Essential Spectrum of Schrödinger Operators with no Periodic Potentials on Periodic Metric Graphs

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The main aim of the talk is the investigation of the essential spectrum of the quantum graphs. For this aim we use the \textit{limit operators method} (see for instance the book)


Earlier this method was successfully applied to the study of the essential spectrum of electromagnetic Schrödinger and Dirac operators on $\mathbb{R}^n$ for wide classes of potentials. In particular, a very simple and transparent proof of the Hunziker-van Winter-Zhislin Theorem (HWZ-Theorem) for multi-particle Hamiltonians has been obtained.

The limit operators method also was applied to the study of the location of the essential spectrum of discrete Schrödinger and Dirac operators on $\mathbb{Z}^n$, and on periodic combinatorial graphs.


We consider a periodic metric graph $\Gamma$ embedded in $\mathbb{R}^n$. We suppose that a graph $\Gamma$ consists of a countably infinite set of vertices $\mathcal{V} = \{v_i\}_{i \in \mathcal{I}}$ and a set $\mathcal{E} = \{e_j\}_{j \in \mathcal{J}}$ of edges connecting these vertices. Each edge $e$ is a line segment $[\alpha, \beta] = \{x \in \mathbb{R}^2 : x = (1 - \theta)\alpha + \theta\beta, \theta \in [0, 1]\} \subset \mathbb{R}^2$ connecting its endpoints (vertices $\alpha, \beta$), and we suppose that for the every pair of vertices $\{\alpha, \beta\}$ there exists not more than one edge connecting this pair. Let $\mathcal{E}_v$ be a set of edges incident to the vertex $v$ (i.e., containing $v$). We will always assume that the degree (valence) $d(v)$ (the number of points of $\mathcal{E}_v$) of any vertex $v$ is finite and positive. Vertices with no incident edges are not allowed.
For each edge $e = [\alpha, \beta]$ we assign its length $l_e = \|\alpha - \beta\|_{\mathbb{R}^n} < \infty$. We also suppose that the graph $\Gamma$ is a connected set. The graph is a metric space with a metric induced by the standard metric of $\mathbb{R}^n$. The topology on $\Gamma$ is induced also by the topology on $\mathbb{R}^n$, and the measure $dl$ on $\Gamma$ is the line Lebesgue measure on every edge.
We suppose that on the graph $\Gamma \subset \mathbb{R}^n$ acts a group $G$ isomorphic to $\mathbb{Z}^m, 1 \leq m \leq n$, that is

$$G = \left\{ g \in \mathbb{R}^n : g = \sum_{j=1}^{m} \alpha_j e_j, \alpha_j \in \mathbb{Z}, e_j \in \mathbb{R}^n \right\}$$

where the system $\{e_1, \ldots, e_m\}$ is linear independent. The group $G$ acts on $\Gamma$ by the shifts

$$G \times \Gamma \ni (g, x) \to g + x \in \Gamma,$$

where $g + x$ is the sum of the vectors in $\mathbb{R}^n$. We suppose that the group $G$ acts freely on $X$, that is if $g + x = x$ for some $x \in \Gamma$, then $g = 0$. Moreover we suppose that the action of $G$ on $\Gamma$ is co-compact, that is the fundamental domain $\Gamma_0 = \Gamma/G$ of $\Gamma$ with respect to the action of $G$ on $\Gamma$ is a compact set in the corresponding quotient topology. Let $G_0 \subset \Gamma$ be a measurable set with the compact closure which contains for every $x \in \Gamma$ exactly one element of the quotient class $x + G \in \Gamma/G$. There exists a natural one-to-one mapping $G_0 \to \Gamma/G$ which is the composition of the inclusion mapping $G_0 \subset \Gamma$ and the canonical projection $\Gamma \to \Gamma/G$. 
Let $G_h = G_0 + h, h \in G$. Then

$$G_{h_1} \cap G_{h_2} = \emptyset \text{ if } h_1 \neq h_2,$$

and

$$\bigcup_{h \in G} G_h = \Gamma.$$ 

We say that the graph $\Gamma$ is \textit{periodic with respect to} $G$ if the above given conditions are satisfied.
We denote by $L^2(\Gamma)$ the space of measurable functions on $\Gamma$ with the norm

