Eigensystem multiscale analysis for Anderson localization in energy intervals II

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Eigensystem multiscale analysis

- We consider the usual Anderson model.

- General strategy: Information about eigensystems at a given scale is used to derive information about eigensystems at larger scales.

- Need to carry over deterministic and probabilistic information since the system is random. The probabilistic part is close to the one in the standard MSA, will not be discussed here.
Level spacing and localization

Definition
A box \( \Lambda_L = [-L, L]^d + x_0 \) is called \( L \)-level spacing for \( H \) if all eigenvalues of \( H_{\Lambda_L} \) are simple, and

\[
|\lambda - \lambda'| \geq e^{-L^\beta} \quad \text{for all} \quad \lambda, \lambda' \in \sigma(H_{\Lambda_L}), \lambda \neq \lambda'.
\]

Definition
Let \( \Lambda_L \) be a box, \( x \in \Lambda_L \), and \( m \geq 0 \). Then \( \varphi \in \ell^2(\Lambda_L) \) is said to be \((x, m)\)-localized if \( \|\varphi\| = 1 \) and

\[
|\varphi(y)| \leq e^{-m\|y-x\|} \quad \text{for all} \quad y \in \Lambda_L \quad \text{with} \quad \|y-x\| \geq L^\tau.
\]
Interval localization (naïve)

Definition (naïve)

Let $I$ be a bounded interval and let $m > 0$. A box $\Lambda_L$ will be called $(m, I)$-localizing for $H$ if

1. $\Lambda_L$ is level spacing for $H$.
2. There exists an $(m, I)$-localized eigensystem for $H_{\Lambda_L}$, that is, an eigensystem $\{(\varphi_\nu, \nu)\}_{\nu \in \sigma(H_{\Lambda_L})}$ for $H_{\Lambda_L}$ such that for all $\nu \in \sigma(H_{\Lambda_L})$ there is $x_\nu \in \Lambda_L$ such that $\varphi_\nu$ is $(x_\nu, m)$-localized.

Level spacing helps to overcome the small denominator problem (resonances), replaces the Wegner estimate.
Failure of naïve approach to EMSA

Consider $\ell \ll L$ and suppose that

- A box $\Lambda_L$ is $L$-level spacing for $H$;
- Any box $\Lambda_\ell \subset \Lambda_L$ is $(m, I)$-localizing for $H$ (in a naïve sense as above).

Can we show that $\Lambda_L$ is $(\hat{m}, \hat{I})$-localizing for $H$ (allowing for small losses in $m$ and $I$)?

The answer is NO.
Failure of naïve approach to EMSA

We don’t know anything about the structure of eigenvectors for $H_{\Lambda_\ell}$ outside $I$. In particular, the quantum tunneling between localized states just inside $I$ for one box $\Lambda_\ell$ and the delocalized states just outside another box $\Lambda'_\ell$ is possible (when we consider $H_{\Lambda_\ell}$ as perturbation of decoupled boxes of size $\ell$).

- This indicates that on the new scale $L$, localization on $I$ is no longer uniform (as far as localization length is concerned): As we approach the edges of $I$, the mass $m$ goes to zero.
- Deep inside $I$ we expect localization to survive, since the quantum tunneling between energetically separated states is suppressed by locality of $H$ (Combes-Thomas estimate).
Correct approach to EMSA
(simplified)

▶ We replace the naive definition with

**Definition**
Let $E \in \mathbb{R}$, $I = (E - A, E + A)$, and $m > 0$. A box $\Lambda_L$ will be called $(m, I)$-localizing for $H$ if

1. $\Lambda_L$ is level spacing for $H$.
2. There exists an $(m, I)$-localized eigensystem for $H_{\Lambda_L}$, i.e. an eigensystem $\{(\varphi_\nu, \nu)\}_{\nu \in \sigma(H_{\Lambda_L})}$ for $H_{\Lambda_L}$ such that for all $\nu \in \sigma(H_{\Lambda_L})$ there is $x_\nu \in \Lambda_L$ such that $\varphi_\nu$ is $(x_\nu, mh_I(\nu))$-localized.

▶ The modulating function $h_I$ satisfies $h_I(E) = 1$ and $h_I(E \pm A) = 0$. 
Key step (simplified version)

Consider $\ell \ll L$ and suppose that

- A box $\Lambda_L$ is $L$-level spacing for $H$;
- Any box $\Lambda_\ell \subset \Lambda_L$ is $(m, I)$-localizing for $H$.

Can we show that $\Lambda_L$ is $(\hat{m}, \hat{I})$-localizing for $H$ for some choice of the modulating function $h_I$, and allowing for small losses in $m$ and $I$?

The answer now is YES.

- Tricky part: Choice of $h_I$ and control over the decay rate.
The general strategy of going from scale $\ell$ to scale $L$ concerns the expansion of a true eigenfunction of $H_{\Lambda L}$ in terms of eigenfunctions of Hamiltonians $H_{\ell}$.

