consider the disconnected subsystem (2). Let $x_i(t) = [x_{i1}(t)\ x_{i2}(t)\ \cdots\ x_{in}(t)]^\top$ be the state of subsystem $i \in \{1, 2, \dots, m\}$. Then, (2) is described as

$$\dot{x}_i(t) = Ax_i(t) + Br_i$$

where $B = [\varepsilon\ 0\ 0\ \cdots\ 0]^\top \in \mathbb{R}^n$ and

$$A = \begin{bmatrix} -\eta & 0 & 0 & \cdots & \eta - \varepsilon \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \in \mathbb{R}^{n \times n}. \tag{3}$$

We analyze the stability of the subsystem (2) by investigating the eigenvalues of the matrix A. First, consider the simplest case that $n = 2$. Then, A is always stable.

Proposition 1: The matrix $A \in \mathbb{R}^{2\times 2}$ given by (3) is stable for any $\varepsilon > 0$ and $\eta \geq 0$.

Proof: The characteristic polynomial of A is given as

$$|\lambda I - A| = \begin{vmatrix} \lambda + \eta & -\eta + \varepsilon \\ -1 & \lambda + 1 \end{vmatrix} = \lambda^2 + (\eta + 1)\lambda + \varepsilon.$$

Its roots are given by

$$\lambda = \frac{-(\eta + 1) \pm \sqrt{(\eta + 1)^2 - 4\varepsilon}}{2}$$

whose real parts are negative for all $\varepsilon > 0$ and $\eta \geq 0$. ■

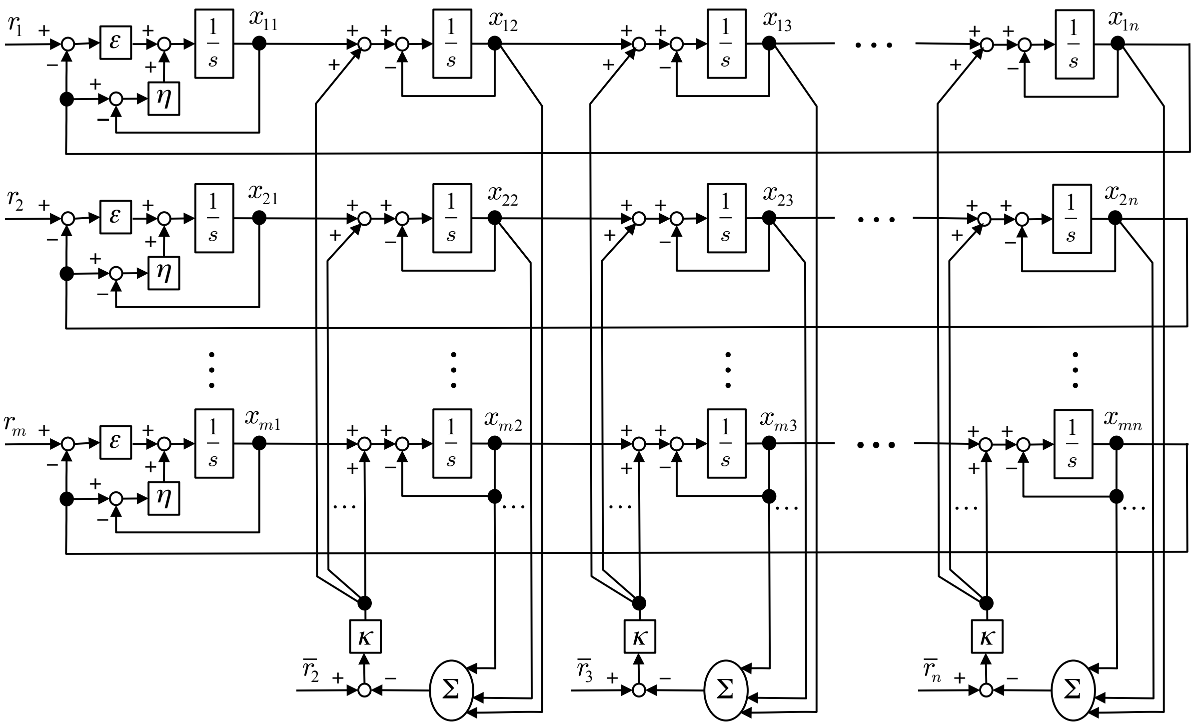

Fig. 2. Block diagram of the target system.