$$
\|u\|_{L^2(\Gamma)} = \left( \int_{\Gamma} |u(x)|^2 \, dx \right)^{1/2} = \left( \sum_{e \in \mathcal{E}} \int_{e} |u(x)|^2 \, dx \right)^{1/2}
$$

and the scalar product

$$
\langle u, v \rangle = \sum_{e \in \mathcal{E}} \int_{e} u(x) \overline{v}(x) \, dx.
$$
Let $\Gamma \subset \mathbb{R}^n$ be a periodic with respect to $G$ metric graph. We denote by $H^s(e), e \in \mathcal{E}, s \in \mathbb{R}$ the Sobolev space on the edge $e$, and let

$$H^s(\Gamma) = \bigoplus_{e \in \mathcal{E}} H^s(e)$$

with the norm

$$\|u\|_{H^s(\Gamma)} = \left(\sum_{e \in \mathcal{E}} \|u_e\|_{H^s(e)}^2\right)^{1/2}.$$

We denote $\mathcal{E}_v$ the set of edges incident $v$, and let $d(v) \in \mathbb{N}$ be a number of the edges in $\mathcal{E}_v$ (The periodicity of the graph $\Gamma$ implies that $d(v + g) = d(v)$ for every $v \in \mathcal{V}$ and $g \in G$).
We consider the Schrödinger operator on $\Gamma$

$$Hu(x) = -\frac{d^2 u(x)}{dx^2} + q(x)u(x), \ x \in \Gamma \setminus \mathcal{V},$$

(1)

where $q \in L^\infty(\Gamma)$. We provide the operator $H$ by the Kirchhoff-Neumann conditions at the every vertex $v \in \mathcal{V}$.

$$u_e(v) = u_{e'}(v), \text{ if } e, e' \in \mathcal{E}_v, \text{ and } \sum_{e \in \mathcal{E}_v} u'_e = 0$$

(2)

where the orientations of the edges $e \in \mathcal{E}_v$ are taken as outgoing from $v$. 
By the usual way we obtain that

\[ \text{Re} \langle Hu, u \rangle \geq m_q \| u \|_{L^2(\Gamma)}^2, \quad u \in \tilde{H}^2(\Gamma), \quad m_q = \inf_{x \in \Gamma} \text{Re} q(x). \]  

(3)

This property implies that the operator \( H \) provided by the Kirchhoff-Neumann conditions (2) defines an unbounded closed operator \( \mathcal{H} \) in \( L^2(\Gamma) \) with the domain \( \tilde{H}^2(\Gamma) \), and \( \mathcal{H} \) is a selfadjoint operator if the potential \( q \) is a real-valued function.
We recall that a closed unbounded operator $A$ acting in the Hilbert space $X$ with dense domain $D_A$ is called a Fredholm operator if $\ker A$ is a finite dimensional sub-space of $X$, $\text{Im} A$ is closed in $X$, and $X / \text{Im} A$ is a finite-dimensional space. We introduce in $X_1 = D_A$ the norm of the graphics
\[
\|u\|_{D_A} = \left( \|u\|_X^2 + \|Au\|_X^2 \right)^{1/2}.
\] (4)
Since $A$ is closed, $X_1$ is a Banach space. Then $A$ is a Fredholm operator as unbounded operator in $X$ if and only if $A : X_1 \to X$ is a Fredholm operator as a bounded operator.
Note that the norm in $\tilde{H}^2(\Gamma)$ equivalents to the graphic norm in $D_H$

$$\|u\|_{D_H} = \left(\|u\|_{L^2(\Gamma)}^2 + \|Hu\|_{L^2(\Gamma)}^2\right)^{1/2}$$

since the potential $q \in L^\infty(\Gamma)$. Hence the Fredholmness of the operator $\mathcal{H}$ as an unbounded operator in $L^2(\Gamma)$ with domain $\tilde{H}^2(\Gamma)$ is equivalent to the Fredholmness of $\mathcal{H}$ as a bounded operator from $\tilde{H}^2(\Gamma)$ into $L^2(\Gamma)$.

