Branching form of the resolvent at threshold for discrete Laplacians

Kenichi ITO (Kobe University)
joint work with
Arne JENSEN (Aalborg University)

9 October 2016
Introduction: Discrete Laplacian

○ Thresholds generated by critical values

For any function $u: \mathbb{Z}^d \to \mathbb{C}$ define $\triangle u: \mathbb{Z}^d \to \mathbb{C}$ by

$$(\triangle u)[n] = \sum_{j=1}^{d} (u[n + e_j] + u[n - e_j] - 2u[n]) \quad \text{for } n \in \mathbb{Z}^d.$$"
Let $\hat{\mathcal{H}} = L^2(\mathbb{T}^d)$, $\mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z})$, and define the Fourier transform $\mathcal{F}: \mathcal{H} \to \hat{\mathcal{H}}$ and its inverse $\mathcal{F}^*: \hat{\mathcal{H}} \to \mathcal{H}$ by

$$(\mathcal{F}u)(\theta) = (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} e^{-in\theta} u[n],$$

$$(\mathcal{F}^*f)[n] = (2\pi)^{-d/2} \int_{\mathbb{T}^d} e^{in\theta} f(\theta) d\theta,$$

and then

$$\mathcal{F}H_0\mathcal{F}^* = \Theta(\theta) = 2d - 2 \cos \theta_1 - \cdots - 2 \cos \theta_d.$$ 

Since $\partial_j \Theta(\theta) = 2 \sin \theta_j$, the critical points of signature $(p, q)$ are

$$\gamma(p, q) = \left\{ \theta \in \{0, \pi\}^d; \# \{\theta_j = 0\} = p, \# \{\theta_j = \pi\} = q \right\}.$$ 

Hence the critical values $0, 4d$ are thresholds of elliptic type, and $4, \ldots, 4(d - 1)$ are those of hyperbolic type.
Purpose: Asymptotic expansion of resolvent

Q. Can we compute an asymptotic expansion of the resolvent
\[ R_0(z) = (H_0 - z)^{-1} \sim ?? \quad \text{as } z \to 4q \text{ for } q = 0, 1, \ldots, d? \]

Note that the resolvent \( R_0(z) \) has a convolution kernel:
\[ R_0(z)u = k(z, \cdot) * u; \quad k(z, n) = (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{e^{in\theta}}{\Theta(\theta) - z} d\theta. \]

A. Yes. By localizing around \( \gamma(p, q) \subset \mathbb{T}^d \) and changing variables the situation reduces to that for an ultra-hyperbolic operator.

- As far as we know, an explicit asymptotics of \( R_0(z) \) around a threshold seems to have been open except for 0 and 4d.
Ultra-hyperbolic operator (a model operator)

Consider an ultra-hyperbolic operator on $\mathbb{R}^d$:

$$\Box = \partial_1^2 + \cdots + \partial_p^2 - \partial_{p+1}^2 - \cdots - \partial_{p+q}^2; \quad p, q \geq 0, \ d = p + q.$$ 

The operator $H_0 = -\Box$ is self-adjoint on $\mathcal{H} = L^2(\mathbb{R}^d)$ with

$$\mathcal{D}(H_0) = \{u \in \mathcal{H}; \ \Box u \in \mathcal{H} \text{ in the distributional sense}\}.$$ 

It has spectrum

$$\sigma(H_0) = \sigma_{\text{ac}}(H_0) = \begin{cases} [0, \infty) & \text{if } (p, q) = (d, 0), \\ (-\infty, 0] & \text{if } (p, q) = (0, d), \\ \mathbb{R} & \text{otherwise}, \end{cases}$$

and a single threshold

$$\tau(H_0) = \{0\}.$$
Using the Fourier transform $\mathcal{F}: \mathcal{H} \to \mathcal{H}$ and its inverse $\mathcal{F}^*: \mathcal{H} \to \mathcal{H}$, we can write

$$\mathcal{F}H_0\mathcal{F}^* = \Xi(\xi) = \xi'^2 - \xi''^2; \quad \xi = (\xi', \xi'') \in \mathbb{R}^p \oplus \mathbb{R}^q.$$ 

The only critical point is $\xi = 0$, and the associated critical value, or a threshold 0 is said to be

1. of elliptic type if $(p, q) = (d, 0)$ or $(0, d)$;

2. of hyperbolic type otherwise.