Although the process itself is very natural, preparations can take some time to explain.

Instead, we will illustrate some ideas of the proof by showing how the eigensystem MSA for energy intervals implies the exponential localization of the Green function (the key player in the usual MSA).

It will also reveal our top secret way of choosing the modulating function $h_I$ mentioned earlier 😊.
EMSA on intervals implies MSA

Let $I = (E - A, E + A)$ with $E \in \mathbb{R}$ and $A > 0$.

Suppose that $\Lambda_L$ is $(m, I)$-localizing for $H$.

Let $\lambda \in I_L$ with $\text{dist}\{\lambda, \sigma(H_{\Lambda_L})\} \geq e^{-L\beta}$.

WTS: For $m$ not too small and not too large,

$$|G_{\Lambda_L}(\lambda; x, y)| \leq e^{-\hat{m}h_I(\lambda)}\|x - y\| \text{ whenever } \|x - y\| \geq L^{\tau'}.$$

Losses in $m$ should be (controllably) small:

$$\hat{m} \geq m(1 - CL^{-\gamma}) \text{ for some } \gamma > 0.$$
Analyticity and localization

- We can try to split \((H_{\Lambda} - \lambda)^{-1}\) into \((H_{\Lambda} - \lambda)^{-1} P_I + (H_{\Lambda} - \lambda)^{-1} \bar{P}_I\).

- \(P_I\) is the spectral projection of \(H_{\Lambda}\) onto \(I\), \(\bar{P}_I = 1 - P_I\).

- We have no information on \(\varphi_v\) for \(v \notin I\), though, and the decay rate of \(\varphi_v\) for \(v \in I\) is not uniform. Not good!

- Gentler approach: Filter out eigenvalues outside \(I\) using an analytic function \(F_I(H_{\Lambda})\) instead of \(P_I\):
\[
(H_{\Lambda} - \lambda)^{-1} = (H_{\Lambda} - \lambda)^{-1} F_I(H_{\Lambda}) + (H_{\Lambda} - \lambda)^{-1} \bar{F}_I(H_{\Lambda}).
\]

- Want (a) \(F_I\) to be exponentially small outside \(I\) and (b) \((z - \lambda)^{-1} \bar{F}_I(z)\) to be analytic in a strip that contains real axis (then Combes-Thomas estimate will kick in, and the corresponding term will be exponentially small too).
Analyticity and localization

To summarize:

1. \[ \left\langle \varphi_{\nu}, (H_{\Lambda L} - \lambda)^{-1} F_I (H_{\Lambda L}) \varphi_{\nu} \right\rangle = (\nu - \lambda)^{-1} F_I (\nu); \]

2. \[ |\varphi_{\nu}(x) \varphi_{\nu}(y)| \leq e^{-m h_I (\nu)} \|x - y\|; \]

3. If \( K(z) = (z - \lambda)^{-1} \bar{F}_I (z) \) is analytic and bounded in the strip \(|\text{Im} \, z| < \eta\) by \( \|K\|_{\infty} \), then (Aizenman-Graf)

\[ |\langle \delta_x, K (H_{\Lambda}) \delta_y \rangle| \leq C \|K\|_{\infty} e^{-\left(\log\left(1 + \frac{\eta}{4d}\right)\right)} \|x - y\|. \]
Analyticity and localization

Let’s start tying up loose ends:

- Combining (1) – (2), we get the uniform exponential decay
  
  \[ \left| \langle \delta_x, (H_{\Lambda} - \lambda)^{-1} F_I (H_{\Lambda}) \delta_y \rangle \right| \quad \text{as long as} \quad (*) \quad F_I (\nu) e^{-m h_I (\nu) \| x - y \|} \leq e^{-m h_I (\lambda) \| x - y \|} 
  
  for all \( \nu \in \sigma (H_{\Lambda}) \).

- (3) yields exponential decay for \( |\langle \delta_x, K (H_{\Lambda}) \delta_y \rangle| \) as long as
  
  \[ (**) \quad \| K \|_{\infty} \leq e^\left( \log \left( 1 + \frac{\eta}{4d} \right) / 2 \right) \| x - y \|. \]

- Are there a filter \( F_I \) and a modulating function \( h_I \) out there that satisfy both (*) and (**)?
End game

The choice

\[ F_I(z) = e^{-t((z-E)^2-(\lambda-E)^2)}; \quad t = \frac{m\|x-y\|}{A^2}, \]

and

\[ h_I(t) = h\left(\frac{t-E}{A}\right) \]

with

\[ h(s) = \begin{cases} 
1 - s^2 & \text{if } s \in [0,1) \\
0 & \text{otherwise}
\end{cases} \]

does the trick! In fact, it turns Eq. (*) into the identity for \( \nu \in I \).
THANKS!