Next, consider the case that $n \geq 3$. In this case, the gains ε and η affect the stability of A. Actually, the following theorem gives a necessary condition for A to be stable, which implies that A can be unstable for large ε.

Theorem 1: For an integer $n \geq 3$ and real numbers $\varepsilon > 0$ and $\eta \geq 0$, the matrix $A \in \mathbb{R}^{n\times n}$ given by (3) is stable only if

$$\varepsilon < \frac{3(1+(n-1)\eta)(2+(n-2)\eta)}{(n-2)(3+\eta(n-3))}. \tag{4}$$

Proof: The characteristic polynomial of A is derived as

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda+\eta & 0 & 0 & \cdots & -\eta+\varepsilon \\ -1 & \lambda+1 & 0 & \cdots & 0 \\ 0 & -1 & \lambda+1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda+1 \end{vmatrix} \\ &= (\lambda+\eta)(\lambda+1)^{n-1} - \eta + \varepsilon \qquad (5) \\ &= (\lambda+\eta)\sum_{i=0}^{n-1} {}_{n-1}\mathrm{C}_i \lambda^i - \eta + \varepsilon \\ &= \lambda^n + \sum_{i=1}^{n-1} \left({}_{n-1}\mathrm{C}_{i-1} + {}_{n-1}\mathrm{C}_i \eta\right)\lambda^i + \varepsilon. \qquad (6) \end{aligned}$$

An eigenvalue $\lambda \in \mathbb{C}$ is not zero for $\varepsilon > 0$. Thus, λ is stable if and only if $\mu = 1/\lambda$ is stable. Because (6) is zero for an eigenvalue λ,

$$\begin{aligned} \frac{1}{\lambda^n}|\lambda I - A| &= \varepsilon\mu^n + \sum_{i=1}^{n-1}\left({}_{n-1}\mathrm{C}_{i-1} + {}_{n-1}\mathrm{C}_i\eta\right)\mu^{n-i} + 1 \\ &= 0 \qquad (7) \end{aligned}$$

is obtained. Then first part of the Routh table of the polynomial (7) is given as

$$\begin{array}{c|ccc} \mu^n & \varepsilon & {}_{n-1}\mathrm{C}_1 + {}_{n-1}\mathrm{C}_2\eta & \cdots \\ \mu^{n-1} & {}_{n-1}\mathrm{C}_0 + {}_{n-1}\mathrm{C}_1\eta & {}_{n-1}\mathrm{C}_2 + {}_{n-1}\mathrm{C}_3\eta & \cdots \\ \mu^{n-2} & b_1 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

where

$$\begin{aligned} b_1 &= \{({}_{n-1}\mathrm{C}_0 + {}_{n-1}\mathrm{C}_1\eta)({}_{n-1}\mathrm{C}_1 + {}_{n-1}\mathrm{C}_2\eta) \\ &\quad -\varepsilon({}_{n-1}\mathrm{C}_2 + {}_{n-1}\mathrm{C}_3\eta)\}/({}_{n-1}\mathrm{C}_0 + {}_{n-1}\mathrm{C}_1\eta). \qquad (8) \end{aligned}$$

From the Routh-Hurwitz stability criteria, (7) has only stable roots μ only if $b_1 > 0$. From (8), $b_1 > 0$ holds if and only if

$$\varepsilon < \frac{({}_{n-1}\mathrm{C}_0 + {}_{n-1}\mathrm{C}_1\eta)({}_{n-1}\mathrm{C}_1 + {}_{n-1}\mathrm{C}_2\eta)}{{}_{n-1}\mathrm{C}_2 + {}_{n-1}\mathrm{C}_3\eta}.$$

The right-side hand of this inequality is reduced to that of (4). ■

For $\eta > 0$, the Taylor series of (4) with respect to n^{-1} around $n^{-1} = 0$ is reduced to

$$\varepsilon < 3\eta + O(n^{-1}).$$

In this case, for a large dimension n of the subsystem (2), the gain ε cannot be larger than 3η. On the other hand, for $\eta = 0$, (4) is reduced to

$$\varepsilon < \frac{2}{n-2}. \tag{9}$$

In this case, ε is monotonically decreasing toward zero as n becomes large.