We recall that the essential spectrum $sp_{ess} \mathcal{H}$ of $\mathcal{H}$ is the set of all $\lambda \in \mathbb{C}$ such that the operator $\mathcal{H} - \lambda I$ is not Fredholm operator as unbounded in $L^2(\Gamma)$ with domain $\tilde{H}^2(\Gamma)$. Note that for a self-adjoint operator $\mathcal{H}$

$$sp_{dis} \mathcal{H} = sp \mathcal{H} \setminus sp_{ess} \mathcal{H}.$$
Let $h \in G$. Then the shift (translation) operators

$$V_h u(x) = u(x - h), x \in \Gamma, h \in G$$

are isometric operators in $L^2(\Gamma)$ and $H^2(\Gamma)$. Moreover if $u \in H^2(\Gamma)$ satisfies the Kirchhoff-Neumann conditions at the every vertex $\nu \in \mathcal{V}$ the function $V_h u$ also satisfies these conditions for every $\nu \in \mathcal{V}$. Hence $V_h$ is an isometric operator in $\tilde{H}^2(\Gamma)$. 
Let $G \ni h_k \to \infty$. We consider the family of operators

$$V_{-h_k} \mathcal{H} V_{h_k} : \tilde{H}^2(\Gamma) \to L^2(\Gamma)$$

defined by the Schrödinger operators

$$V_{-h_k} \mathcal{H} V_{h_k} u(x) = \left( -\frac{d^2 u(x)}{dx^2} + q(x + h_k) \right) u(x), \ x \in \Gamma \setminus \mathcal{V}.$$ 

We say that the potential $q \in L^\infty(\Gamma)$ is rich, if for every sequence $G \ni h_k \to \infty$ there exists a subsequence $G \ni g_k \to \infty$ and a limit function $q^g \in L^\infty(\Gamma)$ such that

$$\lim_{k \to \infty} \sup_{x \in K \subset \Gamma} |q(x + g_k) - q^g(x)| = 0 \quad (5)$$

for every compact set $K \subset \Gamma$. 
Example

Let $q \in C_{b,u}(\Gamma)$ the space of bounded uniformly continuous functions on $\Gamma$. If $q \in C_{b,u}(\Gamma)$ the sequence $\{q(x + h_k), x \in \Gamma, h_k \in G\}$ is uniformly bounded and equicontinuous. Then by Arzela-Ascoli Theorem there exists a subsequence $\{q(x + g_k), x \in \Gamma, g_k \in G\}$ such that (5) holds.
Let \( q \in L^\infty(\Gamma) \) be a potential and a sequence \( G \ni g_k \to \infty \) is such that

\[
\lim_{k \to \infty} \sup_{x \in K \subset \Gamma} |q(x + g_k) - q^g(x)| = 0
\]

for every compact set \( K \subset \Gamma \) and a function \( q^g \in L^\infty(\Gamma) \). Then the unbounded in \( L^2(\Gamma) \) operator \( \tilde{H}^g \) with domain \( \tilde{H}^2(\Gamma) \) generated by the Schrödinger operator

\[
H^g u(x) = -\frac{d^2 u(x)}{dx^2} + q^g(x) u(x), \quad x \in \Gamma \setminus \mathcal{V}
\]

is called the limit operator of \( \mathcal{H} \) defined by the sequence \( G \ni g_k \to \infty \). We denote by \( \text{Lim}(\mathcal{H}) \) the set of all limit operators of the the operator \( \mathcal{H} \).
The main result of the talk is:

**Theorem**

Let $\Gamma$ be a periodic with respect to the group $\mathbb{G}$ metric graph and $\mathcal{H}_q$ be a Schrödinger operator in $L^2(\Gamma)$ with domain $\tilde{H}^2(\Gamma)$ with a rich potential $q \in L^\infty(\Gamma)$. Then

$$sp_{ess} \mathcal{H}_q = \bigcup_{\mathcal{H}_{q}^g \in \text{Lim}(\mathcal{H}_q)} sp\mathcal{H}_q^g.$$
Periodic potentials