$Q'$. Can we compute an asymptotic expansion of the resolvent

$$R_0(z) = (H_0 - z)^{-1} \sim ?? \quad \text{as} \quad z \to 0?$$

$A'$. Yes. In particular, square root, logarithm and dilogarithm branchings show up, depending on parity of $(p, q)$. 

5
Square root, logarithm and dilogarithm

We always choose branches of $\sqrt{w}$ and $\log w$ such that

$$\text{Im } \sqrt{w} > 0 \text{ for } w \in \mathbb{C} \setminus [0, \infty),$$

$$-\pi < \text{Im } \log w < \pi \text{ for } w \in \mathbb{C} \setminus (-\infty, 0],$$

respectively. In addition, let us set for $w \in \mathbb{C} \setminus [1, \infty)$

$$\text{Li}_1(w) = -\log(1 - w), \quad \text{Li}_2(w) = \int_0^w \frac{\text{Li}_1(\lambda)}{\lambda} \, d\lambda,$$

which have the Taylor expansions: For $|w| < 1$

$$\text{Li}_1(w) = \sum_{k=1}^{\infty} \frac{w^k}{k}, \quad \text{Li}_2(w) = \sum_{k=1}^{\infty} \frac{w^k}{k^2}.$$
Elliptic operator in odd dimensional space

**Theorem.** Let $d$ be odd, $(p, q) = (d, 0)$, and $\gamma > 0$. Then

$$k_\gamma(z, x) = \frac{i\pi}{2}(\sqrt{z})^{d-2}e(\sqrt{z}x) - \frac{1}{2} \int_{\tilde{\Gamma}(\gamma)} \frac{\rho^{d-1}e(\rho x)}{\rho^2 - z} d\rho,$$

where $\tilde{\Gamma}(r) = \{re^{i\theta} \in \mathbb{C}; \ \theta \in [0, \pi]\}$.

Elliptic operator in even dimensional space

**Theorem.** Let $d$ be even, $(p, q) = (d, 0)$, and $\gamma > 0$. Then

$$k_\gamma(z, x) = -\frac{1}{2}(\sqrt{z})^{d-2}e(\sqrt{z}x) \text{Li}_1\left(\frac{\gamma^2}{z}\right)$$

$$+ \frac{1}{2} \int_0^\gamma (\sqrt{\lambda})^{d-2}e(\sqrt{\lambda}x) - (\sqrt{z})^{d-2}e(\sqrt{z}x) \frac{d\lambda}{\lambda - z}.$$
Proposition. 1. The function $e(\zeta)$ is even and entire in $\zeta \in \mathbb{C}^d$, and

$$e(\zeta) = \sum_{\alpha \in \mathbb{Z}^d_+} e_\alpha \zeta^{2\alpha}; \quad e_\alpha = \frac{2}{2^d\pi^{d/2} \alpha!\Gamma(|\alpha| + d/2)} \left(-\frac{1}{4}\right)^{|\alpha|}.$$

2. For any $z \in \mathbb{C}$ the function $e(\sqrt{z}x)$ satisfies the eigenequation

$$(-\triangle - z)e(\sqrt{z}x) = 0; \quad \triangle = \Box_{d,0}.$$

Here a branch of $\sqrt{z}$ does not matter, since $e(\zeta)$ is even.
Hyperbolic case with odd-even or even-odd signature

**Theorem.** Let \((p, q)\) be *odd-even or even-odd*, and \(\gamma > 0\). Then

\[
 k_{\gamma}(z, x) = \frac{i\pi}{2} (\sqrt{z})^{d-2} \psi_+ (\sqrt{z}x) + \chi_{\gamma}(z, x),
\]

where

\[
 \chi_{\gamma}(z, x) = -\frac{1}{2} \int_{\Gamma(\gamma)} \frac{\tau^{d-1}\psi_+(\tau x)}{\tau^2 - z} \, d\tau + \frac{1}{2} \int_{\Gamma(\gamma)} \frac{\tau^{d-1}\psi_-(\tau x)}{\tau^2 + z} \, d\tau
\]

\[
 + \frac{1}{4} \int_{i\tilde{\Gamma}(\gamma^2)} \frac{h_{+,\gamma}(\lambda, x)}{\lambda - z} \, d\lambda
\]

with

\[
 h_{\pm,\gamma}(\tau^2, x) = \tau^{d-2} \int_{-\gamma}^{\gamma} \frac{f_{\pm}(\sigma/\tau, \tau x)}{\sigma} \, d\sigma.
\]
Hyperbolic case with even-even signature