The next question is what kinds of gains ε and η guarantee the stability of A. In the case of $\eta > 0$, the following sufficient condition gives a certain criterion of such ε and η.

Theorem 2: For an integer $n \geq 3$ and real numbers $\varepsilon, \eta > 0$, the matrix $A \in \mathbb{R}^{n\times n}$ given by (3) is stable if

$$0 < \varepsilon \leq 2\eta. \tag{10}$$

Proof: This is from the secant condition in [14] or from the Gershgorin circle theorem. ■

In the case of $\eta = 0$, we can just say that A is stable if $\varepsilon > 0$ is sufficiently small.

Theorem 3: For an integer $n \geq 3$, there exists a positive number $\bar{\varepsilon} \leq 2/(n-2)$ such that the matrix $A \in \mathbb{R}^{n\times n}$ given by (3) with $\eta = 0$ is stable for any positive number $\varepsilon < \bar{\varepsilon}$.

Proof: Consider the characteristic polynomial of A given by (5). Let $\lambda(\varepsilon) \in \mathbb{C}$ be a function representing an eigenvalue of A, which explicitly shows the dependency on ε. The function $\lambda(\varepsilon)$ is continuous and its derivative $\lambda'(\varepsilon)$ is measurable [15]. By differentiating (5) with respect to ε, the following equation is obtained for $\eta = 0$.

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|\lambda(\varepsilon)I - A| = \lambda'(\varepsilon)(\lambda(\varepsilon)+1)^{n-2}(n\lambda(\varepsilon)+1)+1 \tag{11}$$

The characteristic polynomial (5) is always zero for any eigenvalue, so is (11). Therefore,

$$\lambda'(\varepsilon) = -\frac{1}{(\lambda(\varepsilon)+1)^{n-2}(n\lambda(\varepsilon)+1)} \tag{12}$$

is achieved as long as $\lambda(\varepsilon)$ is not -1 or $-1/n$.

For $\varepsilon = 0$, $\sigma(A) = \{-1, 0\}$ holds from (5). First, we consider the eigenvalue -1, namely, $\lambda(0) = -1$. Then, $\lambda(\varepsilon)$ is stable for a sufficiently small ε because $\lambda(\varepsilon)$ is a continuous function. Next, we consider the eigenvalue 0, namely, $\lambda(0) = 0$. Then, from (12), $\lambda'(0) = -1 < 0$ holds. Thus, $\mathrm{Re}(\lambda(\varepsilon)) < 0$ holds for a sufficiently small $\varepsilon > 0$. Therefore, A is stable for a sufficiently small $\varepsilon > 0$.

The upper bound of such ε is given by $\bar{\varepsilon} \leq 2/(n-2)$ from (9). The proof is completed. ■

IV. Stability of the Connected Subsystem

Consider the connected subsystem (1), which is described as

$$\dot{x}_i(t) = Ax_i(t) + \sum_{k=1}^{m} \bar{A}x_k(t) + Br_i - \bar{A}\bar{r}$$

for $\bar{r} = [0\ \bar{r}_2\ \bar{r}_3\ \cdots\ \bar{r}_m]^\top$ and

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -\kappa & 0 & \cdots & 0 \\ 0 & 0 & -\kappa & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\kappa \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{13}$$

Then, we have the following collective dynamics

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}r$$

for $x(t) = [x_1(t)^\top\ x_2(t)^\top\ \cdots x_m(t)^\top]^\top$, $r = [r_1\ r_2\ \cdots\ r_m\ \bar{r}^\top]^\top$ and

$$\hat{A} = \begin{bmatrix} A+\bar{A} & \bar{A} & \cdots & \bar{A} \\ \bar{A} & A+\bar{A} & & \bar{A} \\ \vdots & & \ddots & \vdots \\ \bar{A} & \bar{A} & \cdots & A+\bar{A} \end{bmatrix} \in \mathbb{R}^{mn\times mn} \tag{14}$$

$$\hat{B} = \begin{bmatrix} B & 0 & \cdots & 0 & -\bar{A} \\ 0 & B & & 0 & -\bar{A} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B & -\bar{A} \end{bmatrix} \in \mathbb{R}^{mn\times 2m}.$$

Now, the following lemma specifies the eigenvalues of $\hat{A}$ given by (14). Note that this lemma is valid for any matrices A and $\bar{A}$, namely it does not restrict the matrices to (3) and (13).