Let \( \Gamma \) be a graph periodic with respect to the action of the group \( G \)

\[
G = \left\{ g \in \mathbb{R}^n : g = \sum_{j=1}^{m} \alpha_j e_j, \alpha_j \in \mathbb{Z}, e_j \in \mathbb{R}^n \right\},
\]

provided by the Schrödinger operator

\[
H_q u(x) = -\frac{d^2 u(x)}{dx^2} + q(x) u(x), x \in \Gamma \setminus \mathcal{V},
\]

with the potential \( q \in L^\infty(\Gamma) \) periodic with respect to the action of the group \( G \)

\[
q(x + g) = q(x), x \in \Gamma, g \in G.
\]

Since \( \mathcal{H}_q \) is invariant with respect to shifts all limit operators \( \mathcal{H}_q^h \) coincide with \( \mathcal{H}_q \). Hence by Theorem 2

\[
sp_{ess} \mathcal{H}_q = sp \mathcal{H}_q,
\]

and the periodic operator does not have the discrete spectrum.
Let the potential \( q \in L^\infty(\Gamma) \) be a periodic with respect to \( \mathbb{G} \) \textit{real-valued function}. Then \( \mathcal{H}_q \) with domain \( \tilde{H}^2(\Gamma) \) is a self-adjoint operator in \( L^2(\Gamma) \) with the spectrum which has a band structure

\[
\text{sp}\mathcal{H}_q = \text{sp}_{\text{ess}}\mathcal{H}_q = \bigcup_{j=1}^{\infty} [\alpha_j, \beta_j].
\]
Degenerated at infinity perturbations

Let

\[ q = q_0 + q_1, \]

where \( q_0 \in L^\infty(\Gamma) \) is a periodic real-valued function, and \( q_1 \in L^\infty(\Gamma) \) is a real valued functions such that

\[ \lim_{\Gamma \ni x \to \infty} q_1(x) = 0. \]

Then

\[ \mathcal{H}_q^g = \mathcal{H}_{q_0} \]

and hence

\[ sp_{ess} \mathcal{H}_q^g = sp \mathcal{H}_{q_0}. \]

Hence only the discrete spectrum can arise in the gaps of the spectrum of the periodic operator \( \mathcal{H}_{q_0} \) under such sort impurities (perturbations).
We say that a function $a \in C_b(\Gamma)$ is slowly oscillating at infinity and belongs to the class $SO(\Gamma)$ if for every sequence $G \ni g^m \to \infty$

$$\lim_{m \to \infty} \sup_{\{x_1, x_2 \in \Gamma : |x_1 - x_2| \leq 1\}} |a(x_1 + g_m) - a(x_2 + g_m)| = 0. \quad (8)$$

One can prove that $SO(\Gamma) \subset C_{b,u}(\Gamma)$.

Example

Let $f \in C^1_b(\mathbb{R})$, $a(x) = f((1 + |x|)^\alpha)$, $0 < \alpha < 1$, $x \in \mathbb{R}^n$. Then $a \mid_{\Gamma} \in SO(\Gamma)$. 


Let $a \in SO(\Gamma)$. Then every sequence $G \ni h_m \to \infty$ has a subsequence $g_m \in G$ such that for every $x \in \Gamma$ there exists a limit

$$a^g = \lim_{m} a(x + g_m),$$

and $a^g$ independent of $x$. 
We consider potentials of the form

\[ q = q_0 + q_1, \]

where \( q_0 \in L^\infty(\Gamma) \) is a periodic real-valued function, and \( q_1 \) is a real-valued function of the class \( SO(\Gamma) \). Then the potential \( q \) is rich, and all limit operators are of the form

\[ \mathcal{H}^g_q = \mathcal{H}_{q_0 + q_1^g} \]

where \( q_1^g = \lim_{m \to \infty} q(x + g_m) \) and \( q_1^g \in \mathbb{R} \) are independent of \( x \in \Gamma \).
Then

\[ sp\mathcal{H}_q^g = \bigcup_{j=1}^{\infty} \left[ \alpha_j + q_1^g, \beta_j + q_1^g \right]. \]

