Existence of Ground State for the NLS on Star-like Graphs

A joint work in collaboration with D. Finco and D. Noja

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**Metric graph:** Each edge is associated either to a compact interval (if it is finite) or to $[0, +\infty)$ (if it is infinite)

- $E$ : denotes the set of edges of $G$
- $V$ : denotes the set of vertices of $G$

**Assumption 1**

$G$ is a connected graph with a finite number of edges and vertices, and it is composed by at least one infinite edge (one half-line) attached to a compact core.
A Star-Like Graph

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**Notation**

**Hilbert space:** $\Psi \in L^2(G)$ means

$$\Psi = (\psi_1, \psi_2, \ldots, \psi_{|E|}) \quad \psi_e \in L^2(I_e) \quad \forall e \in E$$

**Sobolev spaces:**

$$H^1(G) := \{ \Psi \in L^2(G) | \psi_e \in H^1(I_e) \forall e \in E \text{ and } \Psi \text{ is continuous in the vertices} \}$$

$$H^2(G) := \{ \Psi \in H^1(G) | \psi_e \in H^2(I_e) \forall e \in E \}$$

Scalar products and norms are defined in a natural way:

$$\|\Psi\|_{G}^2 = \sum_{e \in E} \|\psi_e\|_{I_e}^2$$
The Nonlinear Schrödinger Equation

\[ i \frac{d}{dt} \Psi = H \Psi - |\Psi|^{2\mu} \Psi \quad 0 < \mu < 2 \]

**Linear term:** \( H \) is a linear operator with \( \delta \)-interaction in the vertices plus a potential

\[ \mathcal{D}(H) := \left\{ \Psi \in H^2(G) \mid \sum_{e \prec v} \partial_o \psi_e(v) = \alpha(v) \psi_e(v), \ \alpha(v) \in \mathbb{R}, \ \forall v \in V \right\} . \]

\[ H \Psi = -\Psi'' + W \Psi \]

**Nonlinear term:** Focusing powerlike nonlinearity, subcritical

**Componentwise:**

\[ i \frac{d}{dt} \psi_e = -\frac{d^2}{dx_e^2} \psi_e + W_e \psi_e - |\psi_e|^{2\mu} \psi_e \quad \forall e \in E \]

- \(|E|\) scalar equations
- Coupled by the conditions in the vertices
The ground state for the NLS

Nonlinear energy functional: Defined on $H^1(G)$ as

$$E[\Psi] = \|\Psi'\|^2 + (\Psi, W\Psi) + \sum_{v \in V} \alpha(v)|\Psi(v)|^2 - \frac{1}{\mu + 1} \|\Psi\|^{2\mu+2}_{2\mu+2}$$

Ground state: Minimizer of $E[\Psi]$ at fixed mass $m = \|\Psi\|^2$

Problem: Under what conditions on $G$, $H$, $m$ the ground state does/does not exist

or equivalently

Under what conditions on $G$, $H$, $m$ the infimum

$$\inf\{E[\Psi] \mid \Psi \in H^1(G), \|\Psi\|^2 = m\}$$

is/is not attained
Orbital Stability of Ground State

Let \( \hat{\Psi} \) be a ground state, and consider the Cauchy problem:

\[
\begin{aligned}
  i \frac{d}{dt} \Psi &= H \Psi - |\Psi|^{2\mu} \Psi \\
  \Psi|_{t=0} &= \Psi_0
\end{aligned}
\]

(1)

\( e^{i\omega t} \hat{\Psi} \), \( \omega \in \mathbb{R} \), is the stationary solution of (1) with initial datum \( \Psi_0 = \hat{\Psi} \).
Orbital Stability of Ground State

Let $\hat{\Psi}$ be a ground state, and consider the Cauchy problem:

$$\begin{cases} i \frac{d}{dt} \Psi = H \Psi - |\Psi|^{2\mu} \Psi \\ \Psi|_{t=0} = \Psi_0 \end{cases} \tag{1}$$

$e^{i\omega t} \hat{\Psi}$, $\omega \in \mathbb{R}$, is the stationary solution of (1) with initial datum $\Psi_0 = \hat{\Psi}$.

Theorem (Cazenave Lions '82)

Let $0 < \mu < 2$. For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if

$$\|\Psi_0 - \hat{\Psi}\|_{H^1} \leq \delta(\varepsilon)$$

then the corresponding solution of (1) is such that

$$\sup_{t \in \mathbb{R}^+} \inf_{\theta \in \mathbb{R}} \|\Psi(t) - e^{i\theta} \hat{\Psi}\|_{H^1} \leq \varepsilon$$
Main Results: $W = 0$, $\alpha = 0$ [Adami-Serra-Tilli '14 '16]

- Take $W = 0$ and $\alpha(v) = 0$ for all $v \in V$
- Find topological and metric conditions on $\mathcal{G}$ that guarantee existence/nonexistence of the ground state

**Condition H:** From every point of the graph one can get to infinity through two disjoint paths
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**Condition H:** From every point of the graph one can get to infinity through two disjoint paths

Adami-Serra-Tilli ’14: If (H) is satisfied the ground state **does not** exist unless $G$ is isometric to a bubble tower.
Main Results: Star-Graph, $W = 0$, $\alpha < 0$ [Adami, C.C., Finco, Noja ’14]

- Consider a Star-Graph
- Take $W = 0$ and $\alpha(v) < 0$

Adami, C.C., Finco, Noja ’14: There exists $m^* > 0$ such that for $0 < m < m^*$ the ground state exists.

