SPECTRAL STATISTICS OF RANDOM SCHRÖDINGER OPERATORS WITH NON-ERGODIC RANDOM POTENTIAL

Dhriti Ranjan Dolai.
Pontificia Universidad Católica de Chile
Santiago, Chile

Atlanta, 8 October, 2016
On $\ell^2(\mathbb{Z}^d)$ define the operator $\Delta$ by

$$(\Delta u)(n) = \sum_{|k|_+=1} u(n + k) - 2d \, u(n), \quad u \in \ell^2(\mathbb{Z}^d).$$

The random potential $V^\omega$ is the multiplication operator on $\ell^2(\mathbb{Z}^d)$ given by

$$(V^\omega u)(n) = (1 + |n|^\alpha) \, \omega_n \, u(n), \quad \alpha > 0$$

where $(\omega_n)_{n \in \mathbb{Z}^d}$ are iid real random variables uniformly distributed on $[0, 1]$.

Consider the probability space $(\Omega = [0, 1]^{\mathbb{Z}^d}, \mathcal{B}_\Omega, \mathbb{P} = \otimes \mu)$. The random operators $H^\omega$ on $\ell^2(\mathbb{Z}^d)$

$$H^\omega = -\Delta + V^\omega, \quad \omega = (\omega_n)_{n \in \mathbb{Z}^d} \in \Omega.$$
Define \( \{a_j\}_{j \geq 1} \), \( (a_0 = \infty) \) given by

\[
a_j = \inf_{\phi \in C(A_j)} \sum_{(n,m) \mid n-m\!_+=1} |\phi(n) - \phi(m)|^2, \quad \phi \in \ell^2(\mathbb{Z}^d),
\]

\( A_j \subset \mathbb{Z}^d \) with \( \#A_j = j \) and \( A_j \) are connected.

A path between points \( n, m \in \mathbb{Z}^d \) is a sequence of sites

\[
\tau = (n_1, n_2, \cdots, n_k), \quad n_1 = n, \quad n_k = m, \quad |n_{j+1} - n_j|_+ = 1.
\]

\( X \subset \mathbb{Z}^d \) is connected if any two points in \( X \) can be connected with a path which lies within \( X \).

\( \{a_j\}_{j \geq 1} \) is a strictly decreasing sequence.

\[
a_j < a_{j-1}, \quad j = 1, 2, \cdots
\]
- $H^\omega$ has discrete spectrum a.e $\omega$ if and only if $\alpha > d$.
- If $N^\omega(E)$ denotes the number of eigenvalues of $H^\omega$ which are less than $E$ then for $\alpha > d$ and for a.e $\omega$
  
  $$N^\omega(E) = O(E^{\frac{d}{\alpha}}) \text{ as } E \to \infty.$$  

Fixed a $k \in \mathbb{N}$ and $d/k \geq \alpha > d/(k + 1)$ for a.e $\omega$
- $\sigma(H^\omega) = \sigma_{pp}(H^\omega)$
- $\sigma_{ess}(H^\omega) = [a_k, \infty)$,
- $\#\sigma_{disc}(H^\omega) < \infty$. 

Define $H_{L}^{\omega} = \chi_{L} H^{\omega} \chi_{L}$, $\chi_{L}$ is the projection onto $l^{2}(\Lambda_{L})$, $\Lambda_{L}$ is a cube with side length $(2L + 1)$ centered at origin.

$N_{L}^{\omega}(E) = \#\{j : E_{j} \leq E, E_{j} \in \sigma(H_{L}^{\omega})\}$

If $d/k > \alpha > d/(k + 1)$, $k \in \mathbb{N}$ and $E \in (a_{j}, a_{j-1})$, $1 \leq j \leq k$, then

$$\lim_{L \to \infty} \frac{N_{L}^{\omega}(E)}{L^{d-j\alpha}} = N_{j}(E) \ (Non \ random) \ a.e \ \omega,$$

If $\alpha = d/k$ and $E \in (a_{j}, a_{j-1})$, $1 \leq j < k$, the above is valid. If $E \in (a_{k}, a_{k-1})$ then

$$\lim_{L \to \infty} \frac{N_{L}^{\omega}(E)}{lnL} = N_{k}(E) \ (Non \ random) \ a.e \ \omega.$$
For each \( \omega \), \( H^\omega_L \) is a matrix (symmetric) of order \( (2L + 1)^d \).

\[ \#\sigma(H^\omega_L) = (2L + 1)^d. \]

In other words average spacing between two consecutive eigenvalue of \( H^\omega_L \) inside \((a_j, a_{j-1})\) is of order \( L^{-(d-j\alpha)} \).