**Theorem.** Let \((p, q)\) be even-even, and \(\gamma > 0\). Then

\[
k_\gamma(z, x) = -\frac{1}{2} (\sqrt{z})^{d-2} \psi_+ (\sqrt{z}x) \text{Li}_1\left(\frac{\gamma^4}{z^2}\right) + \chi_\gamma(z, x),
\]

where

\[
\chi_\gamma(z, x) = \frac{1}{2} \left( \int_0^\gamma \left( \int_{-\gamma}^0 \frac{(\sqrt{\tau})^{d-2} \psi_+ (\sqrt{\tau}x) - (\sqrt{z})^{d-2} \psi_+ (\sqrt{z}x)}{\tau - z} \right) d\tau 
+ \frac{1}{2} \int_{-\gamma}^\gamma \frac{h_{+,\gamma}(\lambda, x)}{\lambda - z} d\lambda \right).
\]

with

\[
h_{\pm,\gamma}(\tau^2, x) = \tau^{d-2} \int_{\tau}^\gamma \frac{f_{\pm}(\sigma/\tau, \tau x)}{\sigma} d\sigma - \tau^{d-2} \psi_{\pm}(\tau x).
\]
Hyperbolic case with odd-odd signature

**Theorem.** Let \((p, q)\) be odd-odd, and \(\gamma > 0\). Then

\[
k_\gamma(z, x) = -\frac{1}{4}(\sqrt{z})^{d-2\phi_+}(\sqrt{zx})^\gamma \left[ \text{Li}_2\left(\frac{\gamma^2}{z}\right) - \text{Li}_2\left(-\frac{\gamma^2}{z}\right) \right] + \chi_\gamma(z, x),
\]

where

\[
\chi_\gamma(z, x) = \frac{1}{4} \int_0^{\gamma^2} \frac{(\sqrt{\lambda})^{d-2\phi_+}(\sqrt{\lambda} x) - (\sqrt{z})^{d-2\phi_+}(\sqrt{zx})}{\lambda - z} \log\left(\frac{\gamma^2}{\lambda}\right) d\lambda \\
+ \frac{1}{4} \int_{-\gamma^2}^{0} \frac{(\sqrt{\lambda})^{d-2\phi_+}(\sqrt{\lambda} x) - (\sqrt{z})^{d-2\phi_+}(\sqrt{zx})}{\lambda - z} \log\left(-\frac{\gamma^2}{\lambda}\right) d\lambda \\
+ \frac{1}{2} \int_{-\gamma^2}^{\gamma^2} \frac{h_{+,\gamma}(\lambda, x)}{\lambda - z} d\lambda
\]

with \(h_{+,\gamma}(\tau^2, x) = \tau^{d-2} \int_{\tau}^{\gamma} \frac{f_+ (\sigma/\tau, \tau x) - \phi_+ (\tau x)}{\sigma} d\sigma\).
Properties of $\phi_{\pm}(\zeta)$ and $\psi_{\pm}(\zeta)$

The functions $\phi_{\pm}(\zeta)$ and $\psi_{\pm}(\zeta)$ are entire in $\zeta \in \mathbb{C}^d$, and

$$\phi_{\pm}(\zeta) = \sum_{\alpha \in \mathbb{Z}_+^d} \zeta^{2\alpha} \left( \sum_{a \in J_\alpha} f_{\pm,\alpha,a} \right),$$

$$\psi_{\pm}(\zeta) = \sum_{\alpha \in \mathbb{Z}_+^d} \zeta^{2\alpha} \left( \sum_{a \in I_\alpha \setminus J_\alpha} \frac{[i^2|\alpha|-2|a|+d-2-1]}{2|\alpha|-2|a|+d-2} f_{\pm,\alpha,a} \right),$$

where

$$I_\alpha = \left\{ a \in \mathbb{Z}_+^2; \ 0 \leq a \leq (2|\alpha'|+p-1,2|\alpha'''|+q-1) \right\},$$

$$J_\alpha = \left\{ a \in I_\alpha; \ |a| = |\alpha| + (d-2)/2 \right\},$$

$$f_{\pm,\alpha,a} = \frac{(\pm 1)^a (\mp 1)^{a''} \alpha'_e \varepsilon''}{2^{|\alpha|+d-2}} \left( 2|\alpha'| + p - 1 \right) \left( 2|\alpha'''| + q - 1 \right).$$
Properties of $\phi_{\pm}(\zeta)$ and $\psi_{\pm}(\zeta)$, continued