Lemma 1: For matrices $A, \bar{A} \in \mathbb{R}^{n\times n}$ and an integer $m \geq 2$, consider the matrix $\hat{A} \in \mathbb{R}^{nm\times nm}$ given by (14). Then,

$$\sigma(\hat{A}) = \sigma(A) \cup \sigma(A + m\bar{A}) \tag{15}$$

is obtained, which yields

$$\lambda_{\mathrm{Rmax}}(\hat{A}) = \max\{\lambda_{\mathrm{Rmax}}(A), \lambda_{\mathrm{Rmax}}(A+m\bar{A})\}. \tag{16}$$

Proof: Let $I_m \in \mathbb{R}^{m\times m}$ and $\mathbf{1}_m \in \mathbb{R}^m$ be the identity matrix and the vector all whose entries are 1. Then, from (14), $\hat{A}$ is described as follows.

$$\hat{A} = I_m \otimes A + \mathbf{1}_m\mathbf{1}_m^\top \otimes \bar{A} \tag{17}$$

Let $W = [w_1\ W_2] \in \mathbb{R}^{m\times m}$ be an orthogonal matrix for $w_1 = \mathbf{1}_m/\sqrt{m}$ and some matrix $W_2 \in \mathbb{R}^{m\times(m-1)}$. Then, from (17), the following is derived for $\lambda \in \mathbb{C}$.

$$\begin{aligned}(W\otimes I_n)^\top&(\lambda I_{mn} - \hat{A})(W\otimes I_n) \\ &= \begin{bmatrix} \lambda I_n - A - m\bar{A} & 0 \\ 0 & I_{m-1}\otimes(\lambda I_n - A) \end{bmatrix}\end{aligned}$$

Because of this equation and $(W\otimes I_n)^\top(W\otimes I_n) = I_{mn}$, the characteristic polynomial of $\hat{A}$ is reduced to

$$\begin{aligned}|\lambda I_{mn} - \hat{A}| &= |(W\otimes I_n)^\top(\lambda I_{mn} - \hat{A})(W\otimes I_n)| \\ &= |\lambda I_n - A - m\bar{A}||\lambda I_n - A|^{m-1}.\end{aligned}$$

Thus, the set of the eigenvalues of $\hat{A}$ consists of the eigenvalues of $(A + m\bar{A})$ and A. Then, (15) is derived. (16) is directly from (15). The proof is completed. ■

Lemma 1 suggests that in order to investigate the stability of $\hat{A}$, we just have to consider those of A and $(A + m\bar{A})$. The stability of A has been discussed in Section III. Thus, now, we consider $(A + m\bar{A})$ as follows.

Lemma 2: For integers $n, m \geq 2$ and real numbers $\varepsilon, \kappa > 0$ and $\eta \geq 0$, consider the matrices $A, \bar{A} \in \mathbb{R}^{n\times n}$ given by (3) and (13). Then, there exists $c \leq 0$ such that

$$\lambda_{\mathrm{Rmax}}(A + m\bar{A}) \leq \max\{\lambda_{\mathrm{Rmax}}(A), c\}, \tag{18}$$

where c is given by

$$c = \max\left\{-\eta, -\frac{\eta(n-1)+1+m\kappa}{n}\right\}. \tag{19}$$

Proof: The characteristic polynomial of $(A+m\bar{A})$ is given by

$$|\lambda I_n-(A+m\bar{A})|=(\lambda+\eta)(\lambda+1+m\kappa)^{n-1}-\eta+\varepsilon \quad (20)$$

for $\lambda\in\mathbb{C}$ from the same calculation as (5).

We have consider the case that $\eta=\varepsilon$, which can be easily proved from from (5) and (20).