Adami, Noja, Visciglia ’13; Fukuizumi, Ohta, Ozawa ’08: If $|E| = 2$ the ground state exists for any $m > 0$.

Adami, C.C., Finco, Noja ’16: If $|E| \geq 3$ there exists $m^{**} > 0$ such that for $m > m^{**}$ the ground state does not exist.
Main Results: Generic Starlike-Graph [C.C., Finco, Noja prep. ’16]

Assumption 2
\[ W = W_+ - W_- \text{ with } W_\pm \geq 0, \ W_+ \in L^1(G) + L^\infty(G), \text{ and } W_- \in L^r(G) \text{ for some } r \in [1, 1 + 1/\mu]. \]

Assumption 3
\[ \inf \sigma(H) := -E_0, \ E_0 > 0 \text{ and it is an isolated eigenvalue.} \]

Theorem
Let \( 0 < \mu < 2 \). If Assumptions 1, 2, and 3 hold true then
\[ -\infty < \inf \{ E[\Psi] | \Psi \in H^1(G), \|\Psi\|^2 = m \} \leq -E_0 m \]
for any \( m > 0 \). Moreover, there exists \( m^* > 0 \) such that for \( 0 < m < m^* \) the infimum is attained, i.e., the ground state exists.
Main Results: Generic Starlike-Graph [C.C., Finco, Noja prep. '16]

The inequality

$$\inf \left\{ E[\Psi] \mid \Psi \in H^1(G), \|\Psi\|^2 = m \right\} \leq -E_0 m$$

is a direct consequence of

$$E[\Psi] \leq \|\Psi\|^2 + (\Psi, W\Psi) + \sum_{v \in V} \alpha(v)|\Psi(v)|^2$$

for all $\Psi \in H^1(G)$ and of

$$-E_0 m = \inf \left\{ \|\Psi\|^2 + (\Psi, W\Psi) + \sum_{v \in V} \alpha(v)|\Psi(v)|^2 \mid \Psi \in H^1(G), \|\Psi\|^2 = m \right\}$$
Concentration-Compactness

For any $\Psi \in L^2(G)$ define the concentration function

$$\rho(\Psi, s) = \sup_{y \in G} \|\Psi\|_{L^2(B_G(y,s))}^2.$$ 

Let $\{\Psi_n\}_{n \in \mathbb{N}}$ be such that: $\Psi_n \in H^1(G)$,

$$\|\Psi_n\|^2 = m \sup_{n \in \mathbb{N}} \|\Psi'_n\| < C$$

Define the concentrated mass parameter $\tau$ as

$$\tau = \lim_{s \to \infty} \lim_{n \to \infty} \rho(\Psi_n, s).$$

i) (Compactness) If $\tau = m$, at least one of the two following cases occurs:

i$_1$) (Convergence) There exists a function $\Psi \in H^1(G)$ such that $\Psi_n \to \Psi$ in $L^p(G)$ for all $2 \leq p \leq \infty$.

i$_2$) (Runaway) $\|\Psi_n\|_{L^p(B_G(y,s))} \to 0$ for all $2 \leq p \leq \infty$, $y \in G$, $s > 0$.

ii) (Vanishing) If $\tau = 0$, then $\Psi_n \to 0$ in $L^p(G)$ for all $2 < p \leq \infty$.

iii) (Dichotomy) If $0 < \tau < m$, then there exist two sequences $\{R_n\}_{n \in \mathbb{N}}$ and $\{S_n\}_{n \in \mathbb{N}}$ in $H^1(G)$ such that: $\|\Psi_n - R_n - S_n\| \to 0$,

$$\|R_n\|^2 \to \tau \quad \|S_n\|^2 \to m - \tau$$

$$\text{dist}(\text{Supp } R_n, \text{Supp } S_n) \to \infty.$$
Concentration-Compactness

If $\Psi_n$ is a minimizing sequence

- Vanishing and Dichotomy cannot occur
- If $i_2$ (Runaway), then

$$\lim_{n \to \infty} E[\Psi_n] \geq -\gamma \mu m^{1+\frac{2\mu}{2-\mu}}.$$

$-\gamma \mu m^{1+\frac{2\mu}{2-\mu}}$ is the energy of the ground state of mass $m$ of the NLS on the real line

$$-\gamma \mu m^{1+\frac{2\mu}{2-\mu}} = \inf_{\psi \in H^1(\mathbb{R}), \|\psi\|_{L^2(\mathbb{R})} = m} \left( \|\psi\|_{L^2(\mathbb{R})}^2 - \frac{1}{\mu + 1} \|\psi\|_{L^{2\mu+2}(\mathbb{R})}^{2\mu+2} \right).$$
Concentration-Compactness

