Spectral decimation & its applications to spectral analysis on infinite fractal lattices

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Motivation: Analysis on nonsmooth domains
Some fractals are nicer than others

Each of these fractals is obtained from a nested sequence of graphs which has *nice, symmetric* replacement rules.
Spectral decimation (= spectral similarity)

Rammal-Toulouse ‘84, Bellissard ‘88, Fukushima-Shima ‘92, Shima ‘96, etc.
A recursive algorithm for identifying the Laplacian spectrum on highly symmetric, finitely ramified self-similar fractals.
Definition (Malozemov-Teplyaev ’03)

Let $\mathcal{H}$ and $\mathcal{H}_0$ be Hilbert spaces. We say that an operator $H$ on $\mathcal{H}$ is **spectrally similar** to $H_0$ on $\mathcal{H}_0$ with functions $\varphi_0$ and $\varphi_1$ if there exists a partial isometry $U : \mathcal{H}_0 \to \mathcal{H}$ (that is, $UU^* = I$) such that

$$U(H - z)^{-1}U^* = (\varphi_0(z)H_0 - \varphi_1(z))^{-1} =: \frac{1}{\varphi_0(z)}(H_0 - R(z))^{-1}$$

for any $z \in \mathbb{C}$ for which the two sides make sense.

A common class of examples: $\mathcal{H}_0$ subspace of $\mathcal{H}$, $U^*$ is an ortho. projection from $\mathcal{H}$ to $\mathcal{H}_0$. Write $H - z$ in block matrix form w.r.t. $\mathcal{H}_0 \oplus \mathcal{H}_0^\bot$:

$$H - z = \begin{pmatrix} I_0 - z & \bar{X} \\ X & Q - z \end{pmatrix}.$$

Then $U(H - z)^{-1}U^*$ is the inverse of the Schur complement $S(z)$ w.r.t. to the lower-right block of $H - Z$: $S(z) = (I_0 - z) - \bar{X}(Q - z)^{-1}X$.

Issue: There may exist a set of $z$ for which either $Q - z$ is not invertible, or $\varphi_0(z) = 0$. 
Spectral decimation: the main theorem

Spectrum \( \sigma(\Delta) = \{ z \in \mathbb{C} : \Delta - z \) does not have a bounded inverse\}. 

**Definition**

The exceptional set for spectral decimation is

\[
\mathcal{E}(H, H_0) \overset{\text{def}}{=} \{ z \in \mathbb{C} : z \in \sigma(Q) \text{ or } \varphi_0(z) = 0 \}.
\]

**Theorem (Malozemov-Teplyaev '03)**

Suppose \( H \) is spectrally similar to \( H_0 \). Then for any \( z \notin \mathcal{E}(H, H_0) \):

- \( R(z) \in \sigma(H_0) \iff z \in \sigma(H) \).
- \( R(z) \) is an eigenvalue of \( H_0 \) iff \( z \) is an eigenvalue of \( H \). Moreover there is a one-to-one map between the two eigenspaces.
Example: $\mathbb{Z}_+$

Let $\Delta$ be the graph Laplacian on $\mathbb{Z}_+$ (with Neumann boundary condition at 0), realized as the limit of graph Laplacians on $[0, 2^n] \cap \mathbb{Z}_+$.

If $z \neq 2$ and $R(z) = z(4 - z)$, then

- $R(z) \in \sigma(-\Delta) \iff z \in \sigma(-\Delta)$.
- $\sigma(-\Delta) = \mathcal{J}_R$, where $\mathcal{J}_R$ is the Julia set of $R$.
- $\mathcal{J}_R$ is the full interval $[0, 4]$. 
The $pq$-model

A one-parameter model of 1D fractals parametrized by $p \in (0, 1)$. Set $q = 1 - p$.

A triadic interval construction, “next easiest” fractal beyond the dyadic interval.

Earlier investigated by Kigami '04 (heat kernel estimates) and Teplyaev '05 (spectral decimation & spectral zeta function).