Now we want study how the eigenvalues of \( H^\omega_L \) are accumulating (are they following any rule) inside \((a_j, a_{j-1})\), as \( L \) gets large.
Define $\xi_{L,E}(\cdot) = \sum_j \delta_{L^d-\alpha(E_j-E)}(\cdot)$, $E \in (a_1, \infty)$.

- $\xi_{L,E}(I) = \# \{j : E_j \in E + L^{-(d-\alpha)}I \} \quad I \subset \mathbb{R}$
- $\{\xi_{L,E}\}$ is a sequence of integer valued random measure (i.e sequence of Point process).
- We want study the Weak limit of $\xi_{L,E}$
For $d \geq 3$, $\max\{2, \frac{d}{2}\} < \alpha < d$ and $E \in S$ the sequence of point process $\{\xi_{L,E}\}_L$ converges weakly to the Poisson point process with intensity measure $N'_1(E) \, dx$.

$$\lim_{L \to \infty} \mathbb{P}(\omega : \xi_{L,E}(B) = n) = e^{-N'_1(E)|B|} \frac{(N'_1(E)|B|)^n}{n!}, \quad n \in \mathbb{N} \cup \{0\}, \quad B \text{ is bounded Borel set of } \mathbb{R}.$$ 

In above $S \subset (a_1, \infty)$ such that $N'_1(E) > 0, \quad E \in S$
Idea of the proof (Exponential decay of Green’s function)

- We divide $\Lambda_L$ into $N_L^d$ numbers of disjoint cubes $C_p$ with side length $\frac{2L+1}{N_L}$

$$\Lambda_L = \bigcup C_p, \quad |C_p| = \left(\frac{2L+1}{N_L}\right)^d, \quad N_L = O(2L+1)^\epsilon, \quad 0 < \epsilon < 1.$$ 

- Let $H^\omega_p$ be the restriction of $H^\omega$ to $C_p$. Define

$$\eta^\omega_{p,L,E}(\cdot) = \sum_{x \in \sigma(H^\omega_p)} \delta_{Ld-\alpha} (x-E)$$

- Using Aizenman-Molcanov method we showed that

$$\mathbb{E}\left(\left| G^\omega_{\Lambda}(n, m; z) \right|^S \right) \leq Ce^{-r|n-m|}, \quad z \in \mathbb{C}^+$$

$$G^\omega_{\Lambda}(n, m; z) = \langle \delta_n, (H^\omega_{\Lambda} - z)^{-1} \delta_m \rangle, \quad \Lambda \subset \mathbb{Z}^d$$
\[ \mathbb{E}^\omega \left| \int fd\xi_{L,E}^\omega - \sum_{p=1}^{N_L^d} \int fd\eta_{p,L,E}^\omega \right| \xrightarrow{L \to \infty} 0, \ f \in C_c^+ (\mathbb{R}) \]

Since collection of functions of the form \( \frac{1}{x-z} \), \( z \in \mathbb{C}^+ \) is dense in \( C_c^+ \) so, to show the above convergence it is enough to verify the following

\[ \left| \int \frac{1}{x-z} d\xi_{L,E}(x) - \sum_{p=1}^{N_L^d} \int \frac{1}{x-z} d\eta_{p,L,E}(x) \right| \to 0 \text{ as } L \to \infty. \]

Now the above will follows from exponential decay of Green’s functions.
Convergence of uniformly asymptotically negligible triangular array

Now we will show \( \sum_{p=1}^{N_L^d} \eta_{p,E,L}^\omega \) converges weakly to the Poisson point process with intensity measure \( N'_1 \, dx \).

To show this it is enough to verify the following three conditions, for any bounded interval \( I \)

- (Con 1) \( \sup_{1 \leq p \leq N_L^d} \mathbb{P}(\omega : \eta_{p,L,E}(I) > \epsilon) = 0 \) as \( L \to \infty \) \( \forall \) \( \epsilon > 0 \).
- (Con 2) \( \sum_{p=1}^{N_L^d} \mathbb{P}(\omega : \eta_{p,L,E}(I) \geq 2) = 0 \) as \( L \to \infty \)
- (Con 3) \( \sum_{p=1}^{N_L^d} \mathbb{P}(\omega : \eta_{p,L,E}(I) \geq 1) = N'_1(E) |I| \) as \( L \to \infty \).
Minami and Wegner Estimate
(Combes-Germinet-Klein)

- **Wegner Estimate**
  \[ \mathbb{E}( TrE_{H_L^\omega}(l) ) \leq \sum_{n \in \Lambda_L} (1 + |n|^\alpha)^{-1} |l| \]

- **Minami Estimate**
  \[ \mathbb{E}\left( TrE_{H_L^\omega}(l) \left( TrE_{H_L^\omega}(l) - 1 \right) \right) \leq \left( \sum_{n \in \Lambda_L} (1 + |n|^\alpha)^{-1} |l| \right)^2 \]

- \( \nu_L(\cdot) = \frac{1}{L^{d-\alpha}} \mathbb{E}^\omega \left( Tr(E_{H_L^\omega}(\cdot)) \right) \), \( N_1(x) = \nu(a_1, x), \quad x > a_1 \)

- \( \nu_L \xrightarrow{L \to \infty} \nu \) weakly on \( (a_1, \infty) \).