Let

\[ m_{q_1}^\infty = \liminf_{G \ni g \to \infty} q_1(x + g), \quad M_{q_1}^\infty = \limsup_{G \ni g \to \infty} q_1(x + g), \quad x \in \Gamma, \]

where \( m_{q_1}, M_{q_1} \) are independent of the choice of \( x \in \Gamma \).
Let $m > 1$. Then the set of the partial limits of the function $G g 	o q_1(x + g) \in \mathbb{R}$ is a segment $[m_{q_1}^\infty, M_{q_1}^\infty]$. Applying formula

$$sp_{ess} \mathcal{H}_q = \bigcup_{\mathcal{H}_q \in Lim(\mathcal{H}_q)} sp \mathcal{H}_q^g$$

we obtain that

$$sp_{ess} \mathcal{H}_q = \bigcup_{j=1}^{\infty} \left[ \alpha_j + m_{q_1}^\infty, \beta_j + M_{q_1}^\infty \right].$$
In the case $n = 1$ the set of the partial limits has two components $[m_{q_1}^{\pm\infty}, M_{q_1}^{\pm\infty}]$ and we obtain that

$$sp_{ess} \mathcal{H}_q = \bigcup_{j=1}^{\infty} [\alpha_j + m_{q_1}^{+\infty}, \beta_j + M_{q_1}^{+\infty}] \cup [\alpha_j + m_{q_1}^{-\infty}, \beta_j + M_{q_1}^{-\infty}] .$$
We consider the gaps in the essential spectrum of $\mathcal{H}_q$

$$(\beta_j + M_{q_1}^{\infty}, \alpha_{j+1} + m_{q_1}^{\infty}), j = 1, \ldots, \ldots$$

Let

$$\text{osc}_{\infty}(q_1) = M_{q_1}^{\infty} - m_{q_1}^{\infty} > \alpha_{j_0+1} - \beta_{j_0}. \quad (9)$$

Then the gap $(\beta_{j_0} + M_{q_1}^{\infty}, \alpha_{j_0+1} + m_{q_1}^{\infty})$ disappears. If condition (9) is satisfied for all $j \in \mathbb{N}$ all gaps in the essential spectrum of $\mathcal{H}_q$ are disappear and all bands of the $sp_{ess}\mathcal{H}_q$ are overlapping. Hence

$$sp_{ess}\mathcal{H}_q = [\alpha_1, +\infty),$$

and

$$sp_{dis}\mathcal{H}_q \subset (m_q, \alpha_1 + m_{q_1}^{\infty}).$$
Fredholm theory of bounded operators on graphs

Let $\varphi$ be a function defined on $\mathbb{R}^n$. Then we denote by $\hat{\varphi}$ the restriction of $\varphi$ on the graph $\Gamma$.

**Definition**

We say that $A \in \mathcal{B}(L^2(\Gamma))$ belongs to the class $\mathcal{A}(\Gamma)$ if for every function $\varphi \in C_{b,u}(\mathbb{R}^n)$

$$
\lim_{t \to 0} \| [A, \hat{\varphi}_t I] \|_{\mathcal{B}(L^2(\Gamma))} = \lim_{R \to 0} \| A\hat{\varphi}_t I - \hat{\varphi}_t A \|_{\mathcal{B}(L^2(\Gamma))} = 0. \quad (10)
$$

It is easy to prove that $\mathcal{A}(\Gamma)$ is a $C^*$-subalgebra of $\mathcal{B}(L^2(\Gamma))$.

Let $N \in \mathbb{N}$, $[-N, N]_\mathbb{Z} = \{ \alpha \in \mathbb{Z} : |\alpha| \leq N \}$, and

$$
\mathcal{G}_N = \left\{ g \in \mathbb{R}^m : g = \sum_{i=1}^m \alpha_i e_i, \alpha_i \in [-N, N]_\mathbb{Z} \right\}.
$$
We set

$$\Gamma_N = \bigcup_{g \in G_N} G_g$$

and let $\mathbb{P}_N \in \mathcal{B}(L^2(\Gamma))$ be the operator of the multiplication by the characteristic function of $\Gamma_N$, and $Q_N = I - \mathbb{P}_N$. 
Definition