Assign probability weights to the three segments:

$$m_1 = m_3 = \frac{q}{1 + q}, \quad m_2 = \frac{p}{1 + q}$$

Then iterate. Let $\pi$ be the resulting self-similar probability measure.
Spectral decimation for the $pq$-model

The spectral decimation polynomial is $R(z) = \frac{z(z^2 - 3z + (2 + pq))}{pq}$.

$$\sigma(-\Delta_n) = \{0, 2\} \cup \bigcup_{m=0}^{n-1} R^{-m} \{1 \pm q\}$$

\[\text{(0, 0)} \quad \text{max}(p, q) \quad \text{(2, 2)}\]
Spectral decimation for the $pq$-model

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The \( pq \)-model on \( \mathbb{Z}_+ \)

- \( \Delta_p \) is not self-adjoint w.r.t. \( l^2(\mathbb{Z}_+) \), but is self-adjoint w.r.t. the discretization of the aforementioned self-similar measure \( \pi \).
- Let \( \Delta_p^+ = D^* \Delta_p D \), where

\[
D : l^2(\mathbb{Z}_+) \to l^2(3\mathbb{Z}_+), \quad (Df)(x) = f(3x).
\]

Then \( \Delta_p \) is spectrally similar to \( \Delta_p^+ \). Moreover, \( \Delta_p \) and \( \Delta_p^+ \) are isometrically equivalent (in \( L^2(\mathbb{Z}_+) \) or in \( L^2(\mathbb{Z}_+, \pi) \)).
The *pq*-model on \( \mathbb{Z}_+ \)

\[ \text{Spectrum } \sigma(H) = \{ z \in \mathbb{C} : H - z \text{ does not have a bounded inverse} \}. \]

Facts from functional analysis:

- \( \sigma(H) \) is a nonempty compact subset of \( \mathbb{C} \).
- \( \sigma(H) \) equals the disjoint union \( \sigma_{pp}(H) \cup \sigma_{ac}(H) \cup \sigma_{sc}(H) \).

- pure point spectrum \( \cup \) absolutely continuous spectrum \( \cup \) singularly continuous spectrum

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If \( p \neq \frac{1}{2} \), the Laplacian \( \Delta_p \), regarded as an operator on either \( \ell^2(\mathbb{Z}_+) \) or \( L^2(\mathbb{Z}_+, \pi) \), has purely singularly continuous spectrum. The spectrum is the Julia set of the polynomial

\[ R(z) = \frac{z(z^2 - 3z + (2 + pq))}{pq}, \]

which is a topological Cantor set of Lebesgue measure zero.

- One of the simplest realizations of purely singularly continuous spectrum. The mechanism appears to be simpler than those of quasi-periodic or aperiodic Schrodinger operators. (cf. Simon, Jitomirskaya, Avila, Damanik, Gorodetski, etc.)
- See also recent work of Grigorchuk-Lenz-Nagnibeda '14, '16 on spectra of Schreier graphs.
Proof of purely singularly continuous spectrum (when $p \neq \frac{1}{2}$)

1. **Spectral decimation:** $\Delta_p$ is spectrally similar to $\Delta^+_p$, and they are isometrically equivalent. After taking into account the exceptional set, $R(z) \in \sigma(\Delta_p) \iff z \in \sigma(\Delta_p)$. Notably, the repelling fixed points of $R$, $\{0, 1, 2\}$, lie in $\sigma(\Delta_p)$.

2. By 1, $\bigcup_{n=0}^{\infty} R^{\circ-n}(0) \subset \sigma(\Delta_p)$. Meanwhile, since $0 \in J(R)$, $\bigcup_{n=0}^{\infty} R^{\circ-n}(0) = J(R)$. So $J(R) \subset \sigma(\Delta_p)$.

3. If $z \in \sigma(\Delta_p)$, then by 1, $R^{\circ n}(z) \in \sigma(\Delta_p)$ for each $n \in \mathbb{N}$. On the one hand, $\sigma(\Delta_p)$ is compact. On the other hand, the only attracting fixed point of $R$ is $\infty$, so $\mathcal{F}(R)$ (the Fatou set) contains the basin of attraction of $\infty$, whence non-compact. Infer that $z \notin \mathcal{F}(R) = (J(R))^c$. So $\sigma(\Delta_p) \subset J(R)$. 
Proof of purely singularly continuous spectrum (when $p \neq \frac{1}{2}$)

Thus $\sigma(\Delta^p) = \mathcal{J}(R)$. When $p \neq \frac{1}{2}$, $\mathcal{J}(R)$ is a disconnected Cantor set. So $\sigma_{ac}(\Delta^p) = \emptyset$.