Consider the case that $\eta\neq\varepsilon$, then $\lambda+1+m\kappa\neq 0$ holds for any eigenvalues λ. Each eigenvalue of $(A+m\bar{A})$ is regarded as a continuous function $\lambda(\kappa)\in\mathbb{C}$ with the variable κ such that $\lambda'(\kappa)$ is measurable. Because the characteristic polynomial (20) is always zero for the eigenvalue λ, by differentiating (20) with respect to κ, we obtain

$$\mathrm{Re}(\lambda'(\kappa))= \frac{-m(n-1)f(\rho,\omega)}{(n\rho+(n-1)\eta+1+m\kappa)^2+(n\omega)^2} \quad (21)$$

where $\rho=\mathrm{Re}(\lambda(\kappa))$, $\omega=\mathrm{Im}(\lambda(\kappa))$ for certain $\kappa\geq 0$, and

$$f(\rho,\omega)=(\rho+\eta)(n\rho+(n-1)\eta+1+m\kappa)+n\omega^2. \quad (22)$$

From (19), (21) and (22), the expressions

$$\mathrm{Re}(\lambda(\kappa))=\rho>c\Rightarrow f(\rho,\omega)>0\Rightarrow \mathrm{Re}(\lambda'(\kappa))<0 \quad (23)$$

are satisfied for any $\omega\in\mathbb{R}$.

Now, we focus on one of the eigenvalues $\lambda_i(\kappa)$, $i=1,2,\ldots,n$. If $\mathrm{Re}(\lambda_i(0))\leq c$ is satisfied, then $\mathrm{Re}(\lambda_i(\kappa))\leq c$ holds for any κ from (23). On the other hand, if $\mathrm{Re}(\lambda_i(0))>c$, then $\mathrm{Re}(\lambda_i'(\kappa))<0$ holds from (23), and $\mathrm{Re}(\lambda_i(\kappa))$ monotonically decreases as κ increases as long as $\mathrm{Re}(\lambda_i(\kappa))>c$. Therefore, $\mathrm{Re}(\lambda_i(\kappa))\leq\mathrm{Re}(\lambda_i(0))$ is satisfied for any $\kappa>0$. Thus, $\mathrm{Re}(\lambda_i(\kappa))\leq\max\{\mathrm{Re}(\lambda_i(0)),c\}$ is achieved. Then,

$$\mathrm{Re}(\lambda_i(\kappa))\leq\max\left\{\max_{j=1,2,\ldots,n}\mathrm{Re}(\lambda_j(0)),c\right\} \quad (24)$$

is obtained for any $i=1,2,\ldots,n$. Note that $\lambda_i(\kappa)$ and $\lambda_i(0)$ are the eigenvalues of $(A+m\bar{A})$ and A, respectively. Then, (24) is reduced to (18). ∎

From Lemmas 1 and 2, we can specify the maximum real part of the eigenvalues of $\hat{A}$ as follows.

Theorem 4: For integers $m,n\geq 2$ and real numbers $\varepsilon,\kappa>0$ and $\eta\geq 0$, consider the matrices $A,\bar{A}\in\mathbb{R}^{n\times n}$ and $\hat{A}\in\mathbb{R}^{nm\times nm}$ given by (3), (13) and (14). Then,

$$\lambda_{\mathrm{Rmax}}(A)\leq\lambda_{\mathrm{Rmax}}(\hat{A})\leq\max\{\lambda_{\mathrm{Rmax}}(A),c\} \quad (25)$$

is satisfied for c given by (19).

Proof: The first inequality of (25) follows from (16). The second inequality is obtained from (16) and (18). The proof is completed. ∎

Note that if m or κ is large enough, then $c=-\eta$ holds from (19). Then, if $\eta>0$ (i.e. $c<0$), (25) guarantees that $\hat{A}$ is stable if and only if A is stable. However, if $\eta=0$ (i.e. $c=0$), then (25) does not guarantee the stability of $\hat{A}$ even if A is stable. However, also in this case, the stability of $\hat{A}$ and A are equivalent.