Let $A \in \mathcal{B}(L^2(\Gamma))$ and $G \ni h_k \to \infty$. An operator $A^h \in \mathcal{B}(L^2(\Gamma))$ is called a limit operator of $A$ defined by the sequence $h_k \in G$, if for every $N \in \mathbb{N}$

$$\lim_{k \to \infty} \left\| \left( V_{-h_k} AV_{h_k} - A^h \right) \mathbb{P}_N \right\|_{\mathcal{B}(L^2(\Gamma))} = 0,$$

$$\lim_{k \to \infty} \left\| \mathbb{P}_N \left( V_{-h_k} AV_{h_k} - A^h \right) \right\|_{\mathcal{B}(L^2(\Gamma))} = 0.$$

We say that the operator $A$ is rich if every sequence $G \ni h_k \to \infty$ has a subsequence $G \ni g_k \to \infty$ defining a limit operator $A^g$. We denote by $\text{Lim}(A)$ the set of all limit operators of $A$. 

(Institute) Essential Spectrum of Schrödinger Operators
**Definition**

An operator $A \in \mathcal{B}(L^2(\Gamma))$ is called locally invertible at infinity if there exist $R \in \mathbb{N}$ and operators $\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}(L^2(\Gamma))$ such that

$$\mathcal{L}_R A \mathcal{Q}_R = \mathcal{Q}_R, \mathcal{Q}_R A \mathcal{R}_R = \mathcal{Q}_R.$$ 

**Theorem**

*Let $A \in \mathcal{A}(\Gamma)$ and be rich. Then $A$ is locally invertible at infinity if and only if all limit operators $A^h \in \text{Lim}(A)$ are invertible in $L^2(\Gamma)$.***
**Definition**

We say that $A \in \mathcal{B}(L^2(\Gamma))$ is a locally Fredholm operator if for every $R \in \mathbb{N}$ there exits operators $\mathcal{L}_R, \mathcal{R}_R$ such that

$$\mathcal{L}_R A \mathcal{P}_R = \mathcal{P}_R + T^1_R, \quad \mathcal{P}_R A \mathcal{R}_R = \mathcal{P}_R + T^2_R,$$

where $T^j_R \in \mathcal{K}(L^2(\Gamma)), j = 1, 2$.

**Theorem**

Let $A \in \mathcal{A}(\Gamma)$. Then $A$ is a Fredholm operator in $L^2(\Gamma)$ if and only if:

(i) $A$ is a locally Fredholm operator;
(ii) All limit operators $A^h \in Lim(A)$ are invertible.
Corollary

Let \( A \in \mathcal{A}(\Gamma) \), and \( A \) be a locally Fredholm operator. Then

\[
sp_{ess} A = \bigcup_{A^h \in \text{Lim}(A)} sp A^h, \quad (12)
\]

where \( sp_{ess} A \) is the essential spectrum of \( A \) in \( L^2(\Gamma) \) that is the set of \( \lambda \in \mathbb{C} \) such that \( A - \lambda I \) is not Fredholm operator in \( L^2(\Gamma) \).
The proof of the main theorem on the essential spectrum of quantum graphs is reduced to the this corollary. We denote by $\Lambda$ the unbounded operator generated by the Schrödinger operator $-\frac{d^2}{dx^2}$ on $\Gamma \setminus V$ with domain $\tilde{H}^2(\Gamma)$. Note that $\Lambda$ is a nonnegative self-adjoint operator in $L^2(\Gamma)$ and $sp\Lambda \subset [0, \infty)$. Hence the operator $\Lambda_k = \Lambda + k^2 I : \tilde{H}^2(\Gamma) \to L^2(\Gamma)$ is an isomorphism. Then we prove that

$$A = \mathcal{H}_q \Lambda_k^{-1} \in \mathcal{A}(\Gamma), \text{Lim}(A) = \text{Lim}(\mathcal{H}_q),$$

$$sp_{ess} A = sp_{ess} \mathcal{H}_q,$$

and the theorem on the essential spectrum of the operator $\mathcal{H}_q$ as unbounded in $L^2(\Gamma)$ follows from Corollary 10.