If $\Psi_n$ is a minimizing sequence

- Vanishing and Dichotomy cannot occur
- If $i_2$ (Runaway), then

$$\lim_{n \to \infty} E[\Psi_n] \geq -\gamma \mu m^{1 + \frac{2\mu}{2-\mu}}.$$ 

- But for $m$ small enough

$$\inf \left\{ E[\Psi] \mid \Psi \in H^1(G), \|\Psi\|^2 = m \right\} \leq -E_0 m < -\gamma \mu m^{1 + \frac{2\mu}{2-\mu}}$$

- Indeed $m^* = \left( \frac{E_0}{\gamma \mu} \right)^{\frac{1}{\mu}} - \frac{1}{2}$
Bifurcation Analysis

If $E_0$ is a simple eigenvalue one can use bifurcation theory to find a candidate. Consider the stationary equation

$$H\Phi - |\Phi|^{2\mu}\Phi = -\omega\Phi \quad \Phi \in \mathcal{D}(H), \ \omega \in \mathbb{R}$$

Let $H\Phi_0 = -E_0\Phi_0$, with $\|\Phi_0\|^2 = 1$. Then for $\omega > E_0$ there exists a solution

$$\Phi(\omega) = a_*(\omega)\Phi_0 + \Theta_*(a_*(\omega), \omega)$$

such that

$$m(\omega) = \|\Phi(\omega)\|^2 = \left(\frac{\omega - E_0}{\|\Phi_0\|^{2\mu+2}}\right)^{\frac{1}{\mu}} + o\left((\omega - E_0)^{\frac{1}{\mu}}\right)$$

$$\|\Phi(\omega)\|^2$$

$E_0$ $\omega$

$E[\Phi(\omega(m))] = -E_0m + o(m)$
Remarks

- A sufficient condition to have an isolated eigenvalue is

\[ \int_G W dx + \sum_{v \in V} \alpha(v) < 0 \]

- If \( W = 0 \) and \( \alpha(v) \leq 0 \ \forall v \in V \), and strictly negative for at least one vertex. Then \(-E_0\) is a simple eigenvalue [Exner, Jex ’12].

- For compact graphs with \( \delta \)-vertices, simplicity of the spectrum can be achieved by small modification of edge lengths [Berkolaiko, Liu ’16].

- The analysis can be extended in principle to the case in which \(-E_0\) has multiplicity larger than one. One has to use bifurcation analysis in the degenerate case.

- We do not claim that \( \Phi(\omega(m)) \) is the ground state, even though we conjecture that this is true. This can be proved in the case of the Star-Graph with \( W = 0 \) and \( \alpha(v) < 0 \).

- Our result does not cover the case \( W = 0, \alpha(v) = 0 \), treated by Adami, Serra, Tilli. Since in this case there are not isolated eigenvalues, \( \sigma(H) = \sigma_{ess}(H) = [0, +\infty) \).
Global Well-Posedness in $H^1(\mathcal{G})$

**Theorem (Global Well-Posedness)**

Let $0 < \mu < 2$. For any $\Psi_0 \in H^1(\mathcal{G})$, the Cauchy problem

$$
\begin{cases}
    i \frac{d}{dt} \Psi = H\Psi - |\Psi|^{2\mu} \Psi \\
    \Psi|_{t=0} = \Psi_0
\end{cases}
$$

has a unique weak solution $\Psi \in C^0([0, \infty), H^1(\mathcal{G})) \cap C^1([0, \infty), H^1(\mathcal{G}^*)$.

The proof uses the following conservation laws

**Proposition (Conservation laws)**

Let $\mu > 0$. For any weak solution $\Psi \in C^0([0, T), H^1(\mathcal{G})) \cap C^1([0, T), H^1(\mathcal{G}^*)$ to the problem (2), the following conservation laws hold at any time $t$:

$$
\|\Psi(t)\|^2 = \|\Psi_0\|^2, \quad E[\Psi(t)] = E[\Psi_0].
$$

Together with Local Well-posedness (proved by Banach fixed point theorem) and Gagliardo-Nirenberg inequalities
Gagliardo-Nirenberg Inequalities

Proposition

Let $G$ be graph with a finite number of edges and vertices. Then if $p, q \in [2, +\infty]$, with $p \geq q$, and $\alpha = \frac{2}{2+q} (1 - q/p)$, there exists $C$ such that

$$\|\Psi\|_p \leq C \|\Psi\|_{H^1} \|\Psi\|_q^{1-\alpha},$$

for all $\Psi \in H^1(G)$.

If the $G$ has one or more infinite edges one has the stronger inequality

$$\|\Psi\|_p \leq C \|\Psi'\|^{\alpha} \|\Psi\|_q^{1-\alpha},$$

See, e.g., [Mugnolo Springer '14, Adami-Serra-Tilli JFA '16].