The functions $\phi_{\pm}(\zeta)$ and $\psi_{\pm}(\zeta)$ satisfy

1. $\phi_{\pm}(\zeta) = 0$ and $\psi_{\pm}(\zeta) = \psi_{\pm}(-\zeta)$ if $(p, q)$ is odd-even or even-odd;

2. $\phi_{\pm}(\zeta) = 0$ and $\psi_{\pm}(\zeta) = -i^{d-2} \psi_{\mp}(i\zeta)$ if $(p, q)$ is even-even;

3. $\phi_{\pm}(\zeta) = i^{d-2} \phi_{\mp}(i\zeta)$ and $\psi_{\pm}(\zeta) = 0$ if $(p, q)$ is odd-odd.

In addition, for any $z \in \mathbb{C}$

$(-\Box \mp z) \phi_{\pm}(\sqrt{z}x) = 0$,  $(-\Box \mp z) \psi_{\pm}(\sqrt{z}x) = 0$. 


Outline of the results for ultra-hyperbolic operator

**Theorem.** 1. If \((p, q)\) is odd-even or even-odd, there exist operators \(F(z), G(z)\) analytic at \(z = 0\) such that

\[
R_0(z) = F(z)\sqrt{z} + G(z).
\]

2. If \((p, q)\) is even-even, there exist operators \(F(z), G(z)\) analytic at \(z = 0\) such that

\[
R_0(z) = F(z)\text{Li}_1\left(\frac{1}{z}\right) + G(z).
\]

3. If \((p, q)\) is odd-odd, there exist operators \(F(z), G(z)\) analytic at \(z = 0\) such that

\[
R_0(z) = F(z)\left[\text{Li}_2\left(\frac{1}{z}\right) - \text{Li}_2\left(-\frac{1}{z}\right)\right] + G(z).
\]
“Very rough” strategy for proof

The resolvent has a limiting convolution expression

\[
(R_0(z)u)(x) = \lim_{\gamma \to \infty} \int_{\mathbb{R}^d} k_\gamma(z, x - y)u(y) \, dy \quad \text{for } u \in \mathcal{S}(\mathbb{R}^d);
\]

\[
k_\gamma(z, x) = (2\pi)^{-d} \int_{|\xi'| + |\xi''| < \gamma} \frac{e^{ix\xi}}{\xi'^2 - \xi''^2 - z} \, d\xi.
\]

It suffices to expand the kernel \( k_\gamma(z, x) \), since it contains all the singular part of \( R_0(z) \).

If we move on to the spherical or hyperbolic coordinates, a singular part of \( k_\gamma(z, x) \) takes, more or less, the standard form

\[
I = \int_{\gamma}^{\infty} \frac{a(\rho)}{\rho^2 - z} \, d\rho.
\]

There could appear only the following three types of \( a(\rho) \):
• If \( a(\rho) = 2b(\rho^2) \) with \( b \) analytic, then

\[
I = \int_{-\gamma}^{\gamma} \frac{b(\rho^2)}{(\rho - \sqrt{z})(\rho + \sqrt{z})} \, d\rho = i\pi \frac{b(z)}{\sqrt{z}} - \int_{|z|=\gamma, \text{Im } z \geq 0} \frac{b(\rho^2)}{\rho^2 - z} \, d\rho.
\]

• If \( a(\rho) = 2\rho b(\rho^2) \) with \( b \) analytic, then

\[
I = \int_{0}^{\gamma} \frac{b(\lambda)}{\lambda - z} \, d\lambda = \int_{0}^{\gamma} \frac{b(z)}{\lambda - z} \, d\lambda + \int_{0}^{\gamma} \frac{b(\lambda) - b(z)}{\lambda - z} \, d\lambda
\]

• If \( a(\rho) = 2\rho b(\rho^2)(\log \rho^2) \) with \( b \) analytic, then

\[
I = \int_{0}^{\gamma} \frac{b(\lambda) \log \lambda}{\lambda - z} \, d\lambda = \int_{0}^{\gamma} \frac{b(z) \log \lambda}{\lambda - z} \, d\lambda + \int_{0}^{\gamma} \frac{[b(\lambda) - b(z)] \log \lambda}{\lambda - z} \, d\lambda
\]
Precise results for discrete Laplacian