- From Wegner estimate it will follow
  \[ \nu_L(\cdot) \leq C|l| \quad \text{and} \quad \nu(\cdot) \leq |l|. \]
Existence of Intensity and it’s positivity

- Convergence of densities inside $(a_1, \infty)$

\[ f_L(E) = \frac{d\nu_L}{dx}(E) \xrightarrow{\text{uniformly}} N'_1(E) = \frac{d\nu}{dx}(E) \text{ on } [E-\delta, E+\delta], \quad \delta > 0. \]

- To show the above convergence we first showed that \( \psi_L(z) = \frac{1}{L^{d-\alpha}} \sum_{n \in \Lambda_L} \mathbb{E}^\omega \left( G^\omega_{\Lambda_L}(n, n; z) \right) \) is analytic and uniformly bounded on a region \( G(\subset \mathbb{C}) \) which contain \( S \subset (a_1, \infty) \).

- The density \( f_L \) is given by \( f_L(E) = \frac{d\nu_L}{dx}(E) = \frac{1}{\pi} \text{Im}\psi_L(E + i0). \)
We showed that
\[ \nu(a, b) = N_1(b) - N_1(a) > 0, \quad |b - a| > 4d, \]
\[ a, b \in (4d, \infty) \subset (a_1, \infty) \]

The above can be shown using the min-max principal and the operator inequality \( A_{L,0}^\omega \leq H_L^\omega \leq A_{L,4d}^\omega \).

\[ A_{L,0}^\omega = \sum_{n \in \Lambda_L} b_n \omega_n, \quad A_{L,4d}^\omega = 4d + \sum_{n \in \Lambda_L} b_n \omega_n, \quad b_n = 1 + |n|^{-\alpha}. \]
(Con 1) will follow from Wegner estimate.

\[ P(\eta_{\omega}^{\omega \in L, \omega, E}(I) \geq 1) \leq E^{\omega}(\eta_{\omega}^{\omega \in L, \omega, E}(I)) \]

\[ = E^{\omega}[\text{Tr}E_{H_{\omega}^{\omega}}(E + L^{-(d-\alpha)}I)] \]

\[ = O(N_{L}^{-(d-\alpha)}) \]
(Con 2) will follow from Minami estimate.

\[ \sum_{j \geq 2} \mathbb{P}(\eta_{L,p,E}(l) \geq j) \leq \mathbb{E}^\omega \left[ \eta_{L,p,E}(l)(\eta_{L,p,E}(l) - 1) \right] \]

\[ = \mathbb{E}^\omega \left[ \text{Tr} E_\mathcal{H}^\omega \left( E + L^{-(d-\alpha)} l \right) \times \left\{ \text{Tr} E_\mathcal{H}^\omega \left( E + L^{-(d-\alpha)} l \right) - 1 \right\} \right] \]

\[ = O \left( N_{L}^{-(2(d-\alpha))} \right). \]
(Con 3) will follow from the following identity and convergence of density

\[ \mathbb{P}(\eta^\omega_{L,p,E}(l) \geq 1) = \mathbb{E}^\omega [\eta^\omega_{L,p,E}(l)] - \sum_{j \geq 2} \mathbb{P}(\eta^\omega_{L,p,E}(l) \geq j) \]

\[
\lim_{L \to \infty} \sum_p \mathbb{E}^\omega [\eta^\omega_{L,p,E}(l)] = \lim_{L \to \infty} \mathbb{E}^\omega [\xi^\omega_{L,E}(l)]
\]

\[
= \lim_{L \to \infty} \mathbb{E}^\omega \left( \text{Tr} E_{H^\omega_L} \left( E + L^{-(d-\alpha)} I \right) \right)
\]

\[
= \lim_{L \to \infty} L^{d-\alpha} \nu_L(E + L^{-(d-\alpha)})
\]

\[
= \lim_{L \to \infty} L^{d-\alpha} \int_{E + L^{-(d-\alpha)} I} f_L(x) dx
\]

\[
= \lim_{L \to \infty} \int_I f_L(E + L^{-(d-\alpha)} y) dy = N'_1(E) |I|.
\]


Thank You