Find the formal eigenfunctions corresponding to the fixed points of $R$, and show that none of them are in $\ell^2(\mathbb{Z}_+)$ and in $L^2(\mathbb{Z}_+, \pi)$. Thus none of the fixed points lie in $\sigma_{pp}(\Delta^p)$. By spectral decimation, none of the pre-iterates of the fixed points under $R$ are in $\sigma_{pp}(\Delta^p)$. So $\sigma_{pp}(\Delta^p) = \emptyset$.

Conclude that $\sigma(\Delta^p) = \sigma_{sc}(\Delta^p)$. 
The Sierpinski gasket lattice ($SGL$)

Let $\Delta$ be the graph Laplacian on $SGL$. If $z \notin \{2, 5, 6\}$ and $R(z) = z(5 - z)$, then

- $R(z) \in \sigma(-\Delta) \iff z \in \sigma(-\Delta)$.
- $\sigma(-\Delta) = J_R \cup D$, where $J_R$ is the Julia set of $R(z)$ and $D := \{6\} \cup (\bigcup_{m=0}^{\infty} R^{-m}\{3\})$.
- $J_R$ is a disconnected Cantor set.

**Thm.** (Teplyaev ’98)

On $SGL$, $\sigma(\Delta) = \sigma_{pp}(\Delta)$.

Eigenfunctions with finite support are complete.

→ Localization due to geometry.
Localized eigenfunctions on $SGL$
Random potential and Anderson localization

\[ H_\omega = -\Delta + V_\omega(x) : \omega \text{ denotes a realization of the random potential.} \]

**Definition (Anderson localization)**

\( H_\omega \) has **spectral localization** in an energy interval \([a, b]\) if, with probability 1, \( \sigma(H_\omega) \) is p.p. in this interval. Furthermore, \( H_\omega \) has **exponential localization** if the eigenfunctions with eigenvalues in \([a, b]\) decay exponentially.


**Theorem (Aizenman-Molchanov ’93, method of fractional moment of the resolvent)**

Let \( \tau(x, y; z) =: \mathbb{E}[|\langle x | (H_\omega - z)^{-1} | y \rangle|^s] \). If

\[ \tau(x, y; E + i\epsilon) \leq Ae^{-\mu|x-y|} \]

for \( E \in (a, b) \), uniformly in \( \epsilon \neq 0 \) and a suitable fixed \( s \in (0, 1) \), then \( H_\omega \) has exponential localization.

The Aizenman-Molchanov estimate provides proofs of localization in the case of 1) large disorder, or 2) extreme energies.
Anderson localization on SGL

**Theorem (Molchanov ’16)**

On SGL (and many other finitely ramified fractal lattices, \( \sigma_{ac}(H_\omega) = \emptyset \)).

**Proof.** Based on the Simon-Wolff method.

**Theorem (C.-Molchanov-Teplyaev ’16+)**

On SGL, the Aizenman-Molchanov estimate holds, i.e., for \( E \in (a, b) \) and \( E \notin \sigma(-\Delta) \),

\[
\tau(x, y; E + i\epsilon) \leq Ae^{-\mu d(x, y)}
\]

uniformly in \( \epsilon \neq 0 \) and a suitable fixed \( s \in (0, 1) \). ([d(\cdot, \cdot) can be taken to be the graph metric.] As a consequence, \( H_\omega \) has exponential localization on SGL in the case of large disorder or extreme energies.

**Proof.** If \( E < 0 \), then \( \tau(x, y; E + i\epsilon) \) is a suitable Laplace transform of the heat kernel, which has a well-known sub-Gaussian upper estimate that decays exponentially with the graph distance \( d(x, y) \):

\[
\exists C_1, C_2 > 0 : \quad p_t(x, y) \leq C_1 t^{-\alpha} \exp \left( - \left( \frac{d(x, y)^\beta}{t} \right)^{1/(\beta-1)} \right) \quad \forall x, y \in SGL, \quad \forall t > 0,
\]

where \( \alpha = \frac{\log 3}{\log 5} \), and \( \beta = \frac{\log 5}{\log 2} \).

If \( E > 0 \), let \( n(E) \) be the smallest natural number \( n \) such that \( R^n(E) < 0 \), where \( R(z) = z(5-z) \). Use spectral decimation to relate the resolvent at \( E \) to the resolvent at \( R^n(E) \).
Thank you!