○ Localization around critical points

Note that \( R_0(z) \) has a convolution kernel:

\[
k(z, n) = (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{e^{in\theta}}{\Theta(\theta) - z} \, d\theta.
\]

Denote the set of all the critical points of signature \((p, q)\) by

\[
\gamma(p, q) = \{\theta^{(l)}\}_{l=1,...,L}; \quad L = \#\gamma(p, q) = \binom{d}{p} = \binom{d}{q},
\]

Take neighborhoods \( U_l \subset \mathbb{T}^d \) of \( \theta^{(l)} \), and then decompose

\[
k(z, n) = k_0(z, n) + k_1(z, n) + \cdots + k_L(z, n);
\]

\[
k_l(z, n) = (2\pi)^{-d} \int_{U_l} \frac{e^{in\theta}}{\Theta(\theta) - z} \, d\theta \quad \text{for } l = 1, \ldots, L.
\]
We let, e.g.,
\[ \theta^{(1)} = (0, \ldots, 0, \pi, \ldots, \pi) \in \mathbb{T}^p \times \mathbb{T}^q = \mathbb{T}^d. \]
Take local coordinates \( \xi(\theta) = \xi^{(1)}(\theta) \) around \( \theta^{(1)} \in \mathbb{T}^d \):
\[
\xi_j(\theta) = \begin{cases} 
2 \sin(\theta_j/2) & \text{for } j = 1, \ldots, p, \\
2 \cos(\theta_j/2) & \text{for } j = p + 1, \ldots, p + q,
\end{cases}
\]
and set
\[ U_1 = \{ \theta \in \mathbb{T}^d; |\xi'(\theta)| + |\xi''(\theta)| < 2 \}. \]
Then we can write
\[
k_1(z, n) = (2\pi)^{-d} \int_{|\xi'| + |\xi''| < 2} \frac{e^{in\theta(\xi)}}{\xi'^2 - \xi''^2 - (z - 4q)} \prod_{j=1}^{d} (1 - \xi_j^2/4)^{1/2} \, d\xi.
\]
We will do the same construction for \( k_2(z, n), \ldots, k_L(z, n) \).
Theorem. Let \( (p, q) = (d, 0) \).

1. If \( d \) is odd, there exists a function \( \chi(z, n) \) analytic in \( z \in \Delta(4) \) such that
   \[
   k(z, n) = \frac{i\pi}{2} (\sqrt{z})^{d-2} \text{e}(\sqrt{z}, n) + \chi(z, n).
   \]

2. If \( d \) is even, there exists a function \( \chi(z, n) \) analytic in \( z \in \Delta(4) \) such that
   \[
   k(z, n) = -\frac{1}{2} (\sqrt{z})^{d-2} \text{e}(\sqrt{z}, n) \text{Li}_1 \left( \frac{1}{z} \right) + \chi(z, n).
   \]
Proposition. 1. The function $e(\rho, n)$ is analytic in $\rho \in \Delta(2)$, and

$$e(\rho, n) = \frac{2}{2^d \pi^{d/2}} \sum_{k=0}^{\infty} \rho^{2k} \sum_{\alpha \in \mathbb{Z}^d_+, \|\alpha\| = k} \frac{\prod_{j=1}^{d} (1/2 - n_j) \alpha_j (1/2 + n_j) \alpha_j}{4^{\|\alpha\|} \alpha! \Gamma(\|\alpha\| + d/2)}$$

where $(\nu)_k := \Gamma(\nu + k)/\Gamma(\nu)$ is the Pochhammer symbol. In particular, $e(z, n)$ can be expressed by the Lauricella function:

$$e(\rho, n) = \frac{2}{2^d \pi^{d/2} \Gamma(d/2)} F_B^{(d)} \left( \frac{1}{2} - n_1, \ldots, \frac{1}{2} - n_d, \frac{1}{2} + n_1, \ldots, \frac{1}{2} + n_d; \frac{d}{2}, \frac{\rho^2}{4}, \ldots, \frac{\rho^2}{4} \right)$$

