A random matrix approach to many-body localization

François Huveneers (Paris Dauphine)
Collaboration with Wojciech De Roeck (KU Leuven)

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Localized vs Ergodic

Spin chain in d=1, e.g. XXZ chain:

$$H = \sum_{i=1}^{L} h_i S_i^z + J_{\perp} \sum_{i=1}^{L-1} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + J_z \sum_{i=1}^{L-1} S_i^z S_{i+1}^z$$
Many-body localization (MBL)

- Many-body analog of Anderson localization
- Strong breakdown of ergodicity
- No thermalization

E.g.: quantum quench:

\[ E(T_L) \quad \quad E(T_R) \]

wait...

\[ E(T_L) + \mathcal{O}(\varepsilon) \quad \quad E(T_R) + \mathcal{O}(\varepsilon) \]

Perturbation around Anderson insulator: \( J_Z/W \ll 1 \)

See Gornyi, Mirlin, Polyakov ('05), Basko, Aleiner, Altshuler ('06), Oganesyan, Huse ('07), Imbrie ('12), …
Transition expected as \( J/W > \) critical value

Eigenstate Thermalization Hypothesis: each eigenstate is thermal

\[
\langle E | O_i | E \rangle = \langle O_i \rangle_{T(E)} + \mathcal{O}(e^{-cL})
\]

Connection with Random Matrices via Berry’s conjecture: Eigenstates ‘ressemble’ eigenstates of a RM

More details and refinements later
Small bath connected to disordered spins

Finite bath + localized spins

Aim: Find a description of the interface region
Natural set-up

Disorder fluctuations create small baths

Ergodic spots

Stability issue: Could the interface region be of the order of the full volume?
Avalanche scenario

(too) naive scenario:

Ergodic spot thermalizes the near spins...

... and becomes a larger spot

Eventually, the full material becomes thermal!

• d=1: ‘good’ models: Fallacy! — see Imbrie (’12)
• d>1: open question
Our approach

Couple spins one by one to a small bath

Random Matrix + Energy conservation
1 spin coupled to a bath

Single spin coupled to an ergodic Hamiltonian:

\[ H = H_B + H_{BS} + H_S \]

\[ H_B = \text{Ergodic Ham}, \quad H_{BS} = g \sigma_B^{(x)} \sigma_S^{(x)}, \quad H_S = h \sigma_S^{(z)} \]

How do uncoupled states hybridize to form the eigenstates?

\[ |E\rangle = \sum_{b,s} \phi_E(b, s) |b, s\rangle \]
I. Local operators in the bath (ETH)

A local operator is characterized by 3 energy scales:

\[
\langle b' | \sigma_B | b \rangle = \sqrt{\delta} \times \sqrt{v(\omega)} \times \eta_{b,b'}, \quad \omega = E(b') - E(b)
\]

- **Level spacing**
  - (global scale)
- **Structure factor**
  - (scale: energy per site)
- **‘Random’ phases**
  - (scale: level spacing)

\[
\int d\omega \, v(\omega) \sim 1, \quad \text{local interactions: } v(\omega) \sim e^{-|\omega|/\varepsilon}
\]
II. Condition for thermalization

The Bath is not ‘perfect’: strong enough coupling needed

\[ G := \frac{\text{Matrix element}}{\text{Level spacing}} = \frac{\langle b', s' \rvert H_{BS} \rvert b, s \rangle}{\delta/2} \]

\[ \sim \frac{g \times \sqrt{v(\omega)} \times \sqrt{\delta}}{\delta/2}, \quad \omega = E_{b'} + E_{s'} - E_b - E_s \]

\[ G \ll 1 : \text{product structure approximatively preserved} \]

\[ G \gg 1 : \text{hybridization, the spin gets ‘thermalized’} \]
III. Energy conservation

States only hybridize within some energy window

\[ |E_B + E_S - E| \lesssim w \]

Fermi Golden Rule:

\[ w \sim g^2 v(2h) \]

Upper bound on \( w \) can be justified through Feshbach formula
III. Energy conservation (pictorially)
Cartoon for the eigenstates of $H$

$$|E\rangle = \sum_{b,s} \phi_E(b,s)|b,s\rangle$$

$$\phi_E(b,s) = \sqrt{\delta/2} \times \sqrt{k(\omega)} \times \eta_E(b,s)$$

Full level spacing  Hybridization function  ‘Random’ phase

$$\omega = E_B + E_S - E$$

$$k(\omega) \sim \frac{1}{w(1 + (\omega/w)^2)}$$

Some refinements will be needed later…
E.g.: bath with 11 spins
New structure factors

- Crucial to iterate the procedure
- We distinguish between bath / spin operators

Common to both: ETH on the **full** Hilbert space:

\[ \langle E' | \sigma_i | E \rangle = \sqrt{\frac{\delta}{2}} \times \sqrt{v(\omega)} \times \eta_{E,E'}, \quad \omega = E' - E \]

Different for \( i \in B \) or \( i = S \)
In first approximation, structure factors do not change.