Theorem 5: For integers $m,n\geq 2$ and real numbers $\varepsilon,\kappa>0$ and $\eta=0$, consider the matrices $A,\bar{A}\in\mathbb{R}^{n\times n}$ and $\hat{A}\in\mathbb{R}^{nm\times nm}$ given by (3), (13) and (14). Assume that A is stable. Then, the following is satisfied.

$$\lambda_{\mathrm{Rmax}}(A)\leq\lambda_{\mathrm{Rmax}}(\hat{A})<0 \quad (26)$$

Proof: The first inequality of (26) is directly from (16). Let $\lambda(\kappa)$ be the eigenvalues of $(A+m\bar{A})$ for $\eta=0$. If $\mathrm{Re}(\lambda(\kappa))=0$ held for some κ,

$$\mathrm{Re}(\lambda'(\kappa))=\frac{-m(n-1)n\omega^2}{(1+m\kappa)^2+(n\omega)^2}\leq 0 \quad (27)$$

would be satisfied from (21). Because A is stable, any its eigenvalues satisfy $\mathrm{Re}(\lambda(0))<0$. From this and (27), $\mathrm{Re}(\lambda(\kappa))<0$ holds for any $\kappa>0$, which implies the second inequality of (26) with (16). ∎

Theorems 4 and 5 show that the stability of A and $\hat{A}$ are equivalent.

Corollary 1: For integers $m,n\geq 2$ and real numbers $\varepsilon,\kappa>0$ and $\eta\geq 0$, consider the matrices $A,\bar{A}\in\mathbb{R}^{n\times n}$ and $\hat{A}\in\mathbb{R}^{nm\times nm}$ given by (3), (13) and (14). Then, $\hat{A}$ is stable if and only if A is stable.

Corollary 1 shows that the connected systems (1) are stable if and only if the disconnected ones (2) are stable for any κ. However, Theorems 4 and 5 imply that the performance of the connected systems is deteriorated by the network congestion. How much the performance deteriorates can be estimated by c.

Remark 1: Assume that $\eta=0$ and A is stable. Then, although (26) guarantees that $\hat{A}$ is stable, one of its eigenvalues might be in bad condition. Actually, from (20), it is shown that $(A+m\bar{A})$ has an eigenvalue such that $\lambda=O((m\kappa)^{1-n})$ [16]. From (15), $\hat{A}$ has the same eigenvalue. Then, even if A is stable, one of the eigenvalues of $\hat{A}$ is almost zero if m or κ is large.

V. Numerical Example

In this section, numerical examples illustrate the validness of the analysis results. First, consider the disconnected subsystem (2) and verify the conditions in Theorems 1, 2 and 3.

Fig. 3 shows the stability boundary according to the gain ε and the dimension n of the subsystem for $\eta=1$. The solid line is the actual boundary, which shows that ε should be smaller to guarantees the stability as n is larger. The boundary is always under the upper broken line, which is given by the necessary condition (4) in Theorem 1. The boundary is always over the lower broken line $\varepsilon=2$, which is given by the sufficient condition (10) in Theorem 2. It is observed that for small n the upper broken line (i.e. the necessary condition) describes the actual boundary well, and for large n the lower broken line (i.e. the sufficient condition) describes well.

Fig. 4 shows the stability boundary for $\eta=0$. The solid line is the actual boundary, which is always under the broken line given by the necessary condition (4). In this case, the actual boundary converges to zero as n grows, but it is always positive as guaranteed by Theorem 3. Therefore, we have to choose smaller ε as n is larger, which is explained by (9).

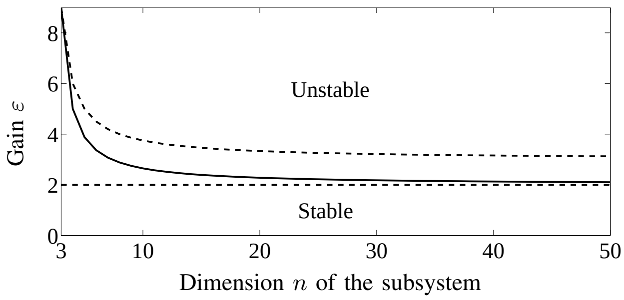

Fig. 3. Stability boundary according to the gain ε and the dimension n of the subsystem for $\eta = 1$. The solid line is the actual boundary. The upper and lower broken lines are from the necessary and sufficient conditions (4) and (10), respectively.