2. For any $z \in \Delta(4)$, as a function in $n \in \mathbb{Z}^d$,

$$(-\Delta - z)e(\sqrt{z}, n) = 0.$$
Theorem. 1. If \((p, q)\) is odd-even or even-odd, then there exists \(\chi(\cdot, n) \in C^\omega(\Delta(4))\) such that

\[
k(w + 4q, n) = \frac{i\pi}{2}(\sqrt{w})^{d-2}\sum_{l=1}^{L} \psi_+^{(l)}(\sqrt{w}, n) + \chi(w, n).
\]

2. If \((p, q)\) is even-even, there exists \(\chi(\cdot, n) \in C^\omega(\Delta(4))\) such that

\[
k(w + 4q, n) = -\frac{1}{2}(\sqrt{w})^{d-2}\text{Li}_1\left(\frac{16}{w^2}\right)\sum_{l=1}^{L} \psi_+^{(l)}(\sqrt{w}, n) + \chi(w, n).
\]

3. If \((p, q)\) is odd-odd, there exists \(\chi(\cdot, n) \in C^\omega(\Delta(4))\) such that

\[
k(w + 4q, n) = -\frac{1}{4}(\sqrt{w})^{d-2}\left[\text{Li}_2\left(\frac{4}{w}\right) - \text{Li}_2\left(-\frac{4}{w}\right)\right]\sum_{l=1}^{L} \phi_+^{(l)}(\sqrt{w}, n) + \chi(w, n).
\]
Properties of $\phi_{\pm}(\tau, n)$ and $\psi_{\pm}(\tau, n)$

The functions $\phi_{\pm}(\tau, n)$ and $\psi_{\pm}(\tau, n)$ are analytic in $\tau \in \Delta(2)$, and

$$\phi_{\pm}(\tau, n) = \sum_{\alpha \in \mathbb{Z}^2_+} \tau^{2|\alpha|} \left( \sum_{a \in J_\alpha} f_{\pm, \alpha, a}[n] \right),$$

$$\psi_{\pm}(\tau, n) = \sum_{\alpha \in \mathbb{Z}^2_+} \tau^{2|\alpha|} \left( \sum_{a \in I_\alpha \setminus J_\alpha} \frac{[i2|\alpha|-2|a|+d-2-1]}{2|\alpha|-2|a|+d-2} f_{\pm, \alpha, a}[n] \right),$$

where

$$I_\alpha = \left\{ a \in \mathbb{Z}^2_+; 0 \leq a \leq (2\alpha' + p - 1, 2\alpha'' + q - 1) \right\},$$

$$J_\alpha = \left\{ a \in I_\alpha; |a| = |\alpha| + (d - 2)/2 \right\}.$$

$$f_{\pm, \alpha, a}[n] = \frac{(\pm 1)^{a'/(\mp 1)^{a''}}}{2^{2|\alpha|+d-2}} \left( \frac{2\alpha' + p - 1}{a'} \right) \left( \frac{2\alpha'' + q - 1}{a''} \right) e'_{\alpha'}[n'] e''_{\alpha''}[n'']. $$

22
Properties of $\phi_{\pm}(\tau, n)$ and $\psi_{\pm}(\tau, n)$, continued

The functions $\phi_{\pm}(\tau, n)$ and $\psi_{\pm}(\tau, n)$ satisfy

1. $\phi_{\pm}(\tau, n) = 0$, $\psi_{\pm}(\tau, n) = \psi_{\pm}(-\tau, n)$ and $\psi_{\pm}(\tau, n) = \psi_{\pm}(\tau, -n)$ if $(p, q)$ is odd-even or even-odd;

2. $\phi_{\pm}(\tau, n) = 0$, $\psi_{\pm}(\tau, n) = -i^{d-2}\psi_{\mp}(i\tau, n)$ and $\psi_{\pm}(\tau, n) = \psi_{\pm}(\tau, -n)$ if $(p, q)$ is even-even;

3. $\phi_{\pm}(\tau, n) = i^{d-2}\phi_{\mp}(i\tau, n)$, $\phi_{\pm}(\tau, n) = \phi_{\pm}(\tau, -n)$ and $\psi_{\pm}(\tau, n) = 0$ if $(p, q)$ is odd-odd.

In addition, for any $w \in \Delta(4)$, as functions in $n \in \mathbb{Z}^d$,

$$(-\triangle \mp w)\phi_{\pm}(\sqrt{w}, n) = 0, \quad (-\triangle \mp w)\psi_{\pm}(\sqrt{w}, n) = 0.$$