!!! Dimension is twice as big as without extra spin:

The bath is now ‘closer to being an ideal bath’
Bath operator (refinement)

This cannot be precisely true… Structure factors have a dynamical meaning:

\[ v(\omega) = \int dt e^{-i\omega t} \langle \sigma_i(t) \sigma_i \rangle_{\beta(E)} \]

Convergence to equilibrium of a bath operator is (weakly) affected by the (weakly) coupled spin.

The back-reaction from the spin to the bath can be estimated. It leads to a ‘small’ correction. Debated issue!
Spin operator

The structure factor exhibits strong frequency dependence

\[ V(\omega) \]

\[ \text{g}^2 / W \]

\[ -2h \quad 2h \]
Coupling a second spin

\[
\text{New Bath} + \text{spin}_2 = \text{Bath} + \text{spin}_1 + \text{spin}_2
\]

Coupling is suppressed due to frequency mismatch:

\[
|E(b') + E(s_2) - E| \lesssim w
\]

\[
w \sim g^2v(2h_2) \quad (v: \text{structure factor of spin 1})
\]
Structure factor of the second spin

Much more narrow than that of the first spin:

\[ v(\omega) \]

\[ g(g/W)^2 \]

(from estimates on the tails of the structure factor of spin 1)
How far does this go \((d=1)\)?

This process stops when the width of the structure factors becomes of the order of the level spacing.

With \(n\) extra spins:

- width: \((g/W)^n = e^{-n/\xi}\)
- level spacing: \(2^{-(L_B+n)}\)

The size of the boundary region is thus given by

\[
n = \frac{\log 2}{\xi^{-1} - \log 2} \times L_B
\]

provided that \(\xi < \left(\log 2\right)^{-1}\)
We imagine coupling layers of spins:

When \( n \) layers are coupled:

Width of the structure factors: \( (g/W)^n =: e^{-n/\xi} \)

Level spacing: \( 2^{-(\ell_B+n)^2} \)

If \( \ell_B > \frac{1}{4\xi \log 2} \), the hybridization process never stops!

And for \( d>1 \) ?

See also Nandkishore et al., PRB 90, 2014
Very long time scales for thermalization

\[ t_{\text{Therm}} \sim e^{d/\xi} \]

\[ d \sim e^{\xi^{-3}} \]

Transport due to this mechanism is astronomically reduced as the bare localization length is sent to 0!
Concrete measurement: IPRs

- Peaks in the structure factors reduce the dimension on the space on which a local operator acts effectively
- IPR: parameter to quantify this effective dimension

Fix an energy $E$:

$$\text{IPR}(O_i) := -\log \sum_{E'} |\langle E'|O_i|E\rangle|^4$$

Ergodic: $\text{IPR}(O_i) \sim L$

MBL: $\text{IPR}(O_i) \sim 1$
Prediction for IPRs

\[ \text{IPR}(O_i) \]

\[ \log 2 (\ell_B + c\ell_B) \]

\[ \log 2 \times \ell_B \]

\[ \ell_B \]

\[ \ell_B + c\ell_B \]

Coupled

Uncoupled
Localized spins already diagonalized (LIOMs): No interaction among themselves, only bath-spin interaction via tails

\[ H = H_B + \sum_{i=1}^{\ell} (g_i \sigma_B^{(x)} \sigma_i^{(x)} + h_i \sigma_i^{(z)}) \]

- **\( H_B \)**: Bath Hamiltonian (Random Matrix or Ergodic)
- **\( g_i \)**: \( g_i = 0.2 \times (0.75)^i \)
- **\( 0.5 < h_i < 1.5 \) (random)**
IPRs for operators in the bath

\[ \text{IPR} \left( \sigma_0^{(x)} \right) \text{ for a 6 spins RM bath} \]

\[ \text{IPR} \left( \sigma_0^{(x)} \right) \text{ for a 1 spin bath} \]

Slopes of the above curves

Theoretical prediction: \[ \log 2 \]
Theory predicts linear decrease of IPR outside the bath
Universal features of thermalization

Thermalization of an external spin by three different baths:

- Random matrix \( \text{dim} = 2^9 \)
- 9 spins bath
- 9 spins bath

To compare them, replace \( g \) by the dimensionless parameter:

\[ g = \frac{\text{Matrix element}}{\text{Level spacing}} \]
Universal feature of thermalization

IPR of $\sigma_B^{(x)}$ - constant

- 4 LIOMs coupled to a 9 spins ergodic ham.
- 1 LIOM coupled to a 9 spins ergodic ham.
- 1 LIOM coupled to a “9 spins” RM bath
- Same as with 3 uncoupled LIOMs

$g = \frac{\text{Matrix element}}{\text{Level spacing}}$
Conclusions

• Qualitative description of the boundary region bath / MBL
• Quantitative description through a simple parameter (IPR)
• Universal features of thermalization
• (In)stability of MBL in $d>1$ or with ‘long range’ hopping?

See W. De Roeck, F. Huveneers, arXiv:1608.01815