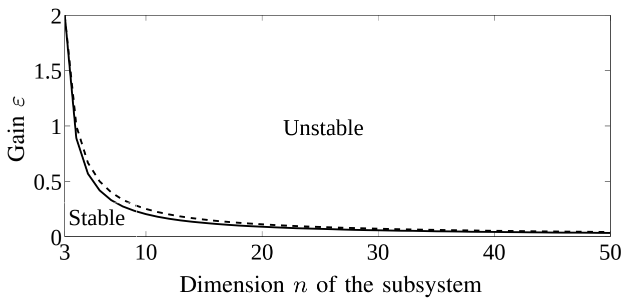

Fig. 4. Stability boundary according to the gain ε and the dimension n of the subsystem for $\eta = 0$. The solid line is the actual boundary and the broken line is from the necessary condition (4).

Next, consider the connected subsystem (1) and verify the correctness of Theorems 4 and 5. For $n = 5$ and $\kappa = 0.1$, we consider the three cases $(\eta, \varepsilon) = (0, 0.09)$, $(0.1, 0.24)$ and $(1, 2)$. In each case, $\lambda_{\mathrm{Rmax}}(A) = -0.19$ is derived for A given in (3).

Fig. 5 depicts the value of $\lambda_{\mathrm{Rmax}}(\hat{A})$ according to the number m of the subsystems. First, the solid line describes the case of $(\eta, \varepsilon) = (0, 0.09)$, which shows that $\lambda_{\mathrm{Rmax}}(\hat{A})$ converges to zero. As stated in Remark 1, although Theorem 5 guarantees the stability of the connected subsystem, one of the eigenvalues has a worse condition as m is larger. Next, the dashed line describes the case of $(\eta, \varepsilon) = (0.1, 0.24)$, which shows that $\lambda_{\mathrm{Rmax}}(\hat{A})$ is always between -0.19 (i.e. $\lambda_{\mathrm{Rmax}}(A)$) and -0.1 (i.e. $-\eta$). This is guaranteed by Theorem 4. Finally, the chain line describes the case of $(\eta, \varepsilon) = (1, 2)$, which shows that $\lambda_{\mathrm{Rmax}}(\hat{A})$ is always -0.19 (i.e. $\lambda_{\mathrm{Rmax}}(A)$). This is because from $c = -\eta < \lambda_{\mathrm{Rmax}}(A)$ in (25), Theorem 4 guarantees that $\lambda_{\mathrm{Rmax}}(\hat{A}) = \lambda_{\mathrm{Rmax}}(A)$.

These numerical examples show the validness of our results, Theorems 1, 2, 3, 4 and 5.

VI. Conclusion

In this paper, we investigated the effect of insufficient time-scale separation between inner and the outer loops in a cascaded, networked system. First, we analyzed the stability of the cascaded system in order to understand the issue of insufficient time-scale separation. Then, it was revealed that the ratio of the outer and inner loop gains is the key of the stability. Next, we investigated the relationship between the stability of the connected system and that of the disconnected systems in order to understand the effect of the network congestion. Then, it was shown that the congestion deteriorates the performance of the networked systems, but it does not harm the stability. These results help us to grasp the phenomena emerging in the network systems from the viewpoint of the control theory.

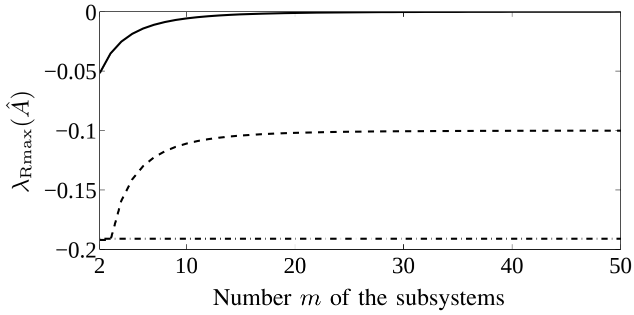

Fig. 5. Maximum real part of the eigenvalues of $\hat{A}$ according to the number m of the subsystems. Solid, dashed and chain lines describe the cases of $(\eta, \varepsilon) = (0, 0.09)$, $(0.1, 0.24)$ and $(1, 2)$, respectively.

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